Discrete Mathematics

2023-24 Semester 1 Assignment 02

Due Time: 5:30pm, 14^{th} December 2023

There are SEVEN questions in this assignment. Answer ALL the questions.

1. (15 points) Drones of eight different colors (red, orange, yellow, green, blue, indigo, violet and black) are chosen for a performance. There are twenty drones for each color.

   (a) How many drones do we need to choose to guarantee that seven drones are of the same color? (Steps required.)

   Solution: Pigeons: N drones
   Pigeonholes: 8 colors

   $\lceil \frac{N}{8} \rceil = 7$
   $7 = \lceil \frac{N}{8} \rceil < \frac{N}{8} + 1$
   $6 < \frac{N}{8}$
   $N > 48$

   By the Generalized Pigeonhole Principle, we need to choose 49 drones to guarantee that seven drones are of the same color.

   (b) How many drones do we need to choose to guarantee that we have 5 red drones, 4 green drones and 3 blue drones? (Steps required.)

   Solution: In the worst case, we need to choose all drones of other colors (i.e., other than red, green and blue). There are $5 \cdot 20 = 100$ drones. Next, we need to choose all the blue drone (i.e., 20) before we have the first red or green drone. Then, we need to choose all the green drone (i.e., 20) before we have the first red drone. At that moment, all the remaining drones will be in red color. We need to choose 5 more drones.

   Thus, we need to choose $100 + 20 + 20 + 5 = 145$ drones to guarantee that we have 5 red drones, 4 green drones and 3 blue drones.

2. (10 points) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{u, v, w, x, y, z\}$. How many onto functions can be formed from A to B? (Steps required.)

   Solution: $S_1 : u$ does not have pre-image
   $S_2 : v$ does not have pre-image

S_3 : w does not have pre-image
S_4 : x does not have pre-image
S_5 : y does not have pre-image
S_6 : z does not have pre-image

No. of functions from A to $B = 6^8$
$|S_1| = |S_2| = |S_3| = |S_4| = |S_5| = |S_6| = 5^8$
$|S_1 \cap S_2| = |S_1 \cap S_3| = |S_1 \cap S_4| = |S_1 \cap S_5| = |S_1 \cap S_6| = |S_2 \cap S_3| = |S_2 \cap S_4| = |S_2 \cap S_5| = |S_2 \cap S_6| = |S_3 \cap S_4| = |S_3 \cap S_5| = |S_3 \cap S_6| = |S_4 \cap S_5| = |S_4 \cap S_6| = |S_5 \cap S_6| = 4^8$
$|S_1 \cap S_2 \cap S_3| = |S_1 \cap S_2 \cap S_4| = |S_1 \cap S_2 \cap S_5| = |S_1 \cap S_2 \cap S_6| = |S_1 \cap S_3 \cap S_4| = |S_1 \cap S_3 \cap S_5| = |S_1 \cap S_3 \cap S_6| = |S_1 \cap S_4 \cap S_5| = |S_1 \cap S_4 \cap S_6| = |S_1 \cap S_5 \cap S_6| = |S_2 \cap S_3 \cap S_4| = |S_2 \cap S_3 \cap S_5| = |S_2 \cap S_3 \cap S_6| = |S_2 \cap S_4 \cap S_5| = |S_2 \cap S_4 \cap S_6| = |S_2 \cap S_5 \cap S_6| = |S_3 \cap S_4 \cap S_5| = |S_3 \cap S_4 \cap S_6| = |S_3 \cap S_5 \cap S_6| = |S_4 \cap S_5 \cap S_6| = 3^8$
$|S_1 \cap S_2 \cap S_3 \cap S_4| = |S_1 \cap S_2 \cap S_3 \cap S_5| = |S_1 \cap S_2 \cap S_3 \cap S_6| = |S_1 \cap S_2 \cap S_4 \cap S_5| = |S_1 \cap S_2 \cap S_4 \cap S_6| = |S_1 \cap S_2 \cap S_5 \cap S_6| = |S_1 \cap S_3 \cap S_4 \cap S_5| = |S_1 \cap S_3 \cap S_4 \cap S_6| = |S_1 \cap S_3 \cap S_5 \cap S_6| = |S_1 \cap S_4 \cap S_5 \cap S_6| = |S_2 \cap S_3 \cap S_4 \cap S_5| = |S_2 \cap S_3 \cap S_4 \cap S_6| = |S_2 \cap S_3 \cap S_5 \cap S_6| = |S_2 \cap S_4 \cap S_5 \cap S_6| = |S_3 \cap S_4 \cap S_5 \cap S_6| = 2^8$
$|S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5| = |S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_6| = |S_1 \cap S_2 \cap S_3 \cap S_5 \cap S_6| = |S_1 \cap S_2 \cap S_4 \cap S_5 \cap S_6| = |S_1 \cap S_3 \cap S_4 \cap S_5 \cap S_6| = |S_2 \cap S_3 \cap S_4 \cap S_5 \cap S_6| = 1^8$
$|S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5 \cap S_6| = 0$

