TUNING OF PID CONTROLLERS BASED ON GAIN AND PHASE MARGIN SPECIFICATIONS

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Abstract. An analytical method is proposed to tune the PI/PID controller to pass through two design points on the Nyquist curve as specified by the gain and phase margins. They are the appropriate use of pole-zero cancellation and algebraic simplification of certain nonlinear functions. The required process parameters can be simply computed on the basis of the ultimate gain, ultimate period and static gain which may be supplied by the well-known relay-feedback or other auto-tuners. The excellent performance of the tuning formulae has been substantiated by extensive simulation.

Key Words. Adaptive control; PID control; gain and phase margins; auto-tuning; step response.

1. Introduction

The recent interest in industrial applications of auto-tuning, adaptive and intelligent PID controllers [1-3] has stimulated renewed research work on controller tuning formulae. The problem is by no means trivial as the PI or PID controller has only two or three parameters whereas the process is usually characterized by high order dynamics, or a combination of low order dynamics, dead time and/or non-minimum phase dynamics. One approach was to critically examine the classical Ziegler-Nichols tuning formula and incorporate refinements based on deep control knowledge and heuristics [4]. It has been found from extensive simulation that the modified Ziegler-Nichols formulae give good responses over a fairly wide range of process dynamics [4]. They are however based largely on heuristics and are difficult to be modified for processes outside the range of dynamics for which it works well, for instance a process with integration.

Another approach was to use the gain margin or phase margin to derive the controller tuning formula analytically [5]. The gain and phase margins are very useful as they can serve as the measures of performance as well as robustness. However, the design to satisfy the gain margin may not assure a good phase margin, and vice versa [5]. A proposed solution [1,5] is to achieve a compromise in phase and gain margins by designing to move the compensated Nyquist curve to pass through a specified design point (e.g. $0.5\angle -\frac{3}{4}\pi$). It has been found subsequently that the satisfaction of one design point often does not provide a good compromise in the required gain and phase margin of the system. For instance, the performance of this tuning method is found to be too conservative when the dead time is large [6] and both the set-point and load disturbance responses get very sluggish.

In this paper, we propose an analytical method to tune the PID controller to pass through two design points on the Nyquist curve as specified by the gain margin (A_m) and phase margin (ϕ_m). This method is based on the measurement of the ultimate gain, ultimate period and the static gain of the process. An intermediate step of computing the simplified process model parameters (gain, time constant and dead-time) is performed and they are then used in the tuning formula. It avoids a comprehensive system identification and hence achieve a faster tuning time. Several simulation examples will be used to substantiate the performance of the design method.

2. Tuning Formula of PID Controller for the Processes with Large Dead Time (Θ>0.3)

It has been found in the analysis that much simplification will be achieved when the controller zero can be used to cancel the major process pole. This approach has not been recommended in the literature [1] because it could lead to poor load disturbance response. However, when the process dead time or non-minimum phase dynamics is significant, for instance, when the normalized dead time Θ (defined as the dead time L normalized by the major time constant τ; i.e. $\Theta = L/\tau$) is larger than 0.3, the load disturbance recovery time is larger than the major time constant of the process and hence it does not matter whether pole-zero cancellation takes place as long as the pole to be canceled is well damped.

2.1 Derivation of Formulae

Denote the process and controller transfer functions as $G_p(s)$ and $G_c(s)$, gain and phase crossover frequencies as ω_g and ω_p, and the specified gain and phase margins as A_m and ϕ_m, respectively. The following set of equations has to be satisfied:

$$\arg G_c(j\omega_p)G_p(j\omega_p) = -\pi \tag{1}$$

$$A_m = \frac{1}{|G_c(j\omega_p)G_p(j\omega_p)|} \tag{2}$$

$$|G_c(j\omega_g)G_p(j\omega_g)| = 1 \tag{3}$$

$$\phi_m = \arg G_c(j\omega_g)G_p(j\omega_g) + \pi \tag{4}$$

The PID controller is given by

$$G_c(s) = k_c(1 + \frac{1}{sT_i} + sT_d) \tag{5}$$

and the process is assumed to be

$$G_p(s) = \frac{k_p}{(1+s\tau_1)^2} e^{-sL_1} \tag{6}$$

Note that the above process model is only used for the purpose of simplified analysis. The actual process may have multiple lags, non-minimum phase zero, etc. but they can all be modeled by (6) provided they give rise to an over-damped response. For the process model of (6), at the phase crossover frequency (which is equal to the ultimate frequency ω_u), the relations of the ultimate gain (k_u), ultimate period (t_u) and the model parameters L_1 and τ_1 can be obtained:

