Intro. characteristic classes are subsets of the cohomology classes of the base space, measure the nontriviality or twisting of the Fibler bundle, obstructions preventing a bundle from being trivial. play central role in index theorems.

Def. de Rham cohomology group

closed forms : $Z^r(M) = \{ d\omega = 0 \mid \omega \in \Omega^r(M) \}$

exact forms : $B^r(M) = \{ \omega = d\eta \mid \omega \in \Omega^r(M), \eta \in \Omega^{r-1}(M) \}$

nilpotent : $d^2 = 0 \Rightarrow \omega \in Z^r$ defined up to $d\eta$.

$B^r(M) \subset Z^r(M)$

$$\Rightarrow H^r(M) \equiv \frac{Z^r(M)}{B^r(M)} = \{ [\omega] \mid \omega \in Z^r, \omega \sim \omega + d\eta \}$$

Cohomology ring :

$$H^*(M) = \bigoplus_{i=0}^{m = \dim M} H^i(M)$$ product : $\wedge : H^*(M) \times H^*(M) \to H^*(M)$

Lemma.

$f^* d\omega = d f^* \omega$

$$f: N \to M \Rightarrow f^*: \Omega^r(M) \to \Omega^r(N) \Rightarrow f^*: H^r(M) \to H^r(N)$$

$$\Rightarrow f^*: H^*(M) \to H^*(N)$$

$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

Def. Invariant polynomials & Polarization.

Vect

$$S^r(M_k(\mathbb{C})) \equiv \{ \tilde{P} : \overset{r}{\otimes} M_k(\mathbb{C}) \to \mathbb{C} \mid \tilde{P} \text{ totally symmetric} \}$$

$$\Rightarrow \qquad S^*(M_k(\mathbb{C})) \equiv \bigoplus_{r=0}^{\infty} S^r(M_k(\mathbb{C})) \qquad \text{algebra}$$

with product : $\tilde{P} \in S^p$, $\tilde{Q} \in S^q$, $\tilde{P} \cdot \tilde{Q} \in S^{p+q}$

$$\tilde{P} \cdot \tilde{Q}(X_1, \dots, X_{p+q}) = \frac{1}{(p+q)!} \sum_{P \in S_{p+q}} \tilde{P}(X_{P(1)}, \dots, X_{P(p)}) \tilde{Q}(X_{P(p+1)}, \dots, X_{P(p+q)})$$

$$\xrightarrow{\text{restrict to Lie algebra}} \qquad S^*(\mathfrak{g}) = \bigoplus_{r=0}^{\infty} S^r(\mathfrak{g})$$

consider. $\tilde{P} \in I^r(G) \subset S^r(\mathfrak{g})$. $\qquad$ $\mathfrak{g}$ is the Lie algebra of G

$$\tilde{P}(Ad_g(A_1), \dots, Ad_g(A_r)) = \tilde{P}(A_1, \dots, A_r)$$

$g \in G$ $\qquad$ $A_i \in \mathfrak{g}$

$$\Rightarrow \quad I^*(G) = \bigoplus_{r=0}^{\infty} I^r(G).$$

$P(A) \equiv \tilde{P}(\underbrace{A, \dots, A}_{r})$ is known as invariant polynomial of degree r

Conversely. $\forall$ invariant $P(A) \Longrightarrow$ polarization of $\tilde{P}(A_1, \dots, A_r) \in I^r$

by expanding $P(t_1 A_1 + \dots + t_r A_r)$ and extract coefficient of $\prod_{i=1}^{r} t_i$

$$\tilde{P}(A_1, \dots, A_r) = \frac{1}{r!}\left[P\left(\sum_{i=1}^{r} t_i A_i\right)\right]_{\prod_{i=1}^{r} t_i}$$

$\in I^r(G)$ $\qquad \in \mathrm{Inv.deg}_r(\mathfrak{g})$.

$\mathfrak{g}$-valued p-forms. $\mathfrak{g} \otimes \Omega^p(M)$

Def. $\tilde{P}(A_1\eta_1, \dots, A_r\eta_r) \equiv \eta_1 \wedge \dots \wedge \eta_r \, \underline{\tilde{P}(A_1, \dots, A_r)}$.

$A_i \in \mathfrak{g}$, $\underline{\eta_i \in \Omega^{p_i}(M)}$ $\qquad \in I^r(G)$.

Theorem. Chern-Weil

Given $P \in \mathrm{Inv.deg}_r(\mathfrak{g} \otimes \Omega^r(M)) = \mathrm{diag}(\tilde{P} \in I^r(G))$

then ① $dP(\mathcal{F}) = 0$

② Given $\mathcal{F}(A)$. $\mathcal{F}'(A')$. $P(\mathcal{F}') - P(\mathcal{F})$ is exact.

Rmk. $\forall$ invariant polynomial can be decomposed into homogeneous polynomials, constraint to $\overset{H}{P_r}(\mathcal{F})$ (deg r. homogeneous).

(1)

identity: $X, X_i \in \mathfrak{g}$

$$\tilde{P}_r^H(\mathrm{ad}_{g_t} X_1, \ldots, \mathrm{ad}_{g_t} X_r) = \tilde{P}_r^H(X_1, \ldots, X_r). \quad g_t = e^{tX}$$

$$\frac{d}{dt}\Big|_{t=0} \Longrightarrow \sum_{i=0}^{r} \tilde{P}_r^H(X_1, \ldots, \underline{[X_i, X]}, \ldots, X_r) = 0 \qquad (*)$$

$\uparrow$ homogeneous

$$e^{-tX} X_i e^{tX} = X_i + t[X_i, X] + O(t^2).$$

generally, $[\Omega_i, A] = \eta_i \wedge \eta [X_i, X] = \eta_i \wedge \eta X_i X - (-1)^{pp_i} \eta \wedge \eta_i X X_i$

