

- Outline :
- I. Topological band theory
  - II.  $\mathbb{Z}_2$  TI in 2D & 3D
  - III. TSC & generalizations
- 

interplay between

symmetry

topology : invariants

classification

invariants

e.g. G-B theorem:  $\int_S k \cdot dA = 4\pi L(g)$   
 $k = \frac{1}{R_1 R_2}$  Gauss curvature

Topological phase — Many body problem

continuous deformation  $\rightarrow$  adiabatic continuation of H

$\Rightarrow$  topological equivalence  
of phase (state)

preserve symmetries of H

changing between topologically distinct states

— phase transition  $\rightarrow$  critical point for  
adiabatic deformation

## Simplification:

single particle topological phases

$H_{\text{many body}}$   $\xrightarrow[\text{mean field theory}]{} \text{noninteracting } H_{\text{eff}}$  of single fermions

ground state of  $H_{\text{many-body}}$   $\xrightarrow[\text{adiabatic continuation}]{} \text{Noninteracting states}$   
(ground states of  $H_{\text{eff}}$ )



may include weak interactions (weakly correlated)

+ Crystalline symmetry

(+ disorder, defects, ...)

$\Rightarrow$  Topological band theory.

Band theory:

$$H|\psi\rangle = E|\psi\rangle \quad \underbrace{[H, T(R)] = 0}$$

$$\Rightarrow T(R)|\psi_k\rangle = e^{ik \cdot R} |\psi_{k+R}\rangle$$

$$|\psi_k\rangle = e^{ik \cdot r} \underbrace{|\psi_k\rangle}_{\text{Bloch, periodic state}}$$

simple unit cell

- Bloch Hamiltonian:

$$H(k)|u_k\rangle = E(k)|u_k\rangle$$

$$k \in \mathbb{BZ} = \mathbb{T}^d$$

parameters  $\Rightarrow$  Berry phase.

$$H(k) = \frac{e^{-ikr}}{2} H e^{ikr}$$

Band structure.

Simplification: gapped

$$H_k \equiv H(k)$$

$$|u_k\rangle \equiv |u(k)\rangle$$

Berry phase in band theory

$$|U_{k\sigma}\rangle \rightarrow e^{\frac{i\phi(k)}{2} \text{not single-valued}} |U_{k\sigma}\rangle$$

\* Berry phase is more general. e.g. spin Berry phase

Berry connection  $A = -i \langle U_k | \nabla_k | U_k \rangle \rightarrow A \in \nabla_k \phi$

Example: 2 band (levels)

$$H(k) = d_0(\vec{k}) + \vec{d}(\vec{k}) \cdot \vec{\sigma}$$

$$H(k) |U_{\pm(k)}\rangle = (d_0 \pm (\vec{d}))_k |U_{\pm(k)}\rangle$$



$$\gamma = \oint_C A \cdot d\ell = \frac{1}{2} \oint_C$$

$$\begin{aligned} \text{Spin} &= \frac{1}{2} \times 2\pi \\ &= \pi \\ &\text{solid angle swept out} \\ &\text{by } d(k) \end{aligned}$$

$$\gamma = \oint_{B2} A \cdot dk \rightarrow \oint_{B2} A \cdot dk + \oint_{B2} \nabla_k \phi_{\text{ext}} \cdot dk.$$

Loop in  $B2$  is not boundary of some interior

Large gauge transformation  $\phi(k) \rightarrow +2\pi$  winding  $B2$

$$\text{e.g. } \phi\left(\frac{\pi}{a}\right) - \phi\left(-\frac{\pi}{a}\right) = 2\pi$$

Electric polarization in 1D

$$\vec{P} = \frac{\vec{p}}{V}$$

Bond charge :

$$p_b = -\nabla \cdot \vec{p}$$

$$Q_b = \vec{p} \cdot \hat{n}$$

Band structure  $\implies$  Berry phase

$\implies$  electric polarization.

$$P = \frac{e}{2\pi} \oint_{BZ} A \cdot d\mathbf{k} \quad r \rightarrow r + 2\pi$$

$P \rightarrow P + e$

↑ not boundary of  
some interior.

Consider  $(\mathbf{k}, \lambda(t))$

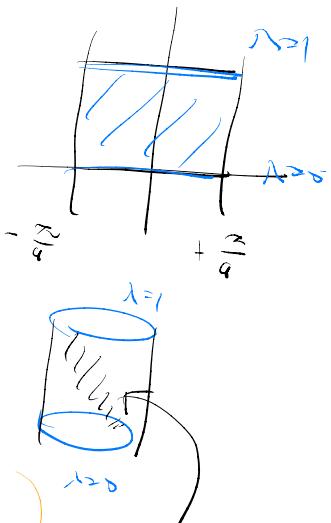
$$\Delta P = P(\lambda=1) - P(\lambda=0)$$

$$= \frac{e}{2\pi} \oint_{C} A \cdot d\mathbf{k} \quad \text{boundary of some interior}$$

$C = C_{\lambda=1} + C_{\lambda=2}$

$$= \frac{e}{2\pi} \left( \oint_{\lambda=1} A \cdot d\mathbf{k} - \oint_{\lambda=0} A \cdot d\mathbf{k} \right)$$

$$= \frac{e}{2\pi} \int_{\substack{\text{interior} \\ \text{surrounded by } \lambda=0}} F \cdot d\mathbf{k} d\lambda \quad \left( \begin{array}{l} \text{well-defined} \\ \text{gauge invariance} \end{array} \right)$$



Large gauge transformation  
on  $\lambda=1, 0$  are similar etc.  
 $P(\lambda)$  will not be smooth inside.

$$P = \frac{e(r)}{\sqrt{V}} \leftarrow \begin{array}{l} \text{Localized states: Wannier states} \\ \text{else } e(r) \text{ is not well-defined.} \end{array}$$

(continuous state)

$$- \langle \phi_R^{(n)} \rangle = \int \frac{dk}{(2\pi)^3} e^{-ik(R-r)} \underbrace{\int_{B_2} e^{i k \cdot r} \rangle}_{\text{Fourier transf}}$$

$\uparrow$   
gauge dependent

Large gauge  
 transf will set  
 $\langle \phi \rangle \rightarrow$  some other atoms  
 localized at  
 point

$$\Rightarrow P \sim e \langle \psi_R \rangle \left| \hat{r} - R \right| \langle \psi_R \rangle.$$

$$\text{def of } \langle \psi_R \rangle = \frac{e}{2\pi} \oint_{B_2} -i \langle \psi_R | D_k | \psi_R \rangle dk.$$

$(r = -r D_k)$

Large gauge transf of  $\langle u \rangle \rightarrow \varphi$

$$\rightarrow \gamma \leftrightarrow p \rightarrow Q_{\text{ext}}$$

(adding  $e$  in unit cell  
 extra filled band)

Krueger prof.

