Part I

Why TQFT:

① describe approximate ground state

② Skelton of more physical/rich QFTs like 2d CFT

③ Some theory have no exact solution, but the topological properties can be computed exactly

(helps understanding D.O.F in system)

④ Toy models for studying geometrical properties of QFT

⑤ connection to deep mathematics

(even $\mathbb{Z}_2$, a nontrivial sign is of great significance)

Defining TQFT:

(1) algebraic: Local operators $\to$ correlation func.

$\to$ Scattering ...

$\Updownarrow$

D.O.F (excitations).

Combined to study D.O.F in theories

(2) geometrical: cut & paste, defect operators, ...

(n+1)d :

dim-n closed (usually compact) space manifold X $\longrightarrow$ Hilbert space $\mathcal{H}_X$

↑ states.

↑

on X we have fields φ_X $\rightsquigarrow$ $\Psi[\varphi_X]$

(fiber bundles)

↑ wave functional.

Axioms: $\mathcal{H}_\emptyset = \mathbb{C}$. $\mathcal{H}_{-X} = \mathcal{H}_X$. $\mathcal{H}_{X\cup Y} = \mathcal{H}_X \otimes \mathcal{H}_Y$

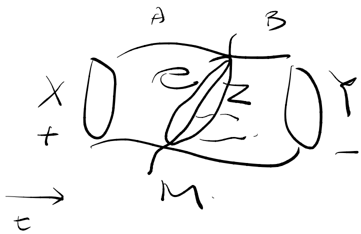
↑ reversed orientation

↑ disjoint. (or connected in pt).

wick rotation → more physical Iso(M)

consider Euclidean here, there's problem with putting on metric.

<u>time evolution:</u>



$\Phi_M : \mathcal{H}_X \to \mathcal{H}_Y$

$\Phi_M = \Phi_A \circ \Phi_B = \Phi_{A\cup B}$

cut into pieces

try to find building blocks

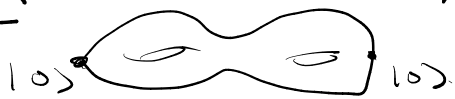
generalization of e^{-tH}, + cut into slices

<u>Partition function (amplitude)</u>

$Z \sim \langle 0| \qquad |0\rangle$

closed M :

$\Phi_M : \mathbb{C} \to \mathbb{C}$

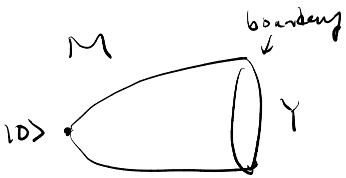


$$\Phi_M = \int D\varphi \, e^{-S[\varphi]} \in \mathbb{C}$$

* general field theory
not necessarily unitary

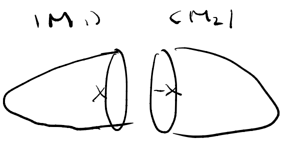
Boundary states

$\Phi_M : \mathbb{C} \to \mathcal{H}_Y \qquad \Phi_M(\alpha) \in \mathcal{H}_Y$



$$\Phi_M \equiv |M\rangle \in \mathcal{H}_Y$$

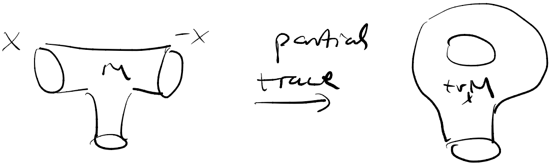
$$\Phi_M[\varphi_Y] = \int_{\varphi_Y \text{ boundary condition}} D\varphi \, e^{-S}$$



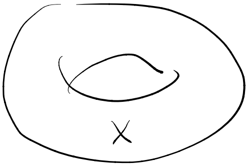
$$\Phi_M = \langle M_2 | M_1 \rangle$$

Trace

general periodic boundary conditions



$$\mathrm{Tr}_{\mathcal{H}_X} \Phi_M$$



$\Phi_M = \mathrm{tr}_X \, e^{-tH}$ partition function.

