Problems and Subjects in Mathematical Analysis

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1 Problems

Problem 1. $\alpha > 1$, then

$$\int_0^\infty \frac{dx}{1+x^\alpha} = \frac{\pi}{\alpha \sin(\pi/\alpha)}.$$

Proof. Let $t = \frac{1}{1+x^\alpha}$, then we have:

$$\int_0^\infty \frac{dx}{1+x^\alpha} = \frac{1}{\alpha}\int_0^1 (1-t)^{\frac{1}{\alpha}-1} t^{-\frac{1}{\alpha}} dt = \frac{1}{\alpha} B\left(\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right) = \frac{\pi}{\alpha \sin(\pi/\alpha)}.$$

□

Problem 2 (A Deeper Discussion of Weierstrass-Stone Theorem). The classic Weierstrass-Stone theorem implies for any $f \in C[a,b]$, there exists a sequence of real polynomials $\{P_n\}_{n\geq 1} \subset \mathbb{R}[X]$, such that P_n converges to f uniformly.

(a) Let $f \in C([a,b])$ be a monotonic increasing function, then there exists a sequence of polynomials with real coefficients $\{P_n\}_{n\geq 1} \subset \mathbb{R}[X]$ such that for all n, P_n is monotonic increasing on $[0,1]$ and that $P_n \rightrightarrows f$.

Proof. Denote $I = [a,b]$. Firstly, if $f \in C^1([a,b])$, w.l.o.g. assume that $f(a) = 0$, then $f(x) = \int_a^x f'(t)dt$. As $f' \in C([a,b])$ is non-negative, one can find a sequence of real polynomials $P_n^0 \rightrightarrows f$. Set $e_n := \inf_{x\in[a,b]} P_n^0(x)$. Trivially, $\liminf e_n \geq 0$, whence $P_n^1 := P_n^0 - \min\{0, e_n\} \rightrightarrows f$. And P_n^1 is non-negative. Then $P_n := \int_a^x P_n^1 dt \rightrightarrows f$ and each P_n is increasing on $[a,b]$. At last one can approximate any increasing continuous function by increasing C^1 function. (For instance, reflect f along $x = b$ and one can get a continuous function with support containing in $[a, 2a-b]$ and then convolve this function with bump functions.) □

(b) (Walsh) Assume that $f \in C([0,1])$, $x_1, \cdots, x_m$ are m points on $[0,1]$, then there exists a sequence of real polynomials $\{P_n\}_{n\geq 1} \subset \mathbb{R}[X]$, such that each P_n coincide with f on each x_i, $i = 1, \cdots, m$ and that $P_n \rightrightarrows f$.

Proof. We have an interpolation $Q \in \mathbb{R}[X]$ such that P coincide with f on each x_i, i.e. one can assume that x_i are all zeros of f. Consider piecewise linear functions l_n that is locally constant at each x_i, i.e. $l_n(x) = f(x_i)$ near x_i. It's trivial that one can always find a sequence $l_n \rightrightarrows f$. So it remains to consider f that is differentiable at each x_i. In this case $\overline{f} := f \cdot \prod_{i=1}^m (x-x_i)^{-1} \in C([a,b])$. Therefore we can find a sequence of polynomials $P_n \rightrightarrows \overline{f}$. Thus $P_n \cdot \prod_{i=1}^m (x-x_i) \rightrightarrows f$. □

(c) (Chudnovsky) Assume that $f \in C(I)$, $I = [a,b] \subset]0,1[$, then there exists a sequence of polynomials $\{P_n\}_{n\geq 1} \subset \mathbb{Z}[X]$ on $\mathbb{Z}$ such that $P_n \rightrightarrows f$.

Proof. Let $p(x) = 2x(1-x)$ and that $p^{\circ n}$ the nth composition of p, then the inductive relation $\|p^{\circ n} - \frac{1}{2}\|_\infty \leq 2\|p^{\circ(n-1)} - \frac{1}{2}\|_\infty^2$ implies $p^{\circ n} \rightrightarrows \frac{1}{2}$ on I. Then for any $k \in \mathbb{Z}$, there exists a sequence of polynomials on $\mathbb{Z}$ that converge uniformly to 2^k on I. Whence there exists a sequence of polynomials on $\mathbb{Z}$ that converges uniformly to any $\alpha \in \mathbb{R}$ on I. That is to say $\mathbb{R}[X]$ is contained in the closure of $\mathbb{Z}[X]$. But Stone-Weierstrass theorem yields that $\mathbb{R}[X]$ is dense in $C([a,b])$, we have $\overline{\mathbb{Z}[X]} = C([a,b])$. □

Reference. Hervé Pépin, Nicolas Tosel, Approximation par des polynômes à coefficients dans $\mathbb{Z}$, RMS, 114 *ème année, 2003-2004*

Remark. One of the key point in the whole proof is that continuous functions on compact sets are uniformly continuous.

Problem 3. Find all $f \in C(\mathbb{R})$ such that for any $x, y \in \mathbb{R}$,

$$f(x)f(y) = \int_{x-y}^{x+y} f(t)dt.$$

Problem 4. Given a sequence of pairwise distinct real numbers $\alpha_1, \cdots, \alpha_{2020}$ and any sequence of non-zero real numbers $a_1, \cdots, a_{2020}$, then

$$a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + \cdots + a_{2020} x^{\alpha_{2020}}$$

has at most 2019 roots on $]0, \infty[$.

Problem 5 (A Soblev inequality). Let $[a, b] \subset \mathbb{R}$ be an arbitrary bounded closed interval, then fro any $\varepsilon > 0$, there exists a constant $D_\varepsilon > 0$ such that

$$\sup_{x\in[a,b]} |f(x)|^2 \leq D_\varepsilon \int_a^b f(x)^2 dx + \varepsilon \int_a^b f'(x)^2 dx,$$

for all $f \in C^1([a, b])$.

Sketch proof. Apply Cauchy-Schwarz inequality and one can get

$$|f(x)^2 - f(a)^2| \leq C_\varepsilon \int_a^b f(x)^2 dx + \varepsilon \int_a^b f'(x)^2 dx, \qquad \forall f \in C^1([a, b])$$

where $C_\varepsilon > 0$ is a constant. □

Problem 6 (Gauss approximation). Let

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n ((x^2 - 1)^n)$$

be the Legendre polynomials and denote the n's real roots of P_n by $-1 < x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} < 1$. Then there exists $\alpha_1^{(n)}, \cdots, \alpha_n^{(n)} \in \mathbb{R}$ such that for any polynomial $Q(x)$ with degree less than or equal to $2n - 1$, one have

$$\int_{-1}^1 Q(x)dx = \sum_{i=1}^n \alpha_i^{(n)} Q(x_i^{(n)}).$$

Moreover for any $\varphi \in C([-1, 1])$,

$$\lim_{n\to\infty} \sum_{i=1}^n \alpha_i^{(n)} \varphi(x_i^{(n)}) = \int_{-1}^1 \varphi(x)dx.$$

Problem 7 (Equidistribution). Let $\{x_k\}_{k\geq1}$ be a sequence in $[0, 1]$ for any $[a, b] \subset [0, 1]$, let $S_n([a, b]) = \#\{x_k \in [a, b] : k \leq n\}$. We say $\{x_k\}_{k\geq1}$ is an **equidistrisbution** on $[0, 1]$ if for any $[a, b] \subset [0, 1]$

$$\lim_{n\to\infty} \frac{S_n([a, b])}{n} = b - a.$$

(a) An equidistribition on $[0, 1]$ is dense but the converse is not valid.

(b) Let

$$D_n^* = \sup_{0<b<1} \left| \frac{S_n([0, b])}{n} - b \right|,$$

then $\{x_k\}_{k\geq1}$ is an equidistribution iff $\lim_{n\to\infty} D_n^* = 0$.

Proof. Sketch proof Let

$$D_n := \sup_{0\leq a<b\leq1} \left| \frac{S_n([a, b])}{n} - (b - a) \right|,$$

then $D_n \leq D_n^* \leq 2D_n$. $\square$

(c) $\{x_k\}_{k\geq 1}$ be an equidistribution on $[0,1]$ iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} f(x_k) = \int_0^1 f(x)dx$$

for any $f \in C([0,1])$.

(d) (Weyl criterion) $\{x_k\}_{k\geq 1}$ is an equidistrubution on $[0,1]$ iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi p x_k} = 0$$

for any $p \in \mathbb{Z}_{\geq 1}$.

(e) Let $\theta > 0$, then $\{\{n\theta\}\}_{n\geq 1}$ is an equidistruibution on $[0,1]$ iff $\theta \notin \mathbb{Q}$, where $\{x\}$ is the fractional part of x.

(f) $\{\{\sqrt{n}\}\}_{n\geq 1}$ is an equidistribution on $[0,1]$.

(g) Given $a \neq 0$, $\sigma \in]0,1[$, then $\{\{an^\sigma\}\}_{n\geq 1}$ is an equidistrubution on $[0,1]$.

(h) $\{\{a \ln n\}\}_{n\geq 1}$ is not an equidistruibution.

Remark. This problem may has something to do with ergodic theory.

Problem 8 (Linking Number of closed plane curve). Let $f : \mathbb{R} \to \mathbb{C}^\times$ be a continuously differentiable function with period 2π, set

$$d(f) := \frac{1}{2\pi i} \int_0^{2\pi} \frac{f'(t)}{f(t)} dt.$$

(a) Such $d(f)$ is well-defined and must be an integer.

(b) If $g : \mathbb{R} \to \mathbb{C}^\times$ is also continuously differentiable with period 2π, and that $\|f - g\|_\infty < \varepsilon$, then $d(f) = d(g)$. And this can be applied to define the linking number of a continuous function with period 2π.

(c) If $F : \mathbb{R} \times [0,1] \to \mathbb{C}^\times$ is continuous and that for any $t \in [0,1]$, $F(x,s)$ is continuously differentiable in x, then

$$d(F(x,0)) = d(F(x,1)).$$

(d) $\mathbb{C}$ is algebraically closed.

Sketch proof. Let $P \in \mathbb{C}[x]$ such that $P(0) \neq 0$, then there respectively exists $\varepsilon_0, R_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[$ and $R \in]R_0, \infty[$, $P(\varepsilon e^{\pi i x})$ and $P(Re^{\pi i x})$ does not admit zeros. $\square$

Problem 9 (Bolzano's Continuous but Nowhere Differentiable Function). Define a sequence of functions on $[0,1]$ inductively. Set $f_0(x) = x$, for $n \geq 0$ and $0 \leq k \leq 3^n$,

$$\begin{cases} f_{n+1}(\frac{k}{3^n}) & = f_n(\frac{k}{3^n}), \\ f_{n+1}(\frac{k}{3^n} + \frac{1}{3^{n+1}}) & = f_n(\frac{k}{3^n} + \frac{2}{3^{n+1}}), \\ f_{n+1}(\frac{k}{3^n} + \frac{2}{3^{n+1}}) & = f_n(\frac{k}{3^n} + \frac{1}{3^{n+1}}), \end{cases}$$

And in the interval of form $[\frac{k}{3^{n+1}}, \frac{k+1}{3^{n+1}}]$ is linear. Then f_n uniformly converges to some continuous function $f \in C([0,1])$ and f is nowhere differentiable.

Problem 10. Let $\{f_n\}_{n\geq 1} \subset C([a,b])$ be a sequence of continuous functions. Assume that

$$\sum_{n=1}^{\infty} \left(\int_1^b |f_n(x)| dx \right) \tag{1}$$

converges. Then $E := \{x_0 \in [a,b] : \sum_{n=1}^\infty |f_n(x_0)| \text{ converges}\}$ is dense in $[a,b]$. Conversely for any dense set $F \subset [a,b]$, there exists a sequence of continuous functions on $[a,b]$ such that (1) converges while $\sum_{n=1}^\infty |f_n(x_0)|$ diverges for any $x_0 \in F$.

Problem 11 (ζ function on $\operatorname{Re} s = \frac{1}{2}$). In this problem $\{x\}$ stands for the fractional part of x.

(a) For any $x \in \mathbb{R}$ and $q \in \mathbb{Z}_{\geq 1}$, let

$$S_q(x) = \sum_{p=1}^{q} \frac{\sin(2p\pi x)}{p\pi} = \int_0^x \frac{\sin((2q+1)\pi u)}{\sin(\pi u)} du - x.$$

Then $S_q(x)$ converges to some $S(x)$ pointwisely. Moreover $S_q \rightrightarrows S$ on every compact subset of $\mathbb{R}$ that does not meet $\mathbb{Z}$.

Sketch proof. $S_q(x)$ is uniformly bounded and

$$S_q(x) + x - \frac{1}{\pi}\int_0^{(2q+1)\pi x} \frac{\sin u}{u} du = O\left(\frac{1}{q}\right).$$

□

(b) Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0, z \neq 1\}$, then for any $z \in \Omega$ and $y \geq n$,

$$\zeta(z) - \sum_{k=1}^{n} k^{-z} = \frac{n^{1-z}}{z-1} - \frac{1}{2}n^{-z} + z\int_y^\infty \left(\frac{1}{2} - \{x\}\right) x^{-z-1} dx + \frac{z}{\pi}\sum_{p=1}^{\infty}\frac{1}{p}\int_n^y x^{-z-1}\sin(2p\pi x) dx.$$

Sketch proof. For any bounded closed interval $[a,b]$ and $f \in C([a,b];\mathbb{C})$, we have

$$\int_a^b f(x)\left(\frac{1}{2} - \{x\}\right) dx = \frac{1}{\pi}\sum_{p=1}^{\infty}\frac{1}{p}\int_a^b f(x)\sin(2p\pi x) dx.$$

□

(c) For any $n \geq 1$ and $t \in [-n, n]$,

$$\zeta(\frac{1}{2} + it) - \sum_{k=1}^{n} k^{-\frac{1}{2}-it} = O\left(\frac{\sqrt{n}}{1+|t|}\right).$$

Sketch proof. Given $p \in \mathbb{Z}_{\geq 1}$ and $t \in [-n, n]$, set

$$g : [n, \infty[\to \mathbb{R}, \qquad x \mapsto \frac{-x^{-\frac{3}{2}}}{2p\pi - \frac{t}{x}},$$

then for any $x \geq n$,

$$g'(x) = O\left(\frac{x^{-\frac{5}{2}}}{p}\right)$$

and for any $y \geq x \geq n$

$$\int_x^y x^{-\frac{3}{2}-it} e^{2ip\pi x} dx = O\left(\frac{1}{n^{\frac{3}{2}}p}\right), \qquad \int_n^y x^{-\frac{3}{2}-it}\sin(2p\pi x) dx = O\left(\frac{1}{n^{\frac{3}{2}}p}\right).$$

□

(d) For any $T > 0$

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt = O(T\ln T).$$

Sketch proof. For $n \geq 2$ we have the following estimates:

$$\sum_{\substack{j,k \\ 1\leq k\leq n, \frac{k}{2}<j<k}} \frac{1}{\sqrt{kj}\ln\left(\frac{k}{j}\right)} \leq \sqrt{2} \sum_{\substack{h,k \\ 1\leq h\leq k\leq n}} \frac{1}{h} = O(n\ln n)$$

$$\sum_{\substack{j,k \\ 1\leq k\leq n, 1\leq j\leq \frac{k}{2}}} \frac{1}{\sqrt{kj}\ln\left(\frac{k}{j}\right)} \leq \frac{1}{\ln 2}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right)^2 = O(n)$$

$$\int_0^n \left|\sum_{k=1}^{n} k^{-\frac{1}{2}-it}\right|^2 dt = n\sum_{k=1}^{n}\frac{1}{k} + i \sum_{\substack{j,k,j\neq k \\ 1\leq j,k\leq n}} \frac{\left(\frac{k}{j}\right)^{-in} - 1}{\sqrt{kj}\ln\left(\frac{k}{j}\right)} = O(n\ln n).$$

□

Problem 12 (Quasi-periodic functions). Let $E := \operatorname{span}_{\mathbb{R}}\{\cos(\omega x), \sin(\omega x) : \omega \in \mathbb{R}\} \subset C(\mathbb{R})$. And let $F = \overline{E}$, then F is a $\mathbb{R}$-algebra.

(a) If $\varphi \in C(\mathbb{R})$ and $f \in F$, then $\varphi \circ f \in F$.

(b) Given any $\omega \in \mathbb{R}$ and $f \in F$, let

$$\widetilde{f}(\omega) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\omega x} f(x)dx.$$

Then for a fixed f, $\widetilde{f}(\omega) = 0$ except for at most countable $\omega \in \mathbb{R}$.

Problem 13 (Korovkin Theorem). Let $E = C([0,1])$. A linear operator T on E is said to be **positive** if $Tf \geq 0$ whenever $f \geq 0$ (i.e. $f(x) \geq 0$) for all $x \in \mathbb{R}$.

(a) A positive operator is bounded, i.e. if T is a positive operator, then there exists a constant C such that

$$\|Tf\|_\infty \leq C\|f\|_\infty.$$

(b) (Korovkin) Let T_n be a sequence of positive operator on E, if $T_n(x^n)$ uniformly converges to x^n for all $n \geq 0$, then for any $f \in E$, $T_n f \rightrightarrows f$.

Sketch proof. For any $\varepsilon > 0$, there exists a constant C_ε such that for any $x, y \in [0,1]$, we have

$$|f(x) - f(y)| \leq \varepsilon + C_\varepsilon(x-y)^2.$$

□

(c) Let

$$B_n : E \to E, \qquad f \mapsto \left(x \mapsto \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k}\right)$$

then $B_n f \rightrightarrows f$.

Problem 14 (Euler expansion of cotangent function). Consider

$$f(x) := \lim \sum_{k=-n}^{n} \frac{1}{x+k}, \qquad x \in \mathbb{R}\backslash\mathbb{Z}.$$

$f(x) - \pi\cot(\pi x)$ has a continuous continuation on $\mathbb{R}$ and therefore $f(x) = \pi\cot(\pi x)$ for any $x \notin \mathbb{Z}$.

Sketch proof. For any $x \in \mathbb{R}\backslash\mathbb{Z}$, $f(-x) = -f(x)$ and $f(x+1) = f(x)$; for any $x \in \mathbb{R}\backslash(\frac{1}{2}\mathbb{Z})$,

$$2f(2x) = f(x) + f\left(x + \frac{1}{2}\right).$$

□

Problem 15 (A Criterion of Uniform Convergence). Let $\{f_n\}$ be a sequence of functions (not necessarily continuous) such that for any convergent sequence $\{x_n\} \subset [0,1]$, the sequence $\{f(x_n)\}$ converges as well. Then f_n uniformly converges to some continuous function on $[0,1]$.

Problem 16 (Pau Lévy). Let $E = \{f \in C([0,1] : f(0) = f(1))\}$. For any $f \in E$, let

$$\Lambda(f) := \{\sigma \in [0,1] : \exists x \in [0,1](f(x+\sigma) = f(x))\}.$$

Then

$$\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{Z}_{\geq 1}\right\} = \bigcap_{f \in E} \Lambda(f).$$

Problem 17. Let $f \in C^\infty(\mathbb{R})$ and assume that $f \geq 0$ and vanishes only at 0. If $f''(0) \neq 0$, then there exists a smooth function $g \in C^\infty(\mathbb{R})$, such that $g^2 = f$.

