§ Quantum CFT: Ward identities & radial quantization

Use physical heuristic approach rather than mathematical rigorous approaches (3 approaches)

(inspired guess, not rigorous yet) (path-integral)

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quantum: PUR of $\boxed{\mathrm{Conf}(\mathbb{R}^{2,0})}$ or more generally $\underline{\mathrm{Vir}_{\mathbb{C}}}$

(analytic continuation) imaginary time / Euclidean (temperature) of (1+1)d

$\mathbb{R}^{2,0}$ (Statistical) Euclidean $\longrightarrow$ Lorentzian ("Real" physics) $\mathbb{R}^{1,1}$

usually constructed

reflection positivity: guarantee this works

Note this is nontrivial, since $\mathrm{Conf}(\mathbb{R}^{2,0}) \mapsto \mathrm{Conf}(\mathbb{R}^{1,1})$ is nontrivial

In statistical physics we really don't have time evolution (equilibrium)

- Imaginary time : $t = -i\beta$

At last

$$e^{-itH} \longrightarrow e^{-\beta H}$$

unitary

not unitary! (statistical physics).

but convergent (there's QFT in $\mathbb{R}^{2,0}$ that is unitary)

easy to do numerical.

In statistical physics, near 2^{nd} phase transition (criticality)

there's large symmetry (conformal)

Digress : what is "physical"

Kinematics : Hilbert space $\mathcal{H} \to \rho(\mathcal{H})$ density operators

Rmk

separable $\mathcal{H} \cong L^2(\mathbb{R})$

observables distinguish different $\mathcal{H}$ than QT.

observables : $O \subset L(\mathcal{H})$ (hermitian)

No time is assumed to be presented

dynamics : M (thing like a manifold)

$\underline{G = Iso(M)}$ (may or may not have "time")

$\Longrightarrow$ PUR of $ISO(M)$ on $\mathcal{H}$ & Local observables ($O_{x \in M}$)

Note, once $Iso(M)$ is represented, the eqn "emerge"

Simplifications (set up).

① $(\sigma^0, \sigma^1) \in \mathbb{R}^{1,1} \to (z, \bar{z}) \in \mathbb{C} \to$ compactification.

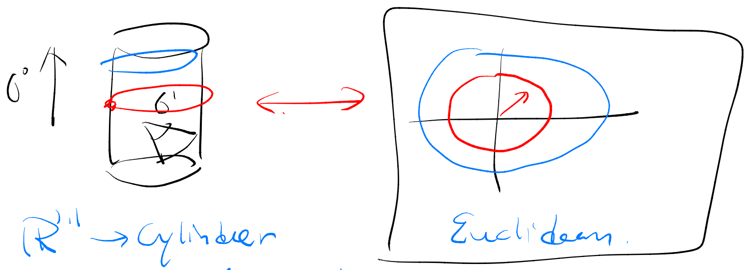
$\boxed{\mathbb{R}^2 \to \text{Cylinder}}$: $\sigma^1 = \sigma^1 + 2\pi$

(avoid infrared divergence)

Correspond systems on a circle

② New coordinates :
$$\begin{cases} z \equiv e^{\sigma^0 + i\sigma^1} \\ \bar{z} \equiv e^{\sigma^0 - i\sigma^1} \end{cases}$$

$z \to 0 \ (\sigma^0 \to -\infty)$



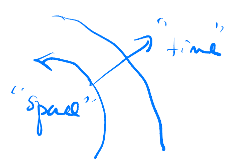
$\mathbb{R}^{1,1} \to$ Cylinder
(compactify space)

time translation $\sigma^0 \to \sigma^0 + a \longrightarrow$ dilations. $z \mapsto e^a z$

(spatial) ... $\sigma^1 \to \sigma^1 + a \longrightarrow$ rotations $z \mapsto e^{ia} z$

goal : study 1+1d CFT on $\mathbb{R}^{1,1} \xrightarrow[\text{space compactified}]{}$ Cylinder.

imaginary time
(changing to $z, \bar{z}$) circles on $\mathbb{C}$ $(\mathbb{R}^{2,0})$



— Technique: Ward identities (for path-integrals).

