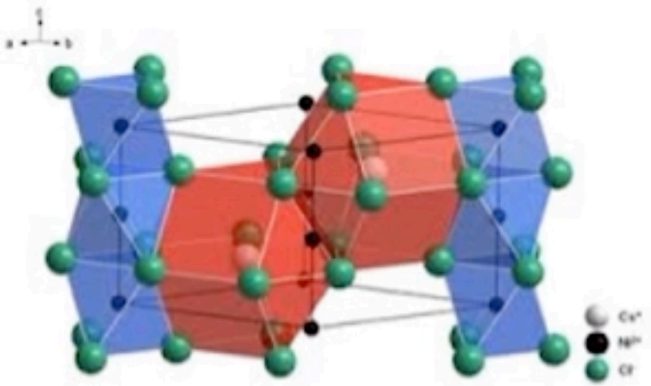
SPT phases in 1D interacting systems

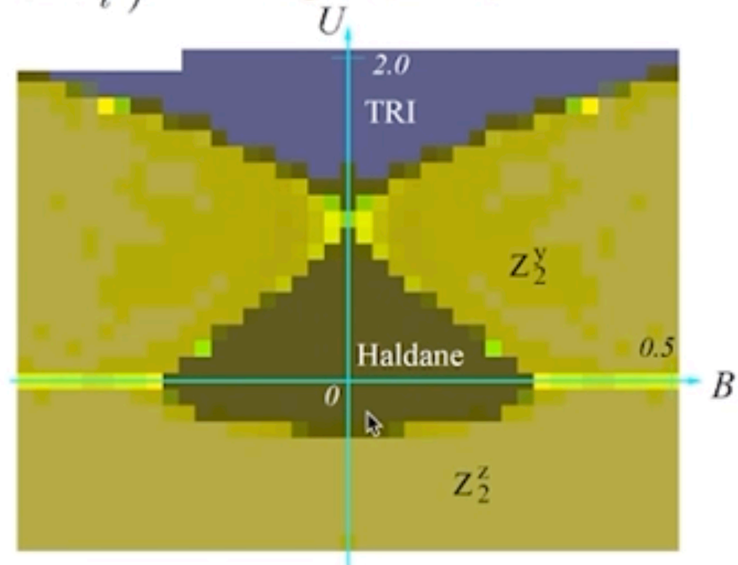
Spin one Haldane chain realizes 1D topological order

$$H = \sum_i \left(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + U(S_i^z)^2 + BS_i^x \right)$$

U_c~1(B=0)



$CsNiCl_3$(U~B~0)



Haldane phase requires symmetry!

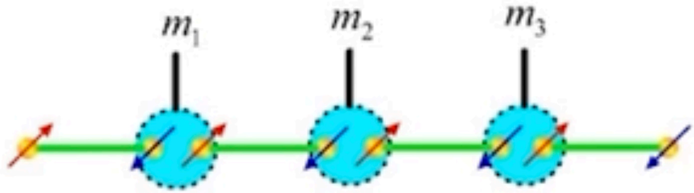
To define a distinct phase of matter, it is important that the characteristic properties are not destroyed by small perturbations. For the Haldane phase this was investigated in 2009 by Gu and Wen

---- Advanced information of 2016 Nobel Prize

The physical nature of Haldane phase

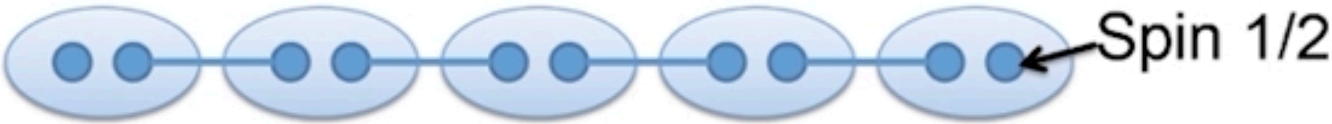
AKLT model realizes Haldane phase (Ian Affleck *etal.*, (1988))

$$H = \sum_i P_2(\mathbf{S}_i + \mathbf{S}_{i+1})$$
$$= \sum_i [\tfrac{1}{2}\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \tfrac{1}{6}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 + \tfrac{1}{3}].$$



$m_i = 0, \pm 1$ $\qquad \frac{1}{2} \otimes \frac{1}{2} = \cancel{0} \oplus 1$

The ground state of AKLT model can be deformed into decoupled spin-1/2 dimers without phase transition!

