Physics-based Animation: A Mathematical Perspective

Based on Ladislav Kavan's CS6660 with annotations from wxgopher

May 15, 2019

Contents

This lecture note is based on Dr. Ladislav Kavan's Physics-based Animation course CS6660, originally taught at the University of Utah. I also try to include some supplementary materials from other resources.

Warning: Although as I strive to make this material useful, there are certain bugs, use this material at your own risk. I would also be grateful to hear feedbacks.

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1 Classical Mechanics

1.1 Basics: a harmonic oscillator

2 Time Integration

3 Optimization

3.1 Implicit Newmark

$$x_{n+1} = x_n + \frac{h}{2}(v_n + v_{n+1}), \tag{1a}$$

$$v_{n+1} = v_n = \frac{h}{2}(f_n + f_{n+1}), \tag{1b}$$

where x and v are positions and velocities, and $f_n \equiv f(x_n)$. Thus we have

$$x_{n+1} - x_n - hv_n = \frac{h^2}{4}(f_n + f_{n+1}). \tag{2}$$

Let $y = x_n + h_n v_n + \frac{h^2}{4}f_n$, $x_{n+1} = x$, note that $\nabla E = -f$, then (2) becomes

$$x - y = \frac{h^2}{4}f \Rightarrow x - y + \frac{h^2}{4}f\nabla E = 0. \tag{3}$$

Let $g(x) = \frac{1}{2}\|x - y\|^2 + \frac{h^2}{4}E$, then solving g equals to solve x for

$$\min_x \quad g(x). \tag{4}$$

3.2 Optimization Problems

Problem formulation:
$$\min_x \quad g(x), \quad x \in R^n, \quad g \in R. \tag{5}$$

Optimization problems can be categorized into constrained or unconstrained problems, or convex or non-convex problems.

Theorem 1. *For a convex problem (where both objective and feasible set are convex), if the objective is C^2, then the Hessian $H \succeq 0$, and the local minimum is the global minimum.*

Definition 1. *A **linear programming (LP)** is a problem with linear objective and linear equality or inequality constraints. A **quadratic programming (QP)** is **the same as LP** except with a quadratic objective.*

Note 1. *Convex QP has polynomial time solver but non-convex QP is NP-hard.*

Example 1. *A non-convex QP:*

$$\min_{x} \quad \frac{1}{2}\|Ax\|^2, \tag{6a}$$

$$s.t. \quad \|x\|_2 = 1. \tag{6b}$$

Note 2. *Software package for solving non-convex problem:* IpOPT, KNITRO, *or NEOS-Guide*

3.2.1 Solving unconstrained problems

There are two ways to solve an unconstrained problem: descent method or trust-region method.

Descent method refers to pick a descent direction d and do an exact or inexact line search (LS) to determine descent distance αd.

Definition 2. *Descent direction:* $\forall \alpha \in (0, \alpha_0)$, $\alpha_0 > 0$, $g(x + \alpha d) < g(x)$.

Definition 3. *Exact LS refers to solving the following problem:*

$$\arg\min_{\alpha > 0} g(x + \alpha d).$$

Backtracking LS

3.2.2 Newton's method

A usual descent direction is called **gradient descent** (GD), denoted by $d = -\nabla g$. It doesn't work well sometime, for example:

Example 2.

$$\min_{x_1, x_2} g(x1, x2) = \frac{1}{2}(x_1^2 + \gamma x_2^2), \quad \gamma > 0. \tag{7}$$

If we do exact LS, we get

$$x_1^k = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \tag{8a}$$

$$x_2^k = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k. \tag{8b}$$

If gamma becomes huge, we can tell GD converges slowly (as two GD directions are perpendicular).

Newton's method refers to do a quadratic approximation to problem (5), given by

$$g(x + p) = g(x) + \nabla g^{\mathsf{T}}(x)p + \frac{1}{2}p^{\mathsf{T}}\nabla^2 g(x)p + O(\|p\|^2) \equiv \tilde{g}(x) + O. \tag{9}$$

Thus we have

$$\frac{\partial \tilde{g}}{\partial p} = 0 \iff \nabla g + \nabla^2 g p = 0.$$

The **Newton descent direction** is given by

$$p = -(\nabla^2 g)^{-1} \nabla g. \tag{10}$$

Note 3. *Newton's method is affine invariant. Let $x = Ty$ with T being a nonsingular matrix, so problem (5) is equivalent to problem*

$$\min_y \tilde{g}(y). \tag{11}$$

We also have

$$\nabla \tilde{g} = T^2 \nabla g, \quad \nabla^2 \tilde{g} = T^T \nabla^2 g T.$$

Newton direction Δy of $\tilde{g}$ and Δx of g is given by

$$\Delta y = -T^{-1} \nabla^2 g \nabla g = T^{-1} \Delta x.$$

Definition 4. *Newton decrement: substitute Newton descent direction (10) into (9), we get the **Newton decrement***

$$\lambda^2 = g - \frac{1}{2} \nabla g^T (\nabla^2 g)^{-1} \nabla g. \tag{12}$$

A stopping criteria for Newton's method would be

$$\frac{\lambda^2}{2} \leq \epsilon.$$

Note that Newton decrement is also affine invariant.