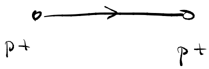
$M = X \times S^1$

$\uparrow$

$X \times I$

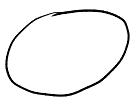
$\Phi_M : \mathcal{H}_X \to \mathcal{H}_X$

QM : 0+1 d $\qquad M = \{*\} \times I$



$\Phi_t = e^{-tH} : \mathcal{H}_* \to \mathcal{H}_*$

S^1



$\Phi_{S^1} = \mathrm{tr}\, e^{-tH}$

Supersymmetric QM :

$X^\mu(t)$ on Σ + fermions

particles

hint: antisymmetrization

wave functions

= differential forms

$\mathcal{H}_\Sigma = \Omega^*(\Sigma)$

$H = -\nabla^2 = -(dd^* + d^*d).$

ground state = harmonic forms, topological QM.

$$\mathrm{Harm}^*(\Sigma) \cong H^*(\Sigma) \quad \text{cohomology}$$

Partition function = Witten index

 $\mathrm{Tr}(-1)^F = \mathrm{Tr}\left((-1)^{\deg} e^{-tH}\right) = \mathrm{Euler}(\Sigma)$

(Euler number!)

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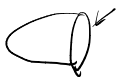
assumption:

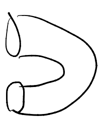
(1+1)d : space : $S^1 \longrightarrow \mathcal{H} = \mathrm{span}\{\phi_i, i=1,\dots,N\}$ finite-dim

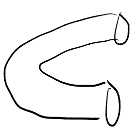
special states:  1D $\phi_0 = |0\rangle = 1 \in \mathcal{H}$.

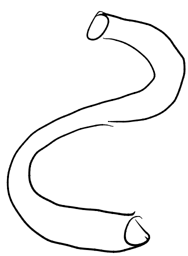
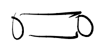
 $\langle 0| = \langle \cdots \rangle_0 \in \mathcal{H}^* : \mathcal{H} \to \mathbb{C}$

Bilinear form:

$$\eta : \mathcal{H}\otimes\mathcal{H} \to \mathbb{C} \qquad \eta_{ij} = \eta(\phi_i, \phi_j)$$

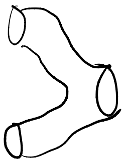


 η^{-1}

 $= \eta^{-1}\eta = $ 

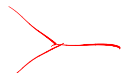
$= \mathrm{id}_H.$

Pair of pants: (building blocks of 2d Riemann surfaces)

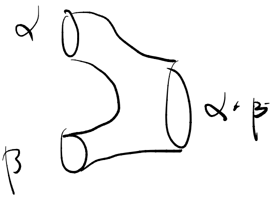


$c : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ multiplication.

$\phi_i \cdot \phi_j = C_{ij}^{k} \phi_k$ (algebra).



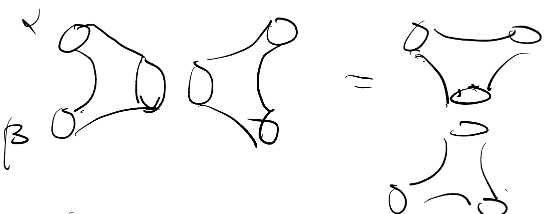
e.g. Feynman diagram.



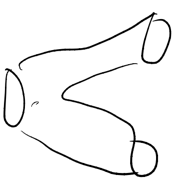
anyon algebra.

Hilbert space = commutative, associative algebra $(C_{ij}^{k} = C_{ji}^{k})$

associative:

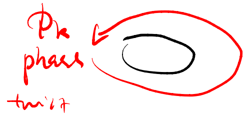


$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.

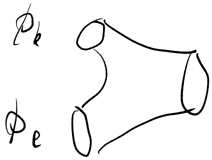


coproduct

Example: $\mathbb{Z}_N$ gauge field



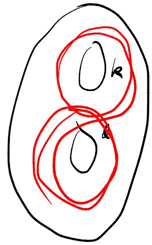
Holonomy around circle: $\phi_k = e^{2\pi i k/N}$



ϕ_{k+l}

↑

product of two phases.

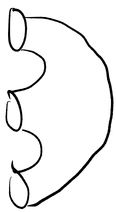


$k+l$.

Frobenius algebra (1+1d).

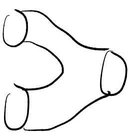
unit  $1 \in \mathcal{H}$ $\quad 1 \cdot \alpha = \alpha$.

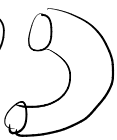
Frobenius algebra



$$C_{ijk} = C_{ij}{}^{n} \eta_{nk} = C_{jki} = \cdots$$

$$\eta(\alpha \cdot \beta, \gamma) = \eta(\alpha, \beta \cdot \gamma)$$





$$\eta(\alpha \cdot \beta) = \langle \alpha \cdot \beta \rangle_0$$

$\exists\, e_i$

If semi-simple, then $\eta(e_i, e_j) = \delta_{ij}$, $e_i \cdot e_j = \lambda_i \delta_{ij} e_i$

(no idempotents $\alpha^n = 0$)

