# Linear Regression

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# Multiple Regression Model: Overview

# Motivation

Often a number of variables affect the response variable in important and distinctive ways such that any single one wouldn't have provided an adequate description. E.g.,

- The weight of a person may be affected by height, gender, age, diet, etc.
- The income of a person may be affected by age, gender, years of education, etc.
- The body fat of a person may be associated with age, gender, weight, height, etc.

# Multiple Regression Model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_{p-1} X_{i,p-1} + \epsilon_i, \quad i = 1, \cdots n$$

- $Y_i$ : value of the response variable in the *ith* case
- $X_{i1}, \cdots, X_{i,p-1}$ : values of the $X$ variables in the *ith* case
- $\beta_0, \beta_1, \cdots, \beta_{p-1}$: regression coefficients
- $\epsilon_i$: random errors

$$E(\epsilon_i) = 0, \quad Var(\epsilon_i) = \sigma^2, \quad Cov(\epsilon_i, \epsilon_j) = 0 \text{ for } i \neq j$$

- *Response function (surface)/ mean response*:

$$E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_{p-1} X_{p-1}$$

# First-Order (Additive) Models

$X_1, \cdots, X_{p-1}$ represent $p-1$ **distinct** predictor variables.

- $\beta_k$ indicates the change in mean response $E(Y)$ with a unit increase in the predictor $X_k$, when all other predictors are held constant.
- This change is the same irrespective of the levels at which other predictors are held.
- **The effects of the predictor variables are additive (without interactions).**

# Models with Interactions

Sometimes the effect of one predictor depends on the value(s) of the other predictor(s), i.e., the effects are **non-additive or interacting**.

- How education level affects income may depend on gender.
- Interactions are often represented by cross product terms among predictors.

# Non-additive Model with Two Predictors

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i1} X_{i2} + \epsilon_i, \quad i = 1, \cdots, n.$$

- This model is in the form of the multiple regression model with $p - 1 = 3$ by defining $X_{i3} := X_{i1} X_{i2}$.
- The mean response $E(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2$ is linear in the parameters $\beta_0, \beta_1, \beta_2$, but is not linear in the original predictors $X_1, X_2$.

# Example

| Brand-liking (Y) | Moisture (X1) | Sweetness (X2) |
|---|---|---|
| 64.0 | 4.0 | 2.0 |
| 73.0 | 4.0 | 4.0 |
| 61.0 | 4.0 | 2.0 |
| 76.0 | 4.0 | 4.0 |
| ... | ... | ... |

Design matrix of a first-order model: $\mathbf{x} = \begin{bmatrix} 1 & 4.0 & 2.0 \\ 1 & 4.0 & 4.0 \\ 1 & 4.0 & 2.0 \\ 1 & 4.0 & 4.0 \\ \vdots & \vdots & \vdots \end{bmatrix}$

Design matrix of a non-additive model: $\mathbf{x} = \begin{bmatrix} 1 & 4.0 & 2.0 & 8.0 \\ 1 & 4.0 & 4.0 & 16.0 \\ 1 & 4.0 & 2.0 & 8.0 \\ 1 & 4.0 & 4.0 & 16.0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

# Polynomial Regression Models

These models contain quadratic and/or higher-order terms of the predictor variable(s), making the response function curvilinear with respect to the predictor(s).

- 2nd-order polynomial regression model with one predictor:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \epsilon_i, \quad i = 1, \cdots, n.$$

  - By defining, $X_{i1} := X_i, X_{i2} := X_i^2$, this model is in the form of the multiple regression model with $p - 1 = 2$.

# Example

| Case | Salary | Experience |
|---|---|---|
| 1 | 71 | 26 |
| 2 | 69 | 19 |
| 3 | 73 | 22 |
| 4 | 69 | 17 |
| 5 | 65 | 13 |
| 6 | 75 | 25 |
| ... | ... | ... |

Design matrix of a 2nd-order polynomial regression model:

$$\mathbf{X} = \begin{bmatrix} 1 & 26 & 26^2 \\ 1 & 19 & 19^2 \\ 1 & 22 & 22^2 \\ 1 & 17 & 17^2 \\ 1 & 13 & 13^2 \\ 1 & 25 & 25^2 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

# Models with Transformed Variables

- Model with logarithm-transformed response variable:

$$\log Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_{p-1} X_{i,p-1} + \epsilon_i, \quad i = 1, \cdots n.$$

  - This model is in the form of the multiple regression model by defining $Y_i^* := \log Y_i$.

# What Makes a LINEAR Regression Model?

The response function is linear in the regression coefficients: $\beta_0, \beta_1, \cdots, \beta_{p-1}$. However, the response function does not need to be linear in the **original predictors**.

- In contrasts, **nonlinear regression models** are nonlinear in the parameters.
- The model below can not be expressed in the form of a linear regression model through transformations or introducing new $X$ variables:

$$Y_i = \beta_0 \exp(\beta_1 X_i) + \epsilon_i, \quad i = 1, \cdots n.$$

# Multiple Regression: Example

# Data

30 cases, one response variable and three predictor variables:

| case | Y | X1 | X2 | X3 |
|---|---|---|---|---|
| 1 | 3.01 | 1.06 | 0.86 | -1.23 |
| 2 | -3.40 | -0.30 | -0.08 | -0.48 |
| 3 | 2.74 | 1.05 | 0.22 | -0.40 |
| ... | ... | ... | ... | ... |
| 30 | -1.42 | 2.12 | -0.8 | -0.62 |

- First examine each variable marginally: Variable type, summary statistics, histogram, boxplot, pie chart, missing values? outliers? etc.
- Then explore their relationships through pairwise scatter plots.

# Scatter Plot Matrix

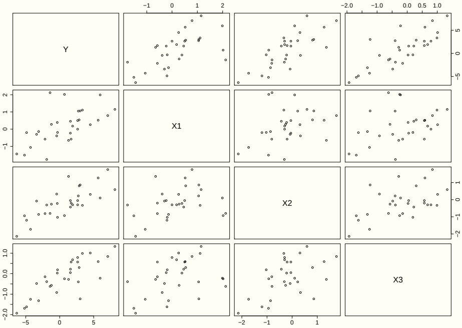



Figure: Pairwise scatter plots

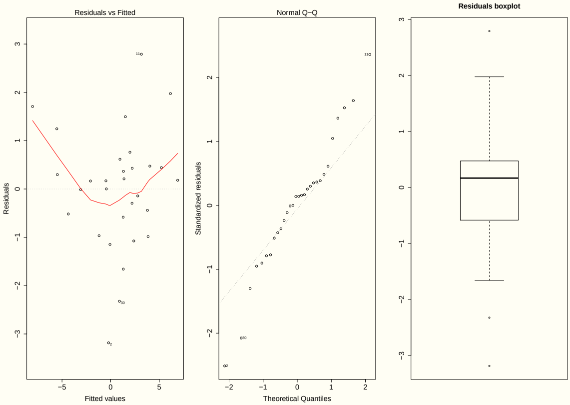
All variables appear to be positively related. No obvious nonlinearity.

