§ Free boson

<u>Rmk</u> using free boson & free fermion, the most examples of CFT can be reconstructed

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = \underline{\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi}$$

(2+0)d.

$[m] \neq 0$ thus not invariant under scale transformation !

$$S = \frac{g}{2}\int d^2x\,(\partial_{x^0} - i\partial_{x^1})\phi\,(\partial_{x^0} + i\partial_{x^1})\phi$$

(integrating by parts).

$$= \frac{1}{2}\int d^2x\, d^2y\;\phi(\underline{x})\,\underline{A(\underline{x},\underline{y})}\,\phi(\underline{y})$$

Gaussian.

$\underline{X} = \vec{x} = (x^0, x^1)$

$$A(\underline{x},\underline{y}) = -g\,\delta^2(\underline{x}-\underline{y})\,\partial^2_{\vec{x}}$$

<u>Path-integral</u> :

Green's function for $A(x,y)$.

$$Z[J] = Z[0]\, e^{\frac{1}{2}\int d^2x\, d^2y\; J(x)\,K(x,y)\,J(y)}$$

$$\left(= \int D\phi\; e^{-S + \int d^2x\, J(x)\phi(x)} \right)$$

Wick rotation

Euclidean. nonunitary.

$$-g\,\partial^2_{\vec{x}}K(\underline{x},\underline{y}) = \delta(\underline{x}-\underline{y}). \Rightarrow \quad K(\underline{x},\underline{y}) = -\frac{1}{2\pi g}\log\left(\sqrt{(x-y)^2}\right)$$

(2+0)d

In higher-dim, $K \sim \frac{1}{r^d}$, In $(2+0)d$, $K(x,y)$ doesn't decay but growth.

$$\langle \hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)} Z[J]\Big|_{J=0}$$

Local fields

$$= \underline{K(x,y)} = -\frac{1}{4\pi g}\log(r^2).$$

Further the separation, stronger the correlation !

Analytically continue: $\begin{cases} z = x^0 + i x^1 \\ w = y^0 + i y^1 \end{cases}$

$$\langle \hat{\phi}(z)\hat{\phi}(w)\rangle = -\frac{1}{4\pi g}\log\left((\bar{z}-\bar{w})(z-w)\right)$$

part of observables in theory

$$= -\frac{1}{4\pi g}\Big[\underbrace{\log(\bar{z}-\bar{w})}_{\text{antiholomorphic}} + \underbrace{\log(z-w)}_{\text{holomorphic}}\Big]$$

Rmk. ① Higher-pt functions are obtained by Wick's theorem in terms of K

② $G(x_1,x_2) \Rightarrow \hat{\phi}(x)$ is not a primary field.

Primary field : $\partial_z \hat{\phi}(z)$ is a candidate of primary field.

$$\begin{cases} \langle \partial_z \hat{\phi}\, \partial_w \hat{\phi} \rangle = -\frac{1}{4\pi g} \frac{1}{(z-w)^2} \\ \langle \partial_{\bar z} \hat{\phi}\, \partial_{\bar w} \hat{\phi} \rangle = -\frac{1}{4\pi g} \frac{1}{(\bar z-\bar w)^2} \end{cases}$$

OPE : $\partial_z \hat{\phi}\, \partial_w \hat{\phi} \sim -\frac{1}{4\pi g} \frac{1}{(z-w)^2}$

conformal transf.
↓
To check if $\partial_z \hat{\phi}(z)$ is a primary field, we need $\hat{T}(z)$.

note: once we solve $\hat{T}(z)$, actually the theory is completely solved (constants of motion).

traceless :
$\langle T^\mu{}_\mu \rangle = 0$

Energy-Momentum tensor :

$$\text{classical } T_{\mu\nu} \overset{\text{Noether}}{=} g\left(\partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}\eta_{\mu\nu}\,\partial_\rho\phi\,\partial^\rho\phi\right)$$

↓

quantum (operator) : quadratic theory (free) suffice to put ^ on upon normal ordering (extract ∞).

change coordinates ⟹

$$\hat{T}(z) = -2\pi T_{zz} = (2\pi)\tfrac{1}{4}\left(T_{00} - 2i T_{10} - T_{11}\right)$$

$$= -2\pi g :\partial_z\phi\,\partial_z\phi:$$

OPE of $\hat{T}(z)$ & $\partial_z\hat{\phi}(z)$:

$$\hat{T}(z)\,\partial_w\hat{\phi}(w) \sim$$

put $(\cdots)$ inside an $\langle \cdots \rangle$
compute it's singular behavior as $z \to w$

