HW3

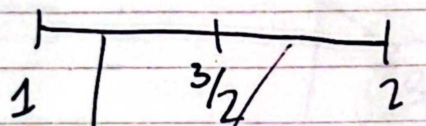
5.1: 2, 6, 7, 8, 9, 26

5.3: 11, ~~13~~ 12

5.4: 2, 3, 5, 6, 7, 9, 12, 16

$\boxed{5.1}$

2. $\int_1^2 \frac{1}{x}\,dx \quad P = \left\{1, \frac{3}{2}, 2\right\}$



$$\frac{1}{2}\underbrace{\left(\frac{3}{2}-1\right)}_{1/2}\underbrace{\left(\frac{1}{1}+\frac{1}{\left(\frac{3}{2}\right)}\right)}_{\substack{1+2/3 \\ 5/3}} = \frac{1}{2}\,\frac{1}{2}\,\frac{5}{3} = \frac{5}{12}$$

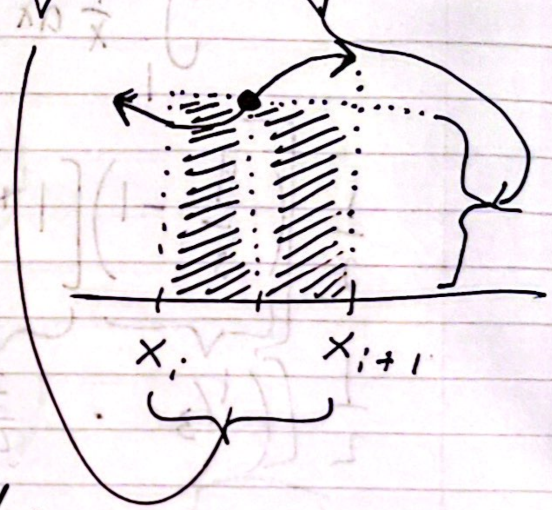
$$\frac{1}{2}\underbrace{\left(2-\frac{3}{2}\right)}_{1/2}\underbrace{\left(\frac{1}{\left(\frac{3}{2}\right)}+\frac{1}{2}\right)}_{\substack{\frac{2}{3}+\frac{1}{2} \\ \frac{4}{6}+\frac{3}{6} \\ \frac{7}{6}}} = \frac{1}{2}\,\frac{1}{2}\,\frac{7}{6} = \frac{7}{24}$$

$$\frac{5}{12} + \frac{7}{24} = \boxed{\frac{17}{24}}$$

6. Establish that the <u>composite midpoint rule</u> for estimating an integral is

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} \underbrace{(x_{i+1}-x_i)}\, f\Big[\underbrace{\tfrac{1}{2}(x_{i+1}+x_i)}\Big]$$

Yes, this makes sense:

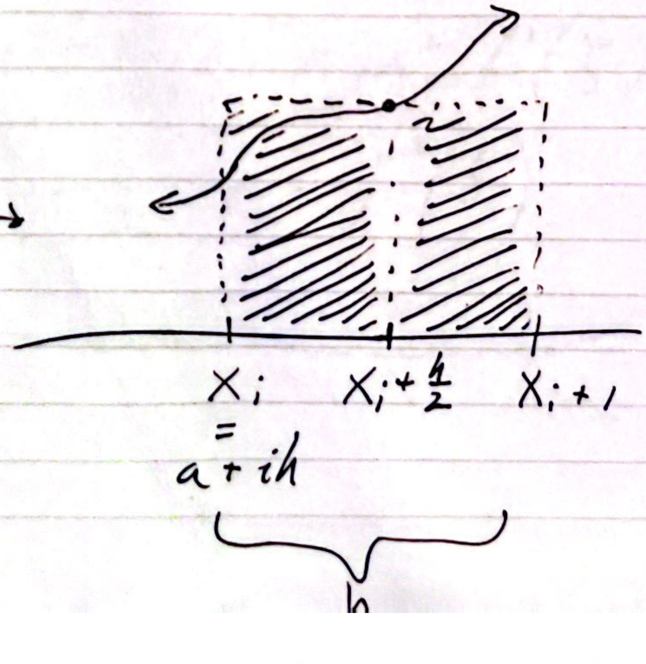


7. with equal subintervals:

$$\int_a^b f(x)\,dx \approx h\sum_{i=0}^{n-1} f\left(x_i + \tfrac{1}{2}h\right)$$

$$h = (b-a)/n,\quad x_i = a + ih,\quad 0 \le i \le n.$$

established! →



8. composite trap. rule of $f(x) = \frac{1}{x}$

pts: $1, \frac{4}{3}, 2.$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{2}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left[f(x_i)+f(x_{i+1})\right]$$