(WLOG, $\underline{\Omega_i = X_i \eta_i} \in \mathfrak{g} \otimes \Omega^{p_i}$
$\underline{A = X\eta} \in \mathfrak{g} \otimes \Omega^p$)

$$\Longrightarrow \tilde{P}_r^H(\Omega_1, \ldots, [\Omega_i, A], \ldots, \Omega_r)$$

$$= \eta_1 \wedge \cdots \wedge (\eta_i \wedge \eta) \wedge \cdots \wedge \eta_r \; \tilde{P}_r^H(X_1, \ldots, X_i X, \ldots, X_r)$$

$$- (-1)^{pp_i} \eta_1 \wedge \cdots \wedge (\eta \wedge \eta_i) \wedge \cdots \wedge \eta_r \; \tilde{P}_r^H(X_1, \ldots, X X_i, \ldots, X_r)$$

$$= \underline{\eta} \wedge (\eta_1 \wedge \cdots \wedge \eta_r) \underline{(-1)^{p(p_1 + \cdots + p_i)}} \; \tilde{P}_r^H(X_1, \ldots, \underline{[X_i, X]}, \ldots, X_r)$$

$$(*) \Longrightarrow \sum_{i=1}^{r} (-1)^{p(p_1 + \cdots + p_i)} \tilde{P}_r^H(\Omega_1, \ldots, [\Omega_i, A], \ldots, \Omega_r) = 0. \qquad (**)$$

$$d\tilde{P}_r^H(\Omega_1, \ldots, \Omega_r) = d(\eta_1 \wedge \cdots \wedge \eta_r) \; \tilde{P}_r^H(X_1, \ldots, X_r)$$

$$= \sum_{i=1}^{r} (-1)^{p_1 + \cdots + p_{i-1}} (\eta_1 \wedge \cdots \wedge d\eta_i \wedge \cdots \wedge \eta_r) \; \tilde{P}_r^H(X_1, \ldots, X_r)$$

$$= \sum_{i=1}^{r} \underline{(-1)^{p_1 + \cdots + p_{i-1}}} \; \tilde{P}_r^H(\Omega_1, \ldots, \underline{d\Omega_i}, \ldots, \Omega_r)$$

take $A \equiv \mathcal{A} \in \mathfrak{g} \otimes \Omega^1(M)$ $\quad (-1)^{(\cdots)} \equiv 0$

$\Omega_i \equiv \mathcal{F} \in \mathfrak{g} \otimes \Omega^2(M)$

$$\Rightarrow \quad d\tilde{P}_r^H(\mathcal{F},\dots,\mathcal{F}) = d P_r^H(\mathcal{F}) = d P_r^H(\mathcal{F}) - \overset{0}{\overset{\shortparallel}{(**)}}$$

$$= \sum_{i=1}^{r} \tilde{P}_r^H(\mathcal{F},\dots,\underset{\underset{i\text{th}}{\uparrow}}{d\mathcal{F}},\dots,\mathcal{F}) + \sum_{i=1}^{r} \tilde{P}_r^H(\mathcal{F},\dots,\underline{[A,\mathcal{F}]},\dots,\mathcal{F})$$

$$\equiv \sum_{i=1}^{r} \tilde{P}_r^H(\mathcal{F},\dots,\underbrace{D\mathcal{F}}_{\equiv 0},\dots,\mathcal{F}) \equiv 0$$

② define $\underline{A_t = A + t\theta}$, $\theta = A' - A$, $0 \le t \le 1$.

$$F_t = dA_t + A_t \wedge A_t \equiv dA + t\,d\theta + \underbrace{(A\wedge\theta + \theta\wedge A)}_{\equiv [A,\theta]} + t^2\,\theta\wedge\theta$$

$$= F + t\,D\theta + t^2\theta^2.$$

$$P_r^H(F') - P_r^H(F) = \int_0^1 dt\, \underline{\frac{d}{dt} P_r^H(F_t)} = \underset{\underset{\text{homogeneous symmetric deg } r.}{\uparrow}}{r} \int_0^1 dt\, \tilde{P}_r^H\left(\frac{d}{dt}F_t, F_t,\dots,F_t\right)$$

$$\underline{\frac{d}{dt} P_r^H(F_t)} = r\,\tilde{P}_r^H\left(\frac{d}{dt}F_t, F_t,\dots,F_t\right)$$

$$= r\,\tilde{P}_r^H(D\theta + 2t\theta^2, F_t,\dots,F_t)$$

$$= r\,\tilde{P}_r^H(D\theta, F_t,\dots,F_t) + 2rt\,\tilde{P}_r^H(\theta^2, F_t,\dots,F_t)$$

<u>identity</u> :

① $DF_t = dF_t + [\underline{A}, F_t] = -[A_t, F_t] + [A, F_t] = t[F_t, \theta]$

$$\uparrow$$

$$D_t F_t = dF_t + [\underline{A_t}, F_t] = 0$$

$$d : (-1)^i$$

② $d\, \tilde{P}_r^H(\theta, F_t, \dots, F_t) = \tilde{P}_r^H(d\theta, F_t, \dots, F_t) - (r-1)\tilde{P}_r^H(\theta, dF_t, \dots)$

$+(**) = \tilde{P}_r^H(D\theta, F_t, \dots, F_t) - (r-1)\tilde{P}_r^H(\theta, DF_t, F_t, \dots)$

$= \tilde{P}_r^H(\underline{D\theta}, F_t, \dots, F_t) - (r-1)\, t\, \tilde{P}_r^H(\theta, [F_t, \theta], F_t, \dots)$

③ $(**) \Rightarrow 2\tilde{P}_r(\underline{\theta^2, F_t}, \dots, F_t) + (r-1)\tilde{P}_r^H(\theta, [F_t, \theta], F_t, \dots, F_t)$
$= 0$