# Model 1: First-order Model

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \epsilon_i, \quad i = 1, \cdots, 30.$$

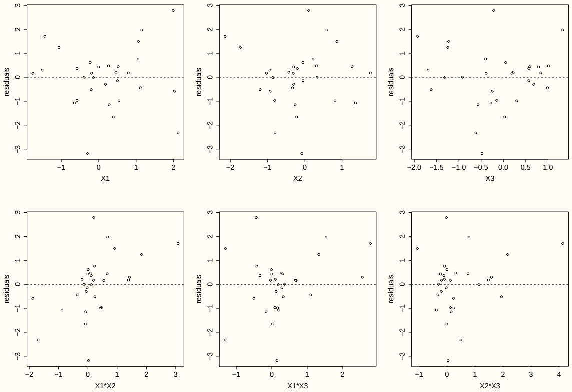
```
Call:
lm(formula = Y ~ X1 + X2 + X3, data = data)

Coefficients:
Estimate  Std. Error t value    Pr(>|t|)
(Intercept)   1.2010     0.2541    4.727   6.91e-05 ***
X1            1.1107     0.2672     4.156   0.000311 ***
X2            1.7978     0.3287     5.469   9.78e-06 ***
X3            1.9596     0.3362     5.829   3.83e-06 ***
---

Residual standard error: 1.299 on 26 degrees of freedom
Multiple R-squared: 0.8883,     Adjusted R-squared: 0.8754
F-statistic: 68.93 on 3 and 26 DF,  p-value: 1.667e-12
```

Figure: Model 1: residual plots




Residual vs. fitted value plot shows nonlinearity. Residual Q-Q plot shows heavy-tail. Residual boxplot shows range from $-3$ to 3.

Figure: Model 1: residual vs. interaction terms



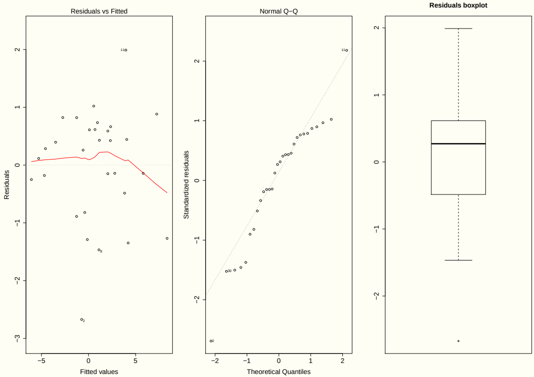

Residual vs. $X_1 X_2$ shows a clear linear pattern $\rightarrow$ this term should be included in the model.

# Model 2: Nonadditive Model with Interaction $X_1X_2$

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i1} X_{i2} + \epsilon_i, \quad i = 1, \cdots, 30.$$

```
Call:
lm(formula = Y ~ X1 + X2 + X3 + X1:X2, data = data)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.8832     0.2153   4.103  0.00038 ***
X1            1.5946     0.2421   6.587 6.69e-07 ***
X2            1.7091     0.2605   6.560 7.16e-07 ***
X3            2.1266     0.2687   7.916 2.85e-08 ***
X1:X2         1.0076     0.2467   4.084  0.00040 ***
---

Residual standard error: 1.026 on 25 degrees of freedom
Multiple R-squared: 0.933,      Adjusted R-squared: 0.9223
F-statistic: 87.04 on 4 and 25 DF,  p-value: 2.681e-14
```

Figure: Model 2: residual plots

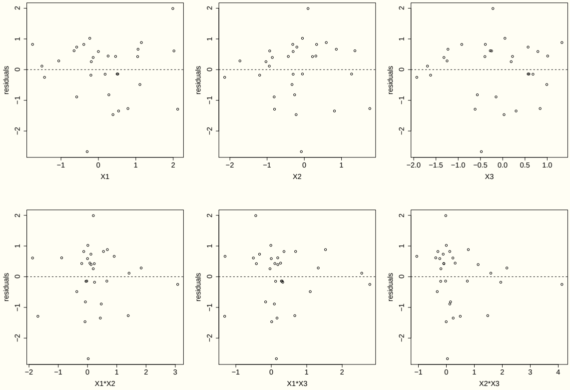



Residual vs. fitted value plot shows no obvious nonlinearity.

Residual Q-Q plot shows no severe deviation from Normality.

Residual boxplot shows range from $-2$ to 2.

Figure: Model 2: residual vs. interaction terms



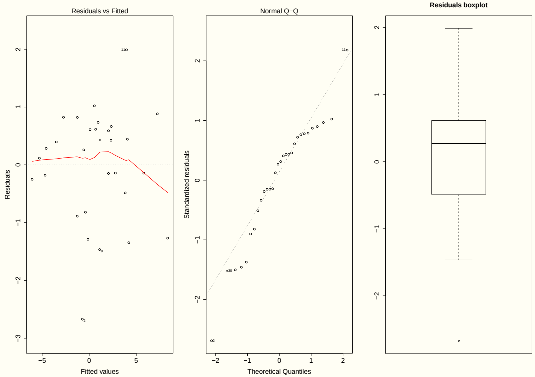

None of these plots shows an obvious pattern → Model 2 appears adequate.

