Gauge change :

	f_1	f_2	f_3	c
A^1_μ	1	-1	0	0
A^2_μ	0	1	-1	0
A^{em}_μ	1/3	1/3	1/3	1

↳ external electromagnetic field.

Now, Let's consider $\boxed{\text{meanfield ansatz.}}$

$$\begin{cases} |\phi_{f_1}\rangle = |LLL\rangle \\ |\phi_{f_2}\rangle = |LLL\rangle \\ |\phi_{f_3}\rangle = |LLL\rangle \end{cases} \Rightarrow \begin{cases} \mathcal{L}^{parton} = \frac{1}{4\pi} \sum_{I=1}^{3} a^I_\mu \partial_\nu a^I_\lambda \epsilon^{\mu\nu\lambda} \\ J^I_\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a^I_\lambda \end{cases}$$

LL: lowest landau level

Add Lagrange's multipliers: A^1_μ, A^2_μ

$$\mathcal{L} = \frac{1}{4\pi}\sum_{I=1}^{3} a^I_\mu \partial_\nu a^I_\lambda \epsilon^{\mu\nu\lambda} + A^1_\mu (J^1_\mu - J^2_\mu) + A^2_\mu (J^2_\mu - J^3_\mu)$$
$$+ \frac{1}{3} A^{em}_\mu (J^1_\mu + J^2_\mu + J^3_\mu)$$

where, $J^I_\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a^I_\lambda$, $I = 1, 2, 3$

Next, we may integrate out A^1_μ & A^2_μ:

$$\Rightarrow J^1_\mu = J^2_\mu = J^3_\mu \xrightarrow{\text{proper gauge}} a^1_\mu = a^2_\mu = a^3_\mu \overset{\text{def.}}{=\!=} a_\mu$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{3}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} A^{em}_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}}$$

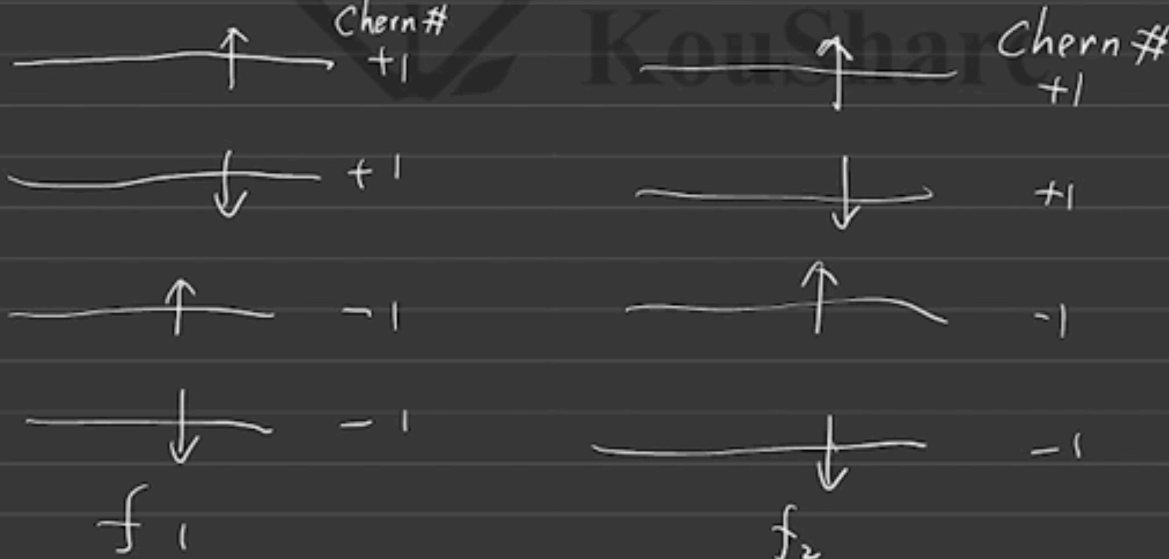
see also the previous lecture (April 17th, 2020)

Physical d.o.f. : 4 different hard-core bosons.