Problem 18. For any $n \in \mathbb{N}$, the following improper integral is 0:

$$\int_0^\infty x^n \sin(x^{1/4}) e^{-x^{1/4}} dx = 0.$$

Problem 19. Let $\{a_n\}$ be a strictly increasing sequence and $a_0 = 0$, if the following series diverges

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots.$$

And if $f \in C([0,1])$ is such that

$$\int_0^1 x^{a_n} f(x) dx = 0 \qquad \forall n \in \mathbb{N}$$

then $f = 0$.

Problem 20. If $f \in C^\infty(\mathbb{R})$ is such that $f(0)f'(0) \geq 0$ and $\lim_{x\to\infty} f(x) = 0$, then there exists $0 \leq x_1 < x_2 < \cdots$ such that

$$f^{(n)}(x_n) = 0.$$

Problem 21 (Isometry on Metric Spaces). Let (X,d) be a compact metric space and $f : X \to X$.

(a) If f is an isometry then f is bijective.

(b) If $d(f(x), f(y)) \geq d(x,y)$ for all $x, y \in X$, then f is an isometry.

(c) If f is surjective and $d(f(x), f(y)) \leq d(x,y)$ for all $x, y \in X$, then f is an isometry.

Problem 22. Let (X,d) be a complete metric space and $f : X \to X$.

(a) If (X,d) is compact and $d(f(x), f(y)) < d(x,y)$ for all $x \neq y \in X$, then f admits a unique fixed point. The compact condition here is necessary.

(b) Let $\omega : [0,\infty[\to \mathbb{R}$ is a right-continuous function and $\omega(0) = 0$. If $0 \leq \omega(t) < t$ for all $t > 0$ and

$$d(f(x), f(y)) \leq \omega(d(x,y)) \qquad \forall x, y \in X,$$

then f admits a unique fixed point.

Problem 23. Let $E := C(\mathbb{R})$ and $F := C^\infty(\mathbb{R})$.

(a) What are the $\mathbb{R}-$subalgebras of E.

(b) Taking derivatives $\frac{d}{dx} : F \to F$ is a linear map, is there a linear map $T : F \to F$ such that $T \circ T = \frac{d}{dx}$?

Problem 24 (e is transcendental). Let $P \in \mathbb{R}[x]$, let

$$I(t) = \int_0^t e^{t-x} P(x) dx = e^t \sum_{i=-}^{n} P^{(i)}(0) - \sum_{i=0}^{n} P^{(i)}(t).$$

If there exists $a_0, \cdots, a_n \in \mathbb{Z}$, $a_0 \neq 0$ such that

$$a_0 + a_1 e + \cdots + a_n e^n = 0.$$

For $p \in \mathbb{N}$, set

$$P_p(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p, \qquad \text{and} \qquad J_p = a_0 I(0) + a_1 I(1) + \cdots + a_n I(n).$$

Then $J_p \in \mathbb{Z}$ and $(p-1)! \mid J_p$. For sufficiently large prime number p, $J_p \neq 0$ whence $J_p \geq (p-1)!$. But $\ln(|J_p|) = O(p)$.

Problem 25. Given $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_n$ and each $\beta_i \neq 0$, then

$$f(x) = \sum_{k=1}^{n} \alpha_k \cos(\beta_k x + \gamma_k)$$

has infinitely many zeros on $\mathbb{R}$.

Problem 26. Let $\lambda_1, \cdots, \lambda_m$ be pairwise distinct real numbers and $c_1, \cdots, c_m$ are complex numbers such that

$$\lim_{x\to\infty} c_1 e^{i\lambda_1 x} + c_2 e^{i\lambda_2 x} + \cdots + c_{2019} e^{i\lambda_{2019} x} = 0$$

then $c_1 = c_2 = \cdots = c_{2019} = 0$.

Proof. We can assume that $\lambda_m = 0$, for each $a < b$, we have:

$$\left| \int_a^b \sum_{k=1}^{m} c_k e^{ix\lambda_k} dx \right| \leq \int_a^b \left| \sum_{k=1}^{m} c_k e^{ix\lambda_k} \right| dx \leq (b-a) \sup_{x\in[a,b]} \left| \sum_{k=1}^{m} c_k e^{ix\lambda_k} \right|$$

On other hand,

$$\left| \int_a^b \sum_{k=1}^{m-1} c_k e^{ix\lambda_k} dx \right| \leq \sum_{k=1}^{m-1} |c_k| \cdot \left| \int_a^b e^{i\lambda_k x} dx \right| \leq \sum_{k=1}^{m-1} \frac{2\pi\, |c_k|}{|\lambda_k|}.$$

Then

$$\left| \int_a^b \sum_{k=1}^{m} c_k e^{ix\lambda_k} dx \right| \geq (b-a)|c_m| - \sum_{k=1}^{m-1} \frac{2\pi\, |c_k|}{|\lambda_k|}$$

Thus,

$$|c_m| \leq \frac{1}{b-a} \sum_{k=1}^{m-1} \frac{2\pi |c_k|}{|\lambda_k|} + \sup_{x\in[a,b]} \left| \sum_{k=1}^{m} c_k e^{ix\lambda_k} \right|$$

Let $b = 2a$ and $a \to \infty$, we have $c_m = 0$. Similarly, $c_1 = \cdots = c_m = 0$. $\square$

Problem 27. Is there a sequence of real numbers $\{a_n\}_{n\geq 1}$ such that the series

$$a_1^l + a_2^l + \cdots + a_n^l + \cdots$$

converges when $l = 5$ while it diverges when l is other positive odd numbers?

Problem 28 (Partial Order). Let $(X, \leq)$ be a finite partially ordered set, a totally ordered set $(X, \preceq)$ is said to be a **total-order extension** if $x \preceq y$ whenever $x \leq y$. Denote by N the number of total-order extension of $(X, \leq)$/ And for $x, y \in X$, denote by $N_{x,y}$ the number of total-order extension $(X, \preceq)$ of $(X, \leq)$ such that $x \preceq y$. Now if $(X, \leq)$ is not a totally ordered set, is there always exists $x, y \in X$ such that

$$\frac{1}{3} \leq \frac{N_{x,y}}{N} \leq \frac{2}{3}.$$

Problem 29 (Reconstruction conjecture). For any $n \in \mathbb{N}$ and $n > 3$, let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two partially ordered set of n elements, say $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_n\}$. If for any $j = 1, \cdots, n$, there exists an order-preserving bijection $\varphi_j : X \backslash \{x_j\} \to Y \backslash \{y_j\}$. Then there is an order-preserving bijection $\varphi : X \to Y$.

Problem 30. Let $I =]a,b[\subset \mathbb{R}$ be an open interval, where $-\infty \le a < b \le \infty$. Assume that $f : I \to \mathbb{R}$ is differentiable of second order and let

$$M_n := \sup_{x \in I} |f^{(n)}(x)| \in [0, \infty], \qquad n = 0, 1, 2.$$

Then there are two well-known result that if M_n are all finite and $a \in \mathbb{R}$, $b = \infty$, then

$$M_1^2 \le 4M_0M_2. \tag{2}$$

Besides if $]a,b[= \mathbb{R}$, then

$$M_1^2 \le 2M_0M_2. \tag{3}$$

Moreover the constant 4 and 2 respectively in (2) and (3) can not be smaller and the inequality can be attained. But if we do not omit the assumption that M_n are all finite, will the previous inequality still hold? In this case we always assume for $C > 0$ that

$$C \cdot \infty = \infty, \qquad \infty \cdot \infty = \infty.$$

(We exclude the meaningless situation, i.e. $M_0 = \infty$ and $M_2 = 0$ whence f is linear.)

Problem 31 (Sarkosky Theorem). Let $f : [0,1] \to [0,1]$, a periodic point x of period n is such that $x = f^{\circ(n)}(x)$ and that $f^{\circ(m)} \ne x$ for all $1 \le m < n$ where $f^{\circ(n)} = f \circ f \cdots \circ f$ is the n-th iteration of f. Now if f is continuous and admits a periodic point of period 3. Then for any $n \in \mathbb{N}^*$, f admits at least on period point of period n.

Sketch proof. The existence of 1-periodic point is trivial by intermediate value theorem. Now if $0 \le a < b < c \le 1$ and $f(a) = b$, $f(b) = c$, $f(c) = a$. Let $I_0 = [a,b]$, $I_1 = [b,c]$, then $I_1 \subset f(I_0)$, $I_0 \cup I_1 \subset f(I_1)$, and therefore f admits a 2-periodic point in I_1. One can further find a sequence of closed interval

$$\cdots \subset I_{n+1} \subset I_n \subset \cdots \subset I_2 \subset I_1$$

by induction such that $f(I_{n+1}) = I_n$ for all $n \in \mathbb{N}^*$. In particular, $f^{\circ(n-1)}(I_n) = I_1$. Now for a fixed $n \ge 4$, there exists closed interval $J_n \subset I_0$ such that $f(J_n) = I_{n-1}$ and closed interval $K_n \subset I_1$ such that $f(K_n) = J_n$. Whence $f^{\circ(n)}(K_n) = I_n$, thus $f^{\circ(n)}$ has a fixed point on I_1 and it is a n-periodic point of f. At last if $0 \le a < b < c \le 1$ and $f(a) = c$, $f(c) = b$, $f(b) = a$, f also admits a n-periodic point. □

Problem 32 (σ-algebra can not be countable). Let $\mathscr{F}$ be a σ-algebra on X, then X is either finite or uncountable.

Problem 33 (Semi-ring and pre-measure). (a) Let $\mathscr{S} \subset \mathcal{P}(X)$ be a semi-ring and $\mu : \mathscr{S} \to [0, \infty]$ is such that $\mu(\varnothing) = 0$ and that μ is finite-additive and sub-additive. Then μ is a pre-measure.

(b) Let $\mathscr{S} \subset \mathcal{P}(X)$ be a semiring and $\mu : \mathscr{S} \to [0, \infty]$ a pre-measure. For a monotonic decreasing sequence $A_n \searrow A$ in $\mathscr{S}$ ($A_n, A \in \mathscr{S}$), if $\mu(A_1) < \infty$, then $\mu(A_n) \to \mu(A)$.

(c) Let $\mathscr{S} \subset \mathcal{P}(X)$ be a semiring and $\mu : \mathscr{S} \to [0, \infty]$ a pre-measure. For a monotonic increasing sequence $A_n \nearrow A$ in $\mathscr{S}$ ($A_n, A \in \mathscr{S}$), one have $\mu(A_n) \to \mu(A)$.

Problem 34. Let $(X, \mathscr{F}, \mu)$ be a measure space and $0 < \mu(X) < \infty$, define a metric ρ on $L^0(X, \mathscr{F}, \mu)$ by

$$\rho(f,g) = \inf\{\mu(\{x \in X : |f(x) - g(x)| > \delta\}) + \delta : \delta > 0\}.$$

Then convergence in measure is nothing but convergence in $(L^0(X, \mathscr{F}, \mu), \rho)$. And $(L^0(X, \mathscr{F}, \mu), \rho)$ is complete. However, the a.e. convergence and a.u. convergence in $L^0(\mathbb{R}, \mathscr{M}, m)$ is not induced by metric.

Problem 35 (Orthonormal basis). (a) Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}((x^2 - 1)^n)$$

form an orthonormal basis of $L^2(-1, 1)$.

(b) Laguerre polynomials
$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n)$$
form an orthonormal basis of $L^2(\,]0,\infty[\,; e^{-x}dx)$.

Problem 36. Let ξ be an irrational number and consider
$$A = \mathbb{Z} + \xi\mathbb{Z}, \qquad B = 2\mathbb{Z} + \xi\mathbb{Z}.$$
Apply axiom of choice to form a set N' consisting of the representation element one for each coset of $\mathbb{R}/A$. Set $M = N' + B$, the for any bounded measurable set E,
$$m_*(M \cap E) = 0, \qquad m^*(M \cap E) = m(E),$$
where m_* is the Lebesgue inner measure and m^* is the Lebesgue outer measure.

Problem 37. (a) Find a continuous function $f : (\mathbb{R}, \mathscr{M}) \to (\mathbb{R}, \mathscr{R})$ such that f is not measurable.
(b) Find a continuous function $\Phi \in C(\mathbb{R})$ and $f \in \mathcal{L}_f(\mathbb{R}, \mathscr{M})$ such that $f \circ \Phi \notin \mathcal{L}_f(\mathbb{R}, \mathscr{R})$.

2 Additional Subjects

2.1 Surreal numbers

Conway combined Cantor's and von Neumann's theory of ordinal numbers and Dedekind cut of real numbers and introduced surreal numbers when researching game theory. The construction is established on the following axioms:

(a) Every surreal number x is consists of two sets of surreal numbers X_L and X_R and is denoted by $(X_L|X_R)$, such that for any $x_L \in X_L$ and $x_R \in X_R$, one wouldn't have $x_R \le x_L$.
(b) There is a relation $\le$ in the system of surreal numbers: For any $x = (X_L|X_R)$ and $y = (Y_L|Y_R)$, $x \le y$ iff the following hold:
  (b1) For any $x_L \in X_L$, $y \le x_L$ does not hold;
  (b2) For any $y_R \in Y_R$, $y_R \le x$ does not hold.

  Similar to partial order, we define
  - $x \ge y$ iff $y \le x$;
  - $x = y$ iff $x \le y$ and $y \le x$;
  - $x < y$ iff $x \le y$ and $y \le x$ does not hold;
  - $x > y$ iff $y < x$.

(c) Addition is defined as following: If $x = (X_L|X_R)$, $y = (Y_L|Y_R)$,
$$x + y = ((X_L + y) \cup (x + Y_L)|(X_R + y) \cup (x + Y_R)).$$
(d) Opposite number is defined by: If $x = (X_L|X_R)$, then $-x = (-X_R| - X_L)$.
(e) Multiplication is defined as following: If $x = (X_L|X_R)$, $y = (Y_L|Y_R)$, then $xy = (Z_L|Z_R)$, where
$$Z_L := \{x_L y + x y_L - x_L y_L : x_L \in X_L, y_L \in Y_L\} \cup \{x_R y + x y_R - x_R y_R : x_R \in X_R, y_R \in Y_R\}$$
$$Z_R := \{x_L y + x y_R - x_L y_R : x_L \in X_L, y_R \in Y_R\} \cup \{x_R y + x y_L - x_R y_L : x_R \in X_R, y_L \in Y_L\}.$$

All definition here is inductive. Form the very beginning we only have $0 := (\varnothing|\varnothing)$. It satisfies condition (a) and $-0 = 0$, $0 + 0 = 0$. Next we can define $1 := (\{0\}|\varnothing)$ and $-1 := (\varnothing|\{0\})$ (note that $(\{0\}|\{0\})$ does not satisfies condition (a)). And $-1 < 0 < 1$, $0 + 1 = 1$, $(-1) + 0 = -1$, $1 + (-1) = 0$.

The class of all surreal numbers (it is a proper class) is an ordered "field" with null element 0 and identity element 1.

Let's see some more examples of surreal numbers. Now we have -1,0,1 and can get 8 sets of surreal numbers. And therefore there are 64 possible surreal numbers, but most of them do not satisfies condition (a). Indeed, we only have
$$2 = (\{1\}|\varnothing), \qquad \frac{1}{2} = (\{0\}|\{1\})$$

and their opposites are new. One can also notice that $0 = (\{-1\}|\{1\})$ etc. Say one surreal number may have different presentations. One can also prove that $1 + 1 = 2$ and $\frac{1}{2} + \frac{1}{2} = 1$ while the latter is not essentially trivial. Next we can have $\frac{1}{4}, \frac{3}{4}, \frac{3}{2}, 3$ and their opposites as new surreal numbers. Keep this process we can get all "binary fractions" $\pm\frac{m}{2^k}$, where $m \in \mathbb{Z}$, $k \in \mathbb{N}$. Given these binary fractions, we can further construct all real numbers with process alike Dedekind cut.

Moreover we get $\omega := (\mathbb{N}|\varnothing)$ and $\omega + 1 = (\{\omega\}|\varnothing)$ etc. And this gives all ordinal numbers (whence the class of all surreal numbers is proper). One can also verify that the ordered "field" of all surreal numbers does not satisfies the Archimedean axiom: For any $n \in \mathbb{N}$, $\omega > n$, i.e. ω is an infinite element. We also have an infinitesimal element $\varepsilon := (\{0\}|\{\frac{1}{n} : n \in \mathbb{N}^*\})$ such that $\varepsilon < \frac{1}{n}$ for all $n \in \mathbb{N}^*$.

2.2 p-adic numbers

Definition. Let $\mathbb{F}$ be a field, an absolute value function on $\mathbb{F}$ is a map $|\cdot| : \mathbb{F} \to \mathbb{R}$ such that

(a) for any $x \in \mathbb{F}$, $|x| \geq 0$ and $|x| = 0$ iff $x = 0$;
(b) for any $x, y \in \mathbb{F}$, $|xy| = |x| \cdot |y|$;
(c) (triangle inequality) for any $x, y \in \mathbb{F}$, $|x + y| \leq |x| + |y|$.

An absolute value function on $\mathbb{F}$ induces a metric $d(x, y) := |x - y|$ on $\mathbb{F}$. Indeed, condition (c) can be substituted with a weaker condition:

(c') For any $x, y \in \mathbb{F}$, $|x + y| \leq 2\max(|x|, |y|)$

i.e. (c) can be derived from (a),(b) and (c'). But if we replace the constant 2 in (c') by 1, we get the strong triangle inequality:

(c+) For any $x, y \in \mathbb{F}$, $|x + y| \leq \max(|x|, |y|)$.

An absolute value function satisfying (c+) is called **non-Archimedean** and the metric induced by a non-Archimedean absolute value function is called an **ultrametric**. In this case we have:

(2.2.1) If $0 < |x| < |y|$, then for any $n \in \mathbb{N}$, $|nx| < |y|$.
(2.2.2) If $|x| < |y|$ then $|x + y| = |y|$.

Definition. Two absolute value function $|\cdot|_1$ and $|\cdot|_2$ are equivalent if $|x|_1 < 1$ iff $|x|_2 < 1$ for all $x \in \mathbb{F}$.

One can verify that equivalent absolute value functions define the same convergent sequence. Except for the familiar absolute value on $\mathbb{R}$ and $\mathbb{C}$, there is an trivial absolute value $|\cdot|_0$ on every fields, which is defined by $|x|_0 = \delta_0$. And this is a non-Archimedean absolute value.

Noe we define the p-adic absolute value on $\mathbb{Q}$, where p is a fixed prime number. Firstly for any $x \in \mathbb{Q}^*$, we can uniquely write it as $x = \pm p^v \frac{m}{n}$, where $m, n \in \mathbb{N}^*$ and are pairwise coprime with p. We define $v_p(x) = v$ and set $v_p(0) = \infty$. Then for any $x, y \in \mathbb{Q}$, we have

$$v_p(xy) = v_p(x) + v_p(y), \qquad v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$$

(i.e. v_p is a valuation on $\mathbb{Q}$ with value group $\mathbb{Z}$). Now we define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ by $|x|_p = p^{-v_p(x)}$, (set $p^{-\infty} = 0$) and this is a non-Archimedean absolute value on $\mathbb{Q}$. Moreover, for any $x \in \mathbb{Q}^*$, we have

$$|x|_\infty \cdot \prod_p |x|_p = 1,$$

where $|x|_\infty$ is the usual absolute value and the product is taken over all prime p. Let d_p be the metric induced by $|\cdot|_p$, then $(\mathbb{Q}, d_p)$ is not complete. In particular, Ostrowski theorem yields that any non-trivial absolute value on $\mathbb{Q}$ is either $|\cdot|_\infty$ or $|\cdot|_p$ for some prime p.

