CPSC 406: Homework 4 — Due March 9

1. Prove that the intersection of convex sets

$$\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \cdots \cap \mathcal{S}_n = \{x \mid x \in \mathcal{S}_1 \text{ and } \mathcal{S}_2 \text{ and } \cdots \text{ and } \mathcal{S}_n\}$$

is itself a convex set.

2. The *second order cone* is the set of tuples

$$\mathcal{S} = \{(x,t) \in \mathbf{R}^n \times \mathbf{R}_+ \mid \|x\|_2 \le t\}.$$

Prove that $\mathcal{S}$ is convex.

3. Consider the convex optimization problem

$$\underset{x}{\text{minimize}} \;\; f(x) \;\; \text{subject to} \;\; x \in \mathcal{C} \tag{1}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is a convex and differentiable function, and $\mathcal{C} \subseteq \mathbf{R}^n$ is a convex set. A point x^* is optimal for (1) if and only if

$$x^* = \mathbf{proj}_{\mathcal{C}}(x^* - \gamma \nabla f(x^*)) \tag{2}$$

for any constant $\gamma > 0$, where the projection operator **proj** itself is an optimization problem:

$$\mathbf{proj}_{\mathcal{C}}(z) = \underset{x \in \mathcal{C}}{\text{argmin}} \; \|x - z\|_2.$$

(a) **Affine set.** For $\mathcal{C} = \{x \mid Ax = b\}$, prove that (2) holds if and only if $\nabla f(x^*)$ is in the range of A^T. Hint: decompose $\nabla f(x)$ into orthogonal components in $\mathbf{range}(A^T)$ and $\mathbf{null}(A)$.

(b) **Nonnegative constraint.** For $\mathcal{C} = \mathbf{R}^n_+$, prove that (2) implies $\nabla f(x^*) \ge 0$.

(c) **Normal cone.** In general, the optimality condition for (1) is that x^* is optimal if and only if

$$\nabla f(x^*)^T (y - x) \ge 0, \; \forall y \in \mathcal{C} \tag{3}$$

In other words, $\nabla f(x^*)$ is in the *normal cone of $\mathcal{C}$ at point x^**. Show that this property is equivalent to (2).