$A = \theta = \Omega_1$

$\Omega_{i \geq 2} = F_t$

$$\Longrightarrow \quad \frac{d}{dt} P_r^H(F_t) \equiv r\, d\, \tilde{P}_r^H(\theta, F_t, \dots, F_t).$$

$$\Longrightarrow \quad P_r^H(F') - P_r^H(F) = \int_0^1 dt\; d\left(r\, \tilde{P}_r^H(\theta, F_t, \dots, F_t)\right)$$

$$= \underline{d\, TP_r^H(A', A)}.$$

transgression: $TP_r^H(A', A) \equiv r \int_0^1 dt\; \tilde{P}_r^H(A' - A, F_t, \dots, F_t).$

$$\uparrow$$

polarization of P_r^H

$$\equiv \frac{1}{r!}\left(P_r^H(t_1\theta + t_2 F_t + \dots + t_r F_t)\right)_{t_1 \cdots t_r}.$$

Def Characteristic class

$$dP(F)=0 \Rightarrow P(F)\in\Omega^r(M).$$

$$P(F')-P(F)=d\,TP(A',A) \Rightarrow P(F) \text{ is determined up to gauge transformation.}$$

$$A\to A' \Rightarrow d\,TP(A',A)+P(F)$$

$$\Rightarrow \underbrace{\left\{ [P(F)]_{\text{gauge}/d\,TP(A',A)} \,\middle|\, P\in \mathrm{Inv}.\deg_r(\mathfrak{g}\otimes\Omega^*(M)) \right\}}_{} \subset H^r(M)$$

$\chi_E(P)$. characteristic class. : ① subset of $H^*(M)$

② gauge (potential) independent

③ P is any invariant polynomial. take value in $\mathfrak{g}\otimes\Omega^*(M)$

$[(A,F)]_{\text{gauge}}$ defined on E (usually vector bundle associated with $P(M,G)$ (same A,F))

Theorem

① $\chi_E : I^*(G) \xrightarrow{\text{hom.}} H^*(M).$

$\qquad P \longrightarrow \chi_E(P).$

$E \xrightarrow{\pi} M$

$(\pi^{-1}(p)\simeq G).$

$\left\{ \begin{array}{l} \text{structure group } G : \text{triv} \\ \text{topology of } M. \end{array} \right.$

$\Longrightarrow$ topology of bundle.

$$
\begin{aligned}
P_r P_s(\mathcal{F}) &\equiv \mathcal{F}^{\alpha_1}\wedge\cdots\wedge\mathcal{F}^{\alpha_r}\wedge\mathcal{F}^{\beta_1}\wedge\cdots\wedge\mathcal{F}^{\beta_s}\left(\frac{1}{(r+s)!}\tilde{P}_r(T_{\alpha_1},\dots,T_{\alpha_r})\cdot\tilde{P}_s(T_{\beta_1},\dots,T_{\beta_s})\right)\\
&= \left(\mathcal{F}^{\alpha_1}\wedge\cdots\wedge\mathcal{F}^{\alpha_r}\right)\wedge\left(\mathcal{F}^{\beta_1}\wedge\cdots\wedge\mathcal{F}^{\beta_s}\right)\frac{1}{(r+s)!}\tilde{P}_r(T_{\alpha_1},\dots,T_{\alpha_r})\cdot\tilde{P}_s(T_{\beta_1},\dots,T_{\beta_s})\\
&= P_r(\mathcal{F})\wedge P_s(\mathcal{F})
\end{aligned}
$$

② $f: N \to M \implies$ <u>Naturality</u>: $\chi_{f^*E} = f^*\chi_E$

$\uparrow$ p-forms

$$
\begin{tikzcd}
\chi_{f^*E} & \chi_E \arrow[l, Leftarrow, "\mathrm{Nat}(f^*)"'] \\
f^*E \arrow[u] \arrow[r, "\tilde{f}=\pi_2"] \arrow[d, "\pi_1"'] & E \arrow[u, "\chi_*"'] \arrow[d, "\pi"] \\
N \arrow[r, "f"] \arrow[d, "\cdot^*"'] & M \arrow[d, "\cdot^*"] \\
H^*(N) & H^*(M) \arrow[l, "f^*"]
\end{tikzcd}
$$

$\chi_* \in [\text{Bundle}, [\text{I-Alg}, \text{H-Alg}]]$

$\mathrm{Nat}(f^*)_p(\chi_E(p)) \overset{\text{induced}}{\equiv} f^*\chi_E(p)$

$= \chi_{f^*E}$

f^*A is the connection on f^*E:

$f^*t_{ij}(p) = t_{ij}(f(p)) = t^*_{ij}(p)$

$\implies f^*A$ transform like a gauge connection.

$\implies \underline{f^*F_i} = d(f^*A_i) + f^*A_i\wedge f^*A_i$

$\implies f^*P(F_i) = P(f^*F_i)$

$\implies f^*\chi_E(P) = \chi_{f^*E}$

<u>Theorem</u> trivial bundle $\implies$ trivial characteristic class.

$E \cong f_0^*E_0 \implies \chi_E = f_0^*\underline{\chi_{E_0}} = 0$

trivial $H^*(\{p\}) = 0$

$$
\begin{tikzcd}
f_0^*E_0 = E \arrow[r] \arrow[d] & E_0 \arrow[d] \\
M \arrow[r, "f_0"'] & \{p\}
\end{tikzcd}
$$

Rmk.

More practically:

$$\chi_*(-) \in [\text{Bundle}, [I^*(G), H^*(M)]]$$

$$\text{or } \in [I^*(G), \underline{[\text{Bundle}, H^*(M)]}]$$

$\chi_*(P)$ is a specific characteristic class

s.t. $\chi_E(P) \equiv P(F) = P([F]) \in H^*(M)$.

$$\begin{aligned} &\parallel \\ P(E, [F]) &\underset{\substack{\uparrow \\ ?\text{ unique}}}{=} P(E). \end{aligned}$$

Def. Universal bundle & classifying space.

$E \xrightarrow{\pi} M$ with $\underline{F \cong \underset{k\text{-plane}}{\mathbb{C}^k \subset \mathbb{C}^n}}$.

$\Longrightarrow$ Grassmann manifold. $G_{k,n}(\mathbb{C}) = \{ k\text{-planes in } \mathbb{C}^n \}$.