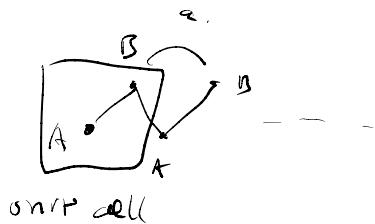
$$\text{varying parameters} \rightarrow \text{linear response.} \rightarrow \int \text{current} = P$$

(Kubo)

SSH model:

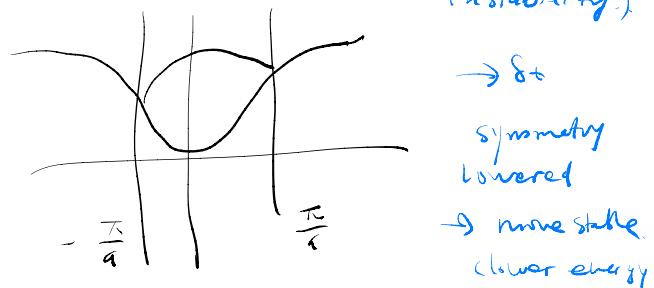
1D tight-binding model:

$$H = \sum_{\text{unit cell}} t_1 c_{iA}^\dagger c_{iB} + t_2 c_{iB}^\dagger c_{iA} + \text{h.c.}$$



(not dimerized)

$t_1 = t_2$ : 1D metal. (Pairing instability.)



$\rightarrow$  stable

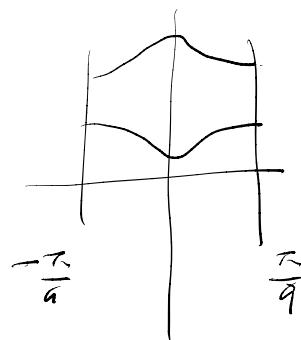
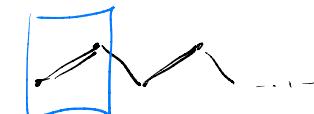
symmetry lowered

$\rightarrow$  more stable  
lower energy

(dimerize)

$t_1 \neq t_2$  open the gap: insulator

$$\left. \begin{cases} t_1 > t_2 \\ t_1 < t_2 \end{cases} \right\}$$



unit cell } Strong bond:  $P \equiv 0$   
weak bond:  $P \neq 0$

$$H(\mathbf{k}) = \sum_{\mathbf{k}} C_{\mathbf{k}B}^+ (t_1 + t_2 e^{i\mathbf{k}\mathbf{a}}) + h.c.$$

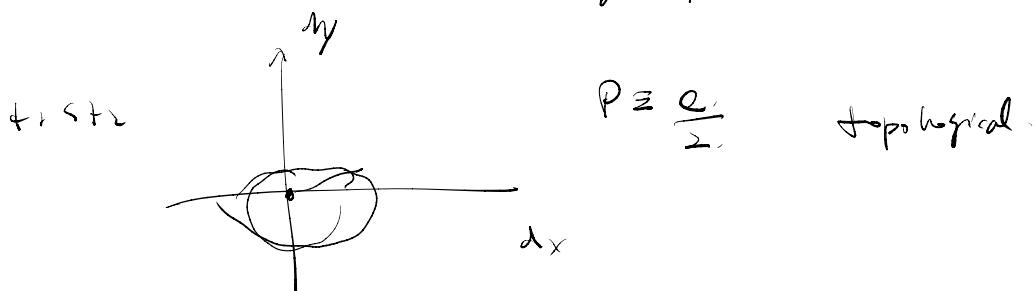
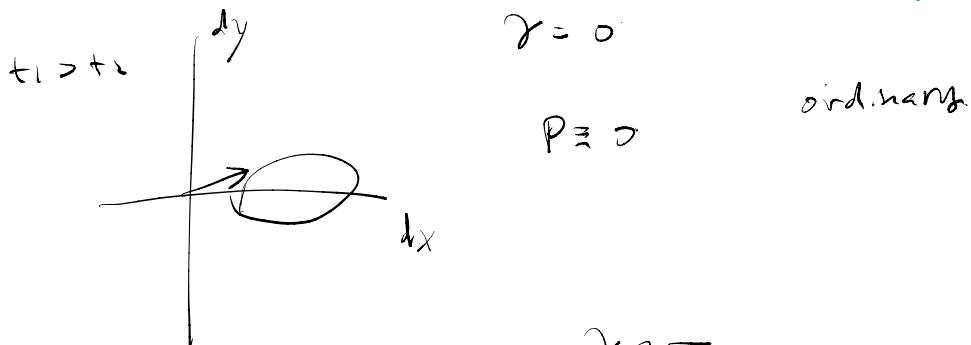
↑

Fourier transform  $= \sum_{\mathbf{k}} h_{ab}(\mathbf{k}) C_{ka}^+ C_{kb}$

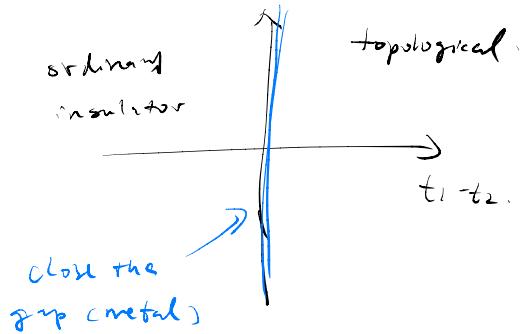
$a.b = A.B$

$$h(\mathbf{k}) = \vec{d}(\mathbf{k}) \cdot \vec{\sigma}$$

$$\left\{ \begin{array}{l} d_x = t_1 + t_2 \cos k_a \\ dy = -t_2 \sin k_a \\ \boxed{d_2 > 0} \end{array} \right. \Rightarrow \text{lead to quantization of } \gamma \text{ protected by } \mathcal{F}$$



phase transition :



