Numerical procedure of shooting with sliding boundary condition using (sine/cosine transformed) spectral decomposition of the resolvent (Green's function) and convolution theory

(The origin of the following equations: A fast diffeomorphic image registration algorithm. Ashburner John)

First determine the Eigenvector and Eigenvalue of the following convolution matrices (which exhibit additive structures-it is not a spectral decomposition) with respect to DCT/DCT (eigenvalues and orthogonal eigenvectors), which can be used later to compute the green kernel of the matrics (regularization operators)

Let U_{ab} be the parameter of the absolute displacement penalty

Membrane energy model:

$$\overbrace{\lambda\delta_2^{-2}}^{\alpha}\underbrace{\begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}}_{L_0} + \overbrace{\lambda\delta_1^{-2} - \lambda\delta_2^{-2}}^{\beta}\underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix}}_{L_1} + \overbrace{2\lambda\delta_2^{-2} + U_{ab}}^{\gamma}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{L_2} =$$

$$\alpha L_0 + \beta L_1 + \gamma L_2 = \begin{pmatrix} 0 & -\lambda\delta_1^{-2} & 0 \\ -\lambda\delta_2^{-2} & 2\lambda(\delta_1^{-2} + \delta_2^{-2}) & -\lambda\delta_2^{-2} \\ 0 & -\lambda\delta_1^{-2} & 0 \end{pmatrix} (Ashburner\ John\ 2007)$$

Bending energy model:

$$\overbrace{\lambda\delta_2^{-4}}^{\alpha}\underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{L_0} + \overbrace{4\lambda\delta_1^{-2}\delta_2^{-2}}^{\beta}\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{L_1} + \overbrace{6\lambda\delta_1^{-4} + U_{ab}}^{\gamma}\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{L_2} +$$

$$\overbrace{2\lambda\delta_1^{-2}\delta_2^{-2}}^{\theta}\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{L_3} + \overbrace{-\lambda\delta_2^{-4} + \lambda\delta_1^{-4}}^{\vartheta}\underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{L_4} + \overbrace{4\lambda\delta_2^{-4} - 4\lambda\delta_1^{-4}}^{\xi}\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{L_5} =$$

$$\alpha L_0 + \beta L_1 + \gamma L_2 + \theta L_3 + \vartheta L_4 + \xi L_5 =$$

$$\begin{pmatrix} 0 & 0 & \lambda\delta_1^{-4} & 0 & 0 \\ 0 & 2\lambda\delta_1^{-2}\delta_2^{-2} & -4\lambda\delta_1^{-2}(\delta_1^{-2} + \delta_2^{-2}) & 2\lambda\delta_1^{-2}\delta_2^{-2} & 0 \\ \lambda\delta_2^{-4} & -4\lambda\delta_2^{-2}(\delta_1^{-2} + \delta_2^{-2}) & \lambda(6\delta_1^{-4} + 6\delta_2^{-4} + 8\delta_1^{-2}\delta_2^{-2}) & -4\lambda\delta_2^{-2}(\delta_1^{-2} + \delta_2^{-2}) & \lambda\delta_2^{-4} \\ 0 & 2\lambda\delta_1^{-2}\delta_2^{-2} & -4\lambda\delta_1^{-2}(\delta_1^{-2} + \delta_2^{-2}) & 2\lambda\delta_1^{-2}\delta_2^{-2} & 0 \\ 0 & 0 & \lambda\delta_1^{-4} & 0 & 0 \end{pmatrix}$$

$$(Ashburner\ John\ 2007)$$

Linear elastic energy model:

$$\overbrace{(2\mu + \lambda)\delta_1^{-2}}^{\alpha}\underbrace{\begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}}_{L_0} + \overbrace{\mu\delta_2^{-2}}^{\beta}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}}_{L_1} + \overbrace{-(2\mu + \lambda)\delta_1^{-2}}^{\gamma}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{L_2} =$$

$$\alpha L_0 + \beta L_1 + \gamma L_2 = \begin{pmatrix} 0 & -(2\mu + \lambda)\delta_1^{-2} & 0 \\ -\mu\delta_2^{-2} & \mu(4\delta_1^{-2} + 2\delta_2^{-2}) + 2\lambda\delta_1^{-2} & -\mu\delta_2^{-2} \\ 0 & -(2\mu + \lambda)\delta_1^{-2} & 0 \end{pmatrix} (Ashburner\ John\ 2007)$$