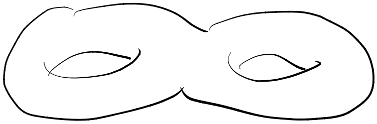
↑

states $\in \mathcal{H}$

$\begin{cases} e_i \cdot e_i = \lambda_i e_i \\ 0 \end{cases}$

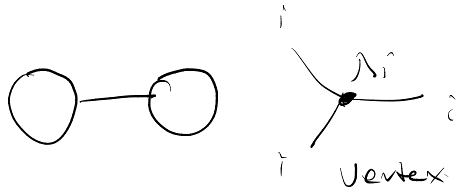
this algebra is simple and can be solved

example :

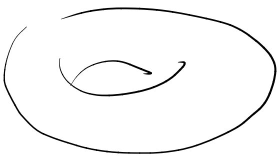


$$\Phi_g = \sum_i \lambda_i^{2g-2}$$

cut into pieces,
gluing together again



$g = 1 \Rightarrow \Phi_g = \sum_{i \in \mathcal{H}} 1$



$S^1 \times S^1 \cong T^2$

$$\underline{\Phi_1 = \mathrm{tr}\, \mathbb{1} = \dim \mathcal{H}}$$

$$\underline{\Phi(T^2) = \dim \mathcal{H}}$$

Example : 2d sigma model with superpotential $W(x_1, \dots, x_n)$

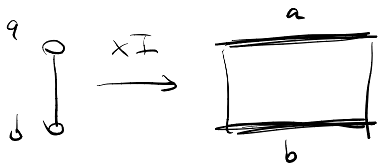
$$\mathcal{H} = \mathbb{C}[x_1, \dots, x_n] / (dW)$$

↑ chiral ring

$$\frac{\partial W}{\partial x_i} = 0$$

semi-simple (TQFT) $\Rightarrow$ massive model = non-deg critical pts.

Boundaries: extended TQFT



$\mathcal{H}$ will depend on boundary conditions!

$\mathcal{H}_{ab}$

L/R boundary conditions

what's the <u>boundary conditions</u>?

$\Rightarrow$ category of boundary conditions $= \mathcal{C}$

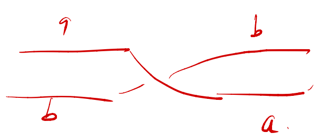
$\mathcal{H}_{ab} \otimes \mathcal{H}_{bc} \to \mathcal{H}_{ac}$.

matching boundary conditions. $\left\{ \begin{array}{l} a \leftrightarrow a \\ b \leftrightarrow b \\ c \leftrightarrow c. \end{array} \right.$

<u>no longer</u> <u>commutative.</u>

(we will have to introduce

<u>twist</u>



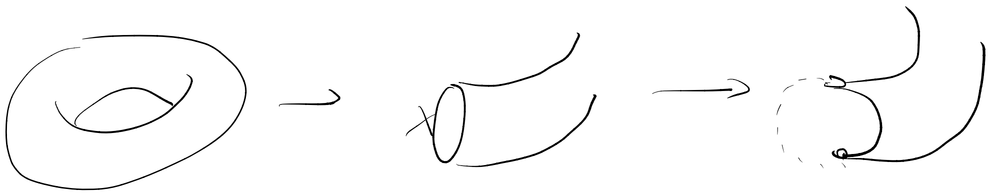
Codim 0 : closed spacetime manifold

Codim 1 : spacelike boundary : Hilbert space.

Codim 2 : corners : category of boundary conditions.

⋮

richer & richer mathematics.

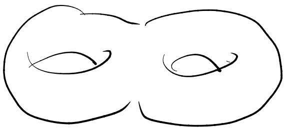


M $\qquad$ $X = \partial M$ $\qquad$ $B = \partial X$

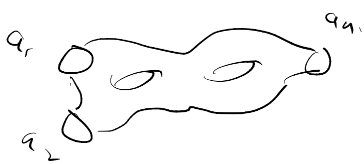
$M^{n+1} \to \Phi_M$ $\qquad$ $X^n \to \mathcal{H}_X$ $\qquad$ $B^{n-1} \to \mathcal{C}_B \in \mathcal{C}at$

$(2+1)d$ $\qquad$ e.g.

space closed
(Rieman surfaces).



+ boundary



Riemann surfaces with holes

boundary conditions : hole $\longleftrightarrow$ anyon !

$$\Rightarrow \quad \mathcal{H}(a_1, \dots, a_n)$$ Hilbert space depend on B.C. ~ Hilbert space of anyon.