No. of onto functions $= 6^8 - \binom{6}{1} \cdot 5^8 + \binom{6}{2} \cdot 4^8 - \binom{6}{3} \cdot 3^8 + \binom{6}{4} \cdot 2^8 - \binom{6}{5} \cdot 1^8 + 0$
$= 191520$

3. (15 points) A pair of fair dice is rolled. Let $X((i,j))$ be the random variable, where i and j are the numbers that appear on the first die and the second die, respectively. $X((i,j))$ is defined as follows:

$$X((i,j)) = \left\{ \begin{array}{ll} 2i + j, & \text{if } i < j; \\ i^2, & \text{if } i = j; \\ 3i + j, & \text{if } j < i. \end{array} \right\}$$

(a) What is the distribution of the random variable X? (Steps required.)

Solution: $X((1,1)) = 1^2 = 1$
$X((1,2)) = 2 \cdot 1 + 2 = 4$
$X((1,3)) = 2 \cdot 1 + 3 = 5$
$X((1,4)) = 2 \cdot 1 + 4 = 6$
$X((1,5)) = 2 \cdot 1 + 5 = 7$
$X((1,6)) = 2 \cdot 1 + 6 = 8$

$X((2,1)) = 3 \cdot 2 + 1 = 7$
$X((2,2)) = 2^2 = 4$

$X((2,3)) = 2 \cdot 2 + 3 = 7$
$X((2,4)) = 2 \cdot 2 + 4 = 8$
$X((2,5)) = 2 \cdot 2 + 5 = 9$
$X((2,6)) = 2 \cdot 2 + 6 = 10$

$X((3,1)) = 3 \cdot 3 + 1 = 10$
$X((3,2)) = 3 \cdot 3 + 2 = 11$
$X((3,3)) = 3^2 = 9$
$X((3,4)) = 2 \cdot 3 + 4 = 10$
$X((3,5)) = 2 \cdot 3 + 5 = 11$
$X((3,6)) = 2 \cdot 3 + 6 = 12$

$X((4,1)) = 3 \cdot 4 + 1 = 13$
$X((4,2)) = 3 \cdot 4 + 2 = 14$
$X((4,3)) = 3 \cdot 4 + 3 = 15$
$X((4,4)) = 4^2 = 16$
$X((4,5)) = 2 \cdot 4 + 5 = 13$
$X((4,6)) = 2 \cdot 4 + 6 = 14$

$X((5,1)) = 3 \cdot 5 + 1 = 16$
$X((5,2)) = 3 \cdot 5 + 2 = 17$
$X((5,3)) = 3 \cdot 5 + 3 = 18$
$X((5,4)) = 3 \cdot 5 + 4 = 19$
$X((5,5)) = 5^2 = 25$
$X((5,6)) = 2 \cdot 5 + 6 = 16$

$X((6,1)) = 3 \cdot 6 + 1 = 19$
$X((6,2)) = 3 \cdot 6 + 2 = 20$
$X((6,3)) = 3 \cdot 6 + 3 = 21$
$X((6,4)) = 3 \cdot 6 + 4 = 22$
$X((6,5)) = 3 \cdot 6 + 5 = 23$
$X((6,6)) = 6^2 = 36$

$X((1,1)) = 1$
$X((1,2)) = X((2,2)) = 4$
$X((1,3)) = 5$
$X((1,4)) = 6$
$X((1,5)) = X((2,1)) = X((2,3)) = 7$
$X((1,6)) = X((2,4)) = 8$
$X((2,5)) = X((3,3)) = 9$
$X((2,6)) = X((3,1)) = X((3,4)) = 10$
$X((3,2)) = X((3,5)) = 11$
$X((3,6)) = 12$
$X((4,1)) = X((4,5)) = 13$
$X((4,2)) = X((4,6)) = 14$