$$\overbrace{(\frac{\mu+\lambda}{4})\delta_1^{-1}\delta_2^{-1}}^{\alpha}\underbrace{\begin{pmatrix}1&0&1\\0&0&0\\1&0&1\end{pmatrix}}_{L_0}=\begin{pmatrix}-(\frac{\mu+\lambda}{4})\delta_1^{-1}\delta_2^{-1}&0&-(\frac{\mu+\lambda}{4})\delta_1^{-1}\delta_2^{-1}\\0&0&0\\-(\frac{\mu+\lambda}{4})\delta_1^{-1}\delta_2^{-1}&0&-(\frac{\mu+\lambda}{4})\delta_1^{-1}\delta_2^{-1}\end{pmatrix}(Ashburner\ John\ 2007)$$

$$\overbrace{\mu\delta_1^{-2}}^{\alpha}\underbrace{\begin{pmatrix}0&-1&0\\-1&2&-1\\0&-1&0\end{pmatrix}}_{L_0}+\overbrace{(2\mu+\lambda)\delta_2^{-2}}^{\beta}\underbrace{\begin{pmatrix}0&0&0\\-1&2&-1\\0&0&0\end{pmatrix}}_{L_0}+\overbrace{\mu\delta_1^{-2}}^{\gamma}\underbrace{\begin{pmatrix}0&0&0\\1&0&1\\0&0&0\end{pmatrix}}_{L_2}=$$

$$\alpha L_0+\beta L_1+\gamma L_2=\begin{pmatrix}0&-\mu\delta_1^{-2}&0\\-(2\mu+\lambda)\delta_2^{-2}&\mu(4\delta_2^{-2}+2\delta_1^{-2})+2\lambda\delta_2^{-2}&-(2\mu+\lambda)\delta_2^{-2}\\0&-\mu\delta_1^{-2}&0\end{pmatrix}(Ashburner\ John\ 2007)$$

DCT/DST formulation of the problem(generic form-using the above matrices and coefficients):

$$K(x-x_0)L(x-x_0)=\underbrace{\delta(x-x_0)}_{\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ip(x-x_0)}dx}\quad,\delta(x)=\delta(-x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}cos(nx)dx,$$

so

$$\delta(x-x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overbrace{\frac{1}{2}[e^{in(x-x_0)}+e^{-in(x-x_0)}]}^{cos(n(x-x_0))}dx$$

and

$$K(x-x_0)L(x-x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}_{dct/dst}\{K(x-x_0)L(x-x_0)\}cos(n(x-x_0))dx$$

thus

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}_{dct/dst}\{K(x-x_0)L(x-x_0)\}cos(n(x-x_0))dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}cos(n(x-x_0))dx\ \Leftrightarrow$$

$$\mathcal{F}_{dct/dst}\{K(x-x_0)L(x-x_0)\}=1 \tag{1}$$

$$L(x-x_0)v(x-x_0)=u(x-x_0) \tag{2}$$

$$(1),(2)\Longrightarrow\mathcal{F}_{dct/dst}\{v(x-x_0)\}=\mathcal{F}_{dct/dst}\{u(x-x_0)\}\odot\mathcal{F}_{dct/dst}\{K(x-x_0)\} \tag{3}$$

$\mathcal{F}_{dct/dst}$ continues cosine/sine transform using trigonometric identities and relations from the properties of the exponential function, and sine and cosine functions. In the case of finding suitable DCT/DST I-IV for the operators $L, L_0, L_1...$ where $L=\alpha L_0+\beta L_1+\gamma L_2+...$ we will have (${\mathcal{F}^{\mathcal{D}}}_{dct/dst}$ discrete cosine/sine transform):

$$\underbrace{\overbrace{\begin{bmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{bmatrix}}^{L}\overbrace{\begin{bmatrix}orthogonaleigenvectors(F_0)&&\\\overbrace{c_0}&\cdot&\cdot\\c_1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{bmatrix}}^{F_{dct/dst}}}_{{\mathcal{F}^{\mathcal{D}}}_{dct/dst}\{L\}=LF,\ LF_0}=\underbrace{\overbrace{\begin{bmatrix}orthogonaleigenvectors(F_0)&&\\\overbrace{c_0}&\cdot&\cdot\\c_1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{bmatrix}}^{F_{dct/dst}}\overbrace{\begin{bmatrix}\lambda_{(F_0)}&\cdot&\cdot\\\cdot&\lambda_{(F_1)}&\cdot\\\cdot&\cdot&\cdot\end{bmatrix}}^{eigenvalues\ of\ F_{dct/dst}}}_{\lambda_{(F_0)}F_0}$$

