# Anyons in discrete gauge theories

Seminar version

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# Contents

# Introduction to anyons

# Motivation

- Why introduce anyon?
  1. Quasi-particles in many-body systems need not to be elementary particles classified by the PUIRs of $ISO^+(1,3)$,they may have exotic statistics,i.e,anyon statistics.
     Example:Fractional quantum hall states:fractional charge,fractional spin
  2. Many-body systems having topological order(long-range entanglement) are proposed to have anyon excitations.
  3. Nonabelian anyons have extra degeneracies which can be used to realize topological quantum computation
     Example:Kiteav's quantum double model
  4. Anyons related to TQFTs may be used to calculate topological invariants of knots/links.

(1) $SU(2)_2$ **class**. For these, the Kauffman bracket invariant gives the quantum amplitude of a process by using the value $A = ie^{-i\pi/(2(2+2))} = i^{3/4}$. This is also known as "Ising" anyons[10]. Possibly physical realizations include

- $\nu = 5/2$ Fractional Quantum Hall Effect (2D electrons at low temperature in high magnetic field). See chapters ***.
- 2D p-wave superconductors.
- 2D Films of $^3$HeA superfluid[11].
- A host of "engineered" structures that are designed to have these interesting topological properties. Typically these have a combination of spin-orbit coupling, superconductivity, and magnetism of some sort. Recent experiments have been quite promising. See chapter ***?

(2) $SU(2)_3$ **class**. For this, the Kauffman bracket invariant gives the quantum amplitude of a process by using the value $A = ie^{-i\pi/(2(2+3))} = i^{4/5}$. The only physical system known in this class is the $\nu = 12/5$ fractional quantum hall effect.

(3) $SU(2)_4$ **class**. For this, the Kauffman bracket invariant gives the quantum amplitude of a process by using the value $A = ie^{-i\pi/(2(2+4))} = i^{5/6}$. It is possible that $\nu = 2 + 2/3$ Fractional quantum hall effect is in this class.

(4) $SU(2)_1$ **class** Also known as semions. These are proposed to be realized in rotating boson fractional quantum Hall effect (See comments in chapter 39). This corresponds to a fairly trivial knot invariant as we will see later in section ***.

(5) $SU(3)_2$ **class**. This corresponds to a case of the HOMFLY knot invariant rather than the Kauffman bracket invariant. It is possible that the unpolarized $\nu = 4/7$ fractional quantum hall effect is in this class.

Figure: These are closely related to $SU(N)_k$ Chern-Simons theory.This figure is from 'Topological quantum' by H.Simon

# Origin of anyon statistic

**Configuration space**

$$\mathcal{C}_n(\Sigma) = \frac{\Sigma^n - \Delta}{\mathcal{P}} \tag{1}$$

$\Sigma$ is the space manifold,$\mathcal{P} = S_n$ for n identical particle

- Wavefunction:

$$\Psi(K = \{x_i, i = 1, ..., n\} \in \mathcal{C}_n(\Sigma); t) \in \mathbb{C}^k \tag{2}$$

Exchanging two particle have no physical consequnces:

**Remark**

$\Psi$ furnish OUIR of all $\tau_i$.
These $\tau_i$ have further relations which can be derived using path-integral picture.

## Path integral

$$Z(t_f, K_f; t_i, K_i) = \langle t_f, K_f | t_i, K_i \rangle = \langle K_f | U(t_f, t_i) | K_i \rangle \\ \sim \sum_{[\gamma] \in \pi_1(\mathcal{C}_n(\Sigma))} U([\gamma]) \sum_{p \in [\gamma]} e^{iS[p]} \tag{3}$$

- The evolution of the system can be considered as braiding of worldlines in spacetime,forming a path in configuration space classified by $\pi_1(\mathcal{C}_n(\Sigma))$
- For the path integral to be well-defined,the matrice U must furnish OUIR of $\pi_1(\mathcal{C}_n(\Sigma))$
- Consider $\tau_i$ as special paths we have:

## Braid group

$$\pi_1(\mathcal{C}_n(\Sigma)) = B_n = \langle \tau_i, i = 1, ..., n-1 | \begin{cases} \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \\ \tau_i \tau_j = \tau_j \tau_i, |i-j| \geq 2 \end{cases} \rangle \tag{4}$$

Special $\Sigma$ may lead to extra relations.

- When $dim\Sigma \geq 3$,the following relation always holds:

$$\tau_i = \tau_i^{-1} \tag{5}$$

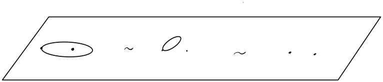

Figure: $\tau_i^2$ is homotopic to the identity

- This fact have deep mathmatical origin:In 4d space M,all knots(embedding of $S^1$ into M) are homeomorphic to the trivial embedding $S^1$ itself,while in 3d space there's no such homeomorphism,i.e,we can form nontrivial knots.

