WU TIANWEI

Question 3:

Solution:

(i): $A=\begin{bmatrix}0 & 1\\ -4 & -5\end{bmatrix}$; $B=\begin{bmatrix}0\\ 1\end{bmatrix}$; $T=0.5\ \text{sec}$

$$\Rightarrow [sI-A]^{-1}=\begin{bmatrix}\frac{s+5}{(s+1)(s+4)} & \frac{1}{(s+1)(s+4)}\\ \frac{-4}{(s+1)(s+4)} & \frac{s}{(s+1)(s+4)}\end{bmatrix}$$

$$\Rightarrow \phi(t)=\mathcal{L}^{-1}\{[sI-A]^{-1}\}=\begin{bmatrix}\frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t} & \frac{1}{3}e^{-t}-\frac{1}{3}e^{-4t}\\ -\frac{4}{3}e^{-t}+\frac{4}{3}e^{-4t} & -\frac{1}{3}e^{-t}+\frac{4}{3}e^{-4t}\end{bmatrix}$$

$$\Rightarrow \phi(T)=\begin{bmatrix}\frac{4}{3}e^{-\frac{1}{2}}-\frac{1}{3}e^{-2} & \frac{1}{3}e^{-\frac{1}{2}}-\frac{1}{3}e^{-2}\\ -\frac{4}{3}e^{-\frac{1}{2}}+\frac{4}{3}e^{-2} & -\frac{1}{3}e^{-\frac{1}{2}}+\frac{4}{3}e^{-2}\end{bmatrix}=\begin{bmatrix}0.7636 & 0.1571\\ -0.6283 & -0.0217\end{bmatrix}$$

$$\Theta(T)=\left[\int_0^T \phi(t)dt\right]\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0.0591\\ 0.1571\end{bmatrix}$$

$$\Rightarrow Ad=\begin{bmatrix}0.7636 & 0.1571\\ -0.6283 & -0.0217\end{bmatrix},\quad Bd=\begin{bmatrix}0.0591\\ 0.1571\end{bmatrix}$$

$$\Rightarrow x(k+1)=\begin{bmatrix}0.7636 & 0.1571\\ -0.6283 & -0.0217\end{bmatrix}x(k)+\begin{bmatrix}0.0591\\ 0.1571\end{bmatrix}u(k)$$

(ii): $x(k+1)=Ad\,x(k)+Bd\,u(k)$

Taking z-transform and with $x(0)=0$,

$$zX(z)=AdX(z)+BdU(z)$$

$$\Rightarrow X(z)=[zI-Ad]^{-1}Bd\,U(z)$$

$$\Rightarrow \frac{X(z)}{U(z)}=[zI-Ad]^{-1}Bd=\begin{bmatrix}\frac{(4e^2-e^{\frac{1}{2}}-3e^{\frac{5}{2}})z-e^2+4e^{\frac{1}{2}}-3}{-12e^{\frac{5}{2}}z^2+(12e^2+12e^{\frac{1}{2}})z-12}\\ \frac{-(e^2-e^{\frac{1}{2}})z+e^2-e^{\frac{1}{2}}}{-3e^{\frac{5}{2}}z^2+(3e^2+3e^{\frac{1}{2}})z-3}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{-8.64z-3.7942}{-146.1899z^2+108.4533z-12}\\ \frac{-5.7403z+5.7403}{-36.5475z^2+27.1133z-3}\end{bmatrix}=\begin{bmatrix}\frac{0.0591z+0.026}{z^2-0.7419z+0.0821}\\ \frac{0.1571z-0.1571}{z^2-0.7419z+0.0821}\end{bmatrix}$$