For equation (1) in discrete case we have

$${\mathcal{F}^{\mathcal{D}}}_{dct/dst}\{K(x-x_0)L(x-x_0)\}=K(x-x_0)\overbrace{L(x-x_0)F_{dct/dst}}^{\lambda_L^{dct/dst}F_{dct/dst}}=\lambda_L^{dct/dst}\underbrace{K(x-x_0)F_{dct/dst}}_{{\mathcal{F}^{\mathcal{D}}}_{dct/dst}\{K(x-x_0)\}}=1,\ \Longrightarrow$$

$${\mathcal{F}^{\mathcal{D}}}_{dct/dst}\{K(x-x_0)\}=\frac{1}{\lambda_L^{dct/dst}}=\frac{1}{[(\alpha\lambda_{L_0}^{dct/dst}+\beta\lambda_{L_1}^{dct/dst}+\gamma\lambda_{L_2}^{dct/dst}+...),...]} \tag{4}$$

Substituting (4) in (3) and applying inverse discrete cos/sine transform ${\mathcal{F}^{\mathcal{D}}}^{-1}_{dct/dst}$we get (for one dimensional problem)

$$v(x-x_0)={\mathcal{F}^{\mathcal{D}}}^{-1}_{dct/dst}\{\mathcal{F}_{dct/dst}\{u(x-x_0)\}\odot\frac{1}{[(\alpha\lambda^{dct/dst}_{L_0}+\beta\lambda^{dct/dst}_{L_1}+\gamma\lambda^{dct/dst}_{L_2}+...),...]}\} \tag{5}$$

For two dimensional problem we have

$$K(x-x_0)L(x-x_0)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{K(x-x_0)L(x-x_0)\}\}cos(n(x-x_0))cos(m(x-x_0))dx^2$$

thus

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{K(x-x_0)L(x-x_0)\}\}cos(n(x-x_0))cos(m(x-x_0))dx^2=$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\overbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}cos(n(x-x_0))dx}^{\delta(x-x_0)}\,cos(m(x-x_0))dx\Leftrightarrow$$

$$\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{K(x-x_0)L(x-x_0)\}\}=1 \tag{6}$$

and equations (2),(6) imply

$$\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{v(x-x_0)\}\}=\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{u(x-x_0)\}\}\odot\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{K(x-x_0)\}\}$$

For equation (6) in discrete case we have

$$\mathcal{F^D}_{dct/dst}\{\mathcal{F^D}_{dct/dst}\{K(x-x_0)L(x-x_0)\}\}=[K(x-x_0)\underbrace{\overbrace{L(x-x_0)F_{dct/dst}}^{\lambda_L^{dct/dst}F_{dct/dst}}]F^{/}_{dct/dst}}_{LF_{dct/dst}F^{/}_{dct/dst}}=$$

$$\lambda_L^{dct/dst}\underbrace{K(x-x_0)F_{dct/dst}F^{/}_{dct/dst}}_{\mathcal{F^D}_{dct/dst}\{\mathcal{F^D}_{dct/dst}\{K(x-x_0)\}\}}=1$$

so we need to determine $\lambda_L^{dct/dst}$ such that (eigenvalue of multiple eigenvector/eigenmatrix)

$$\overbrace{LF_{dct/dst}}^{M}F^{/}_{dct/dst}=\overbrace{\lambda_L^{dct/dst}F_{dct/dst}}^{\lambda_M}F^{/}_{dct/dst}$$