$$k_u = \frac{1+\omega_u^2\tau_1^2}{k_p} \tag{7}$$

$$t_u = \frac{2\pi}{\omega_u} \tag{8}$$

$$\omega_u L_1 = \pi - 2\arctan(\sqrt{k_u k_p - 1}) \tag{9}$$

and L_1 and τ_1 can be estimated from Eqns. (7)-(9):

$$\tau_1 = \frac{t_u}{2\pi}\sqrt{k_u k_p - 1} \tag{10}$$

$$L_1 = \frac{t_u}{2\pi}(\pi - 2\arctan\frac{2\pi\tau_1}{t_u}) \tag{11}$$

From (5) and (6) and making the choice of $T_d = T_i/4$, we have

$$G_c(s)G_p(s) = \frac{k_c k_p (1+\frac{1}{2}sT_i)^2}{sT_i(1+s\tau_1)^2}e^{-sL_1} \tag{12}$$

Choosing $T_i = 2\tau_1$ to achieve pole-zero cancellation, the above greatly simplifies to:

$$G_c(s)G_p(s) = \frac{k_c k_p}{sT_i}e^{-sL_1}$$

Substituting into Eqns. (1) to (4), we obtain

$$\omega_p L_1 = \frac{\pi}{2} \tag{13}$$

$$A_m k_c k_p = \omega_p T_i \tag{14}$$

$$k_c k_p = \omega_g T_i \tag{15}$$

$$\phi_m = \frac{\pi}{2} - \omega_g L_1 \tag{16}$$

Eliminating ω_p and ω_g using (13) and (16), we obtain

$$\boxed{A_m(\frac{\pi}{2} - \phi_m) = \frac{\pi}{2}} \tag{17}$$

Eqn. (17) gives the constraint for applying pole-zero cancellation in that the gain and phase margins have to be specified accordingly, such as the following pairs:

ϕ_m	= 72°	67.5°	60°	45°	30°
A_m	= 5	4	3	2	1.5

The pair (60°; 3) is found to be most appropriate experimentally and will be the default values assumed.

Using Eqns. (7), (8), (9), (13) and (14), we obtain the following simple tuning formulae:

$$k_c = \frac{\pi\tau_1}{A_m k_p L_1} \tag{18}$$

$$T_i = 2\tau_1 \tag{19}$$

$$T_d = 0.5\tau_1 \tag{20}$$

The above can be easily implemented with on-line measurement of k_p, k_u, and t_u provided by the relay auto-tuner or self-tuning controller. Although a process model of Eqn. (6) was assumed in the analysis, the tuning formulae are applicable to any over-damped process of having higher order or non-minimum phase dynamics.

2.2 Simulation Results

With $A_m = 3, \phi_m = 60°$, the following controller tuning results shown in Table 1 are obtained for different processes.

The resultant values of A_m, ϕ_m, ω_p and ω_g, (marked by *) are fairly close to the theoretical values of $A_m=3$ and $\phi_m=60°$ for the second order plus dead time model. The formulae also give satisfactory results for the higher order or non-minimum phase processes which are modeled by the second order plus dead time model. Typical simulation results shown in Fig. 1 demonstrate the good set-point and load disturbance responses of the processes with wide range dynamics. The default amplitude of the load disturbances is chosen as 0.5 in the simulations.