i.e.
$$\begin{cases} b_1 = f_1^{\uparrow} f_2^{\uparrow} \\ b_2 = f_1^{\uparrow} f_2^{\downarrow} \\ b_3 = f_1^{\downarrow} f_2^{\uparrow} \\ b_4 = f_1^{\downarrow} f_2^{\downarrow} \end{cases} \Longrightarrow \quad n_{f_1} = n_{f_2} \text{ at any site}$$

$$n_{f_1} = n^{\uparrow}_{f_1} + n^{\downarrow}_{f_2}, \qquad n_{f_2} = n^{\uparrow}_{f_1} + n^{\downarrow}_{f_2}$$

Meanfield ansatz : Several Chern bands



Therefore, physical state $= P_G \,|\text{meanfields}\rangle$

P_G: $n_{f_1} = n_{f_2}$ at any site

$|\text{meanfields}\rangle$: Chern bands.

To study the physical nature of this many-body state,

define $J^I_\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a^I_\lambda$, $I = 1, 2, \cdots, 7, 8$

$I = 1, 2, 3, 4$ are currents of four Chern bands

$\therefore J^{f_1}_\mu = J^1_\mu + J^2_\mu + J^3_\mu + J^4_\mu$ occupied by f_1.

$I = 5, 6, 7, 8$ are currents of four Chern bands

$\therefore J^{f_2}_\mu = J^5_\mu + J^6_\mu + J^7_\mu + J^8_\mu$ occupied by f_2.

1, Projective construction and field theory

$$\mathcal{L}^{\text{parton}} = \frac{K^{IJ}}{4\pi}\sum_{I,J}^{8} a^I_\mu \partial_\nu a^J_\lambda \epsilon^{\mu\nu\lambda} + \sum_{I}^{8} q^I_c A^{em}_\mu J^I_\mu + \sum_{I}^{8} q^I_s A^s_\mu J^I_\mu$$

charge vector spin vector

$$K^{IJ} = \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && -1 &&&&& \\ &&& -1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& -1 & \\ &&&&&&& -1 \end{bmatrix}_{8\times 8}, \quad q^c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{8\times 1} \quad q^s = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}_{8\times 1}$$

Projection is done by $n_{f_1} = n_{f_2}$ at each site

$$\Longrightarrow \quad J^{f_1}_\mu = J^{f_2}_\mu$$: The worldlines of f_1 and f_2 are tied together

$$\mathcal{L} = \mathcal{L}^{\text{parton}} + A_\mu\left[\underbrace{(J^1_\mu + J^2_\mu + J^3_\mu + J^4_\mu)}_{J^{f_1}_\mu} - \underbrace{(J^5_\mu + J^6_\mu + J^7_\mu + J^8_\mu)}_{J^{f_2}_\mu}\right]$$

Integrate out $A_\mu \Rightarrow J^1 + J^2 + J^3 + J^4 = J^5 + J^6 + J^7 + J^8$

$$\xrightarrow{\text{proper gauge}} a^1_\mu + a^2_\mu + a^3_\mu + a^4_\mu = a^5_\mu + a^6_\mu + a^7_\mu + a^8_\mu$$

$\therefore$ The eight gauge fields are not linearly independent

Without loss of generality we choose to eliminate a^8_μ

by
$$a^8_\mu = a^1_\mu + a^2_\mu + a^3_\mu + a^4_\mu - a^5_\mu - a^6_\mu - a^7_\mu$$

We end up a new CS theory:

$$\mathcal{L} = \frac{1}{4\pi} K^{IJ} a^I_\mu \partial_\nu a^J_\lambda \epsilon^{\mu\nu\lambda} + \sum_I^7 q_c^I A^{em}_\mu \frac{\partial_\nu a^I_\lambda \epsilon^{\mu\nu\lambda}}{2\pi}$$

$$+ \sum_I^7 q_s^I A^s_\mu \frac{\partial_\nu a^I_\lambda \epsilon^{\mu\nu\lambda}}{2\pi}$$

where,

$$K^{IJ} = \begin{bmatrix} 0 & -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -2 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -2 \end{bmatrix}_{7\times 7} \quad q^c = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{7\times 1} \quad q^s = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}_{7\times 1}$$

We note that $|\det K| = 0$

$\therefore$ there must be zero eignvalues.

