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Question 1:

Solution:

$y(k) = c(k), x_1(k) = c(k), x_2(k) = c(k+1), x_3(k) = c(k+2)$

$\Rightarrow y(k) = x_1(k)$

$x_1(k+1) = c(k+1) = x_2(k)$

$x_2(k+1) = c(k+2) = x_3(k)$

$x_3(k+1) = c(k+3) = -7x_1(k) - 5x_2(k) - 3x_3(k) + 9u(k)$

$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}, \quad C = [1 \ 0 \ 0], d = 0$

Therefore, the state-space representation for the system is

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} u(k)$$

$$y(k) = [1 \ 0 \ 0] x(k)$$

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Question 2:

Solution:

If $x_3(k) = u(k)$, we have

$$0.1x_3(k+1) + 0.2x_3(k) = 0.3e(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0.9 \end{bmatrix} x_3(k)$$

$$\Rightarrow \quad x_1(k+1) = 0.4x_1(k) + 0.5x_2(k) + 0.8x_3(k)$$

$$x_2(k+1) = 0.6x_1(k) + 0.7x_2(k) + 0.9x_3(k)$$

$$x_3(k+1) = -2x_3(k) + 3e(k)$$

$$y(k) = [1 \quad 2] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$\Rightarrow \quad y(k) = [1 \quad 2 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 0.4 & 0.5 & 0.8 \\ 0.6 & 0.7 & 0.9 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \quad C = [1 \quad 2 \quad 0], \quad d = 0$$

$$\Rightarrow \quad x(k+1) = \begin{bmatrix} 0.4 & 0.5 & 0.8 \\ 0.6 & 0.7 & 0.9 \\ 0 & 0 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} e(k); \quad x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

$$y(k) = [1 \quad 2 \quad 0]\, x(k)$$

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Question 3:

Solution:

(i): $A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $T = 0.5$ sec

$$\Rightarrow [sI-A]^{-1} = \begin{bmatrix} \frac{s+5}{(s+1)(s+4)} & \frac{1}{(s+1)(s+4)} \\ \frac{-4}{(s+1)(s+4)} & \frac{s}{(s+1)(s+4)} \end{bmatrix}$$

$$\Rightarrow \phi(t) = \mathcal{L}^{-1}\{[sI-A]^{-1}\} = \begin{bmatrix} \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} \\ -\frac{4}{3}e^{-t} + \frac{4}{3}e^{-4t} & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix}$$

$$\Rightarrow \phi(T) = \begin{bmatrix} \frac{4}{3}e^{-\frac{1}{2}} - \frac{1}{3}e^{-2} & \frac{1}{3}e^{-\frac{1}{2}} - \frac{1}{3}e^{-2} \\ -\frac{4}{3}e^{-\frac{1}{2}} + \frac{4}{3}e^{-2} & -\frac{1}{3}e^{-\frac{1}{2}} + \frac{4}{3}e^{-2} \end{bmatrix} = \begin{bmatrix} 0.7636 & 0.1571 \\ -0.6283 & -0.0217 \end{bmatrix}$$

$$\Theta(T) = \left[\int_0^T \phi(\tau)d\tau\right]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.0591 \\ 0.1571 \end{bmatrix}$$

$$\Rightarrow A_d = \begin{bmatrix} 0.7636 & 0.1571 \\ -0.6283 & -0.0217 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.0591 \\ 0.1571 \end{bmatrix}$$

$$\Rightarrow x(k+1) = \begin{bmatrix} 0.7636 & 0.1571 \\ -0.6283 & -0.0217 \end{bmatrix} x(k) + \begin{bmatrix} 0.0591 \\ 0.1571 \end{bmatrix} u(k)$$

(ii): $x(k+1) = A_d x(k) + B_d u(k)$

Taking z-transform and with $x(0) = 0$,

$$z X(z) = A_d X(z) + B_d U(z)$$

$$\Rightarrow X(z) = [zI - A_d]^{-1} B_d U(z)$$

$$\Rightarrow \frac{X(z)}{U(z)} = [zI - A_d]^{-1} B_d = \begin{bmatrix} \frac{(4e^2 - e^{\frac{1}{2}} - 3e^{\frac{5}{2}})z - e^2 + 4e^{\frac{1}{2}} - 3}{-12e^{\frac{5}{2}}z^2 + (12e^2 + 12e^{\frac{1}{2}})z - 12} \\ \frac{-(e^2 - e^{\frac{1}{2}})z + e^2 - e^{\frac{1}{2}}}{-3e^{\frac{5}{2}}z^2 + (3e^2 + 3e^{\frac{1}{2}})z - 3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-8.64z - 3.7942}{-146.1899z^2 + 108.4533z - 12} \\ \frac{-5.7403z + 5.7403}{-36.5475z^2 + 27.1133z - 3} \end{bmatrix} = \begin{bmatrix} \frac{0.0591z + 0.026}{z^2 - 0.7419z + 0.0821} \\ \frac{0.1571z - 0.1571}{z^2 - 0.7419z + 0.0821} \end{bmatrix}$$

