Aristides A. G. Requicha
Production Automation Project
University of Rochester
Rochester, New York 14627

Toward a Theory of Geometric Tolerancing

Abstract

Manual drafting is rapidly being replaced by modern, computerized systems for defining the geometry of mechanical parts and assemblies, and a new generation of powerful systems, called geometric (solid) modeling systems (GMSs), is entering industrial use. Solid models are beginning to play an important role in off-line robot programming, model-driven vision, and other industrial robotic applications.

A major deficiency of current GMSs is their lack of facilities for specifying tolerancing information, which is essential for design analysis, process planning, assembly planning for tightly toleranced components, and other applications of solid modeling. This paper proposes a mathematical theory of tolerancing that formalizes and generalizes current practices and is a suitable basis for incorporating tolerances into GMSs.

A tolerance specification in the proposed theory is a collection of geometric constraints on an object's surface features, which are two-dimensional subsets of the object's boundary. An object is in tolerance if its surface features lie within tolerance zones, which are regions of space constructed by offsetting (expanding or shrinking) the object's nominal boundaries.

1. Introduction

The traditional media for specifying mechanical parts and assemblies are blueprints (engineering drawings), which contain graphic descriptions of *nominal* or ideal parts plus tolerancing information that defines allowable departures from the nominal objects. Tolerances are specified by designers and should ensure that parts "in spec" are functionally equivalent and interchangeable in assembly. Tolerancing information is essential for planning part manufacture and tight assembly operations, for part inspection, and for other design and production activities.

Manual drafting is rapidly being replaced by modern, computerized systems for defining the geometry of mechanical parts and assemblies, and a new generation of powerful systems, called *geometric (solid) modeling systems* (GMSs), is entering industrial use (Requicha and Voelcker 1982). Solid models generated by computer-aided design (CAD) are expected to become the primary sources of geometric information in integrated design and production systems, and solid-modeling technology is beginning to play an important role in off-line robot programming, model-driven vision, and other industrial robotic applications.

Current GMSs lack tolerancing facilities and therefore can neither support fully automatic manufacturing planning nor perform some of the spatial reasoning required for assembly planning. For example, analysis of manipulator and sensor inaccuracies must be supplemented with analysis of *part* inaccuracies for tight-fitting assembly tasks (Brooks 1982). To incorporate tolerancing information in GMSs and use it in automatic analysis and planning, the semantics of tolerances (i.e., their geometric meaning) should be defined mathematically. General definitions, which can be "understood" by general programs, are preferable to special-case definitions, which are difficult to embed in programs and lead to large amounts of possibly inconsistent code. Unfortunately, current industrial tolerancing practices, described in standards (ANSI 1973) and texts (Levy 1974), are defined informally, mainly for special situations.

The goal of this paper is to propose a theory of tolerancing that formalizes and generalizes current practices and is a suitable basis for incorporating tolerances into GMSs. The theory was designed to follow established tolerancing practices closely, but some departures seemed desirable and others unavoid-

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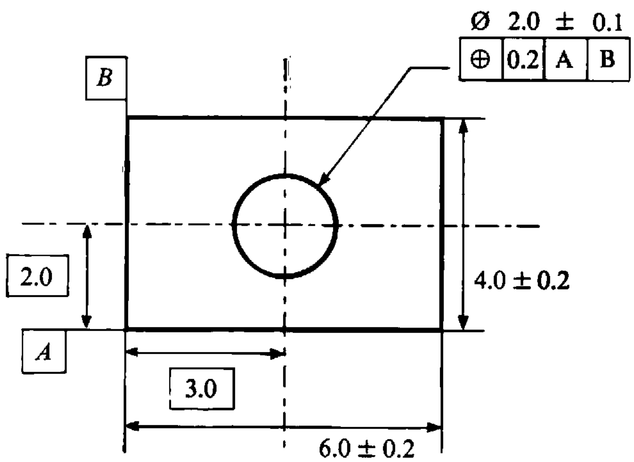
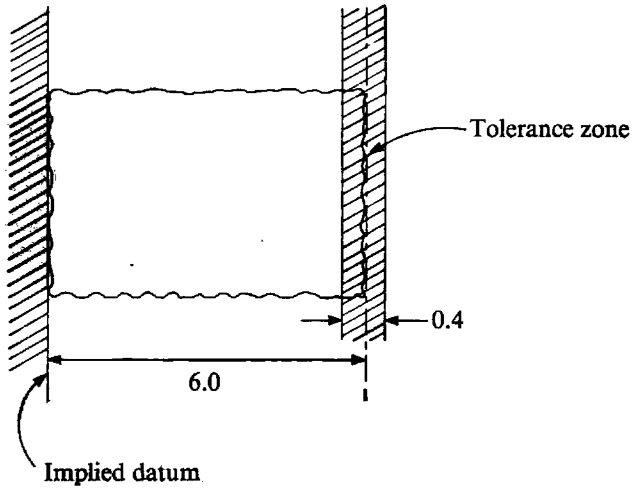
Fig. 1. A simple example of current tolerancing practices. The rectangle and circle dimensions are specified by conventional ± tolerances. The symbol block associated with the circle specifies a "true-position" tolerance with respect to datums A *and* B, *which are the surfaces labeled in the drawing.*

Fig. 2. Interpretation based on an "implied datum."





able. The paper also explores briefly the representational implications of the theory, but does not attempt to discuss its algorithmic implications. Algorithms for processing tolerancing information in solid modelers are just beginning to be studied, and it is too early to draw conclusions. The representational implications of the theory are straightforward but important, because they establish the data requirements for the tolerancing facilities of GMSs. These requirements differ markedly for alternative theoretical approaches (Requicha 1983).

2. Current Practices and Prior Work

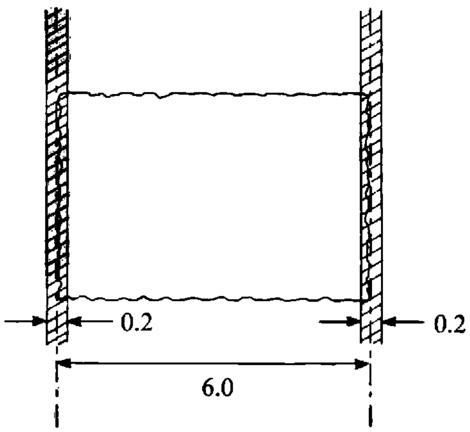
Current industrial tolerancing practices use a mix of "geometric tolerances," sometimes called "modern" or "true-position" tolerances, with "conventional" (±) tolerances (see Fig. 1 for a very simple example). The current trend in industrial practice is toward an increased use of geometric tolerancing.

I have not found any standards or texts that define carefully the meaning of conventional tolerances. There appear to be inherent ambiguities that experienced humans usually resolve by appealing to "implicit datums" and other information not explicitly stated in the specification (e.g., the mechanical function of the component). For example, the dimensional constraint 6.0 ± 0.2 in Fig. 1 can be interpreted as shown in Figs. 2 and 3, and perhaps in other ways. In the interpretation of Fig. 2, a plane tangent to the left face of an actual object is constructed and treated as a datum. The right face must lie within the tolerance zone shown, which is bounded by two lines parallel to the datum. In the interpretation of Fig. 3 no datums are assumed, and the vertical faces are required to lie within the two tolerance zones shown. Observe that no ambiguities would arise if the actual object's faces were perfect, parallel lines; one would simply measure the distance between them to determine if it fell in the specified interval (5.8, 6.2).