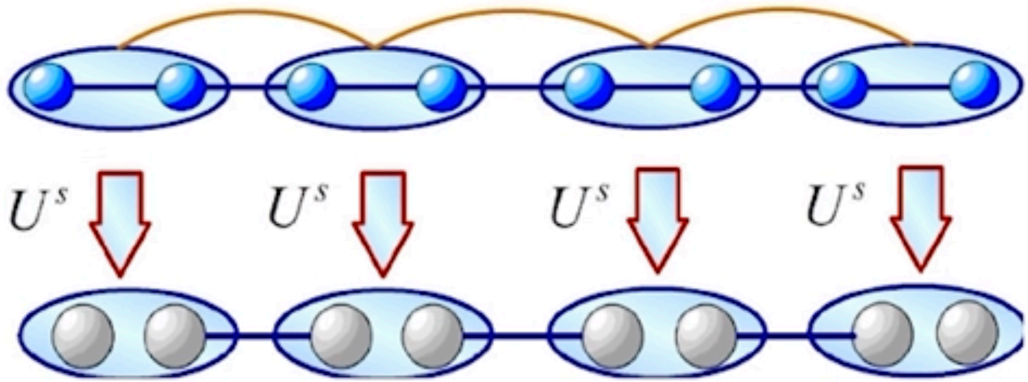


The key observation: edge states form projective representation of the symmetry group!

RG Fixed point wavefunction

SPT phase is a gapped quantum phase with unique ground state

correlation length $\xi = 2$



Each dimer is a symmetric maximally entangled state.

Each dimer end carry projective representation.

$$\boxed{u_L(g_1)u_L(g_2) = \omega(g_1, g_2)u_L(g_1g_2)}$$

$$\boxed{u_R(g_1)u_R(g_2) = \omega(g_1, g_2)^{-1}u_R(g_1g_2)}$$

Classification of 1D bosonic SPT phases

- A condition on the omega factor

$$u(g_1)[u(g_2)u(g_3)] = \omega(g_2,g_3)u(g_1)u(g_2g_3) = \omega(g_2,g_3)\omega(g_1,g_2g_3)u(g_1g_2g_3)$$

$$[u(g_1)u(g_2)]u(g_3) = \omega(g_1,g_2)u(g_1g_2)u(g_3) = \omega(g_1,g_2)\omega(g_1g_2,g_3)u(g_1g_2g_3)$$

$$\boxed{\omega(g_2,g_3)\omega(g_1,g_2g_3) = \omega(g_1,g_2)\omega(g_1g_2,g_3)}$$

- An equivalent relation

$$\boxed{u_{L(R)}(g) \ \sim \ \beta_{L(R)}(g)u_{L(R)}(g); \quad \beta_{L(R)}(g) \in U(1)}$$

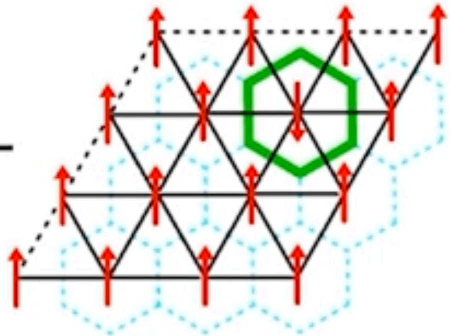
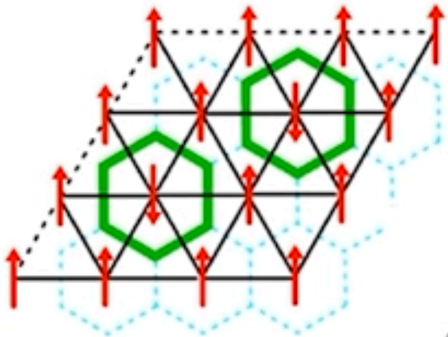
Projective representation which is classified by second group cohomology $H^2(G, U(1))$!

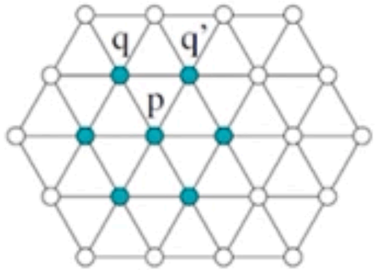
$$\begin{aligned}
\mathcal{Z}^2[G,U(1)] &= \{\omega \in U(1) | \omega(g_2,g_3)\omega(g_1,g_2g_3) = \omega(g_1,g_2)\omega(g_1g_2,g_3)\} \\
\mathcal{B}^2[G,U(1)] &= \{\omega \in U(1) | \omega(g_1,g_2) = \beta(g_1)\beta(g_2)/\beta(g_1g_2); \beta \in U(1)\} \\
\mathcal{H}^2[G,U(1)] &= \mathcal{Z}^2[G,U(1)]/\mathcal{B}^2[G,U(1)]
\end{aligned}$$

An example of Ising SPT phase in 2D

How many different paramagnetic phases? Two!

