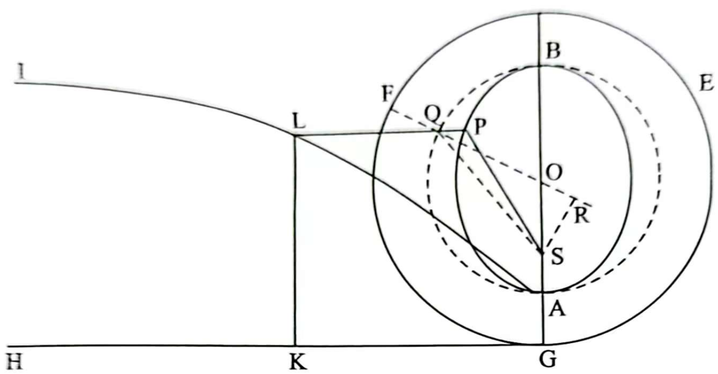
Numerical Analysis – Lab 3

1 Newton's method

Two-body problem. In his *Principia*, Isaac Newton (1643–1727) describes the Problem 23:"*To find the place of a body moving in a given ellipse at any assigned time.*" Consider a planet orbiting the sun. Johannes Kepler (1571–1630) discovered that the orbit is an ellipse and the sun is at a focus of the ellipse. Kepler found that the travel time from one point to another on the ellipse is proportional to the area swept by the line segment connecting the planet and the sun, known as Kepler's second law. Suppose at the initial time 0 the planet is at the principal vertex A of the ellipse, and T the period (the travel time to complete the ellipse once). Let O be the centre of the ellipse, S the location of the sun, and P the place of the planet at the time $t > 0$ (moved clockwise from A). Newton first gives a geometric solution by drawing graphs; see the figure below.



Produce OA to G so that $OG : GA = OA : OS$. Erect the perpendicular GH; and about O, with the the radius OG, describe the circle GEF; and on the ruler GH, as a base, suppose the wheel GEF to move forwards, revolving about its axis, and in the meantime by its point A describing the cycloid ALI. This done, take GK to the perimeter $GEFG$, in the ratio of $t : T$. Erect the perpendicular KL meeting the cycloid in L; then LP drawn parallel to KG will meet the ellipse in P, the required place of the body.

He then says "*But since the description of this curve is difficult, a solution by approximation will be preferable.*" Thereafter, Newton gives his method to *proceed in infinitum* for solving the equation

$$\theta - e \sin\theta = 2\pi \frac{t}{T}$$

where $\theta = \angle AOQ$ is the unknown (from which P can be determined because QP is perpendicular to the principal axis), and $e = OS : OA$ is the eccentricity of the ellipse. Indeed, the equation is according to Kepler's second law:

$$\frac{\text{area of } SAP}{\text{area of ellipse}} = \frac{t}{T}$$

with AP the elliptic arc, and

$$\text{area of } SAP = \text{area of } OAP - \text{area of } OPS = \frac{1}{2}ab\theta - \frac{1}{2}abe\sin\theta, \quad \text{area of ellipse} = \pi ab,$$

where $a = OA$ is the semi-major axis and b the semi-minor axis of the ellipse. The earth orbiting the sun has $T = 365.26$ days and $e = 0.0167$. On the 4th of July, 2022, the earth was at the principal vertex (aphelion). Can you find where we are today on the earth's orbit?

```
>> e = 0.0167;
>> T = 365.26;
>> t = 235;
>> f = @(x) x - e*sin(x) - 2*pi*t/T
>> df = @(x) 1 - e*cos(x)
>> newton(f,df,pi,1e-11,20)
```

Bond yields[1]. A bond is a financial instrument where the holder of the bond receives periodic payments until the maturity of the bond. Let $c_i, i = 1, .., n$ be all the cash payments received by the holder of the bond, and let t_i be the time when payment c_i is made. The yield of the bond y is the internal rate of return of the bond, and is related to the value B of the bond, i.e.,

$$B = \sum_{i=1}^{n} c_i \mathrm{e}^{-y t_i}.$$

If the price of the bond and all its future cash flows are known, finding the yield y of the bond means solving the equation for y. For example, a 34 months semiannual coupon bond with coupon rate 8% is priced at $B = 105$. The face value of the bond is assumed to be 100. Then what is the yield of the bond?

For a semiannual bond, one coupon payment is made every six months. Since the final payment will be made in 34 months, when the bond expires, we conclude that there are $n = 6$ coupon payments made in 4, 10, 16, 22, 28, and 34 months, corresponding to

$$(t_i)_{i=1}^{6} = \begin{pmatrix} \frac{4}{12} & \frac{10}{12} & \frac{16}{12} & \frac{22}{12} & \frac{28}{12} & \frac{34}{12} \end{pmatrix}.$$