Perform a $GL(7,N)$ rotation to change basis:

$|\det W| = 1$

$$W = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & 0 & & \\ & & 1 & & & & \\ & 0 & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}_{7\times 7} \in GL(7,N)$$

$$K \to W^T K W = \begin{bmatrix} 0 & -1 & -1 & -1 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & -2 & -1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{7\times 7}$$

zero eign value has been separated out.

$$q_c \longrightarrow W q_c = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad q_s \longrightarrow W q_s = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, in the new basis, CS term of a^7_μ is gone,

the only sub-leading term for a^7_μ is Maxwell term $\sim \frac{1}{g^2}(\partial a^7)^2$

Two possibilities:

- if both q^7_c and q^7_s are zero, Polyakov told us instanton proliferation leads to a spectral gap for photon.
- if either q^7_c or q^7_s is nonzero, instanton is not allowed by symmetry.

Since, $q_s^7 = q_c^7 = 0$, we may completely remove a_μ^7.

$$K_{7\times 7} \longrightarrow K_{6\times 6} = \begin{bmatrix} 0 & -1 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & -2 & -1 & 1 & 1 \\ -1 & -1 & -1 & -2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & -1 & 0 \end{bmatrix} ; \quad q_c = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} ; \quad q_s = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

This CS theory is the effective theory of projected state.

1° Diagonal of K : all even integer $\Rightarrow$ bosonic state

2° $|\det K| = 1$: GSD on a torus is unique.

$\Downarrow$

no topological order

3° # of positive eigen values of K = # of negative eigen values of K

$\Longrightarrow$ non chiral. $(c^- = 0)$

4°. Hall conductance: $\sigma^c = \frac{1}{2\pi} q^{cT} K^{-1} q^c = 0$

$$\sigma^s = \frac{1}{2\pi} q^{sT} K^{-1} q^s = 0$$

$$\sigma^{cs} = \frac{1}{2\pi} q^{sT} K^{-1} q^c = 0$$

∴ We end up with a completely trivial state. trivial $U(1)\times U(1)$ SP

☆☆☆ Remark:

However, suppose, in some cases, $(q_c^7 \neq 0,\ q_s^7 = 0)$

As we explained, instanton configurations are not allowed by symmetry:

$$\mathcal{L} = \frac{q_7^c}{2\pi} A_\mu \partial_\nu a_\lambda^7 \epsilon^{\mu\nu\lambda} + \frac{1}{g^2}(\partial_\mu a_\nu^7)^2$$

$\Updownarrow$ Particle-vortex duality

$$\mathcal{L} = g^2(\partial_\mu \theta + q_7^c A_\mu^c)^2$$

: charged superfluid. A_μ^c is non-dynamical field.

SPTs: 1° Absence of Ginzburg-Landau order parameter

2° bulk excitations are gapped and trivial (non-fractionalized)

3° If symmetry-breaking perturbation is added,

non-trivial SPTs $\xrightarrow{\text{reduced to}}$ trivial SPTs

4° Gauging "symmetry group" in SPTs.

$\Big\downarrow$ leads to

An intrinsic topological order state.

e.g. $\begin{cases} \text{trivial } \mathbb{Z}_2 \text{ SPT in 2D} \longrightarrow \text{toric code } (\mathbb{Z}_2 \text{ topo. order.}) \\ \text{Nontrivial } \mathbb{Z}_2 \text{ SPT in 2D} \longrightarrow \text{double-semion state } (\text{twisted } \mathbb{Z}_2) \end{cases}$

Two field-theoretical approaches.

① NLσM with topological θ-term : Cenke Xu, etc.

Start with well-defined Neel local order near critical point. Then push to SPT fixed point.

② Topological gauge theories at "Level-1" : PY, ZCGu, ... YMLu, ...

Start with "trivial" TQFT

Then study how symmetry transformations are defined.