- canonical quantization:

continuous symmetry $\rightarrow$ conserved classical charges $\{ Q_i, Q_j \} = i f_{ij}{}^{k} Q_k$

$\xrightarrow[\text{canonical}]{\text{"quantization"}}$ $[\hat{Q}_i, \hat{Q}_j] = i f_{ij}{}^{k} \hat{Q}_k +$

$\uparrow$

subproblem: constructing $\mathcal{H}$

expressing $\hat{Q} = f(\hat{\phi})$

Rmk. canonical quantization works fine for free theories, but not well for complex theories.

- Path-integrals: tools for guessing quantum theory.

more practical, more easy, more physical.

Classical: $S[\phi]$ invariant under $\phi'(x) = \phi(x) - i\omega^a G_a \phi(x)$

vector fields.

Pathintegral:

classical

$$\underbrace{\langle T\{\phi(x_1)\dots\phi(x_n)\}\rangle}_{\text{quantum}} = \lim_{T\to\infty(1-i\epsilon)} \frac{\int D\phi\, \overbrace{\phi(x_1)\cdots\phi(x_n)}\, e^{iS}}{\int D\phi\, e^{iS}}$$

Assume $\underline{D\phi = D\phi'}$ (no anomaly!)

$$\langle \hat{X} \rangle = \int D\phi' \,(X + \delta X) \exp\left(i\left[S(\phi') + \int dx\, \underline{\partial_\mu j^\mu_a}\, \underline{\omega_a(x)}\right]\right)$$

$$\left(\phi'(x) = \phi(x) - i\,\underline{\omega_a(x)}\, G_a \phi(x)\right)$$

action not invariant, this obtain $\hat{J}_{Noether}$ directly

$$X + \delta X = T\{\phi(x_1)\cdots\phi(x_n) + i\omega_a(x_1)\, G_a(x_1)\phi(x_1)\cdots\phi(x_n) + (\dots) + O(\omega^2)\}$$

up to $O(\omega)$ on LHS & RHS

$$\Longrightarrow \langle \delta X \rangle_T = \int d^dx\, \partial_\mu \langle \hat{j}^\mu(x)\, \hat{X} \rangle\, \omega_a(x)$$

$\Rightarrow$ Constraints must be satisfied at quantum level (for CFTs).

Rmk: this can also be obtained from canonical quantization, which is complex.

Ward's identity (Local arguments)

$\hat{j}^\mu = j^\mu_{\text{Noether}}(\hat{\phi})$

$$\frac{\partial}{\partial x^\mu} \langle \hat{j}^\mu_a(x) \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) \rangle_T$$

$$= -i \sum_{j=1}^{n} \delta(x - x_j) \langle \hat{\phi}(x_1) \cdots \left[G_a \hat{\phi}(x_j) \right] \cdots \hat{\phi}(x_n) \rangle_T \qquad (*)$$

$$(*) \xrightarrow[\text{over spacetime}]{\text{integral}} \quad \delta_\omega \langle \phi(x_1) \cdots \phi(x_n) \rangle_T = -i\omega_a \sum_{j=1}^{n} \langle \cdots \rangle \equiv 0$$

assume decay properties LHS = 0

invariant under conformal symmetry (quantum!)

Integrate $(*)$ over a thin box $t_- < \underset{\substack{\shortparallel \\ x_1^0}}{t} < t_+$:

$$\langle \hat{Q}_a(t_+) \hat{\phi}(x_1) \hat{Y} \rangle_T - \langle \hat{Q}_a(t_-) \hat{\phi}(x_1) \hat{Y} \rangle_T$$

$$= -i \langle G_a \hat{\phi}(x_1) \hat{Y} \rangle_T$$

$\beta \to \infty$; in limit $t_- \to t_+$, assume $x_j^0 > x_i^0$

$$\Longrightarrow \langle 0 | [\hat{Q}_a, \hat{\Phi}(x_i)] \hat{Y} | 0 \rangle_T = -i \langle 0 | G_a \hat{\Phi}(x_i) \hat{Y} | 0 \rangle_T$$

equal time slices.

other fields

True for all $\hat{Y}$ differential operator in classical mp

$$\Longrightarrow [\hat{Q}_a, \hat{\Phi}] = -i G_a \hat{\Phi}$$ equal-time

QM, hermitian.