Action of diffeomorphisms

$$\varphi : X \to X \qquad \varphi \in \mathrm{Diff}(X)$$

$$\Rightarrow \quad \hat{\varphi} : \mathcal{H}_X \to \mathcal{H}_X \qquad \underline{\hat{\varphi} \in \mathrm{Aut}(\mathcal{H}_X)}$$

(1+1d).
2D CFT :

TQFT depend only on topology. not on metric, ...

CFT depend on metric up to scale

trace anomaly $\longleftarrow$ (traceless $\underline{T^{\mu}{}_{\mu} = 0}$)

<u>chiral stress tensors</u> : Virasoro algebra.

$$T(z) = T_{zz} \qquad \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}} \qquad \bar{\partial} T = \partial \bar{T} = 0$$

<u>holomorphic fields.</u>

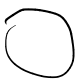
only sensitive to complex structure on Riemann surface.

Laurent series.

$$T(z) = \sum L_n z^{-n-2} \Longrightarrow [L_m, L_n] = (n-m) L_{n+m} + \frac{c}{12}(m^3 - m)\,\delta_{m,-n}$$

Category point of view

$\times I$ ↑ S_1

 $\rightarrow \mathcal{H}_{S^1}$

↑ diffeomorphism of S^1

↺ representation on $\mathcal{H}_{S^1}$

infinite-dim representation.

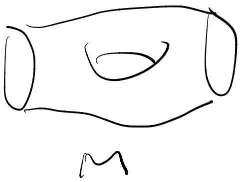
$$Vir \otimes \overline{Vir} \xrightarrow{PUR} Diff(S^1)$$

constructed on $\mathcal{H}$.

$\Longrightarrow$ Action of $Vir \otimes \overline{Vir}$ on $\mathcal{H}$ reproduce $Diff(S^1)$

(endowed with complex structure)

Consider Riemann surface with genus g and



$n = p + q$ punctures. (boundary condition/components)

$$\Phi_M : \mathcal{H}^{\otimes p} \longrightarrow \mathcal{H}^{\otimes q}$$

Finite number of $3g-3+n$ of moduli

Finite dimension moduli

space of complex structure.

complex structures

$\Downarrow$

(non rigorously: space of all metrics up to local rescaling)

(CFT).

finite dim.

space of all metric infinite dim.

CFT: $[\text{metric}] \longleftrightarrow$ complex structures

equivalent classes.

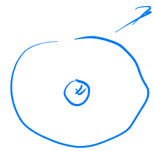
Cylinder, plane, sphere:

eg. CFT on S^1

$\tau \to -\infty$  $t \to \infty$

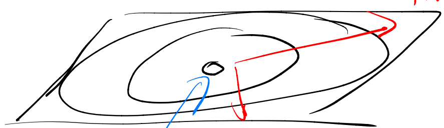
$\downarrow \; w = \tau + i\sigma$

$z = e^{w}$

crush = 

plane.

time (radial).



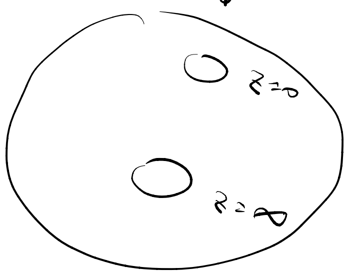
$z \in \mathbb{C}$

space (S^1).

singularity

add a singularity in ∞

Riemann sphere



$z=0$

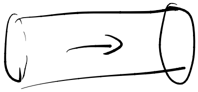
$z=\infty$

here all Euclidean (Riemann).

Operators and states

(this picture only in CFT
different picture have different metric
but equivalent! (same complex structure))

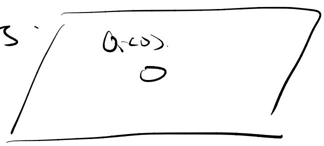
States.

$|i\rangle$ 

States at $t=0$

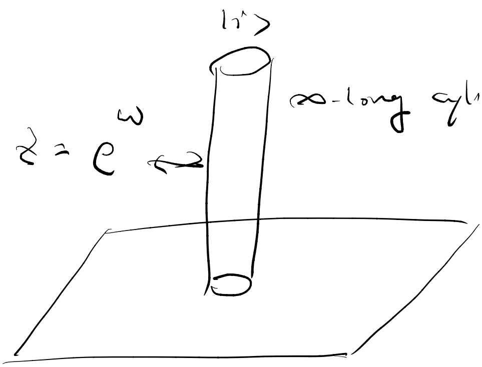
operator-state equivalence.

Local operators:



put a boundary condition at origin $O_i(0)$

tube algebra ?