Comment :

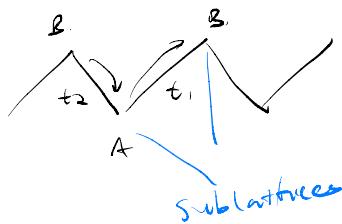
1. the choice of unit cell is also a choice of gauge ( $t_1 + t_2 e^{i\phi}$ )

this choice makes  $b(\mathbf{k})$  periodic in  $B_2$ .

but this is not necessary, since it's up to gauge transform

2.  $d_2(\mathbf{k}) \equiv 0$  is protected by symmetry.

Symmetries:



$$H(\mathbf{k}) = \vec{d}(\mathbf{k}) \cdot \vec{\sigma}$$
$$\underline{d_z = 0} \quad \begin{cases} d_x = t_1 + t_2 \cos k_x \\ d_y = -t_2 \sin k_x \end{cases}$$

↓ anti-commuting.

- chiral symmetry:  $\{ H(\mathbf{k}), \underline{S_z} \} = 0$

$$(d_z = 0)$$

net constraint to one sublattice

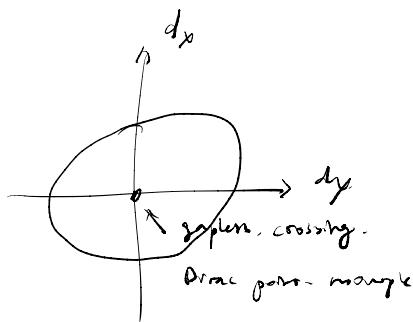
+ the transformation:

works for  
one body

$$\begin{cases} c_A \rightarrow c_A \\ c_B \rightarrow -c_B \end{cases}$$

↑  
change of phase.

this is not a symmetry  
if  $d_z \neq 0$



topological invariant:

winding number,

$$= 0 \text{ or } 1$$

$$\frac{D=1, d_z \neq 0}{\text{chiral symmetry}}$$

only defined  
with  $\hat{S}$  symmetry

↔ Berry phase

$$0 \text{ or } \pi$$

particle-hole symmetric spectrum:

chiral symmetry  $\rightarrow$   $\underline{G_2(\gamma_E) \propto (\gamma_{-E})}$

\* not particle-hole symmetry in  $H_{BdG}$

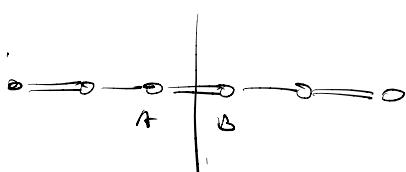
analog with:

Dirac eqn:  $\{\gamma^5, \gamma^\mu\} = 0$

↑  
chiral symmetry generator -

(this is the real case, ssm model is toy)

• Reflection symmetry:



$d_2 \text{ may } \neq 0$   
(no chiral symmetry)

Mirror exchanging A, B sublattices

$\Rightarrow$  constraint on  $H$ :

$$H(-k) = \sigma_x H(k) \sigma_x$$

$$\Rightarrow \begin{cases} d_x(k) = d_x(-k) \\ d_{y,z}(k) = -d_{y,z}(-k) \end{cases}$$

$\Rightarrow \mathbb{Z}_2$  invariant

group

mirror

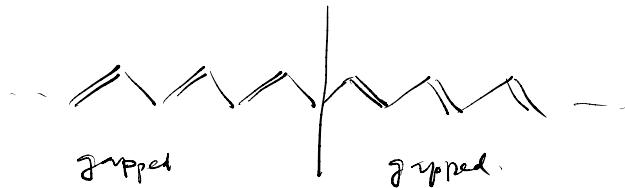
physical interpretation:  $P = -P \pmod{e}$

$$\mathbb{Z}_2 \left\{ \begin{array}{l} P=0 \\ P=\frac{e}{2} \end{array} \right.$$

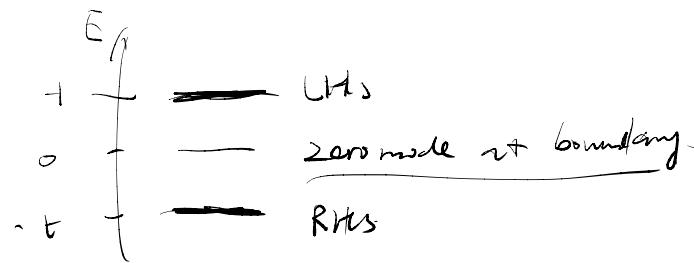
Consequences of bulk topologys

boundary = Domain Walls

• SSVF =



dimerization :  $t_{\text{strong}} = t$ ,  $t_{\text{weak}} = 0$   
( limit )



implications of chiral symmetry :

$$E(k) \leftrightarrow -E(k)$$

$$\text{zero mode} : E(k) = -E(k) \quad \text{special}$$

even if  $t_{\text{min}}$  is turned on, this zero mode  
is still stable (protected by  $\hat{S}$ ).

this or proceed as long as the gap is not closed.

gap  $\rightarrow$  correlation : states inside the length. gap decay  $e^{-\frac{L}{\text{gap}}}$

$\Rightarrow$  assumption of boundary condition

chain Length  $\Rightarrow$  correlation.