$$\underbrace{\overbrace{\begin{bmatrix}d_0 & d_1. & .\\ . & . & .\\ . & . & .\end{bmatrix}}^{orth.eigenvectors}\overbrace{\begin{bmatrix}. & . & .\\ . & . & .\\ . & . & .\end{bmatrix}}^{L}\overbrace{\begin{bmatrix}c_0 & . & .\\ c_1. & . & .\\ . & . & .\end{bmatrix}}^{orth.eigenvectors}}_{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{L\}\}}=\underbrace{\overbrace{\begin{bmatrix}d_0 & d_1. & .\\ . & . & .\\ . & . & .\end{bmatrix}}^{orth.eigenvectors}\overbrace{\begin{bmatrix}c_0 & . & .\\ c_1. & . & .\\ . & . & .\end{bmatrix}}^{orth.eigenvectors}\overbrace{\begin{bmatrix}\lambda_1 & . & .\\ . & \lambda_2. & .\\ . & . & .\end{bmatrix}}^{eigenvalues}}$$

Using Kronecker tensor product/sum, we have:

$$L=\alpha(\overbrace{L_0}^{InclNeumannBC}\otimes I+I\otimes\underbrace{L_0}_{InclDirichletBC})+\beta(\overbrace{L_1}^{InclNeumannBC}\otimes I+I\otimes\underbrace{L_1}_{InclDirichletBC})+$$

$$\gamma(\overbrace{L_2}^{InclNeumannBC}\otimes I+I\otimes\underbrace{L_2}_{InclDirichletBC})+...$$

thus

$$\underbrace{K(x-x_0)F_{dct/dst}F^{/}_{dct/dst}}_{\mathcal{F^D}_{dct/dst}\{\mathcal{F^D}_{dct/dst}\{K(x-x_0)\}\}} = \tag{7}$$

$$\frac{1}{\begin{bmatrix} [\alpha(\lambda_{L_0}^{F_{dct/dst}}+\lambda_{L_0}^{F^{/}_{dct/dst}})+\beta(\lambda_{L_1}^{F_{dct/dst}}+\lambda_{L_1}^{F^{/}_{dct/dst}})+\gamma(\lambda_{L_2}^{F_{dct/dst}}+\lambda_{L_2}^{F^{/}_{dct/dst}})+...] & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}}$$

and finally (for two dimensional problem)

$$v(x-x_0) = \mathcal{F^D}^{-1}_{dct/dst}\{\mathcal{F^D}^{-1}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{u(x-x_0)\}\}\odot$$

$$\frac{1}{\begin{bmatrix} [\alpha(\lambda_{L_0}^{F_{dct/dst}}+\lambda_{L_0}^{F^{/}_{dct/dst}})+\beta(\lambda_{L_1}^{F_{dct/dst}}+\lambda_{L_1}^{F^{/}_{dct/dst}})+\gamma(\lambda_{L_2}^{F_{dct/dst}}+\lambda_{L_2}^{F^{/}_{dct/dst}})+...] & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}}\}\} \tag{8}$$

For three dimensional problem we have

$$K(x-x_0)L(x-x_0) = \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{$$

$$K(x-x_0)L(x-x_0)\}\}\}cos(n(x-x_0))cos(m(x-x_0))cos(l(x-x_0))dx^3$$

thus

$$\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{K(x-x_0)L(x-x_0)\}\}\}$$

$$cos(n(x-x_0))cos(m(x-x_0))cos(l(x-x_0))dx^3 =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\overbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}cos(n(x-x_0))dx\,cos(m(x-x_0))}^{\delta(x-x_0)}dx\,cos(l(x-x_0))}_{\mathcal{F}_{dct/dst}\{\delta(x-x_0)\}}dx \Leftrightarrow$$

$$\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{K(x-x_0)L(x-x_0)\}\}\} = 1 \tag{9}$$

