Topics : Ensembles of Hamiltonians

— Periodic table of TI . TSC

— Random matrix theory

History : n + Heavy nuclei → — . . 1950s.

Wigner : study statistics of random H.

↓

Dyson : 1962 " 3-fold way "

the natural probability distribution

is determined by symmetry

Usually : $H \to$ symmetry of H

Dyson : start with symmetry $\longrightarrow$ Find H.

§ · states : $\psi \in \mathcal{H}$, $\|\psi\| = 1$.

Rank-1 projection operator :

$$P_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

$$\operatorname{Im} P_\psi = l_\psi = \{ z\psi \mid z \in \mathbb{C} \} \subset \mathcal{H}.$$

$$\left\{ \begin{matrix} \text{lines/rays in} \\ \mathcal{H} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Rank 1} \\ \text{projectors} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{states in} \\ \mathcal{H}/\mathbb{C}^* \end{matrix} \right\}$$

$\equiv \mathbb{P}\mathcal{H}$.

· probability $\dfrac{|\langle\psi_1|\psi_2\rangle|^2}{\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle} = \operatorname{Tr}_{\mathcal{H}} P_{\psi_1} P_{\psi_2}$

$\mathcal{O} : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to [0,1]$ overlap function.

$\mathbb{P}\mathcal{H}$ has a natural Riemannian metric d

$$\mathcal{O}(P_1, P_2) = \cos^2 \frac{d(P_1, P_2)}{2}$$

e.g. $\mathcal{H} = \mathbb{C}^2$ Bloch ball.

$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $\|\psi\| = |z_1|^2 + |z_2|^2 = 1$

$$S_1\mathcal{H} = \{\psi \mid \|\psi\| = 1\} \cong S^3$$

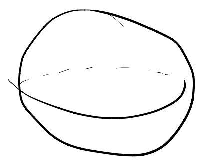
General density matrix:

$$\mathrm{Herm}_{2\times 2}(\mathbb{C}) = \{a \cdot \mathbb{I} + \vec{b}\cdot\vec{\sigma} \mid a, b_i \in \mathbb{R}\}$$

$$\rho : \quad \rho \geq 0, \quad \mathrm{tr}\,\rho = 1$$

$$\Rightarrow \rho = \frac{1}{2}(1 + \vec{x}\cdot\vec{\sigma}) \qquad |\vec{x}| \leq 1$$

the space of all $\rho \cong D^3$



Pure state

$$\rho^2 = \frac{1}{2}\left(\frac{1+\vec{x}^2}{2} + \vec{x}\cdot\vec{\sigma}\right) = \rho$$

$$\Rightarrow \vec{x}^2 = 1$$

$$\Rightarrow \mathbb{P}\mathcal{H} \cong S^2$$

$$\begin{array}{ccc} S_1\mathcal{H} & \cong & S^3 \\ \downarrow & & \downarrow \\ \mathbb{P}\mathcal{H} & \cong & S^2 \end{array}$$

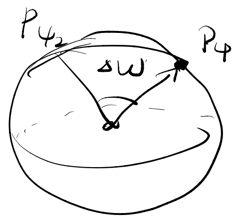
specifically,

$$\psi = u \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u = e^{-i\frac{\phi}{2}\sigma^3} e^{-i\frac{\theta}{2}\sigma^2} e^{-i\frac{\psi}{2}\sigma^3}$$

$$= \begin{pmatrix} e^{-i\frac{\psi+\phi}{2}} \cos\frac{\theta}{2} \\ e^{-i\frac{\psi-\phi}{2}} \sin\frac{\theta}{2} \end{pmatrix}$$

$$P_\psi = u \, \frac{1}{2}(1+\sigma^3)\, u^\dagger = \frac{1}{2}(1+\hat{n}\cdot\vec{\sigma})$$

rotation



overlap : $\mathrm{Tr}\, P_{\psi_1} P_{\psi_2} = \cos^2 \frac{\Delta W}{2}$

$$\underline{\Delta W = d(P_1, P_2).}$$

§ Quantum Symmetry:

Def. Grp of Quantum Automorphisms

is the set $\mathrm{Aut}_{qm}(\mathbb{P}\mathcal{H}) = \{ f : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H} \ \underset{\text{preserving overlaps}}{} \}$

e.g. $\mathcal{H} = \mathbb{C}^2 \quad \mathrm{Aut}_{qm}(\mathcal{H}) = \mathrm{Iso}(S^2) = O(3)$

Wigner's theorem:

every quantum automorphism is induced by

(not the same)

(anti) unitary operators on $\mathcal{H}$.

$$\left.\begin{array}{l} u \in U(\mathcal{H}) \\ a \in AU(\mathcal{H}) \end{array}\right\} \quad f_u : P \to u P u^{\dagger}$$

unique topological construction.

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r, "i"] \arrow[dd, equal] & G \arrow[r, "\pi'"] \arrow[d, "\phi''"] \arrow[dd, bend right, "\rho"'] & \bar{G} \arrow[dl, "\phi'''"'] \arrow[dd, "\bar{\rho}"] \\
 & & \mathbb{Z}_2 & \\
1 \arrow[r] & U(1) \arrow[r, hook, "i"] & \mathrm{Aut}_{\mathbb{R}}(\mathcal{H}) \arrow[r, two heads, "\pi"] \arrow[u, "\phi"] & \mathrm{Aut}_{gm}(\mathbb{P}\mathcal{H}) \arrow[r] \arrow[ul, "\phi'"'] & 1 \\
 & & U(\mathcal{H}) \arrow[u] & 1 \arrow[l] &
\end{tikzcd}$$

connected to $\mathbb{1}$ or not.

unitary v.s. antiunitary

$\Longrightarrow$

$$\mathrm{Ker}\,\pi = \{ z : \psi \to z\psi \} \cong U(1).$$

$$G = \{ (\bar{g}, \sigma) \mid \bar{\rho}(\bar{g}) = \pi(\sigma) \} \subset \bar{G} \times \mathrm{Aut}_{gm}(\mathbb{P}\mathcal{H})$$

- $(G, \phi : G \to \mathbb{Z}_2)$ is a $\mathbb{Z}_2$-graded group

ϕ-twisted central extension by $U(1)$.

"almost", z may not commute with antiunitary ones.

"ϕ-twisted" central extensions by $U(1)$

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & G \arrow[rr] \arrow[d, "\rho"] \arrow[dd, bend left, "\phi''"] & & \bar{G} \arrow[r] \arrow[d, "\bar{\rho}"] \arrow[ddll, "\phi'''"'] & 1 \\
1 \arrow[r] & U(1) \arrow[r] & \mathrm{Aut}_{B}(\mathcal{H}) \arrow[rr, "\pi"] \arrow[d, "\phi"] & & \mathrm{Aut}_{qm}(P\mathcal{H}) \arrow[r] \arrow[dll, "\phi'"] & 1 \\
 & & \mathbb{Z}_2 & & &
\end{tikzcd}$$

$$\left\{ \begin{array}{l} \mathcal{O} z = z^{\phi(\mathcal{O})} \mathcal{O} = \begin{cases} z\mathcal{O} & \phi(\mathcal{O}) = +1 \\ z^{-1}\mathcal{O} & \phi(\mathcal{O}) = -1 \end{cases} \Big\} \text{ (anti)unitary, } \mathbb{Z}_2 \text{ graded.} \\ g \cdot z = z^{\phi''(g)} g = \begin{cases} zg & \phi''(g) = +1 \\ z^{-1}g & \phi''(g) = -1. \end{cases} \Big\} \ \mathbb{Z}_2 \text{ graded.} \end{array} \right.$$

multiply

ϕ-twisted: up to ϕ. $U(1) \subset Z(G)$

e.g. $\bar{M}_2 = \{1, T\}$. $T^2 = 1$ $\phi(T) = -1$

$$1 \to U(1) \to M_2^{\pm} \to \bar{M}_2 \to 1$$

$$T \to \mp$$

$$z = T^2 \to 1$$

Im ker

$$T^3 = T\,T^2 = T z = z^{-1} T$$

$$= T^2 T = z T$$

$$\Longrightarrow z = \pm 1$$

$$M_2^{\pm} = \{ z \cdot T \mid \underbrace{T z = z^{-1} T}, \; T^2 = \pm 1 \}$$

$\Longrightarrow$ Grp in QM preserving probability (quantum automorphism/symmetry) are $\underline{\mathbb{Z}_2\text{-graded}}$ and act on $\mathcal{H}$ through $\underline{\phi\text{-twisted } U(1) \text{ central extensions}}$.