$X((4,3)) = 15$
$X((4,4)) = X((5,1)) = X((5,6)) = 16$
$X((5,2)) = 17$
$X((5,3)) = 18$
$X((5,4)) = X((6,1)) = 19$
$X((6,2)) = 20$
$X((6,3)) = 21$
$X((6,4)) = 22$
$X((6,5)) = 23$
$X((5,5)) = 25$
$X((6,6)) = 36$

$P(X = 1) = \frac{1}{36}$
$P(X = 4) = \frac{2}{36}$
$P(X = 5) = \frac{1}{36}$
$P(X = 6) = \frac{1}{36}$
$P(X = 7) = \frac{3}{36}$
$P(X = 8) = \frac{2}{36}$
$P(X = 9) = \frac{2}{36}$
$P(X = 10) = \frac{3}{36}$
$P(X = 11) = \frac{2}{36}$
$P(X = 12) = \frac{1}{36}$
$P(X = 13) = \frac{2}{36}$
$P(X = 14) = \frac{2}{36}$
$P(X = 15) = \frac{1}{36}$
$P(X = 16) = \frac{3}{36}$
$P(X = 17) = \frac{1}{36}$
$P(X = 18) = \frac{1}{36}$
$P(X = 19) = \frac{2}{36}$
$P(X = 20) = \frac{1}{36}$
$P(X = 21) = \frac{1}{36}$
$P(X = 22) = \frac{1}{36}$
$P(X = 23) = \frac{1}{36}$
$P(X = 25) = \frac{1}{36}$
$P(X = 36) = \frac{1}{36}$

(b) What is the expected value X? Correct the answer to 4 decimal places. (Steps required.)

Solution: The expected value of X= $E(X)$=
$\frac{1}{36} \cdot 1 + \frac{2}{36} \cdot 4 + \frac{1}{36} \cdot 5 + \frac{1}{36} \cdot 6 + \frac{3}{36} \cdot 7 + \frac{2}{36} \cdot 8 + \frac{2}{36} \cdot 9 + \frac{3}{36} \cdot 10 + \frac{2}{36} \cdot 11 + \frac{1}{36} \cdot 12$
$+ \frac{2}{36} \cdot 13 + \frac{2}{36} \cdot 14 + \frac{1}{36} \cdot 15 + \frac{3}{36} \cdot 16 + \frac{1}{36} \cdot 17 + \frac{1}{36} \cdot 18 + \frac{2}{36} \cdot 19 + \frac{1}{36} \cdot 20 + \frac{1}{36} \cdot$
$21 + \frac{1}{36} \cdot 22 + \frac{1}{36} \cdot 23 + \frac{1}{36} \cdot 25 + \frac{1}{36} \cdot 36$
$= 13.2222$

(c) What is the variance of X? Correct the answer to 4 decimal places. (Steps required.)

Solution: $\frac{1}{36} \cdot 1^2 + \frac{2}{36} \cdot 4^2 + \frac{1}{36} \cdot 5^2 + \frac{1}{36} \cdot 6^2 + \frac{3}{36} \cdot 7^2 + \frac{2}{36} \cdot 8^2 + \frac{2}{36} \cdot 9^2 + \frac{3}{36} \cdot 10^2 + \frac{2}{36} \cdot 11^2 + \frac{1}{36} \cdot 12^2 + \frac{2}{36} \cdot 13^2 + \frac{2}{36} \cdot 14^2 + \frac{1}{36} \cdot 15^2 + \frac{3}{36} \cdot 16^2 + \frac{1}{36} \cdot 17^2 + \frac{1}{36} \cdot 18^2 + \frac{2}{36} \cdot 19^2 + \frac{1}{36} \cdot 20^2 + \frac{1}{36} \cdot 21^2 + \frac{1}{36} \cdot 22^2 + \frac{1}{36} \cdot 23^2 + \frac{1}{36} \cdot 25^2 + \frac{1}{36} \cdot 36^2$
$= 223.6111$

$V(X) = E(X^2) - (E(X))^2$
$= 48.7840$

4. (15 points) Prove by Mathematical Induction, or disprove, that $4^p + 7^p \leq 11^p$ for all natural numbers $p \geq 1$.

Solution: Basis step:
When $p = 1$, L.H.S. $= 4^1 + 7^1 = 11$
R.H.S. $= 11^1 = 11$
The statement is true when $p = 1$.