Following same rule as two dimensional problem we get

$$\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{v(x-x_0)\}\}\} = \mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{u(x-x_0)\}\}\}\odot$$

$$\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{K(x-x_0)\}\}\}$$

Again using Kronecker tensor property we get

$$L = \alpha(\overbrace{L_0}^{Incl Neumann BC}\otimes I\otimes I + I\otimes\underbrace{L_0}_{Incl Dirichlet BC}\otimes I + I\otimes I\otimes\overbrace{L_0}^{third Dim}) + \beta(\overbrace{L_1}^{Incl Neumann BC}\otimes I\otimes I +$$

$$I\otimes\underbrace{L_1}_{Incl Dirichlet BC}\otimes I + I\otimes I\otimes\overbrace{L_1}^{third Dim})+$$

$$\gamma(\overbrace{L_2}^{IclNeumannBC} \otimes I \otimes I + I \otimes \underbrace{L_2}_{InclDirichletBC} \otimes I + I \otimes I \otimes \overbrace{L_2}^{thirdDim}) + ...$$

thus

$$\underbrace{K(x-x_0)F_{dct/dst}F^{/}_{dct/dst}F^{//}_{dct/dst}}_{\mathcal{F^D}_{dct/dst}\{\mathcal{F^D}_{dct/dst}\{\mathcal{F^D}_{dct/dst}\{K(x-x_0)\}\}\}} = \tag{10}$$

$$\frac{1}{\begin{bmatrix} \cdots \begin{bmatrix}\begin{bmatrix} [\alpha(\lambda_{L_0}^{F_{dct/dst}} + \lambda_{L_0}^{F^{/}_{dct/dst}} + \lambda_{L_0}^{F^{//}_{dct/dst}}) + \beta(\lambda_{L_1}^{F_{dct/dst}} + \lambda_{L_1}^{F^{/}_{dct/dst}} + \lambda_{L_1}^{F^{//}_{dct/dst}}) + ...] & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}\end{bmatrix} \cdots \end{bmatrix}}$$

Finally (for three dimensional problem)

$$v(x-x_0) = \mathcal{F^D}^{-1}_{dct/dst}\{\mathcal{F^D}^{-1}_{dct/dst}\{\mathcal{F^D}^{-1}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{\mathcal{F}_{dct/dst}\{u(x-x_0)\}\}\}\odot$$

$$\frac{1}{\begin{bmatrix} \cdots \begin{bmatrix}\begin{bmatrix} [\alpha(\lambda_{L_0}^{F_{dct/dst}} + \lambda_{L_0}^{F^{/}_{dct/dst}} + \lambda_{L_0}^{F^{//}_{dct/dst}}) + \beta(\lambda_{L_1}^{F_{dct/dst}} + \lambda_{L_1}^{F^{/}_{dct/dst}} + \lambda_{L_1}^{F^{//}_{dct/dst}}) + ...] & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix}\end{bmatrix} \cdots \end{bmatrix}}\}\}\}$$

Following general computation technique to determine the eigenvector and eigenvalue of the given matrix with respect to DCT/DST I-IV(based on first, second, third, fourth ... approximated derivative and boundary conditions) hold (only if DCT/DST column/row vectors are the orthogonal eigenvectors of the given matrices)

$$\underbrace{\overbrace{\begin{pmatrix} \otimes & \otimes & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \otimes & \otimes \end{pmatrix}}^{L=\frac{\partial^2}{\partial x^2}} \begin{pmatrix} . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\ cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\ cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}}_{-cos(j-\frac{1}{2})\frac{k\pi}{N}+2cos(j+\frac{1}{2})\frac{k\pi}{N}-cos(j+\frac{3}{2})\frac{k\pi}{N}=2(1-cos\frac{k\pi}{N})cos(j+\frac{1}{2})\frac{k\pi}{N}} =$$

$$\begin{pmatrix} . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\ cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\ cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \underbrace{\begin{pmatrix} 2(1-cos\frac{k\pi}{N}) & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix}}_{\lambda} \tag{11}$$

$$
\text{If } L = \left(\frac{\partial^2}{\partial x^2}\right)^2 \text{ then } \underbrace{\begin{pmatrix}
[2(1-cos\frac{k\pi}{N})]^2 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & .
\end{pmatrix}}_{\lambda}
$$

$$
\underbrace{\overbrace{\begin{pmatrix}
\oslash & \oslash & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & . & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \oslash & \oslash
\end{pmatrix}}^{L=Id}
\begin{pmatrix}
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}}_{cos(j+\frac{1}{2})\frac{k\pi}{N} = 1*cos(j+\frac{1}{2})\frac{k\pi}{N}} =
$$

$$
\begin{pmatrix}
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}
\underbrace{\begin{pmatrix}
1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}}_{\lambda} \tag{12}
$$