Examples: in 2D bosonic integer quantum Hall effect
(2D SPT with $U(1)$ symmetry)

$$\mathcal{L} = \frac{1}{4\pi} K^{IJ} a^I_\mu \partial_\nu a^J_\lambda \epsilon^{\mu\nu\lambda} + \frac{q^I}{2\pi} A_\mu \partial_\nu a^I_\lambda \epsilon^{\mu\nu\lambda}$$

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ n \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Hall conductance $\sigma_{xy} = \frac{1}{2\pi} q^T K^{-1} q = \frac{1}{2\pi}(1 . n)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ n \end{pmatrix}$

$$= 2n \frac{e^2}{h}$$

$\therefore$ σ_{xy} is quantized at even integer.

$$\sigma_{xy} = \frac{e^2}{h} \times 2, 4, \ldots$$

Each even integer "uniquely" labels a distinct $U(1)$ SP

In (2+1)D, we may use CS term

$$\frac{1}{4\pi} K^{IJ} a^I_\mu \partial_\nu a^J_\lambda \epsilon^{\mu\nu\lambda} \quad \text{with } K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma$$

to describe most examples of SPTs

We can also consider "twisted term" "$a_1 \wedge a_2 \wedge a_3$" to get more complete classification of Abelian SPTs

Questions: What kinds of topological terms can we express in (3+1) spacetime?

The most simplest topological term is BF term

$$S = \frac{k}{2\pi}\int B\wedge dA \quad (\text{BF theory at level-}k)$$

Remarks: 1° $B = \frac{1}{2!} B_{\mu\nu}\, dx_\mu dx_\nu$ is a 2-form $U(1)$ gauge field. $A = A_\mu dx_\mu$ is a usual 1-form $U(1)$ gauge field.

2° Gauge transformations are defined as:

$$\left\{\begin{array}{l} B \to B + dV \\ A \to A + d\theta \end{array}\right\} \longrightarrow k \in \mathbb{Z}$$

exercise

level quantization

gauge invariance on closed manifold.

where, V and θ are 1-form and 0-form respectively

3° Physically, BF term at level-k describes the topological limit of deconfined phase of $\mathbb{Z}_k$ gauge theory.

$$\frac{k}{2\pi} B\, dA \xleftrightarrow{\text{duality}} (\partial_\mu \theta + k A_\mu)^2$$ where A is dynamical.

i.e. charge-k condensate coupled to $U(1)$ dynamical gauge field.

4° Braiding data: particle-loop braiding.

$\Rightarrow$ Aharonov-Bohm phase $e^{i\frac{2\pi}{N}\mathcal{L}}$

where $\mathcal{L}$ is linking number

5° When $k=1$, all braiding data are trivialized. $\Longrightarrow$ A good starting point of SPTs

In addition to BF terms, we can also consider "twisted" topological terms involving 1-form gauge fields of different U(1) gauge groups.

PY, Z-C. Gu, PRB (2016)

BF term

twisted terms.

$\{b^I\}$: A set of 2-form gauge fields

$\{a^I\}$: A set of 1-form gauge field.

Symmetry	Topological Quantum Field Theory
$\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$	$\frac{i}{2\pi}\int\sum_I^2 b^I\wedge da^I+ip_1\int a^1\wedge a^2\wedge da^2\ (\mathbb{Z}_{N_{12}})$;
	$\frac{i}{2\pi}\int\sum_I^2 b^I\wedge da^I+ip_2\int a^2\wedge a^1\wedge da^1\ (\mathbb{Z}_{N_{12}})$
$\prod_I^3\mathbb{Z}_{N_I}$	$\frac{i}{2\pi}\int\sum_I^3 b^I\wedge da^I+ip_1\int a^1\wedge a^2\wedge da^3\ (\mathbb{Z}_{N_{123}})$;
	$\frac{i}{2\pi}\int\sum_I^3 b^I\wedge da^I+ip_2\int a^2\wedge a^3\wedge da^1\ (\mathbb{Z}_{N_{123}})$
$\prod_I^4\mathbb{Z}_{N_I}$	$\frac{i}{2\pi}\int\sum_I^4 b^I\wedge da^I+ip\int a^1\wedge a^2\wedge a^3\wedge a^4\ (\mathbb{Z}_{N_{1234}})$
$\mathbb{Z}_2^T$	$\frac{iK^{IJ}}{2\pi}\int b^I\wedge da^J\ (\mathbb{Z}_2)$ [54];
	$\frac{iK^{IJ}}{2\pi}\int b^I\wedge da^J+\frac{i\Lambda^{IJ}}{4\pi}\int b^I\wedge b^J\ (\mathbb{Z}_2)$[54]
$\mathrm{U}(1)\rtimes\mathbb{Z}_2^T$	$\frac{iK^{IJ}}{2\pi}\int b^I\wedge da^J\ (\mathbb{Z}_2)$[54]
$\mathrm{U}(1)\times\mathbb{Z}_2^T$	$\frac{iK^{IJ}}{2\pi}\int b^I\wedge da^J\ (\mathbb{Z}_2)$; $\frac{iK^{IJ}}{2\pi}\int b^I\wedge da^J\ (\mathbb{Z}_2)$
$\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\times\mathrm{U}(1)$	$\frac{i}{2\pi}\int\sum_I^3 b^I\wedge da^I+ip\int a^1\wedge a^2\wedge da^3\ (\mathbb{Z}_{N_{12}})$

