CFT (10 Lectures)

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Motivation

I. provide good effective description of the system at or near criticality (2nd phase transition)

e.g. Long-range correlations.

scaling invariance

II. the only true QFTs (Cut-off independent)

"All" QFTs are perturbations of CFT

(the perspective from RG)

$\boxed{\text{CFT correspond to RG fixed point}}$

III. CFT can be made by & large rigorous

at least at 1+1 d. (Quantum fluctuation very drastic)

* Note, there're now 3 mathematical formulations

haven't unified yet!

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§ What's CFT

- It's quantum field theory (local)
- Invariant under the symmetry group:

conformal group

classical: leave S invariant (eqn)

Quantum:

Wigner's theorem:

Transformation ~ (anti)unitary operators

the $\hat{O}$ commute with $\hat{H}$

A Quantum theory is symmetric under G $\Updownarrow$ Furnish projective representation of G QM: projective $\mathcal{H}$ Group cohomology & extension

§ Conformal transformations in d dimensions.

Flat

Def $M = \mathbb{R}^{p,q}$, $\mathbb{R}^{d=p+q}$ with metric $g_{\mu\nu} = \mathrm{diag}(\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q})$

$= \eta_{\mu\nu}$

* AdS/CFT have something to do with $q=2$.

- Conformal transf. leave the metric invariant up to a <u>scale factor</u> ~ angles.

$$M \ni x \xmapsto[\text{transf}]{\text{conformal}} x'(x)$$

<u>smooth</u> change of coordinates

note this is a <u>global</u> coordinate system.

such that, $g_{\mu\nu} \xmapsto[\text{transf.}]{\text{conformal}} g'_{\mu\nu}(x') = \dfrac{\partial x^\alpha}{\partial x'^\mu}\dfrac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$

$\propto \eta_{\mu\nu} \quad (= g_{\mu\nu})$

$$\Longrightarrow \quad \underline{g'_{\mu\nu}(x') = \Omega(x)\, g_{\mu\nu}(x), \quad \Omega(x) > 0}$$

avoid singularity.

though distoring the coordinate system, the angles are preserved by conformal transf.

$$\angle\theta = \frac{u\cdot v}{|u|\,|v|} = \frac{u\cdot v}{\sqrt{u^2 v^2}}$$

$$u\cdot v = g(u,v) = g_{\mu\nu}u^\mu v^\nu$$

Def. the conformal grp $\mathrm{Conf}(M)$

the connected component containing $\mathbb{1}$ (identity) is the grp of all conformal transfs. of M.

allow infinitesemal version
not including ones linked by
P, T etc. (Lie algebra).

* Note, connectedness of group of transfs. are usualy in compact-open cases.

* Note the definition of general M, i.e. including (pesudo-)Riemanian manifold is straight forward. $g_{\mu\nu} \rightarrow$ a varying tensor field.

QM:

$$G \xrightarrow{\text{representation}} \Pi(G)$$

$\mathcal{H}$

over projective Hilbert space

(projective) unitary representation

The interesting part of CFT is finding these PUIRs in local QFTs

making things complicated.

Note, most of (may be all) PUIRs of Conf(M) have been found and classified by mathematician. at least for specific M.

tension between $\begin{cases} \text{Locality} \\ \text{Unitarity.} \end{cases}$

Nontrivial physics:

The hard part is to encode the UIRs using local constructions, like what happens to relativistic field theory and Lorentz group / Poincaré group.

§ Classification of $G = \mathrm{Conf}(M = \mathbb{R}^{p,q})$ ⟶ flat here.

For large p,q. the G usually turns out to be ordinary Lie groups. but for small p,q like 2d case, things are complicate and interesting

Not all will be Lie algebra.

↓

Using infinitesimal arguement (local algebra):

$$x \mapsto x + \varepsilon(x) \longrightarrow O(\mathbb{1})$$

condition for $\varepsilon(x)$:

$g_{\mu\nu}$ independent of x (flat).

↓

$$g'_{\mu\nu} = g_{\mu\nu} + (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) + O(\varepsilon^2).$$

$$= \Omega(x) g_{\mu\nu} \propto \eta_{\mu\nu} \text{ (flat).}$$

$$\Longrightarrow \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \propto \eta_{\mu\nu}$$

take trace $\longrightarrow$ $\boxed{\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}(\partial \cdot \varepsilon)\eta_{\mu\nu}}$ $(*)$

$$\Longrightarrow \quad \Omega(x) = 1 + \frac{2}{d}(\partial \cdot \varepsilon) + O(\varepsilon^2).$$

d=2 is special.