$$
\underbrace{\overbrace{\begin{pmatrix}
\oslash & \oslash & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & . & 0 & . & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & . & 0 & . & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \oslash & \oslash
\end{pmatrix}}^{L=-\frac{\partial^2}{\partial x^2}+2*Id}
\begin{pmatrix}
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}}_{cos(j-\frac{1}{2})\frac{k\pi}{N}+cos(j+\frac{3}{2})\frac{k\pi}{N} = 2cos(\frac{k\pi}{N})cos((j+\frac{1}{2})\frac{k\pi}{N})} =
$$

$$
\begin{pmatrix}
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}
\underbrace{\begin{pmatrix}
2cos(\frac{k\pi}{N}) & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & .
\end{pmatrix}}_{\lambda} \tag{13}
$$

$$
\underbrace{\overbrace{\begin{pmatrix}
\ogreaterthan & \ogreaterthan & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & . & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & . & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & . & 0 & . & 0 & . & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \olessthan & \olessthan
\end{pmatrix}}^{L=(\frac{\partial^2}{\partial x^2})^2-4*\frac{\partial^2}{\partial x^2}+3*Id}
\begin{pmatrix}
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
cos(j-\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{5}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}}_{cos(j-\frac{3}{2})\frac{k\pi}{N}+cos(j+\frac{1}{2})\frac{k\pi}{N}+cos(j+\frac{5}{2})\frac{k\pi}{N}=(2cos(\frac{2k\pi}{N})+1)cos((j+\frac{1}{2})\frac{k\pi}{N})}
=
$$

$$
\begin{pmatrix}
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
cos(j-\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{5}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}
\underbrace{\begin{pmatrix}
(2cos(\frac{2k\pi}{N})+1) & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}}_{\lambda}
\tag{14}
$$

Alternative approaches (simplification)

Now assuming voxel size of the lattice being equal in all dimensions, namely $\delta_1 = \delta_2 = \delta_3$(a natural choice which would not have any significant negative effect on registration/shooting problems at all), the above complex formulation to find the eigenvalues of the kernels with respect to DST/DCT can be simplified by the following formulations (it works specifically for Membrane energy model).

Membrane energy model:

$$
\begin{pmatrix}
0 & -\lambda\delta_1^{-2} & 0 \\
-\lambda\delta_2^{-2} & 2\lambda(\delta_1^{-2}+\delta_2^{-2}) & -\lambda\delta_2^{-2} \\
0 & -\lambda\delta_1^{-2} & 0
\end{pmatrix}
= \lambda\delta_1^{-2}
\begin{pmatrix}
0 & -1 & 0 \\
-1 & 4 & -1 \\
0 & -1 & 0
\end{pmatrix},
$$

thus

$$
\underbrace{\lambda\delta_1^{-2}\overbrace{\begin{pmatrix}
\ogreaterthan & \ogreaterthan & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & . & . & . & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \olessthan & \olessthan
\end{pmatrix}}^{Membrane energy model}
\begin{pmatrix}
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}}_{\lambda\delta_1^{-2}\{-cos(j-\frac{1}{2})\frac{k\pi}{N}+4cos(j+\frac{1}{2})\frac{k\pi}{N}-cos(j+\frac{3}{2})\frac{k\pi}{N}\}=\lambda\delta_1^{-2}\{2(2-cos\frac{k\pi}{N})cos(j+\frac{1}{2})\frac{k\pi}{N}\}}
=
$$

$$
\begin{pmatrix}
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
cos(j-\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{1}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
cos(j+\frac{3}{2})\frac{k\pi}{N} & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}
\underbrace{\overbrace{\begin{pmatrix}
2\lambda\delta_1^{-2}(2-cos\frac{k\pi}{N}) & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & .
\end{pmatrix}}^{eigenvalues}}_{\lambda}
\tag{15}
$$