Consider the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ under Q_p. We can embed $\mathbb{Q}$ into $\mathbb{Q}_p$ canonically. Then $\mathbb{Q}_p$ is also a field with addition and multiplication induced from the original one on $\mathbb{Q}$. The metric on it is also given by an absolute value function $|\cdot|_p$ and the restriction of this absolute value on $\mathbb{Q}$ is also the same as the original one.

Let $\mathbb{Z}_p = \{x \in \mathbb{Q} : |x|_p \leq 1\}$, then $\mathbb{Z} \subset \mathbb{Z}_p$ and $\mathbb{Z}_p$ is nothing but the completion of $\mathbb{Z}$ w.r.t. the $(p)-$topology (see Atiyah Chapter 10 for precise definition). $\mathbb{Z}_p$ is both open and closed in $\mathbb{Q}_p$. For any $x \in \mathbb{Z}_p$, we can find a sequence $\{a_n\}_{n=0^\infty}$ which takes value in $\{0, 1, \cdots, p-1\}$ such that $\lim_{n\to\infty} \sum_{k=0}^n a_k p^k = x$. We denote $x = \sum_{n=0}^\infty a_n p^n$ as the $p-$adic expansion of x. (This expansion is essentially trivial by definition of p-adic completion).

2.3 Period 3 means chaos

As a matter of fact, Sarkosky proved a stronger proposition as following: Define a total order on N^* by

$$\begin{aligned} 3 \prec 5 \prec 7 \prec \cdots \prec (2n+1) \prec \cdots \\ \prec 3\times 2 \prec 5\times 2 \prec 7\times 2 \prec \cdots \prec (2n+1)\times 2 \prec \cdots \\ \prec 3\times 2^2 \prec 5\times 2^2 \prec 7\times 2^2 \prec \cdots \prec (2n+1)\times 2^2 \prec \cdots \\ \prec \cdots \prec \\ \prec \cdots 2^n \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1 \end{aligned}$$

(it is not a well-order). Then if $f:[0,1]\to[0,1]$ admits a m-periodic point and $m \prec n$, the f admits at least one n-periodic point. In particular, if f admits a 5-periodic point then f must admit all periodic points except for 3-periodic point. Here is an example: $F:[1,5]\to[1,5]$, $F(1)=3$, $F(2)=5$, $F(3)=4$, $F(4)=2$, $F(5)=1$ and F is linear on each $[n,n+1]$, $n=1,2,3,4$. Then F admits a 5-periodic point and no 3-periodic point.

Li-Yorke proved the following result: If $f:[0,1]\to[0,1]$ admits a 3-periodic point, then there exists a uncountable subset $S\subset[0,1]$ containing no periodic point and that

(1) for any $p,q\in S$, $p\neq 1$, we have
$$\limsup_{n\to\infty}|f^{\circ(n)}(p)-f^{\circ(n)}(q)|>0$$
while
$$\liminf_{n\to\infty}|f^{\circ(n)}(p)-f^{\circ(n)}(q)|=0.$$

(2) For any $p\in S$ and a periodic point q, we have
$$\limsup_{n\to\infty}|f^{\circ(n)}(p)-f^{\circ(n)}(q)|>0.$$

2.4 Space of continuous functions

Let (X,d) be a metric space and consider

$$C_b(X):=\{f\in C(X): f \text{ is bounded}\},\qquad C_0(X):\{f\in C(X):\forall\varepsilon>0(f^{-1}(\mathbb{R}\backslash B_0(\varepsilon))) \text{ is compact}\}.$$

If X is compact then $C(X)=C_b(X)=C_0(X)$. In the following discuss we assume that X is locally compact. Then $(C_b(X),\|\cdot\|_{C(X)})$ is a Banach space, where

$$\|f\|_{C(X)}:=\sup_{x\in X}|f(x)|.$$

$C_0(X)$ is a closed (linear) subspace of $C_b(X)$, whence $C_0(X)$ is also a Banach space. If $X=\mathbb{N}$, we denote $C_b(\mathbb{N})$ and $C_0(\mathbb{N})$ respectively by $l^\infty(\mathbb{N})$ and $c_0(\mathbb{N})$. There is a multiplication on $C_b(X)$ and $C_0(X)$ so that they are (commutative) Banach algebra:
$$\|fg\|\le\|f\|\|g\|.$$

In particular, multiplication is continuous. If we consider the corresponding complex value function

$$C_b(X,\mathbb{C}):=\{f\in C(X):|f| \text{ is bounded}\},\qquad C_0(X):\{f\in C(X):\forall\varepsilon>0(|f|^{-1}(\mathbb{R}\backslash B_0(\varepsilon))) \text{ is compact}\}.$$

There is an involution on these space, i.e. conjugation map $*: f\mapsto f^*$, where $f^*(x)=\overline{f(x)}$. It satisfies the following properties:

(a) $\forall (f^*)^*=f$;
(b) $\forall f,g(f+g)^*=f*+g^*$;
(c) $\forall f\forall\lambda\in\mathbb{C}(\lambda f)^*=\overline{\lambda}f^*$;
(d) $\forall fg,(fg)^*=g^*f^*$

whence $C_b(X,\mathbb{C})$ and $C_0(X,\mathbb{C})$ are Banach $*$-algebra. Moreover we have

(e) $\|f*f\|=\|f^*\|\cdot\|f\|$,

a involution satisfies (a)-(e) on a Banach algebra makes it a C^*-algebra. The above definition of Banach algebra, Banach $*$-algebra and C^*-algebra commonly do not require the multiplication to be commutative but to be associative and distributive to addition. A classical non-commutative example is the matrix algebra $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$. Every finite dimensional C^*-algebra is isomorphic to some $M_n(\mathbb{C})$.

Next we consider the (commutative) Banach algebra $\mathcal{A} = C(X)$ with identity, where X is a compact metric space i.e. $\mathcal{A}$ it self is a topological ring. We know $\mathrm{Max}(C(X)) \cong X$ (homeomorphic as topological spaces) where $\mathrm{Max}(C(X))$ is the maximal spectrum of $C(X)$ (see Atiyah Chapter 1).

A character of a Banach algebra $\mathcal{A}$ is a homomorphism of Banach algebras $\varphi : \mathcal{A} \to \mathbb{C}$. Denote the set of all characters on $\mathcal{A}$ by $\mathrm{Spec}(\mathcal{A})$ and is called the **Gelfand spectrum**. For any $\varphi \in \mathrm{Spec}(\mathcal{A})$, $\varphi^{-1}(0) := N(\varphi)$ is a maximal ideal of $\mathcal{A}$. Conversely if I is a maximal ideal, then $\mathcal{A}/I \cong \mathbb{C}$, i.e. we have a quotient map $\varphi : \mathcal{A} \to \mathbb{C}$ and $\varphi \in \mathrm{Spec}(\mathcal{A})$. Therefore, for compact metric space (X, d), $\mathrm{Spec}(\mathcal{A}) \cong X$ (homeomorphic as topological spaces).

If X is not compact, e.g. $X = \mathbb{N}$, $C_b(X) = l^\infty$. There are a lot of maximal ideals except for the common maximal ideal $I_n = \{(a_n)_{n\in\mathbb{N}} : a_n = 0\}$, $n \in \mathbb{N}$, e.g. c_0 is a maximal ideal of l^∞. In fact, $\mathrm{Spec}(l^\infty)$ corresponds to a compactification $\beta\mathbb{N}$ of $\mathbb{N}$ which is called the Stone-Čech compactification.

The above discussion is a basis to study the structure of X by the algebraic structure of $C(X)$, we can further consider non-standard C^*-algebra and regard them as "space of continuous functions" on "non-commutative" topological spaces. This is one of the most fundamental opinion of non-commutative algebra.

2.5 Non-standard Analysis

2.5.1 Filter, Ultrafiter

Definition. For a non-empty set I, let $\mathcal{P}(I)$ be the power set of I. A filter $\mathcal{F}$ is a non-empty subset of $\mathcal{P}(I)$ satisfies the following conditions:

(a) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
(b) if $A \in \mathcal{F}$, and $A \subset B \subset I$, then $B \in \mathcal{F}$.

Moreover if:

(c) $\varnothing \notin \mathcal{F}$;
(d) for any $A \subset I$, exactly one of A and $A^{\complement}$ belongs to $\mathcal{F}$.

Then we say $\mathcal{F}$ is an ultrafilter.

Here are some fundamental examples:

- $\mathcal{P}(X)$ is the only filter that contains $\varnothing$, all other filters are called **proper**;
- $\{I\}$ is the smallest filter and every filter contains I.
- For any $a \in I$, $\mathcal{F}^a := \{A \subset I : a \in A\}$ is the **principal filter** generated by a adn it is a ultrafilter. Note that if I is finite, then every ultrafilter is a principal filter.
- The Fréchet filter, or the cofinite filter on I is
$$\mathcal{F}^{co} = \{A \subset I : \#(A^{\complement}) < \infty\}.$$
$\mathcal{F}^{co}$ is proper iff I is finite; $\mathcal{F}^{co}$ is an ultrafilter.
- Set $\varnothing \neq \mathcal{H} \subset \mathcal{P}(I)$, the filter generated by $\mathcal{H}$ is the minimal filter generated containing $\mathcal{H}$, i.e.
$$\mathcal{F}^{\mathcal{H}} = \{A \subset I : A \supset B_1 \cap \cdots \cap B_n, n \in \mathbb{N}, B_j \in \mathcal{H}\}.$$
If $\mathcal{H} = \varnothing$, denote $\mathcal{F}^{\mathcal{H}} = \{I\}$.

The following are some basic propositions of filter and ultrafilter:

(1) If $\mathcal{F}$ is an ultrafilter, $\{A_1, \cdots, A_n\}$ is a sequence of pairwise disjoint sets such that $A_1 \cup \cdots \cup A_n \in \mathcal{F}$, then exactly one of $A_j \in \mathcal{F}$.
(2) If an ultrafilter contains a finite set then it must be principal. So every non-principal ultrafilter must contain every all cofinte sets.
(3) $\mathcal{F}$ is an ultrafilter on I iff $\mathcal{F}$ is a maximal proper filter.

(4) We say $\mathcal{H} \subset \mathcal{P}(I)$ possesses finite intersection property, if any non-empty finite subset of $\mathcal{H}$ has a non-empty intersection. The filter $\mathcal{F}^{\mathcal{H}}$ is proper iff $\mathcal{H}$ possesses finite intersection property.

(5) By Zorn's lemma, if $\mathcal{H} \subset \mathcal{P}(I)$ possesses finite intersection property, then one can extend to an ultrafilter on I. As a corollary, any infinite set has non-principal ultrafilter.

2.5.2 Hyperreal numbers

Now we consider the ring of real sequence $\mathbb{R}^{\mathbb{N}}$, where addition $\oplus$ and multiplication $\odot$ are termwise addition and multiplication with null element $\mathbf{0} = (0, 0, \cdots)$ and identity element $\mathbf{1} = (1, 1, \cdots)$. Take a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ and define an equivalence relation $\equiv$ on $\mathbb{R}^{\mathbb{N}}$ by

$$\langle r_n \rangle \equiv \langle s_n \rangle \Leftrightarrow [[r = s]] := \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}.$$

We usually say that $\langle r_n \rangle$ and $\langle s_n \rangle$ coincide on a big set or are the same for almost all n, or are almost the same modulo $\mathcal{F}$. We have:

(1) If $r \equiv s$ and $r' \equiv s'$ then $r \oplus r' \equiv s \oplus s'$, $r \odot r' \equiv s \odot s'$.

(2) $\varepsilon := \langle 1, \frac{1}{2}, \frac{1}{3}, \cdots \rangle \not\equiv \mathbf{0}$.

Let ${}^*\mathbb{R} = \mathbb{R}/\equiv$ and denote the equivalence class of r by $[r]$. We can define addition $+$, multiplication $\cdot$ and order $\leq$ on ${}^*\mathbb{R}$:

- $[r] + [s] = [r \oplus s]$;
- $[r] \cdot [s] = [r \odot s]$;
- $[r] \leq [s] \Leftrightarrow [[r \leq s]] := \{n \in \mathbb{N} : r_n \leq s_n\} \in \mathcal{F}$.

In particular, $({}^*\mathbb{R}, +, \cdot, \leq)$ is an ordered field with null element $[\mathbf{0}]$ and identity element $[\mathbf{1}]$. And we can embed $\mathbb{R}$ into ${}^*\mathbb{R}$ by $a \mapsto \mathbf{a} := \langle a, a, \cdots \rangle =^* a$ (this map is order-preserving). Moreover, we have an infinitesimal element $[\varepsilon]$ such that ${}^*0 < [\varepsilon] <^* a$ for all $a \in \mathbb{R}_{>0}$ and an infinite element $[\omega] = [\langle 1, 2, 3, \cdots \rangle] = [\varepsilon]^{-1}$ such that ${}^*a < [\omega]$ for all $a \in \mathbb{R}$. As $[\varepsilon], [\omega] \in^* \mathbb{R} \backslash \mathbb{R}$, ${}^*\mathbb{R}$ is a proper extension of $\mathbb{R}$. ${}^*\mathbb{R}$ does not satisfies Archimedean axiom neither: $n[\varepsilon] > 1$ for all $n \in \mathbb{N}$.

For $A \subset \mathbb{R}$, the extension ${}^*A \subset^* \mathbb{R}$ is defined as follow: For any $r \in \mathbb{R}^{\mathbb{N}}$,

$$[r] \in^* A \Leftarrow [[r \in A]] := \{n \in \mathbb{N} : r_n \in A\} \in \mathcal{F}.$$

The elements in ${}^*A \backslash A$ is called the "non-standard" element of A. For example

(1) $[\omega] \in^* \mathbb{N}$ is a non-standard nature number or hypernature number.

(2) any infinite subset of $\mathbb{R}$ admits non-standard element while every finite subset does not.

(3) ${}^*\digamma$ is a subring of ${}^*\mathbb{R}$, and the element in ${}^*\mathbb{Z}$ are called non-standard integers or hyperintegers and elements in ${}^*\mathbb{Q}$ are called non-standard rational numbers or hyperrational numbers.

(4) ${}^*(\mathbb{R}_{>0}) = ({}^*\mathbb{R})_{>^*0}$.

For any $f : \mathbb{R} \to \mathbb{R}$ we can extend to ${}^*f :^* \mathbb{R} \to^* \mathbb{R}$: For any $r \in \mathbb{R}^{\mathbb{N}}$, denote $f \circ r := \langle f(r_n) \rangle$ and let ${}^*f([r]) = [f \circ r]$. Some more subjects of real numbers can be extend to ${}^*\mathbb{R}$ analogously. Even though the structure of ${}^*\mathbb{R}$ is independent of the choice of the non-principal ultrafilter $\mathbb{F}$, whether some sentence is true may has something to do with the choice, e.g. ${}^*\sin([\omega]) \geq 0$?

The **Transfer principal** yields that some sentence of $\mathbb{R}$ can be transferred to an equivalent sentence on ${}^*\mathbb{R}$ via $*$-transference. For instance, Archimedeam axiom does not hold in ${}^*\mathbb{R}$ but the $*$-version Archimedean axiom holds:

(*A) For any $x \in^* (\mathbb{R}_{>0})$ and $y \in^* \mathbb{R}$, there exists $n \in^* \mathbb{N}$ such that $nx > y$.

2.5.3 Non-standard Analysis

Since we have infinitesimal element and infinite element on ${}^*\mathbb{R}$, we can discuss many problems in analysis instead of introducing the concept of limit. The analysis on ${}^*\mathbb{R}$ is the **non-standard analysis**, which is another standardization of the calculus at Newton, Leibniz and Euler's age. Here is an example of derivative: For $f(x) = x^2$, Leibnize said that the derivative is the infinitesimal difference

$$f'(x) = \frac{\Delta(x^2)}{\Delta(x)} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x.$$

The Δx is "infinitesimal", so $f'(x) = 2x$. In this process we first admit that $\Delta x \neq 0$ and therefore we can do division and then we ignore it can attain the final answer. To standardize the process, we introduce the following definition:

- $x \in^* \mathbb{R}$ is called infinitesimal if for any $a \in \mathbb{R}_{>0}$, $|x| <^* a$. Denote the set of infinitesimal hyperreal numbers by $\mathbb{I}$.
- $x \in^* \mathbb{R}$ is called limitable if there exists $a, b \in \mathbb{R}$ such that ${}^*a < x <^* b$. Denote the set of all limitable hyperreal numbers by $\mathbb{L}$.
- For any $S \subset^* \mathbb{R}$, denote $S_\infty := S \backslash \mathbb{L}$ as the set of unlimitable elements in S. In particular ${}^*\mathbb{R}_\infty$ is the set of all unlimitable (or infinite) elements.
- For $x, y \in^* \mathbb{R}$, if $x - y \in \mathbb{I}$, then we say they are infinitesimally approximate and denote by $x \approx y$. This is an equivalence relation. The equivalence class $x + \mathbb{I}$ of $x \in^* \mathbb{R}$ is called the halo of x, denoted by $\mathrm{halo}(x)$.
- If $x \in^* \mathbb{R}$ and $\mathrm{halo} \cap \mathbb{R} \neq \varnothing$, then $\mathrm{halo} \cap \mathbb{R}$ has a unique element. Such element is called the shadow of x and is denoted by $\mathrm{shad}(x)$.
- For $x \in^* \mathbb{R}$, $x + \mathbb{L}$ is called the galaxy of x and is denoted by $\mathrm{gal}(x)$.

We have some basic properties: If $\delta, \varepsilon \in \mathbb{I}\backslash\{{}^*0\}$, $a, b \in \mathbb{L}\backslash\mathbb{I}$, $x, y \in^* \mathbb{R}_\infty$, then

(a) $\delta + \varepsilon$, $\delta\varnothing$, a/x, δ/x all belong to $\mathbb{I}$;
(b) $a + \delta$, $a + b$, ab, a/b all belong to $\mathbb{L}$;
(c) $x + \delta$, $x + a$, xy, ax, $|x| + |y|$, a/δ, x/δ all belong to ${}^*\mathbb{R}_\infty$.

We also have

(1) $x \in^* \mathbb{R}$ has a shadow iff $x \in \mathbb{L}$ and $\mathrm{shad} : \mathbb{L} \to \mathbb{R}$ is an order-preserving ring epimorphism with kernel $\mathbb{I}$. Whence $\mathbb{L}/\mathbb{I}$ is isomorphic to $\mathbb{R}$ as ordered ring.
(2) Every halo $\mathrm{halo}(x)$ contains a nonstandard rational number $y \in^* \mathbb{Q}$. $\mathbb{R} \cong (\mathbb{L} \cap^* \mathbb{Q})/(\mathbb{I} \cap^* \mathbb{Q})$.
(3) For every hyperinteger $x \in^* \mathbb{Z}$, $\mathrm{gal}(x) \cap^* \mathbb{Z} = x + \mathbb{Z}$. Whence ${}^*\mathbb{Z}$ is the disjoint union of all $s + \mathbb{Z}$, where s is taken over all hyperintegers. There always exists $\mathrm{gal}\left(\frac{s+t}{2}\right) \cap \mathbb{Z}$ lies between two distinct $s + \mathbb{Z}$ and $t + \mathbb{Z}$.
(4) There always be a galaxy lies between two distinct galaxy.
(5) The order on ${}^*\mathbb{N}$ is not a well-order.