§ $\mathbb{R}, \mathbb{C}, \mathbb{H}$ - vector spaces.

Def. $V \in \mathbb{R}$-vect, then linear map: $I : V \to V$, $I^2 = -1$

is a complex structure.

$$(V, I) \Longrightarrow \underbrace{z \cdot v}_{(V, \mathbb{C})} = xv + y\, I(v)$$

without loss of generality: $\begin{cases} V \text{ Eucl metric} \\ I \in O(V) \end{cases}$

$$I_0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \qquad V = \mathbb{R}^{2n}.$$

$$\forall I, \exists S \in O(2n) \quad S I S^{-1} = I_0$$

not unique complex structure.

$$\mathbb{C}\text{-str}(\mathbb{R}^{2n}) = O(2n)/K = O(2n)/U(n)$$

$$K = \mathrm{Stab}(I_0) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right\} \underset{\text{unitary transf}}{\sim} \left\{ \begin{pmatrix} \underline{A+iB} \in U(n) & \\ & A-iB \end{pmatrix} \right\}.$$

$$\cong U(N)$$

Def. $(V, \mathbb{C})$ $\quad V_{\mathbb{R}}$ = same set as a real vect

$$\dim_{\mathbb{R}} V_{\mathbb{R}} = 2 \dim_{\mathbb{C}} V$$

$\mathbb{C}$-antilinear operator $O: V \to V$. $O^2 = +1$

$\Rightarrow$ $(V_{\mathbb{C}}, O)$ is a real structure

$$V_+ = \mathrm{Fix}(O) = \{ v \mid O(v) = v \} \qquad (V_+, \mathbb{R})$$

$$\mathbb{R}\text{-str}(\mathbb{C}^n) = U(n)/O(n).$$

Quaternions

Def. Algebra over k is a k-vect with k-bilinear

$\mu: A \times A \to A$

$\mu(A_1, A_2) = A_1 A_2$

A^{opp} : $\quad \mu^{opp}(V_1, V_2) = \mu(V_2, V_1)$

Def. quaternian algebra $\mathbb{H} = \mathbb{R}^4$ as $\mathbb{R}$-vect

basis : $\{1, i, j, k\}$

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji \quad + \text{cycl} \dots$$

- associative but not commutative

$$q = x_1 i + x_2 j + x_3 k + x_4 1$$

$$\bar{q} = -x_1 i - x_2 j - x_3 k + x_4 1$$

$$q\bar{q} = \bar{q}q = \left(\sum x_\mu x^\mu\right) \cdot 1$$

- $U(n, \mathbb{H}) = \{ u \in \mathrm{Mat}_n(\mathbb{H}) \mid u^\dagger u = 1 \}$

$$\cong USp(2n) = \{ u \in U(2n, \mathbb{C}) \mid u^* = J u J^{-1} \}$$

$$u^{tr} J u = J$$

$$J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

- $\mathbb{H}$ act on itself :

$$L(q) = q' \to q \cdot q' \qquad \{L(q)\} \cong \mathbb{H}$$

$$R(q) = q' \to q' \cdot q \qquad \{R(q)\} \cong \mathbb{H}^{opp} \cong \mathbb{H}$$

$$\mathbb{H} \otimes \mathbb{H}^{opp} \cong \mathrm{Mat}_4(\mathbb{R}).$$

Def. $\mathbb{H}$-vect is a $\mathbb{R}$-vect V with 3 complex structures. $I^2 = J^2 = K^2 = -1$ $IJ + JI = 0$ $(+ \dots)$

$$\mathbb{H}^{\oplus n} \cong \mathbb{R}^{4n}$$

by $L(i), L(j), L(k)$

Def. $\mathbb{H}$-structure on $\mathbb{C}$-vect is a $\mathbb{C}$ antilinear $J : V \to V$ s.t. $J^2 = -1$