The meaning of modern geometric tolerances is reasonably well described in the standards, but there are significant gaps, centered on the notions of "feature" and "size." Geometric tolerances apply to features, but standards and texts provide no precise definition of feature. Similarly, the concept of size is never defined (and has no obvious meaning for features of imperfect shape, e.g., for a hole that is not perfectly cylindrical), but is used extensively to describe the semantics of the various tolerance specifications.

There are about a dozen types of tolerances applicable to features: cylindricity, roundness, position, perpendicularity, and so on. Readers who are not familiar with current tolerancing practices can gain a reasonable idea of what they are by reading the formalizations I propose in Section 3, but I strongly recommend a careful reading of the standards to anyone seriously interested in the problems discussed in this paper.

Fig. 3. Interpretation based on "size."



Published studies of tolerancing in the context of geometric modeling are vey few. My own report (Requicha 1977*a*) addressed both conventional and geometric tolerancing but provided no clues on how to deal with both simultaneously. Hillyard's and Hoffman's work (Hillyard 1978; Hillyard and Braid 1978*a*; 1978*b*; Hoffman 1982) appears to be concerned exclusively with conventional tolerances and does not address the issues discussed in this paper. Work on geometric constraints and object parameterization (e.g., Brooks 1981; Fitzgerald 1981; Lin, Gossard, and Light 1981) also is relevant to certain "parametric" approaches to tolerancing (see Requicha 1983). The approach that I shall discuss below evolved from my earlier research (Requicha 1977*a*) and is not related to any of the other published work cited above.

3. A Theory of Tolerancing

The following subsections propose a mathematical theory of tolerancing. (This theory provides the theoretical basis for the tolerancing facilities currently being implemented in the PADL-2 modeler [Brown 1982]). Two of the main, and perhaps surprising, tenets of the theory are the following:

Conventional tolerances are subsumed into geometric tolerances that contain no implicit datums or unstated constraints.[1]

There is no formal notion of a "measured size," although "size tolerances" and similar concepts are defined mathematically.

I shall present first a rough outline of the theory and then embark on a detailed exposition.

3.1. Outline

Consider initially a solid cylinder in three-dimensional euclidean space (E^3). The size of the cylinder obviously is well defined and can be described by its radius and height. We can also agree on an unambiguous way, and a corresponding set of parameters (distances and angles), to define the cylinder's position, that is, its location and orientation.

Consider now a family of (perfect-form) cylinders with sizes and positions close to those of the initial cylinder. It is clear that such a family can be defined in terms of a range of allowable values for the agreed size and positional parameters of cylinders. This seems to be the spirit of conventional tolerancing practices and also of the geometric-modeling literature cited in Section 2. Specifically, a manufactured solid is in spec if its defining parameters are within specified ranges.

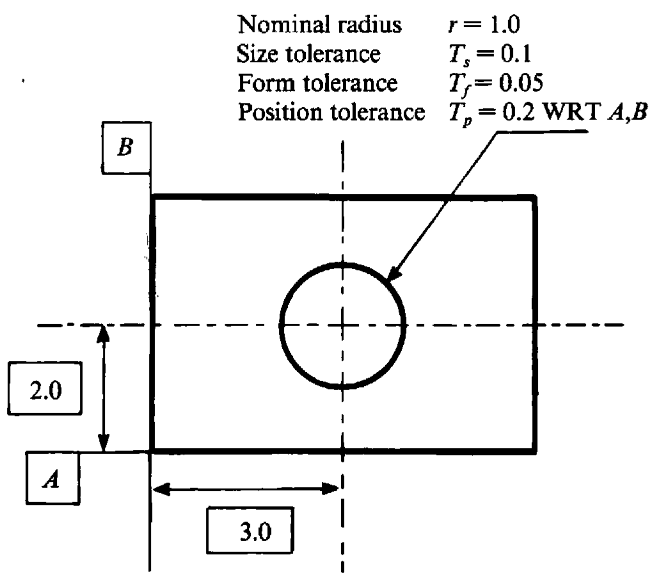
Manufacturing processes, however, do not produce perfect cylinders. Therefore, let us consider an object that results from a small but random deformation of our initial cylinder. It should be clear that the notions of size and position for such an object have no obvious meanings. A possible way out of this impasse consists of defining the size and position parameters of an imperfect-form object as the corresponding parameters of an associated perfect-form object, generated by surface-fitting or similar techniques. This approach is sometimes useful, for example, for defining datums (see Section 3.4), but does not seem applicable to most of the modern tolerancing practices.

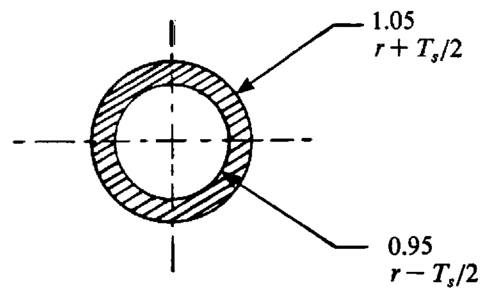
The approach I shall describe does not attempt to assign well-defined sizes or positions to imperfect-form objects. It considers an imperfect-form object to be in spec when its boundary is within "tolerance zones," which are defined in terms of perfect-form objects. To flesh out the approach, we need general mathematical procedures for building tolerance zones around objects and for specifying how such zones are located and oriented.

1. This does not mean that human users of GMSs must specify explicitly all constraints—good user interfaces can take care of many of these through defaults.

Fig. 4. A two-dimensional hole with a size tolerance $T_s = 0.1$, form tolerance $T_f = 0.05$, and position tolerance $T_p = 0.2$.

Fig. 5. Tolerance zone for the size specification of Fig. 4. The position of the tolerance zone is arbitrary.





As a preface to the detailed discussion that follows, consider the simple two-dimensional example shown in Fig. 4, and focus on the circular hole of nominal radius $r = 1$, with associated tolerance specifications of size, form, and position. (This example is a slight modification of the specification shown by standard drafting conventions in Fig. 1.)

1. Size tolerance $T_s = 0.1$. A hole (generally not a perfect circle) satisfies this specification if its boundary lies entirely within an annular tolerance zone defined by two concentric circles of radii $r + T_s/2$ and $r - T_s/2$ (see Fig. 5). The location and orientation of the tolerance zone are arbitrary, that is, can be adjusted to ensure (hopefully) that the surface fits in the zone. Thus we can imagine sliding the zone of Fig. 5 over the actual part to determine whether there is a position in which the annulus encloses the hole boundary. If so, the hole satisfies the size constraint; if no such position can be found, the hole is not acceptable.
2. Form tolerance $T_f = 0.05$. The tolerance zone is an arbitrarily positioned annulus defined by two concentric circles of radii r_1 and r_2 that are unrelated to r but satisfy $r_1 - r_2 = T_f$ (see Fig. 6). Observe that a feature that satisfies a size tolerance T_s also satisfies a form tolerance $T_f = T_s$, but the converse is not generally true. (For example, a perfect circle with radius 5 obviously satisfies the form-tolerance specification but is too big.)
3. Position tolerance $T_p = 0.2$ with respect to a coordinate system defined by A and B. The tolerance zone is the annulus defined by two concentric circles of radii $r + T_p/2$ and $r - T_p/2$, correctly located and oriented with respect to the given coordinate system, as shown in Fig. 7. A position tolerance implies size and form tolerances, but the converses generally are not true. (There are less restrictive position tolerances—see Section 3.7.)