Bound charge at Domain Wall

$$\frac{e}{z} + \sigma = \frac{e}{z}$$

$$2 \times \frac{e}{z}$$

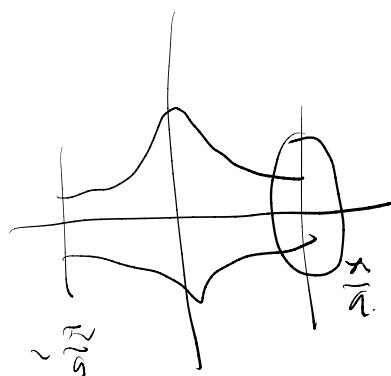
real case (spin-doubling)  $\Rightarrow e$

when gap is small, we can form

low-energy continuous QFT description

low energy theory

$$t_1 - t_2 \ll t_i$$



$$k = \frac{\pi}{a} \approx q$$

$$\left\{ \begin{array}{l} dx \gg t_1 - t_2 \sim 0 \\ dy = +t_2 a q \\ dz = 0 \end{array} \right.$$

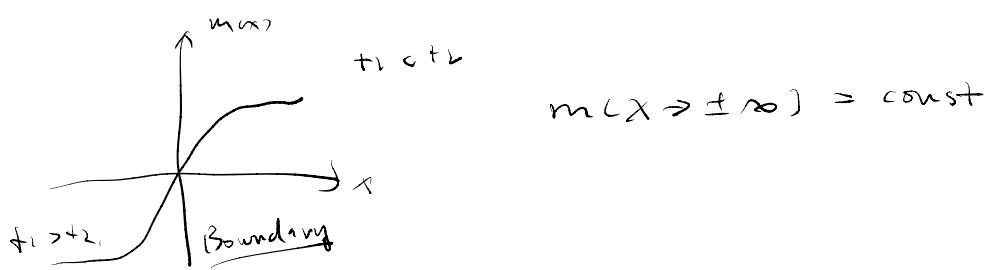
$$H_{\text{FQD}} = \underbrace{v_F q \sigma_y}_{v_F \gg t_2 a} + m \sigma_x$$

$$v_F \gg t_2 a \quad m = t_1 - t_2$$

$$E_{(g)} = \pm \sqrt{v_F^2 q^2 + m^2}$$

energy FQP

Domain Wall (Jackiw & Rebbi) :



$$m(x \rightarrow \pm \infty) = \text{const}$$

$$\hat{g} \rightarrow -i \frac{\partial}{\partial x}$$

$$H = -i v_F \sigma_y \partial_x + m \sigma_x$$

k-p approximation

zero mode.

$$(-i v_F \sigma_y \partial_x + m \sigma_x) \psi_{0x} \Rightarrow$$

↑

two-component  
wave function.

(describing bound charge).



$$\sigma_z = \pm 1$$

$$\psi_{\pm} = e^{\mp \int_0^x \frac{m \sigma_z}{v_F} dx} \quad \phi_{\pm}$$

$$\phi_{\pm} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$m \sigma_z$  have right sign for

$\psi_{\pm}$  to be normalizable.

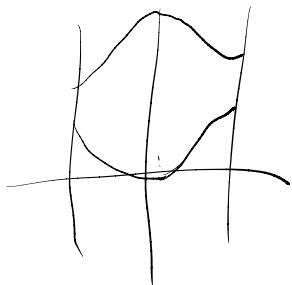
$$\left\{ \begin{array}{l} \nearrow \psi_+ \\ \searrow \psi_- \end{array} \right.$$

## • Thouless charge pump

Adiabatic cycle of no insulator.

$$H(\mathbf{k}, t+T) = H(\mathbf{k}, t)$$

Nearly free electron



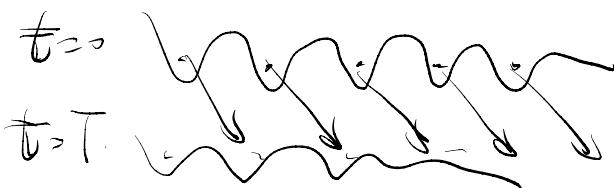
periodic potential

$$V_G = V_0 e^{i \phi_{kz}} \quad \phi_{(t+T)} \\ - \phi_{(t)} = 0$$

$$\text{mod } 2\pi$$

$$V_G \stackrel{+}{\leftarrow} C_{k+G} C_{k+G}$$

• electron is locked inside the potential.



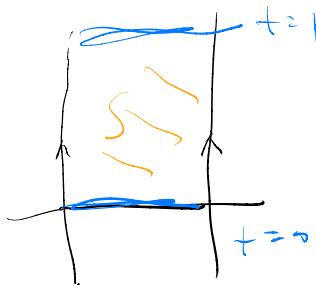
changed the  
polarization.

$$P \Rightarrow \rightarrow P = e$$

topological.

For any periodic cycle  $\Delta P = ne$

$$\frac{n \in \mathbb{Z}}{\downarrow}$$



$$\Delta P = \frac{e}{2\pi} \left[ \oint_A (A(k, \tau) dk - \oint_{A(k, 0)} dk) \right]$$

$$= \frac{e}{2\pi} \int_S dt \cdot dk F$$

$$H(k, t) \text{ periodic} \Rightarrow \boxed{(k, \tau) = \tau^2}$$

$$|u(k, \tau)\rangle = e^{i\langle k \rangle} |u(k, 0)\rangle$$

$$\oint (A(k, \tau) - A(k, 0)) dk = \underbrace{\oint 2\pi \tau dk}_{\text{winding number}} = 2\pi n$$

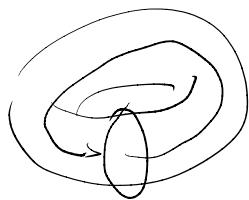
= Chern-number

$$n = \frac{1}{2\pi} \int_{T^2} F \in \mathbb{Z}$$

chern  
number

obstruction to define eigenvector

smoothly through  $T^2$ .  
(continuous) gauge.