Theorem

① $\exists\ \bar{E}$. $E \oplus \bar{E} \cong M \times \mathbb{C}^n$.

② $\exists f : M \to G_{k,n}(\mathbb{C})$, $E \cong f^* L_{k,n}(\mathbb{C})$

$L_{k,n}(\mathbb{C}) \xrightarrow{\pi} G_{k,n}(\mathbb{C})$ has fiber $\mathbb{C}^k \cong F_p$ of E.

③ $f \underset{\text{homotopic}}{\sim} g : M \to G_{k,n}(\mathbb{C})$.

$\Longrightarrow$ $f^* L_{k,n}(\mathbb{C}) \cong g^* L_{k,n}(\mathbb{C})$

$\Longrightarrow$ homotopy class of E $\underset{\text{part of}}{\longleftarrow}$ homotopy class of $f : M \to G_{k,n}(\mathbb{C})$

Caution : For different Fiber bundle, n may be different !

$\Longrightarrow$ universal classifying space : $G_k(\mathbb{C}) = \bigcup_{n=k}^{\infty} G_{k,n}(\mathbb{C})$

universal bundle : $L_k(\mathbb{C}) \longrightarrow G_k(\mathbb{C})$, $F \cong \mathbb{C}^k$.

$\forall$ $E \xrightarrow{\pi} M$ with $F \cong \mathbb{C}^k$, $\exists f : M \to G_k(\mathbb{C})$, $E \cong f^* L_k(\mathbb{C})$

How to classify $[f]$?

Practically. characteristic classes : $\chi(f^* E) = f^* \chi(E)$.

$E \cong E' \not\Leftarrow\!\!\Rightarrow \chi(E) = \chi(E')$

not inverse ! (not complete).

$[f] : M \to Gr_k(\mathbb{C}) \longleftrightarrow f^* : H^*(Gr_k(\mathbb{C})) \to H^*(M)$

given $\chi(Gr_k)$, we construct $\chi(E)$. then use $\chi(E)$

to distinguish non equivalent bundles (different $[f]$)

not inverse !

Rmk $F \cong \mathbb{C}^k$.

Def Chern class $E \xrightarrow{\pi} M \begin{cases} \dim M = m \\ \dim_{\mathbb{C}} E = k \end{cases}$ $H^{\ell > m}(M) \cong 0$

① $c(f^*E) = f^* c(E)$

② $c(E) = c_0(E) \oplus c_1(E) \oplus \cdots \oplus c_k(E)$

$c_i(E) \in H^{2i}(M)$, $c_{i>k}(E) = 0$

③ $c(E \oplus F) = c(E) \wedge c(F)$.

④ $c(L) = 1 + x$. , $x \in H^2(M)$.

complex Line bundle over $\mathbb{C}P^n \cong L_1$.

total chern class

invariant polynomial

$$\Longrightarrow \quad c(E) = c(F) = \det\left(I + \frac{iF}{2\pi}\right) \in H^*(M).$$

uniquely.

$$= c_0(F) + c_1(F) + \cdots + c_k(F)$$

$c_0(F) = 1$, $c_1(F)$: j-th chern-class, $c_k(F) = \det\left(\frac{iF}{2\pi}\right)$.

Since $\det(1 + A)$ is invariant

$\Longrightarrow$ diagonalization : $\quad g^{-1}\left(i\frac{F}{2\pi}\right) g = A = \operatorname{diag}(x_1, \cdots, x_k)$

$x_1 \in \Omega^2(M).$

$$\det(I+A) = \prod_{j=1}^{k}(1+x_j) \equiv \det\left(I + \frac{iF}{2\pi}\right) = c(F).$$

$\Longrightarrow$

$$c_0(F) = S_0(x_j) = 1$$

$$c_1(F) = \sum_{j=1}^{k} x_k = \operatorname{tr} A = \operatorname{tr}\left(\frac{iF}{2\pi}\right).$$

$$c_2(F) = \sum_{i<j=1}^{k} x_i x_j = \frac{1}{2}\left[(\operatorname{tr} A)^2 - \operatorname{tr} A^2\right]$$

$$= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2\left[\operatorname{tr} F \wedge \operatorname{tr} F - \operatorname{tr}(F \wedge F)\right]$$

$\vdots$

$$c_k(F) = \det A = \det\left(\frac{i}{2\pi} F\right) = \left(\frac{i}{2\pi}\right)^k \det F$$

Lemma

① $c(f^*E) = c(F_{f^*E}) \equiv c(f^*F) = \det\left(I + f^* \frac{i}{2\pi} F\right)$

$= f^* \det\left(I + \frac{i}{2\pi} F\right) = f^* c(F) = f^* c(E).$

② $c(E \oplus F) = c(E) \wedge c(F).$ $\quad (F_{E\oplus F} = \operatorname{diag}(F_E, F_F))$

③ Splitting principle:

L_i : Complex line bundle

$\tilde{E} = L_1 \oplus L_2 \oplus \cdots \oplus L_k$

$\Longrightarrow \quad c(\tilde{E}) = c(L_1) \wedge \cdots \wedge c(L_k)$

$$= \prod_{i=1}^{k} (1+x_i) = c(E)$$

$c(E)$ can't distinguish between general $\dim E = k$ bundle and whitney sum of k complex line bundle:

upon digonalization: $\mathbb{C}^k \sim \underset{\uparrow \atop x_i}{\mathbb{C}} \underbrace{\oplus \cdots \oplus \mathbb{C}}_{k}$

not observed by $c(E)$.