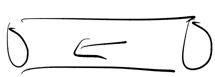


$$\lim_{z\to 0} O_i(z)|0\rangle = |i\rangle$$

Vacuum = Disk.

 $|0\rangle \in \mathcal{H}$.

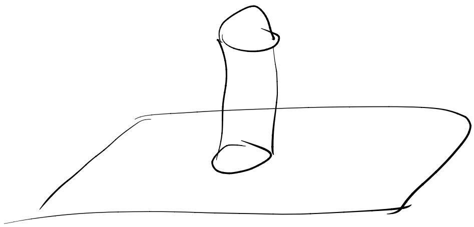
Vacuum amplitude:

$\langle 0|$  $|0\rangle$

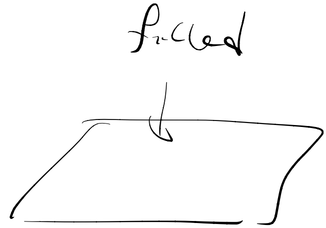
$\simeq$  $= S^2.$

$$\langle 0|0\rangle \equiv \Phi_{S^2}.$$

$|0\rangle$ state:



$\cong$

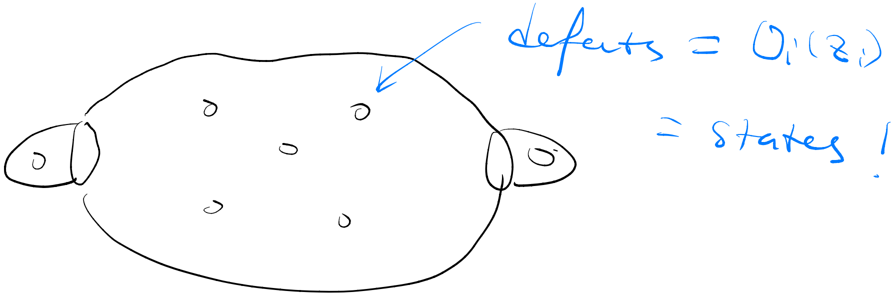


$= $ put $\mathbb{1} = O_i(z=0)$ $\quad (\mathbb{1}|0\rangle = |0\rangle)$

generally

$$\langle 0| O_{i_1}(z_1) \cdots O_{i_n}(z_n) |0\rangle$$

$$\|$$



Space $\longrightarrow$ cut into pieces

(local $\mathcal{H}$).

stick together $\longrightarrow$ $\mathcal{H}_{global}$.

$\uparrow$

Boundary conditions

twisting, ..., entanglement

tensor product states

String theory

B.C. $\longrightarrow$ D-brane, ...

for open string

more richer

further than

boundary conditions

(category!).

Codim $\uparrow$ complex $\uparrow$

math

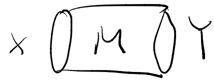
Finer

boundary $\uparrow$

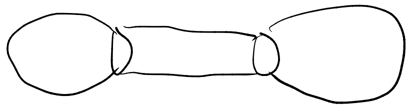
condition

Part II

ob : Manifold with boundaries



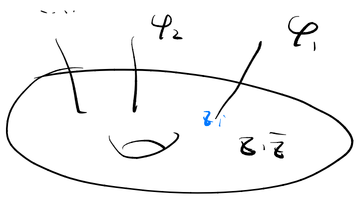
morphism : Cobordism



2D CFT

2-dim manifold with complex structure $(z, \bar{z})$

operators inserted



holomorphic

$$\Rightarrow \quad \Phi_M = \langle \cdots \rangle = \sum_{\substack{\alpha \\ \text{(indices)}}} |\Psi_\alpha(\bar{z})|^2.$$

wave functional of C-S

(building blocks)

Free bosons φ

{ holomorphic : L-moving (chiral half)
(anti) -- : R-moving }

$$S = \frac{1}{4\pi}\int \partial\varphi\,\bar\partial\varphi \overset{\text{equ.}}{\Longrightarrow} \partial\bar\partial\varphi = 0$$

modes decomposition : $z = e^{\omega}$ $\quad \omega = \sigma^0 + i\sigma^1$

$$\partial\varphi(z) = \sum \alpha_n\, (z)^{-n-1} \qquad \bar\partial\varphi = \sum \bar\alpha_n\, (\bar z)^{-n-1}$$

canonical quantization $\Longrightarrow$ $[\alpha_n, \alpha_m] = n\,\delta_{n+m,0}$