# Model 3: Nonadditive Model with All Two-way Interactions

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i1} X_{i2} + \beta_5 X_{i1} X_{i3} + \beta_6 X_{i2} X_{i3} + \epsilon_i, \quad i = 1, \cdots, 30.$$

```
Call:
lm(formula = Y ~ X1 + X2 + X3 + X1:X2 + X1:X3 + X2:X3, data = data)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.8927     0.2278   3.920 0.000687 ***
X1            1.7179     0.2819   6.095 3.24e-06 ***
X2            1.5828     0.2925   5.411 1.69e-05 ***
X3            1.9982     0.3041   6.571 1.05e-06 ***
X1:X2         1.1925     0.3368   3.541 0.001744 **
X1:X3         0.2227     0.4009   0.555 0.583989
X2:X3        -0.4403     0.3675  -1.198 0.243074
---

Residual standard error: 1.038 on 23 degrees of freedom
Multiple R-squared: 0.937,      Adjusted R-squared: 0.9205
F-statistic: 56.99 on 6 and 23 DF,  p-value: 1.172e-12
```

Figure: Model 3: residual plots

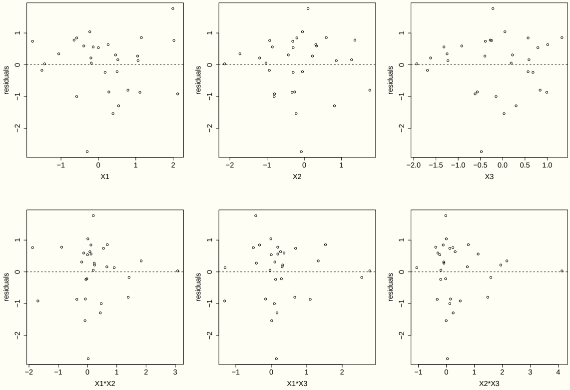



Residual vs. fitted value plot shows no obvious nonlinearity.

Residual Q-Q plot shows no severe deviation from Normality.

Residual boxplot shows range from $-2$ to 2.

Figure: Model 3: residual vs. interaction terms




None of these plots shows an obvious pattern → Model 3 appears adequate, but there is also no obvious improvement over Model 2.

# Multiple Regression:

# Matrix Form

# Model Equations

$$\underset{n\times 1}{\mathbf{Y}} = \underset{n\times p}{\mathbf{X}}\ \underset{p\times 1}{\boldsymbol{\beta}} + \underset{n\times 1}{\boldsymbol{\epsilon}}$$

$$\underset{n\times p}{\mathbf{X}} := \begin{bmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1,p-1} \\ 1 & X_{21} & X_{22} & \cdots & X_{2,p-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & X_{i1} & X_{i2} & \cdots & X_{i,p-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & X_{n1} & X_{n2} & \cdots & X_{n,p-1} \end{bmatrix}, \quad \underset{p\times 1}{\boldsymbol{\beta}} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{bmatrix}.$$

Each row of **X** corresponds to a case and each column of $X$ corresponds to an $X$ variable.

# Model Assumptions

$$\mathbf{E}\{\boldsymbol{\epsilon}\} = \mathbf{0}_n, \quad \boldsymbol{\sigma}^2\{\boldsymbol{\epsilon}\} = \sigma^2\mathbf{I}_n.$$

- In terms of the observations:

$$\mathbf{E}\{\mathbf{Y}\} = \mathbf{X}\boldsymbol{\beta}, \quad \boldsymbol{\sigma}^2\{\mathbf{Y}\} = \sigma^2\mathbf{I}_n.$$

- Under the Normal error model, $\boldsymbol{\epsilon}$ and $\mathbf{Y}$ are vectors of independent normal random variables.

# Least Squares Estimators

- Least squares criterion:

$$
\begin{aligned}
Q(\mathbf{b}) &= \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_{i1} - \cdots - b_{p-1} X_{i,p-1})^2 \\
&= (\mathbf{Y} - \mathbf{X}b)' (\mathbf{Y} - \mathbf{X}b), \quad \underset{p\times 1}{\mathbf{b}} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-1} \end{bmatrix}.
\end{aligned}
$$

- Differentiate $Q(\cdot)$ and set the gradient to zero $\Longrightarrow$ normal equation: $\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{Y}$.

LS estimators are solutions of the normal equation:

$$\underset{p\times 1}{\hat{\boldsymbol{\beta}}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_{p-1} \end{bmatrix} = \underset{p\times p}{(\mathbf{X}'\mathbf{X})^{-1}} \underset{p\times n}{\mathbf{X}'} \underset{n\times 1}{\mathbf{Y}}.$$

- $\hat{\boldsymbol{\beta}}$ is an unbiased estimator for $\boldsymbol{\beta}$:

$$\mathbf{E}\{\hat{\boldsymbol{\beta}}\} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{E}\{\mathbf{Y}\} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \boldsymbol{\beta}.$$

- Variance-covariance matrix of $\hat{\boldsymbol{\beta}}$:

$$\boldsymbol{\sigma}^{\mathbf{2}}\{\boldsymbol{\beta}\} = \sigma^2 \underset{p\times p}{(\mathbf{X}'\mathbf{X})^{-1}}.$$

# Fitted Values and Residuals

$$\underset{n\times 1}{\widehat{\mathbf{Y}}} := \begin{bmatrix} \widehat{Y}_1 \\ \widehat{Y}_2 \\ \vdots \\ \widehat{Y}_n \end{bmatrix} = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \mathbf{H}\mathbf{Y}, \quad \underset{n\times 1}{\mathbf{e}} := \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \mathbf{Y} - \widehat{\mathbf{Y}} = (\mathbf{I}_n - \mathbf{H})\mathbf{Y}.$$

$$\mathbf{E}\{\widehat{\mathbf{Y}}\} = \mathbf{X}\boldsymbol{\beta} = \mathbf{E}\{\mathbf{Y}\}, \quad \boldsymbol{\sigma}^{\mathbf{2}}\{\widehat{\mathbf{Y}}\} = \sigma^2\mathbf{H}.$$

$$\mathbf{E}\{\mathbf{e}\} = \mathbf{E}\{\mathbf{Y}\} - \mathbf{E}\{\widehat{\mathbf{Y}}\} = \mathbf{0}_n, \quad \boldsymbol{\sigma}^{\mathbf{2}}\{\mathbf{e}\} = \sigma^2(\mathbf{I}_n - \mathbf{H}).$$

- Linear transformations of the observations vector **Y**
- Under the Normal error model, they are normally distributed

# Hat Matrix

$$\underset{n\times n}{\mathbf{H}} := \underset{n\times p}{\mathbf{X}} \underset{p\times p}{(\mathbf{X}'\mathbf{X})^{-1}} \underset{p\times n}{\mathbf{X}'}.$$

- $\mathbf{H}$ and $\mathbf{I}_n - \mathbf{H}$ are projection matrices: symmetric and idempotent.
- $rank(\mathbf{H}) = p, \quad rank(\mathbf{I}_n - \mathbf{H}) = n - p$.
- $\mathbf{H}$ is the projection matrix to $\mathrm{col}(\mathrm{X})$:
    - Fitted value vector $\widehat{\mathbf{Y}} = \mathbf{HY}$ is the projection of the observations vector $\mathbf{Y}$ to $\mathrm{col}(\mathrm{X})$.
    - Residual vector $\mathbf{e} = (\mathbf{I}_n - \mathbf{H})\mathbf{Y}$ is orthogonal to $\mathrm{col}(\mathrm{X})$.