1D there's no

large gauge transform

thus no problems

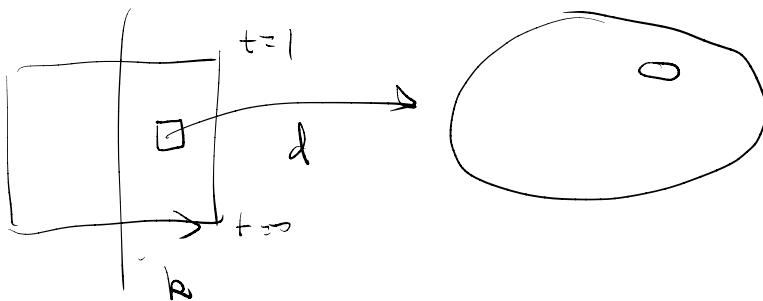
of choosing

continuous gauge

solving: large gauge transformation

→ 2 level system

$H(k, t)$ .



$$\text{Berry phase} \sim \frac{\int dk dt}{\text{solid angle}} = \frac{i}{2} \underbrace{\int dk dt}_{=}$$

$$\equiv \frac{1}{2} d \cdot (\hat{\sigma}_k \hat{d} \times \hat{\sigma}_t \hat{d}) dk dt$$

$$h = \frac{1}{2\pi} \int F \delta k dt = \frac{1}{4\pi} \int \underbrace{\hat{d}(\partial_k \hat{d} \times \partial_d \hat{d})}_{dk dt} \overbrace{dt}^{\Omega_{\text{total}}}$$

wrapping number:

$$w(d) \quad d: T^2 \rightarrow S^2$$

Different wrapping numbers are topologically distinct ( $H(k, t) \rightarrow w(d)$ ).

Chern number is more general.

For 2 band system, it can be interpreted as  $w(d)$ .

# IQHE

$$\sigma_{xy} \neq 0$$

$$\vec{J} = \sigma_{xy} \hat{z} \times \vec{E}$$

Take  $R \rightarrow \infty$ , the surface is 2D IQHE.  $B$  is perturbative

Laughlin argument?

$$E_y = \frac{d\phi}{dt}$$



$$\phi(t=0) = 0, \phi(t=T) = \frac{\hbar}{e} = \phi_0$$

like Thouless charge pump

For  $\phi_0$ , the spectrum will be the same but the state will be "transported"

$\underline{\Phi} = \phi_0$  can be eliminated by

large gauge transformation

in real space

$$\psi_m \rightarrow \psi_m e^{i\theta(r)}$$

$$A_{em} \rightarrow A_{em} + \frac{i}{e} \nabla_r \theta(r)$$

$$\underline{\Phi} = \oint A \cdot d\vec{r} \rightarrow \underline{\Phi} + \frac{2\pi i}{e} \frac{\phi_0}{\epsilon}$$

$$\Rightarrow H(t) = U^\dagger H(0) U$$

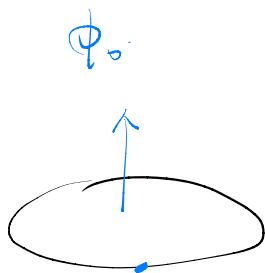
↑  
Unitary  
gauge transf.

$$\Delta P = \int d\tau I_{WJ} = \sigma_{xy} \int d\tau \frac{d\phi}{d\tau}$$

$$= \sigma_{xy} \frac{\hbar}{e} = ne.$$

$$\sigma_{xy} = \frac{e^2}{h} n$$

↑  
Chern number.



$\Rightarrow$  same spectrum  
but the state are  
changed (gauge transformed)

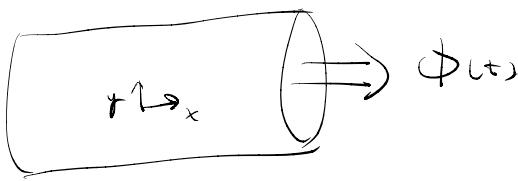
topological degeneracy

\* FQH:  $n \notin \mathbb{Z}$ , the flaws:

that the ground state is not unique

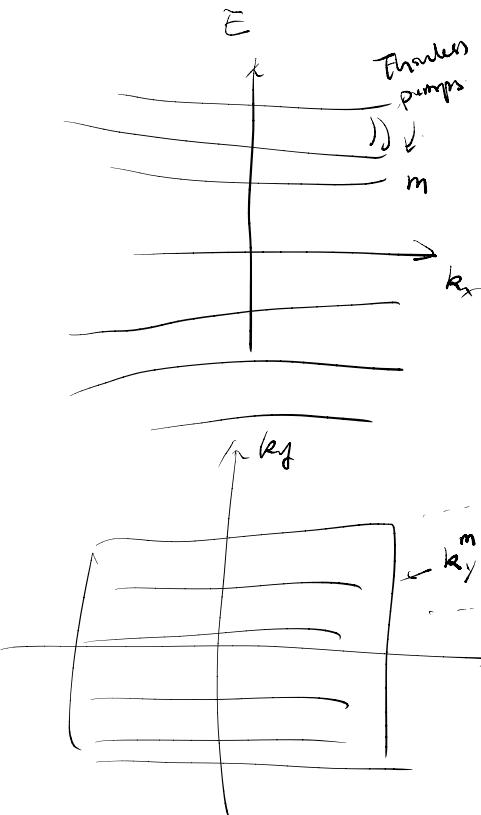
In band theory:  $G.S = \text{Filled valence bands}$

↑ adiabatic  
with charge  
states.



on the surface : 2D band structure  $H(k_x, k_y)$   
 (not necessarily  
 Landau levels)

$$k_y = \frac{1}{R} \left( m + \frac{\phi}{\phi_0} \right) \quad \begin{matrix} \leftarrow \text{changing boundary condition} \\ \text{shift of state} \end{matrix}$$



like Thouless ( $\rightarrow$ )

$$\Delta Q = \frac{e}{2\pi} \int_{\text{bands}} dk_x \int_{B_2} d\phi F(k_x, \phi).$$

$$= \frac{e}{2\pi} \int_{B_2} dk_x dk_y F(k_x, k_y)$$

$\Rightarrow Qn$

$\dagger$

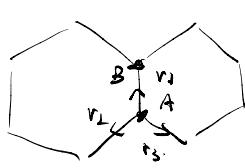
Chem. number

characterizes 2D gapped  
 band structure

Kubo

$\rightarrow \sigma_{xy}$   
formula

- Model system: Graphene



$$H = -t \sum_{\langle r_{ij} \rangle} c_{iA}^+ c_{jB} + \sum_{\mathbf{k}} c_{ka}^+ c_{kb} h_{ab}(\mathbf{k})$$

$$h_{ab} = \begin{pmatrix} 0 & \sum_i e^{i\mathbf{k} \cdot \mathbf{r}_i} \\ \sum_i e^{-i\mathbf{k} \cdot \mathbf{r}_i} & 0 \end{pmatrix} = \vec{d} \cdot \vec{\sigma}$$

$$\left\{ \begin{array}{l} dx = -t \sum_i \cos \mathbf{k} \cdot \mathbf{r}_i \\ dy = -t \sum_j \sin \mathbf{k} \cdot \mathbf{r}_j \\ dz = 0 \end{array} \right.$$