this is defines generator $\hat{Q}_a$, we construct theorie give rise to this. this also give rise to Lie algebra

Ward identities for conformal symmetries

Rmk. $\langle \quad \rangle \equiv \langle 0 | T\{ \cdots \} | 0 \rangle$

translation :

$$\partial_\mu \langle T^\mu{}_\nu(x) \hat{X} \rangle = - \sum_j \delta(x - x_j) \partial_{x_j}^\nu \langle \hat{X} \rangle$$

energy momentum tensor.

rotations : $j^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$

$$\langle (\hat{T}^{\rho\nu}_{(x)} - \hat{T}^{\nu\rho}_{(x)}) \hat{X} \rangle = -i \sum_j \delta(x-x_j)\, S_j^{\nu\rho} \langle \hat{X} \rangle$$

$S_j^{\nu\rho}$ is the spin matrix rep

(In this last $S \equiv 1$ (only free bosons))

dilations : $D = -i x^\nu \partial_\nu \underline{- i \Delta}$

$$\langle \hat{T}^\mu_\mu{}_{(x)} \hat{X} \rangle = - \sum_j \delta(x-x_j)\, \Delta_j \langle \hat{X} \rangle$$

Note $\hat{T}^\mu_\mu \neq 0$

$\langle \hat{T}^\mu_\mu(x)\, \hat{T}^\mu_\mu(0) \rangle \equiv 0$

Ward identities in (2+0)d

changing variables : $ds^2 = dx^2 + dy^2 = dz\, d\bar{z}$ $\qquad g|_{\mathbb{C}^2} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}_{(z,\bar{z})}$

$$F = F_x \partial_x + F_y \partial_y = F_z \partial_z + F_{\bar{z}} \partial_{\bar{z}}$$

$$\Rightarrow \begin{cases} F_z = F_x + i F_y \\ F_{\bar{z}} = F_x - i F_y \end{cases}$$

$$T_{\mu\nu} dx^\mu dx^\nu = T_{..} dz\, d\bar{z} + \cdots$$

$$\Rightarrow \begin{cases} T_{zz} = \frac{1}{4}\left(T_{xx} - 2i T_{yx} - T_{yy}\right) \\ T_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{xx} + 2i T_{yx} - T_{yy}\right) \\ T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}\left(T^x{}_x + T^y{}_y\right) \end{cases}$$

Rmk.

Other generators are constructed out of T & x

$$\hat{X} = \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) = \hat{\phi}(x_1)\, \hat{Y}$$

$\Rightarrow$ Ward identities :

$$\int_M d^2x\, \partial_\nu f^\nu = \frac{1}{2} i \oint_{\partial M} \left(-dz\, F_{\bar{z}} + d\bar{z}\, F_z\right)$$

$$\Downarrow$$

$$\delta(x) = \frac{1}{\pi} \partial_{\bar{z}} \left(\frac{1}{z}\right)$$

$$w_j = x_j + i y_j$$

(1)
$$\begin{cases} 2\pi \partial_z \langle \hat{T}_{\bar{z}z} \hat{X} \rangle + 2\pi \partial_{\bar{z}} \langle \hat{T}_{zz} \hat{X} \rangle = -\sum_{j=1}^{n} \partial_{\bar{z}} \left(\frac{1}{z - w_j}\right) \partial_{w_j} \langle \hat{X} \rangle \\ 2\pi \partial_z \langle \hat{T}_{\bar{z}\bar{z}} \hat{X} \rangle + 2\pi \partial_{\bar{z}} \langle \hat{T}_{z\bar{z}} \hat{X} \rangle = -\sum_{j=1}^{n} \partial_z \left(\frac{1}{\bar{z} - \bar{w}_j}\right) \partial_{\bar{w}_j} \langle \hat{X} \rangle \end{cases}$$