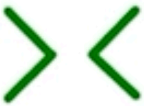
(M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012))

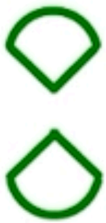
$|\Psi_1\rangle = -$  $+$  $+ \dots$



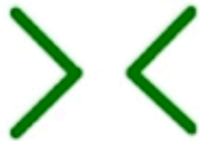
$$H_1 = -\sum_p B_p \,, \quad B_p = -\sigma_p^x \prod_{\langle pqq' \rangle} i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}} \qquad [B_p, B_{p'}] = 0$$

Domain deformation rule

 $= -$ 

 $= -$ 

But why not?

 $= e^{i\theta}$ 

Duality between Toric code model and Ising model

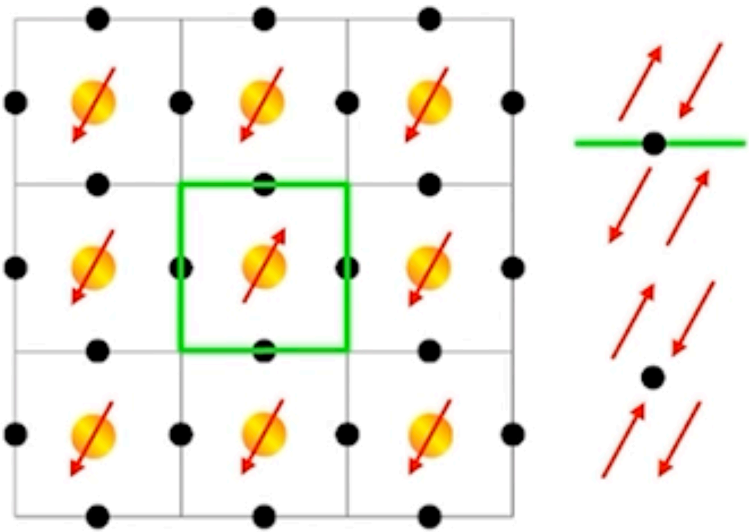
$$H_{Z_2} = U\sum_v \left(1 - \prod_{l\in v}\tau_l^z\right) - t\sum_p\prod_{l\in p}\tau_l^x \quad \Longleftrightarrow \quad H_{\text{Ising}} = -\sum_p \sigma_p^x$$

$$\begin{aligned} \tau_l^z &= \sigma_p^z\sigma_q^z \\ \sigma_p^x &= \prod_{l\in p}\tau_l^x \end{aligned}$$

$$H'_{Z_2} = U\sum_v \left(1 - \prod_{l\in v}\tau_l^z\right) - t\sum_p\prod_{l\in p}\tau_l^x - h\sum_l \tau_l^z$$

$$\Updownarrow \qquad \Updownarrow$$

$$H = -t\sum_p \sigma_p^x - h\sum_{\langle pq\rangle}\sigma_p^z\sigma_q^z$$

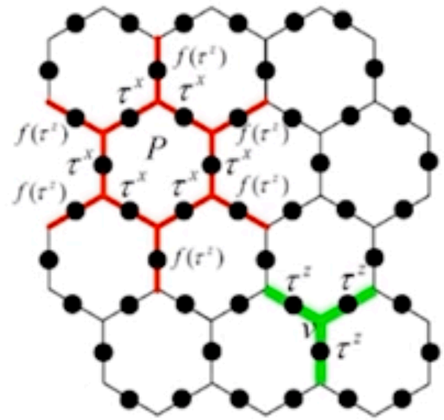


Duality map requires Z_2 symmetry preserving!

The twisted toric code: double semion model

(M. Levin and X.G. Wen 2005)

$$H_{dsemion} = U\sum_v \left(1-\prod_{l\in v}\tau_l^z\right) - t\sum_p \left(\prod_{l\in p}\tau_l^x \prod_{l\in \text{leg of } p} i^{\frac{1+\tau_l^z}{2}}\right)$$



$$|\Psi_{\text{dsemion}}\rangle = \sum_{X\,\text{closed}} (-)^{n(X)}|X\rangle$$

$$f(x) = \frac{1+x}{2}$$

Double semion model describes a different intrinsic topological order

$$L = \frac{1}{4\pi}K_{IJ}a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda}, \qquad I,J=1,2 \qquad K=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$$

- End of string is a semion.

Quasi-particle types in double semion model:

1, s, $\underline{s}$, b=s$\underline{s}$

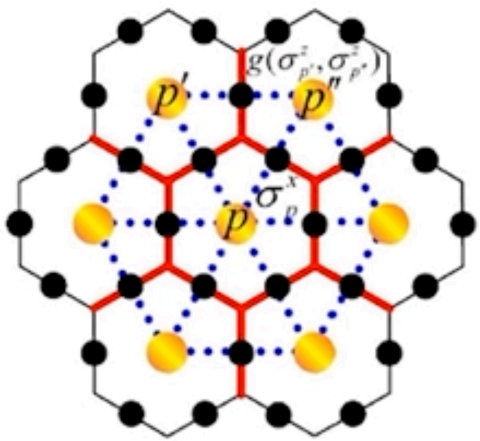
The dual theory of double semion model

$$H_{dsemion} = U\sum_v \left(1 - \prod_{l\in v}\tau_l^z\right) - t\sum_p \left(\prod_{l\in p}\tau_l^x \prod_{l\in \text{leg of } p} i^{\frac{1+\tau_l^z}{2}}\right)$$

$$\tau_l^z = \sigma_{p'}^z\sigma_{q'}^z$$

$$\sigma_p^x = \prod_{l\in p}\tau_l^x$$

$$H_{\text{twistIsing}} = -\sum_p \tilde{\sigma}_p^x = -\sum_p \left(\sigma_p^x \prod_{\text{sites}\in \text{p}} i^{\frac{1+\sigma_{p'}^z\sigma_{p''}^z}{2}}\right)$$



The dual theory of double semion model is an SPT ordered phase!

$$H = -\alpha\sum_p \sigma_p^x - (1-\alpha)\sum_p \tilde{\sigma}_p^x$$

$$H' = -U\sum_v \prod_{i\in v}\sigma_i^z - \alpha\sum_p\prod_{i\in p}\sigma_i^x - (1-\alpha)\sum_p\left(\prod_{i\in p}\sigma_i^x \prod_{\text{legs of} p} i^{\frac{1+\sigma_i^z}{2}}\right)$$

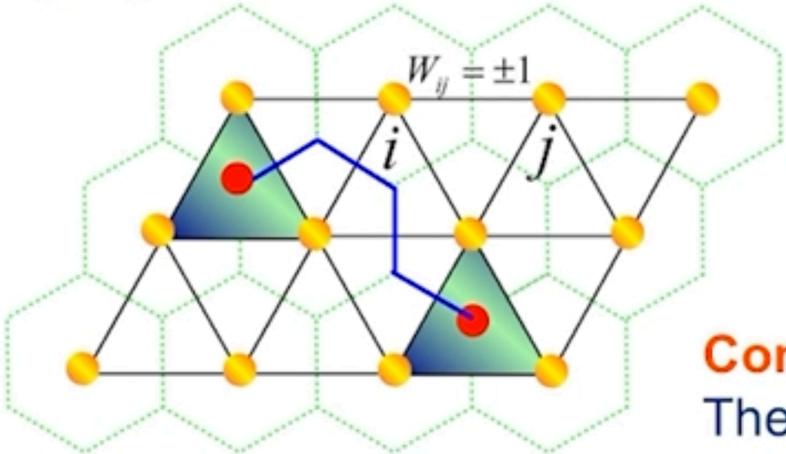
Different SPT orders

Different intrinsic topological orders

Anomaly detecting and the nature of gapless edge

Assume that Ising spins carry Z2 gauge charge and can couple to background Z2 gauge field

Z2 gauge flux carries semion statistics!



$$\widetilde{W}_\beta|0\rangle = |0\rangle$$

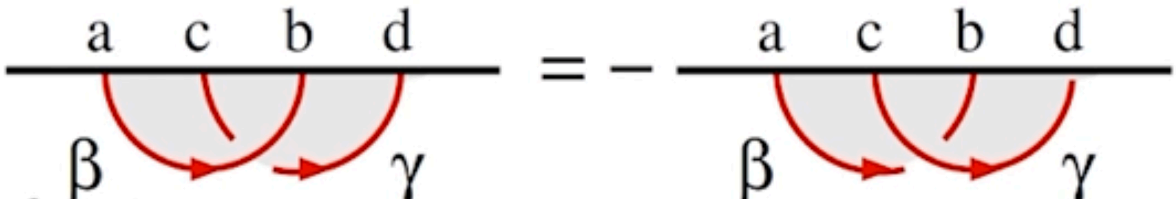
$$\widetilde{W}_\gamma|0\rangle = |0\rangle$$

$$\tilde{W}_\beta\tilde{W}_\gamma = -\tilde{W}_\gamma\tilde{W}_\beta$$

Contradiction

There is No 1D representation!

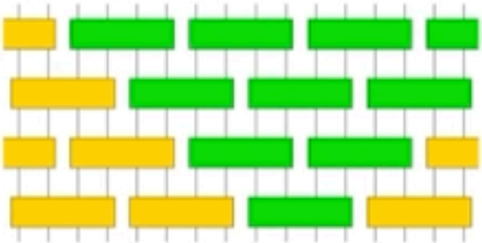
Non-trivial statistics of flux leads to degenerate edge states!