It can be verified that from (*)

$$(**) \qquad \boxed{\left(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu\right)(\partial\cdot\varepsilon) = 0} \qquad \Box = \partial^2 = \eta_{\mu\nu}\partial^\mu\partial^\nu$$

d=2 less constraining.

Note, $\varepsilon(x)$ is C^∞ (smooth)

Q: Are these two condition sufficient to classify all ε ?

Classification due to (**) & (*) : (of $\varepsilon(x)$).

d>2 :

(*), (**) $\Longrightarrow$ all 3rd derivatives vanishes

$\Longrightarrow$ $\varepsilon(x)$ is at most quadratic in x

(Global) Poincare

$\underline{a}$. $\varepsilon^\mu = a^\mu$ constants. <u>spacetime translations</u>.

including <u>boosts</u>

linear in x: $\underline{b}$. $\varepsilon^\mu = \omega^\mu{}_\nu x^\nu$ $\omega^{\mu\nu}$ <u>antisymmetric</u> "<u>rotations</u>"

linear in x: $\underline{c}$. $\varepsilon^\mu = \underline{\lambda} x^\mu$ $(\lambda > 0)$ <u>scale transformations</u>.

constant (linear in x)

quadratic in x $\underline{d}$.

$$\varepsilon^\mu = b^\mu x^2 - 2x^\mu (b\cdot x)$$

<u>special conformal transformation</u>. (SCT)

(rigid/global)

<u>Theorem</u>. $\forall$ conformal transf. φ acts on connected subspace $U \subset \mathbb{R}^{p+q}$ $\underline{d=p+q>2}$, is a composition of <u>exponentials</u> (<u>Finite</u> opposed to infinitesimal) of $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ types. (generators)

— <u>translation</u>: $x \to x + c$ $c \in \mathbb{R}^d$

— <u>orthogonal transf</u>: $x \to \Lambda x$ $\Lambda \in O(p,q)$ ($SO^+(p,q)$ actually)

— <u>dilations</u>: $x \to \lambda x$. $\lambda \in \mathbb{R}^+$

— <u>special conformal transf:</u>

$$x^\mu \longrightarrow \frac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2 x^2} \qquad b \in \mathbb{R}^d$$

$\mu = 1, \ldots, d$

0

these SCT bring ∞ to some finite point and some finite point to ∞ (according to b^μ).

Rmk.

∞ as an extra point

By compactification $\mathbb{R}^{p,q} \cup \{\infty\}$. we get a group, else some SCT is not allowed (ill-defined)

*Note, conformally infinity is interesting topic of (general) relativity. and AdS/CFT correspondence.

$d=2$

For $\underline{g_{\mu\nu} = \delta_{\mu\nu}}$: ($= \mathrm{diag}(+1, +1)$)

$(*)$ became the Cauchy-Riemann eqns : analytic functions.

$$\partial_1 \varepsilon_1 = \partial_2 \varepsilon_2$$

$$\& \quad \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1$$

complex coordinates : $z = x^1 + i x^2$

$$\varepsilon(z) = \varepsilon^1 + i \varepsilon^2$$

looks like infinitesimal analytic functions.

(much more possibilities than $d>2$)

! not the same meaning in gauge theory

(finite, no singularity when exponentiated)

$\Rightarrow$ 2d global conformal transformations. (after exponentiating)

(extend to finite distance from id, not infinitesimal).

corresponds to entire holomorphic functions.

(differentiable functions without singularity)

$z \longmapsto f(z)$ with holomorphic inverse $f^{-1}(z)$

to form a group.

they have form : $\boxed{f(z) = \alpha z + \beta.}$

There're more transformations allowed locally (near id) (can't be exponentiated)

singular transf. → exponentiated)
(having poles when exponentiated)

Q: Is singularity in finite transf related to anomaly?

The 2d theories having global conformal symmetry is richer than locally conformal invariant ones since there're less symmetries thus less constraints.