$$\Longrightarrow \boxed{\begin{cases} \mathbb{C}\text{-str}(\mathbb{R}^{2n}) = O(2n)/U(n) \\ \mathbb{R}\text{-str}(\mathbb{C}^{n}) = U(n)/O(n) \\ \mathbb{H}\text{-str}(\mathbb{C}^{2n}) = U(2n)/USp(2n) \\ \mathbb{C}\text{-str}(\mathbb{H}^{n}) = USp(2n)/U(n) \end{cases}}$$

these're examples of Cartan Symmetric spaces

they play important role in 10-fold way (6 more to obtain)

Tangent spaces $T_{p=\text{origin}}$ of these spaces = Linear space of Free fermion H.

§ Rep-Theory of (G, ϕ)

<u>Def</u>. A (G,ϕ)-rep (V,ρ)

$\mathbb{C}$-vect V, $\rho : G \to \mathrm{Aut}(V_{\mathbb{R}})$.

$$\rho(g) = \begin{cases} \mathbb{C}\text{-lin} & \phi(g) = +1 \\ \mathbb{C}\text{-antilin} & \phi(g) = -1 \end{cases}$$

Intertwiner : $\mathcal{O}$, <u>$\mathbb{C}$-linear</u>

G-invariant (equivariant).

$$\begin{tikzcd} (V_1, \rho_1) \arrow[r, "\mathcal{O}"] \arrow[d, "\rho_1(g)"] & (V_2, \rho_2) \arrow[d, "\rho_2(g)"] \\ (V_1, \rho_1) \arrow[r, "\mathcal{O}"'] & (V_2, \rho_2) \end{tikzcd}$$

$\rho_2(g)\mathcal{O}(v_1) = \mathcal{O}(\rho_1(g) v_1)$

$\mathrm{Hom}^G_{\mathbb{C}}(V_1, V_2) = \{\text{intertwiners}\}$

is a <u>$\mathbb{R}$-vect</u>

$H \sim$ intertwiners.

e.g.

$$1 \to U(1) \to M_2^{\pm} \longrightarrow \mathbb{Z}_2 \to 1$$

$+$: $T^2 = 1$ $\qquad \phi(T) = -1$.

$$\downarrow$$

$V = \mathbb{C}$-vect. $\quad \rho(T)$ antilinear

$$\rho(T)^2 = \rho(T^2) = \rho(1) = 1.$$

$=$ Real structure on $V_{\mathbb{C}}$

$-$: $\quad \rho(T)$ antilinear $\quad \rho(T)^2 = \rho(T^2) = \rho(-1)$

$$= -1$$

Quaternionic structure on $V_{\mathbb{C}}$

$\Longrightarrow \quad V \cong \mathbb{C}^{2n}$ $\qquad$ (Kramer's degeneracy).

Schur's lemma

A division algebra A
(if $a \neq 0$, then a^{-1} exist)

Theorem :

1. an intertwiner between two (G, ϕ) reps is 0 or isomorphism.

2. $\mathrm{Hom}_{\mathbb{C}}^{G}(V, V)$ = Real division algebra if V is irreducible/invariant

$\exists$ precisely 3 real associative division algebras :

$\mathbb{R}, \mathbb{C}, \mathbb{H}$

$\Longrightarrow$ 3-fold way !

e.g. M_2^+: $V = \mathbb{C}$ $\quad \rho(T) = \overset{\text{complex conjugation}}{\overset{\downarrow}{C}}$ $\quad \mathrm{Hom}(\cdots) \cong \mathbb{R}$

M_2^-: $V = \mathbb{C}^2$

$$\begin{cases} \rho(e^{i\theta}) \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} e^{i\theta} z_1 \\ e^{i\theta} z_2 \end{pmatrix} \\ \rho(T) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_2^* \\ z_1^* \end{pmatrix} \end{cases}$$

(just like spin-$\frac{1}{2}$).