(2)
$$\begin{cases} 2 \langle \hat{T}_{z\bar{z}} \hat{X} \rangle + 2 \langle \hat{T}_{\bar{z}z} \hat{X} \rangle = -\sum_{j=1}^{n} \delta(x - x_j)\, \Delta_j \langle \hat{X} \rangle \\ -2 \langle \hat{T}_{z\bar{z}} \hat{X} \rangle + 2 \langle \hat{T}_{\bar{z}z} \hat{X} \rangle = -\sum_{j=1}^{n} \delta(x - x_j)\, S_j \langle \hat{X} \rangle \end{cases}$$

by some subtractions additions to each other $\Rightarrow$

$\Delta_j = h_j + \bar{h}_j$

compact form : $\langle \hat{T}_{z\bar{z}} \sim \operatorname{tr} \hat{T} \rangle = 0$

$$\begin{cases} T(z, \bar{z}) = -2\pi T_{zz}(z, \bar{z}) \\ \bar{T}(z, \bar{z}) = -2\pi T_{\bar{z}\bar{z}}(z, \bar{z}). \end{cases}$$

$\delta_j = h_j - \bar{h}_j$

$$(*)\begin{cases} 0 = \partial_{\bar z}\left\{ \langle \hat{T}(z,\bar z)\hat{X}\rangle - \sum_{j=1}^{n}\left[\frac{1}{z-w_j}\partial_{w_j}\langle\hat{X}\rangle + \frac{h_j}{(z-w_j)^2}\langle\hat{X}\rangle\right]\right\} \\ 0 = \partial_{z}\left\{ \langle \hat{\bar{T}}(z,\bar z)\hat{X}\rangle - \sum_{j=1}^{n}\left(\frac{1}{\bar z-\bar w_j}\partial_{\bar w_j}\langle\hat{X}\rangle + \frac{\bar h_j}{(\bar z-\bar w_j)^2}\langle\hat{X}\rangle\right)\right\} \end{cases}$$

holomorphic

anti holomorphic

implications

$$\Longrightarrow \quad \langle T(z)\hat{X}\rangle = \sum_{j=1}^{n}\frac{1}{(z-w_j)}\partial_{w_j}\langle\hat{X}\rangle + \frac{h_j}{(z-w_j)^2}\langle\hat{X}\rangle + (\text{regular functions of } z) \qquad (*)$$

holomorphic

independent of $\bar z$

Another way to capture Ward identities:

$$\begin{cases} x \longmapsto x+\varepsilon \\ \delta_{\varepsilon,\bar\varepsilon}\langle\hat{X}\rangle = \int_M d^2x\,\partial_\mu\langle\hat{T}^{\mu\nu}(x)\,\varepsilon_\nu(x)\hat{X}\rangle. \end{cases}$$

use the fact that (conformal)

$$\begin{cases} \frac{1}{2}(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu) = \frac{1}{2}(\partial_\rho\varepsilon^\rho)\eta_{\mu\nu} \\ \frac{1}{2}(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu) = \frac{1}{2}\,\varepsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta \end{cases}$$

integral over $\int d^2x \xRightarrow{\text{Gauss's theorem}}$

$$\delta_{\varepsilon,\bar\varepsilon}\langle\hat{X}\rangle = \frac{1}{2}i\oint_C\left\{-d\bar z\,\langle\hat{T}^{\bar z\bar z}\varepsilon_{\bar z}\hat{X}\rangle + dz\,\langle\hat{T}^{zz}\varepsilon_z\hat{X}\rangle\right\} \qquad (**)$$

Rmk. If we construct conformal field theories, they must obey Ward identities (*) & (**)

(correlation functions)

(integrated form)

to obtain correlation functions, we use path-integral formalism (quantization of theories)

(building models)

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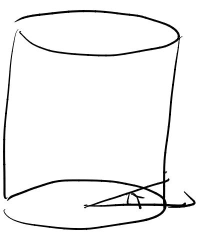
§ Radial quantization.

Changing coordinates

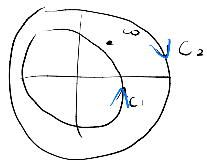


- space $\to S^1 \leftrightarrow \underline{\mathrm{diff}(S^1)}$
- Time ordering $\to$ Radial ordering R

$$\langle A B \rangle = \langle 0 | R\{A(z)B(w)\} | 0 \rangle$$

- equal time $[\ ,\ ] \longrightarrow$ equal radius $[\ ,\ ]$
- $\int d\vec{x} \longrightarrow \oint dz$