Let's first consider SPTs with $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$

Take the following action as an example:

$$\boxed{S = \frac{1}{2\pi}\sum_{I=1}^{2}\int b^I \wedge da^I + ip\int a^1 \wedge a^2 \wedge da^2}$$

Remarks: 1° The level of BF terms $\Rightarrow$ trivial topological order in the bulk.

2° $\{a^I\}$ and $\{b^I\}$ are 1-form and 2-form gauge fields with Dirac quantization conditions:

$$\frac{1}{2\pi}\int_{M^2} da^I \in \mathbb{Z}, \quad \frac{1}{2\pi}\int_{M^3} db^I \in \mathbb{Z}$$

3°. Exotic gauge transformation:

$$\begin{cases} a^I \longrightarrow a^I + d\chi^I \\ b^I \longrightarrow b^I + dV^I \boxed{-2\pi p\,\epsilon^{123}\chi^I da^2} \end{cases}$$

gauge parameters: $\{\chi^I\}$: scalars (0-form)

$\{V^I\}$: vectors (1-form)

Nontrivial Winding number

$$\begin{cases} \frac{1}{2\pi}\int_{M^1} d\chi^I \in \mathbb{Z} \\ \frac{1}{2\pi}\int_{M^2} dV^I \in \mathbb{Z} \end{cases}$$

☆: when winding number is nonzero, the corresponding gauge transformations are said to be "Large".

"large gauge transformation".

4° Gauge invariant operators: ("modified" Wilson loops and Wilson surfaces)

Wilson loops: $e^{i\int_{M^1} a^I}$

Wilson surfaces: $e^{i\int_{M^2} b^2 - i2\pi p\int_{V^3} \epsilon^{IJ3} a^J \wedge da^2}$

where, $M^2 = \partial V^3$

Now, let's consider how to impose global symmetry,

$$G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$$

Physically, we may define particle currents in (3+1)D

$$J^I_\mu = \frac{1}{4\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\nu b^I_{\lambda\rho} \xrightarrow{\text{symbolically}} *J^I = \frac{1}{2\pi} db^I$$

In continuous spacetime, we should start with a continuous symmetry group $U(1)$ and its probe field A^I

$$S \to S + \sum_{I=1}^{2} \frac{1}{2\pi}\int A^I db^I \qquad I=1,2$$

The true symmetry group $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ is encoded by $\int_{M^1} A^I = \frac{2\pi}{N^I} \times 0, \pm 1, \pm 2, \ldots$, where, $I = 1, 2$.

Next, we study the quantization and the period of p in the presence of global symmetry $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$.

The result is: $\boxed{p = \frac{k}{4\pi^2} \frac{N_1 N_2}{N_{12}} \quad , k \in \mathbb{Z}_{N_{12}}}$.

Remarks. 1° N_{12} is greatest common divisor of N_1 and N_2 (GCD)

2° Distinct $k \bmod N_{12}$ labels distinct SPT.