$|0\rangle$: $\quad \alpha_n|0\rangle = 0 \quad n \geq 0$

Fock space :

$$F_0 = \mathrm{Span}\{\alpha_{-n_1}\cdots\alpha_{-n_s}|0\rangle\}$$

$$\mathcal{H} = F_0 \otimes \overline{F_0} \qquad (L \otimes R)$$

currents : $J(z) = \partial\varphi$

mode-0 : $\alpha_0 = \frac{1}{2\pi}\oint_\gamma J(z)$

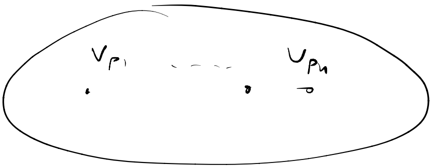
Other states :

$$\alpha_0|p\rangle = p|p\rangle$$

"charge"

$$\simeq V_p = e^{ip\varphi}$$

Vertex operator



$$= \langle \prod V_{p_i}(z_i) \rangle_0$$

0 (free boson) use $\langle \varphi(z)\varphi(w)\rangle = -\log|z-w|^2$

$$= \Big| \prod_{i<j} (z_i - z_j)^{p_i p_j} \Big|^2$$

holomorphic current = Stress tensor

$$T(z) = -\tfrac{1}{2}(\partial\varphi)^2$$

For free bosons:

extended chiral algebra: expanded by $\begin{cases} T(z) = -\frac{1}{2}(\partial\varphi)^2 \to \text{Virasoro} \\ J(z) = \partial\varphi \end{cases}$

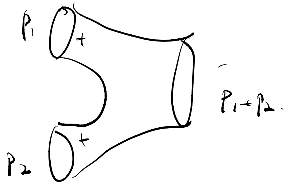
$\downarrow$ (Kac-Moody). at level 1.

$$\mathcal{H} = \oint dp \, F_p \otimes \bar{F}_p$$

$\uparrow$ states $|p\rangle$

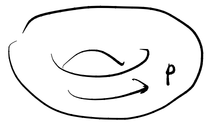
$$p_i = \frac{1}{2\pi}\oint \frac{\partial\varphi}{2\pi}$$

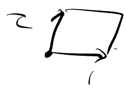
building blocks of 2D CFT



example:

$$q = e^{2\pi i \tau}$$

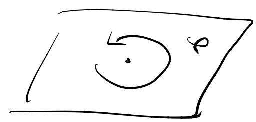




$$\Phi_{T^2} = \int dp \left| \frac{q^{\frac{p^2}{2}}}{\prod_n (1-q^n)} \right|^2$$

Consider more generally $\begin{cases} \alpha_0 = P_L \\ \bar{\alpha}_0 = P_R \end{cases}$

compact scalar: $\varphi \cong \varphi + 2\pi R$



$$\oint (\partial\varphi + \bar{\partial}\varphi) = \oint d\varphi = 2\pi m R$$

winding

$$p = \frac{P_L + P_R}{2} = \frac{n}{R}$$

$$\Longrightarrow \quad P_L, P_R = \frac{n}{R} \pm mR$$

$\boxed{\text{extended}}$ chiral algebra :

$V_{p_R=0}(z)$ for $\frac{n}{R} - mR = 0$, $\Rightarrow \boxed{R^2 = \frac{m}{n} \in \mathbb{Q}}$

$k = m \cdot n$

the chiral algebra is bigger (more chiral operators) :

$e^{i\sqrt{k}\varphi}$ is chiral.

$\Rightarrow$ representation : $e^{i\frac{n\varphi}{\sqrt{k}}}$ $n = 0, \ldots, k-1$

$\boxed{\mathbb{Z}_k \text{ symmetry}}$

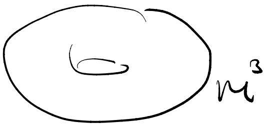
A simple case of Rational CFT.

RCFT : have a large enough chiral algebra that have only a finite number of representations.

Chern-Simons theory:

$$S = \frac{k}{4\pi}\int_M \mathrm{tr}\left(A\,dA + \frac{2}{3}A^3\right), \quad \text{gauge group } G$$

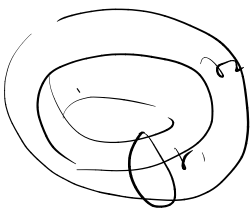
$\delta S = 0 \implies F = 0$, the gauge (G) bundle is flat



G-bundle over M^3

$\implies$ Flat connections

holonomy: $\xi_\gamma = P\exp\left(\oint_\gamma A\right) \in G$



there's homomorphism

$$\pi_1(M) \mapsto G$$
$$\gamma \qquad\quad \xi_\gamma$$

up to gauge transform

$$\implies \mathrm{Hom}\{\pi_1(M) \to G\}/G = \mathcal{A}_{flat}/\mathcal{G}$$

action

$$\boxed{\xi_\gamma \to g\,\xi_\gamma\,g^{-1}}$$

space of flat gauge field.