# Multiple Regression:

# ANOVA

# Decomposition of Total Variation

$$SSTO = SSE + SSR$$

- **Total sum of squares**:

$$SSTO = \sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \mathbf{Y}'\left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right)\mathbf{Y}, \;\; d.f.(SSTO) = rank(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n) = n - 1.$$

- **Error sum of squares**:

$$SSE = \sum_{i=1}^{n}(Y_i - \widehat{Y}_i)^2 = \mathbf{Y}'(\mathbf{I}_n - \mathbf{H})\mathbf{Y}, \;\; d.f.(SSE) = rank(\mathbf{I}_n - \mathbf{H}) = n - p.$$

- **Regression sum of squares**:

$$SSR = \sum_{i=1}^{n}(\widehat{Y}_i - \overline{Y})^2 = \mathbf{Y}'(\mathbf{H} - \frac{1}{n}\mathbf{J}_n)\mathbf{Y}, \;\; d.f.(SSR) = rank(\mathbf{H} - \frac{1}{n}\mathbf{J}_n) = p - 1.$$

# Sampling Distributions of Sums of Squares

Under the Normal error model:

- $SSE \sim \sigma^2 \chi^2_{(n-p)}$
- $SSE$ and $SSR$ are independent.
- If $\beta_1 = \cdots = \beta_{p-1} = 0$, then $SSR \sim \sigma^2 \chi^2_{(p-1)}$.

# Mean Squares

- MSE: an unbiased estimator of the error variance $\sigma^2$

$$MSE = \frac{SSE}{n-p}, \quad E(MSE) = \sigma^2.$$

- $MSR = \frac{SSR}{p-1}$ :

$$E(MSR) = \begin{cases} \sigma^2 & \text{if} \quad \beta_1 = \cdots = \beta_{p-1} = 0 \\ > \sigma^2 & \text{if} \quad \text{otherwise} \end{cases}$$

# F Test for Regression Relation

Test whether the response variable and the set of $X$ variables are related:

- $H_0 : \beta_1 = \cdots = \beta_{p-1} = 0$ vs. $H_a$: not all $\beta_k$s equal to zero
- F ratio and its null distribution:

$$F^* = \frac{MSR}{MSE}, \quad F^* \sim_{H_0} F_{p-1,n-p},$$

  where $F_{p-1,n-p}$ denotes the F distribution with $(p-1, n-p)$ degrees of freedom.

- Decision rule at level $\alpha$: reject $H_0$ if $F^* > F(1-\alpha; p-1, n-p)$.

# ANOVA Table

| Source of Variation | SS | d.f. | MS | $F^*$ |
|---|---|---|---|---|
| Regression | $SSR = \mathbf{Y}'(\mathbf{H} - \frac{1}{n}\mathbf{J}_n)\mathbf{Y}$ | $p - 1$ | $MSR = \frac{SSR}{p-1}$ | $F^* = \frac{MSR}{MSE}$ |
| Error | $SSE = \mathbf{Y}'(\mathbf{I}_n - \mathbf{H})\mathbf{Y}$ | $n - p$ | $MSE = \frac{SSE}{n-p}$ | |
| Total | $SSTO = \mathbf{Y}'\left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right)\mathbf{Y}$ | $n - 1$ | | |

# Example: Model 2

| Source of Variation | SS | d.f. | MS | $F^*$ |
|---|---|---|---|---|
| Regression | $SSR = 366.4846$ | 4 | $MSR = 91.62116$ | $F^* = 87.03703$ |
| Error | $SSE = 26.31672$ | 25 | $MSE = 1.052669$ | |
| Total | $SSTO = 392.8013$ | 29 | | |

Pvalue $= P(F_{4,25} > 87.037) \approx 0$, so there is a significant regression relation between $Y$ and $X_1, X_2, X_3, X_1X_2$.

# Multiple Regression: Coefficient of Multiple Determination

# Coefficient of Multiple Determination

$$R^2 := \frac{SSR}{SSTO} = 1 - \frac{SSE}{SSTO}$$

- $R^2$ is the proportion of total variation in $Y$ that may be explained by the $X$ variables .
- $0 \le R^2 \le 1$
- Adding more $X$ variables to the model will never decrease $R^2$:
  - (i) $SSTO$ remains the same.
  - (ii) $SSE$ will not increase $\leftrightarrow$ $SSR$ will not decrease.

# Use As Many $X$ Variables As Possible?

- With more $X$ variables, the model does fit the observed data better, indicated by smaller $SSE$.
- However, a model with many $X$ variables that are unrelated to the response variable and/or are highly correlated with each other
  - tends to **overfit** the observed data and often do a poor job for prediction due to increased sampling variability.
  - makes interpretation more difficult.
  - makes model maintenance more costly.

# Adjusted Coefficient of Multiple Determination

Adjust for the number of $X$ variables in the model:

$$R_a^2 = 1 - \frac{MSE}{MSTO} = 1 - \frac{n-1}{n-p}\frac{SSE}{SSTO}$$

- $R_a^2 \leq R^2$
- It's possible for $R_a^2$ to decrease when adding more $X$ variables into the model:
  - decrease in $SSE$ may be more than offset by the loss of degrees of freedom in $SSE$.

# Example

- Model 1: $Y \sim X_1, X_2, X_3$

$$R^2 = 0.8883, \quad R_a^2 = 0.8754$$

- Model 2 : $Y \sim X_1, X_2, X_3, X_1X_2$

$$R^2 = 0.933, \quad R_a^2 = 0.9223$$

- Model 3: $Y \sim X_1, X_2, X_3, X_1X_2, X_1X_3, X_2X_3$

$$R^2 = 0.937, \quad R_a^2 = 0.9205$$

# Multiple Regression: Inference of Regression Coefficients

# LS Estimator: Standard Error

$$\underset{p\times 1}{\hat{\boldsymbol{\beta}}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_{p-1} \end{bmatrix} = \underset{p\times p}{(\mathbf{X}'\mathbf{X})^{-1}} \underset{p\times n}{\mathbf{X}'} \underset{n\times 1}{\mathbf{Y}}.$$

$$\underset{p\times 1}{\mathbf{E}\{\hat{\boldsymbol{\beta}}\}} = \boldsymbol{\beta}, \quad \underset{p\times p}{\boldsymbol{\sigma}^{\mathbf{2}}\{\hat{\boldsymbol{\beta}}\}} = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}.$$

$s(\hat{\beta}_k)$ – the standard error of $\hat{\beta}_k$ – is the positive square-root of the $(k+1)th$ diagonal element of $MSE(\mathbf{X}'\mathbf{X})^{-1}$.