Wick's theorem

$$-2\pi g\Big(:\partial_z\hat{\phi}\,\partial_z\hat{\phi}:\,\partial_w\hat{\phi}$$

$$+ :\partial_z\hat{\phi}\,\partial_z\hat{\phi}:\,\partial_w\phi \Big).$$

$$= -4\pi g\,\partial_z\hat{\phi}\,\langle \partial_z\hat{\phi}\,\partial_w\hat{\phi} \rangle$$

$$= \frac{\partial_z\hat{\phi}}{(z-w)^2} \overset{\text{Taylor}}{=} \frac{\left(\partial_w\phi + (z-w)\partial_w^2\phi + \text{regular}\right)}{(z-w)^2}$$

$$= \frac{\partial_w\phi}{(z-w)^2} + \frac{\partial_w^2\phi}{(z-w)}$$

$\Longrightarrow$ $\partial_z\phi$ is indeed a primary field.

OPE of $\hat{T}(z)\,\hat{T}(w)$:

Physical meaning: do local conformal transf "twice". what's the composed local conformal transf.

$$\hat{T}(z)\,\hat{T}(w) = 4\pi^2 g^2 :\partial_z\hat{\phi}\,\partial_z\phi::\partial_w\hat{\phi}\,\partial_w\phi:$$

$$\sim \overset{\text{Wick}}{=} \frac{1/2}{(z-w)^4} - 4\pi g \frac{:\partial_z\hat{\phi}\,\partial_w\hat{\phi}:}{(z-w)^2}$$

$$\overset{\text{Taylor}}{=} \frac{1/2}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w\hat{T}(w)}{(z-w)}$$

OPE of $\hat{T}(z)\hat{T}(w)$ in general:

this expose the quantum algebra of local conformal transfs.

central charge

$$\hat{T}(z)\hat{T}(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{A\,\hat{T}(w)}{(z-w)^2} + \frac{B\,\partial_w\hat{T}(w)}{(z-w)}$$

T is of rank 2

$$T(z) \to \frac{1}{\lambda^2} T(\lambda z)$$

$$\partial_z T(z) \to \frac{1}{\lambda^3} \partial_{\lambda z} T(\lambda z)$$ (no possible $\frac{1}{(z-w)^3}$ term)

c : central extension of Witt algebra.

(Conformal group be PUR).

$$\langle \hat{T}(z)\hat{T}(w)\rangle \geq 0 \Rightarrow c \geq 0$$

§ The Virasoro algebra.

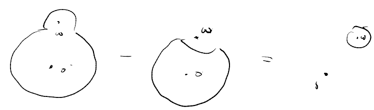
Make mode expansion of $\hat{T}(z)$ (Laurent series.)

$$\hat{T}(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2}\hat{L}_n \,, \quad \hat{\bar{T}}(\bar{z}) \equiv \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\hat{\bar{L}}_n$$

$$\Longrightarrow \begin{cases} \hat{L}_n \equiv \oint \dfrac{dz}{2\pi i}\, \hat{T}(z)\, z^{n+1} \\ \hat{\bar{L}}_n \equiv \oint \dfrac{d\bar{z}}{2\pi i}\, \hat{\bar{T}}(\bar{z})\, \bar{z}^{n+1} \end{cases}$$

compute $[\,,\,]$ of these operators.

$$[\hat{A}, \hat{B}] = \oint_0 dw \oint_w dz \; \hat{a}(z)\hat{b}(w)$$



$$[\hat{L}_n, \hat{L}_m] = \frac{1}{(2\pi i)^2}\oint_0 dw\, w^{m+1} \oint_w dz\, z^{n+1}$$

$$\times \left\{ \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w(T(w))}{(z-w)} \right\}$$

Residue

$$= \frac{1}{2\pi i} \oint_0 dw\, w^{n+1} \left\{ \frac{c}{12} (n+1) n (n-1) w^{n-2} + 2(n+1) w^{n} \hat{T}(w) + w^{n+1} \partial_w \hat{T}(w) \right\}$$

$$= \frac{c}{12} n(n^2-1)\, \delta_{n+m,0} + 2(n+1) \hat{L}_{n+m} - \frac{1}{2\pi i} \oint_0 dw\, (n+m+2)\, w^{n+m+2} \hat{T}(w).$$

$$= \underline{\frac{c}{12} n(n^2-1)\, \delta_{n+m,0}} + (n-m) \hat{L}_{m+n}$$

Similar for $\hat{\bar{L}}_n$'s. $[L_n, \bar{L}_m] = 0$

central extension.