Table 1. Controller parameters based on Eqns. (18)-(20)

Process Model	k_c	T_i	T_d	A_m^*	ϕ_m^*	ω_p^*	ω_g^*
$\frac{e^{-0.5s}}{(1+s)^2}$	2.09	2.00	0.50	3.06	58.5	3.20	1.10
$\frac{e^{-1.0s}}{(1+s)^2}$	1.05	2.00	0.50	3.06	58.5	1.60	0.55
$\frac{e^{-5.0s}}{(1+s)^2}$	0.21	1.99	0.50	3.03	58.4	0.32	0.11
$\frac{1}{(1+s)^5}$	1.14	3.77	0.94	3.38	62.5	0.90	0.33
$\frac{1-s}{(1+s)^3}$	0.67	2.00	0.50	3.00	52.4	1.00	0.34

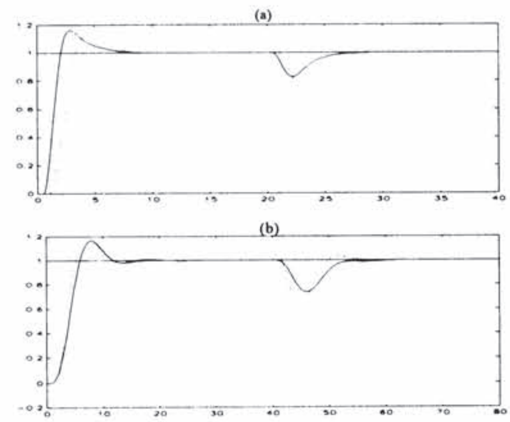

Fig. 1 Set-point and Load Disturbance Responses of PID Control Using Formula for Large Θ

(a) $G_p(s) = e^{-sL_2}/(1+s)^2$: $L_2 = 0.5$ (—); $L_2 = 1.0$ (.....).

(b) $G_p(s) = 1/(1+s)^5$ (—) and $G_p(s) = (1-s)/(1+s)^3$ (.....)

3. Tuning Formula of PID Controller for the Processes with Small Dead Time (Θ<0.3)

When the normalized dead time is small, the above pole-zero cancellation will result in poor load responses. However, the simulation study shows that the performance is satisfactory if only one process pole is canceled. In this section, we shall derive the tuning formulae for the controller for the case where Θ<0.3. The tuning formula is also based on the model as Eqn. (6) and derived for the series form of PID controller for the reason of simplification. It is easy to convert the PID controller parameters of series form to the parallel form [9]. The series form of PID controller is given by

$$G_c(s) = \frac{k_c'(1+sT_i')(1+sT_d')}{sT_i'} \tag{21}$$

3.1 Derivation of Formula

Choosing $T_d' = \tau_1$, thus canceling one process pole by one controller zero, we get

$$G_c(s)G_p(s) = \frac{k_c'(1+sT_i')}{sT_i'}\frac{k_p e^{-sL_1}}{1+s\tau_1} \tag{22}$$

Substituting (22) into Eqns. (1) to (4), we obtain

$$\frac{\pi}{2} + \arctan\omega_p T_i' - \arctan\omega_p\tau_1 - \omega_p L_1 = 0 \tag{23}$$

$$A_m k_c' k_p = \omega_p T_i' \sqrt{\frac{\omega_p^2 \tau_1^2 + 1}{\omega_p^2 T_i'^2 + 1}} \qquad (24)$$

$$k_c' k_p = \omega_g T_i' \sqrt{\frac{\omega_g^2 \tau_1^2 + 1}{\omega_g^2 T_i'^2 + 1}} \qquad (25)$$

$$\phi_m = \frac{\pi}{2} - \omega_g L_1 + \arctan\omega_g T_i' - \arctan\omega_g \tau_1 \qquad (26)$$

For $\Theta < 0.3$, numerical solution shows that $x > 1$, where x is one of $\omega_p T_i'$, $\omega_p \tau_1$, $\omega_g T_i'$ and $\omega_g \tau_1$. The arctan function can be approximated as follows:

$$\arctan x \approx \frac{\pi}{2} - \frac{\pi}{4x}, \quad x > 1 \qquad (27)$$

Since $x >> 1$, we can obtain the simplification of Eqns (23) to (26) and get

$$k_c' = \frac{\omega_p \tau_1}{A_m k_p}$$

$$T_i' = \frac{1}{2\omega_p - \frac{4\omega_p^2 L_1}{\pi} + \frac{1}{\tau_1}}$$

where τ_1, L_1 can be obtained from Eqns (10), (11) respectively and

$$\omega_p = \frac{A_m \phi_m + \frac{\pi}{2} A_m (A_m - 1)}{(A_m^2 - 1) L_1}$$

Converting the parameters of series PID form to those of parallel PID form we have