Symmetry:

2D



1. chiral symmetry:  $\{H, \sigma_2\} = 0$

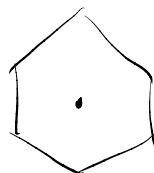


not real (toy model)

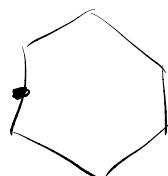
real graphene can have

2<sup>nd</sup> Hopfing terms, violating  
this symmetry

2. inversion: P:



or



interchanging sublattices

$$\Rightarrow H(-k) = \sigma_x H(k) \sigma_x$$

3. time reversal:

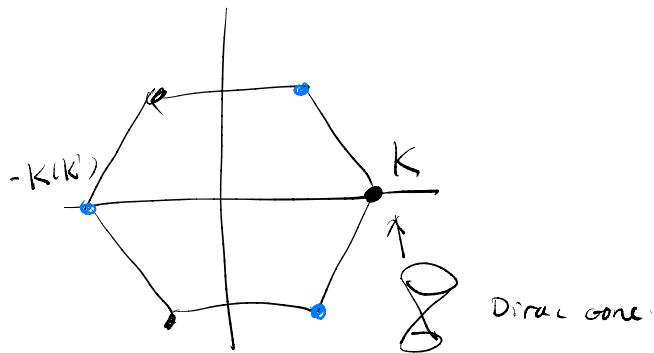
toy model ignore spin  $T \sim \text{complex } \star$

$$H(-k) = H(k)^*$$

PT:  $H(k) \rightarrow \sigma_x H(k)^* \sigma_x \Rightarrow d_8 = 0$

Allow point defects in  $k$  space.

$\Rightarrow$  Dirac point



Dirac Point :  $\vec{d}(\pm k) \Rightarrow$  singularity  
 $\sim$  wrapping  $2\pi$   
 $\sim \gamma = \pi$

Low energy theory -

$$E(\pm k + g) = v_F (\pm \sigma_x g_x + \sigma_y g_y)$$

$C_3$  symmetry force Dirac points to be

at corners

(protected Dirac points)

changing  $\begin{pmatrix} +1 \\ +3 \\ -2 \end{pmatrix}$  will move Dirac point around

but not opening the gap

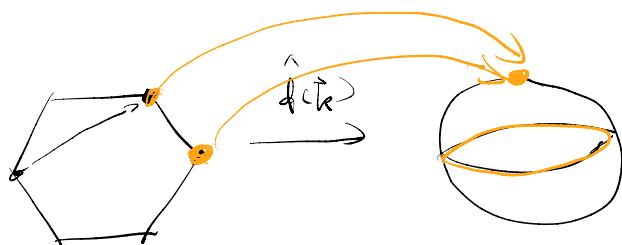
Lowering symmetries to open gap:

$$\underbrace{+ m \pm \sigma_2 \text{ term.}} \quad E(g) = \pm \sqrt{v_F^2 g^2 + m_{\pm}^2}$$

Chern number:

Degree of  $\frac{\partial \psi(x, k_y)}{\partial k_x}$  on  $S^2$   
 $\frac{T^2}{T^2}$ .

1. break P:  $\Delta H = m \sigma_2 \quad \underline{m_+ = m_-}$   
e.g. BN

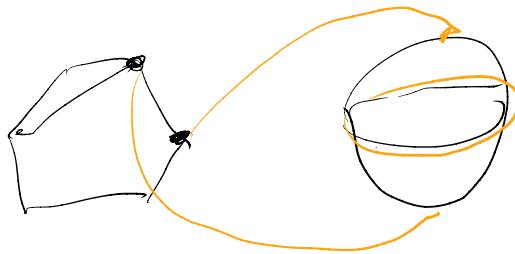


Degree = 0      trivial  
insulator

2. break  $T$ :

Haldane model:

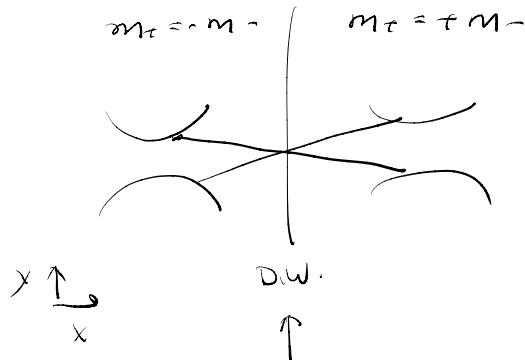
$$m_+ = -m_-$$



$$\text{Degree} = \pm 1 \quad n=1$$

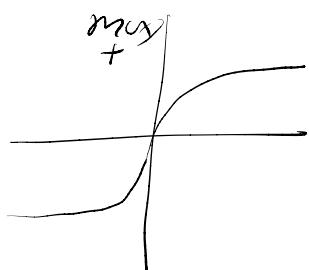
I  $\otimes$  H.