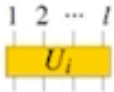
SPT phases defined via equivalence classes of symmetric local unitary transformation

- Two states describe the same symmetry protected topological phase iff they are connected by finite depth (generalized) symmetric local unitary(SLU) transformation.

$$|\Phi(1)\rangle \sim |\Phi(0)\rangle \text{ iff } |\Phi(1)\rangle = U^M_{circ}|\Phi(0)\rangle \qquad U^M_{circ} = U^{(1)}_{pwl} U^{(2)}_{pwl} \cdots U^{(M)}_{pwl}$$



(a)



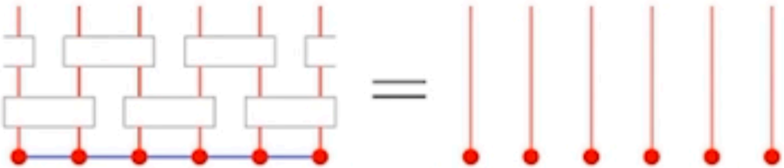
(b)

$$U_{pwl} = \prod_i U(g_{i0}, g_{i1}, g_{i2}, \cdots)$$

- U_i is l-local

$$U(gg_{i0}, gg_{i2}, gg_{i3}, \cdots) = U(g_{i0}, g_{i1}, g_{i2}, \cdots) \;\; g \in G$$

SPT state is short-range entangled:



$$|\mathrm{SPT}\rangle = U^M_{circ}|\mathrm{Trivial}\rangle$$

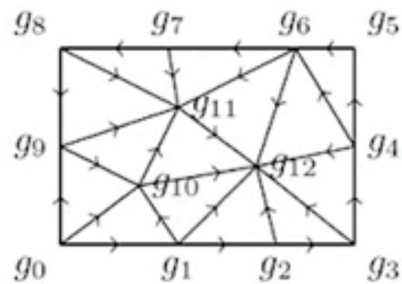
SPT phases are classified by equivalence class of generalized SLU transformation with one dimensional support space!

A general fixed point wavefunction for 2D SPT state

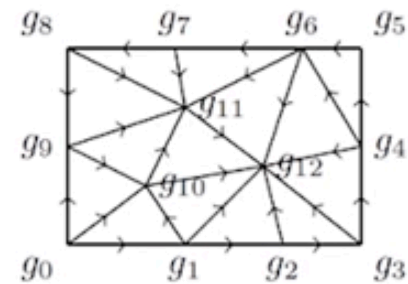
- As group element basis gives rise to the faithful representation of a symmetry group, we use this basis to describe a symmetric wavefunction.

- Fixed-point state is a superposition of those basis states on all possible triangulations with a branching structure.

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi\left(\right.$$



$$\left.\right)\Bigg|$$

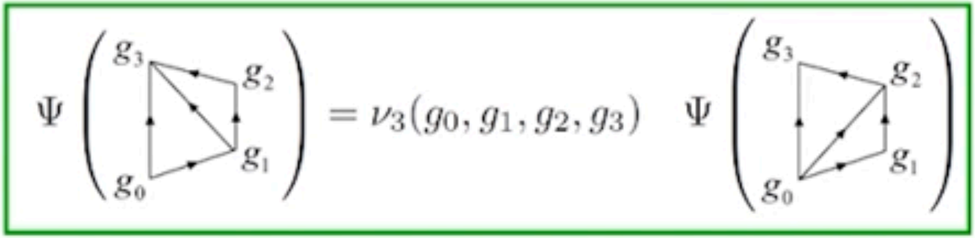


$$\Bigg\rangle$$

- Branching structure is implemented as an arrow on each link, and there is no oriented loops on any triangle.

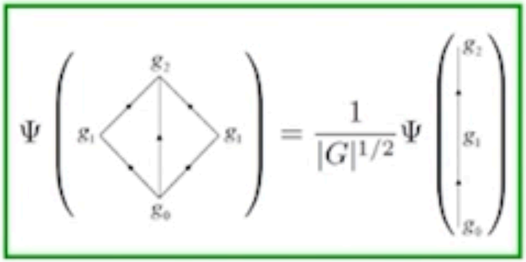
Basic wavefunction renormalization moves with symmetric local unitary transformation

The standard 2-2 move (The analogy of F-move)



$\nu_3(gg_0, gg_1, gg_2, gg_3) = \nu_3(g_0, g_1, g_2, g_3)$ is a symmetric U(1) phase

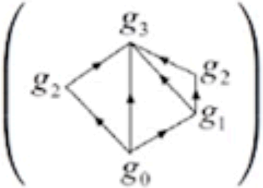
The standard 2-0/0-2 move (The analogy of O/Y-move)

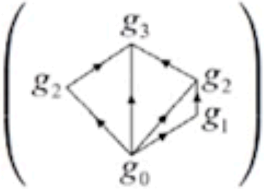


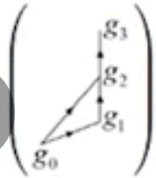
With the above two types of basic renormalization moves, all other renormalization moves can be generated!