$$k_c = k_c'(1 + \frac{T_d'}{T_i'}) \qquad (28)$$

$$T_i = T_i' + T_d' \qquad (29)$$

$$T_i = \frac{T_i' T_d'}{T_i' + T_d'} \qquad (30)$$

3.2 Simulation Examples

Numerical results and the simulations for $G_p(s) = e^{-0.1s}/(1+s)^2$ are shown in Table 2 and Fig. 2.

Table 2. Controller parameters based on Eqns. (28)-(30)

Specified		*Resultant*						
A_m	ϕ_m	k_c	T_i	T_d	A_m^*	ϕ_m^*	ω_p^*	ω_g^*
3	45	18.85	1.35	0.26	2.91	41.6	14.5	5.45
5	45	11.31	1.35	0.26	4.84	46.5	14.5	3.65
3	60	10.47	2.00	0.50	3.01	59.9	15.8	5.25
5	60	8.70	1.54	0.35	4.95	58.5	15.2	3.35

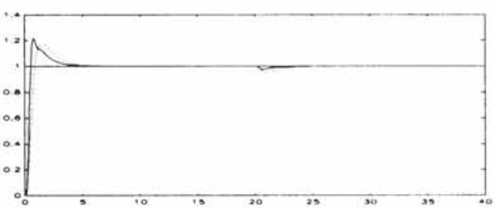

Fig. 2 Set-point and Load Disturbance Responses of PID Control Using Formula for Small Θ: $G_p(s) = e^{-0.1s}/(1+s)^2$

($A_m = 3$; $\phi_m = 45^o$, —) and ($A_m = 5$; $\phi_m = 60^o$, ...)

4. Formula of Tuning PI Controller

PI control is a very common control structure in industries. Formulae of tuning PI controller based on the first order plus dead time model of (31) have also been derived. When the actual process is of higher order dynamics or non-minimum phase, a second order plus dead time model of (6) can be used to model it and be approximated to the form of (31) in the further step as discussed in section 4.3. The details can be found in [10]. A brief presentation is given in the follows.

4.1 Formula for Processes with Large Θ

The first order plus dead time process is given by

$$G(s) = \frac{k_p}{1 + s\tau} e^{-sL} \qquad (31)$$

The PI controller takes the form

$$G_c(s) = k_c(1 + \frac{1}{sT_i}) \qquad (32)$$

Substituting Eqns. (31) and (32) into Eqns. (1) to (4) and introducing pole-zero cancellation we obtain the following simple tuning formulae:

$$k_c = \frac{\pi\tau}{2 A_m k_p L} \qquad (33)$$

$$T_i = \tau \qquad (34)$$

where

$$\tau = \frac{t_u}{2\pi}\sqrt{k_u^2 k_p^2 - 1}$$

$$L = \frac{t_u}{2\pi}(\pi - \arctan\frac{2\pi\tau}{t_u})$$

When we use the above formulae the gain and phase margins have to be specified accordingly. The constraint is given by Eqn. (17) as in the case of PID control.

With $A_m = 3, \phi_m = 60°$, the following controller tuning results shown in Table 3 are obtained for different processes .