The three tolerancing constraints are to be tested independently, and an acceptable hole must pass the three tests simultaneously. Observe that the specification is not redundant because $T_f < T_s < T_p$, and therefore the constraints do not imply one another.

The tolerance zones of the example introduced in Fig. 4 were obtained by "offsetting" the perfect-form surface.[2] The particular offsetting technique used in the proposed theory is described in Section 3.5. It is general, very simple, and always produces well-defined tolerance zones. However, much of the theory is independent of the specific offsetting procedures used (see Requicha 1983).

The following subsections elaborate and formalize the rough outline presented above. The concepts introduced through the simple two-dimensional exam-

2. Offsetting seems to be a very useful concept in geometric modeling. It has found applications in cutter-path generation for machine tools, mass-property calculation (Lee and Requicha 1982), and trajectory planning for robots (Lozano-Perez and Wesley 1979).

Fig. 6. Tolerance zone for the form tolerance of Fig. 4. The position of the zone is arbitrary, and the radii r_1 *and* r_2 *of the two circles must satisfy* $r_1 - r_2 = T_f$ *but are otherwise arbitrary.*

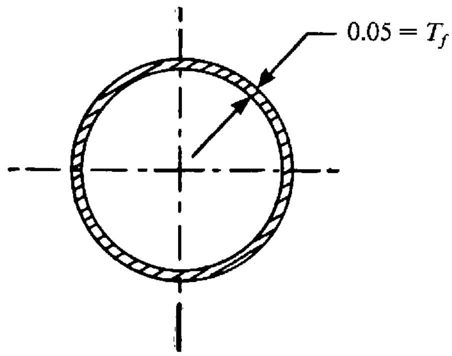
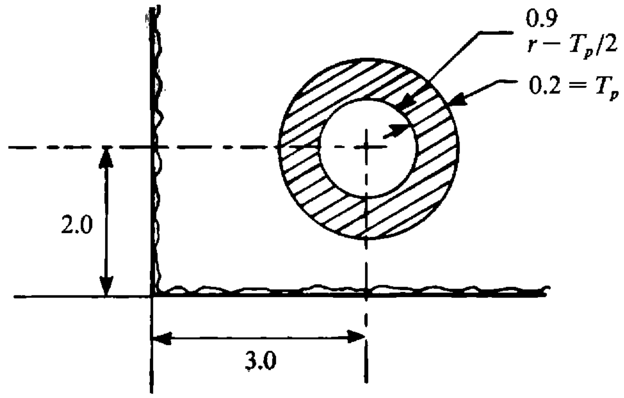


Fig. 7. Tolerance zone for the position tolerance of Fig. 4. The position of the tolerance zone is fixed with respect to the coordinate system defined by A *and* B.



ple must be generalized to complex three-dimensional features, and we must address issues such as the following:

How are three-dimensional zones to be constructed? For example, should a tolerance zone for a cylindrical hole have nominal depth? Should it be bounded at all?
How is the boundary of an object to be decomposed into features? (An imperfectly shaped object does not have neat, sharp edges to delimit its features.)
How are tolerance zones to be defined when features are geometrically complex?

3.2. Variational Classes and Tolerance Specifications

The goal of a tolerance specification is to define a class of objects that are (1) interchangeable in assembly operations and (2) functionally equivalent. Such classes of objects will be called in this paper *variational classes* (Requicha 1977*a*). A *tolerance specification* is an entity of a computational nature: it is a *representation* of a variational class. (See Requicha 1980 for basic representational concepts.) I propose in this paper tolerance specifications that consist of

1. An unambiguous representation for a nominal solid S (an *r*-set [Requicha 1977*b*; 1980])
2. A representation for a decomposition of ∂S (the boundary of S) into subsets F_i, called *nominal surface features,* which are homogeneously two-dimensional (Requicha 1977*b*) and whose union is ∂S.
3. A collection of geometric assertions A_{ij} about S's nominal surface features

The specific schemes used to represent solids and their surface features are unimportant for the purposes of the theory, but solids are assumed to be definable through regularized Boolean operations and rigid motions on a finite set of *primitive half-spaces,* whose boundaries are called *primitive surfaces* (Requicha 1977*b*; 1980). Surface features and assertions must possess certain properties discussed later in this paper. (Some two-dimensional subsets of ∂S cannot be features.)

The semantics (geometric meaning) of a tolerance specification will be defined by a mathematical rule—called a *theoretical inspection procedure*—for answering the following question: Given a tolerance specification $T = (S, \{F_i\}, \{A_{ij}\})$ and a subset P of E^3 that models an actual (physical) manufactured part, is P in the variational class defined by the specification T?

A (model of a) part P satisfies a tolerance specification T if and only if there exists a decomposition of ∂P into subsets G_i, called *actual surface features,* such that

1. $\cup G_i = \partial P$.
2. There exists a one-to-one correspondence between G_i and the nominal surface features F_i of S.
3. Each G_i satisfies the assertions A_{ij} associated with its corresponding F_i.

Observe that this definition ensures that the entire boundary of a part P is taken into consideration but

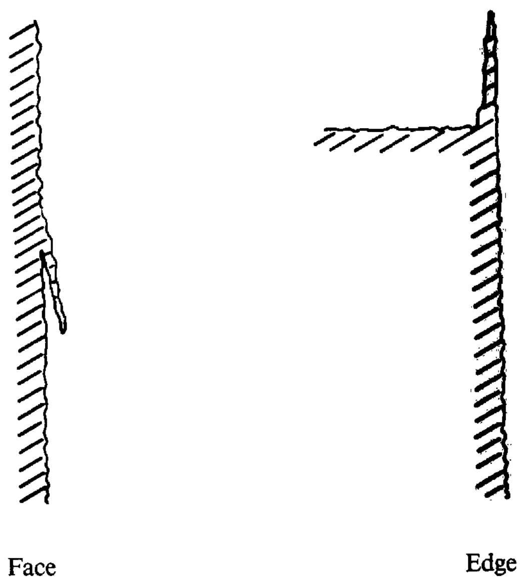


Fig. 8. Violations of the slow-variation assumption in a "face" and at an edge.