gauge transf

characteristic class

$$\mathcal{M}^{flat} = \mathrm{Hom}(\pi_1(M), G)/G\text{-}\boxed{\text{conjugation}}$$

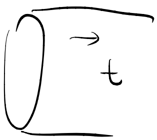
space of flat bundles

$(\delta S = 0)$

this is the classical configuration space of Chern-Simons gauge theory (flat).

quantization :

$A_0 = 0$ (temporal gauge) $\Rightarrow F_{ij} = 0$



X^2

$$S = \frac{k}{4\pi}\int d^2x\,dt\ \mathrm{tr}\left(\varepsilon^{ij} A_i \frac{dA_j}{dt}\right)$$

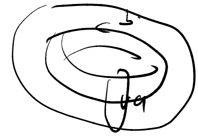
canonical conjugation : $A_2 = i\hbar \frac{\delta}{\delta A_1}$ $\quad \hbar = \frac{2\pi}{k}$

phase space : $\mathcal{M}^{flat}(X^2)$

natural symplectic form : $\int_X \mathrm{tr}(\delta A \wedge \delta A')$

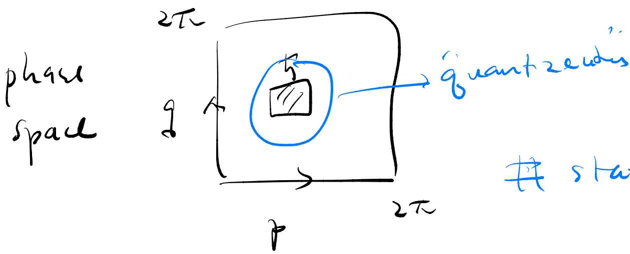
quantization with symplectic form, $(\hbar = \frac{2\pi}{k})$

Example: $G = U(1)$, $X^2 = T^2$



$$p = \underbrace{\oint_a A}_{}, \quad q = \oint_b A \quad \text{mod } 2\pi$$

coordinates of BZ (dual T^2).



$$\# \text{ states} = \frac{(2\pi)^3}{h}$$

$$= k \quad \text{(level)}$$

$$p = -i\hbar \frac{\partial}{\partial q} = n\hbar = \frac{2\pi n}{k}, \quad n \in \mathbb{Z}_k$$

$|p\rangle = |0\rangle, \ldots, |k-1\rangle$ $\quad$ $\mathbb{Z}_k$ structure of $\mathcal{H}$

$(2+1)$d	$(1+1)$d
U(1) Chern-Simons $\longleftrightarrow$	Compactified boson (CFT)
k	$\mathbb{Z}_k$

Finite dim Hilbert space

(Chern-Simons)

<u>Finite groups</u>

For finite G, there's no curvature thus any G-bundle is flat (with flat connection pure gauge)

ξ : G-bundle $\to M$.

$\Rightarrow \quad \mathrm{Hom}(\pi_1(M), G)/G$ is finite

<u>Partition function</u> :

$$\Phi_M = \sum_{\xi \in M} 1$$

try.

$$\Phi_M = \frac{1}{|G|} \sum_{\xi \in \pi_1 \to G} 1$$

G-action.

$$\mathrm{Stab}_G(\xi) = \{ g \; \xi_\gamma g^{-1} = \xi_\gamma \}$$

symmetry group of flat bundle

$$= \mathrm{Aut}(\xi).$$

particular bundle (mod gauge transf).

dividing by equivalence classes

$$\Phi_M = \sum_{\xi \in \mathcal{A}/\mathcal{G}} \frac{1}{|\mathrm{Aut}(\xi)|} \in \mathbb{Q}$$

gauge equivalence bundles.

Example:

$M = S^n$, $\Phi_M = \frac{1}{|G|}$

$\times$ 

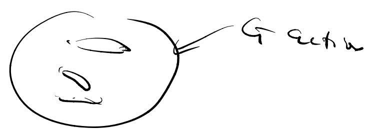
phase space: $\mathcal{M}_X = \mathrm{Hom}(\pi_1(X), G)/G$

Finite set

$\downarrow$

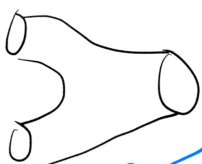
Hilbert space $\mathcal{H}_X = \mathbb{C}^{N = |\mathcal{M}_X|}$

Hilbert space.