Inductive Hypothesis: $4^k + 7^k \leq 11^k$

When $p = k + 1$,
L.H.S. $= 4^{k+1} + 7^{k+1}$
$= 4 \cdot 4^k + 7 \cdot 7^k$
$< 7 \cdot 4^k + 7 \cdot 7^k$
$= 7 \cdot (4^k + 7^k)$
$\leq 7 \cdot 11^k$
$< 11 \cdot 11^k$
$= 11^{k+1}$

So, It is true when $p = k + 1$.

Thus, by Mathematical Induction, $4^p + 7^p \leq 11^p$ for all natural numbers $p \geq 1$.

5. (15 points) Prove by Mathematical Induction, or disprove, that any natural number a can be expressed as $\sum_{j=0}^{n_a} c_{a,j} 2^j$, where n_a is a non-negative integer and $c_{a,j} \in \{0, 1\}$.

Solution: Basis step:
When $a = 1$, $1 = \sum_{j=0}^{0} 1 \cdot 2^j$, where $n_1 = 0$ and $c_{1,0} = 1$.

Inductive Hypothesis: Any natural number b can expressed as $\sum_{j=0}^{n_b} c_{b,j}2^j$, where n_b is a non-negative integer and $c_{b,j} \in \{0, 1\}$ and $b \leq k$.

Consider the case when $a = k + 1$.

Case 1: ($k + 1$ is odd)
$k + 1$ is odd.
$\Rightarrow k$ is even.
$\Rightarrow k = \sum_{j=0}^{n_k} c_{k,j}2^j$ where $c_{k,0} = 0$.
$\Rightarrow k + 1 = \sum_{j=0}^{n_{k+1}} c_{k+1,j}2^j$, $n_{k+1} = n_k$, $c_{k+1,0} = 1$ and $c_{k+1,j} = c_{k,j}, \forall j\ 1 \leq j \leq n_k$.

Case 2: ($k + 1$ is even)
$\Rightarrow \frac{k+1}{2} = \sum_{j=0}^{n_{(k+1)/2}} c_{(k+1)/2,j}2^j$
$\Rightarrow k+1 = \sum_{j=0}^{n_{k+1}} c_{k+1,j}2^j$, $n_{k+1} = n_{(k+1)/2}+1$, $c_{k+1,0} = 0$ and $c_{k+1,j} = c_{(k+1)/2,j-1}, \forall j\ 1 \leq j \leq n_{k+1}$.

So, It is true when $a = k + 1$.

Thus, by Mathematical Induction, any natural number a can be expressed as $\sum_{j=0}^{n_a} c_{a,j}2^j$, where n_a is a non-negative integer and $c_{a,j} \in \{0, 1\}$.

6. (15 points) (a) $60 \in S$.
If $x \in S$, then $x - 15 \in S$.
Based on the above recursive definition, describe S in at most two sentences.

Solution: S is a set of multiples of 15 which are smaller than or equal to 60.

(b) $\lambda \in T$.
If $y \in T$, then "00"$y \in T$ and y"11" $\in T$.
Write down any five non-empty strings in T.
Use at most two sentences to describe the elements of T.

Solution: $00, 11, 0000, 0011, 1111$.
Elements of T are bit strings of even length. The bit string starts with an even number of "00" and ends with an even number of "11".

7. (15 points) Consider the poset $(\{4, 5, 8, 10, 20, 24, 25, 30, 32, 40\}, |)$.

(a) Write down all the maximal element(s). (Write down 'Nil' if there isn't any maximal element.)

Solution: 24, 25, 30, 32, 40.

(b) Write down all the minimal element(s). (Write down 'Nil' if there isn't any minimal element.)

Solution: 4, 5.

(c) Write down all the upper bound(s) of $\{8, 10\}$. (Write down 'Nil' if there isn't any upper bound.)

Solution: Upper bound: 40.

(d) Write down all the lower bound(s) of $\{20, 40\}$. (Write down 'Nil' if there isn't any lower bound.)

Solution: Lower bounds: 4, 5, 10, 20.

Reminders:

- No mark will be given to late assignments.
- This is an individual assignment. You are encouraged to discuss with your classmates. But, please ensure that you will use your own words to answer the questions.
- Do NOT post/broadcast your answers on any shared platform. Zero mark will be given to those students who violate this regulation.
- Collusion and plagiarism are serious offences and may result in disciplinary action. A mark of zero will be given for the piece of coursework and; in addition, the final grade of the course may be affected (for example, it may be lowered from D to F). Please carefully read the section about Collusion and Plagiarism in your Student Handbook.
- Assignment (hard copy) should be put into the assignment box before the deadline AND soft copy should be submitted via the SOUL platform before the deadline.