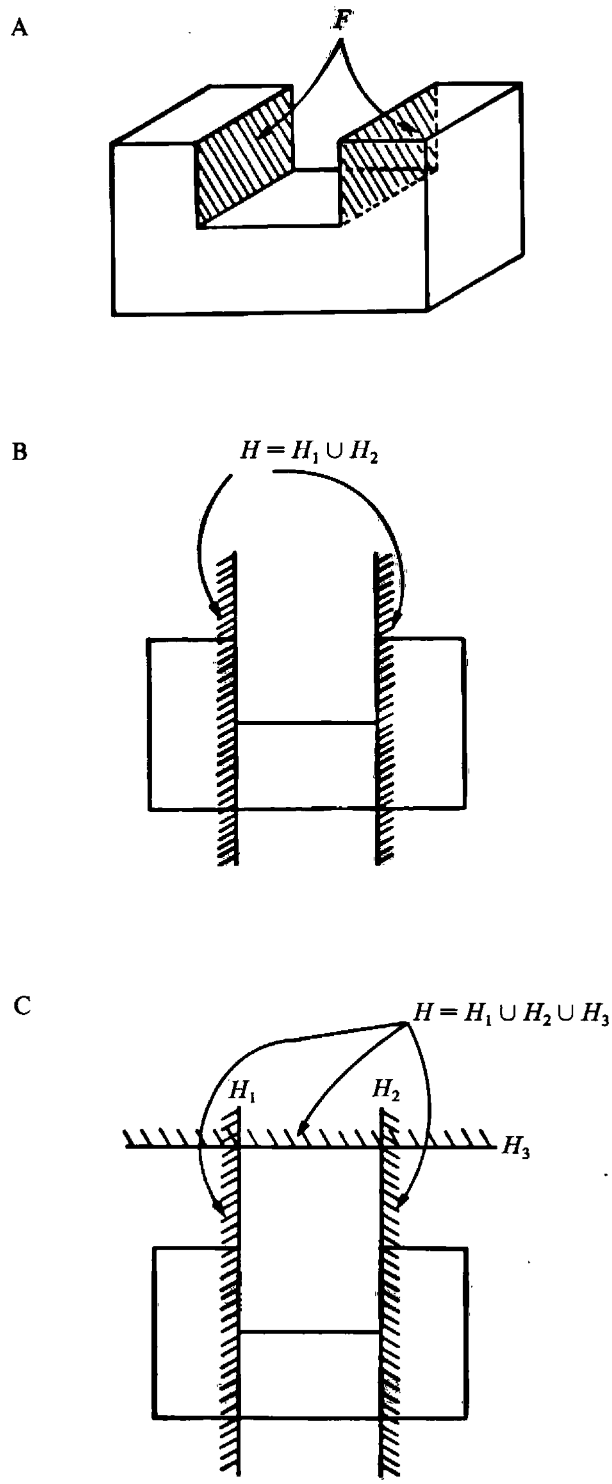


Fig. 9. A. A slot. B. An associated extended feature. C. A set that violates the defining conditions for extended features.

does not prescribe a unique segmentation of P's boundary into actual features.

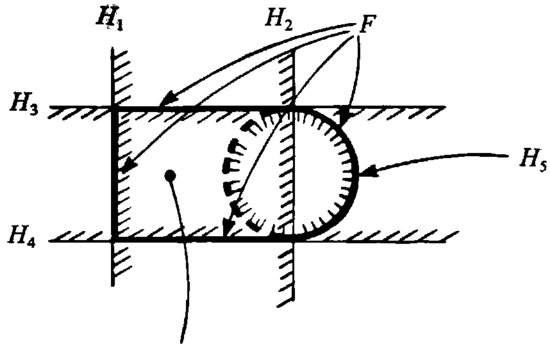
A *simple* nominal feature is the lowest-level geometric entity in a tolerance specification; each simple nominal feature must lie in a single primitive surface. A feature is *composite* if it is the union of simple features. Note that the simple features that contribute to a composite feature need not lie in distinct primitive surfaces (although they usually do). Actual features that correspond to simple nominal features also are called simple, and actual features that correspond to composite nominal features are called composite.

It will be assumed throughout the paper that the bounding surfaces of manufactured objects satisfy a "slow-variation" constraint that ensures that a surface does not vary too rapidly (at a scale comparable to the applicable tolerance values) and therefore does not create "nicks" or thin "slivers," as shown in Fig. 8. (I have not tried to formalize this constraint; a formal definition should exclude slivers but consider low-amplitude, high-frequency variations acceptable.)

3.3. Extended and Symmetric Features; Associated Measured Entities

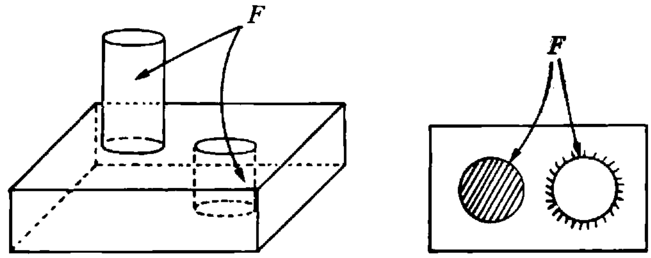
Tolerancing assertions often establish constraints on the geometric relationships between bounded portions

Fig. 10. The planar half-space H_2 *contributes neither to* F *nor to the boundary of* H, *yet it cannot be omitted in the Boolean composition that defines* H.



$$H = (H_1 \cap H_2 \cap H_3 \cap H_4) \cup H_5$$

Fig. 11. Two cylindrical surfaces that have no corresponding extended feature and therefore are not a nominal surface feature.



of a physical object's boundary and unbounded surfaces, centerplanes, axes, and similar geometric entities, which are associated with surface features. These notions are formalized in this section.

Extended features are used in this section to define symmetric features and in later sections to construct tolerance zones. I use the notion of extended feature as a theoretical device to resolve such issues as the following:

Can one associate an axis of "symmetry" with a cylinder that is "interrupted" by a keyway or oil hole?

What is the axial dimension of a tolerance zone associated with the cylindrical surface of a hole? (Answer in this theory: infinite.)

After defining extended features, I define symmetry in terms of them. Finally, I introduce the surface-fitting concepts of this theory—the *measured* planes, axes, or centers (of symmetry). Measured entities are used in later sections to define *datums* (roughly, coordinate systems), and as entities that can be constrained directly through tolerancing assertions.

Let us define extended features initially through an example. Consider the nominal surface feature that consists of the two faces of the slot shown in Fig. 9A. The extended feature corresponding to F consists of the two unbounded planes shown in projection in Fig. 9B, together with information on "where the material is." More precisely, the extended feature is an unbounded solid defined as the union of the planar half-spaces shown in Fig. 9B.

A general definition follows. An extended feature H associated with a nominal surface feature F is a solid (possibly unbounded) defined as a Boolean composition of half-spaces H_i and satisfying the following conditions:

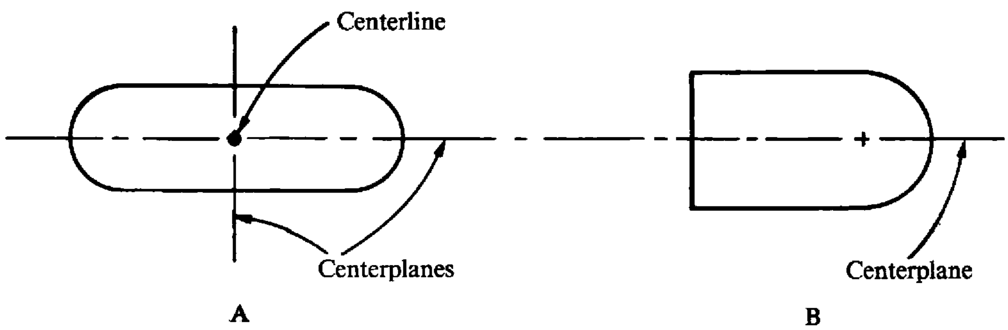
1. $\partial H \supset F$.
2. H must not contain in its definition half-spaces H_i that contribute two-dimensional subsets to ∂H but not to F.
3. Define the neighborhood of p with respect to S as $N(p, S; R) = B(p; R) \cap S$, where $B(p; R)$ is a ball of radius R centered on p, and define similarly a neighborhood with respect to H (Tilove 1980). Then, for every point p in the (two-dimensional) interior of the feature F, and for R sufficiently small, $N(p, S; R) = N(p, H; R)$. (This condition says essentially that the "material sides" of S and H must agree.)

