1. **Standard form.** Convert this problem into standard form:

$$\begin{array}{ll} \text{minimize} & -2x_1 - x_2 \\ \text{subject to} & x_1 - x_2 \le 2 \\ & x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0. \end{array}$$

   Cnstruct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

   Ans. The corresponding basic feasible solution (including slack/surplus variables) is

$$(x_1, x_2, x_3, x_4) = (0, 0, 2, 6).$$

2. **Simplex.**

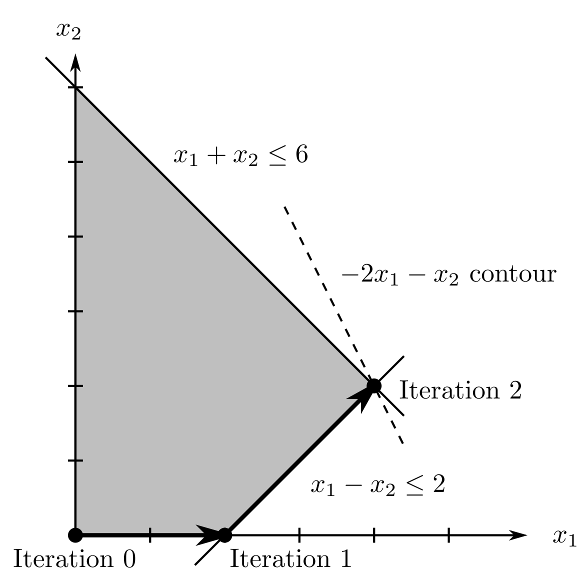
   (a) Carry out the steps of the simplex method, starting with the basic feasible solution in the first question. At each iteration, show the basic index set, the reduced costs, and which variables enter and leave the basis. Determine the optimal value, optimal solution, and optimal basis.

   Ans. Begin with the obvious basis:

   - Iteration 0:
$$B = \{3, 4\}, \quad z = (-2, -1, 0, 0).$$
   - Iteration 1: 3 leaves and 1 enters, so that
$$B = \{1, 4\}, \quad z = (0, -3, 2, 0).$$
   - Iteration 2: 4 leaves and 2 enters, so that
$$B = \{1, 2\}, \quad z = (0, 0, 1/2, 3/2),$$
     which is is optimal. Therefore $x^* = (4, 2, 0, 0)$ and the optimal value is -10.

   (b) Draw a graphical representation of the problem in terms of the original variables x_1, x_2, and indicate the path taken by the simplex algorithm.

   Ans.



3. **Robust optimization** We've already seen the linear least-squares, which minimize the 2-norm of the residual $r = b - Ax$. But other norms are possible. In this exercise, you'll consider these three different residual objectives:

$$\|Ax - b\|_1 = \sum_k |a_k^T x - b_k| \tag{1}$$

$$\|Ax - b\|_\infty = \max_k |a_k^T x - b_k| \tag{2}$$

$$\|Ax - b\|_2 = \sqrt{\sum_k |a_k^T x - b_k|^2} \tag{3}$$

In the next two questions, you're asked to write the problem of minimizing the 1- and ∞-norm objections as a linear program with the form

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & A_{\text{in}} x \le b_{\text{in}} \\ & A_{\text{eq}}\, x = b_{\text{eq}} \\ & \ell \le x \le u. \end{array} \tag{4}$$

(a) Show that (1) is a linear program by writing it in form (4), i.e., find c, A_{in}, b_{in}, A_{eq}, b_{eq}, ℓ, and u.
Ans. For (1), introduce a dummy variable y where $y_k = |a_k^T x - b_k|$. Then (1) can be relaxed to

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & y^T e \\ \text{subject to} & \max\{(Ax-b)_k, -(Ax-b)_k\} \le y_k,\ k = 1, \dots \end{array}$$

where we know that at optimality, the inequality is tight. This can be rewritten as

$$\begin{array}{ll} \underset{x\in\mathbf{R}^n,\ y\in\mathbf{R}^n}{\text{minimize}} & \begin{bmatrix} \mathbf{0} \\ e \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} b \\ -b \end{bmatrix}. \end{array}$$

(b) Show that (2) is a linear program by writing it in the form (4).
Ans. The conversion is almost exactly the same, except we use a dummy scalar variable $y = \|Ax - b\|_\infty$. Then the problem may be relaxed to

$$\begin{array}{ll} \underset{x,y}{\text{minimize}} & y \\ \text{subject to} & \max\{(Ax-b)_k, -(Ax-b)_k\} \le y,\ k = 1, \dots \end{array}$$

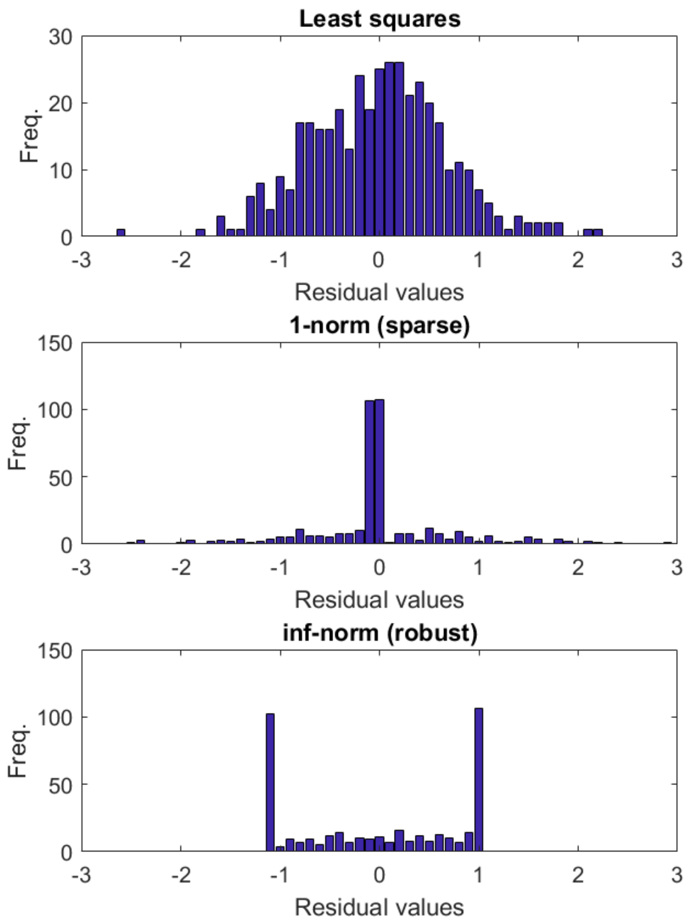
which we can rewrite as

$$\begin{array}{ll} \underset{x\in\mathbf{R}^n,\ y\in\mathbf{R}}{\text{minimize}} & \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} A & -e \\ -A & -e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

(c) Generate the data as follows:

```
m, n = 100, 30
A = randn(m, n)
b = randn(m)
```

Use Julia's JuMP optimization package to solve the LP formulations you derived for minimizing the 1- and ∞-norms. Also solve the 2-norm problem. Plot histograms of the residuals $r = Ax^* - b$ for the solutions obtained for these problems.
Ans. The plot should look something like this:



(d) What conclusions can you draw about each norm? Which norm do you choose if you're determined to have uniformly small residuals, e.g., if the residuals correspond the distance of an autonomous vehicle to the edge of the road? Which norm would you choose if you're happy to have most of the residuals small, but don't mind a few being large?

Ans. The ∞ norm (2), since it minimizes worst case error.

Ans. The 1 norm (1), since it optimizes for many 0's but allows for a few (very bad) outliers.