Edge states :



$$H = v_F \left( -i\sigma_x \partial_x + b_y \sigma_y \right) + m(x) \sigma_z$$

closing band  
band inversion

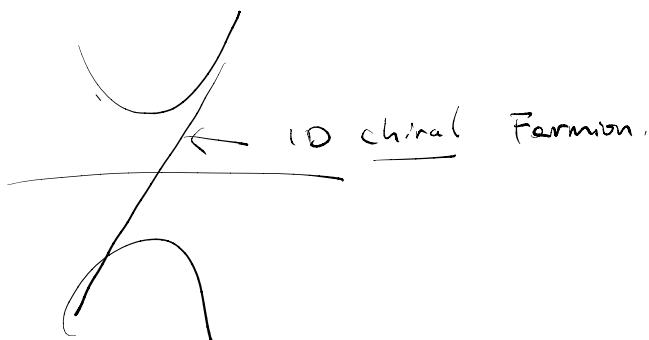


Same as Jackiw Rebbi

$$|\psi_{k_0} \rangle = e^{i k_0 x} e^{i \int_0^x \frac{m(x')}{v} dx'} |\sigma_y + \rangle$$

zero modes eigenstate of  $\sigma_y$

$$H|\psi\rangle = v_F k_y |\psi\rangle$$



Fermion Doubling theorem

chiral states on boundary is separated

thus interesting

• 4D QM E

$$\left\{ \begin{array}{l} A_{ij} = \langle u_i | \nabla_k | u_j \rangle dk \\ F = dA + A \wedge A \end{array} \right.$$

$$n = \frac{1}{8\pi^2} \int_{T^4} \text{Tr}[F \wedge F]$$

invariant  
under  $\mathcal{F}$

2nd. Chern number.  
(no 4-forms in  
less than 4-dim!)

boundary:

3+1 D chiral Dirac fermion,  
(Weyl points).

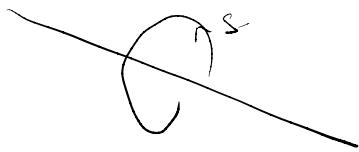
$$\int_S \text{Tr}[F \wedge F] = \int_S dQ_3 = \int_{\partial S} Q_3$$

Chern-Simons 3-form,

## Topological defects

Band structure varies slowly in space.

$$H(\vec{q}, s) \rightarrow \text{Land Chern number.}$$



← chiral fermion along  
topological defect

Energy gap in graphene :

$$H = v (\sigma_x \tau_2 g_x + \sigma_y g_y) + V$$

↑    ↑  
Valley  $\pm K$                               involve  $\sigma_2$  to  
open the gap at Dirac pt

1.  $V = m_I \sigma_2 \Rightarrow$  break P. trivial insulator

$$\underline{m_K = m_{-K}}$$

2.  $V = m_{II} \sigma_2 \tau_2 \Rightarrow$  IQHE break T

3. spin-orbit coupling: (<sup>(in real graphene, this is very small)</sup>)

$$V = m_s \sigma_2 \tau_2 S_z \quad \underline{\text{respect all symmetries}}$$

$\uparrow$  1st order degenerate perturbation.

Microscopic :  $U_{SC} = (\vec{s} \times \vec{p}) \cdot \vec{\tilde{E}}$

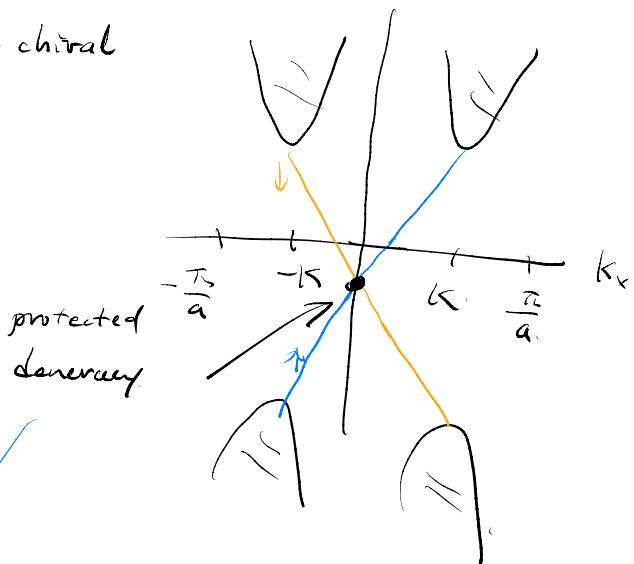
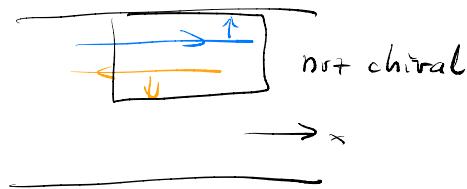
$$\vec{\tilde{E}} = \nabla U_{atom}$$

$S_z$  is conserved

$\Rightarrow$  Haldane model doubled.

$$H = \begin{pmatrix} H_\uparrow & \\ & H_\downarrow \end{pmatrix} = \begin{pmatrix} H_{\text{Haldane}} & \xrightarrow{\sigma_{xy} > 0} \\ \xleftarrow{\sigma_{xy} < 0} & H_{\text{Haldane}}^\dagger \end{pmatrix}$$

No Hall current, but have spin current.  
( $S_z$  is conserved)



even if  $S_z$  conservation  
is violated by perturbation  
(spin orbit)

but this degeneracy is protected  
from T.

T-symmetry:

$$\textcircled{H} \quad \psi = e^{i\pi S_y} \psi^*$$

spin  $\frac{1}{2}$  :

$$\textcircled{H} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \begin{pmatrix} \psi_\downarrow^* \\ -\psi_\uparrow^* \end{pmatrix}$$

$$\Rightarrow \textcircled{H}^2 = -\mathbb{1}$$

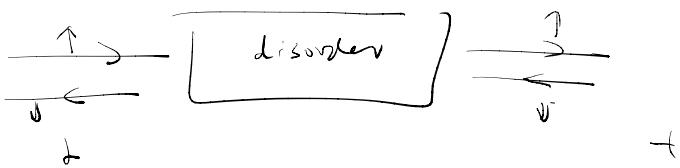
back scattering forbidden by T:

reverse  $S_z \Rightarrow$  inverse  $S_z$

which is not invariant  
under T

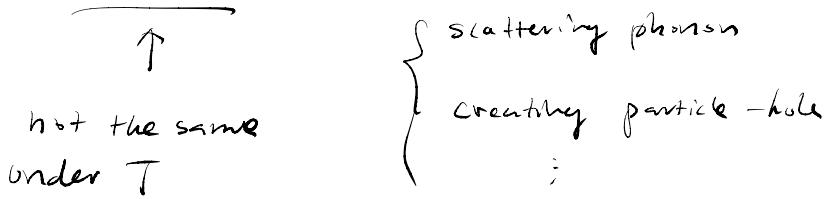
All states are extended even for strong disorder.