The solid H shown in Fig. 9C violates condition 2 above because H_3 contributes a "face" to ∂H but not to F. Note, however, that an extended feature may contain half-spaces that contribute neither to ∂H nor to F. The half-space H_2 of Fig. 10 provides an example.

It can be shown that, under mild assumptions, the half-spaces that do contribute two-dimensional subsets to ∂H—called *b*-components (Requicha 1977*a*)—are uniquely defined (up to complementation) by the feature F (Silva 1981). Half-spaces that are not *b*-components are not unique, but this does not lead to any difficulties in the development of the theory.

Observe that certain two-dimensional subsets of a solid's boundary cannot be associated with an extended feature. The union of the cylindrical surfaces of the boss and hole shown in Fig. 11 provides an example. Nominal surface features are required to be homogeneously two-dimensional and to have associated

Fig. 12. *Two symmetric features (shown in two-dimensional projection).*



extended features; F in Fig. 11 therefore is *not* an acceptable feature.

Extended features that exhibit symmetry are especially interesting because they can be used to establish datums, as explained below, and because they may have identifiable centerplanes, centerlines, or centers, which can be constrained by assertions.

For simplicity, the definitions below apply to symmetry about the principal planes, axes, and origin of a master coordinate system, but generalizations to other coordinate systems are obvious. An extended feature H is symmetric with respect to the xy plane if it is invariant under the reflection $z \rightarrow -z$; the plane xy is then called a *centerplane* for H (and also for its associated nominal surface feature F). H is symmetric with respect to the z axis if it is invariant under the transformation $x \rightarrow -x$, $y \rightarrow -y$; the z axis is the feature's *centerline,* or *axis.* Finally, H is symmetric with respect to the origin O—called the feature's *center*—if it is invariant under the transformation $x \rightarrow -x$, $y \rightarrow -y$, $z \rightarrow -z$.

Figure 12 shows centerplanes and centerlines for two features. Observe that the features shown in the figure also have an infinite number of centerplanes parallel to the paper.

Experienced humans can infer from blueprints which axes or centerplanes are relevant to a tolerance specification, but in computerized systems it may be more reasonable to require that the axes or centerplanes be specified explicitly, through such statements as: The centerplane of feature F that passes through point O and is normal to vector V.

For symmetric features, we can define the concepts of *measured axes* (or centerplanes, or centers) by formalizing the current practice of using expanding mandrels to determine hole centers. Specifically, let us consider an actual subset G of a part's boundary and its corresponding surface feature F, extended feature H, and its centerplane C_p, which we assume to be the xy plane. The measured centerplane of G corresponding to C_p is defined as follows. Consider a family of solids H_k such that each H_k is obtained from H by a one-dimensional scaling transformation $x \rightarrow x$, $y \rightarrow y$, $z \rightarrow kz$.[3] We seek a solid H_k with minimal k and such that a congruent version H'_k of H_k completely encloses G. The measured centerplane of G is the appropriate centerplane of H'_k. Figure 13 illustrates the concept for a slot. Observe that the location and orientation of H'_k are selected to minimize k.

Measured centerlines are defined similarly by using two-dimensional scalings of the form $x \rightarrow kx$, $y \rightarrow ky$, $z \rightarrow z$, where the axis is assumed to be the z axis. For measured centers, we use three-dimensional scalings $x \rightarrow kx$, $y \rightarrow ky$, $z \rightarrow kz$, where the center is assumed to be the origin.

Readers should convince themselves that the above definitions capture mathematically current procedures for such simple features as holes and slots and apply to any symmetric feature, no matter how complex.

Features that lie in a single planar surface (henceforth called simply *planar features*) are not symmetric in the sense defined above. For such features, I

3. Scaling transformations seem the most reasonable formalization for the notion of an expanding feature. Because the effects of scaling depend on the choice of origin or axes, I required features to be symmetric so as to have "natural" origins and axes. A general theory for nonsymmetric features could be constructed by specifying explicit origins and axes. An alternative theory, not requiring symmetry, could also be constructed through the concept of offsetting, which is described in Section 3.5.

Fig. 13. *The measured centerplane for an actual slot feature.*

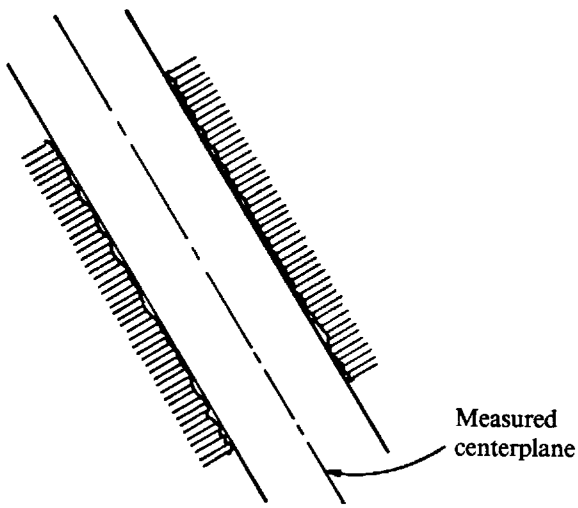
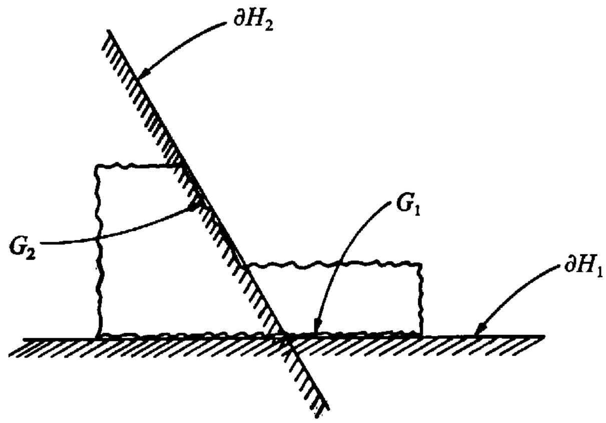


Fig. 14. *Two planar features* G_1 *and* G_2 *and associated measured planes* ∂H_1 *and* ∂H_2.



introduce the notion of a *measured plane,* which is analogous to a measured centerplane for a symmetric feature. Specifically, the measured plane associated with an actual planar feature G is ∂H, where H is a planar half-space that encloses G completely and whose boundary is closest to G in some appropriate sense. In current practice, ∂H is selected so as to just rest on or be tangent to G, and sometimes points or lines where the actual feature should contact an ideal plane are specified explicitly. A more precise requirement might be that the (integral) sum of the distances of all points of G to ∂H be minimal. Figure 14 shows two examples.

Finally, it should be noted that measured entities (planes, centerlines, and so on) sometimes are not uniquely defined. When there is ambiguity, a tolerance specification will always be interpreted to mean that there is (at least one) measured entity for which the appropriate assertions are satisfied. With this understanding, for simplicity of language I shall henceforth refer to *the measured entity* as if it were unique.