Rmk. Chern character is important in index theorem

Def. Chern character

total chern character:

$(ch_{j>2m}(F) = 0)$

$$ch(F) = \operatorname{tr}\exp\left(\frac{iF}{2\pi}\right) = \sum_{j=1}^{2m} \frac{1}{j!} \operatorname{tr}\left(\frac{iF}{2\pi}\right)^j = \sum_{j=1}^{2m=2\dim M} \underset{\uparrow}{ch_j(F)}$$

j-th Chern character.

diagonalize.
$$= \sum_{j=1}^{k=\dim E} \exp(x_j)$$

$$\Rightarrow \begin{aligned} ch_0(F) &= k = \dim E \\ ch_1(F) &= c_1(F) = \operatorname{tr}\left(\frac{iF}{2\pi}\right) \\ ch_2(F) &= \frac{1}{2}\left[c_1(F)^2 - 2c_2(F)\right] = \frac{1}{2}\operatorname{tr}\left(\frac{iF}{2\pi}\right)^2 \end{aligned}$$

Lemma

① $\mathrm{ch}(f^*E) = f^*\mathrm{ch}(E)$

② $\mathrm{ch}(E\otimes F) = \mathrm{ch}(E)\wedge\mathrm{ch}(F)$

$\mathrm{ch}(E\oplus F) = \mathrm{ch}(E)\oplus\mathrm{ch}(F)$

$$\mathrm{ch}(B\otimes C) = \sum_j \frac{1}{j!}\left(\frac{i}{2\pi}\right)^j \mathrm{tr}(B\otimes I + I\otimes C)^j$$
$$= \sum_j \frac{1}{j!}\left(\frac{i}{2\pi}\right)^j \sum_{m=1}^{j} \mathrm{tr}B^m\,\mathrm{tr}C^{j-m}\binom{j}{m}$$
$$= \sum_m \frac{1}{m!}\mathrm{tr}\left(\frac{iB}{2\pi}\right)^m \sum_n \frac{1}{n!}\mathrm{tr}\left(\frac{iC}{2\pi}\right)^n$$
$$= \mathrm{ch}B\wedge\mathrm{ch}C$$

$$\mathrm{ch}(B\oplus C) = \sum_j \frac{1}{j!}\left(\frac{i}{2\pi}\right)^j(\mathrm{tr}B^j + \mathrm{tr}C^j) = \mathrm{ch}B\oplus\mathrm{ch}C$$

③ Splitting principle:

$$\mathrm{ch}(E) = \prod_{j=1}^{k}\exp(x_j) = \mathrm{ch}(L_1)\oplus\cdots\oplus\mathrm{ch}(L_k)$$
$$= \mathrm{ch}(L_1\oplus\cdots\oplus L_k).$$

Rmk

the characteristic classes themselves are local, measuring trivial local pieces, thus can't distinguish bundles with same M, F

but their integral over M is global and nontrivial.

Rmk

In physics (gauge theory): TQFT, topological configurations (monopole, instantons, ...).

anomaly. the highest chern character usually appear.

① classifying the bundle.
(homotopy class)

② denoting topological charge (sector) in path-integral.

③ manifesting the violation of current conservation...

⋮

Def Todd classes

$$Td(F) = \prod_j \frac{x_j}{1-e^{-x_j}} \qquad A = g\frac{iF}{2\pi}g^{-1} = diag(x_1, \dots, x_k).$$

Bernoulli number

$$= \prod_j \left(1 + \frac{1}{2}x_j + \sum_{k\geq 1} (-1)^{k-1} \frac{B_k}{(2k)!} x_j^{2k}\right)$$

$$= 1 + \frac{1}{2}c_1(F) + \frac{1}{12}\left[c_1^2(F) + c_2(F)\right] + \cdots$$

$$Td(E \oplus F) = Td(E) \wedge Td(F)$$

Rmk $F \cong \mathbb{R}^k$.

Def Pontryagin class

$\dim M = m$, $\dim_{\mathbb{R}} E = k$, with a fiber metric $\Longrightarrow$ orthonormal frames,

$F \cong \mathbb{R}^k$, $G = O(k)$

$$\underline{\underline{\mathfrak{g} = o(k)}}_{\text{skew-symmetric}} \Longrightarrow A^t = -A \xrightarrow{\text{"diagonalize"}} \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix} = A_{\mathbb{R}}$$

$$[k/2] \begin{cases} k \text{ even} & = \begin{pmatrix} 0 & \lambda_{k/2} \\ -\lambda_{k/2} & 0 \end{pmatrix} \\ k \text{ odd} & = 0 \end{cases}$$

total Pontryagin class :

$$p(F) = \det\left(I + \frac{1}{2\pi}F\right) = \det\left(I + \frac{1}{2\pi}F^t\right)$$

$$= \det\left(I - \frac{1}{2\pi}F\right) = 1 + p_1(F) + p_2(F) + \cdots$$

$\mathbb{R}$-Mod

$\deg p_j(F) = 2j$, $p_j(F) \in H^{4j}(M;\mathbb{R})$

($p_j(F) = 0$ if $2j > k = \dim E$

or $4j > \dim M$

note $p_{2n}(F) = 0 \not\Rightarrow p_m(B) = 0$)

$$\frac{F}{2\pi} \longrightarrow A_{\mathbb{C}} = \begin{pmatrix} -ix_1 & & & & \\ & ix_1 & & & \\ & & -ix_2 & & \\ & & & ix_2 & \\ & & & & \ddots \end{pmatrix}$$

$x_k = \frac{-1}{2\pi}\lambda_k$. $\lambda_k \in \mathrm{Spec}(F)$.

$$p(F) = \det\left(I + \frac{F}{2\pi}\right) = \det(I + A_{\mathbb{R}})$$

$$= \prod_{i=1}^{[\frac{k}{2}]} (1 + x_i^2) \quad (\pm i A_{\mathbb{C}})_i^2$$

$$\Rightarrow p_j(F) = \sum_{i_1 < i_2 < \cdots < i_j}^{[k/2]} x_{i_1}^2 \cdots x_{i_j}^2$$