Under Normal error model:

- $(1-\alpha)100\%$-confidence interval of $\beta_k$:

$$\hat{\beta}_k \pm t(1-\alpha/2; (n-p))s\{\hat{\beta}_k\}.$$

- T statistic:

$$T^* = \frac{\hat{\beta}_k - \beta_k^0}{s\{\hat{\beta}_k\}} \underset{H_0}{\sim} t_{(n-p)}.$$

- Two-sided T-Test: $H_0 : \beta_k = \beta_k^0 \ \ vs. \ \ H_a : \beta_k \neq \beta_k^0$.

  At level $\alpha$, the decision rule is to reject $H_0$ if and only if

  $|T^*| > t(1-\alpha/2; (n-p))$.

# Example: Nonadditive Model with Interaction $X_1X_2$

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i1} X_{i2} + \epsilon_i, \quad i = 1, \cdots, 30.$$

```
Call:
lm(formula = Y ~ X1 + X2 + X3 + X1:X2, data = data)

Coefficients:
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Residual standard error: 1.026 on 25 degrees of freedom
Multiple R-squared: 0.933,     Adjusted R-squared: 0.9223
F-statistic: 87.04 on 4 and 25 DF,  p-value: 2.681e-14
```

Test whether there is an interaction between $X_1$ and $X_2$ at significance level 0.01.

- $H_0 : \beta_4 = 0, \quad vs., \quad H_a : \beta_4 \neq 0.$
- $T^* = \frac{1.0076-0}{0.2467} = 4.084.$
- $n = 30, p = 5,\ t(0.995; 25) = 2.787.$
- Since $|4.084| > 2.787$, reject the null hypothesis and conclude that there is a significant interaction effect between $X_1$ and $X_2$.
- Alternatively, pvalue=$P(|t_{(25)}| > |4.084|) = 0.00040 < 0.01$, so reject $H_0$.

# Multiple Regression: Estimation of Mean Response

# Mean Response

For a given set of $X$ values:

$$\underset{p\times 1}{\mathbf{X}_h} = \begin{bmatrix} 1 \\ X_{h1} \\ \vdots \\ X_{h,p-1} \end{bmatrix},$$

the corresponding mean response is:

$$E(Y_h) = \mathbf{X}_h'\boldsymbol{\beta} = \beta_0 + \beta_1 X_{h1} + \cdots + \beta_{p-1} X_{h,p-1}.$$

- $\widehat{Y}_h := \mathbf{X}_h'\hat{\boldsymbol{\beta}}$ is an unbiased estimator of $E(Y_h)$:

$$E(\widehat{Y}_h) = E(\mathbf{X}_h'\hat{\boldsymbol{\beta}}) = \mathbf{X}_h'\mathbf{E}\{\hat{\boldsymbol{\beta}}\} = \mathbf{X}_h'\boldsymbol{\beta} = E(Y_h)$$

$$\sigma^2\{\widehat{Y}_h\} = \mathbf{X}_h'\boldsymbol{\sigma}^{\mathbf{2}}\{\hat{\boldsymbol{\beta}}\}\mathbf{X}_h = \sigma^2\left(\mathbf{X}_h'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h\right)$$

- Standard error of $\widehat{Y}_h$:

$$s(\widehat{Y}_h) = \sqrt{MSE\left(\mathbf{X}_h'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h\right)}$$

- $(1-\alpha)100\%$-confidence interval of $E(Y_h)$:

$$\widehat{Y}_h \pm t(1-\alpha/2; n-p)s(\widehat{Y}_h)$$

# Example: Nonadditive Model with Interaction $X_1 X_2$

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i1} X_{i2} + \epsilon_i, \quad i = 1, \cdots, 30.$$

```
Call:
lm(formula = Y ~ X1 + X2 + X3 + X1:X2, data = data)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.8832     0.2153   4.103  0.00038 ***
X1            1.5946     0.2421   6.587 6.69e-07 ***
X2            1.7091     0.2605   6.560 7.16e-07 ***
X3            2.1266     0.2687   7.916 2.85e-08 ***
X1:X2         1.0076     0.2467   4.084  0.00040 ***
---

Residual standard error: 1.026 on 25 degrees of freedom
Multiple R-squared: 0.933,      Adjusted R-squared: 0.9223
F-statistic: 87.04 on 4 and 25 DF,  p-value: 2.681e-14
```

Estimate the mean response when $X_1 = 0.8, X_2 = 0.5, X_3 = -1$:

- $\mathbf{X}_h' = \begin{bmatrix} 1 & 0.8 & 0.5 & -1 & 0.8 \times 0.5 \end{bmatrix}$
- Estimator $\widehat{Y}_h = \mathbf{X}_h' \hat{\beta} = 1.290$ :

$$\mathbf{X}_h'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h = 0.170, \quad \sqrt{MSE} = 1.026$$

$$s(\widehat{Y}_h) = 1.026 \times \sqrt{0.170} = 0.423$$

- $n = 30, p = 5$: $t(0.995; 25) = 2.787$
- A 99%-confidence interval of $E(Y_h)$:

$$1.290 \pm 2.787 \times 0.423 = [0.111, 2.469]$$

# Multiple Regression: Prediction

# Prediction of a New Observation

- $Y_{h(new)} = \mathbf{X}'_h \boldsymbol{\beta} + \epsilon_h$: independent with the observations $Y_i$s.
- Predicted value: $\widehat{Y}_h := \mathbf{X}'_h \hat{\boldsymbol{\beta}}$

$$\sigma^2\{pred_h\} := Var(\widehat{Y}_h - Y_{h(new)}) = \sigma^2\{\widehat{Y}_h\} + \sigma^2\{Y_{h(new)}\} = \sigma^2 \mathbf{X}'_h(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h + \sigma^2.$$

- Standard error of prediction:

$$s(pred_h) = \sqrt{MSE\left[1 + \mathbf{X}'_h(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h\right]}.$$

- $(1-\alpha)100\%$-prediction interval of $Y_{h(new)}$:

$$\widehat{Y}_h \pm t(1-\alpha/2; n-p)s(pred_h).$$

# Example: Nonadditive Model with Interaction $X_1 X_2$

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i1} X_{i2} + \epsilon_i, \quad i = 1, \cdots, 30.$$

```
Call:
lm(formula = Y ~ X1 + X2 + X3 + X1:X2, data = data)

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X1:X2         1.0076     0.2467   4.084  0.00040 ***
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Residual standard error: 1.026 on 25 degrees of freedom
Multiple R-squared: 0.933,      Adjusted R-squared: 0.9223
F-statistic: 87.04 on 4 and 25 DF,  p-value: 2.681e-14
```

