Discrete Mathematics

2023-24 Semester 1 Course Quiz

Time: 3:00-4:30pm, 4 Nov. 2023
Class: 02 (Mon) / 03 (Wed) / 01 (Fri)

Name: ______________________ Student number: ______________________

There are FOUR questions. Answer ALL questions.

1. (25 points) (a) (10 points) Prove by contradiction, or disprove, that the product of one odd number and one even number is even.
   (Remark: To prove by contradiction, the assumption should be written down clearly.)

   Solution: Assumption: The product of one odd number and one even number is odd.

   The product of one odd number and one even number is odd.
   $\Rightarrow (2k_1 + 1) \cdot 2k_2$ is odd, $k_1, k_2 \in \mathbb{Z}$
   $\Rightarrow 2k_2(2k_1 + 1)$ is odd.
   $\Rightarrow 2k_2(2k_1 + 1)$ is even.
   $\Rightarrow$ Contradiction!

   Thus, by contradiction, the product of one odd number and one even number is even.

   (b) (15 points) Prove by contradiction, or disprove, that if a mod $3 = 1$ and b mod $6 = 3$, then $(b - a)$ mod $3 = 2$, where $a, b \in \mathbb{Z}^+$ and $b > a$.
   (Remark: To prove by contradiction, the assumption should be written down clearly.)

   Solution: Assumption: a mod $3 = 1$ and b mod $6 = 3$ and $(b-a)$ mod $3 \neq 2$.

   a mod $3 = 1$ and b mod $6 = 3$ and $(b - a)$ mod $3 \neq 2$.
   $\Rightarrow a = 3k_1 + 1$ and $b = 6k_2 + 3$ and $(b - a)$ mod $3 \neq 2$, where $k_1, k_2 \in \mathbb{Z}$
   $\Rightarrow b - a = 6k_2 + 3 - 3k_1 - 1$ and $(b - a)$ mod $3 \neq 2$
   $\Rightarrow b - a = 3(2k_2 - k_1) + 2$ and $(b - a)$ mod $3 \neq 2$
   $\Rightarrow (b - a)$ mod $3 = 2$ and $(b - a)$ mod $3 \neq 2$
   $\Rightarrow$ Contradiction!

   Thus, by contradiction, if a mod $3 = 1$ and b mod $6 = 3$, then $(b - a)$ mod $3 = 2$.

2. (25 points) $P = \{\text{p, q, r, s, t}\}$, $S = \{\text{s, t, u, v, w}\}$ and $M = \{1, 4, 7, 10\}$.

(a) (7 points) $A = (P \cup S) - M$. Write down A. How many non-empty binary relations can be formed on A? (Steps required)

> **Solution:** $P \cup S = \{p, q, r, s, t, u, v, w\}$
> $(P \cup S) - M = \{p, q, r, s, t, u, v, w\}$
> No. of subsets of $A \times A = 2^{8^2}$.
> No. of non-empty binary relations which can be formed on $S = 2^{64} - 1$.

(b) (8 points) Construct an injective function from M to S. (Write down "Nil" if it is impossible.)

> **Solution:** $f(1) = \text{s}$
> $f(4) = \text{t}$
> $f(7) = \text{u}$
> $f(10) = \text{v}$

(c) (10 points) Write down one binary relation, in the form of a set of ordered pairs, on M which is not symmetric, not antisymmetric and not transitive. (The relation should not contain more than eight elements.) (Write down "Nil" if it is impossible.)

> **Solution:** $R = \{(1,\ 4),\ (4,\ 1),\ (1,\ 7)\}$

3. (25 points) (a) (10 points) Construct a truth table for the following statement:

$$(a \oplus c) \rightarrow (b \vee \neg c)$$

(All cases should be shown in the truth table.)

Solution: Truth table:

a	b	c	$a \oplus c$	$\neg c$	$b \vee \neg c$	$(a \oplus c) \rightarrow (b \vee \neg c)$
T	T	T	F	F	T	T
T	T	F	T	T	T	T
T	F	T	F	F	F	T
T	F	F	T	T	T	T
F	T	T	T	F	T	T
F	T	F	F	T	T	T
F	F	T	T	F	F	F
F	F	F	F	T	T	T

(b) (15 points) Determine the true values of the following two expressions, where $x \in \mathbb{Z}, y \in \mathbb{Z}^+$:

- $\forall x\ \exists y\ (x - y)^2 = 0$
- $\exists x\ \forall y\ x + y > 0$

Explain your answers.

Solution: $\forall x\ \exists y\ (x - y)^2 = 0$
Set $x = -1$. $(-1 - y)^2 = (y + 1)^2 > 0$, for all $y \in \mathbb{Z}^+$.
Thus, it is false.

$\exists x\ \forall y\ x + y > 0$
Set $x = 1$. $1 + y > 1 > 0$, for all $y \in \mathbb{Z}^+$.
Thus, it is true.

4. (25 points) Let $S(x, y)$ be the predicate which "x is a student in the class y", $P(x)$ be the predicate which "x has used the mobile app POE", where the domains for x and y consist of all students in the college and all classes in the college respectively. Express the following statements by using quantifiers.
(In the statements, you can only use $\forall, \exists, \vee, \wedge, \neg$, and/or $\rightarrow$.)

(a) (8 points) All students in the "Music" class have not used POE.

Solution: $\forall x\ (S(x, \text{“}\texttt{Music}\text{”}) \rightarrow \neg P(x))$

(b) (17 points) The "Engineering" class is the only class in which all students have used POE.

Solution: $\forall x\ (S(x, \text{“}Engineering\text{”}) \rightarrow P(x)) \wedge \forall y\ ((y \neq \text{“}Engineering\text{”}) \rightarrow \exists x\ (S(x, y) \wedge \neg P(x)))$

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