$G =$ 

$C_B \ni g_B$

$C_A \ni g_A$



$g_C \in C_C$ $\Longrightarrow$ $\{C_A\}$ conjugacy classes

multiplication (up to conjugation)

$\Rightarrow$ states may be labelled by

C_A conjugacy classes of G

quantized Hilbert space

Recall: (1+1)d = Frobenius algebra.

the algebra :

group algebra : $\mathbb{C}[G]$ but not

commutative

instead consider :

$$\phi_A = \sum_{g \in C_A} g \qquad \text{then} \quad \phi_A \phi_B = \phi_B \phi_A$$

this is the center of the group algebra.

$Z[\mathbb{C}(G)]$

$$\phi_A \cdot \phi_B = \sum N_{AB}^{C} \phi_C .$$

$$\downarrow$$

$$\# \{ g_A \cdot g_B = g_C \} / G$$

$$g_i \in C_i$$

Another perspective:

R_α rreps of G equivalent R_α

$\Updownarrow$

conjugacy classes

C_A

character: $\chi_\alpha(g) = \mathrm{tr}(R_\alpha(g))$

$= \chi_\alpha^A$ ← conjugacy class

$$\chi_\alpha(1) = \dim R_\alpha = d_\alpha$$

$$\Longrightarrow \qquad e_\alpha = \sum_{g \in G} \chi_\alpha(g)\, g$$

$$\underline{e_\alpha e_\beta = \delta_{\alpha,\beta} \lambda_\alpha e_\alpha.} \qquad \lambda_\alpha = \frac{|G|}{d_\alpha}$$

Group cohomology : $H^n(G, U(1))$

meaning : $n=1 \Rightarrow$ α is 1D rep

$$\alpha(gh) = \alpha(g)\alpha(h).$$

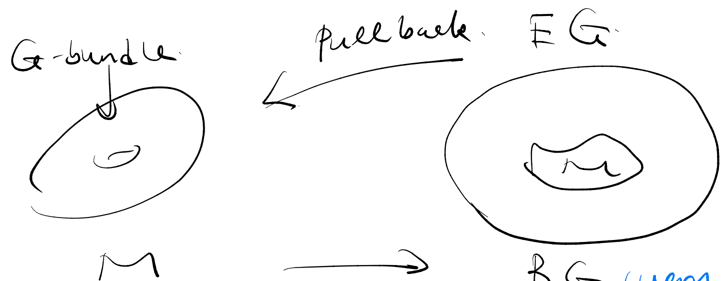
$n=2. \Rightarrow$ projective rep / central extension

$$R(gh) = R(g)R(h)\,\alpha(g,h).$$

2-cocycle: $\alpha(g,h)\,\alpha(gh,k)$

$= \alpha(h,k)\,\alpha(g,hk)$

geometrically view : classifying space,



(very big to "include" all bundle possible)

$$BU(1) = \mathbb{RP}^\infty$$

Facts :

$$H^n(G, U(1)) \cong H^n(BG, U(1))$$

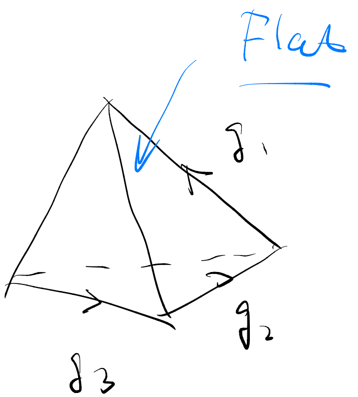
physically :

$$M \xrightarrow{\xi} BG \quad \underline{\forall\, \alpha \in H^n(BG, U(1))}$$

$$\text{"}e^{iS_{cs}}\text{"} = \langle M, \xi^* \alpha \rangle$$

$$\underline{\in H^n(G, U(1))}$$

Lattice model :



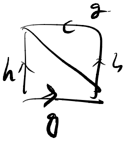
$$e^{iS} = \sum_{g_i \in G} \prod_{\triangle} \alpha(\triangle)^{\pm 1}$$

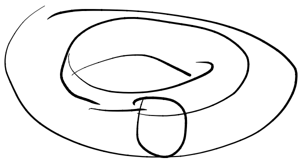
orientation.

cocycle condition $\Longleftrightarrow$ P-moves (independent of triangulation).

Example

1+1d

$T^2 =$ 



$$= \dim \mathcal{H}^{\alpha}_{S^1} = \frac{1}{|G|} \sum_{\substack{gh=hg \\ (\text{flat})}} \frac{\alpha(g,h)}{\alpha(h,g)}$$

$$= \# \, \mathrm{PUIR}(G)$$