Convergence analysis of the Newton's method is given as follows:

Theorem 2. *Kantorovich: if g is strictly convex and $\nabla^2 g$ is Lipschitz continuous, then $\exists \epsilon > 0, \eta > 0, \gamma > 0$, s.t.*

Damped phase I *If $\|\nabla g\| \geq \eta$, then $\|g^{k+1}\| \leq \|g^k\| - \gamma$;*

Quadratic phase II *If $\|\nabla g\| < \eta$, then $\|g^{k+1}\| \leq c\|g^k\|^2$.*

Some cases where Newton's method doesn't work well:

1. If g is highly nonlinear, the damped phase will be long;

2. If g is non-convex, solution might not converge to a minimum.

In practical implementations, the Hessian might not always be positive definite. A possible Hessian modification is:

Note 4. *Hessian modification: Consider $A = \nabla^2 g + cI$, where $c > 0$. It is shown that if $\nabla^2 g = Q \Lambda Q^T$ ($\nabla^2 g$ is symmetric, hence it is always diagonalizable), then $A = Q(\Lambda + cI)Q^T$, hence A is positive definite given sufficiently large $c > 0$.*

Solving Newton's direction (10) refers to solving a linear system. We introduce several techniques to tackle this.

3.3 Numerical Linear Algebra

We consider solving a linear system $Ax = b$.

3.3.1 Direct solvers

We can prefactor A into $A = UV^T$, with U and V being low rank matrices.

Common factorization methods are

LU If A is nonsingular, then there exist L, P, U, where they are lower-triangular matrix, permutation matrix (hence orthogonal), and upper-triangular matrix, respectively, s.t. $A = PLU$.

Sparse LU $A = P_1 LU P_2$, where P_1 and P_2 are permutations to utilize sparse information of A.

Cholesky For symmetric positive-definite matrix, we have $A = LL^T$ with unique L. Note that for positive semi-definite matrix, Cholesky is not unique.

LDLT

3.3.2 Iterative solvers

Iterative solvers includes:

1. Classical method: Jacobian, Gauss-Seidel, SOR (super over-relaxation), etc.;

2. Krylov subspace method: CG (conjugate gradient)[1], GMRES (generalized minimal residual), etc.;

3. multigrid method.

Classical method Suppose P is easy to invert, let $A = P + A - P$, then

$$Ax = b \quad \Leftrightarrow \quad Px = (P - A)x + b.$$

$P = diag(A)$ gives us Jacobian method, and it's GPU-friendly; $P = lower(A)$ gives us GS method; $P = diag(A) + w \cdot lower(A)$ gives us SOR method where $w < 2$.

The residual term follows

$$r_k \equiv A(x^\star - x_k) = b - Ax_k \quad \Leftrightarrow \quad r_{k+1} = (I - P^{-1}A)r_k.$$

To make classical method converge, we need $\rho(I - P^{-1}A) < 1$, where $\rho(\cdot)$ is the spectrum radius.

Krylov subpace method Assuming $A^T = A$ and $A \succeq 0$, then

$$\arg\min_x g(x) = \frac{1}{2}x^T A x - x^T b \quad \Leftrightarrow \quad solve \ Ax = b. \tag{13}$$

Let $\nabla g_k = Ax_k - b = -r_k$[2] as a descent direction, the line search for (13) is as follows:

1. Solve α from $\arg\min_\alpha (\frac{1}{2}(x + \alpha r)^T A(x + \alpha r) - (x + \alpha r)^T b \equiv \frac{1}{2}\|x + \alpha r\|_A^2 - (x + \alpha r)^T b)$ by letting $\frac{\partial E}{\partial \alpha} = 0$, where E is the objective;

2. α follows $\alpha = -\frac{\|r\|^2}{\|r\|_A^2}$;

3. $x_{k+1} = x_k + \alpha_k r_k$.

This is the gist of CG from gradient descent.

CG Assuming solving problem (13).

Definition 5. *A-conjugate* A set of vectors $\{p_i\}$ are A-conjugate iff $\forall i, j, i \neq j, p_i^T A p_j = 0$.

The CG algorithm is as follows:

1. We define $r_k = Ax_k - b$;

2. Let $p_1 = r_1$[3];

3. $p_k = r_k + \beta_k p_{k-1}, \beta_k = \frac{r_k^T A p_{k-1}}{p_{k-1}^T A p_{k-1}}$;

4. Do LS along direction p_k, we have $\alpha_k = \frac{r_k^T p_k}{p^T A p_k}$;

5. $x_{k+1} = x_k + \alpha_k p_k$.

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[1] A good reference on CG is [4].
[2] Ax is computed on-fly, we don't need the whole A.
[3] r_1 is usually chosen as a GD step

Preconditioning CG   If A is ill-conditioned, we consider adding preconditioner M to A, s.t. $cond(MA) < cond(A)$.

The Jacobian preconditioner is given by $M = diag(A)^{-1}$. The incomplete Cholesky is given by prefactoring A into $L_a L_a^T$ where $-L_a$ is an approximation of L of the Cholesky decomposition of A given by $A = LL^T$.

4 Elastic Materials and Finite Element Simulation

Consider kinetic energy

$$k = \frac{1}{2} v^T M v.$$

Assuming unit mass, we have

$$k = \frac{1}{2} \int_\Omega \rho(p) \|v(p)\|^2 \mathrm{d}p, \tag{14}$$

where $p \in \Omega$ is the material point. We discretize Ω, and for every node p_i, we define piece-wise shape function $s_i : \Omega \mapsto [0, 1]$, s.t. $\sum_i s_i = 1$. We have

$$r_t(p) = \begin{cases} 1 & p \in E_t \\ 0 & otherwise \end{cases}$$

5 Fluids and Partial Differential Equations

6 Something more...

7 Papers and books

Numerical optimization can refer to [2] [3].

[1] is a great reference for continuum mechanics and FEM.

References

[1] Javier Bonet and Richard D Wood. *Nonlinear continuum mechanics for finite element analysis*. Cambridge university press, 1997.

[2] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[3] Jorge Nocedal and Stephen Wright. *Numerical optimization*. Springer Science & Business Media, 2006.

[4] Jonathan Richard Shewchuk et al. An introduction to the conjugate gradient method without the agonizing pain, 1994.