$p(E \oplus F) = p(E) \wedge p(F)$

From $\quad \operatorname{tr}\left(\frac{F}{2\pi}\right)^{2j} = \operatorname{tr} A_{\mathbb{C}}^{2j} = 2(-1)^j \sum_{i=1}^{[k/2]} x_i^{2j}$

$$\Longrightarrow \left\{ \begin{array}{l} p_1(F) = \sum_i x_i^2 = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \operatorname{tr} F^2 \\ p_2(F) = \sum_{i<j} x_i^2 x_j^2 = \frac{1}{2}\left[\left(\sum_i x_i^2\right)^2 - \sum_i x_i^4\right] \\ \quad\vdots \qquad\qquad\qquad = \frac{1}{8}\left(\frac{1}{2\pi}\right)^4 \left[(\operatorname{tr}F^2)^2 - 2\operatorname{tr}F^4\right] \\ p_{[k/2]}(F) = x_1^2 x_2^2 \cdots x_{[k/2]}^2 = \left(\frac{1}{2\pi}\right)^k \det F \end{array} \right.$$

In terms of Chern class :

$$\det(I + iA) = \prod_{i=1}^{[k/2]} (1 - x_i^2) = 1 - p_1(A) + p_2(A) - \cdots$$

$$\Longrightarrow \quad p_j(E^{\mathbb{R}}) = (-1)^j c_{2j}(E^{\mathbb{C}}) \quad \leftarrow \text{complexification}$$

Def. Euler classes

Given $\dim M = 2l$, orientable, Riemannian with curvature.

$R \equiv F$

$$\underbrace{e(A)}_{\Omega^{2l}}\ e(A) = \underbrace{pe(A)}_{\in \Omega^{2l}}$$

A skew-symmetric $2l \times 2l$.

$\Omega^{2l} \downarrow$ volumn element of M.

For odd $\dim M$, $e(M) \equiv 0$

Theorem Gauss-Bonet

compact orientable M.

$$\int_M e(M) = \chi(M)$$

$\chi \uparrow$ Euler character

Rmk.

For $A^t = -A$: $\det A = Pf(A)^2$.

$$Pf(A) = \frac{(-1)^l}{2^l\, l!} \sum_p \sigma(p)\, A_{p(1)p(2)} \cdots A_{p(2l-1)p(2l)}$$

$$\Longrightarrow \quad e(M) = Pf\left(\frac{R}{2\pi}\right)$$

$$= x_1 x_2 \cdots x_l = \prod_{i=1}^{l} x_i \qquad x_i = -\frac{\lambda_i}{2\pi}$$

$\lambda_i = (A)_i \quad A \leftarrow \frac{F}{2\pi}$

Def Hirzebruch L-polynomial.

$$L(x) = \prod_{j=1}^{k} \frac{x_j}{\tanh x_j} = \prod_{j=1}^{k} \left(1 + \sum_{n \geq 1} (-1)^{n-1} \frac{2^{2n}}{(2n)!} B_n \, x_j^{2n} \right)$$

$$= 1 + \frac{1}{3} p_1 + \frac{1}{45}(-p_1^2 + 7p_2) + \cdots$$

$$L(E \oplus F) = L(E) \wedge L(F)$$

$\longrightarrow$ Hirzebruch signature theorem.

Def. $\hat{A}$ genus. (Dirac genus)

$$\hat{A}(F) = \prod_{j=1}^{k} \frac{x_j/2}{\sinh(x_j/2)} = \prod_{j=1}^{k} \left(1 + \sum_{n \geq 1} (-1)^{n} \frac{(2^{2n} - 2)}{(2n)!} B_n \, x_j^{2n} \right)$$

$$= 1 - \frac{1}{24} p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots$$

$$\hat{A}(E \oplus F) = \hat{A}(E) \wedge \hat{A}(F).$$

Def Chern-Simons form.

$\forall$ $2j$-form characteristic class. $P_j(F)$

$$dP_j(F) = 0 \xrightarrow[\text{Lemma}]{\text{Poincare's}} P_j(F) = d\,Q_{2j-1}(A,F) \quad \text{Locally}$$

$Q_{2j-1}(A,F) \in \mathfrak{g} \otimes \Omega^{2j-1}(M)$ is known as

Chern-Simons form of $P_j(F)$.

Explicitly. $\begin{cases} \theta = A \\ F_t = 0 + tDA + t^2 A^2 \end{cases}$ $\quad \begin{cases} A_0 = 0 & A_1 = A \\ F_0 = 0 & F_1 = F \end{cases}$

$$Q_{2j-1}(A,F) = TP_j(A,0) = j\int_0^1 dt\ \tilde{P}_j(A, F_t, \dots, F_t)$$

Theorem $\dim M = m = 2l$. even-dimensional

$$\underset{\text{usually } \in \mathbb{Z}}{\underbrace{\int_M P_l(F)}} = \int_M dQ_{m-1}(A,F) = \underset{\text{characteristic class for odd-dim } \partial M}{\underbrace{\int_{\partial M \neq \emptyset} Q_{2l-1}(A,F)}}$$

even-dim

Theorem

For $ch_j(F) = P_j(F)$, $\quad A_t = tA$, $\quad F_t = t\,dA + t^2A^2$

$$= tF + (t^2 - t)A^2$$

$\Longrightarrow$

$$Q_{2j-1}(A, F) = \frac{1}{(j-1)!}\left(\frac{i}{2\pi}\right)^j \int_0^1 dt \; \mathrm{str}(A, F_t^{\,j-1})$$

symmetric trace.

$$\mathrm{str}(A_1, \ldots, A_r) = \frac{1}{r!}\sum_p \mathrm{tr}(A_{p(1)} \ldots A_{p(r)})$$

$\Longrightarrow$

$$Q_1(A, F) = \frac{i}{2\pi}\int_0^1 dt \; \mathrm{tr}\,A = \frac{i}{2\pi}\mathrm{tr}\,A$$

$$Q_3(A, F) = \left(\frac{i}{2\pi}\right)^2 \int_0^1 dt \; \mathrm{str}(A, t\,dA + t^2A^2)$$

$$= \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}\left(A\,dA + \frac{2}{3}A^3\right)$$

$$Q_5(A, F) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^3 \int_0^1 dt \; \mathrm{str}(A, (t\,dA + t^2A^2)^2)$$

$$= \frac{1}{6}\left(\frac{i}{2\pi}\right)^3 \mathrm{tr}\left[A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5\right]$$