Elastic ( $T=0$ )



$r$  is odd under T  $\Rightarrow r = 0$

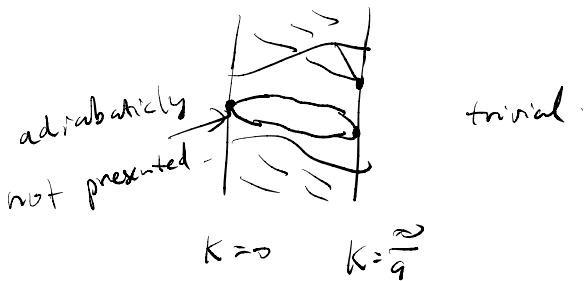
For  $T \neq 0$  inelastic scattering can happen:



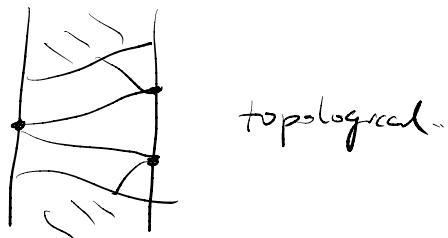
For  $\underline{H_L(k) = \bigcirc H(k) \bigcirc^{-1}}$  constraint,

+ there's  $\underline{\mathbb{Z}_2 \text{ top. inv.}}$

the two possibility:



trivial

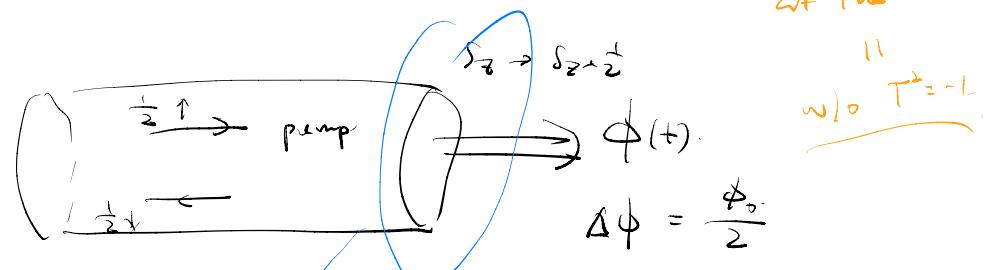


crossing even  
with  
 $\tilde{E}_F$

odd

variant on Laughlin:

change in Fermion parity  
at the end

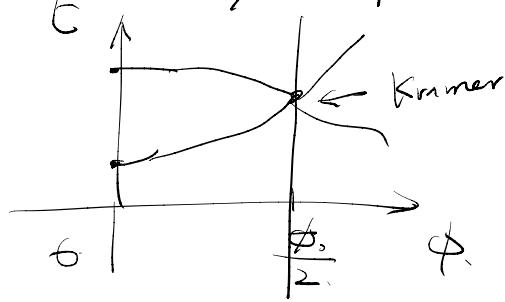


w/o  $T^2 = -L$

$$S_z \rightarrow S_z - \frac{1}{2}$$

For  $\phi = 0$  or  $\frac{\phi_0}{2}$ , there's  
time reversal symmetry

Many body spectrum



$\uparrow$   
no Kramer  $\underline{\text{spin}=\frac{1}{2}}$

$\underline{\text{spin}=\frac{1}{2}}$

change in Local fermion parity

trivial case :

$$\overbrace{\quad}^{\uparrow \downarrow} \Rightarrow \begin{cases} \text{no spin change} \\ \text{at boundary} \end{cases}$$

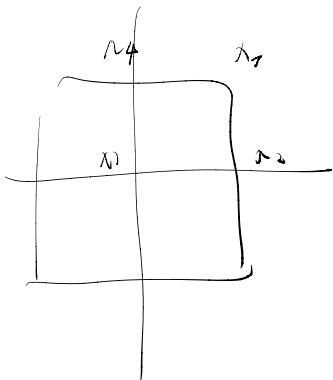
physical in bulk : computation of  $\mathbb{Z}_2$  inv.

1.  $\{u_{n(k)}\}$  ( $N$  bands) - Chem number = 0  
 $\Rightarrow \{u_{n(k)}\}$  defined  
continuously through  $B_2$   
(gauge)

To calculate on computer, we need to fix a continuous gauge.

$$W_{mn}(k) \equiv \langle u_{mk} | \textcircled{H} | u_{nl-k} \rangle \in U(N)$$

$$\textcircled{H}^2 = -1 \Rightarrow W(k) = -W(-k)^T$$



For  $k = \lambda_a$   $W(\lambda_a) = -W(\lambda_a)$

$$\text{Pfaffian} : \det \begin{bmatrix} W(\lambda_a) \end{bmatrix} = (\text{Pf}[W(\lambda_a)])^2$$

$$S(\lambda_a) = \frac{\text{Pf}[W(\lambda_a)]}{\sqrt{\det(W(\lambda_a))}} = \pm 1$$

choice of branch  $\sqrt{\cdot}$  is fixed globally.

even  $\prod_a S(\lambda_a)$  is independent of  $\sqrt{\cdot}$  ambiguity

but not gauge invariant under 'Large'

gauge transformation.

symmetry to P

$$v = \frac{4}{\pi} S(\lambda_a) = \pm 1$$

$a=1$   
↑

in 2D, there's always  $4 \lambda$ .

with symmetries:  $v$  is easier to calculate.

- $S_2$  conservation :  $\sum$  Chern number even or odd
- P :  $S(\lambda_a) = \prod_m \delta_m$   
 ↑  
 parity eigenvalues of states.

3D Topological insulator