In another perspective, locally conformal invariant 2d theories are highly constraint

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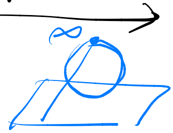
e.g.

$$\mathbb{R}^{2,0} \xrightarrow{\sim} \mathbb{C} \xrightarrow{\text{compactification}} \mathbb{C}\cup\{\infty\} = S^2$$

Riemann sphere.



global: $\mathrm{Conf}(\mathbb{C})$
$= \{$ linear complex variable functions $\}$

Larger global $\mathrm{conf}(M)$

$z \in S^2$

$= \mathrm{Diff}(S^2)$

$$\mathrm{Conf}(\mathbb{C}\cup\{\infty\}) = \left\{ f(z) \,\middle|\, f(z) = \frac{\alpha z+\beta}{\gamma z+\delta},\ \alpha,\beta,\gamma,\delta\in\mathbb{C},\ \alpha\delta-\beta\gamma\neq 0 \right\}$$

= group of Möbius transformations

§ More on $d=2$ case: local conformal transformations (infinitesimal) singular finite ones.

Local is less constrained since it only ask it be be conformal transfs up to $O(\varepsilon)$. thus there's more "vector field" $X : \mathbb{R}^{2,0} \to \mathbb{R}^{2,0}$

(like generators in Lie algebra)

$$\phi = id + \underline{\varepsilon} X : \mathbb{R}^{2,0} \longrightarrow \mathbb{R}^{2,0}$$

need not to be well-defined on all the manifold of finite transfs. (upon exponentiation)

up to all orders $O(\varepsilon^k)$

(these doesn't form a group! algebra!)

usually, we'll have ∞-dim algebras near $\mathbb{1}$

For infinitesimal conformal transf. we $\boxed{\text{focus on the algebras}}$ rather than groups.

(usually ∞-dim algebras)

(infinitesimally, only for continuous groups).

conformal algebra is not (not all can be exp).

like G = Lie group $\longrightarrow$ Lie algebra $\mathfrak{g}$

rep of $\mathfrak{g}$: $(\mathcal{H}, \pi : \mathfrak{g} \longrightarrow \mathcal{L}(\mathcal{H}))$

$\mathcal{L}(\mathcal{H})$: linear operators, not necessarily bounded especially when $\mathcal{H}$ is ∞-dim.

$\downarrow$ exp

$\exp(iX\theta) \Rightarrow \pi(g \in G) = \mathcal{U}(\mathcal{H})$

X: elements in $\mathfrak{g}$

$\mathcal{U}(\mathcal{H})$: unitary, since $\mathbb{1}$ is

This picture already assume well-defined G, but for local conformal transfs. there's more in algebra. (infinitesimal)

$\mathfrak{g}$ ∞ dimensional : $\exp(isX)$ may not be continuous in s at some points.

Work with "Lie" algebra (infinitesimal transf), rather than finite group (not well-defined for all generators)

(no group of finite transf).

§ local algebra : $(1+1d)$

$$z = x^1 + i x^2$$

$$\varepsilon = \varepsilon_1 + i\varepsilon_2$$

Global : $z \overset{\text{holomorphic}}{\xmapsto{f}} f(z)$

local : $z \longmapsto z + \varepsilon(z)$ $\quad \varepsilon$ is holomorphic

$\bar{z} \underset{\phi}{\longmapsto} \bar{z} + \bar{\varepsilon}(\bar{z})$ $\quad$ i.e. ε depends only on z, not $\bar{z}$

A convenient basis : $\varepsilon_n(z) = -\varepsilon z^{n+1}$, $n \in \mathbb{Z}$, $\varepsilon \sim O(1)$

$\uparrow$ ∞-dim.

Corresponding vector field:

vector field $\in T_z(M = \mathbb{C})$

$$\phi(z) (\equiv z + \varepsilon_n(z)) = \exp(\epsilon X)$$

diffeomorphism of $\mathbb{C}$

$\Longrightarrow X(z) = l_n = -z^{n+1}\partial_z \,,\quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$

depend on z

Differential operator / field representation of the (Lie) algebra

(like $p_\mu \to -i\partial_\mu$)

the Lie brackets :

$$[l_n, l_m] = (m-n)\, l_{m+n}$$

$\forall m, n \in \mathbb{Z}$ $\quad [\bar{l}_n, \bar{l}_m] = (m-n)\, \bar{l}_{m+n}$

∞-dim Lie algebra.

$[\bar{l}_n, l_n] = 0$

∞-dim Lie algebra is more subtle & consequential.

this is called the Witt algebra : $\boxed{\underline{\text{Witt}} = A \oplus \bar{A}}$

$A \to \text{Span}\{l_n\}$, $\bar{A} \to \{\bar{l}_n\}$

Though the exp of l_ns may not be well-defined, $\underline{\text{Witt}}$ is indeed a Lie algebra of some group but not Conf(M)

(Many structure are defined via algebra like ~~quantum group~~ (group).)