$$\frac{1}{2}\left[\underbrace{\left(\frac{4}{3}-1\right)}\underbrace{\left[1+\frac{3}{4}\right]}+\underbrace{\left(2-\frac{4}{3}\right)}\underbrace{\left[\frac{3}{4}+\frac{1}{2}\right]}\right]$$

$$\frac{1}{2}\left[\left(\frac{1}{3}\cdot\frac{7}{4}\right)+\left(\frac{2}{3}\cdot\frac{5}{4}\right)\right]$$

$$\frac{1}{2}\left[\frac{7}{12}+\frac{10}{12}\right]=\frac{17}{24}$$

5.1 cont.

9 compute $\int_0^1 \frac{1}{(x^2+1)}dx$

$$\tan^{-1}(x)\Big|_0^1 = \boxed{\frac{\pi}{4}}$$

$p = (0, \frac{1}{2}, 1)$ I'm using uniform spacing

$$\frac{\left(\frac{1}{2}\right)}{2}\left(f(0)+f(1)\right) + \frac{1}{2}\left(f\left(\frac{1}{2}\right)\right)$$

$$\Rightarrow \frac{1}{4}\left(1+\frac{1}{2}\right) + \frac{1}{2}\left(\frac{4}{5}\right)$$

$$\Rightarrow \frac{3}{8} + \frac{2}{5} = \frac{15}{40} + \frac{16}{40} = \boxed{\frac{31}{40}} = 0.775$$

approximation : 0.775

exact : $\frac{\pi}{4} \approx 0.785398$

error : $-\frac{1}{12}(b-a)h^2 f''(\xi)$

$$-\frac{1}{12}\cdot\frac{1}{4}\cdot f''(\xi)$$

5.1 cont.

26. Show that there exist coefficients $w_0, w_1, \ldots, w_n$ depending on $x_0, x_1, \ldots, x_n$ & on a, b s.t.

$$\int_a^b p(x)\,dx = \sum_{i=0}^{n} w_i\, p(x_i)$$

for all polynomials p of degree $\leq n$.

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

$$p_n(x) = y_0 \frac{(x-x_1)(x-x_{\ldots})(x-x_n)}{(x_0-x_1)(x_0-x_{\ldots})(x_0-x_n)} + y_1$$

$$w_i = \int_a^b l_i(x)\,dx$$

$p(x) = \sum_{i=0}^{n} l_i\, p(x_i)$ where l_i is one of these

$$\int_a^b p(x)\,dx = \int_a^b \sum_{i=0}^{n} l_i\, p(x_i)\,dx$$

$$= \sum_{i=0}^{n} p(x_i) \underbrace{\int_a^b l_i\,dx}_{\text{this is } w_i}$$

$\boxed{5.3}$

11. $\int_0^1 f(x)\,dx \approx \alpha \overbrace{f(0)}^{1 \text{ or } 0} + \beta \overbrace{f(1)}^{1 \text{ or } 1}$

$f(x) = 1 \to 1$

$\alpha + \beta = 1$

$\alpha \to \frac{2}{3}$

$f(x) = x^2$

$\to \int_0^1 x^2\,dx$

$\frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$\beta = \frac{1}{3}$

$$\boxed{\int_0^1 f(x)\,dx \approx \frac{2}{3} f(0) + \frac{1}{3} f(1)}$$

$$\int_0^1 x\,dx = \frac{1}{2} \overset{?}{\approx} \frac{1}{3}$$

no, it doesn't work for $f(x) = x$

5.3 cont.

12. $\int_{-1}^{1} f(x)\,dx \approx \alpha f(-1) + \beta f(0) + \gamma f(1)$

$f(x)$	x	x^2	x^3
$\int_{-1}^{1} f(x)\,dx$	$\frac{x^2}{2}\Big\|_{-1}^{1} = 0$	$\frac{x^3}{3}\Big\|_{-1}^{1} = \frac{2}{3}$	0
	$-\alpha + \gamma = 0$	$\alpha + \gamma = \frac{2}{3}$	$-\alpha + \gamma = 0$

$\alpha = \frac{1}{3} \qquad \gamma = \frac{1}{3}$

$$\boxed{\frac{1}{3} f(-1) + 0 f(0) + \frac{1}{3} f(1)}$$

Does it work with...

$x \mapsto 1$ no

$x \mapsto x^4$ ~~no~~ no

$x \mapsto x^5$ yes

5.4) 2. Show directly that the Gaussian quadrature rule is exact polynomials

$1, x, x^2, \ldots, x^{2n+1}$

when:

a. $n=1$

$1, x, x^2, x^3$