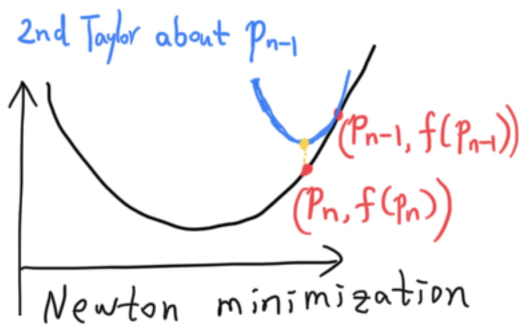
Each coupon payment is equal to the face value (100) times the coupon rate (0.08) divided by two since two payments are made every year for a semiannual bond. At maturity, both a coupon payment is made, and the face value is returned to the holder of the bond. Therefore,

$$(c_i)_{i=1}^{6} = \begin{pmatrix} 4 & 4 & 4 & 4 & 4 & 104 \end{pmatrix}.$$

```
>> t = [4  10  16  22 28  34]'/12;
>> c = [4   4   4   4  4 104];
>> B = 105;
>> f = @(y) c*exp(-y*t) - B;
>> df = @(y) -c*(t.*exp(-y*t));
>> newton(f,df,0.1,1e-11,20)
```

where `c` is a row vector, `t` is a column vector, and the row-column product `*` gives the required function. The result is

n	p_n	f(p_n)
0	0.1000000000	-9.1128036763
1	0.0627262417	0.4794462888
2	0.0644976319	0.0011580797
3	0.0645019314	0.0000000068
4	0.0645019314	0.0000000000
5	0.0645019314	0.0000000000



So the yield of the bond is 6.4502%.

Newton's method for optimization (optional). Suppose $f \in C^2(\mathbb{R})$ and the minimum of f is attainable at $p \in \mathbb{R}$ with $f''(p) > 0$. It follows that $f'(p) = 0$. We can use a simple fixed point iteration to solve the equation, known as the gradient descent method, namely

$$p_n = p_{n-1} - \alpha_n f'(p_{n-1})$$

where $\alpha_n > 0$ is called the step size or learning rate, and the minus sign before α_n is chosen to ensure $f(p_n) \leq f(p_{n-1})$ for sufficiently small $\alpha_n > 0$. The method moves from p_{n-1} to p_n in the decreasing direction of f at p_{n-1}. We can also use Newton's method to solve the equation, which gives

$$p_n = p_{n-1} - \frac{f'(p_{n-1})}{f''(p_{n-1})}.$$

[1] The material is taken from *A Primer for the Mathematics of Financial Engineering.*

Indeed, Newton's method for finding the minimum[2] of f can be understood as approximating f by the second Taylor polynomial about p_{n-1} and finding the minimum point p_n of the polynomial (assuming $f''(p_{n-1}) > 0$); see the figure above. More precisely,

$$f(x) \approx T_2(x) := f(p_{n-1}) + f'(p_{n-1})(x - p_{n-1}) + \frac{1}{2} f''(p_{n-1})(x - p_{n-1})^2$$

and

$$\arg\min_x T_2(x) = p_{n-1} - \frac{f'(p_{n-1})}{f''(p_{n-1})} \quad \text{because } T_2'(x) = f'(p_{n-1}) + f''(p_{n-1})(x - p_{n-1}).$$

Let us try Newton's method for the optimal daily exercise problem from Lab 2, for which

$$f(x) = 120x + g(x)(1 - x), \quad g(x) = 50 + 30\mathrm{e}^{-100x}.$$

We have

$$f'(x) = 120 - g(x) + g'(x)(1 - x), \quad g'(x) = -3000\mathrm{e}^{-100x}$$

and

$$f''(x) = -2g'(x) + g''(x)(1 - x), \quad g''(x) = 3 \times 10^5 \mathrm{e}^{-100x}.$$

```
>> g = @(x) 50 + 30*exp(-100*x)
>> dg = @(x) -3000*exp(-100*x)
>> d2g = @(x) 3e5*exp(-100*x)
>> df = @(x) 120-g(x)+dg(x)*(1-x)
>> d2f = @(x) -2*dg(x) + d2g(x)*(1-x)
>> newton(df,d2f,0,1e-11,20)
```

Try to use different initial points e.g. 0.1 in the above and find in which domain the initial point needs to be for convergence of Newton's method. How do you compare Newton's method and the golden section method from Lab 2?

Let us try also the gradient descent method with a constant learning rate:

```
>> eta = 0.0001;
>> F = @(x) x-eta*df(x);
>> fixedpoint(F, 0, 1e-11, 200)
```

Try to use different values of the learning rate `eta` and different initial points in the above. How do you compare Newton's method and the gradient descent method?

2 The Secant method

Singular zero point. Let $f(x) = \sqrt[3]{1 - \frac{3}{4x}}$ with the zero point $p = \frac{3}{4}$. Note that $f'(p) = \infty$. Can Newton's method work in this case? We use

```
>> f = @(x)nthroot(1-3/4/x,3)
>> df = @(x)1/4/x^2/nthroot(1-3/4/x,3)^2
>> newton(f,df,0.76,1e-10,50)
```

where `nthroot` gives a real number if possible (while `(-1)^(1/3)` gives a complex number). Can the Secant method work?

```
>> secant(f,0.76,0.77,1e-10,50)
```

Replace Newton's method. Redo the experiments in the section 1 but using the Secant method instead of Newton's method. How do you compare the Secant method with Newton's method? List two advantages of the Secant method. List one advantage of Newton's method. Is the Secant method or Newton's method more sensitive to the initial points?

[2]To find the *maximum* point, we need $f''(p) < 0, f''(p_{n-1}) < 0$, and p_n is the maximum point of the second Taylor.