$\therefore$ $S = \int \frac{1}{2\pi} \sum_I b^I \wedge db^I + p \int a^1 \wedge a^2 \wedge da^2$ gives $\mathbb{Z}_{N_{12}}$ classification

In addition, we can also consider $p' \int a^1 \wedge a^2 da^1$, $p' \in \mathbb{Z}_{N_{12}}$

$\therefore$ Total classification is $(\mathbb{Z}_{N_{12}})^2$

Field theories
of SPTs with mixture of space symmetry (e.g. C_4 rotation) and on-site symmetry (e.g., spin)

B. Han, H. Wang, PY. PRB (2019)

TABLE I. Generalized Wen-Zee terms as response actions of SPT phases jointly protected by both spatial and internal symmetries. Here A^I are different flavors of U(1) or $\mathbb{Z}_N$ 1-form gauge fields and ω is the $SO(2)$ spin connection, which is an effective U(1) or $\mathbb{Z}_N$ 1-form gauge field. $N_{ij\ldots k} \equiv \mathrm{GCD}[N_i, N_j, \ldots, N_k]$ where GCD stands for greatest common divisor. The coefficient k or k of each topological response action is quantized to be integral, giving the classification of the corresponding topological phase. It is taken from the finite group in the irreducible classification. For real crystalline materials, the point group is restricted to C_2, C_3, C_4, and C_6 due to lattice structures in two spatial dimensions. In our discussion of classifications, we relax it to arbitrary finite group, assuming that this finite group symmetry is reduced from some continuum medium. Classifications of SPT phases protected by both spatial and internal symmetries in 3+1 dimensions (denoted by *), with higher form internal symmetries [48] are also presented. For higher form symmetries, the corresponding gauge connections are higher form gauge fields rather than 1-form fields. Here A, B are U(1) or $\mathbb{Z}_N$ 1- and 2-form gauge fields, respectively. In the main text, we omit the wedge product symbol in order to simplify the notation. "Irreducible" means that any nontrivial subgroup of the total symmetry group would contribute to protecting the SPT phases. Otherwise, it is said to be "reducible". The BF terms are implicit. For instance, for $G_s = C_{N_0}$ and $G_i = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, there are extra terms $\frac{1}{2\pi}(N_0 B^0 \wedge d\omega + \sum_{I=1}^{2} N_I B^I \wedge dA^I)$. The angular momentum is the response charge to the ω field defined by $\mathcal{J} = \frac{1}{N_0}\int d^D x \frac{\delta S}{\delta \omega_0}$ for $G_s = C_{N_0}$ and $\mathcal{J} = \int d^D x \frac{\delta S}{\delta \omega_0}$ for $G_s = SO(2)$, where D is the spatial dimension.