To find intertwiner: consider

$$\mathbb{C}^2 \leftrightarrow \mathbb{H}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \leftrightarrow z_1 + z_2 \hat{j}$$

then

$$\rho(e^{i\theta}) = \cos\theta + \sin\theta \, L(i)$$

$$\rho(T) = L(j).$$

the intertwiner must commute with $\rho(G) = \{L(g)\}$.

$\Longrightarrow$ $R(g)$s.

$$\mathrm{Hom}_{\mathbb{C}}^{G}(V, V) \cong \mathbb{H}^{opp} \cong \mathbb{H}$$

§ $(G, \phi: G \to \mathbb{Z}_2) \longrightarrow$ Rep: (V, ρ)

and. Schur: $\underline{\mathrm{Hom}^G_{\mathbb{C}}(V,V)}$ = real division algebra.

$= \mathrm{End}^G_{\mathbb{C}}(V) \xrightarrow{\text{Frobenius}} \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$

- complete reducibility:

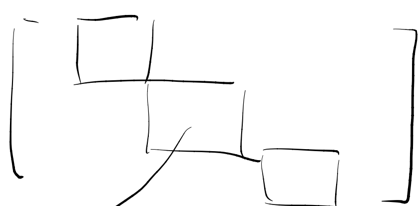
G compact, $\forall$ (G, ϕ) rep finite dim (V, ρ) is reducible to $\{(V_\lambda, \rho_\lambda)\}$s.

$$\underset{\rho(g)}{V} \cong \bigoplus_\lambda \Big(\underset{\text{Id}}{S_\lambda} \otimes \underset{\rho_\lambda(g)}{V_\lambda} \Big)$$

degeneracy space

$S_\lambda \cong \mathbb{R}^D$ and represent degeneracies.

$\Longleftrightarrow$ Block diagonalization:

$\rho(g) =$ 

$\mathbb{1}_{\dim S_\lambda} \otimes \rho_\lambda(g)$

U

$$\underbrace{\begin{bmatrix} \rho_\lambda(g) & & \\ & \ddots & \\ & & \rho_\lambda(g) \end{bmatrix}}_{\dim S_\lambda \text{ blocks}}$$

space of <u>intertwiners</u>

$$\mathrm{End}_{\mathbb{C}}^{G}(V) \cong \bigoplus_{\lambda} \mathrm{End}(S_\lambda) \otimes \underbrace{\mathrm{End}_{\mathbb{C}}^{G}(V_\lambda)}$$

$$\cong \bigoplus_{\lambda} \mathrm{Mat}_{\dim S_\lambda}(D_\lambda)$$

Schur: real division algebra D_λ

$$\rho(G) \subset \mathrm{End}\, V_{\mathbb{R}}$$

$$\mathrm{End}_{\mathbb{C}}^{G}(V) \cong Z(V, \rho)$$

commutant of $\rho(G)$

notation: $K(m) \equiv \mathrm{Mat}_m(K)$

$$mK = \mathrm{diag}(k, \ldots, K) \cong K$$

· Weyl's theorem:

$$\rho(G) \cong \bigoplus_\lambda S_\lambda D_\lambda(\tau_\lambda)$$

$$Z(V, \rho) \cong \bigoplus_\lambda \tau_\lambda D_\lambda^{\mathrm{opp}}(S_\lambda)$$

$$\tau_\lambda = \begin{cases} 2d_\lambda & D=\mathbb{R} \\ d_\lambda & D=\mathbb{C} \\ \frac{d_\lambda}{2} & D=\mathbb{H} \end{cases} \qquad d_\lambda = \dim_{\mathbb{C}} V_\lambda$$

<u>Rmk</u>. For Crystallographic

G = Lattice L.

Bloch's theorem:

$$\bigoplus_\lambda \xrightarrow[\text{B.T.}]{} \oint dk.$$

§ In QM, $\tau : G \to \mathbb{Z}_2$

$\uparrow$

wether reverse time or not

suppose time translation invariance:

$$U(t) = \exp\left(-i\frac{t}{\hbar}H\right)$$

Def. (G, ϕ) is compatible with dynamics if (Quantum symmetry)

$$\rho(g)U(t)\rho(g^{-1}) = U(\tau(g)t)$$

$$\Longrightarrow \quad \rho(g)H\rho(g^{-1}) = \chi(g)H$$

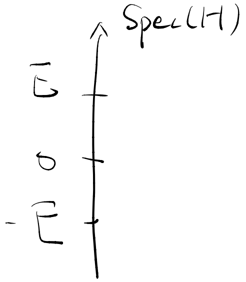
$$\chi(g) = \phi(g)\tau(g), \quad \underbrace{\phi \cdot \tau \cdot \chi = 1}_{\text{only two independent}}$$