$f : S \to \mathbb{R}$ is said to be continuous at $a \in S$, if for any $x \in^* S \cap \mathrm{halo}({}^*a)$, ${}^*f(x) \in \mathrm{halo}({}^*(\mathrm{f(a)}))$. The familiar proposition holds: If $f, g : S \to \mathbb{R}$ is continuous at $x \in S$, then so is $d + g, fg$ and f/g (if $g(x) \neq 0$).

If $a, b \in \mathbb{R}$ and $a < b$, $f :]a, b[\subset \mathbb{R} \to \mathbb{R}$ is said to be differentiable at $c \in]a, b[$, if there exists $d \in \mathbb{R}$ such that for any $\delta \in \mathbb{I}\backslash\{{}^*0\}$, we have

$$\frac{{}^*f({}^*c + \delta) -^* (f(c))}{\delta} \in \mathrm{halo}({}^*d).$$

In this case we denote $f'(c) = d$. In other words,

$$f'(c) = \mathrm{shad}\left(\frac{{}^*f({}^*c + \delta) -^* (f(c))}{\delta}\right).$$

We also have the familiar proposition: If $f, g :]a, b[\to \mathbb{R}$ are differentiable at $c \in]a, b[$, then f, g are continuous at c; $f + g, fg$ and f/g (if $g(c) \neq 0$) are differentiable at c. Moreover $(f + g)'(c) = f'(c) + g'(c)$; $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$ and $(f/g)'(c) = (f'(c)g(c) - f(c)g'(c))/g(c)^2$. In particular, if $f(x) = x^2$, $x \in \mathbb{R}$, $\delta \in \mathbb{I}\backslash\{{}^*0\}$,

$$\frac{{}^*f({}^*c + \delta) -^* (f(c))}{\delta} = \frac{({}^*x + \delta)^2 -^* (x^2)}{\delta} = 2^*x + \delta$$

whence

$$f'(x) = \mathrm{shad}\left(\frac{{}^*f({}^*c + \delta) -^* (f(c))}{\delta}\right) = 2x.$$

2.6 The indefinite integral of elementary functions

Liouville gave a condition to decide whether the indefinite integral of an elementary function is still elementary.

Definition. Let $\mathbb{F}$ be a field and $D : \mathbb{F} \to \mathbb{F}$ is a derivative if $D(a+b) = D(a)+D(b)$ and $D(ab) = aD(b)+bD(a)$ for all $a, b \in \mathbb{F}$. An element $c \in \mathbb{F}$ is said to be constant if $D(c) = 0$. And if $a, b \in \mathbb{F}$, $a \neq 0$ and $D(b) = D(a)a^{-1}$, then we say a is an exponent of b and b is a logarithm of a.

Definition. A differential extension $(\mathbb{F}', D')$ of a differential field $(\mathbb{F}, D)$ is a differential field containing $\mathbb{F}$ such that $D'|_{\mathbb{F}} = D$. An elementary extension of $(\mathbb{D}, D)$ is $\mathbb{F}(f_1, \cdots, f_n)$ such that each f_j satisfies the one of the following conditions:

(a) f_j is algebraic over $\mathbb{F}(f_1, \cdots, f_{j-1})$;
(b) f_j is an exponent of some $f \in \mathbb{F}(f_1, \cdots, f_{j-1})$;
(c) f_j is a logarithm of some $f \in \mathbb{F}(f_1, \cdots, f_{j-1})$.

The elementary functions we usually concern about are holomorphic functions on an open subset of $\mathbb{C}$ and we can take $\mathbb{F} = \mathbb{C}(x, f_1, \cdots, f_n)$ as an elementary extension of $\mathbb{C}(x)$ that contains f: On $\mathbb{C}$, every elementary function is a finite combination of rational functions, exponential function and logarithm function.

Theorem 1 (Liuoville)**.** Let $\mathbb{F}$ be a differential field with character 0, $\alpha \in \mathbb{F}$. If $D(y) = \alpha$ has a solution in some elementary extension $\mathbb{F}'$ of $\mathbb{F}$ and if $\mathbb{F}$' and $\mathbb{F}$ share the same subfield of constants, then there exists constant $c_1, \cdots, c_n \in \mathbb{F}$ and $u_1, \cdots, u_n, v \in \mathbb{F}$ such that

$$\alpha = \sum_{j=1}^{n} c_j D(u_j) u_j^{-1} + D(v).$$

As a corollary, if $f, g \in \mathbb{C}(x)$ are rational functions, then the primitive function of $f(x)e^{g(y)}$ is elementary iff there exists rational function $R \in \mathbb{C}(x)$ such that $R' + g'R = f$. Whence the primitive function of e^x/x and e^{x^2} are not elementary.

Definition. A Liouville extension of a differential field $(\mathbb{F}, D)$ is $\mathbb{F}(f_1, \cdots, f_n)$ such that each f_j satisfies the one of the following conditions:

(a) f_j is algebraic over $\mathbb{F}(f_1, \cdots, f_{j-1})$;
(b) f_j is an exponent of some $f \in \mathbb{F}(f_1, \cdots, f_{j-1})$;
(c) f_j is a primitive function of some $f \in \mathbb{F}(f_1, \cdots, f_{j-1})$, i.e. $D(f_j) = f$.

Theorem 2 (Rosenlicht)**.** Let $\mathbb{F}$ be a differential field with character 0, $\mathbb{F}'$ a Liouville extension of $\mathbb{F}$ such that $\mathbb{F}$' and $\mathbb{F}$ share the same subfield of constants. If $y_1, \cdots, y_n, z_1, \cdots, z_n \in \mathbb{F}'$ are such that $D(z_j) = D(y_j)y_j^{-1}$, $j = 1, \cdots, n$ and $\mathbb{F}(y_1, \cdots, y_n, z_1, \cdots, z_n)$ is algebraic over it's subfields $\mathbb{F}(y_1, \cdots, y_n)$ and $\mathbb{F}(z_1, \cdots, z_n)$. Then $y_1, \cdots, y_n$ are algebraic over $\mathbb{F}$.

2.7 Convergence of nets

Definition. Let D be a non-empty set, $\prec$ a relation on D, $(D, \prec)$ is set to be directed if

(a) $\forall \alpha \in D$, $\alpha \prec \alpha$;
(b) if $\alpha \prec \beta$ and $\beta \prec \gamma$, then $\alpha \prec \gamma$;
(c) for any $, \beta \in D$, there exists $\gamma \in D$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$.

Here are some basic example:

- Total ordered sets are directed, e.g. $(\mathbb{N}, \leq)$;
- The power set $(\mathcal{P}(X), \subset)$ of a non-empty X is a directed set.
- In the definition of Riemann integral, we consider the set $\mathcal{S}'(I)$ of the tagged partition (σ, ξ) of $I = [a, b]$. $\mathcal{S}'(I)$ becomes a directed set with the following relation: $(\sigma, \xi) \prec (\sigma', \xi')$ iff σ' is a refinement of σ. This is an example of a directed set that is not a partial ordered set.
- In a topological space X, the set $\mathcal{N}_x \subset \mathcal{P}(X)$ of all neighborhood of a given point $x \in X$ is a directed with the relation $\supset$.

Definition. A net of X is a map form a directed set $(D, \prec)$ to X, usually denoted by $(x_\alpha : \alpha \in D)$ or $(x_\alpha)_{\alpha \in D}$. A net $(x_\alpha)_{\alpha \in D}$ converge to $x \in X$, if for all neighborhood U of x, there exists $\beta \in D$ such that $x_\alpha \in U$ whenever $\beta \prec \alpha$ and is denoted by $x = \lim_D x_\alpha$. Generally, x is a cluster point of $(x_\alpha)_{\alpha \in D}$ if for any neighborhood U of x and any $\beta \in D$, there exists $\beta \prec \alpha$ such that $x_\alpha \in U$.

For instance,

- Net is a straight forward generalization of sequence: If $(D, \prec) = (\mathbb{N}, \leq)$, it is the definition of convergence of sequence;

- The limit of a function can be described by some proper net, e.g. the limit $x \to x_0$ be defined by the net defined by the distance to x_0 on $\mathbb{R}$;
- An equivalence relation definition of Riemann integral is the limit of the net of Riemann sum $S(f;\sigma,\xi)$.

Proposition 2.1. Let X, Y be two topological space and $f : X \to Y$. f is continuous iff for any net $(x_\alpha)_{\alpha\in D}$ on X such that $\lim_D x_\alpha = x$ one have $\lim_D f(x_\alpha) = f(x)$.

Definition. $x \in X$ is a cluster point of a base $\mathcal{B}$ if every neighborhood of x meets every set in $\mathcal{B}$.

Proposition 2.2. The following are equivalent:

(1) A topological space is compact;
(2) every net admits a convergent sub-net;
(3) every base admits a cluster point.

Bases and nets can describe convergence in general topological spaces and they are equivalent. Specifically, base is a structure over sets while net is a structure over map.

2.8 Euler-Boole formula and Euler-Maclaurin formula

We define Euler number e_n and Euler polynomial $E_n(x)$ by

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} z^n, \qquad e_n := 2^n E_n\left(\frac{1}{2}\right).$$

For example,

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \cdots$$
$$e_0 = 1, \quad e_1 = 0, \quad e_2 = -1, \quad e_3 = 0, \quad e_4 = 5, \quad e_5 = 0, \quad e_6 = -61, \cdots$$

We also have Bernoulli number b_n and Bernoulli polynomial $B_n(x)$

$$\frac{e^{xz}}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \qquad b_n := B_n\left(\frac{1}{2}\right).$$

For example,

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \cdots$$
$$b_0 = 1, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{6}, \quad b_3 = 0, \quad b_4 = -\frac{1}{30}, \quad b_5 = 0, \quad b_6 = \frac{1}{42}, \cdots$$

Proposition 2.3. (B1)

$$\sum_{k=0}^{n} \binom{n+1}{k} b_k = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases}$$

and

$$B_n(x) = \sum_{k=0}^{n} b_k x^{n-k}, \qquad n \in \mathbb{N}^*.$$

In particular, $B_n(x)$ is a polynomial of degree n and $b_n \in \mathbb{Q}$.

(B2) For $k \in \mathbb{N}^*$, $(-1)^{k+1} b_{2k} > 0$, $b_{2k+1} = 0$ and when $k \to \infty$

$$(-1)^{k+1} b_{2k} \sim \frac{2(2k)!}{(2\pi)^{2k}} \sim 4\sqrt{\pi k}\left(\frac{k}{\pi e}\right)^{2k}.$$

(B3) For $n \in \mathbb{N}$, $B_n(\frac{1}{2}) = (2^{1-n} - 1) b_n$.
(B4) For $n \in \mathbb{N}$, $B_n(x+1) = B_n(x) + nx^{n-1}$, $B_n(1-x) = (-1)^n B_n(x)$.
(B5) For $n \in \mathbb{N}$, $B'_{n+1}(x) = (n+1) B_n(x)$.
(B6) For $k \in \mathbb{N}^*$, $B_{2k}(x)$ has exactly two zeros in [0,1] x_{2k} and x'_{2k} such that $x_{2k} + x'_{2k} = 1$; $B_{2k+1}(x)$ has exactly 3 zeros in [0,1], i.e. $0, \frac{1}{2}, 1$.

Proposition 2.4. (E1)

$$\sum_{k=0}^{n}\binom{2n}{2k}e_k = 0, \qquad n \in \mathbb{N}^*$$

and

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{e_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}, \qquad n \in \mathbb{N}^*.$$

In particular, $E_n(x)$ is a polynomial of degree n and $e_n \in \mathbb{Z}$.

(B2) For $k \in \mathbb{N}^*$, $(-1)^k e_{2k} > 0$, $e_{2k+1} = 0$ and when $k \to \infty$

$$(-1)^k b_{2k} \sim \frac{2^{2k+2}(2k)!}{\pi^{2k+1}} \sim 8\sqrt{\frac{k}{\pi}}\left(\frac{4k}{\pi e}\right)^{2k}.$$

(B3) For $n \in \mathbb{N}*$, $E_n(0) = -E_n(1) = -\frac{2}{n+1}(2^{n+1}-1)b_n$.

(B4) For $n \in \mathbb{N}$, $E_n(x+1) + E_n(x) = 2x^n$, $E_n(1-x) = (-1)^n E_n(x)$.

(B5) For $n \in \mathbb{N}$, $E'_{n+1}(x) = (n+1)E_n(x)$.

(B6) For $k \in \mathbb{N}^*$, $E_{2k}(x)$ has exactly two zeros in [0,1], i.e. 0 and 1; $E_{2k+1}(x)$ has exactly 1 zero in [0,1], i.e. $\frac{1}{2}$.

Bernoulli number and Euler number appears in some Taylor expansions and power summations:

Proposition 2.5. For $p \in \mathbb{N}^*$

$$\begin{aligned}
\tan x &= \sum_{n=1}^{\infty}\frac{b_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1}, && |x| < \frac{\pi}{2}\\
\cot x &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n b_{2n}(2x)^{2n}}{(2n)!}, && 0 < |x| < \pi\\
\sec x &= \sum_{n=0}^{\infty}\frac{e_{2n}}{(2n)!}x^{2n}, && |x| < \frac{\pi}{2}\\
\csc x &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}2(2^{2n-1}-1)b_{2n}x^{2n}}{(2n)!}, && 0 < |x| < \pi.
\end{aligned}$$

And for $p \in \mathbb{N}^*$

$$\begin{aligned}
\sum_{k=1}^{n}k^p &= \frac{1}{p+1}(B_p(n+1) - b_{p+1});\\
\sum_{k=1}^{n}(-1)^k k^p &= \frac{1}{2}((-1)^n E_p(n+1) + E_p(0)).\\
\zeta(2m) &= (-1)^{m-1}\frac{(2\pi)^{2m}}{(2m)!}b_{2m};\\
\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}} &= (-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n)!}e_{en}.
\end{aligned}$$

Theorem 3 (Euler-Maclaurin formula)**.** Let $\widetilde{B}_n(x) = B_n(x - \lfloor x \rfloor)$, (for $n \geq 2$, $\widetilde{B}_n \in C^{n-2}(\mathbb{R})$), then for $a, b \in \mathbb{Z}$, $a < b$, $f \in C^{2m+1}([a,b])$, $m \in \mathbb{N}^*$, we have

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x) + \frac{f(a)+f(b)}{2} + \sum_{k=1}^{m}\frac{b_{2k}}{(2k)!}f^{(2k-1)(x)}|_{x=a}^{b} + \frac{1}{(2m+1)!}\int_a^b \widetilde{B}_{2m+1}(x)f^{(2m+1)}(x)dx.$$

Theorem 4 (Euler-Boole formula)**.** Let $\widetilde{E}_n(x) = (-1)^{\lfloor x \rfloor}E_n(x - \lfloor x \rfloor)$, then for $a, b \in \mathbb{Z}$, $a < b$, $f \in C^m([a,b])$, $m \in \mathbb{N}^*$ and $h \in [0,1]$, we have

$$\sum_{k=a}^{b-1}(-1)^k f(k+h) = \frac{1}{2}\sum_{k=0}^{m-1}\frac{E_k(h)}{k!}((-1)^{b-1}f^{(k)}(b) - (-1)^a f^{(k)}(a)) + \frac{1}{(2m+1)!}\int_a^b \widetilde{E}_{m-1}(h-x)f^{(m)}(x)dx.$$

3 Henstock-Kurzweil integral

3.1 Gauge function and partition

Recall that in the definition of Riemann integral, we have

- A partition of a bounded closed interval $I = [a, b]$ i.e. $\sigma : a = a_0 < a_1 < \cdots < a_n = b$. The set of all partitions is denoted by $\mathcal{S}(I)$.
- A partition with nodes of I, i.e. (σ, ξ), where σ is as above and $\xi = (\xi_1, \cdots, \xi_n)$, $\xi_j \in [a_{j-1}, a_j]$, $j = 1, \cdots, n$. The set of all tagged partitions is denoted by $\mathcal{S}'(I)$.
- For $f : I = [a, b] \to \mathbb{R}$, and $(\sigma, \xi) \in \mathcal{S}'(I)$ as above, the corresponding Riemann summation is defined by

$$S(f; \sigma, \xi) = \sum_{j=1}^{n} (a_j - a_{j-1}) f(\xi_j).$$

Now we define gauge function and partition with nodes associated with some gauge function:

- A gauge function on $I = [a, b]$ is a positive real valued function $\delta : [a, b] \to]0, \inf[$. The set of all gauge functions is denoted by $\mathcal{G}(I) :=]0, \infty[^I$.
- We say a partition $(\sigma, \xi) \in \mathcal{S}'(I)$ with nodes on I is associated with (or belongs to) a gauge function $\delta \in \mathcal{G}(I)$, if $\sigma : a = a_0 < a_1 < \cdots < a_n = b$ and $\xi = (\xi_1, \cdots, \xi_n)$ are such that

$$[a_{j-1}, a_j] \subset]\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)[, \qquad j = 1, 2, \cdots, n.$$

  The set of partitions with nodes that is associated with δ by $\mathcal{S}'_\delta(I)$.

Theorem 5 (Cousin)**.** For any $\delta \in \mathcal{G}(I)$, $\mathcal{S}'_\delta(I) \neq \varnothing$.