Predict a new observation when $X_1 = 0.8, X_2 = 0.5, X_3 = -1$:

- Predicted value $\widehat{Y}_h = \mathbf{X}_h'\hat{\beta} = 1.290$ :

$$\mathbf{X}_h'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h = 0.170, \quad \sqrt{MSE} = 1.026$$

$$s(pred) = 1.026 \times \sqrt{1 + 0.170} = 1.1098$$

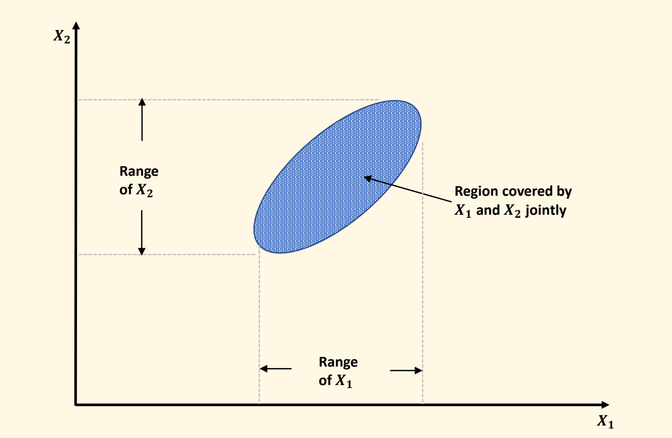
- A 99%-prediction interval of $Y_{hnew}$:

$$1.290 \pm 2.787 \times 1.1098 = [-1.803, 4.383]$$

# Hidden Extrapolations

- Extrapolation occurs when predicting the response variable for values of the $X$ variable(s) lying outside the range of the observed data.
- With more than one $X$ variables, the levels of all $X$ variables jointly define the region of the observations.

With two $X$ variables, we can look at their scatter plot to determine the region of observations.



# Multiple Regression: Extra Sum of Squares

# Notation

- $\mathcal{I}$: an index set
- $X_{\mathcal{I}} := \{X_i : i \in \mathcal{I}\}$
- Example: $\mathcal{I} = \{2, 3\}$, $X_{\mathcal{I}} = \{X_2, X_3\}$
- $SSE(X_{\mathcal{I}})$ and $SSR(X_{\mathcal{I}})$ denote the error sum of squares and regression sum of squares, respectively, under the regression model with $X_{\mathcal{I}} := \{X_i : i \in \mathcal{I}\}$ being the set of $X$ variables.

# Extra Sum of Squares

$$SSR(X_{\mathcal{J}}|X_{\mathcal{I}}) := SSE(X_{\mathcal{I}}) - SSE(X_{\mathcal{I}}, X_{\mathcal{J}}),$$

where $\mathcal{I}$ and $\mathcal{J}$ are two **non-overlapping** index sets.

- It is the reduction in error sum of squares by adding $X_{\mathcal{J}}$ to the model where $X_{\mathcal{I}}$ is the set of $X$ variables.
- degrees of freedom: the number of additional $X$ variables being added: $d.f.(SSR(X_{\mathcal{J}}|X_{\mathcal{I}})) = |\mathcal{J}|$
- Mean squares:

$$MSR(X_{\mathcal{J}}|X_{\mathcal{I}}) := \frac{SSR(X_{\mathcal{J}}|X_{\mathcal{I}})}{d.f.(SSR(X_{\mathcal{J}}|X_{\mathcal{I}}))}$$

# Properties

- $SSR(X_{\mathcal{J}}|X_{\mathcal{I}}) \geq 0$
- In general, $SSR(X_{\mathcal{J}}|X_{\mathcal{I}}) \neq SSR(X_{\mathcal{I}}|X_{\mathcal{J}})$
- $SSR(X_{\mathcal{J}}|X_{\mathcal{I}}) = SSR(X_{\mathcal{I}}, X_{\mathcal{J}}) - SSR(X_{\mathcal{I}})$, so it is also the marginal increase of the regression sum of squares by adding $X_{\mathcal{J}}$ to the model.

# Multiple Regression: ESS

# Examples

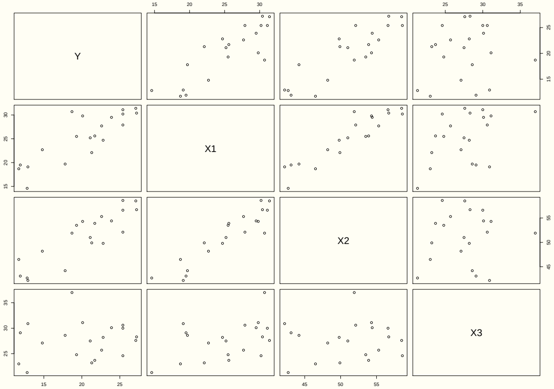
# Body Fat

A researcher measured the amount of body fat ($Y$) of 20 healthy females 25 to 34 years old, together with three (potential) predictor variables, triceps skinfolds thickness ($X_1$), thigh circumference ($X_2$), and midarm circumference ($X_3$). The amount of body fat was obtained by a cumbersome and expensive procedure requiring immersion of the person in water. Thus it would be helpful if a regression model with some or all of these predictors could provide reliable estimates of body fat as these predictors are easy to measure.

A snapshot of the data.

| case | X1 | X2 | X3 | Y |
|---|---|---|---|---|
| Triceps | Thigh | MidArm | BodyFat | |
| 1 | 19.5 | 43.1 | 29.1 | 11.9 |
| 2 | 24.7 | 49.8 | 28.2 | 22.8 |
| 3 | 30.7 | 51.9 | 37.0 | 18.7 |
| 4 | 29.8 | 54.3 | 31.1 | 20.1 |
| 5 | 19.1 | 42.2 | 30.9 | 12.9 |
| 6 | 25.6 | 53.9 | 23.7 | 21.7 |
| ... | ... | ... | ... | ... |

Figure: Scatter plot matrix




No obvious nonlinearity

Correlation matrix

| | X1 | X2 | X3 | Y |
|---|---|---|---|---|
| X1 | 1.0000000 | 0.9238425 | 0.4577772 | 0.8432654 |
| X2 | 0.9238425 | 1.0000000 | 0.0846675 | 0.8780896 |
| X3 | 0.4577772 | 0.0846675 | 1.0000000 | 0.1424440 |
| Y | 0.8432654 | 0.8780896 | 0.1424440 | 1.0000000 |

$X_1$ and $X_2$ are strongly correlated, $X_1$ and $X_3$ are moderately correlated, $X_2$ and $X_3$ are weakly correlated. Moreover, $X_1, X_2$ are strongly correlated with $Y$ and $X_3$ is weakly correlated with $Y$.