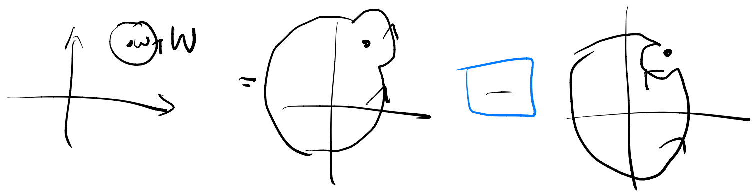
evaluate $[\hat{A}, b(w)]$: $A = \oint dz\, \hat{a}(z)$.



different orientation : "−"

C_1 close enough to C_2.

$$\Rightarrow \quad [\hat{A}, \hat{b}(w)] = \oint_w dz\, \hat{a}(z)\hat{b}(w)$$



Rmk. this is valid inside $\langle | R\{ \quad \} | 0 \rangle$

Let $\Phi(w,\bar{w})$ be a primary field.

Conformal Ward identity :

$$\begin{cases} T(z) = -2\pi T_{zz} \\ \bar{T}(\bar{z}) = -2\pi T_{\bar{z}\bar{z}} \end{cases}$$

$$\delta_{\varepsilon,\bar{\varepsilon}} \hat{\Phi}(w,\bar{w}) = \frac{1}{2\pi i} \oint \left[dz\, \varepsilon(z) R\{\hat{T}(z)\hat{\Phi}(w,\bar{w})\} + d\bar{z}\, \bar{\varepsilon}(\bar{z}) R\{\hat{\bar{T}}(\bar{z})\hat{\phi}(w,\bar{w})\} \right]$$

$$= \frac{1}{2\pi i} \oint_{C_1} - \oint_{C_2} (\cdots)$$

use the metric g_{**}

this is $(**)$.

$$= \frac{1}{2\pi i} \oint dz\, \varepsilon(z) \left[\hat{T}(z), \hat{\Phi}(w,\bar{w}) \right] + d\bar{z}\, \bar{\varepsilon}(\bar{z}) \left[\hat{\bar{T}}(\bar{z}), \hat{\Phi}(w,\bar{w}) \right]$$

$$\hat{Q} \equiv \frac{1}{2\pi i} \oint \left(dz\, \hat{T}(z)\, \varepsilon(z) + d\bar{z}\, \hat{\bar{T}}(\bar{z})\, \bar{\varepsilon}(\bar{z}) \right)$$

From def. of primary field, $\delta_{\varepsilon,\bar{\varepsilon}} \phi = (\cdots)$

$\Longrightarrow$

together with Ward identities

$$\begin{cases} R\{ \hat{T}(z) \hat{\Phi}(w,\bar{w}) \} = \frac{h}{(z-w)^2} \hat{\Phi}(w,\bar{w}) + \frac{1}{z-w} \partial_w \hat{\Phi}(w,\bar{w}) \\ \qquad + (\text{holomorphic}) \\ R\{ \hat{\bar{T}}(\bar{z}) \hat{\Phi}(w,\bar{w}) \} = \frac{\bar{h}}{(\bar{z}-\bar{w})^2} \hat{\Phi}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \partial_{\bar{w}} \hat{\Phi}(w,\bar{w}) \\ \qquad + (\text{anti-holomorphic}) \end{cases}$$

this is just ⊛ from Ward identity

equal up to expressions which are regular as $z \to w$ and under $\langle 0 | R\{\ \} | 0 \rangle$

$$\Longrightarrow \quad \underset{\substack{\uparrow \\ \text{energy-momentum tensor}}}{\hat{T}(z)}\ \underset{\substack{\uparrow \\ \text{primary field}}}{\hat{\Phi}(w,\bar{w})} \sim \underbrace{\overset{\substack{\text{conformal weight} \\ \downarrow}}{\frac{h}{(z-w)^2}} \hat{\Phi}(w,\bar{w}) + \frac{1}{z-w} \partial_w \hat{\Phi}(w,\bar{w})}_{\text{singular}}$$

this tells about short-distance behavior of

$\hat{T}(z)$ & $\Phi(w,\bar{w})$ as $z \to w$

This kinds of expressions are known as operator product expansions (OPE)

In general:

$$\hat{A}(z)\,\hat{B}(w) \sim \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}$$

$\{AB\}_n$ composite fields (non-singular)