Proof. Consider

$$A = \{x \in]a, b] : \mathcal{S}'_\delta([a, x]) \neq \varnothing\}$$

and set $x = \min\{b, a + \frac{\delta(a)}{2}\} \in]a, b]$. Consider the partition $\sigma : a = a_0 < a_1 = x$ and the node $\xi = (a)$, one trivially get $x \in A$. As A is bounded from above by definition, $m := \sup A \in]a, b]$. Indeed the previous construction (of partition with nodes) implies that $m = b$. On the other hand, for any $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that $x_\varepsilon > m - \varepsilon$. In particular, let $x \in A$ and $x > m - \delta(m)$, namely there exists some $\sigma : a = a_0 < a_1 < \cdots < a_n = x$ and $\xi_j \in [a_{j-1}, a_j]$ such that $[a_{j-1}, a_j] \subset]\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)[$, $j = 1, 2, \cdots, n$. Then take $\sigma' : a = a_0 < a_1 < \cdots < a_n = x < a_{n+1} = m$ and $\xi' = (\xi_1, \cdots, \xi_n, m)$. Directly by definition $(\sigma', \xi') \in \mathcal{S}'_\delta([a, x])$, so $m = b \in A$. And therefore $\mathcal{S}'_\delta(I) \neq \varnothing$. □

3.2 Definition of gauge integral

Now we can simulate the definition of Riemann integral to define Henstock-Kurzweil integral (i.e. gauge integral): $f : I = [a, b] \to \mathbb{R}$ is said to be gauge integrable if there exists $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists $\delta \in \mathcal{G}(I)$ such that

$$|S(f; \sigma, \xi) - L| < \varepsilon, \qquad \forall (\sigma, \xi) \in \mathcal{S}'_\delta(I).$$

In this case we define L to be the gauge integral of f on I and denote by

$$L := \int_I^{\mathcal{HK}} f(x) dx.$$

Such L is unique. Indeed, if $|S(f; \sigma', \xi') - L'| < \varepsilon$ for all $(\sigma', \xi') \in \mathcal{S}'_{\delta'}(I)$. One have $\mathcal{S}_{\delta''}(I) \subset \mathcal{S}'_\delta(I) \cap \mathcal{S}'_{\delta'}(I)$, where $\delta'' = \min\{\delta, \delta'\}$ is another gauge function. Then

$$|S(f; \sigma, \xi) - L| < \varepsilon, \qquad \text{and} \qquad |S(f; \sigma, \xi) - L'| < \varepsilon \qquad \forall (\sigma, \xi) \in \mathcal{S}'_{\delta''}(I).$$

By letting $\varepsilon \to 0$, $L = L'$. The set of all gauge integrable function on I is denoted by $\mathcal{HK}(I)$. To distinguish from Riemann integral and Lebesgue integral, we denote them respectively by

$$\int_I^{\mathcal{R}} f(x) dx \qquad \text{and} \qquad \int_I f(x) dx.$$

Cauchy's criterion holds for gauge integral:

Proposition 3.1. $f : I \to \mathbb{R}$ is gauge integrable iff for any $\varepsilon > 0$, there exists $\delta \in \mathcal{G}(I)$ such that for any (σ, ξ) and $(\sigma', \xi') \in \mathcal{S}'_\delta(I)$, one have

$$|S(f; \sigma, \xi) - S(f; \sigma', \xi')| < \varepsilon. \tag{4}$$

Proof. (4) is trivial if f is gauge integral. Conversely, let $\delta_n \in \mathcal{G}(I)$ be a sequence of gauge function such that

$$|S(f; \sigma, \xi) - S(f; \sigma', \xi')| < \frac{1}{n}, \qquad \forall (\sigma, \xi), (\sigma', \xi') \in \mathcal{S}'_{\delta_n}(I).$$

Consider $\delta'_n = \min_{k \le n}\{\delta_k\}$, then it's trivial that $\delta'_n \in \mathcal{G}(I)$ and $\mathcal{S}'_{\delta'_n}(I) \subset \bigcap_{k \le n} \mathcal{S}'_{\delta_k}(I)$. Denote

$$a_n := \inf_{(\sigma,\xi) \in \mathcal{S}'_{\delta'_n}(I)} S(f; \sigma, \xi), \qquad b_n := \sup_{(\sigma,\xi) \in \mathcal{S}'_{\delta'_n}(I)} S(f; \sigma, \xi).$$

As $\delta'_{n+1} \le \delta'_n$, $\mathcal{S}'_{\delta'_{n+1}}(I) \subset \mathcal{S}'_{\delta_n}(I)$, and therefore $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$. One the other hand, by the construction of δ'_n, $|a_n - b_n| < \frac{1}{n}$, whence a_n, b_n converges to some $L \in \mathbb{R}$ (by the nested closed interval lemma). And this implies that f is gauge integrable. $\square$

Similar to other integrals, $\mathcal{HK}(I)$ is a vector space and $\int_I^{\mathcal{HK}} : \mathcal{HK}(I) \to \mathbb{R}$ is a (monotonic) linear map:

Proposition 3.2. (a) For any $f, g \in \mathcal{HK}(I)$, we have $f + g \in \mathcal{HK}(I)$ and

$$\int_I^{\mathcal{HK}} f(x) + g(x)dx = \int_I^{/\mathcal{HK}} f(x)dx + \int_I^{\mathcal{HK}} g(x)dx.$$

(b) For any $c \in \mathbb{R}$, $f \in \mathcal{HK}(I)$, we have $cf \in \mathcal{HK}(I)$, and

$$\int_I^{\mathcal{HK}} cf(x)dx = c\int_I^{\mathcal{HK}} f(x)dx.$$

(c) If $f, g \in \mathcal{HK}(I)$ and $f(x) \le g(x)$, then

$$\int_I^{\mathcal{HK}} f(x)dx \le \int_I^{\mathcal{HK}} g(x)dx.$$

Proof. Essentially trivial by definition. $\square$

Gauge integral is finitely additive w.r.t. integral region:

Proposition 3.3. (a) If $I' \subset I$ is a closed subinterval, $f \in \mathcal{HK}(I)$, then $f|_{I'} \in \mathcal{HK}(I')$.

(b) If $a < c < b$, and $f : [a, b] \to \mathbb{R}$ is such that $f|_{[a,c]} \in \mathcal{HK}([a, c])$ and that $f|_{[c,b]} \in \mathcal{HK}([c, b])$, then $f \in \mathcal{HK}([a, b])$ and

$$\int_{[a,b]}^{\mathcal{HK}} f(x)dx = \int_{[a,c]}^{\mathcal{HK}} f(x)dx + \int_{[c,b]}^{\mathcal{HK}} f(x)dx.$$

Proof. (a) Assume that $I' = [a', c']$. For any $\varepsilon > 0$, take $\delta' = \delta|_{I'}$ where for any $(\sigma_1, \xi_1), (\sigma_2, \xi_2) \in \mathcal{S}'_\delta(I)$,

$$|S(f; \sigma_1, \xi_1) - S(f; \sigma_2, \xi_2)| < \varepsilon.$$

Then for any $(\sigma', \xi'), (\sigma'', \xi'') \in \mathcal{S}_{\delta'}(I')$, set $\sigma_1 = \sigma^0 \cup \sigma' \cup \sigma^1$, $\xi_1 = (\xi^0, \xi', \xi^1)$ and $\sigma_2 = \sigma^0 \cup \sigma'' \cup \sigma^1$, $\xi_2 = (\xi^0, \xi'', \xi^1)$. Where $(\sigma^0, \xi^0) \in \mathcal{S}_\delta([a, a_1])$ and $(\sigma^1, \xi^1) \in \mathcal{S}_\delta([c_1, c])$ (Cousin theorem guarantees that they are non-empty). Obviously $(\sigma_1, \xi_1), (\sigma_2, \xi_2) \in \mathcal{S}_\delta(I)$, whence

$$|S(f|_{I'}; \sigma', \xi') - S(f|_{I'}; \sigma'', \xi'')| = |S(f; \sigma_1, \xi_1) - S(f; (\sigma_2, \xi_2))| < \varepsilon.$$

(b) For any $\varepsilon > 0$, consider the gauge function

$$\delta = \begin{cases} \min\{\delta_1, \delta_2, |x - c|\}, & x \ne c, \\ \min\{\delta_1(c), \delta_2(c)\}, & x = c. \end{cases}$$

Where δ_1 and δ_2 are respectively gauge functions such that

$$\left|S(f|_{[a,c];\sigma_1,\xi_1}) - \int_{[a,c]}^{\mathcal{HK}} f(x)dx\right| < \frac{\varepsilon}{2} \qquad \forall(\sigma_1,\xi_1) \in \mathcal{S}'_{\delta_1}([a,c]);$$

$$\left|S(f|_{[c,b];\sigma_2,\xi_2}) - \int_{[c,b]}^{\mathcal{HK}} f(x)dx\right| < \frac{\varepsilon}{2} \qquad \forall(\sigma_2,\xi_2) \in \mathcal{S}'_{\delta_2}([c,b]).$$

Note that for all $x \neq c$, $|x-c| \geq \delta(x)$ and therefore $c \notin B_x(\delta(x))$, that is to say for any partition with nodes (σ,ξ) associated with δ, c must be one of the tags. Therefore one can split the subinterval, say $[a_i, a_{i+1}]$ that c belongs to into two subintevals $[a_i, c]$ and $[c, a_{i+1}]$ with shared node c to get a new tagged partition (σ', ξ') such that

$$S(f;\sigma,\xi) = S(f;\sigma',\xi').$$

Namely one may assume that every tagged partition (σ,ξ) associated with δ takes form $\sigma : a = a_0 < a_1 < \cdots < a_m = c < a_{m+1} < \cdots < a_n = b$. Set $\sigma_1 : a = a_0 < \cdots < a_m = c$, $\sigma_2 : c = a_m < \cdots < a_n$ and $\xi_1 = (\xi_1, \cdots, \xi_m)$, $\xi_2 = (\xi_{m+1}, \cdots, \xi_n)$, then $(\sigma_1,\xi_1) \in \mathcal{S}'_\delta([a,c])$ and $(\sigma_2,\xi_2) \in \mathcal{S}'_\delta([c,b])$. As $\delta \leq \delta_1, \delta_2$, we have $\mathcal{S}'_\delta([a,c]) \subset \mathcal{S}'_{\delta_1}([a,c])$ and $\mathcal{S}'_\delta([c,b]) \subset \mathcal{S}'_{\delta_2}([c,b])$. Therefore

$$\begin{aligned}&\left|S(f;\sigma,\xi) - \int_{[a,c]}^{\mathcal{HK}} f(x)dx - \int_{[c,b]}^{\mathcal{HK}} f(x)dx\right| \\ &\qquad = \left|S(f|_{[a,c]};\sigma_1,\xi_1) + S(f|_{[c,b]};\sigma_2,\xi_2) - \int_{[a,c]}^{\mathcal{HK}} f(x)dx - \int_{[c,b]}^{\mathcal{HK}} f(x)dx\right| < \varepsilon.\end{aligned}$$

□

Let's see some fundamental examples (the key point of most proofs is about finding appropriate gauge function).

Proposition 3.4. $\mathcal{R}(I) \subset \mathcal{HK}(I)$ and for all $f \in \mathcal{R}(I)$,

$$\int_I^{\mathcal{HK}} f(x)dx = \int_I^{\mathcal{R}} f(x)dx.$$

Proof. Consider constant gauge functions. □

Problem 38. What is the relationship between $\mathcal{L}(I)$ and $\mathcal{HK}(I)$?

Proposition 3.5. (a) If f is equivalently zero except for a countable set, then $f \in \mathcal{HK}(I)$ and

$$\int_I^{\mathcal{HK}} f(x)dx = 0.$$

In particular, Dirichlet functions is gauge integrable on every finite closed interval (while it is not Riemann integrable).

(b) (Fundamental theorem of calculus, the first form) If $F : [a,b] \to \mathbb{R}$ is differentiable in $]a,b[$ and admits one sided derivative at a and b, then F' is gauge integrable and

$$\int_{[a,b]}^{\mathcal{HK}} F'(x)dx = F(b) - F(a).$$

Proof. (a) For any $\varepsilon > 0$, take

$$\delta(x) = \begin{cases} 1, & f(x) = 0; \\ \dfrac{2^{-(n+1)}\varepsilon}{|f(x)|}, & f(x) \neq 0. \end{cases}$$

(b) As F is differentiable, for any $x \in [a,b]$, there exists $h_x > 0$ such that

$$|F'(x)(y-x) - (F(y) - F(x))| < |y-x|\varepsilon,$$

whenever $y \in B_x(h_x) \cap [a,b]$. Then consider $\delta(x) = h_x$.

$\square$

Problem 39.

(a) If f is identically zero except for a null set, will f be gauge integrable?

(b) Will the fundamental theorem still hold for a.e. differentiable functions (or for absolutely continuous functions)?

Corollary 3.1. If $f \in \mathcal{HK}(I)$ and $g : I \to \mathbb{R}$ agrees with f except for a countable set, then $g \in \mathcal{HK}(I)$ and

$$\int_I^{\mathcal{HK}} f(x)dx = \int_I^{\mathcal{HK}} g(x)dx.$$

Example 3.1. (a)

$$F(x) = \begin{cases} x^2 \sin \dfrac{1}{x^2}, & x \in]0,1], \\ 0, & x = 0. \end{cases}$$

Then $F' : [0,1] \to \mathbb{R}$ is gauge integrable but it is note Riemann integrable.

(b)

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & .x \in]0,1], \\ 0, & x = 0 \end{cases}$$

is gauge integrable.

(c)

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \in]0,1], \\ 0, & x = 0 \end{cases}$$

is not gauge integrable.

Proof. (a) is a direct corollary of the fundamental theorem. For (b), note that $f|_{[2^{-(n+1)},2^{-n}]} \in \mathcal{R}([2^{-(n+1)}, 2^{-n}])$. For any $\varepsilon > 0$, one can find $\delta_n > 0$ such that

$$\left| S(f|_{[2^{-(n+1)},2^{-n}]}; \sigma, \xi) - \int_{[2^{-(n+1)},2^{-n}]}^{\mathcal{R}} f(x)dx \right| < 2^{-n}\varepsilon, \qquad n = 0, 1, 2, \cdots,$$

whenever $(\sigma, \xi) \in \mathcal{S}'_{\delta_n}([2^{-(n+1)}, 2^{-n}])$ (i.e. $\|\sigma\| < \delta_n$). Now consider the gauge function

$$\delta(x) := \begin{cases} \min\{2^{-n} - x, x - 2^{-(n+1)}, \delta_n\}, & x =\in]2^{-(n+1)}, 2^{-n}[, \\ \min\{2^{-1}, \delta_0\}, & x = 1, \\ \min\{2^{-(n+1)}, \delta_n, \delta_{n-1}\}, & x = 2^{-n}, n \geq 1 \\ 2^{-m}, & x = 0, \end{cases}$$

where $m \in \mathbb{N}^*$ is such that $2^{-\frac{m}{2}+1} < \varepsilon$. Similar to the proof of Proposition 3.3-(b), for any partition with nodes (σ, ξ) associated with δ, if $\sigma : 0 = a_0 < a_1 < \cdots$ and $\lceil \log_2 a_1 \rceil = N$ (then one must have $N > m$), there exists $(\sigma_n, \xi) \in \mathcal{S}'_\delta([2^{-(n+1)}, 2^{-n}])$, $n = 0, 1, \cdots, N-1$ such that

$$S(f; \sigma, \xi) = S(f|_{[0,2^{-N}]}; \sigma', \xi') + \sum_{n=0}^{N-1} S(f|_{[2^{-(n+1)},2^{-n}]}; \sigma_n, \xi_n),$$

where σ' has only 1 or 2 subintervals. Indeed, if $a_1 \neq 2^{-N}$, $\sigma' : 0 = a_0 < a_1 < a_2 = 2^{-N}$ and $\xi' = (0, 2^{-N})$; if $a_1 = 2^{-N}$, $\sigma' : 0 = a_0 < a_1 = 2^{-N}$ and $\xi' = (0)$. As $\delta \leq \delta_n$ on each $[2^{-(n+1)}, 2^{-n}]$, one have $\mathcal{S}'_\delta([2^{-(n+1)}, 2^{-n}]) \subset$

$\mathcal{S}'_{\delta_n}([2^{-(n+1)}, 2^{-n}])$. Therefore

$$\begin{aligned}
|S(f;\sigma,\xi)-2| &\le S(f|_{[0,2^{-N}]};\sigma',\xi') + 2^{-\frac{N}{2}+1} + \sum_{n=0}^{N-1} |S(f|_{[2^{-(n+1)},2^{-n}]};\sigma_n,\xi_n) - (2^{-\frac{n}{2}+1} - 2^{-\frac{n+1}{2}+1})| \\
&\le (2^{-N}-a_1)\frac{1}{\sqrt{a_1}} + 2^{-\frac{m}{2}+1} + \sum_{n=0}^{N-1} 2^{-n}\varepsilon \\
&< 2^{-\frac{N+1}{2}} + 3\varepsilon < 4\varepsilon.
\end{aligned}$$

As for (c), suppose for the contrary, one have $f|_{[\varepsilon,1]} \in \mathcal{HK}([\varepsilon,1])$ and $f|_{[0,\varepsilon]} \in \mathcal{HK}([0,\varepsilon])$ by Proposition 3.3-(a) and Proposition 3.4

$$\int_{[0,1]}^{\mathcal{HK}} f(x)dx = \int_{[0,\varepsilon]}^{\mathcal{HK}} f(x)dx + \int_{[\varepsilon,1]}^{\mathcal{HK}} f(x)dx = \int_{[0,\varepsilon]}^{\mathcal{HK}} f(x) - \ln\varepsilon.$$

As $f \ge 0$, one get $\int_{[0,1]}^{\mathcal{HK}} f(x) \ge -\ln\varepsilon \to \infty$, a contradiction. □

3.3 Further discussion

In order to investigate gauge integral deeper, we introduce some more general concepts: $I = [a,b]$ is a closed interval

- A tagged subinterval of I is $(J=[c,d],\xi)$ where $\xi \in J$; (J,ξ) is said to be associated with (or belongs to) $\delta \in \mathcal{G}(I)$, if $J \subset]\xi-\delta(\xi), \xi+\delta(\xi)[$.
- Finitely many tagged subintervals are $(J_k=[c_k,d_k],\xi_k)$, $k=1,\cdots,n$. If the interior of these J_k are pairwise disjoint, we say they are non-overlapping. We use $\tau = \{(J_k,\xi_k)\}$ to denote such a set consisting of non-overlapping finite tagged subintervals and denote the set of such τ by $\mathcal{P}(I)$. We can regard $\mathcal{S}'(I)$ as a subset of $\mathcal{P}(I)$.
- For $\delta \in \mathcal{G}(I)$ and $\tau \in \mathcal{P}(I)$, if every element in τ belongs to δ, then we say τ belongs to δ. Denote the set of such τ by $\mathcal{P}_\delta(I)$.
- For any $\tau = \{(J_k=[c_k,d_k],\xi_k)\}_{k=1}^n \in \mathcal{P}(I)$ and $f: I \to \mathbb{R}$, we can define Riemann summation by

  $$S(f;\tau) = \sum_{k=1}^n (d_k - c_k) f(\xi_k),$$

  and

  $$V(f;\tau) = \sum_{k=1}^n (f(d_k) - f(c_k)).$$

Proposition 3.6. If $f: [a,b] \to \mathbb{R}$ is gauge integrable. Then

$$F(x) = \int_{[a,x]}^{\mathcal{HK}} f(t)dt$$

is continuous.

Proof. For any $\varepsilon > 0$, by definition, we have $\delta \in \mathcal{G}(I)$ such that

$$|S(f;\sigma,\xi) - F(b)| < \varepsilon$$

whenever $(\sigma,\xi) \in \mathcal{S}'_\delta(I)$. Then for any $\tau \in \mathcal{P}_\delta(I)$, by Cousin theorem, one may assume that τ is a subset of a tagged partition (σ,ξ). Denote $\tau' := (\sigma,\xi)\backslash\tau = \{([c'_m,d'_m],\xi'_m)\}_{m=1}^{n'}$. For any $\varepsilon' > 0$ and on each subinterval $[c'_k,d'_k]$, consider a gauge function $\delta'_k (\le \delta)$ such that

$$\left| S(f|_{[c'_k,d'_k]};\sigma'_k,\xi'_k) - \int_{[c'_k,d'_k]} f(x)dx \right| < \varepsilon',$$

whenever (σ'_k,ξ'_k) belongs to δ'_k. Now consider the tagged partition (σ',ξ') containing all the (σ'_k,ξ'_k) and τ.