- Model 1: regression of $Y$ on $X_1$

$$Y_i = \beta_0 + \beta_1 X_{i1} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Model 2: regression of $Y$ on $X_2$

$$Y_i = \beta_0 + \beta_2 X_{i2} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Model 3: regression of $Y$ on $X_1$ and $X_2$

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Model 4: regression of $Y$ on $X_1, X_2$ and $X_3$

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \epsilon_i, \quad i = 1, \cdots, 20.$$

# Boy Fat: Model 1

```
Call:
lm(formula = Y ~ X1, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)  -1.4961     3.3192  -0.451    0.658
X1            0.8572     0.1288   6.656 3.02e-06 ***
---
Residual standard error: 2.82 on 18 degrees of freedom
Multiple R-squared: 0.7111,     Adjusted R-squared: 0.695
F-statistic:  44.3 on 1 and 18 DF,  p-value: 3.024e-06

Analysis of Variance Table
Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 352.27  352.27  44.305 3.024e-06 ***
Residuals 18 143.12    7.95
```

# Boy Fat: Model 2

```
Call:
lm(formula = Y ~ X2, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) -23.6345     5.6574  -4.178 0.000566 ***
X2            0.8565     0.1100   7.786  3.6e-07 ***
---
Residual standard error: 2.51 on 18 degrees of freedom
Multiple R-squared: 0.771,      Adjusted R-squared: 0.7583
F-statistic: 60.62 on 1 and 18 DF,  p-value: 3.6e-07

Analysis of Variance Table
Response: Y
Df Sum Sq Mean Sq F value  Pr(>F)
X2         1 381.97  381.97  60.617 3.6e-07 ***
Residuals 18 113.42    6.30
```

# Boy Fat: Model 3

```
Call:
lm(formula = Y ~ X1 + X2, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) -19.1742     8.3606  -2.293   0.0348 *
X1            0.2224     0.3034   0.733   0.4737
X2            0.6594     0.2912   2.265   0.0369 *
---
Residual standard error: 2.543 on 17 degrees of freedom
Multiple R-squared: 0.7781,     Adjusted R-squared: 0.7519
F-statistic:  29.8 on 2 and 17 DF,  p-value: 2.774e-06

Analysis of Variance Table
Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 352.27  352.27 54.4661 1.075e-06 ***
X2         1  33.17   33.17  5.1284    0.0369 *
Residuals 17 109.95    6.47
```

# Boy Fat: Model 4

```
lm(formula = Y ~ X1 + X2 + X3, data = fat)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept)  117.085     99.782   1.173    0.258
X1             4.334      3.016   1.437    0.170
X2            -2.857      2.582  -1.106    0.285
X3            -2.186      1.595  -1.370    0.190
---
Residual standard error: 2.48 on 16 degrees of freedom
Multiple R-squared: 0.8014,     Adjusted R-squared: 0.7641
F-statistic: 21.52 on 3 and 16 DF,  p-value: 7.343e-06

Analysis of Variance Table
Response: Y
Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 352.27  352.27 57.2768 1.131e-06 ***
X2         1  33.17   33.17  5.3931   0.03373 *
X3         1  11.55   11.55  1.8773   0.18956
Residuals 16  98.40    6.15
```

# Body Fat: ESS

- From Model 1, $SSE(X_1) = 143.12$ and from Model 3, $SSE(X_1, X_2) = 109.95$:

$$SSR(X_2|X_1) = SSE(X_1) - SSE(X_1, X_2) = 143.12 - 109.95 = 33.17$$

- From Model 2, $SSE(X_2) = 113.42$:

$$SSR(X_1|X_2) = SSE(X_2) - SSE(X_1, X_2) = 113.42 - 109.95 = 3.47$$

- The reduction of SSE by adding $X_2$ to the model with $X_1$ is much more than the reduction of SSE by adding $X_1$ to the model with $X_2$.

- From Model 4, $SSE(X_1, X_2, X_3) = 98.40$:

$$\begin{aligned} SSR(X_3|X_1, X_2) &= SSE(X_1, X_2) - SSE(X_1, X_2, X_3) \\ &= 109.95 - 98.40 = 11.55 \end{aligned}$$

- Moreover,

$$SSR(X_2, X_3|X_1) = SSE(X_1) - SSE(X_1, X_2, X_3) = 143.12 - 98.40 = 44.72,$$

$$SSR(X_1, X_3|X_2) = SSE(X_2) - SSE(X_1, X_2, X_3) = 113.42 - 98.40 = 15.02.$$

- These two extra sums of squares have degrees of freedom 2:

$MSR(X_2, X_3|X_1) = 44.72/2 = 22.36,$

$MSR(X_1, X_3|X_2) = 15.02/2 = 7.51$

# Multiple Regression: Decomposition of SSR

# Decomposition of SSR into ESS

For a model with multiple $X$ variables, the regression sum of squares (SSR) can be expressed as the sum of several extra sums of squares.

- $SSR(X_1, X_2) = SSR(X_1) + SSR(X_2|X_1)$ : $SSR(X_1)$ measures the contribution by having $X_1$ alone in the model, whereas $SSR(X_2|X_1)$ measures the additional contribution when $X_2$ is added, given that $X_1$ is already in the model.
- However, such decomposition is usually not unique: $SSR(X_1, X_2) = SSR(X_2) + SSR(X_1|X_2)$.

- More $X$ variables, more decompositions.
- For example, with three $X$ variables:

$$
\begin{aligned}
SSR(X_1, X_2, X_3) &= SSR(X_1) + SSR(X_2|X_1) + SSR(X_3|X_1, X_2) \\
SSR(X_1, X_2, X_3) &= SSR(X_2) + SSR(X_1|X_2) + SSR(X_3|X_1, X_2) \\
SSR(X_1, X_2, X_3) &= SSR(X_1) + SSR(X_2, X_3|X_1), \quad \cdots, \cdots
\end{aligned}
$$

# Body Fat

- From Model 1, $SSR(X_1) = 352.27$; Also $SSR(X_2|X_1) = 33.17$ and $SSR(X_3|X_1, X_2) = 11.55$. So

$$SSR(X_1, X_2, X_3) = 352.27 + 33.17 + 11.55 = 396.99.$$

- From Model 2, $SSR(X_2) = 381.97$; Also $SSR(X_1|X_2) = 3.47$. So

$$SSR(X_1, X_2, X_3) = 381.97 + 3.47 + 11.55 = 396.99.$$

# R output: anova()

```
lm(formula = Y ~ X1 + X2 + X3, data = fat)
> anova(fit4)
Analysis of Variance Table
Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 352.27  352.27 57.2768 1.131e-06 ***
X2         1  33.17   33.17  5.3931   0.03373 *
X3         1  11.55   11.55  1.8773   0.18956
Residuals 16  98.40    6.15
```