Lemma Cartan's homotopy formula

$$A_t = A_0 + t(A_1 - A_0) = A_0 + t\theta, \quad F_t = dA_t + A_t^2$$
$$= F_0 + tD\theta + t^2\theta^2.$$

def
$$\begin{cases} l_t A_t \equiv 0 \\ l_t F_t = \delta t \, (A_1 - A_0) \end{cases} \qquad l_t(\eta_p \omega_q) = (l_t \eta_p)\omega_q + (-1)^p \eta_p (l_t \omega_q)$$
$$\eta_p \in \Omega^p(M)$$

$$(d l_t + l_t d) A_t = l_t (F_t - A_t^2) = \delta t \, (A_1 - A_0) = \delta t \frac{\partial A_t}{\partial t}$$
$$(d l_t + l_t d) F_t = d[\delta t (A_1 - A_0)] + l_t[\underbrace{D_t F_t}_{0} - A_t F_t + F_t A_t]$$
$$= \delta t [d(A_1 - A_0) + A_t(A_1 - A_0) + (A_1 - A_0) A_t]$$
$$= \delta t \, D_t (A_1 - A_0) = \delta t \frac{\partial F_t}{\partial t}$$

$$\Rightarrow (d l_t + l_t d) S(A_t, F_t) = \delta t \frac{\partial}{\partial t} S(A_t, F_t)$$

$$\Rightarrow S(A_1, F_1) - S(A_0, F_0) = (d k_{01} + k_{01} d) S(A_t, F_t)$$
$$k_{01} S(A_t, F_t) = \int_0^1 \delta t \, l_t S(A_t, F_t).$$

$\Rightarrow$ For $S(A, F) = ch_{j+1}(F)$. $A_1 = A$, $A_0 = 0$

$$ch_{j+1}(F) = (d k_{01} + k_{01} d) ch_{j+1}(F_t) \underset{\uparrow \; d\,ch_{j+1}(F) = 0}{=} d[k_{01} ch_{j+1}(F_t)]$$

$$k_{01} ch_{j+1}(F_t) = \frac{1}{(j+1)!} k_{01} \mathrm{tr}\left(\frac{iF}{2\pi}\right)^{j+1}$$
$$= \frac{1}{(j+1)!}\left(\frac{i}{2\pi}\right)^{j+1} \int_0^1 \delta t \, l_t \mathrm{tr}(F_t^{j+1}).$$
$$= \frac{1}{j!}\left(\frac{i}{2\pi}\right)^{j+1} \int_0^1 \delta t \, \mathrm{str}(A, F_t^j) = Q_{2j+1}(A, F).$$

Theorem Gauge transformation of Chern-Simons form

$$A \to A^g = g^{-1}(A+d)g$$

$$F \to F^g = g^{-1}Fg$$

consider

$$A_t^g = t g^{-1}Ag + g^{-1}dg \quad \begin{cases} A_0^g = g^{-1}dg \\ A_1^g = A^g \end{cases}$$

$$F_t^g = dA_t^g + A_t^{g\,2} = g^{-1}F_t g \quad \begin{cases} F_0^g = 0 \\ F_1^g = F^g \end{cases}$$

$$\Rightarrow Q_{2j+1}(A^g, F^g) - Q_{2j+1}(g^{-1}dg, 0) = (dk_{01} + k_{01}d)\, Q_{2j+1}(A_t^g, F_t^g)$$

take $d\,Q_{2j+1}(A_t^g, F_t^g) = ch_{j+1}(F_t^g) = ch_{j+1}(F_t)$

$$k_{01}\, d\, Q_{2j+1}(A_t^g, F_t^g) = k_{01}\, ch_{j+1}(F_t^g) = k_{01}\, ch_{j+1}(F_t)$$

$$= Q_{2j+1}(A, F)$$

$$\Longrightarrow Q_{2j+1}(A^g, F^g) - Q_{2j+1}(A, F) = Q_{2j+1}(g^{-1}dg, 0) + d\,\alpha_{2j}$$

① $\alpha_{2j}(A, F, v) = k_{01}\, Q_{2j+1}(A_t^g, F_t^g)$

$$= k_{01}\, Q_{2j+1}(A_t + v, F_t)$$

$$v = (dg)g^{-1}$$

② $$Q_{2j+1}(g^{-1}dg, 0) = \frac{1}{j!}\left(\frac{i}{2\pi}\right)^{j+1}\int_0^1 \delta t \; \mathrm{tr}\left[g^{-1}dg\,\{(t^2-t)(g^{-1}dg)^2\}^j\right]$$

$$= \underline{\frac{1}{j!}}\left(\frac{i}{2\pi}\right)^{j+1}\mathrm{tr}\left[(g^{-1}dg)^{2j+1}\right]\underline{\int_0^1 \delta t\,(t^2-t)^j}$$

$$= \underline{(-1)^j\frac{j!}{(2j+1)!}}\left(\frac{i}{2\pi}\right)^{j+1}\mathrm{tr}\left[(g^{-1}dg)^{2j+1}\right]$$

M orientable, $\dim M = m$.

Def Spin bundle:

$TM \xrightarrow{\pi} M$ $\xrightarrow{\text{associated}}$ (principle bundle), frame bundle LM. $G = F = SO(m)$.

with $G = SO(m)$ $\qquad\qquad\qquad\qquad \uparrow$ t_{ij}.

A spin structure on M is defined as:

$$\tilde{t}_{ij} \in \underline{Spin(m)} = \overline{SO(m)} \text{ (double covering)}$$

$\varphi(\tilde{t}_{ij}) = t_{ij}$, consistency of $\tilde{t}_{ij}$ conditions

$\uparrow$
double covering.

$\Longrightarrow$ $PS(M)$ spin bundles over M.

Rmk. ① M may admit many spin structure depending on $\tilde{t}_{ij}$

② M may even admit no spin structure, this can be manifested by Stiefel–Whitney class.