Clearly $(\sigma', \xi') \in \mathcal{S}'_\delta(I)$. Then, if $\tau = \{([c_k, d_k], \xi_k)\}_{k=1}^n$

$$\begin{aligned}
|S(f;\tau) - V(F;\tau)| &= \left| \sum_{k=1}^n \left((d_k - c_k) f(\xi_k) - \int_{[c_k,d_k]}^{\mathcal{HK}} f(x)dx \right) \right| \\
&\leq \left| S(f;\sigma',\xi') - \int_I^{\mathcal{HK}} f(x)dx \right| + \sum_{k=1}^{n'} \left| S(f|_{[c'_k,d'_k]};\sigma'_k,\xi'_k) - \int_{[c'_k,d'_k]}^{\mathcal{HK}} f(x)dx \right| \\
&\leq \varepsilon + n'\varepsilon'.
\end{aligned}$$

But $\varepsilon' > 0$ is arbitrary, one get

$$|S(f;\tau) - V(F;\tau)| \leq \varepsilon. \tag{5}$$

In particular, take $\tau = \{([x, x+r], x)\}$ (or $\{([x-r, x], x)\}$), where $0 < r < \delta_\varepsilon$ and δ_ε is a constant such that $|f(x)|\delta_\varepsilon < \varepsilon$. By (5), one immediately get

$$|F(y) - F(x)| \leq \varepsilon + |f(x)|r < 2\varepsilon, \qquad \forall y \in I, |y - x| < \delta_\varepsilon,$$

namely F is continuous at any $x \in I$. $\square$

Problem 40 (Fundamental theorem, second form). F is *a.e.* differentiable and $F' = f$, *a.e.*

Proposition 3.7. If $f|_{[a,c]} \in \mathcal{HK}([a,c])$ for any $c \in [a,b[$, and

$$\lim_{c \to b-} \int_{[a,c]}^{\mathcal{HK}} f(x)dx$$

exists (and is finite), then $f \in \mathcal{HK}([a,b])$ and

$$\int_{[a,b]}^{\mathcal{HK}} f(x)dx = \lim_{c \to b-} \int_{[a,c]}^{\mathcal{HK}} f(x)dx.$$

In particular, if $f : [a,b[\to \mathbb{R}$ is improper integrable, then $f \in \mathcal{HK}([a,b])$ and its gauge integral equals its improper integral.

Proof. Set $c_i = b - 2^{-i}(b-a)$, by the proof of (5), one can find a gauge function $\delta_i : [c_{i-1}, c_i] \to \mathbb{R}_{>0}$ such that

$$|S(f;\tau_i) - V(F;\tau_i)| < 2^{-i}\varepsilon$$

whenever $\tau \in \mathcal{P}_{\delta_i}([c_{i-1}, c_i])$. Next consider $\delta : [a,b[\to \mathbb{R}$ by

$$\delta(x) = \begin{cases} \min\{x - c_{n-1}, c_n - x, \delta_n(x)\}, & x \in]c_{n-1}, c_n[; \\ \min\{2^{-(n+1)}(b-a), \delta_n(x), \delta_{n+1}(x)\}, & x = c_n, n \neq 0; \\ \min\{2^{-1}(b-a), \delta_1(x)\}, & x = c_0 = a. \end{cases}$$

Now for any $c \in [a,b[$, say $c \in [c_{n-1}, c_n[$, then for any $\tau_i := (\sigma_i, \xi_i) \in \mathcal{S}'_\delta([c_{i-1}, c_i]) \subset \mathcal{S}'_{\delta_i}([c_{n-1}, c_i]) \subset \mathcal{P}_{\delta_i}([c_{i-1}, c_i])$,

$$\left| S(f|_{[c_{i-1},c_i]};\sigma_i,\xi_i) - \int_{[c_{i-1},c_i]}^{\mathcal{HK}} f(x)dx \right| = |S(f;\tau_i) - V(F;\tau_i)| < 2^{-i}\varepsilon.$$

Besides, for any $\tau_n := (\sigma_n, \xi_n) \in \mathcal{S}'_\delta([c_{n-1}, c]) \subset \mathcal{S}'_{\delta_n}([c_{n-1}, c]) \subset \mathcal{P}_{\delta_n}([c_{n-1}, c_n])$, so

$$\left| S(f|_{[c_{n-1},c]};\sigma_n,\xi_n) - \int_{[c_{n-1},c]}^{\mathcal{HK}} f(x)dx \right| = |S(f;\tau_n) - V(F;\tau_n)| < 2^{-n}\varepsilon.$$

However, with the same argument of Example 3.1-(b), we for all $(\sigma_c, \xi_c) \in \mathcal{S}'_\delta([a,c])$, one can find $(\sigma_i, \xi_i) \in \mathcal{S}'_\delta([c_{i-1}, c_i])$, $i = 1, \cdots, n-1$ and $(\sigma_n, \xi_n) \in \mathcal{S}'_\delta([c_{n-1}, c])$ such that

$$S(f;\sigma_c,\xi_c) = \sum_{i=1}^{n-1} S(f|_{[c_{i-1},c_i]};\sigma_i,\xi_i) + S(f|_{[c_{n-1},c]};\sigma_n,\xi_n).$$

Then

$$\left|S(f;\sigma_c,\xi_c)-\int_{[a,c]}^{\mathcal{HK}} f(x)\right| \le \sum_{i=1}^{n-1}\left|S(f|_{[c_{i-1},c_i]};\sigma_i,\xi_i)-\int_{[c_{i-1},c_i]}^{\mathcal{HK}} f(x)dx\right| + \left|S(f|_{[c_{n-1},c]};\sigma_n,\xi_n)-\int_{[c_{n-1},c]}^{\mathcal{HK}} f(x)dx\right|$$
$$\le \sum_{i=1}^{n} 2^{-i}\varepsilon < \varepsilon \tag{6}$$

for all $c\in[a,b[$ and $(\sigma_c,\xi_c)\in\mathcal{S}'_\delta([a,c])$. Now denote $L := \lim_{c\to b-}\int_{[a,c]}^{\mathcal{HK}} f(x)dx$, we have $c_\varepsilon < b$ such that

$$\left|\int_{[a,c]}^{\mathcal{HK}} f(x)dx - L\right| < \varepsilon \tag{7}$$

whenever $c\in]c_\varepsilon,b[$. Consider a new gauge function $\delta:[a,b]\to\mathbb{R}_{>0}$ by

$$\delta'(x) := \begin{cases}\min\{\delta(x),b-x\}, & x\neq b\\ d, & x=b\end{cases}$$

where d is a constant such that $d\le b-c_\varepsilon$ and that $|f(b)|d\le\varepsilon$. It's easy to see that b is a node for any $(\sigma,\xi)\in\mathcal{S}'_{\delta'}([a,b])$. Indeed, if $\sigma: a_0=a<\cdots<a_{n-1}<a_n=\xi_n=b$, then $a_{n-1}>b-\delta(b)\ge c_\varepsilon$ and $\tau := (\sigma,\xi)\backslash\{([a_{n-1},b],b)\}\in\mathcal{S}'_{\delta'}([a,a_{n-1}])\subset\mathcal{S}'_\delta([a,a_{n-1}])$. Thus (by (6) and the definition of $\delta(b)$)

$$\left|S(f|_{[a,a_{n-1}]};\tau)-\int_{[a,a_{n-1}]}^{\mathcal{HK}} f(x)dx\right|<\varepsilon, \qquad |f(b)|(b-a_{n-1})<|f(b)|\delta(b)\le\varepsilon.$$

Combine (7), one immediately get ($c:=a_{n-1}$)

$$|S(f;\sigma,\xi)-L| \le \left|\int_{[a,a_{n-1}]}^{\mathcal{HK}} f(x)dx - L\right| + \left|S(f|_{[a,a_{n-1}]};\tau)-\int_{[a,a_{n-1}]}^{\mathcal{HK}} f(x)dx\right| + (b-a_{n-1})|f(b)| < 3\varepsilon.$$

□

Proposition 3.8. If $f,g\in\mathcal{HK}([a,b])$ and $|f|\le|g|$, then $|f|\le\mathcal{HK}([a,b])$.

Proof. Let

$$F(x) = \int_{[a,x]}^{\mathcal{HK}} f(t)dt,$$

then for any $\varepsilon>0$ there exists a gauge function $\delta:[a,b]\to\mathbb{R}$ such that

$$|S(f;\sigma,\xi)-F(b)|<\varepsilon$$

whenever $(\sigma,\xi)\in\mathcal{S}'_\delta([a,b])$. We can go further from (5). Say for $\tau=\{(J_k=[c_k,d_k],\xi_k)\}_{k=1}^n\in\mathcal{P}_\delta([a,b])$, assume that $(d_k-c_k)f(\xi_k)-F(d_k)+F(c_k)\ge0$ for $k=1,\cdots,m$ and $(d_k-c_k)f(\xi_k)-F(d_k)+F(c_k)<0$ for $k=m+1,\cdots,n$. Then by (5), we have

$$0\le-\sum_{k=m+1}^{n}\big((d_k-c_k)f(\xi_k)-(F(d_k)-F(c_k))\big)<\varepsilon;$$
$$0\le\sum_{k=1}^{m}\big((d_k-c_k)f(\xi_k)-(F(d_k)-F(c_k))\big)<\varepsilon.$$

That is to say

$$\sum_{k=1}^{n}|(d_k-c_k)f(\xi_k)-(F(d_k)-F(c_k))|$$
$$=\sum_{k=1}^{m}\big((d_k-c_k)f(\xi_k)-(F(d_k)-F(c_k))\big)-\sum_{k=m+1}^{n}\big((d_k-c_k)f(\xi_k)-(F(d_k)-F(c_k))\big)$$
$$\leq 2\varepsilon. \tag{8}$$

We first prove that $F \in BV([a,b])$. For any partition $\sigma \in \mathcal{S}([a,b])$, if $\sigma' \prec \sigma$ (i.e. σ' is a refinement of σ), then $V(F;\sigma;) \geq V(F;\sigma)$. By (8), we can find a gauge function $\delta_f : [a,b] \to \mathbb{R}_{>0}$ such that

$$\sum_{i=1}^{n}|f(\xi_i)(a_i-a_{i-1})-(F(a_i)-F(a_{i-1}))| \leq 2\varepsilon$$

whenever $(\sigma_f,\xi_f) \in \mathcal{S}'_{\delta_f}([a,b])$, where $\sigma_f : a_0 = a < \cdots < a_n = b$ and $\xi_f = (\xi_1,\cdots,\xi_n)$. And since $g \in \mathcal{HK}([a,b])$, there exists another gauge function δ_g such that

$$S(g;\sigma_g,\xi_g) \leq \int_{[a,b]}^{\mathcal{HK}} g(x)dx + \varepsilon, \qquad \forall(\sigma_g,\xi_g) \in \mathcal{S}'_{\delta_g}([a,b]).$$

Let's consider $\delta_1 := \min\{\delta_f,\delta_g\}$. For a partition $\sigma \in \mathcal{S}([a,b])$, Cousin theorem implies that we can find a refinement $(\sigma',\xi') \in \mathcal{S}'_{\delta_1}([a,b])$. If $\sigma' : a'_0 = a < \cdots < a'_n = b$, then

$$2\varepsilon \geq \sum_{i=1}^{n'}|f(\xi'_i)(a'_i-a'_{i-1})-(F(a'_i)-F(a'_{i-1}))| \geq \sum_{i=1}^{n'}\big(|F(a'_i)-F(a'_{i-1})|-|f(\xi'_i)|(a'_i-a'_{i-1})\big). \tag{9}$$

Moreover we have

$$3\varepsilon + \int_{[a,b]}^{\mathcal{HK}} g(x)dx > 2\varepsilon + S(g;\sigma',\xi') \geq 2\varepsilon + S(|f|;\sigma',\xi') \geq V(F;\sigma') \geq V(F;\sigma).$$

As $\varepsilon > 0$ is arbitrary, one get $F \in BV([a,b])$. Next we prove $|f| \in \mathcal{HK}([a,b])$. For any $\varepsilon > 0$, there exists $\sigma_0 \in \mathcal{S}[a,b]$ such that $V_I(F) \geq V(F;\sigma_0) > V_I(F) - \varepsilon$, assume that $\sigma_0 : a_0^{(0)} = a < \cdots < a_{n_0}^{(0)} = b$. Let

$$\delta_0(x) = \begin{cases} \min\{\delta_f(x), a_i^{(0)}-x, x-a_{i-1}^{(0)}\}, & x \in]a_{i-1}^{(0)}, a_i^{(0)}[; \\ \min\{a_{i+1}^{(0)}-x, x-a_{i-1}^{(0)}, \delta_i(x)\}, & x = a_i^{(0)}, i \neq 0, n_0; \\ \min\{a_1^{(0)}-a, \delta_f(a)\}, & x = a; \\ \min\{b-a_{n_0-1}^{(0)}, \delta_f(b)\}, & x = b. \end{cases}$$

Again with the same argument of Example 3.1-(b), one may assume that $(\sigma^0,\xi^0) \in \mathcal{S}'_{\delta_0}([a,b])$ implies $\sigma^0 \prec \sigma_0$. Therefore $V(F;\sigma^0) \in [V(F;\sigma_0), V_{[a,b]}(F)] \subset [V_{[a,b]}(F)-\varepsilon, V_{[a,b]}(F)]$. Meanwhile $\mathcal{S}'_{\delta_0}([a,b]) \subset \mathcal{S}_{\delta_f}([a,b])$, therefore for $\sigma^0 : a_0^0 = a < \cdots < a_{n^0}^0 = b$ and $\xi^0 = (\xi_1^0,\cdots,\xi_{n^0}^0)$ by (9)

$$2\varepsilon \geq \left|\sum_{i=1}^{n^0}|f(\xi_i^0)|(a_i^0-a_{i-1}^0) - V(F;\sigma^0)\right|$$

whence

$$V_{[a,b]}(F) + 2\varepsilon \geq V(F;\sigma^0) + 2\varepsilon \geq S(|f|;\sigma^0,\xi^0) \geq V(F;\sigma^0) - 2\varepsilon > V_{[a,b]}(F) - 3\varepsilon.$$

That is to say $|S(|f|;\sigma^0,\xi^0) - V_{[a,b]}(F)| \leq 3\varepsilon$. □

3.4 McShane integral

We then introduce the concept of free tagged subintervals.

- A free tagged subinterval I, i.e. $(J = [c,d], \xi^*)$, where $J \subset I$, $\xi^* \in I$. But we no longer require $\xi^* \in J$. We say (J,ξ^*) belongs to (or associated with) $\delta \in \mathcal{G}(I)$ if $J \subset]\xi^*, \delta(\xi^*), \xi^* + \delta(\xi^*)[$.

- Simulate the previous definition, we can define a finite set of free tagged non-overlapping subintervals $\tau^* \in \mathcal{P}^*(I)$ and free tagged partition $(\sigma, \xi^*) \in \mathcal{S}^*(I)$. For $\delta \in \mathcal{G}(I)$, we can define the sets $\mathcal{P}^*_\delta(I)$ and $\mathcal{S}^*_\delta(I)$ of elements belonging to δ.
- Given $f : [a,b] \to \mathbb{R}$, for $\tau^* \in \mathcal{P}^*(I)$ and $(\sigma, \xi^*) \in \mathcal{P}^*(I)$, we can define Riemann summation $S(f;\tau^*)$ and $S(f;\sigma,\xi^*)$ analogously.
- At last, resemble the definition of gauge integral, we can define free gauge integral, i.e. the McShane integral. The set of all the McShane integrable function is denoted by $\mathcal{M}(I)$, the McShane integral of f is denoted by

$$\int_{[a,b]}^{\mathcal{M}} f(x)dx.$$

Analogous statement of Proposition 3.1-3.3 and (5), (8) also holds for McShane integral.

Proposition 3.9. $\mathcal{M}(I) \subset \mathcal{HK}(I)$, and for any $f \in \mathcal{M}(I)$,

$$\int_I^{\mathcal{HK}} f(x)dx = \int_I^{\mathcal{M}} f(x)dx.$$

Proof. Trivial, as $\mathcal{S}'_\delta(I) \subset \mathcal{S}^*_\delta(I)$. □

Proposition 3.10. $C(I) \subset \mathcal{M}(I)$ and for any $f \in C(I)$,

$$\int_I^{\mathcal{M}} f(x)dx = \int_I^{\mathcal{R}} f(x)dx.$$

Proof. As f is uniformly continuous, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon,$$

whenever $x, y \in I$ are such that $|x-y| < 2\delta$ and that

$$\left| S(f;\sigma,\xi) - \int_I^{\mathcal{R}} f(x)dx \right| < \varepsilon$$

whenever $\|\sigma\| < 2\delta$. Consider the constant gauge function δ. Then for any $(\sigma,\xi^*) \in \mathcal{S}^*_\delta(I)$, $\sigma : a_0 = a < a_1 < \cdots < a_n = b$, one have $a_i - a_{i-1} < 2\delta$ and $a_i - \xi_i^* < 2\delta$, hence

$$|f(\xi_i^*) - f(a_i)| < \varepsilon, \qquad \left| S(f;\sigma,(a_1,\cdots,a_n)) - \int_I^{\mathcal{R}} f(x)dx \right| < \varepsilon.$$

Therefore

$$\begin{aligned}\left| S(f;\sigma,\xi^*) - \int_I^{\mathcal{R}} f(x)dx \right| &\le \sum_{i=1}^n (a_i - a_{i-1})|f(\xi^*) - f(a_i)| + \left| S(f;\sigma,(a_1,\cdots,a_n)) - \int_I^{\mathcal{R}} f(x)dx \right| \\ &\le \varepsilon(b-a) + \varepsilon.\end{aligned}$$

□

Example 3.2. For $n \in \mathbb{N}^*$, let $I_n = [2^{-n}, 2^{-n+1}]$, define $f : [0,1] \to \mathbb{R}$ to be

$$f(x) = \begin{cases} (-1)^{n-1}\dfrac{2^n}{n}, & x \in I_n^\circ, n \in \mathbb{N}^* \\ 0, & x \notin \bigcup_{n=1}^\infty I_n^\circ. \end{cases}$$

Then

(a) f is gauge integrable.
(b) f is not McShane integrable.
(c) $|f|$ is not gauge integrable.

Proof. f is improper integrable whence is gauge integrable. If f is McShane integrable, one have

$$\int_{[0,1]}^{\mathcal{M}} f(x)dx = \int_{[0,1]}^{\mathcal{HK}} f(x)dx = \ln 2.$$

So there exists a gauge function $\delta \in \mathcal{G}([0,1])$ such that

$$|S(f;\sigma,\xi^*) - \ln 2| < 1, \qquad \forall(\sigma,\xi^*) \in \mathcal{S}_\delta^*([0,1]).$$

Suppose $\delta(0) \in [2^{-N}, 2^{-N+1}]$. By Cousin theorem one have a tagged partition $(\sigma_0,\xi_0) \in \mathcal{S}_\delta^*([2^{-N},1])$ and for any $2n > N$ one have $(\sigma_n,\xi_n) \in \mathcal{S}_\delta([2^{-2n}, 2^{-2n+1}])$, say $\sigma_n : a_{n,0} = 2^{-2n} < \cdots < a_{n,m_n} = 2^{-2n+1}$, $\xi_n = (\xi_{n,1}, \cdots, \xi_{n,m_n})$, thus

$$S(f;\sigma_n,\xi_n) = \sum_{k=1}^{m_n} f(\xi_{n,k})(a_{n,k} - a_{n,k-1}) = \frac{2^{2n}}{2n}\sum_{k=1}^{m_n}(a_{n,k} - a_{n,k-1}) = \frac{1}{2n}.$$

As for $[2^{-2n-1}, 2^{-2n}]$, we have $([2^{-2n-1}, 2^{-2n}], 0) \in \mathcal{S}_\delta^*([2^{-2n-1}, 2^{-2n}])$, i.e. for any $M > N$ we have

$$(\sigma,\xi^*) := \left(\bigcup_{M\geq n>N}(\sigma_n,\xi_n)\right) \bigcup \left(\bigcup_{M\geq n>N}([2^{-2n-1}, 2^{-2n}], 0)\right) \bigcup \big((\sigma_0,\xi_0) \cup ([0, 2^{-M}, 0])\big) \in \mathcal{S}_\delta^*([0,1]).$$

And then

$$\ln 2 + 1 > S(f;\sigma,\xi^*) = S(f;\sigma_0,\xi_0) + \sum_{M\geq 2n>N} S(f;\sigma_n,\xi_n) = S(f;\sigma_0,\xi_0) + \sum_{M\geq 2n>N}\frac{1}{2n} \to \infty$$

a contradiction. And if $|f| \in \mathcal{HK}([0,1])$, we have

$$\int_{[0,1]}^{\mathcal{HK}} |f(x)|dx = \int_{[0,2^{-n}]}^{\mathcal{HK}} |f(x)|dx + \sum_{i=1}^{n}\int_{[2^{-i},2^{-i+1}]}^{\mathcal{R}} |f(x)|dx \geq \sum_{i=1}^{n}\frac{1}{i} \to \infty,$$

a contradiction. □

Proposition 3.11. If $f \in \mathcal{M}([a,b])$, then $|f| \in \mathcal{M}([a,b])$.