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(iii): $x(k+1) = A_d x(k) + B_d u(k)$

$\Rightarrow$ the characteristic equation is

$\det(\lambda I - A_d) = 0$

$$\Rightarrow \begin{vmatrix} \lambda - \frac{4}{3}e^{-\frac{1}{2}} + \frac{1}{3}e^{-2} & -\frac{1}{3}e^{-\frac{1}{2}} + \frac{1}{3}e^{-2} \\ \frac{4}{3}e^{-\frac{1}{2}} - \frac{4}{3}e^{-2} & \lambda + \frac{1}{3}e^{-\frac{1}{2}} - \frac{4}{3}e^{-2} \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (e^{-\frac{1}{2}} + e^{-2})\lambda + e^{-\frac{5}{2}} = 0$$

$$\Rightarrow (\lambda - e^{-\frac{1}{2}})(\lambda - e^{-2}) = 0$$

$$\Rightarrow \lambda_1 = e^{-\frac{1}{2}}, \lambda_2 = e^{-2}$$

So, the eigenvalues of A_d are $\lambda_1 = e^{-\frac{1}{2}}$ and $\lambda_2 = e^{-2}$

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Question 4:

Solution:

(a): $x(k+1) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} x(k) + \begin{bmatrix} 3 \\ 4 \end{bmatrix} u(k)$

$y(k) = \begin{bmatrix} 5 & 6 \end{bmatrix} x(k)$

$\Rightarrow A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, C = \begin{bmatrix} 5 & 6 \end{bmatrix}, d = 0$

$\Rightarrow |\lambda I - A| = \lambda^2 - 3\lambda + 2 - 2 = \lambda^2 - 3\lambda$

$\Rightarrow a_0 = 0, a_1 = -3$

$\Rightarrow W_o = \begin{bmatrix} 5 & 6 \\ 11 & 22 \end{bmatrix}, A_o = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}, C_o = \begin{bmatrix} 0 & 1 \end{bmatrix}, d_o = d = 0$

$\Rightarrow \tilde{W}_o = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$; $W_o^{-1} = \frac{1}{110-66} \begin{bmatrix} 22 & -6 \\ -11 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{22} \\ -\frac{1}{4} & \frac{5}{44} \end{bmatrix}$

$\Rightarrow Q = W_o^{-1} \tilde{W}_o = \begin{bmatrix} -\frac{3}{22} & \frac{1}{11} \\ \frac{5}{44} & \frac{1}{11} \end{bmatrix}$

$\Rightarrow Q^{-1} = \frac{1}{\frac{1}{11}(-\frac{3}{22} - \frac{5}{44})} \begin{bmatrix} \frac{1}{11} & -\frac{1}{11} \\ -\frac{5}{44} & -\frac{3}{22} \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 5 & 6 \end{bmatrix}$

$\Rightarrow B_o = Q^{-1} B = \begin{bmatrix} -4 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 39 \end{bmatrix}$

$\Rightarrow$ The state-space representation in the OCF form is

$w(k) = \begin{bmatrix} -4 & 4 \\ 5 & 6 \end{bmatrix} x(k)$

$w(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} w(k) + \begin{bmatrix} 4 \\ 39 \end{bmatrix} u(k)$

$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} w(k)$

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(b): From (a), we have

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix},\ B = \begin{bmatrix} 3 \\ 4 \end{bmatrix},\ C = [5 \quad 6],\ d = 0$$

$a_0 = 0,\ a_1 = -3$

$$\Rightarrow W_c = \begin{bmatrix} 3 & 11 \\ 4 & 11 \end{bmatrix},\ A_c = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix},\ B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ d_c = d = 0$$

$$\Rightarrow \tilde{W}_c = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \Rightarrow \tilde{W}_c^{-1} = -1 \times \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow P = W_c \tilde{W}_c^{-1} = \begin{bmatrix} 3 & 11 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

$$\Rightarrow C_c = CP = [5 \quad 6] \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = [4 \quad 39]$$

$\Rightarrow$ The state-space representation in the CCF form is

$$w(k) = P^{-1} x(k) = \begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \end{bmatrix} x(k)$$

$$w(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} w(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [4 \quad 39] w(k)$$

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Question 5:

Solution:

(a): According to the system given, we have

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{2}{125}x_1(t) + \frac{1}{625}u(t)$$

$$y_1(t) = x_1(t)$$

$$\Rightarrow \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{2}{125} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{625} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