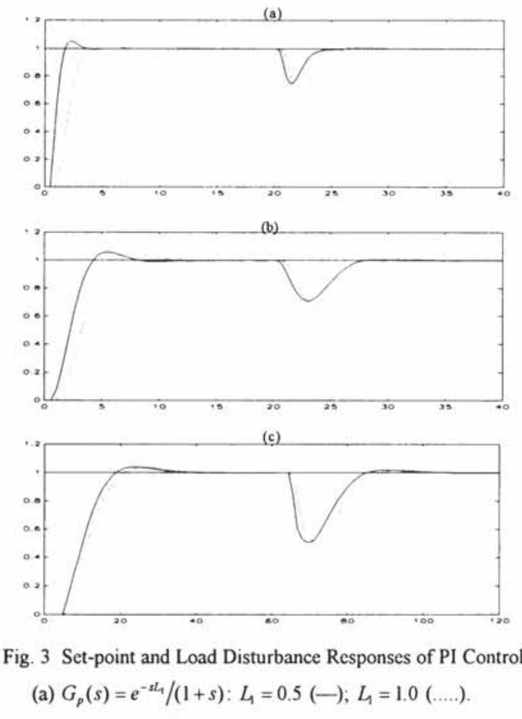

Fig. 3 Set-point and Load Disturbance Responses of PI Control

(a) $G_p(s) = e^{-sL_1}/(1+s)$: $L_1 = 0.5$ (—); $L_1 = 1.0$ (.....).

(b) $G_p(s) = e^{-sL_2}/(1+s)^2$: $L_2 = 0.5$ (—); $L_2 = 1.0$ (.....).

(c) $G_p(s) = e^{-5s}/(1+s)^n$: $n = 1$ (—); $n = 2$ (.....).

The resultant values of A_m, ϕ_m, ω_p and ω_g (marked by *) are fairly close to the theoretical values for A_m=3 and ϕ_m=60°. The excellent set-point and load disturbance responses are shown in Fig. 3 (a). When the dead time of the process is very large, the tuning formula still gives very good performance shown as Fig. 3 (c). Using the approximation of Section 4.3 we can design the

Table 3. Controller parameters based on Eqns. (33) and (34)

Process model	k_c	T_i	A_m^*	ϕ_m^*	ω_p^*	ω_g^*
$\frac{e^{-0.5s}}{1+s}$	1.05	1.00	3.00	60.0	3.15	1.05
$\frac{e^{-1.0s}}{1+s}$	0.52	1.00	3.00	60.0	1.56	0.53
$\frac{e^{-5.0s}}{1+s}$	0.10	1.00	3.01	59.9	0.32	0.11

PI controller for higher order processes. The examples are shown in Fig. 3 (b) and 3 (c).

4.2 Formula for Processes with Small Θ

Substituting Eqns. (31) and (32) into (1) to (4), we obtain the following tuning formulae with the approximation of arctan function as Eqn. (27) and assumptions of large values for $\omega_p T_i$, $\omega_p \tau$, $\omega_g \tau$ and $\omega_g T_i$.

$$k_c = \frac{1}{A_m k_p}\omega_p \tau \tag{35}$$

$$T_i = \frac{1}{(2\omega_p - \frac{4\omega_p^2 L}{\pi} + \frac{1}{\tau})} \tag{36}$$

where

$$\omega_p = \frac{A_m \phi_m + \frac{\pi}{2} A_m (A_m - 1)}{(A_m^2 - 1)L}$$

Results for $G_p(s) = e^{-0.1s}/(1+s)$ are shown in Table 4 and the performances are shown in Fig. 4.

Table 4. Controller parameters based on Eqns. (38) and (39)

Specified		*Resultant*					
A_m	ϕ_m	k_C	T_i	A_m^*	ϕ_m^*	ω_p^*	ω_g^*
2.5	45	5.98	0.41	2.44	41.9	14.76	6.33
5	45	2.95	0.35	4.83	46.6	14.46	3.62
3	60	5.24	1.00	3.00	60.0	15.71	5.24
5	60	3.05	0.54	4.94	58.5	15.16	3.35

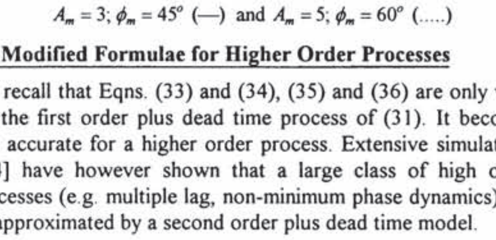

Fig. 4 Set-point and Load Disturbance Responses of PI Control Using Formula for Small Θ $G_p(s) = e^{-0.1s}/(1+s)$: $A_m = 3$; $\phi_m = 45^\circ$ (—) and $A_m = 5$; $\phi_m = 60^\circ$ (.....)

4.3 Modified Formulae for Higher Order Processes

We recall that Eqns. (33) and (34), (35) and (36) are only valid for the first order plus dead time process of (31). It becomes less accurate for a higher order process. Extensive simulations [3,4] have however shown that a large class of high order processes (e.g. multiple lag, non-minimum phase dynamics) can be approximated by a second order plus dead time model.