Bending energy model:

$$\begin{pmatrix} 0 & 0 & \lambda\delta_1^{-4} & 0 & 0 \\ 0 & 2\lambda\delta_1^{-2}\delta_2^{-2} & -4\lambda\delta_1^{-2}(\delta_1^{-2}+\delta_2^{-2}) & 2\lambda\delta_1^{-2}\delta_2^{-2} & 0 \\ \lambda\delta_2^{-4} & -4\lambda\delta_2^{-2}(\delta_1^{-2}+\delta_2^{-2}) & \lambda(6\delta_1^{-4}+6\delta_2^{-4}+8\delta_1^{-2}\delta_2^{-2}) & -4\lambda\delta_2^{-2}(\delta_1^{-2}+\delta_2^{-2}) & \lambda\delta_2^{-4} \\ 0 & 2\lambda\delta_1^{-2}\delta_2^{-2} & -4\lambda\delta_1^{-2}(\delta_1^{-2}+\delta_2^{-2}) & 2\lambda\delta_1^{-2}\delta_2^{-2} & 0 \\ 0 & 0 & \lambda\delta_1^{-4} & 0 & 0 \end{pmatrix} =$$

$$\lambda\delta_1^{-4}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -8 & 2 & 0 \\ 1 & -8 & 20 & -8 & 1 \\ 0 & 2 & -8 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \lambda\delta_1^{-4}\{\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 \\ 1 & -8 & 20 & -8 & 1 \\ 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + 2\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{eigenvaluesastheabove}\}$$

thus

$$\underbrace{\lambda\delta_1^{-4}\overbrace{\begin{pmatrix} \oslash & \oslash & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -8 & 20 & -8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -8 & 20 & -8 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \oslash & \oslash \end{pmatrix}}^{Bendingenergymodel(firstpart)}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j-\frac{3}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j-\frac{1}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j+\frac{1}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j+\frac{3}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j+\frac{5}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}}_{\lambda\delta_1^{-4}\{cos(j-\frac{3}{2})\frac{k\pi}{N}-8cos(j-\frac{1}{2})\frac{k\pi}{N}+20cos(j+\frac{1}{2})\frac{k\pi}{N}-8cos(j+\frac{3}{2})\frac{k\pi}{N}+cos(j+\frac{5}{2})\frac{k\pi}{N}\}} =$$

where $\lambda\delta_1^{-4}\{cos(j-\frac{3}{2})\frac{k\pi}{N}-8cos(j-\frac{1}{2})\frac{k\pi}{N}+20cos(j+\frac{1}{2})\frac{k\pi}{N}-8cos(j+\frac{3}{2})\frac{k\pi}{N}+cos(j+\frac{5}{2})\frac{k\pi}{N}\} = \lambda\delta_1^{-4}\{14-8cos\frac{k\pi}{N}+(2-2cos\frac{k\pi}{N})^2\}cos(j+\frac{1}{2})\frac{k\pi}{N}$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j-\frac{3}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j-\frac{1}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j+\frac{1}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j+\frac{3}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ cos(j+\frac{5}{2})\frac{k\pi}{N} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\underbrace{\overbrace{\begin{pmatrix} \lambda\delta_1^{-4}\{14-8cos\frac{k\pi}{N}+(2-2cos\frac{k\pi}{N})^2\} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}}^{eigenvalues}}_{\lambda}$$

Linear elastic energy model:

$$\begin{pmatrix} 0 & -(2\mu+\lambda)\delta_1^{-2} & 0 \\ -\mu\delta_2^{-2} & \mu(4\delta_1^{-2}+2\delta_2^{-2})+2\lambda\delta_1^{-2} & -\mu\delta_2^{-2} \\ 0 & -(2\mu+\lambda)\delta_1^{-2} & 0 \end{pmatrix} = (2\mu+\lambda)\delta_1^{-2}\overbrace{\begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}}^{eigenvaluesastheabove} + (\mu+\lambda)\delta_1^{-2}\overbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}}^{eigenvl.astheabove}$$

$$\begin{pmatrix} 0 & -\mu\delta_1^{-2} & 0 \\ -(2\mu+\lambda)\delta_2^{-2} & \mu(4\delta_2^{-2}+2\delta_1^{-2})+2\lambda\delta_2^{-2} & -(2\mu+\lambda)\delta_2^{-2} \\ 0 & -\mu\delta_1^{-2} & 0 \end{pmatrix} = (2\mu+\lambda)\delta_1^{-2}\overbrace{\begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}}^{eigenvaluesastheabove} + (\mu+\lambda)\delta_1^{-2}\overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}}^{eigenvl.astheabove}$$