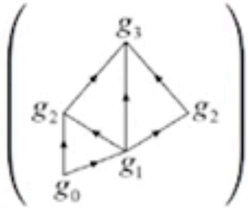
Example: other wavefunction renormalization move generated by basic moves

Another 2-2 move (The analogy of H-move)

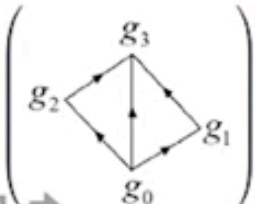
Ψ 

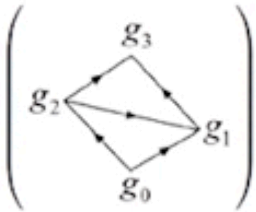
$= \nu_3(g_0, g_1, g_2, g_3)$ Ψ 

$= \frac{1}{|G|^{1/2}} \nu_3(g_0, g_1, g_2, g_3)$ Ψ 

$= \nu_3(g_0, g_1, g_2, g_3)$ Ψ 

- Compare the different part

Ψ 

$= \nu_3(g_0, g_1, g_2, g_3)$ Ψ 

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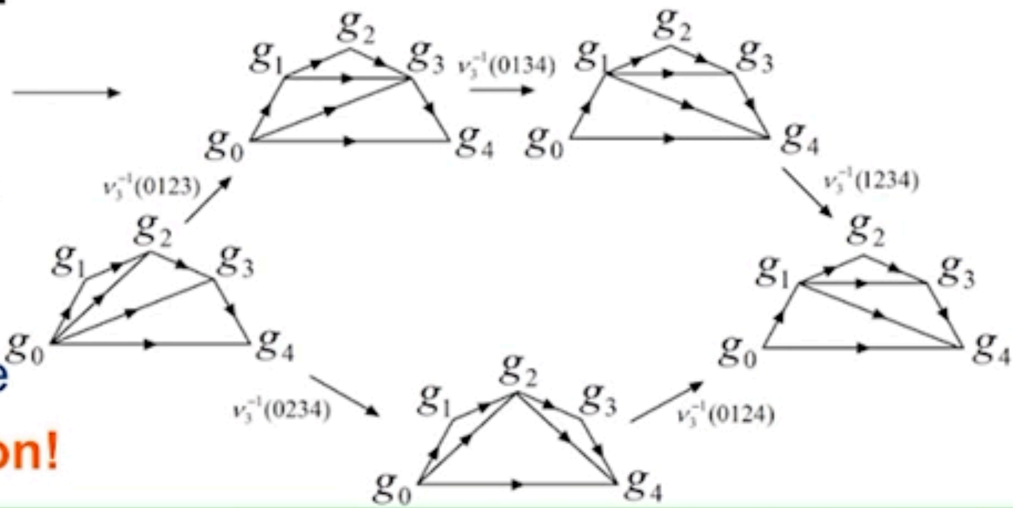
Coherent condition and classification of 2D bosonic SPT phases

As a fixed point wavefucntion, different symmetric LU transformation must give rise to the same amplitude



3-cocycle condition!

$$(\mathrm{d}\nu_3)(g_0,g_1,g_2,g_3,g_4) \equiv \frac{\nu_3(g_1,g_2,g_3,g_4)\nu_3(g_0,g_1,g_3,g_4)\nu_3(g_0,g_1,g_2,g_3)}{\nu_3(g_0,g_2,g_3,g_4)\nu_3(g_0,g_1,g_2,g_4)} = 1$$

Equivalent solutions

$$|\{g_l\}\rangle' = U_{\mu_2}|\{g_l\}\rangle = \prod_{\langle ijk\rangle} \mu_2(g_i,g_j,g_k)^{s_{\langle ijk\rangle}}|\{g_l\}\rangle$$

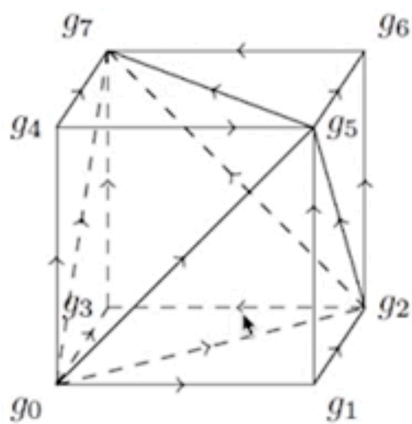
$$\nu_3'(g_0,g_1,g_2,g_3) = \nu_3(g_0,g_1,g_2,g_3)\frac{\mu_2(g_1,g_2,g_3)\mu_2(g_0,g_1,g_3)}{\mu_2(g_0,g_2,g_3)\mu_2(g_0,g_1,g_2)}$$

Fixed point wavefunction of 3D bosonic SPT states

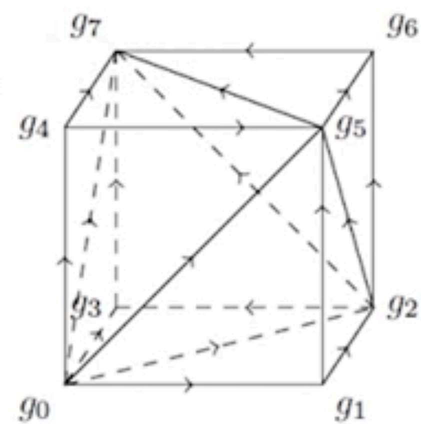
- Again, we use group element basis to describe a symmetric wavefunction.