3.4. Datum Systems

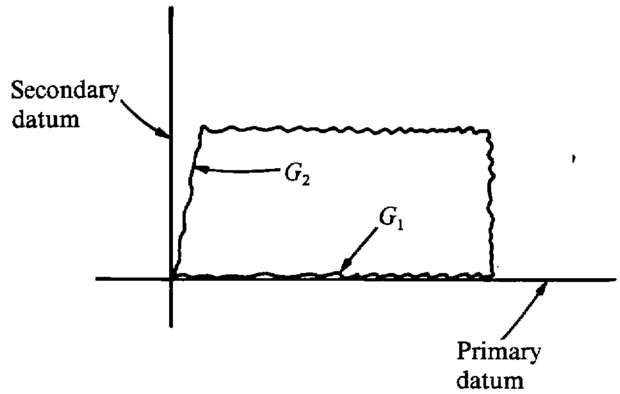
The notions of datum and datum systems are well described in current standards and texts (ANSI 1973). In essence, a datum is a perfect-form geometric entity that is associated with an imperfectly shaped, actual object feature by agreed-upon rules. The datums are used to construct partial (e.g., single-plane) or complete coordinate systems, which in turn are used as positional references for other object features. In the context of this paper, a *datum* is a measured axis, centerplane, or center associated with a symmetric feature, or a measured plane associated with a planar feature.

The location and orientation (attitude) of features is controlled by assertions involving the geometric relationships of a feature to a datum or system of datums. Suppose, for example, that a datum D_1 is associated with a feature F_1, and another feature F_2 is related to D_1 via an assertion. Given actual features G_1 and G_2 corresponding to F_1 and F_2, we construct from G_1 the measured centerplane (or axis, etc.) as explained in Section 3.3 and use it as a reference to check whether G_2 satisfies the assertion.

Ordered datum systems can be accommodated by modifying slightly the measuring procedure of Section 3.3. Consider the example of Fig. 15. The measured plane corresponding to G_1—the primary datum—is constructed as in Section 3.3, and the secondary datum corresponding to G_2 is constructed by a similar procedure but is constrained to be normal to the first. Observe that inverting the order of the datums yields a different coordinate system.

The discussion above pertains to so-called RFS datums. (RFS stands for "regardless of feature size" [ANSI 1973].) *Floating* (maximum-material-condition) *datums* are also useful and can be accommodated essentially as explained in the standards (ANSI 1973).

Fig. 15. An ordered datum system.



The proposed theory departs from current practice in that it requires that each part have an explicitly defined and complete (e.g., a three-plane) *master datum system*, and that all features be located from the master datum system or from explicitly specified datum systems related to the master. For example, we can locate a feature with respect to the master datum system, use this feature to construct a new datum, use the new datum to locate another feature, and so on. Therefore, what I propose is something akin to a tree of datum systems, rooted at the master datum system. (The datum graph is not a tree because there are datums that are not part of complete datum systems, and so forth.)

3.5. Offset Solids; Size Tolerances; MMC and LMC

This section contains the proposed theory's major departures from current practices. In essence, I reject the notion of a measured size and provide instead mathematical rules for deciding whether a feature satisfies a size tolerance specification. The rules are based on notions of maximum material condition (MMC) and least material condition (LMC).

A few definitions are needed. First, recall that the distance of a point p of E^3 to a subset S of E^3 is (Nadler 1978)

$$\text{Dist}(p, S) = \min_{q \in S} \text{Dist}(p, q),$$

where Dist (p, q) is the ordinary E^3 point distance.[4] Observe that the definition above implies that all points of S are at zero distance from S, and that when S is a solid (r-set), the distance between an external point and the solid is the same as the distance between the point and the boundary of the solid, that is,

$$\text{Dist}(p, S) > 0 \Rightarrow \text{Dist}(p, S) = \text{Dist}(p, \partial S).$$

Now let D be a positive number and S a solid, and define the corresponding (positive) *single-offset solid* $O(D; S)$ as

$$O(D; S) = \{p : \text{Dist}(p, S) \leq D\}.^5$$

For a negative offset D, a (negative) single-offset solid is defined as

$$O(D; S) = S -^* O(|D|; c^* S),$$

where $-^*$ and c^* denote regularized difference and complement. (Regularized set operators are slightly modified versions of their usual counterparts [Requicha 1977*b*; 1980].) Given a nominal feature F, its associated extended feature H, and two numbers $D_p > 0$ and $D_n < 0$ called the *positive and negative offsets*, the MMC and LMC solids are defined as the appropriate offset solids, namely,

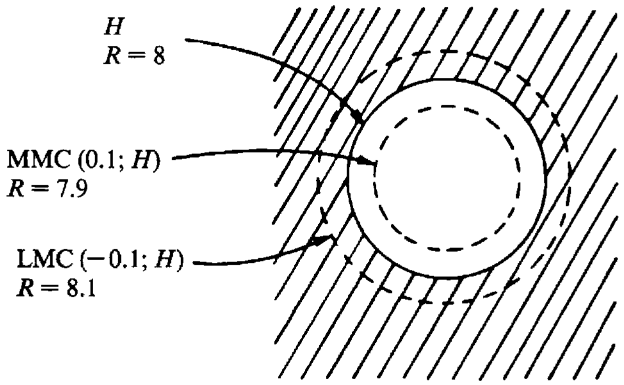
$$\text{MMC}(D_p; H) = O(D_p; H),$$
$$\text{LMC}(D_n, H) = O(D_n; H).$$

Figure 16 provides a simple example. For a cylindrical hole feature of nominal radius 8, positive offset 0.1 and negative offset −0.1, the corresponding MMC and LMC solids are unbounded cylindrical half-spaces of radius 7.9 and 8.1, respectively. Intuitively, the MMC solid is the result of adding to the extended feature H

4. If S is not sufficiently smooth, we must replace the minimum in the above definition by the infimum, or greatest lower bound.

5. Positive single-offset solids are sometimes called *generalized balls* in the mathematical literature (Nadler 1978) and are closely related to Minkowski sums, which are used, for example, in geometric probability theory.

Fig. 16. A cylindrical hole and its associated MMC and LMC solids.



a layer of material of thickness D_p, and the LMC solid is the result of subtracting from H a layer of thickness $|D_n|$.

More generaly, a *multiple-offset solid* can be defined as follows. Consider N nominal simple features F_i, with corresponding extended features H_i, and such that $F = \cup F_i$ is a composite feature with a corresponding extended feature H. Denote by E_i the homogeneously two-dimensional subset of ∂H_i that also is a subset of ∂H. (Intuitively, E_i is the contribution of F_i to the boundary of H.) Associate with each F_i a positive number D_i. The corresponding (positive) multiple-offset solid is

$$O(D_1, D_2, \ldots ; H) = \bigcup_i \{p : p \in cH \text{ and } \mathrm{Dist}(p, E_i) \leq D_i\} \cup H.$$