Decomposition of *SSR* into single d.f. ESS, by order of the *X* variables entering the model:

| Source of Variation | SS | d.f. | MS |
|---|---|---|---|
| Regression | 396.99 | 3 | 132.33 |
| $X_1$ | 352.27 | 1 | 352.27 |
| $X_2 \| X_1$ | 33.17 | 1 | 33.17 |
| $X_3 \| X_1, X_2$ | 11.55 | 1 | 11.55 |
| Error | 98.40 | 16 | 6.15 |
| Total | 495.39 | 19 | |

- $SSR(X_2, X_3|X_1) = SSR(X_2|X_1) + SSR(X_3|X_1, X_2) = 33.17 + 11.55 = 44.72.$
- How to get $SSR(X_2|X_1, X_3)$ from the R output? Enter the $X$ variables in a different order, i.e., $X_1, X_3, X_2$:

```
lm(formula = Y ~ X1 + X3 + X2, data = fat)
> anova(fit4.alt2)
Analysis of Variance Table
Df Sum Sq Mean Sq F value    Pr(>F)
X1         1 352.27  352.27 57.2768 1.131e-06 ***
X3         1  37.19   37.19  6.0461   0.02571 *
X2         1   7.53    7.53  1.2242   0.28489
Residuals 16  98.40    6.15
```

- $SSR(X_2|X_1, X_3) = 7.53$

# Multiple Regression:

# General Linear Tests

# General Linear Tests

$\mathcal{I}$ and $\mathcal{J}$ are two non-overlapping index sets:

- **Full model**: with both $X_{\mathcal{I}}$ and $X_{\mathcal{J}}$
- **Reduced model**: with only $X_{\mathcal{I}}$
- Test whether $X_{\mathcal{J}}$ may be dropped out of the full model:

$$H_0 : \beta_j = 0, \text{ for } \textbf{all } j \in \mathcal{J} \quad vs. \quad H_a : \text{not all } \beta_j : j \in \mathcal{J} \text{ is zero}$$

- $H_0$ corresponds to the reduced model with only $X_{\mathcal{I}}$.

# F Test

Compare $SSE$ under the full model with $SSE$ under the reduced model by an F ratio:

$$F^* = \frac{\frac{SSE(R)-SSE(F)}{df_R - df_F}}{\frac{SSE(F)}{df_F}} = \frac{MSR(X_{\mathcal{J}}|X_{\mathcal{I}})}{MSE(F)}$$

- Under $H_0$ (i.e., the reduced model):

$$F^* \sim_{H_0} F_{df_R - df_F, df_F}$$

- Reject $H_0$ at level $\alpha$ iff the observed $F^* > F(1-\alpha; df_R - df_F, df_F)$.

# Multiple Regression:

# General Linear Tests

# Examples

# F-test for Regression Relation

- Full model with $X_1, \cdots, X_{p-1}$:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_{p-1} X_{i,p-1} + \epsilon_i, \quad i = 1, \cdots n$$

- Reduced model with no $X$ variable:

$$Y_i = \beta_0 + \epsilon_i, \ \ i = 1, \cdots, n, \quad SSE(R) = SSTO, \ df_R = n - 1$$

- $SSE(R) - SSE(F) = SSTO - SSE(F) = SSR(F)$, and $df_R - df_F = (n-1) - (n-p) = p - 1 = d.f.(SSR(F))$
- $F^* = \frac{SSR(F)/(p-1)}{SSE(F)/(n-p)} = \frac{MSR(F)}{MSE(F)}$

# Test whether a Single $\beta_k = 0$

Body Fat: for the model with all three predictors, test whether the midarm circumference ($X_3$) can be dropped.

- Full model: $SSE(F) = 98.40$ with d.f. 16:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Reduced model: $SSE(R) = 109.95$ with d.f. 17:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- $F^* = \frac{11.55/1}{98.40/16} = 1.88$; Pvalue=$P(F_{1,16} > 1.88) = 0.189$, so $X_3$ can be dropped.

# Equivalence between F-test and T-test

- $H_0 : \beta_k = 0 \;\; vs. \;\; H_a : \beta_k \neq 0$

- T-test:

$$T^* = \frac{\hat{\beta}_k}{s\{\hat{\beta}_k\}} \underset{H_0}{\sim} t_{(n-p)},$$

  where $\hat{\beta}_k$ is the LS estimator of $\beta_k$ and $s\{\hat{\beta}_k\}$ is its standard error. At level $\alpha$, reject $H_0$ when $|T^*| > t(1-\alpha/2; n-p)$.

- $F^* = (T^*)^2$ and $F(1-\alpha; 1, n-p) = (t(1-\alpha/2; n-p))^2 \rightarrow$ F-test and two-sided T-test are equivalent.

*For one-sided alternatives, we still need the T-tests.*

# Test whether Several $\beta_k = 0$

Body Fat: Test whether both $X_2$ and $X_3$ can be dropped from the model with all three predictors:

- Full model: $SSE(F) = 98.40$ with d.f. 16:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- Reduced model: $SSE(R) = 143.12$ with d.f. 18:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \epsilon_i, \quad i = 1, \cdots, 20.$$

- $F^* = \frac{44.72/2}{98.40/16} = 3.635$; Pvalue$= P(F_{2,16} > 3.635) = 0.0499$

# Test Equality of Several $\beta_k$s

- Full model: $Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_{p-1} X_{i,p-1} + \epsilon_i$
- For $q \le p - 1$: $H_0 : \beta_1 = \cdots = \beta_q$ *vs.* $H_a : \beta_1, \cdots, \beta_q$ are not all equal
- Reduced model: $Y_i = \beta_0 + \beta_c(X_{i1} + \cdots + X_{iq}) + \cdots + \beta_{p-1} X_{i,p-1} + \epsilon_i$
- $\beta_c$ denotes the common value of $\beta_1, \cdots, \beta_q$ under $H_0$, and $X_1 + \cdots + X_q$ is the corresponding (new) $X$ variable. $SSE(R)$ has d.f. $n - (p - q + 1)$.
- $F^* = \frac{(SSE(R) - SSE(F))/(q-1)}{SSE(F)/(n-p)} \underset{H_0}{\sim} F_{q-1,n-p}$