<u>Subalgebra correspond to global transfs:</u> Lie(Conf(M))

Vector field : $v(x) = \tilde{v}(z) = -\sum_{n=-\infty}^{+\infty} v_n l_n = \sum_n v_n z^{n+1} \partial_z$

when does $v(x)$ exponentiated to be a <u>holomorphic function</u>.

<u>necessary</u>

$\Rightarrow$ nonsingular as $z \to 0$

$\Rightarrow$ $\underline{v_n = 0 \quad n < -1}$

<u>inverse</u> exp to be holomorphic $\Rightarrow$ non singular as $\underline{z \to \infty}$

on Riemann sphere.

$\Rightarrow$ If we don't compactification, then $v_n = 0 \quad n > 0$

we end up with linear maps $(f(z) = \alpha z + \beta)$

If we compactificated, then we demand holomorphism

on Riemann sphere, we have more possibilities

$\Rightarrow \quad v_n = 0 \quad n > 1$ only l_{-1}, l_0, l_1 left

& $(\bar{l}_{-1}, \bar{l}_0, \bar{l}_1)$

these 6 generators close to form a subalgebra under $[\cdot,\cdot]$

Note that for $\mathbb{R}^{2,0}$ compactificated to $\mathbb{C} \cup \{\infty\}$, the global conformal transfs. consists of Möbius transf. discussed above

Thus, this subalgebra (6 generators) of Witt generate subgroup of $\mathrm{Conf}(\mathbb{C} \cup \{\infty\})$

$$\Rightarrow \left\{ \text{the linear fractional transformations}: \quad z \longmapsto \frac{az+b}{cz+d} \right\}$$

$$\& \quad ad - cb = 1$$

$$= \boxed{PSL(2, \mathbb{C})} \subset \{\text{Möbius transfs}\} = \mathrm{Conf}(\mathbb{C} \cup \{\infty\})$$

$\supset$?

explicitly :

Matrix form (representation on $\mathbb{C}$)

$$z \longmapsto \frac{az+b}{cz+d} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{act on } \mathbb{C}$$

- $\exp(s\,l_{-1}) : z \longrightarrow z - s$ translation
- $\exp(s(l_0 + \bar{l}_0)) : z \longmapsto e^{-s} z$ dilation
- $\exp(is(\bar{l}_0 - l_0)) : z \longmapsto e^{is}$ rotations
- $\exp(s l_1) : z \longmapsto \dfrac{z}{1+sz}$ SCT.

- $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} e^{i\frac{\theta}{2}} & \\ & e^{-i\frac{\theta}{2}} \end{pmatrix}$
- $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$

Claim. For $\mathbb{R}^{p,0}$, $\boxed{p>2}$,

global ones $\longleftrightarrow$ local ones.

imaginary time = temperature.

Physical interpretation :

($\mathbb{R}^{2,0}$ is good for Statistical Mechanics. by Wick rotation)

Minkowski space : $d=2$, $\mathbb{R}^{1,1}$

$\mathrm{Conf}(\mathbb{R}^{1,1})$ is special !

$(x,y) \longmapsto (u(x,y), v(x,y))$

Theorem : A smooth map $\varphi : M \to \mathbb{R}^{1,1}$ from a subset $M \subset \mathbb{R}^{1,1}$ is conformal iff

(pull back the metric to a scalar multiplication of the metric)

$\varphi^* g_{\mu\nu} \propto g_{\mu\nu}$

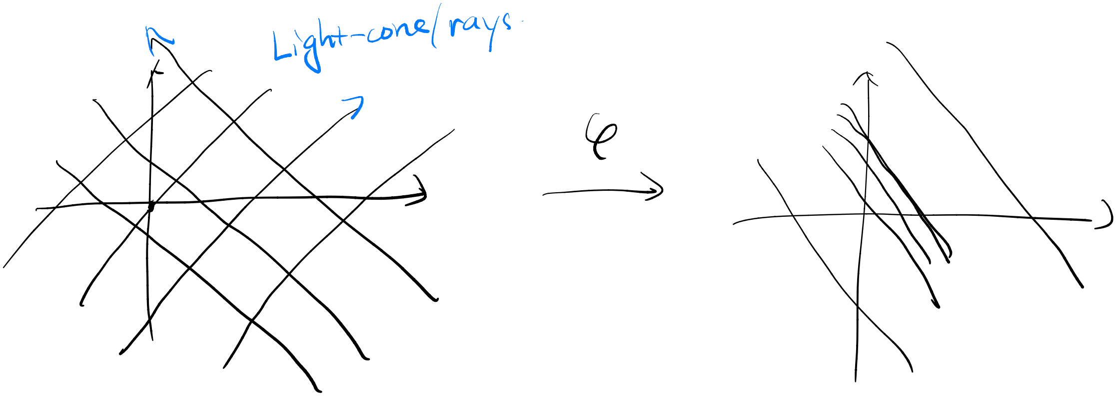
Arbitrary conformal transformation of $\mathbb{R}^{1,1}$