- Fixed-point state is a superposition of those basis states on all possible triangulations with a branching structure.

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi\left(\right.$$



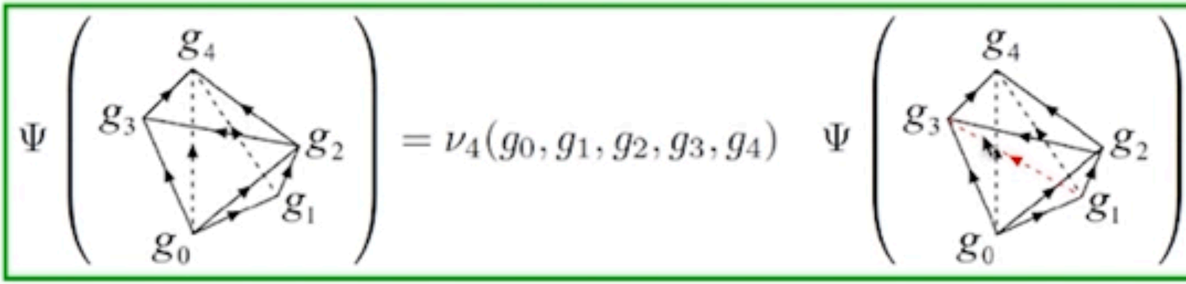
$$\left. \right) \Bigg|$$



$$\Bigg\rangle .$$

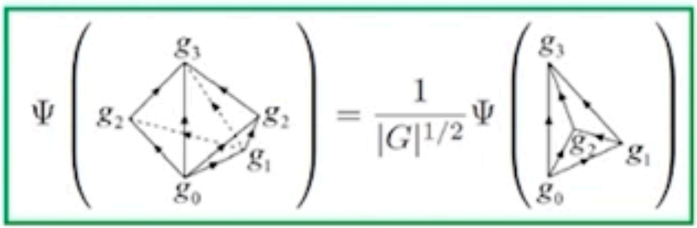
Basic wavefunction renormalization moves with symmetric local unitary transformation

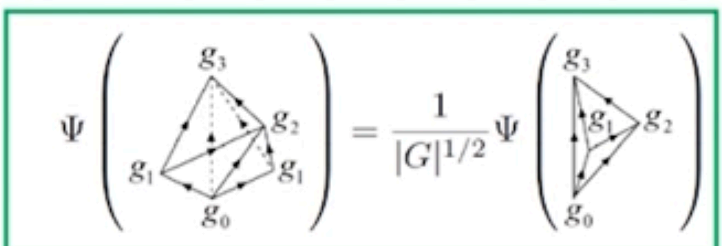
The standard 2-3/3-2 move



$\nu_4(g_0, g_1, g_2, g_3, g_4) = \nu_4(gg_0, gg_1, gg_2, gg_3, gg_4)$ is a symmetric U(1) phase

