1. Prove that the log-sum-exp function

$$f(x) = \log \sum_{i=1}^{m} \exp(a_i^T x)$$

is convex.

2. Suppose that the scalar random variable x takes values $\{a_1, a_2, \dots, a_n\}$ with probability $\mathbf{prob}(x = a_i) = p_i$ for $i = 1, \dots, n$. Is the variance

$$\mathbf{var}\ x = \mathbb{E}x^2 - (\mathbb{E}x)^2$$

a convex or concave function in the probabilities $p = (p_1, \dots, p_n)$? Prove your answer.

3. Prove that the intersection of convex sets $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n$ is a convex set.

4. Show that the *second-order cone*

$$\mathcal{S} = \{(x, t) \in \mathbf{R}^n \times \mathbf{R}_+ \mid \|x\|_2 \le t\}$$

is convex.

5. Consider the convex optimization problem

$$\underset{x}{\text{minimize}}\ \ f(x)\ \ \text{subject to}\ \ x \in \mathcal{C}, \tag{1}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is a convex and differentiable function, and $\mathcal{C} \subseteq \mathbf{R}^n$ is a convex set. A point x^* is optimal for (1) if and only if

$$x^* = \mathbf{proj}_{\mathcal{C}}(x^* - \gamma \nabla f(x^*)) \tag{2}$$

for any constant $\gamma > 0$, where the projection operator **proj** itself is an optimization problem:

$$\mathbf{proj}_{\mathcal{C}}(z) = \underset{x \in \mathcal{C}}{\operatorname{argmin}}\ \|x - z\|_2.$$

   (a) **Affine set.** For $\mathcal{C} = \{x \mid Ax = b\}$, prove that (2) holds if and only if $\nabla f(x^*)$ is in the range of A^T.
   Hint: decompose $\nabla f(x)$ into orthogonal components in $\mathbf{range}(A^T)$ and $\mathbf{null}(A)$.

   (b) **Nonnegative constraint.** For $\mathcal{C} = \mathbf{R}^n_+$, prove that (2) implies $\nabla f(x^*) \ge 0$.

   (c) **Normal cone.** In general, x^* is optimal for (1) if and only if

$$-\nabla f(x^*) \in \mathcal{N}(x^*), \tag{3}$$

   where $\mathcal{N}(x^*)$ is the normal cone of $\mathcal{C}$ at point x^*. Show that this property is equivalent to (2).