Proof. By Cauchy's criterion, it will suffices to prove that for any $\varepsilon > 0$, one can find $\delta \in \mathcal{G}([a,b])$ such that

$$S(|f|;\sigma,\xi^*) - S(|f|;\widetilde{\sigma},\widetilde{\xi}^*) < 2\varepsilon, \qquad \forall(\sigma,\xi^*),(\widetilde{\sigma},\widetilde{\xi}^*) \in \mathcal{S}_\delta^*(I).$$

As f is gauge integrable, we have a gauge function δ such that

$$\left|S(f;\sigma,\xi^*) - \int_I^{\mathcal{M}} f(x)dx\right| < \varepsilon$$

whenever $(\sigma,\xi^*) \in \mathcal{S}_\delta^*(I)$. Assume that $(\sigma,\xi^*) = \{(J_k,\xi_k^*)\}_{k=1}^n$ and $(\widetilde{\sigma},\widetilde{\xi}^*) = \{(\widetilde{J}_l,\widetilde{\xi}_l^*)\}_{l=1}^m$. Consider the refinement $\overline{\sigma} := \{J_k \cap \widetilde{J}_l\} \in \mathcal{S}(I)$. Set

$$\overline{\xi}_{kl}^* = \begin{cases} \xi_k^*, & f(\xi_k^*) \geq f(\widetilde{\xi}_l^*) \\ \widetilde{\xi}_l^*, & f(\xi_k^*) < f(\widetilde{\xi}_l^*), \end{cases} \qquad \overline{\widetilde{\xi}}_{kl}^* = \begin{cases} \xi_k^*, & f(\xi_k^*) < f(\widetilde{\xi}_l^*) \\ \widetilde{\xi}_l^*, & f(\xi_k^*) \geq f(\widetilde{\xi}_l^*). \end{cases}$$

So

$$f(\overline{\xi}_{kl}^*) - f(\overline{\widetilde{\xi}}_{kl}^*) = |f(\xi_k^*) - f(\widetilde{\xi}_l^*)|.$$

Trivially, we have $(\overline{\sigma}, \overline{\xi}^*), (\overline{\sigma}, \overline{\overline{\xi}})^* \in \mathcal{S}_\delta^*(I)$, where $\overline{\xi}^* = (\overline{\xi}_{kl}^*)$ and $\overline{\overline{\xi}^*} = (\overline{\overline{\xi}}_{kl}^*)$, then we have

$$
\begin{aligned}
2\varepsilon &> \left| S(f; \overline{\sigma}, \overline{\xi}^*) - \int_I^{\mathcal{M}} f(x)dx \right| + \left| S(f; \overline{\sigma}, \overline{\xi}^*) - \int_I^{\mathcal{M}} f(x)dx \right| \\
&\geq S(f; \overline{\sigma}, \overline{\xi}^*) - S(f; \overline{\sigma}, \overline{\overline{\xi}}^*) = \sum_{k=1}^n \sum_{l=1}^m (f(\overline{\xi}_{kl}^*) - f(\overline{\overline{\xi}}_{kl}^*)) m(J_k \cap \overline{J}_l) \\
&= \sum_{k=1}^n \sum_{l=1}^m |f(\xi_k^*) - f(\widetilde{\xi}_l^*)| m(J_k \cap \overline{J}_l) \\
&\geq \sum_{k=1}^n \sum_{l=1}^m (|f(\xi_k^*)| - |f(\widetilde{\xi}_l^*)|) m(J_k \cap \overline{J}_l) \\
&= S(|f|; \sigma, \xi^*) - f(\widetilde{\sigma}, \widetilde{\xi}^*),
\end{aligned}
$$

where m is the Lebesgue measure. □

Similar to the fundamental theorem of calculus, we introduce the concept of strong differentiable: $F : [a, b] \to \mathbb{R}$, $c \in [a, b]$, if there exists $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, we have $\delta > 0$ such that

$$
\left| \frac{F(x) - F(y)}{x - y} - L \right| < \varepsilon
$$

whenever $[x, y] \in]c - \delta, c + \delta[\cap [a, b]$, then we say F is strong differentiable at c and has strong derivative $F'_{\mathrm{st}}(c) = L$. Trivially, F is differentiable at c and $F'(c) = F'_{\mathrm{st}}(c)$.

Example 3.3. Consider $F : [0, 1] \to \mathbb{R}$

$$
F(x) := \begin{cases} x^2 \sin \dfrac{1}{x^2}, & x \in]0, 1], \\ 0, & x = 0. \end{cases}
$$

Then F is differentiable but not strong differentiable at 0.

Proof. If F is strong differentiable at 0, $F'_{\mathrm{st}}(0) = 0$. For any $\delta > 0$, set $x = (k\pi)^{-1/2} < \delta$, $k \in \mathbb{N}^*$ then $|F'(x)| = 2(k\pi)^{1/2} \geq 2\sqrt{\pi}$. So we have $\delta_x > 0$ such that $\left| \frac{F(y) - F(x)}{y - x} - F'(x) \right| < \sqrt{\pi}$ whenever $0 < |y - x| < \delta_x$ and $y \in [0, 1]$. Take $x' \in [0, 1] \cap B_x(\delta_x) \cap B_0(\delta) \backslash \{x\} \neq \varnothing$, then

$$
\left| \frac{F(x') - F(x)}{x' - x} \right| = \left| \frac{F(x') - F(x)}{x' - x} - F'(x) + F'(x) \right| \geq \sqrt{\pi},
$$

a contradiction. □

Proposition 3.12. If $F : [a, b] \to \mathbb{R}$ is strong differentiable at every $c \in [a, b]$, then $F' \in \mathcal{M}([a.b])$ and

$$
\int_{[a,b]}^{\mathcal{M}} F'(x)dx = F(b) - F(a).
$$

Proof. For any $\varepsilon > 0$, $x \in I$, we have $\delta_x > 0$ such that

$$
\left| \frac{F(y) - F(x)}{y - x} - F'(x) \right| < \varepsilon,
$$

whenever $[y, z] \subset B_x(\delta_x) \cap I$. Now take $\delta(x) := \delta_x$. For $(\sigma, \xi^*) \in \mathcal{S}_\delta^*(I)$, $\sigma : a_0 = a < \cdots < a_n = b$,

$[a_{i-1}, a_i] \subset]\xi_i^* - \delta(\xi_i^*), \xi_i^* + \delta(\xi_i^*)[$, hence

$$\begin{aligned}|S(F';\sigma,\xi^*) - F(b) + F(a)| &= \left|\sum_{i=1}^{n} F'(\xi_i^*)(a_i - a_{i-1}) - F(b) + F(a)\right| \\ &\leq \sum_{i=1}^{n} |F'(\xi_i^*)(a_i - a_{i-1}) - F(a_i) + F(a_{i-1})| \\ &< \sum_{i=1}^{n}(a_i - a_{i-1})\varepsilon = (b-a)\varepsilon.\end{aligned}$$

□

4 Reconstruction of measure theory

4.1 Vector lattice and basic integral

We define a lattice L on a non-empty set X to be a linear space of real valued functions on X that is closed under ther following operations:

$$(f \wedge g)(x) := \max\{f(x), g(x)\}, \qquad (f \vee g)(x) = \min\{f(x), g(x)\}.$$

Proposition 4.1. A vector space L of real valued function on X is a lattice iff $|f| \in L$ whenever $f \in L$.

Now we define integral on vector lattice, in the following by f_n converges decreasingly to f we refer to pointwise convergence: $f_n \searrow f$ i.e. for every $x \in X$, when $n \to \inf$, $f_n(x)$ converges to $f(x)$ decreasingly; $f_n \nearrow f$ is defined similarly. A basic integral on a vector lattice L on X is a linear map $I : L \to \mathbb{R}$ satisfying

- I is non-negative, i.e. for every $f \in L$, if $f \geq 0$, then $I(f) \geq 0$.
- I is continuously decreasing at 0, i.e. for any decreasing sequence $f_n \in L$, if $f_n \searrow 0$, then $I(f_n) \searrow 0$.

And we call such (X, L, I) an basic integration space.

Proposition 4.2.

(a) Monotonicity: If $f, g \in L$ and $f \leq g$, then $I(f) \leq I(g)$.

(b) Monotonic convergence theorem: For an arbitrary monotonic non-negative sequence $f_n \in L$, if $f = \lim f_n \in L$ (i.e. $f_n \nearrow f$ or $f_n \searrow f$), then $I(f) = \lim I(f_n)$.

Proposition 4.3. For an arbitrary sequence of increasing non-negative functions $f_n \in L$, and $f \in L$, if $f \leq \lim f_n$ (in this case the limit may not lie in L), then $I(f) \leq \lim I(f_n)$,

Proof. Let $g_n := f \vee f_n \in L$, then $g_n \geq f_n$, $g_n \geq f$ and g_n is increasing. Obviously $I(f) \leq \lim I(g_n)$. Meanwhile, as $\lim f_n \geq f$, we have $g_n - f_n \searrow 0$ and therefore $I(f) \leq \lim I(g_n) = \lim I(f_n)$. Indeed, if $f(x) = \lim f_n(x)$, we have $g_n(x) = f(x)$ and $g_n(x) - f_n(x) \to 0$, else $f(x) < \lim f_n(x)$, for sufficiently large n, we must have $f_n(x) > f(x)$ and therefore $g_n(x) - f_n(x) = 0$. Besides, if $g_{n+1}(x) - f_{n+1}(x) \neq 0$, $f(x) > f_{n+1}(x) \geq f_n(x)$ and therefore $(g_n - f_n)(x) \geq (g_{n+1} - f_{n+1})(x)$, say $g_n - f_n$ is decreasing. □

Corollary 4.1. If $f_n, g_n \in L$ are such that $f_n \nearrow f$, $g_n \nearrow g$ and $f \leq g$ (f, g may not belong to L), then

$$\lim_{n\to\infty} I(f_n) \leq \lim_{n\to\inf} I(g_n).$$

Proof. We have $f_n \leq f \leq g$, by previous proposition, $I(f_n) \leq \lim I(g_n)$. Then as f_n are increasing, $\lim I(f_n) \leq \lim I(g_n)$. □

Example 4.1 (Integral of simple functions)**.** $\mathscr{A}$ is an algebra on X, $\mu : \mathscr{A} \to [0, \infty[$ is a pre-measure, the simple functions on $(X, \mathscr{A})$ are still defined to be finite linear combination of indicator functions of sets in $\mathscr{A}$:

$$f = \sum_{j=1}^{n} a_j \chi_{A_j}, \qquad a_j \in \mathbb{R}, A_j \in \mathscr{A}.$$

Then by Proposition 4.1, it's easy to see that all simple functions form a vector lattice $L = \mathcal{SP}(X, \mathscr{A})$. Define simple integral $I_\mu : L \to \mathbb{R}$ by

$$I_\mu(f) = \sum_{j=1}^{n} a_j \mu(A_j).$$

Then I_μ is independent of the choice of a_j, A_j and (X, L, I_μ) is a basic integration space.

Proof. We only prove that I_μ satisfies the monotonic convergence theorem: For a decreasing sequence of functions $f_n \searrow 0$ and any $\varepsilon > 0$, set $A_n(\varepsilon) := \{f < \frac{\varepsilon}{\mu(X)}\}$, so $A_n(\varepsilon) \nearrow X$. By continuity, $\mu(X) = \lim_{n\to\infty} \mu(A_n(\varepsilon))$. Let $N \in \mathbb{N}$ such that $\mu(A_n^{\complement}(\varepsilon)) < \varepsilon / \max f_1(x)$ whenever $n > N$, then

$$I_\mu(f_n) \le \max f_1(x)\mu(A_n(\varepsilon)^{\complement}) + \frac{\varepsilon}{\mu(X)}\mu(A_n(\varepsilon)) < 2\varepsilon.$$

So $I_\mu(f_n) \searrow 0$. $\square$

Example 4.2 (Riemann integral of continuous functions)**.** X is a bounded closed interval, $L = C(X)$ is the vector space of all real valued continuous functions on X, $I : C(X) \to \mathbb{R}$ is the Riemann integral. Then by Dini theorem, (X, L, I) is a basic integration space.

4.2 The first continuation of basic integral

If (X, L, I) is a basic integration space, we define the space of all limits of increasing sequences in L to be

$$L^\bullet := \{f : X \to]-\infty, \infty] : \exists f_n \in L (f_n \nearrow f)\}$$

and define $I^\bullet :\to]-\infty, \infty]$ to be

$$I^\bullet(f) := \lim_{n\to\infty} I(f_n), \qquad \text{if } f_n \in L, f_n \nearrow f.$$

Analogously, we can define $I_\bullet : L_\bullet \to [-\infty, \infty[$ to be

$$L_\bullet := \{f : X \to [-\infty, \infty[: \exists f_n \in L(f_n \searrow f)\}$$
$$I_\bullet := \lim_{n\to\infty} I(f_n), \qquad \text{if } f_n \in L, f_n \searrow f.$$

By Corollary 4.1, $I_\bullet, I^\bullet$ are independent of f_n. Here are some basic properties (some of are sated only for $L^\bullet$ and analogous properties holds for $L_\bullet$ too):

Proposition 4.4. (a) $f \in L_\bullet$ iff $-f \in L^\bullet$ and $I_\bullet(f) = -I^\bullet(-f)$;

(b) $L \subset L_\bullet \cap L^\bullet$ and for any $f \in L$, $I^\bullet(f) = I_\bullet(f) = I(f)$;

(c) $L^\bullet$ is closed under addition and scale multiplication of non-negative constant: If $f_1, f_2 \in L^\bullet$, $\alpha_1, \alpha_2 \in [0, \infty[$, then $\alpha_1 f + \alpha_2 f \in L^T able$ and

$$I^\bullet(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 I^\bullet(f_1) + \alpha_2 I^\bullet(f_2).$$

(d) $L^\bullet$ is closed under $\wedge, \vee$: If $f_1, f_2 \in L^\bullet$, then $f_1 \wedge f_2, f_1 \vee f_2 \in L^\bullet$;

(e) Monotonicity: If $f, g \in L^\bullet$ and $f \le g$, then $I^\bullet(f) \le I^\bullet(g)$;

(f) If $f \in L_\bullet$, $g \in L^\bullet$ and $f \le g$, then $I_\bullet(f) \le I^\bullet(g)$;

(g) Monotonic convergence theorem: If $f_n \in L^\bullet$ and $f_n \nearrow f$, then $f \in L^\bullet$ and $I^\bullet(f) = \lim I^\bullet(f_n)$.

Proof. We only prove (f) and (g). For (f), suppose $f_n \searrow f$, $g_n \nearrow g$, $f_n, g_n \in L$. Set $F_n = f_n \vee g_n$ and $G_n = f_n \wedge g_n$. Then $f_n - F_n \searrow 0$ and therefore $\lim I(F_n) = \lim I(f_n) = I_\bullet(f)$. Similarly, $I^\bullet(g) = \lim I(G_n) \ge \lim I(F_n) = I_\bullet(f)$. Indeed if $f_n(x) \le g_n(x)$, then $f_{n+1}(x) \le f_n(x) \le g_n(x) \le g_{n+1}(x)$, thus $(f_{n+1} - F_{n+1})(x) - (f_n - F_n)(x) = 0$. If $f_n(x) > g_n(x)$, $f_{n+1}(x) \le g_{n+1}(x)$, then $(f_{n+1} - F_{n+1})(x) - (f_n - F_n)(x) = g_n(x) - f_n(x) < 0$. If $f_n(x) > g_n(x)$ while $f_{n+1}(x) > g_{n+1}(x)$, $(f_{n+1} - F_{n+1})(x) - (f_n - F_n)(x) = (f_{n+1} - f_n)(x) - (g_{n+1} - g_n)(x) < 0$.