$u_x^2 > v_x^2$ & $u_x = v_y$, $u_y = v_x$

$\partial_x u$

or $u_x = -v_y$, $u_y = -v_x$

Using lightcone coordinates to make the form of these C.T. clear:



dilate-contrate of the lightcone coordinates preserve angles.

two independent series.

$(\mathbb{R}^{1,1})$

All possible conformal transfs. take these form.

Theorem: For $f \in C^\infty(\mathbb{R})$ let $f_\pm \in C^\infty(\mathbb{R}^2, \mathbb{R})$ defined by

$$f_\pm(x, y) = f(x \pm y)$$

$\mathbb{R}^2$ coordinates

lightcone coordinates (two axes are distinguished) (independent).

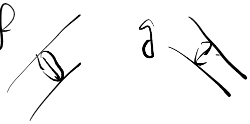
construct dilations, contractions of each axis

go to Light-cone coordinates.

the map $\Phi : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}^2, \mathbb{R}^2)$

$$(f_{(x)}, g_{(y)}) \longmapsto \frac{1}{2}(\underbrace{f_+ + g_-}_{u(x,y)}, \underbrace{f_+ - g_-}_{v(x,y)})$$

f g



use f&g, dilate/contract one axis correspondingly

has properties: vector functions. necessary for C.T.

(1) $\mathrm{Im}\,\Phi = \{(u,v) \mid u_x = v_y,\ u_y = v_x\}.$

(2) $\Phi(f,g)$ is conformal transf. iff $f' > 0$ & $g' > 0$

or $f' < 0$ & $g' < 0$

(3) $\Phi(f,g)$ bijective iff f, & g bijective.

(4) $\Phi(f \cdot h, g \cdot k) = \Phi(f, g) \cdot \Phi(h, k)$

(a homomorphism.)

The two theorems

$\Longrightarrow$ the group of orientation-preserving transformations of

$M = \mathbb{R}^{1,1}$ is isomorphic to (M hasn't compactified)

together orientation-perserving

$$\boxed{\left(\underbrace{\mathrm{Diff}_+(\mathbb{R})}_{\text{orientation-preserving}} \times \mathrm{Diff}_+(\mathbb{R}) \right)} \sqcup \left(\underbrace{\mathrm{Diff}_-(\mathbb{R})}_{\text{orientation reversing}} \times \mathrm{Diff}_-(\mathbb{R}) \right)$$

Subgroup in interest

Dropped usually

Study $\mathrm{Diff}_+(\mathbb{R})$ (finding UIRs) and

construct tensor product representations. (to obtain $\mathrm{Conf}(\mathbb{R}^{1,1})$)

Compactification: $\mathbb{R}^{1,1} \longrightarrow S^{1,1}$

$$\subset \mathbb{R}^{2,0} \times \mathbb{R}^{0,2}$$

$\Longrightarrow$ .... is isomorphic to

$(\underline{\underline{\mathrm{Diff}_+(S^1)}} \times \mathrm{Diff}_+(S^1)) \cup \underline{(\ldots\ldots)}$

not interested.

$\cong \mathrm{Conf}(\mathbb{R}^{1,1}_{\text{compactified}})$ since not connected to $\mathbb{1}$

Focus on $\boxed{\mathrm{Diff}_+(S^1)}$ (chiral half)

$\downarrow$

Q: why $\mathrm{Diff}_+(S^1)$ have ∞-dim Lie(G) while $\mathrm{Conf}(\cdots)$ have finite dim

Lie algebra $\cong$ Witt (same for $\mathbb{R}^{2,0}$ case)

Note, UIRs of this group ($\mathrm{Diff}_+(S^1)$) actually won't lead to sensibale Quantum theorys, some of them don't give H bounded below (unstable Quantum theories)

Note, all PUIRs of this group correspond to the number of central charge, use PUIR to construct sensible QFTs