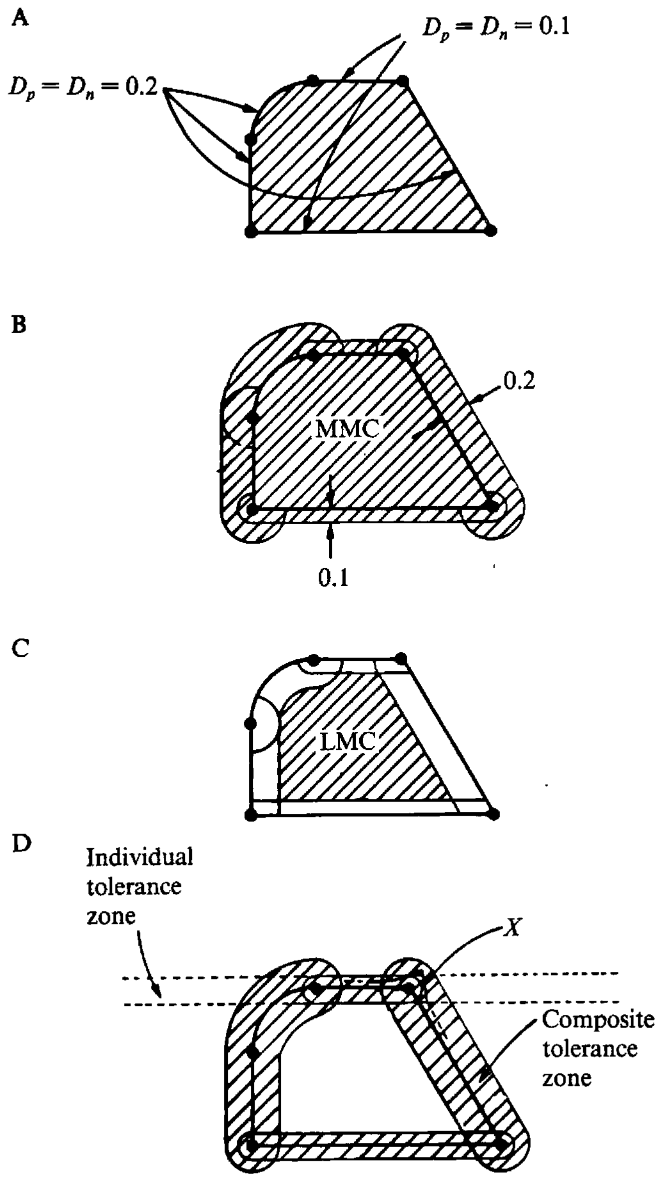
Intuitively, the multiple-offset solid is the result of adding to H a layer of variable thickness; the thickness of the layer adjacent to feature F_i is D_i. When all the D_i are negative, a (negative) multiple-offset solid is defined per

$$O(D_1, D_2, \ldots ; H) = H -^* O(|D_1|, |D_2|, \ldots ; c^*H).$$

which is a direct generalization of the earlier definition. Multiple-offset solids can be used to define multiple-offset MMC and LMC solids by considering vectors (arrays) of positive and negative offsets associated with features and interpreting D_p and D_n as vectors in the definitions of MMC and LMC given above. Figure 17A is a graphic multiple-offset specification for a simple solid, and Figs. 17B and C show the corresponding MMC and LMC solids.

Armed with precise notions of MMC and LMC, we

Fig. 17. A multiple-offset size specification (A), its associated MMC (B) and LMC (C) objects, and tolerance zone (D).



can now define size tolerances. Let G be an actual feature with corresponding nominal feature F and extended feature H. G satisfies a size tolerance with single offsets D_p, D_n if and only if there is a congruent instance $H' = R(H)$, where R is a rigid motion, such that

$$G \subset \text{MMC}\ (D_p; H') -^* \text{LMC}\ (D_n; H').$$

In essence, G must be within the tolerance zone that lies between an MMC and an LMC solid. The location and orientation of this tolerance zone are unspecified and can be selected so as to (hopefully) force G to fit in it. When $D_p = -D_n = T_s/2$, we say simply "a size tolerance T_s."

When D_p and D_n are vector (multiple) offsets, G satisfies the size tolerance if it lies in the composite tolerance zone delimited by the MMC and LMC solids, as in the single-offset definition above, *and* if each individual simple G_i satisfies the single size tolerance that corresponds to F_i. Figure 17D shows the composite tolerance zone that corresponds to the specification of Fig. 17A and also the individual tolerance zone that corresponds to the top face of the feature.

The definitions above apply to *any* feature and therefore are extremely general. They largely agree with current practice for such simple features as cylindrical holes, but depart from current practice for more complex features, as in Fig. 17D. Whether such departures are practically important remains to be seen. Note, however, that undesirable behavior such as that of surface X in Fig. 17D cannot occur, because X cannot be segmented so as to satisfy simultaneously the individualize size tolerances of the features and the slow variation constraint discussed in Section 3.2.

I chose to abandon in the definitions above the so-called Taylor principle sometimes used in current practice, because I was unable to generalize it to arbitrary features. My definition is somewhat more restrictive than Taylor's, which defines lower size limits by means of a two-dimensional tolerance zone that can "float" within an MMC solid.

It is worth remarking that size tolerances, as defined above, are applicable also to planar features. For such features, size-tolerance specification is essentially what is called *flatness* in current practice.

3.6. Form, Orientation, and Runout Tolerances

I propose to replace various special-case tolerances used in current practice (e.g., cylindricity, flatness) with a single-form tolerance that applies to all features. In essence, an actual feature G satisfies a *surface-form* tolerance specification with tolerance value T_f if it lies in a tolerance zone of "width" T_f and arbitrary "size" and orientation. A precise definition follows. Let H be the extended feature that corresponds to G. Then, G satisfies a form tolerance with value T_f if there is a congruent instance H' of H and two numbers D_1 and D_2 such that

$$G \subset O(D_1; H') -^* O(D_2; H'),$$
$$D_1 > D_2,$$
$$D_1 - D_2 \leq T_f.$$

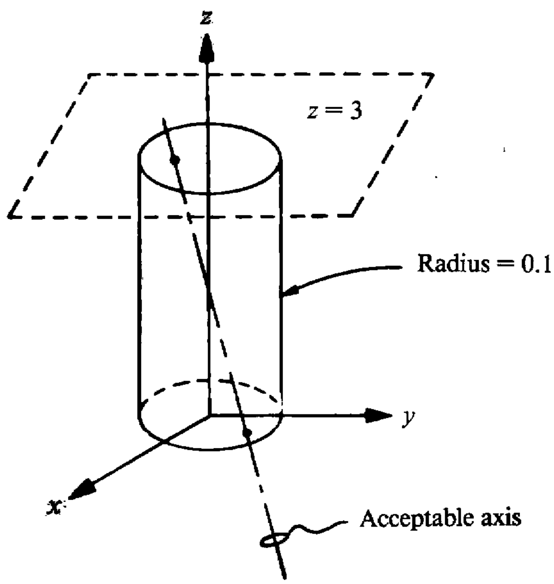
For example, a cylindrical hole feature G satisfies a form-tolerance specification (called cylindricity in current practice) if G lies in a cylindrical annulus defined by two coaxial cylinders; the radii of the two cylinders are unspecified and need not equal the nominal, MMC, or LMC radii, but they must differ by the specified form-tolerance value.

Surface-orientation tolerances are similar to surface-form tolerances, but require datum specifications and imply tolerance zones correctly oriented with respect to the datums. Similarly, *surface-* (or total-) *runout tolerances* require datum specifications and imply tolerance zones correctly located and oriented with respect to the datums. (Note that this definition of runout applies also to features that are not rotationally symmetric.)

The form and orientation tolerances described above apply to a surface as a whole. There are also similar tolerances, called *curve tolerances* in this theory, that apply to curves lying on surfaces. They are the counterparts of roundness, straightness, circular runout, and similar constraints of current practice.[6]

6. Details may be found in Production Automation Project Tech. Memo. No. 40, which is an earlier version of this paper.

Fig. 18. An RFS tolerance zone for an axis.