Def. Čech cohomology group

multiplicative

Čech r-cochain : $f(i_0, i_1, \dots, i_r) \in \mathbb{Z}_2$ defined on $\underline{U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_r} \neq \emptyset}$

<u>totally symmetric</u>

$C^r(M; \mathbb{Z}_2)$ be <u>multiplicative</u> group of r-cochains.

coboundary operator $\quad \delta : C^r(M; \mathbb{Z}_2) \to C^{r+1}(M; \mathbb{Z}_2)$

$$(\delta f)(i_0, \dots, i_{r+1}) = \prod_{j=0}^{r+1} f(i_0, \dots, \hat{i}_j, \dots, i_{r+1})$$

omitted

$$\Longrightarrow \delta^2 f(i_0, \dots, i_{r+2}) = \prod_{j \cdot k = 1}^{r+1} f(i_0, \dots, \hat{i}_j, \dots, \hat{i}_k, \dots, i_{r+2}) = 1$$

$\underline{\delta^2 = 1}$

-1 appear even times due to symmetry of f.

$\Longrightarrow$ Čech cohomology group

$$H^r(M; \mathbb{Z}_2) = \frac{\ker \delta_r}{\operatorname{im} \delta_{r-1}} = \frac{Z^r(M; \mathbb{Z}_2)}{B^r(M; \mathbb{Z}_2)}$$

Def. Stifel–Whitney class

Assume $\{U_i\}$ is simple : $U_i \cap U_j = \emptyset$ or contractible

$\{e_{i\alpha}\}$ $(1 \le \alpha \le m = \dim M)$ be local orthonormal fram of TM (over U_i).

$$e_{i\alpha} = t_{ij} e_{j\alpha} \qquad t_{ij} : U_i \cap U_j \to O(m)$$

Čech 1-cochain

$$f(i,j) = \det(t_{ij}) = \pm 1$$
$(= f(j,i))$

: $t_{ij} t_{jk} t_{ki} = I$ (consistency conditions).

$\Rightarrow$ cocycle condition

$$\delta f(i,j,k) = \det(t_{ij}) \det(t_{jk}) \det(t_{ki})$$
$$= 1.$$

$\Rightarrow$ $f \in Z^1(M; \mathbb{Z}_2)$. $\Rightarrow$ $[f] \in H^1(M; \mathbb{Z}_2)$

the coboundary transformation is induced by change of local fram ($[f]$ independent of frame)

$$\bar{e}_{i\alpha} = h_i e_{i\alpha} \quad h_i \in O(m), \quad \bar{e}_{i\alpha} = \bar{t}_{ij} \bar{e}_{j\alpha}$$

$\Rightarrow$ $\bar{t}_{ij} = h_i t_{ij} h_j^{-1}$ $\Rightarrow$ $\tilde{f}(i,j) = \delta f_0(i,j) f(i,j)$. $f_0(i) = \det h_i$

1st Stiefel-Whitney class:

$$w_1(M) = [f] \in H^1(M; \mathbb{Z}_2)$$

$$f(i,j) = \det(t_{ij})$$

Theorem. $TM \xrightarrow{\pi} M$ with fiber metric.

$$M \text{ orientable} \iff w_1(M) \text{ trivial.}$$

M orientable $\Rightarrow G = SO(m) \Rightarrow f(i,j) \equiv 1$, $w_1(M) = 1$

$w_1 = 1 \Rightarrow f = \delta f_0$, $f_0(i) = \pm 1$ we can always choose

$h_i \in O(m)$, s.t. $\det(h_i) = f_0(i)$.

define $\tilde{e}_{i\alpha} = h_i e_{i\alpha} \Rightarrow \det(\tilde{t}_{ij}) \equiv 1$, M orientable

1st S-W is the obstruction to orientability.

Def. 2^{nd} S-W class

$$M \text{ orientable} \Rightarrow G = SO(m).$$

consider lifting (projective-rep): $t_{ij} \to \tilde{t}_{ij} \in Spin(m)$

double covering. $\to \varphi(\tilde{t}_{ij}) = t_{ij}$

$\tilde{t}_{ji}^{-1} = \tilde{t}_{ij}$

$$\varphi(\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki}) = t_{ij}t_{jk}t_{ki} = 1 \Rightarrow \tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} \in \ker\varphi = \mathbb{Z}_2.$$

$\Rightarrow$ Čech 2-cochain :

$$f_{(i,j,k)} I = \tilde{t}_{ij} \tilde{t}_{jk} \tilde{t}_{ki}$$

$\uparrow$

symmetric and closed (cocycle condition).

$\Rightarrow$ 2nd S-W class : independent of local form.

$$w_2 = [f] \in H^2(M; \mathbb{Z}_2).$$

Theorem. $TM \xrightarrow{\pi} (M, \text{orientible})$,

$\exists$ nontrivial $PS(M)$ $\Longleftrightarrow$ $w_2(M)$ trivial.

$\exists$ $PS(M)$ $\Rightarrow$ $\tilde{t}_{ij} \tilde{t}_{jk} \tilde{t}_{ki} = I$ $\Rightarrow$ $w_2(M)$ trivial.

$w_2(M)$ trivial $\Rightarrow$ $f(i,j,k) = \delta f_1(i,j,k)$

take $f_1(i,j) = \pm$ of $\pm \tilde{t}_{ij}$ lift of t_{ij}

$\Rightarrow$ $\tilde{t}_{ij}' = \tilde{t}_{ij} f_1(i,j)$

$\Rightarrow$ $\tilde{t}_{ij}' \tilde{t}_{jk}' \tilde{t}_{ki}' = [\delta f_1(i,j,k)]^2 \equiv I.$

Theorem

① $w_1(\mathbb{C}P^m) = 1 \qquad w_2(\mathbb{C}P^m) = \begin{cases} 1 & m \text{ odd.} \\ x & m \text{ even} \end{cases}$

generator of $H^2(M; \mathbb{Z}_2)$

② $w_1(S^m) = w_2(S^m) = 1.$

③ $w_1(\Sigma_g) = w_2(\Sigma_g) = 1.$