Hilbert space of QFT

Hilbert space
↓
Note, there's an axiomatic way to construct $\mathcal{H}$. for any field theory by **Wightman functions**.

n-pt correlation functions $\Longrightarrow$ linear space & inner product (Hilbert space).
↑
but for general QFT, n-pt is not well-understood

$$W_n = \langle 0 | \hat{\phi}_1(x_1) \cdots \hat{\phi}_n(x_n) | 0 \rangle$$

Def. Hermitian

$$|x_1, \ldots, x_n\rangle \equiv \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) |0\rangle$$

distributions (in general not normalizable)

$$\langle x_1, \ldots, x_n | y_1, \ldots, y_m\rangle = \langle 0 | \underbrace{\phi(x_1) \cdots \phi(x_n) \phi(y_1) \cdots \phi(y_m)}_{W_n(x_1, \ldots, x_n; y_1, \ldots, y_m)} | 0 \rangle$$

$\Rightarrow$ (become normalizable) smearing functions
↓

$$\langle f_1 \ldots f_n | g_1 \ldots g_m \rangle = \int dx_1 \ldots dy_1 \ldots \bar{f}_1(x_1) \cdots f_n(x_n) \times g_1(y_1) \cdots g_m(y_m) \times W_n(x_1, \ldots, x_n; y_1, \ldots, y_m)$$

Let $\hat{A}(z,\bar{z})$ be a field, associate to $\hat{A}$ we have the

in-state :

$$|A_{in}\rangle = \lim_{\sigma^0 \to -\infty} \hat{A}(\sigma^0,\sigma^1)|0\rangle$$

$$= \lim_{z,\bar{z}\to 0} \hat{A}(z,\bar{z})|0\rangle$$

only one state, independent of both σ^0, σ^1, since we compactify the space (Riemann sphere).

out-state :

$$\langle A_{out}| = \lim_{w,\bar{w}\to 0} \langle 0| \hat{\tilde{A}}(w,\bar{w}) \quad , w = \frac{1}{z}$$

$$\tilde{\phi}(w,\tilde{w}) = \phi(f(w),\bar{f}(w))\, \partial f(w)^{h}\, \bar{\partial}\bar{f}(\bar{w})^{\bar{h}}$$

$\tilde{A}(w,\bar{w}) = (\cdots)\hat{A}(z,\bar{z})$: primary field :

factors due to $z \to w = \frac{1}{z}$

$$\langle \phi_{out}| = \lim_{z,\bar{z}\to 0} \langle 0| \hat{\phi}\left(\frac{1}{z},\frac{1}{\bar{z}}\right) \frac{1}{z^{2h}} \frac{1}{\bar{z}^{2\bar{h}}}$$

definition of adjoint

$$= \lim_{z,\bar{z}\to 0} \langle 0| \hat{\phi}^{\dagger}(\bar{z},z)$$

$$= \left[\lim_{z,\bar{z}\to 0} \hat{\phi}(\bar{z},z)|0\rangle\right]^{\dagger}$$

no difference in limit

$$= |\phi_{in}\rangle^{\dagger}$$

Rmk : this is not the same meaning of scattering in/out states.

Inner product:

$$\langle \phi_{out} | \phi_{in} \rangle = \lim_{z,\bar{z},w,\bar{w} \to 0} \langle 0 | \underbrace{\hat{\phi}(z,\bar{z})^{+}}_{z,\bar{z}\to 0} \hat{\phi}(w,\bar{w}) | 0 \rangle$$

$$= \lim_{\cdots} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0 | \phi(\tfrac{1}{\bar{z}}, \tfrac{1}{z}) \phi(w,\bar{w}) | 0 \rangle$$

$$= \lim_{\bar{\zeta},\zeta \to 0} \bar{\zeta}^{2h} \zeta^{2\bar{h}} \langle 0 | \phi(\bar{\zeta}, \zeta) \phi(0,0) | 0 \rangle$$

(independent of $\bar{\zeta}, \zeta$).