Eqns (10) and (11) can be used to obtain τ_1 and L_1 from k_u and t_u. As the tuning formulae of Eqns. (33) and (34), (35) and (36) are in terms of the first order plus dead time model parameters τ and L, a conversion step is required. The least-squares-fitting method of Sundaresan and Krishnaswamy [7,8] can be used to fit the step responses of the models of Eqn. (31) and Eqn. (6) to give τ and L. The conversion formulae are:

$$L = 1.3t_1 - 0.29t_2 \tag{37}$$

$$\tau = 0.67(t_2 - t_1) \tag{38}$$

where t_1 and t_2 are the time taken for the unit step response of the process of Eqn. (6) to reach 0.35 and 0.85 respectively. Both t_1 and t_2 can be solved analytically from Eqn. (6) and hence we can get τ and L from Eqns. (37) and (38).

5. Discussion

When the normalized dead time Θ is small, choosing the $A_m \sim \phi_m$ pairs showed in Eqn. (17) will lead to pole-zero cancellation, in this case those $A_m \sim \phi_m$ pairs are not suggested to use. The results also show that the tuning formulae of PI and PID controller can be used in the range of $2 \le A_m \le 5$ and $45^\circ \le \phi_m \le 75^\circ$. In this region the formula gives satisfactory approximation.

It was suggested that PI control is sufficient when the process dynamics is essentially of the first order [1]. In our study it shows that a PID controller cannot improve both the set-point and load disturbance responses more than a PI controller with the same gain and phase margin specification for the first order with dead time process. Therefore PI control is suggested to be used when the process dynamics is essentially of the first order plus dead time.

6. Conclusion

The tuning formulae have been derived for both PI and PID controllers to achieve specified gain and phase margins of the controlled system. They give the analytical relations between controller parameters and specifications. The simplified formulae which are based on pole-zero cancellation may be applied for a process with moderate and large normalized dead time. When the process is with small normalized dead time pole-zero cancellation is not advisable. Another two sets of formulae have been derived for tuning PI and PID respectively. For higher order processes, a set of conversion formulae has been proposed to overcome the modeling approximation problem. Simulation results have been presented to substantiate the excellent results achieved.

Reference

[1] Åström K. J. and T. Hägglund (1988): *Automatic Tuning of PID Controllers*, Instrument Society of America.

[2] Åström K. J. and B. Wittenmark (1989): *Adaptive Control*, Addison-Wesley.

[3] Åström K.J., C.C. Hang, P.Persson and W.K.Ho (1991): *Towards Intelligent PID Control*, Automatica, Vol. 28, No.1, pp. 1-9.

[4] Hang C. C., K. J. Åström and W. K. Ho (1991): *Refinements of the Ziegler-Nichols Tuning Formula*, IEE Proceedings-D, Vol. 138, No. 2, pp. 111-118.

[5] Åström K. J. and T. Hägglund (1984): *Automatic Tuning of Simple Regulators with Specifications on Phase and Ampli-tude Margins*, Automatica, Vol. 20, No. 5, pp. 645-651.

[6] Hang C. C. and K. K. Sin (1991): *A Comparative Performance Study of PID Auto-tuners*, IEEE Control Systems, Vol. 11, No. 5, pp. 41-47.

[7] Seborg D.E., T. F. Edgar and D. A. Mellichamp (1989): *Process Dynamics and Control*, John Wiley & Sons.

[8] Sundaresan K. R. and P. R. Krishnaswamy (1978): *Estimation of Time Delay Time Constant Parameters in Time, Frequency and Laplace Domains*, The Canadian Journal of Chemical Engrg., Vol. 56, pp. 257-262.

[9] Corrpio (1990): *Tuning of Industrial Control Systems*, Instrument Society of America.

[10] Ho W. K., C. C. Hang and L. S. Cao (1992): *Tuning of PI Controllers Based on Gain and Phase Margin Specifications*, Proceedings of the IEEE International Symposium on Industrial Electronics, Vol. 2, pp. 879-882, China.