(b): $A = \begin{bmatrix} 0 & 1 \\ -\frac{2}{125} & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{625} \end{bmatrix}$, $C = [1 \ 0]$, $d = 0$, $T = 0.5$ sec

$$\Rightarrow \phi(t) = \mathcal{L}^{-1}\{[sI-A^{-1}]\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{125s}{125s^2+2} & \frac{125}{125s^2+2} \\ -\frac{2}{125s^2+2} & \frac{125s}{125s^2+2} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \cos\frac{\sqrt{10}}{25}t & \frac{5}{2}\sqrt{10}\sin\frac{\sqrt{10}}{25}t \\ -\frac{\sqrt{10}}{25}\sin\frac{\sqrt{10}}{25}t & \cos\frac{\sqrt{10}}{25}t \end{bmatrix}$$

$$\Rightarrow \phi(T) = \phi(0.5) = \begin{bmatrix} 0.998 & 0.4997 \\ -0.008 & 0.998 \end{bmatrix} = Ad$$

$$\theta(T) = \left[\int_0^{0.5}\phi(\tau)d\tau\right]\begin{bmatrix} 0 \\ \frac{1}{625} \end{bmatrix} = \begin{bmatrix} 1.999\times10^{-4} \\ 7.995\times10^{-4} \end{bmatrix} = Bd$$

$\Rightarrow$ The ZOH equivalent of the continuous-time system is

$$x(k+1) = \begin{bmatrix} 0.998 & 0.4997 \\ -0.008 & 0.998 \end{bmatrix} x(k) + \begin{bmatrix} 1.999\times10^{-4} \\ 7.995\times10^{-4} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

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(c): $[zI - A_d + B_d K] =$

$$\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} \cos\frac{\sqrt{10}}{50} & \frac{5}{2}\sqrt{10}\sin\frac{\sqrt{10}}{50} \\ -\frac{\sqrt{10}}{25}\sin\frac{\sqrt{10}}{50} & \cos\frac{\sqrt{10}}{50} \end{bmatrix} + \begin{bmatrix} -\frac{1}{10}\cos\frac{\sqrt{10}}{50} + \frac{1}{10} \\ \frac{1}{250}\sqrt{10}\sin\frac{\sqrt{10}}{50} \end{bmatrix} [k_1 \ \ k_2]$$

$$= \begin{bmatrix} z - \cos\frac{\sqrt{10}}{50} + k_1\left(\frac{1}{10} - \frac{1}{10}\cos\frac{\sqrt{10}}{50}\right) & -\frac{5}{2}\sqrt{10}\sin\frac{\sqrt{10}}{50} + k_2\left(\frac{1}{10} - \frac{1}{10}\cos\frac{\sqrt{10}}{50}\right) \\ \frac{\sqrt{10}}{25}\sin\frac{\sqrt{10}}{50} + \frac{1}{250}\sqrt{10}\sin\frac{\sqrt{10}}{50}\cdot k_1 & z - \cos\frac{\sqrt{10}}{50} + \frac{1}{250}\sqrt{10}\sin\frac{\sqrt{10}}{50}\cdot k_2 \end{bmatrix}$$

$$\Rightarrow \alpha(z) = z^2 + \left[\left(\frac{1}{10} - \frac{1}{10}\cos\frac{\sqrt{10}}{50}\right)k_1 + \frac{\sqrt{10}}{250}\sin\frac{\sqrt{10}}{50}\cdot k_2 - 2\cos\frac{\sqrt{10}}{50}\right] z +$$

$$\left[1 - \frac{\sqrt{10}}{250}\sin\frac{\sqrt{10}}{50}\cdot k_2 - \left(10\cos\frac{\sqrt{10}}{50} - \frac{1}{10}\right)k_1\right] \equiv \alpha_c(z) = z^2$$

$$\Rightarrow K = \left[\frac{10\cos\frac{\sqrt{10}}{50} - 5}{1 - \cos\frac{\sqrt{10}}{50}}, \ \frac{25\sqrt{10} + 50\sqrt{10}\cos\frac{\sqrt{10}}{50}}{2\sin\frac{\sqrt{10}}{50}}\right] = [2490.8 \ \ 1873.7]$$

$$\Rightarrow x(k+1) = [A - BK]\, x(k)$$

$$= \begin{bmatrix} 0.5 & 0.125 \\ -2 & -0.5 \end{bmatrix} x(k)$$

The response to any $x(0) = [a \ \ b]^T$:

$$\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.125 \\ -2 & -0.5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0.5a + 0.125b \\ -2a - 0.5b \end{bmatrix}$$

$$\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.125 \\ -2 & -0.5 \end{bmatrix} \begin{bmatrix} 0.5a + 0.125b \\ -2a - 0.5b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$\Rightarrow x(k) = 0$, $k = 2, 3, 4, \cdots$

So, the response is clearly deadbeat.