The Hilbert space is generated by acting on L_0, L_{-1}, L_{+1}.

(forming $SL(2,\mathbb{C})$ subalgebra)

Ensure $|0\rangle$ is invariant under $SL(2,\mathbb{C})$ by

demanding $T(z)|0\rangle$ & $\bar{T}(\bar{z})|0\rangle$ are regular.

(as $z, \bar{z} \to 0$).

$$\Rightarrow \quad T(z)|0\rangle = \sum_{n\in\mathbb{Z}} \hat{L}_n z^{-n-2} |0\rangle$$

$$\Rightarrow \quad \underbrace{\hat{L}_n|0\rangle = 0}_{\text{invariance}} \quad \forall n \geq -1 \quad (\text{especially } -1, 0, +1)$$

$$\& \quad \hat{\bar{L}}_n |0\rangle = 0 \quad \forall n \geq -1$$

Primary fields create eigenstate of $\hat{H} = \hat{L}_0 + \hat{\bar{L}}_0$

$$[L_n, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, \hat{T}(z)\phi(w,\bar{w}).$$

$$\overset{OPE}{=} \frac{1}{2\pi i}\oint_{\varepsilon} dz\, z^{n+1}\left[\frac{h\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\phi(w,\bar{w})}{z-w} + \text{reg}\right]$$

(*) $$= h(n+1)w^n\,\hat{\phi}(w,\bar{w}) + w^{n+1}\partial_w\phi(w,\bar{w}).$$

<u>Define</u> asymptotic in state : (eigenstate of $\hat{H}$)

$$|h,\bar{h}\rangle \equiv \phi(0,\bar{0})|0\rangle$$

use (*) $\hat{L}_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle$. $\quad \hat{\bar{L}}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle$

$$\hat{L}_n|h,\bar{h}\rangle = \hat{\bar{L}}_n|h,\bar{h}\rangle = 0, \quad \underline{\forall n > 0}$$

Excited states are created by modes of primary field.

$$\hat{\phi}(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}} z^{-m-h}\,\bar{z}^{-n-\bar{h}}\,\phi_{m,n}$$

$$\Rightarrow \phi_{m,n} = \left(\frac{1}{2\pi i}\oint dz\, z^{m+h-1}\right)\left(\frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+\bar{h}-1}\,\hat{\phi}(z,\bar{z})\right)$$

$$\left(\phi^{+}_{m,n} = \phi_{-m,-n} \text{ on } z = \bar{z} \right).$$

For in/out states to be well-defined: $\phi_{m,n}|0\rangle = 0$

$m > -h,\ n > -\bar{h}$

drop $\bar{z}$ dependence

$$\phi(z,\bar{z}) \longmapsto \phi(z) = \sum_{n \in \mathbb{Z}} z^{-n-h} \hat{\phi}_n$$

$$\phi_m = \frac{1}{2\pi i} \oint dz\, z^{m+h-1} \phi(z).$$

$$[L_n, \phi_m] \overset{(*)}{=} [n(h-1) - m] \hat{\phi}_{n+m}$$

$\Longrightarrow$ ϕ_{-m} increase conformal dimension by m.

$\hat{L}_{-m}$ also raise dimensions $[\hat{L}_0, \hat{L}_{-m}] = m \hat{L}_{+m}$

descendants: $\hat{L}_{-k_1} \dots \hat{L}_{-k_n} |h\rangle$

have eigenvalue

$$h' = h + k_1 + \dots + k_n = h + \underset{\text{level}}{\underline{N}}$$

The subspace of full Hilbert space generated by $|h\rangle$ & descendants is a <u>module</u> ("rep") of Virasoro

(closed/invariant under the algebra)

Using $L_n^\dagger = L_{-n}$ $L_0|h\rangle = h|h\rangle$, $\hat{L}_n|h\rangle = 0$ $(n>0)$ & Virasoro

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

$$\Rightarrow \langle h|[L_{-n}^\dagger, L_{-n}]|h\rangle = \langle h|[\hat{L}_n, L_{-n}]|h\rangle$$

$$= \left(2nh + \frac{c}{12}(n^3-n)\right)\langle h|h\rangle$$

$\Rightarrow c \geq 0$, & $n=1$ $h \geq 0$

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