General principle of QM

$\Longrightarrow$ $\boxed{\text{"symmetry group" is always } \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ graded.}}$

Rmks

① if $\chi(g) = -1$ for some $g \in G$



② H > below then $\chi = 1$

③ $\exists$ physical systems with $\chi \neq 1$

For $\chi = 1 \Longrightarrow$ 3-fold way

$\chi \neq -1 \Longrightarrow$ 10-fold way

④ $1 \to \ker \pi = G_0 \to G \xrightarrow{\pi} \mathbb{Z}_2 \to 1$

$G = \mathbb{Z}_2$ extension of G_0 $\sim H^2(G_0, \mathbb{Z}_2)$

$\omega \in H^2(G_0, \mathbb{Z}_2) = 1 \xrightarrow{\text{split}}$ $G = G_0 \rtimes_\omega \mathbb{Z}_2$

Symmorphic magnetic crystalline group.

other's all "nonsymmorphic"

general result: 230 3D space group

$\downarrow$

1651 3D MSG

§ Dyson's 3 fold way : $\underline{\chi = 1}$

Dyson problem:

given a (G, ϕ) rep $\mathcal{H} = \bigoplus_\lambda S_\lambda \otimes V_\lambda$.

what's the ensemble of $\underline{\text{compatible}}$ Hs.

$$Z(\mathcal{H}, \rho) \cong \bigoplus \mathrm{Mat}_{\dim S_\lambda}(D_\lambda)$$

$$\cup$$

$$H \in \mathcal{E} = \bigoplus \mathrm{Herm}_{\dim S_\lambda}(D_\lambda).$$

3-fold way

$$\mathrm{Herm}_{S_\lambda}(D_\lambda) = \begin{cases} \text{real symmetric} & D = \mathbb{R} \\ \text{Hermitian} & D = \mathbb{C} \\ \text{Quat-Hermitian} & D = \mathbb{H} \end{cases}$$

each ensemble have natural $\underline{U(N \circ D) \text{ invariant}}$ measure.

§ Random matrix theory:

A-Z: e^- transport, hybraid systems

$\chi \neq 1$:

Def. supervector space over k

?

$$V = \underbrace{V^0}_{\text{even}} \oplus \underbrace{V^1}_{\text{odd}}$$

homogeneous vectors $\in V^0$ or V^1 entirely.

and $\deg v = |v| = 0$ or $1 \mod 2$.

e.g. • $P: V \to V$ $P^2 = 1$. $P = (-1)^F$

• V. $\mathbb{R}$-vect, with complex str. I.

End(V) has a $\mathbb{Z}_2$-grading $A \to IAI^{-1}$

• $k = \mathbb{C} \cdot \mathbb{R}$

$$k^{\underline{n_+|n_-}} = \underbrace{k^{n_+}}_{\text{even}} \oplus \underbrace{k^{n_-}}_{\text{odd}}$$

superdimension.

given 2 super vect. V, W

$\mathrm{Hom}(V, W)$ is also a super vect ...

$$\begin{cases} \mathrm{Hom}(V,W)^0 = \mathrm{Hom}(V^0, W^0) \oplus \mathrm{Hom}(V^1, W^1) \\ \mathrm{Hom}(V,W)^1 = \mathrm{Hom}(V^1, W^0) \oplus \mathrm{Hom}(V^0, W^1) \end{cases}$$

even: $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ odd: $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$

<u>Def.</u> superalgebra: A super vect over k.

$A \times A \to A$ is even.

