Discrete Mathematics

2023-24 Semester 1 Assignment 01

Due Time: 5:30pm, 7^{th} Nov. 2023

There are SIX questions in this assignment. Answer ALL the questions.

1. (20 points) The binary relation S is defined on $\mathbb{Z}$ such that $p\ S\ q$ if and only if $(p+q) \bmod 5 = 0$.
   Determine whether S is reflexive, symmetric, antisymmetric and/or transitive. (Steps required.)

Solution: $(1+1) \bmod 5 = 2$
$\Rightarrow (1,1) \notin S$
Thus, S is not reflexive.

$(p,q) \in S$
$\Rightarrow (p+q) \bmod 5 = 0$
$\Rightarrow (q+p) \bmod 5 = 0$
$\Rightarrow (q,p) \in S$
$\Rightarrow S$ is symmetric.

$(1,4) \in S$ since $(1+4) \bmod 5 = 0$
$(4,1) \in S$ since $(4+1) \bmod 5 = 0$
$(1,4) \in S$ and $(4,1) \in S$, but $1 \neq 4$.
Thus, S is not antisymmetric.

$(1,4) \in S$ since $(1+4) \bmod 5 = 0$
$(4,6) \in S$ since $(4+6) \bmod 5 = 0$
$(1,6) \notin S$ since $(1+6) \bmod 5 = 2$
Since $(1,4) \in S$ and $(4,6) \in S$ but $(1,6) \notin S$, S is not transitive.

2. (10 points) Without drawing a Venn diagram, prove or disprove the following statement:

$$(A-B)-C = A-(B-C),$$

   for any non-empty sets A, B and C. (Steps required.)

Solution: Set $A = \{1,2,3,5\}, B = \{1,2,4,6\}$ and $C = \{1,3,4,7\}$.

$A - B = \{3,5\}$
$(A-B) - C = \{5\}$

$B - C = \{2, 6\}$
$A - (B - C) = \{1, 3, 5\}$

$(A - B) - C \neq A - (B - C)$

Thus, the statement is false.

3. (15 points) Determine whether each of the following functions (i.e., $f_1()$, $f_2()$ and $f_3()$) is injective and/or surjective. (Explanations required.)
   (a) $f_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ $f_1((a, b)) = 4b + 1$
   (b) $f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $f_2((a, b)) = \dfrac{a + b}{2}$
   (c) $f_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $f_3((a, b)) = \lceil a \rceil - b$

Solution:

(a) Not injective: $f_1((0, 0)) = f_1((1, 0)) = 1$
Not surjective: 0 does not have any pre-image.

(b) Not injective: $f_2((2, 0)) = f_2((0, 2)) = 1$
Surjective: Every real number x has a pre-image $(2x, 0)$.

(c) Not injective: $f_3((0.1, 0)) = f_3((0.2, 0)) = 1 - 0 = 1$
Surjective. Every real number x has a pre-image $(0, -x)$.

4. (15 points) Prove by contradiction, or disprove, the following two statements.
   (a) a is even or b is even or ab is odd, where $a, b \in \mathbb{Z}$;
   (b) For any non-empty binary relation R on the set $P = \{p, q, r\}$, a sufficient condition for R to be reflexive is that R is transitive and symmetric.

   (To prove by contradiction, the assumption should be written down clearly.)

Solution:

(a) a is even or b is even or ab is odd.
$\Rightarrow$ (a is even or b is even) or ab is odd.
$\Rightarrow \neg$ (a is odd and b is odd) $\vee$ ab is odd.
$\Rightarrow \neg$ (a and b are odd) $\vee$ ab is odd.
$\Rightarrow$ If (a and b are odd) then ab is odd.

Assumption: a is odd and b is odd and ab is even.

a is odd and b is odd and ab is even.
$\Rightarrow a = 2k + 1$ and $b = 2h + 1$ and ab is even, where $h, k \in \mathbb{Z}$
$\Rightarrow ab = (2k + 1)(2h + 1)$ and ab is even.
$\Rightarrow ab = 2(2hk + h + k) + 1$ and ab is even.
$\Rightarrow ab$ is odd and ab is even.
$\Rightarrow$ Contradiction!

Thus, by contradiction, a is even or b is even or ab is odd.

(b) A sufficient condition for R to be reflexive is that R is transitive and symmetric.
It is identical to "If R is transitive and symmetric, then R is reflexive".

Consider $R = \{(p, q), (q, p), (p, p), (q, q)\}$.
R is transitive and symmetric.
However, R is not reflexive.

Thus, the statement is false.

5. (20 points) Let $C(x, y)$ be the predicate "Student x is a student of the class y", $G(x)$ be the predicate "Student x has used a grocery delivery platform", $R(x)$ be the predicate "Student x has used a ride-hailing platform", where the domain for x contains all students in the college and y contains all classes in the college.

   (a) Translate the following logical expression into an English statement:
   "$\exists x\ (C(x, \text{"}Sports\text{"}) \land \neg R(x))$"

   Solution: At least one student of the "Sports" class has not used any ride-hailing platform.

   (b) Express the following statement by a logical expression with quantifiers:
   "Exactly one student of the "Smart" class has used both a grocery delivery platform and a ride-hailing platform."
   (In the expression, you can only use $\forall, \exists, \lor, \land, \neg$, and/or $\rightarrow$.)

   Solution: $\exists x\ ((C(x, \text{"}Smart\text{"}) \land G(x) \land R(x)) \land \forall y\ ((y \neq x) \land C(y, \text{"}Smart\text{"}) \rightarrow (\neg G(y) \lor \neg R(y))))$

6. (20 points) (a) On a fictional island, all inhabitants are either knights, who always tell the truth, or knaves, who always lie. Assume that you are a visitor to the island. Based on what the inhabitants say, determine who they are. (Steps required.)
   P says, "At least one of Q and R is a knave".

Q says, "P is a knight".
R says, "Exactly one of three of us is a knave".

Solution:

P	Q	R	P's statement	Q's statement	R's statement
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	T	F	F

Thus, there is no solution.

(b) Let's assume that there are three kinds of people on the island. Besides knights and knaves, there are spies who can either lie or tell the truth. Now, you encounter A, B, and C. Each of them knows the type of person each of the other two is. Moreover, one of them is a knight, one is a knave, and one is a spy. Based on what they say, determine who they are. (Steps required.)
A says, "B is not a spy".
B says, "C is not a knave".
C says, "A is not a knight".

Solution:

A	B	C	A's statement	B's statement	C's statement
T	S	F	F	F	F
T	F	S	T	T	F
F	T	S	T	T	T
F	**S**	**T**	**F**	**T**	**T**
S	T	F	T	F	T
S	F	T	T	T	T

Therefore, A is a knave, B is a spy and C is a knight.

Reminders:

- No mark will be given to late assignments.
- This is an individual assignment. You are encouraged to discuss with your classmates. But, please ensure that you will use your own words to answer the questions.
- Do NOT post/broadcast your answers on any shared platform. Zero mark will be given to those students who violate this regulation.
- Collusion and plagiarism are serious offences and may result in disciplinary action. A mark of zero will be given for the piece of coursework and; in addition, the final grade of the course may be affected (for example, it may be lowered from D to F). Please carefully read the section about Collusion and Plagiarism in your Student Handbook.
- Assignment (hard copy) should be put into the assignment box before the deadline AND soft copy should be submitted via the SOUL platform before the deadline.