3.7. Position Tolerances

An unqualified position tolerance with value $T_p > 0$ and relative to a datum system defines a tolerance zone of width T_p about a nominal feature. More precisely, an actual feature G corresponding to a nominal feature F and extended feature H satisfies an unqualified position tolerance T_p if

$$G \subset O(T_p/2; H') -^* O(-T_p/2; H'),$$

where H' is a congruent instance of H that is positioned correctly, that is, whose location and orientation relative to the datum system are precisely the same as those of H relative to the nominal features that define the datum system. Observe that if G satisfies a position tolerance T_p, then it also satisfies a size tolerance with positive and negative offsets $T_p/2$ and $-T_p/2$, a form tolerance T_p, and an orientation tolerance T_p. Unqualified position tolerances are quite strict and do not reflect certain part-mating relationships that are important in practice. The two types of qualified position tolerances described below also are necessary.

Given an actual feature G corresponding to a nominal feature F with size tolerances characterized by offsets D_p and D_n, an *MMC position tolerance* T_p (with $T_p \geqslant 0$) relative to some datum system has the following meaning. First, construct an instance H' of the extended feature H at the appropriate location and orientation relative to the datum system. G satisfies its position constraints if $G \subset O(T_p + D_p; H')$. When D_p is a vector of offsets, T_p should also be a vector, usually with all components equal to a single positive number.

RFS position tolerances have different semantics. They can be applied only to planar or symmetric features (i.e., features that can be associated with datums) and define constraints on the *measured* entities of the features (see Section 3.3). Specifically, an RFS position tolerance is an assertion on a planar or symmetric feature that defines a solid tolerance zone within which a measured entity of the feature must lie. The tolerance specification consists of a tolerance value together with the name of the entity to which it applies and two bounding planes. Consider, for example, a feature symmetric about the z axis. A tolerance specification consisting of (axis = z, value = 0.1, $Z_{max} = 3$, $Z_{min} = 0$) defines a cylindrical tolerance zone as shown in Fig. 18.

An actual feature G will satisfy this specification if and only if its *measured axis* L (see Section 3.3) satisfies

$$L \cap (P_{max} \cap^* P_{min}) \subset \text{tolerance zone},$$

where P_{max} and P_{min} are the planar half-spaces whose intersection defines the "slab" of interest.

RFS tolerances that apply to planes and centers are defined similarly and correspond to cuboid and spherical tolerance zones, respectively.[7] (Specifications for spherical tolerance zones do not include bounding-plane data.)

Current ANSI standards state that position tolerances may not be applied to planar features, but I see no reason for this restriction.

7. I find the semantics of RFS position tolerances discussed above somewhat undesirable because it involves *bounded* tolerance zones, instead of (generally) unbounded zones defined by offset solids. Unfortunately, I have not been able to provide a reasonable counterpart for current RFS practices via offset solids.

3.8. Validity of Tolerance Specifications

A tolerance specification is *valid* if it defines an acceptable variational class. Unfortunately, a formal characterization of what constitutes an acceptable variational class is unknown, and therefore the validity of tolerance specifications is largely an open problem.[8]

The remainder of this section presents some informal thoughts on validity but does not attempt to settle the issues. Two validity conditions seem clear: (1) the nominal representation must itself be valid and (2) the union of all the nominal features must equal the nominal object's boundary. But what assertions must be required for validity? Can single assertions be unsatisfiable? Can collections of assertions have contradictions and therefore be unsatisfiable?

Observe that the theory prescribes tolerance zones that are always well defined, and are also "thick enough" to contain a nominal surface feature as well as an infinite family of features "close" to the nominal. This implies that single assertions are always satisfiable, provided that some relatively trivial conditions are met. (For example, datum specifications must be valid, and symmetric-feature tolerances must not be applied to asymmetric features.) Observe also that it is impossible to construct contradictory assertions. Suppose, for example, that a feature has both a size tolerance T_s and a form tolerance T_f. If $T_f > T_s$ then the form constraint is redundant (and can be ignored) because it is implied by the size tolerance; but if $T_f < T_s$, then both tolerances apply independently. In either case, there is no contradiction.[9]

The conclusions are that an object cannot be overconstrained and (with minor qualifications, as noted above) any collection of assertions is satisfiable. But can an object be underconstrained? The answer clearly is yes. Conditions to ensure that constraints are sufficient are unknown. Intuitively, conditions should ensure that each feature lies in a tolerance zone of finite thickness and restricted position with respect to some datum system that is related to the object's master datum system. The important tolerances for specification validity appear to be size and position tolerances; others are optional and not needed for validity. My current conjecture is that a specification is valid if each feature's constraints imply a strict (unqualified) position tolerance with some finite, nonnull tolerance value (see Section 3.7).

8. A conjecture: variational classes are regular closed sets in a hyperspace (Nadler 1978) whose elements are *r*-sets (Requicha 1980).

9. When a *nominal* object is defined via geometric constraints (e.g., by requiring that certain distances or angles have given values) the situation is quite different, and it is easy to construct constraint sets that are unsatisfiable (Requicha 1983).

4. Conclusions

Current tolerancing standards and practices must be tightened (formalized) considerably if we are to represent tolerancing information in computer-based geometric modeling systems in a form suitable for automatic tolerance analysis; automatic planning of manufacturing, assembly, and inspection operations; and other design and production activities.

The theory outlined in this paper is a step toward such formalization. The theory almost surely needs refinement, may require substantial modifications, and may even prove totally inadequate. However, it illustrates the level of precision required, and it shows that one can construct a reasonable theory of tolerancing that is based on a few general concepts rather than on an extensive set of special-case considerations. The theory occasionally departs from current practices, but such departures may be unimportant.

The theory states that an object satisfies a tolerance specification if the object's bounding surfaces are within suitably defined regions of space called tolerance zones. Tolerance zones used for so-called RFS positioning are bounded cylinders, parallelepipeds, or spheres, within which axes, centerplanes, or centers of symmetric features must lie. General tolerance zones, used for all tolerancing except RFS positioning, are constructed via offset solids, and are usually unbounded. The locations and orientations of general tolerance zones may be related to datums. The distinctions between size, position, and form-related tolerances lie in the specific rules for constructing the zones. (For example, is the location and orientation of a zone fixed, or can it float?)

Offset solids are defined in the theory in a specific way, but much of the theory is applicable with any reasonable concept of offsetting. Parameterized, constructive solid geometry (CSG) or boundary representations (Requicha 1980) presumably could provide alternative offsetting methods, but neither has been thoroughly investigated and both seem to raise delicate problems (Requicha 1983).

Conventional ± tolerances are not supported directly; they must be replaced with functionally equivalent (or nearly so) constraints expressible in the theory. Usualy, conventional tolerances can be replaced either by size tolerances or by position tolerances with explicit, rather than implied, datums.

Computational implementation of representation schemes based on the theory is relatively straightforward. In essence, one need only provide in a GMS facilities for (1) defining and "naming" nominal surface features and associated entities (e.g., axes), (2) establishing relations between them, and (3) assigning attributes to them. (Offsetting semantics different from those described in Section 3.5 may pose quite different representational requirements [Requicha 1983].)

An experimental software system for representing tolerancing ("variational") information is currently being implemented in the PADL-2 CSG-based geometric modeler at the University of Rochester (Brown 1982). This subsystem is designed to support the tolerancing semantics described in this paper but provides "escape mechanisms" for user-defined semantics because it is unreasonable to expect a rapid change of tolerancing practices.

Two major questions must be addressed by future research:

1. Does the theory adequately satisfy industrial requirements? (One should resist the temptation of "enriching" the theory to cater to special cases unless a critical analysis of industrial requirements shows that such cases are truly important.)
2. Is the theory effective for applications such as tolerance analysis and manufacturing and assembly planning? That is, are such applications mathematically and computationally tractable in terms of the theory?

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