$\deg(aa') = \deg(a) + \deg(a') \pmod 2$

↑ homogeneous

- graded commute:

$$aa' = (-1)^{|a||a'|} a'a.$$

- $Z_s(A) = \{ a \mid ab = (-1)^{|a||b|} ba \ \forall b \in A \}$

graded tensor product

$A_1 \hat{\otimes} A_2$: $(a_1 \hat{\otimes} b_1) \cdot (a_2 \hat{\otimes} b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \hat{\otimes} b_1 b_2$.

e.g. V super algebra, $\mathrm{End}(V)$ is super algebra.

Matrix superalgebra

"Morita equivalence" $A_1 \sim A_2$

if $A_1 \cong A_2 \hat{\otimes} \mathrm{End}(V)$

$$\underline{[A_1] \cdot [A_2] = [A_1 \hat{\otimes} A_2]}$$

§ Real & complex Clifford algebra :

$\mathbb{C}l_n$: $\{e_1, \dots, e_n\}$

$$e_i e_j + e_j e_i = 2\delta_{ij}$$

Real

Cl_n : $\{e_1, \dots, e_n\}$ $\quad e_i e_j + e_j e_i = 2\delta_{ij}$ $\quad \underline{n > 0}$

in general different, we can't multiply e_i with $\underline{i}$ (complex)

$\underline{\underline{Cl_{-n}}}$: $\{e_1, \dots, e_n\}$ $\quad e_i e_j + e_j e_i = -2\delta_{ij}$

$$\dim_{\mathbb{C}} \mathbb{C}l_n = (2^{n-1} | 2^{n-1})$$

$$\dim_{\mathbb{R}} Cl_{\pm n} = (2^{n-1} | 2^{n-1})$$

$$\mathbb{C}l_n = \mathbb{C}l_1 \hat{\otimes} \cdots \hat{\otimes} \mathbb{C}l_1$$

$$Cl_{\pm n} = Cl_{\pm 1} \hat{\otimes} \cdots \hat{\otimes} Cl_{\pm 1}$$

$\mathbb{C}l_0 = \mathbb{C}$, $\mathbb{C}l_1 = \mathbb{C} + \mathbb{C}e$ not a matrix superalgebra

⋮

	Graded	ungraded
$\mathbb{C}l_{2k}$	$\mathrm{End}(\mathbb{C}^{2^{k-1}\|2^{k-1}})$	$\mathbb{C}(2^k)$
$\mathbb{C}l_{2k+1}$	$\mathrm{End} \otimes \mathbb{C}l_1$	$\mathbb{C}(2^k) \oplus \mathbb{C}(2^k)$.

$\{[\mathbb{C}l_0], [\mathbb{C}l_1]\} = \mathbb{Z}_2$ $\dashrightarrow$ K-theory

Bott periodicity.

$Cl_{-1} \to \mathbb{C}$

$Cl_{-2} \to \mathbb{H}$

$Cl_0 \to \mathbb{R}$

$Cl_{-3} \to \mathbb{H} \oplus \mathbb{H}$

$\mathbb{R} \oplus \mathbb{R}$

$Cl_{-4} \to \mathbb{H}$

$\mathbb{R}$

$\cdots \to \mathbb{C}$

§ the general Dyson problem:

$$\rho(g) H = \chi(g) H^{\phi(g)}$$

<u>Def</u>. (G, ϕ, χ)-rep

G super vect V. $\rho(g) \begin{cases} u & \phi = +1 \\ a & \phi = -1. \end{cases}$

$\begin{cases} \text{even} & \chi(g) = +1 \\ \text{odd} & \chi(g) = -1. \end{cases}$

Given a <u>$\mathbb{Z}_2 \times \mathbb{Z}_2$ graded group</u> and a (G, ϕ, χ)-rep

what is the ensemble of compatible H's?

⋮

<u>Conclusion</u>:

$\exists$ precisely 10 <u>$\mathbb{R}$ superdivision algebras</u>.

$$\left.\begin{array}{l} \bullet\ \ \mathbb{R},\ \mathbb{C},\ \mathbb{H} \\[1em] \bullet\ \ \mathbb{C}l_1,\ Cl_{\pm 1},\ Cl_{\pm 2},\ Cl_{\pm 3} \end{array}\right\} \text{ Morita equivalence classes of } \mathbb{R} \text{ and } \mathbb{C} \text{ Clifford algebras.}$$