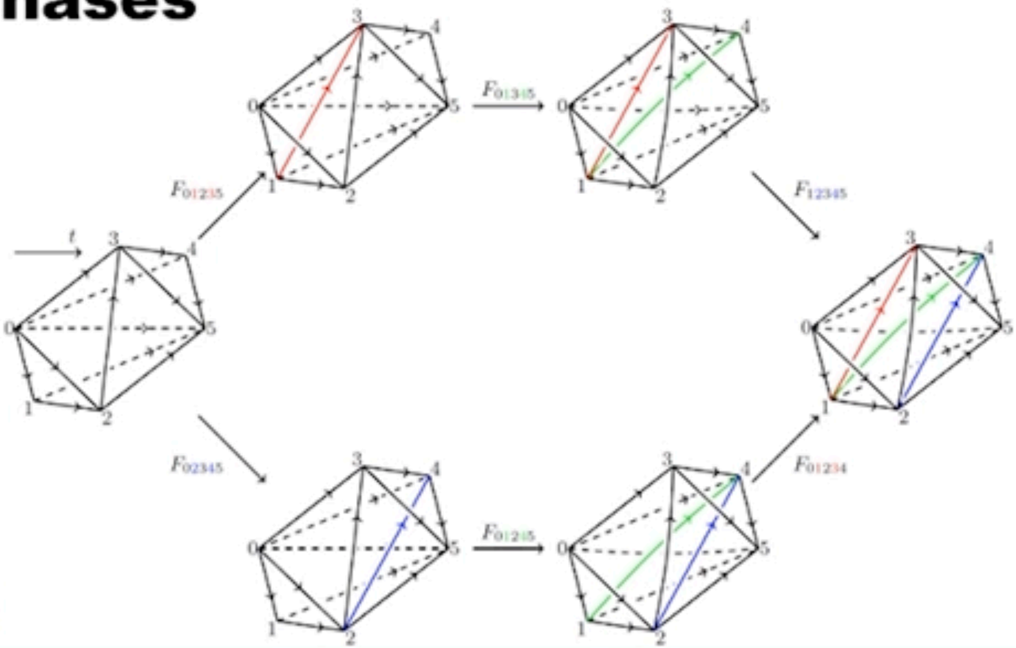
The 2-0/0-2 moves





Coherent condition and classification of 3D bosonic SPT phases

Again, as a fixed point wavefucntion, different symmetric LU transformation must give rise to the same amplitude



4-cocycle condition!

$$(\mathrm{d}\nu_4)(g_0, g_1, g_2, g_3, g_4, g_5) \equiv \frac{\nu_4(g_1, g_2, g_3, g_4, g_5)\nu_4(g_0, g_1, g_3, g_4, g_5)\nu_4(g_0, g_1, g_2, g_3, g_5)}{\nu_4(g_0, g_2, g_3, g_4, g_5)\nu_4(g_0, g_1, g_2, g_4, g_5)\nu_4(g_0, g_1, g_3, g_2, g_4)} = 1$$

Equivalent solutions: $|\{g_l\}\rangle' = U_{\mu_3}|\{g_l\}\rangle = \prod_{\langle ijk\rangle} \mu_2(g_i, g_j, g_k)^{s\langle ijk\rangle}|\{g_l\}\rangle,$

$$\nu_4'(g_0, g_1, g_2, g_3, g_4) = \nu_4(g_0, g_1, g_2, g_3, g_4)\frac{\mu_3(g_1, g_2, g_3, g_4)\mu_3(g_0, g_1, g_3, g_4)\mu_3(g_0, g_1, g_2, g_3)}{\mu_3(g_0, g_2, g_3, g_4)\mu_3(g_0, g_1, g_2, g_3)},$$

Classifications of SPT phases in interacting boson systems

Symm. group	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen (Science 338, 1604 (2012))

- **Almost all SPT phases in interacting bosonic systems are classified by (1+d) group cohomology $\mathcal{H}^{1+d}[G, U(1)]$ in d spacial dimension.**

- Each element gives rise to an exactly solvable hermitian Hamiltonian with a unique ground state on closed manifold.

Extended reading: classification of SPT phases in interacting fermion systems

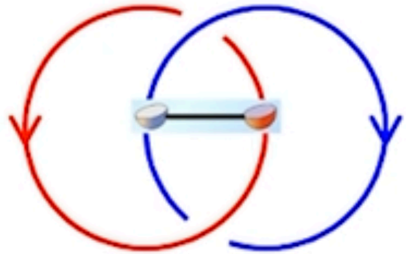
- 1D fermionic systems can be mapped to bosonic systems with an additional unbroken fermion parity symmetry. (Xie Chen, Z C Gu, X G Wen, Phys. Rev. B 84, 235128 (2011))

- The statistics of the gauge flux is still a good way to understand the 2D classification. (Z.-C. Gu, M. Levin,Phys. Rev. B 89, 201113(R) (2014) M. Cheng, Z. Bi. Y. You, Z. C. Gu. arXiv:1501.01313(2015))

- A group super-cohomology theory is developed to classify SPT phases in interacting fermion systems(Physically, such a theory can be regarded as decorating complex fermion on intersection points of symmetry domain walls and decorating Kitaev Mjorana chain on symmetry domain walls in 2D and intersection lines of, symmetry domain walls in 3D). (Z.-C. Gu, X.-G. Wen, Phys. Rev. B 90, 115141 (2014), QR Wang, ZC Gu, Phys. Rev. X 8, 011055 (2018), QR Wang, ZC Gu, arXiv:1811.00536).

Connection to fundamental physics

- Gauging the global symmetry for 3D FSPT states with Majorana chain decoration leads to (Non-Abelian) Ising three loop braiding process, with (minimal) simplest example $G_b=Z_4*Z_4$ or $G_b=Z_2*Z_8$ (JR Zhou, QR Wang, C Wang, ZC Gu, arXiv:1912.13505)



- Condensing loop-like objects leads to the emergence of Einstein equation! (Z.-C. Gu, arXiv:1709.09806)

- Conjecture: Elementary particle can be regarded as tiny linked loop attached by Majorana zero modes.

A natural explanation for three generations!
(Z.-C. Gu, arXiv:1308.2488)