Space-time Dimension	Spatial Symmetry G_s	Internal Symmetry G_i	Irreducible Generalized Wen-Zee terms	Angular Momentum (spin) $\mathcal{J}$
(2+1)D	$SO(2)$	$U(1)$	$\frac{k}{2\pi}\int \omega\wedge dA,\ k\in\mathbb{Z}$	$\frac{k}{2\pi}\int_{M^2} dA$
	C_{N_0}	$\mathbb{Z}_{N_1}$	$\frac{k}{2\pi}\int \omega\wedge dA,\ k\in\mathbb{Z}_{N_{01}}$,	$\frac{k}{2\pi}\int_{M^2} dA$
	C_{N_0}	$\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$	$k\frac{N_1N_2}{(2\pi)^2N_{012}}\int \omega\wedge A^1\wedge A^2,\ k\in\mathbb{Z}_{N_{012}}$	$k\frac{N_1N_2}{(2\pi)^2N_{012}}\int_{M^2} A^1\wedge A^2$
(3+1)D	C_{N_0}	$\mathbb{Z}_{N_1}$	$k\frac{N_0N_1}{(2\pi)^2N_{01}}\int \omega\wedge A\wedge dA,\ k\in\mathbb{Z}_{N_{01}}$	$k\frac{N_1}{(2\pi)^2N_{01}}\int_{M^3} A\wedge dA$
	C_{N_0}	$\mathbb{Z}_{N_1}$	$k\frac{N_0N_1}{(2\pi)^2N_{01}}\int A\wedge\omega\wedge d\omega,\ k\in\mathbb{Z}_{N_{01}}$	$k\frac{N_1}{(2\pi)^2N_{01}}\int_{M^3} A\wedge d\omega$
	C_{N_0}	$\mathbb{Z}_{N_1}\times U(1)$	$k\frac{N_0N_1}{(2\pi)^2N_{01}}\int \omega\wedge A^1\wedge dA^2,\ k\in\mathbb{Z}_{N_{01}}$	$k\frac{N_1}{(2\pi)^2N_{01}}\int_{M^3} A^1\wedge dA^2$
	$SO(2)$	$\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$	$k\frac{N_1N_2}{(2\pi)^2N_{12}}\int A^1\wedge A^2\wedge d\omega,\ k\in\mathbb{Z}_{N_{12}}$	$k\frac{N_1N_2}{(2\pi)^2N_{12}}\int_{M^3} d(A^1\wedge A^2)$
	C_{N_0}	$\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$	$k\frac{N_0N_1}{(2\pi)^2N_{01}}\int \omega\wedge A^1\wedge dA^2,\ k\in\mathbb{Z}_{N_{012}}$	$k\frac{N_1}{(2\pi)^2N_{01}}\int_{M^3} A^1\wedge dA^2$
	C_{N_0}	$\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$	$k\frac{N_0N_2}{(2\pi)^2N_{02}}\int \omega\wedge A^2\wedge dA^1,\ k\in\mathbb{Z}_{N_{012}}$	$k\frac{N_2}{(2\pi)^2N_{02}}\int_{M^3} A^2\wedge dA^1$
	C_{N_0}	$\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\times\mathbb{Z}_{N_3}$	$k\frac{N_0N_1N_2N_3}{(2\pi)^3N_{0123}}\int \omega\wedge A^1\wedge A^2\wedge A^3,\ k\in\mathbb{Z}_{N_{0123}}$	$k\frac{N_1N_2N_3}{(2\pi)^3N_{0123}}\int_{M^3} A^1\wedge A^2\wedge A^3$
(3+1)D (*)	$SO(2)$	$U(1)$	$\frac{k}{2\pi}\int \omega\wedge dB,\ k\in\mathbb{Z}$	$\frac{k}{2\pi}\int_{M^3} dB$
	C_{N_0}	$\mathbb{Z}_{N_1}$	$\frac{k}{2\pi}\int \omega\wedge dB,\ k\in\mathbb{Z}_{N_{01}}$	$\frac{k}{2\pi}\int_{M^3} dB$
	C_{N_0}	$\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$	$k\frac{N_0N_1N_2}{(2\pi)^2N_{012}}\int \omega\wedge A\wedge B$ [33], $k\in\mathbb{Z}_{N_{012}}$	$k\frac{N_1N_2}{(2\pi)^2N_{012}}\int_{M^3} A\wedge B$

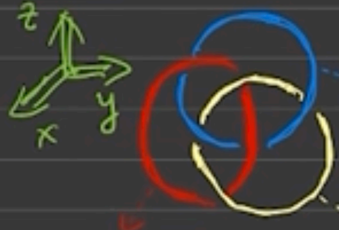
Field theories of "Borromean-Rings braiding" in 3D

Topological orders and SPTs with higher-form global symmetry

A.P.O. Chan, PY, S. Ryu, PRL (2018)

$$S = \sum_{I}^{3} \frac{N^I}{2\pi} \int B^I dA^I + \boxed{p A^1 \wedge A^2 \wedge B^3}, \quad p = \frac{N_1 N_2 N_3 k}{(2\pi)^3 N_{123}}.$$

$$k \in \mathbb{Z}_{N_{123}}$$



trajectory of particle excitation ($\mathbb{Z}_{N_3}$ gauge charge)

static loop excitation ($\mathbb{Z}_{N_2}$ flux)

static loop excitation ($\mathbb{Z}_{N_1}$ flux)

$\Updownarrow$ DUALITY

probed by 1-form A

SPTs protected by $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \tilde{\mathbb{Z}}_{N_3}$ → probed by 2-form B

carried by particles carried by loops

References:

Ye, Acta Physica Sinica (物理学报) **69** 077102 (2020)

Thanks!