As for (g), suppose $L \ni f_n^k \nearrow f_n$. Let $f^k := \max_{n\le k} f_n^k \in L$. Then the sequence f^k is increasing. For any $x \in X$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $f(x) \ge f_n(x) > f(x) - \varepsilon$ whenever $n > N$. There also exists. $N \le M \in \mathbb{N}$ such that $f_N(x) \ge f_N^m(x) > f_N(x) - \varepsilon$ whenever $m > M$. Then for all $k > M$,

$$F(x) \ge \max_{n\le k} f_n(x) \ge f^k(x) \ge f_N^k(x) > f_N(x) - \varepsilon > f(x) - 2\varepsilon.$$

Thus $f^k \nearrow f$ and therefore $f \in L^\bullet$ and by monotonicity, $I^\bullet(f) \geq \lim I^\bullet(f_n)$. Meanwhile

$$I^\bullet(f) = \lim_{n\to\infty} I(f^k) \leq \lim_{n\to\infty} I^\bullet(f_n).$$

Thus $I^\bullet(f) = \lim I^\bullet(f_n)$. $\square$

4.3 Upper integral, lower integral and integrable functions

For any $f : X \to [-\infty, \infty]$, the upper/lower integral of it is defined to be

$$\overline{I}(f) := \inf\{I^\bullet(g) : g \in L^\bullet, g \geq f\}$$
$$\underline{I}(f) := \sup\{I_\bullet(g) : g \in L_\bullet, g \leq f\}.$$

Where we take

$$\inf \varnothing = \infty, \qquad \sup \varnothing = -\infty.$$

Trivially, we have $\overline{I}(f) \geq \underline{I}(f)$. If

$$-\infty < \underline{I}(f) = \overline{I}(f) < \infty,$$

then we say f is (L, I)-integrable and denote the set of all (L, I)-integrable functions by $\mathcal{L}^1 = \mathcal{L}^1(X, L, I)$ and define the integral of $f \in \mathcal{L}^1(X, L, I)$ to be

$$\mathcal{I}(f) = \overline{I}(f) = \underline{I}(f).$$

We have Cauchy criterion:

Proposition 4.5. *$f \in \mathcal{L}^1(X, L, I)$ iff for any $\varepsilon > 0$, there is $g \in L_\bullet, h \in L^\bullet$ such that $g \leq f \leq h$ and that $I^\bullet(h - g) < \varepsilon$.*

Proof. The "only if" part is trivial by definition. Conversely, if such g, h exists, from $I^\bullet(h - g) < \varepsilon$, one immediately get $I^\bullet(h) < \infty$ and $I_\bullet(g) > -\infty$. As

$$I_\bullet(g) \leq \underline{I}(f) \leq \overline{I}(f) \leq I^\bullet(f),$$

one then have $\overline{I}(f) - \underline{I}(f) < \varepsilon$. But $\varepsilon > 0$ is arbitrary, f is integrable. $\square$

We have the following basic properties:

Proposition 4.6. (a) *$L \subset \mathcal{L}^1(X, L, I)$ and for any $f \in L$, $\mathcal{I}(f) = I(f)$.*

(b) *Monotonicity: If $f, g \in \mathcal{L}^1(X, L, I)$ and $f \leq g$, then $\mathcal{I}(f) \leq \mathcal{I}(g)$.*

(c) *$\mathcal{L}^1(X, L, I)$ is closed under $\wedge$ and $\vee$.*

(d) *All finite valued (L, I)-integrable functions (i.e. $\mathrm{ran} f \subset \mathbb{R}$) form a linear space $\mathcal{L} := \mathcal{L}^1_f(X, L, I)$ and $\mathcal{I} : \mathcal{L} \to \mathbb{R}$ is a linear map.*

(e) *The set $\mathcal{L}^1_+(X, L, I)$ of all non-negative valued (L, I)-integrable functions (f can assume ∞) is closed under addition and non-negative scale multiplication: If $f_1, f_2 \in \mathcal{L}^1_+(X, L, I)$, $\alpha_1, \alpha_2 \in [0, \infty[$, then $\alpha_1 f_1 + \alpha_2 f_2 \in \mathcal{L}^1_+(X, L, I)$ and*

$$\mathcal{I}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{I}(f_1) + \alpha_2 \mathcal{I}(f_2).$$

(f) *$f \in \mathcal{L}^1(X, L, I)$ iff $f_\pm \in \mathcal{L}^1_+(X, L, I)$ and*

$$\mathcal{I}(f) = \mathcal{I}(f_+) - \mathcal{I}(f_-).$$

In particular, if $f \in \mathcal{L}^1(X, L, I)$, then $|f| \in \mathcal{L}^1_+(X, L, I)$ and

$$|\mathcal{I}(f)| \leq \mathcal{I}(|f|).$$

Proof. For (c) (the "$\vee$" part) one only need to notice that $f_1 \vee g_1 - f_2 \vee g_2 \leq (f_1 - f_2) \vee (g_1 - g_2) \leq (f_1 - f_2) + (g_1 - g_2)$, where $f_2 \leq f_1, g_2 \leq g_1$ and $f_1, g_1 \in L^\bullet$, $f_2, g_2 \in L_\bullet$ and the $\wedge$ part is similar. $\square$

Theorem 6 (Monotonic convergence theorem)**.** *If $f_n \in \mathcal{L}^1_+(X, L, I)$ are such that $f_n \nearrow f$, $\mathcal{I}(f_n) \nearrow M \in [0, \infty]$, then $\overline{I}(f) = \underline{f} = M$. Particularly, if $M < \infty$, then $f \in \mathcal{L}^1(X, L, I)$, $\mathcal{I}(f) = M$. Therefore we get integration continuation theorem: $(X, \mathcal{L}, \mathcal{I})$ is a basic integration space.*

Proof. For any $\varepsilon > 0$, let $f_n \geq \underline{f}_n \in L_\bullet$ such that $\mathcal{I}(f_n) - 2^{-n}\varepsilon < I_\bullet(\underline{f}_n) \leq \mathcal{I}(f_n)$. Therefore $I_\bullet(f_n) \to M$. As $f \geq f_n \geq \underline{f}_n$, $\underline{I}(f) \geq M$. Let $L^\bullet \ni \overline{f}_n \geq f_n$ be such that $\mathcal{I}(f_n) \leq I^\bullet(\overline{f}_n) < \mathcal{I}(f_n) + 2^{-n}\varepsilon$, then $I^\bullet(f_n) \to M$. Take $\widetilde{f}_n := \bigvee_{k=1}^n \overline{f}_k \in L^\bullet$. then $\widetilde{f}_n$ is increasing. Then $\lim_{n\to\infty} \widetilde{f}_n \in L^\bullet$. Let $f'_n := \bigvee_{k=1}^n \underline{f}_n \in L_\bullet$. By definition, $\widetilde{f}_n \geq \overline{f}_n \geq f_n = \bigvee_{k=1}^n f_k \geq \bigvee_{k=1}^n \underline{f}_k = f'_k$. Note that if $\widetilde{f}_n(x) = \overline{f}_i(x)$, then

$$(\widetilde{f}_n - f'_n)(x) = \overline{f}_i(x) - f'_n(x) \leq (\overline{f}_i - \underline{f}_i)(x) \leq \bigvee_{k=1}^n (\overline{f}_k - \underline{f}_k)(x) \leq \sum_{k=1}^n (\overline{f}_k - \underline{f}_k)(x).$$

Therefore

$$I^\bullet(\widetilde{f}_n - f'_n) \leq \sum_{k=1}^n I^\bullet(\overline{f}_k - \underline{f}_k) < 2\varepsilon$$

and

$$\mathcal{I}(f_n) = \overline{I}(f_n) \leq I^\bullet(\widetilde{f}_n) < 2\varepsilon + I_\bullet(f'_n) \leq 2\varepsilon + \underline{I}(f_n) = 2\varepsilon + \mathcal{I}(f_n). \tag{10}$$

From (10) and Proposition 4.4-(g) one immediately get

$$M + 2\varepsilon > I^\bullet(\lim_{n\to\infty} \widetilde{f}_n) \geq \overline{I}(f).$$

But $\varepsilon > 0$ is arbitrary, we have $\overline{I}(f) = \underline{I}(f) = M$. $\square$

If we start form $(X, \mathcal{L}, \mathcal{I})$ and define

$$\mathcal{L}^\bullet = \{f : \exists f_n \in \mathcal{L}, f_n \nearrow f\}, \qquad \mathcal{L}_\bullet = \{f : \exists f_n \in \mathcal{L}, f_n \searrow f\}.$$

And set $\mathcal{I}^\bullet$ and $\mathcal{I}_\bullet$ analogously, then $\mathcal{L}^\bullet \cap \mathcal{L}_\bullet = \mathcal{L}$. And we would reach the same basic integration space $(X, \mathcal{L}, \mathcal{I})$.

Indeed, if $f \in \mathcal{L}^\bullet \cap \mathcal{L}_\bullet$ and if $f(X) \subset \mathbb{R}$. The same argument of the last proof yields that for $\mathcal{L} \ni \overline{f}_n \searrow f$ and $\mathcal{L} \ni \underline{f}_n \nearrow f$, we have $\overline{I}(f) = \underline{I}(f) = \lim_{n\to\infty} \mathcal{I}(\overline{f}_n) \neq \infty$ and $\overline{I}(f) = \underline{I}(f) = \lim_{n\to\infty} \mathcal{I}(\underline{f}_n) \neq -\infty$. Therefore $f \in \mathcal{L}$ and $\mathcal{L} = \mathcal{L}^\bullet \cap \mathcal{L}_\bullet$.

Denote $\mathcal{L}^1(X, \mathcal{L}, \mathcal{I}) := \mathscr{L}^1$ and $\mathcal{L}^1_f(X, \mathcal{L}, \mathcal{I}) := \mathscr{L}$, then $\mathcal{L} \subset \mathscr{L}$. If $f \in \mathscr{L}$, we have $h_n \geq f \geq g_n$, $h_n \in \mathcal{L}^\bullet$, $g_n \in \mathcal{L}_\bullet$ such that $\mathcal{I}^\bullet(h_n) < \overline{\mathcal{I}}(f) + \frac{1}{n}$ and that $\mathcal{I}_\bullet(g_n) > \underline{\mathcal{I}}(f) - \frac{1}{n}$. Suppose $\mathcal{L} \ni h^k_n \nearrow h_n$, $\lim_{k\to\infty} \mathcal{I}(h^k_n) = \mathcal{I}^\bullet(h_n) \neq \infty$. By theorem 6, $h_n \in \mathcal{L}^1$. Similarly, $g_n \in \mathcal{L}^1$. Take $\widetilde{h}_n = \bigwedge_{k=1}^n h_k \in \mathcal{L}^1$, $\widetilde{h}_n$ is monotonic decreasing and $\widetilde{h}_n \geq f$, therefore $\mathcal{I}^\bullet(\widetilde{h}_n) = \mathcal{I}(\widetilde{h}_n) \in [\overline{\mathcal{I}}(f), \bigwedge_{k=1}^n \mathcal{I}(h_k)[\subset [\overline{\mathcal{I}}(f), \overline{\mathcal{I}}(f) + \frac{1}{n}[$. Again by theorem 6, $\lim \widetilde{h}_n \in \mathcal{L}^1$ and $\mathcal{I}(\lim \widetilde{h}_n) = \overline{\mathcal{I}}(f)$. Similarly, $\widetilde{g_n} := \bigwedge_{k=1}^n g_k$, $\lim \widetilde{g}_n \in \mathcal{L}^1$ and $\underline{\mathcal{I}}(f) = \mathcal{I}(\lim \widetilde{g}_n)$. Moreover

$$\overline{\mathcal{I}(f)}\mathcal{I}(\lim_{n\to\infty} \widetilde{h}_n) = \overline{I}(\lim_{n\to\infty} \widetilde{h}_n) \geq \overline{I}(f) \geq \underline{I}(f) \geq \underline{I}(f) \geq \underline{I}(\lim_{n\to\infty} \widetilde{g}_n) = \underline{\mathcal{I}}(f).$$

As $f \in \mathscr{L}$, $\overline{\mathcal{I}}(f) = \underline{\mathcal{I}}(f) \neq \infty$ so $\overline{I}(f) = \underline{I}(f) \neq \pm\infty$. Thus $f \in \mathcal{L}$ and $\mathcal{L} = \mathscr{L}$. The above process also implies that the integral on $\mathscr{L}$ is nothing but $\mathcal{I}$, say one would reach exactly the same basic integration space form $(X, \mathcal{L}, \mathcal{I})$.

4.4 The reconstruction

Now we reconstruct measure theory from the above theory of integral, **in this process we CANNOT use the reulf form measure theory**. We first define null sets, whence we can introduce the concept of "almost everywhere":

- A subset A of X is said to be (L, I)-null, if $\chi_A \in \mathcal{L}^1(X, L, I)$, and $\mathcal{I}(\chi_A) = 0$.
- We say a property w.r.t. $x \in X$ holds (L, I)-almost everywhere, denoted by (L, I)-a.e., if the set of points that violate this property is (L, I)-null.

In the following, we omit "(L, I)-" in the notation.

Proposition 4.7. (a) $A \subset X$ is null iff for any $\varepsilon > 0$, there exists $f \in L^\bullet$ such that $\chi_A \leq f$ and that $I^\bullet(f) < \varepsilon$.

(b) The subset of a null set is null.

(c) Countable union of null sets is null.

Proof. We only prove (c). Let $\{A_n\}_{n=1}^\infty$ be a sequence of null set. By (a) we can find $L^\bullet \ni f_n \geq \chi_{A_n}$ such that $I^\bullet(f_n) < 2^{-n}\varepsilon$. We have $\sum_{k=1}^n f_k \nearrow \sum_{k=1}^\infty f_k$. By proposition 4.4 $\sum_{k=1}^\infty k_k \in L^\bullet$ and $I^\bullet(\sum_{k=1}^\infty f_k) = \lim_{n\to\infty} I^\bullet(\sum_{k=1}^n f_k) < \varepsilon$. Meanwhile, we have $\sum_{k=1}^\infty f_k \geq \sum_{k=1}^\infty \chi_{A_n} \geq \chi_{\cup A_n}$, then apply (a). ☐

Proposition 4.8. (a) If $f = g$ a.e. and $f \in \mathcal{L}^1(X, L, I)$, then $g \in \mathcal{L}^1$ and $\mathcal{I}(f) = \mathcal{I}(g)$.
(b) If $f \in \mathcal{L}^1$, then f is finite a.e. namely $\{x : |f(x)| = \infty\}$ is null.

Proof. For (a), passing to $g - f$, one may assume that f is equivalently zero and by proposition 4.6, one can further assume that g is non-negative. Let $A = \{x : g(x) \neq 0\}$, by assumption, A is a null set. Take $g_n := n\chi_A$, then $g_n \in \mathcal{L}^1$ and $\mathcal{I}(g_n) = 0$. By theorem 6, $g_n \nearrow \infty\chi_A \in \mathcal{L}^1$ and $\mathcal{I}(\infty\chi_A) = 0$. Clearly $g \leq \infty\chi_A$, thus $g \in \mathcal{L}^1$ and $\mathcal{I}(g) = 0$.

As for (b), we know $|f| \in \mathcal{L}^1$ by proposition 4.6. Then for any $\varepsilon > 0$, one can find $h \geq |f| \geq g$, $h \in L^\bullet$, $g \in L^\bullet$ such that $I^\bullet(h - g) < \varepsilon$. If $|f(x)| = \infty$, $h(x) = \infty$ and therefore $(h - g)(x) = \infty$. Clearly we have $\chi_A \leq h - g$. Now as $\varepsilon > 0$ is arbitrary, proposition 4.7-(a) yields that A is null. ☐

Next we can define measurable functions:

- We say $f : X \to [-\infty, \infty]$ is (L, I)-measurable, if for any $g, h \in \mathcal{L}^1_+(X, L, I)$, $(f \wedge g) \vee (-h) \in \mathcal{L}^1(X, L, I)$.
- Denote the space of all (L, I)-measurable functions by $\mathcal{M} = \mathcal{M}(X, L, I)$.

Proposition 4.9. (a) $\mathcal{L}^1 \subset \mathcal{M}$.
(b) $\mathcal{M}$ is closed under $\wedge$ and $\vee$.

Proof. (b) can be deduced from the distribution law of $\wedge$ and $\vee$ ☐

Proposition 4.10. (a) If $f = g$ a.e. and $f \in \mathcal{M}$, then $g \in \mathcal{M}$.
(b) If $f_n \in \mathcal{M}$ and $f_n \to f$ a.e. then $f \in \mathcal{M}$.
(c) If $f, g \in \mathcal{M}$ don't assume $\pm\infty$, $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in \mathcal{M}$.

Proof. (a) is a corollary of proposition 4.8. (b) If $f_n \nearrow (\searrow) f$, one have $(f_n \wedge (\vee) g) \nearrow (\searrow) f \wedge (\vee) g$. Thus $\mathcal{M} \ni f_n \nearrow f$ implies $\mathcal{L}^1 \ni (g \wedge f_n) \wedge (-h) \nearrow (f \wedge g) \vee (-h)$. Meanwhile

$$\mathcal{I}((f_n \wedge g) \vee (-h)) = \mathcal{I}((f_n \vee (-h)) \wedge (g \vee (-h))) \leq \mathcal{I}(g \vee (-h));$$
$$\mathcal{I}((f_n \wedge g) \vee (-h)) \geq \mathcal{I}(-h).$$

Thus $(f \wedge g) \vee (-h) \in \mathcal{L}^1$ and $f \in \mathcal{M}$. So if $\mathcal{M} \ni f_n \to f$ a.e. $\vee_{k=m}^{m+n} f_k \in \mathcal{M}$ and $\vee_{k=m}^{m+n} f_k \nearrow \sup_{k\geq m} f_k \in \mathcal{M}$. Similarly, $\sup_{k\geq m} \searrow \limsup f_n \in \mathcal{M}$. However $f = \limsup f_n$ a.e. whence $f \in \mathcal{M}$.

For (c), denote the set of finite valued measurable function by $\mathcal{M}_f$ and we prove

- $\alpha \in \mathbb{R}$, $f \in \mathcal{M}_f$ then $\alpha f \in \mathcal{M}$.
  $\alpha = 0$ is trivial. If $\alpha > 0$, $((\alpha f) \wedge g) \vee (-h) = \alpha(f \wedge (\alpha^{-1} g)) \vee (-\alpha^{-1} h) \in \mathcal{L}^1$. And if $\alpha < 0$, $((\alpha f) \wedge g) \vee (-h) = \alpha((f \wedge (-\alpha^{-1} h)) \vee (-(-\alpha^{-1} g))) \in \mathcal{L}^1$.

- $f_1, f_2 \in \mathcal{M}_f$, then $f_1 + f_2 \in \mathcal{M}$.
  Note that for any $l \in \mathcal{L}^1_+$,

$$(f \wedge (nl)) \vee (-nl) = \begin{cases} f(x), & |f(x)| \leq nl(x); \\ \pm nl(n), & |f(x)| \geq nl(x). \end{cases} \to \begin{cases} f(x), & l(x) \neq 0; \\ 0, & l(x) = 0. \end{cases}$$

  Denote the set of zero of h as N_h, then $(f \wedge (nl)) \vee (-nl) \to f \mathbb{1}_{N_l^\complement}$. Now if $f_1, f_2 \in \mathcal{M}$, for all $l \in \mathcal{L}^1_+$, $(f_i \wedge (nl)) \vee (-nl) \in \mathcal{L}^1$, $i = 1, 2$. So $(f_1 \wedge (nl)) \vee (-nl) \in \mathcal{L}^1 + (f_2 \wedge (nl)) \vee (-nl) \in \mathcal{L}^1 \in \mathcal{L} \subset \mathcal{M}$. Therefore by previous proposition, $(f_1 + f_2)\mathbb{1}_{N_l^\complement} \in \mathcal{M}$. Next or any $g, h \in \mathcal{L}^1_+$, take $l = g \vee h$. If $g(x) = h(x) = 0$, i.e. $l(x) = 0$, then $f \wedge g(x) \leq 0$ and $(f \wedge g) \vee (-h)(x) = 0$. So

$$((f_1 + f_2) \wedge g) \vee (-h) = ((f_1 + f_2)\mathbb{1}_{N_l^\complement} \wedge g) \vee (-h).$$

  As $(f_1 + f_2)\mathbb{1}_{N_l^\complement} \in \mathcal{M}$, we have $((f_1 + f_2) \wedge g) \vee (-h) \in \mathcal{L}^1$ and therefore $f_1 + f_2 \in \mathcal{M}$.

☐

Finally, let's define measurable sets and their measure:

- We naturally define $E \subset X$ is measurable iff $\chi_E \in \mathcal{M}(X, L, I)$;
- And denote the collection of all measurable sets by $\mathscr{F}$.

In order to ensure the measurability of X, we need to introduce an additional condition:

$$f \in L \quad \Rightarrow \quad f \wedge 1 \in L. \tag{11}$$

And now we define a non-negative function $\mu : \mathscr{F} \to [0, \infty]$ on $\mathscr{F}$ by:

- If $\chi_A \in \mathcal{L}$, then $\mu(A) = \mathcal{I}(\chi_A)$;
- If $\chi_A \in \mathcal{M} \backslash \mathcal{L}$, then $\mu(A) = \infty$.

Problem 41. If (X, L, I) is a basic integration space, satisfies condition (11), then $(X, \mathscr{F}, \mu)$ is a complete measure space. $f \in \mathcal{L}^1(X, L, I)$ iff f is integrable w.r.t. μ and in this case

$$\mathcal{I}(f) = \int_X f d\mu.$$

Moreover, if constant function $1 \in L$, then $(X, \mathscr{F}, \mu)$ is a finite measure space.

Problem 42. What measure space will one get from example 4.1 of simple functions?