# Classification of particle statistics

- $dim\Sigma \geq 3$:$\pi_1(\mathcal{C}_n(\Sigma)) = S_n$.There are exactly two 1D UIRs of $S_n$:
  1. Boson:$U(\tau_i) = 1$
  2. Fermion:$U(\tau_i) = -1$
- $dim\Sigma = 2$:$\pi_1(\mathcal{C}_n(\Sigma)) = B_n$:
  1. 1D UIRs of $B_n$ takes the form:$U(\tau_i) = e^{i\Theta}$,these are known as abelian anyons.The wavefunction $\Psi$ is 1D,and accumulate a phase when two particles are exchanged.
  2. Higher-dim UIRs of $B_n$ are known as nonabelian anyons.The wavefunction $\Psi$ is higher-dim, and transformed by unitary matrices when two particles are exchanged.
- Higher-dim UIRs of $S_n$ are known as parastatistics,but they are not compatible with locality.[1]
- The extra degeneracies and unitary transformations related to the nonabelian anyon can be turned into elements(protected space;gates) in quantum computation.

[1] The proofs(both field theoretically and categorically) are very tedious.

# Remark

- The anyon statistics can be fully understood as representations of quotient groups of $B_n$.Though it is propsoed that all anyon theories can be described using tensor category theory,the explicit construction of anyon models are various.
- Some types of anyon model appear in the literature of special CFTs & TQFTs,which have a rather clear algebraic structure,which implies that these anyon models are fully described by the algebraic structure.
  Example:$SU(N)_k$Chern-Simon theories,where $SU(N)_k$ are Drinfeld quantum groups(k-deformations of Lie bialgebras,special cases of Hopf algebra)

### An important example:Quasi-quantum double

Holomorphic rational CFTs & Dijkgraaf-Witten TQFT are shown to have algebraic structure known as quasi/twisted-quantum double $D^{\omega}(H)$,a special case of quasi-Hopf algebra.For trivial $\omega$,it's known as quantum double of finite groups,a special case of Hopf algebra.Here finite groups are gauge groups.

- In my bachelor thesis,the algebraic structure of $D^{\omega}(H)$ is reviewed and two physical models(discrete gauge theories) that give rise to anyons described by $D^{\omega}(H)(Rep[D^{\omega}(H)])$ are briefly discussed.

# Anyon diagrammatics

- An anyon theory is fully determined by the following three elements:
  1. Fusion ring:$a \times b = \sum_c N_{ab}^c c$
  2. F-symbol
  3. R-symbol
- All other informations including:$d_a$(quantum dimension);$\mathcal{D}$(total dimension):$\theta_a$(twist factor);$c_a$(central charge);S,T,C-matrices can be derived from them.
- Loosely speaking[2],give certain fusion ring and F-symbol we can form an unitary fusion category.If appropriate R-symbol exists,then we obtain an unitary braided/ribbon fusion category.With more symmetry restrictions[3] we can obtain special categories like spherical category,tetrahedron category,etc

[2] There are more requirements

[3] Relations satisfied by the three elements

- Objects and morphisms can be represented as anyon diagrams.

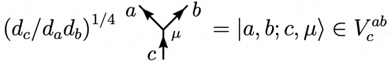


Figure: Split

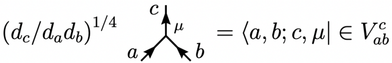


Figure: Fusion

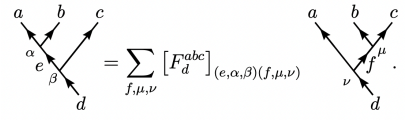


Figure: F-symbol

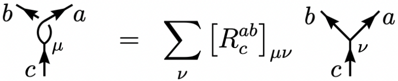


Figure: R-symbol

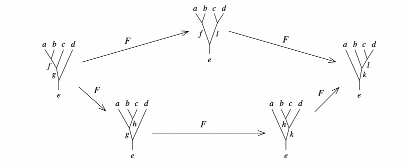

Figure: Pentagon equation

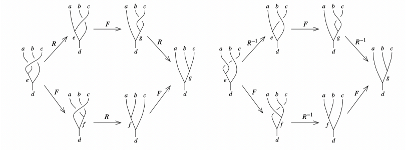

Figure: Hexagon equation

# *Explainations

- Fusion ring:$a \times b = \sum_c N_{ab}^c c$;Split space:$V_{ab}^c$;Fusion space:$V_c^{ab} = V_{ab}^{c*}$.
- F-symbol:Associativity of the fusion ring lead to:

$$V_d^{abc} \cong_1 \oplus_e V_e^{ab} \otimes V_d^{ec} \cong_2 \oplus_f V_d^{af} \otimes V_f^{bc}$$

$$|a, b; e, \alpha\rangle |e, c; d, \beta\rangle = \sum_{f,\mu,\nu} (F_d^{abc})_{(e,\alpha,\beta),(f,\mu,\nu)} |b, c; f, \mu\rangle |a, f; d\nu\rangle$$

For more particles,consistency lead to P-eqn:

$$\sum_{\delta} (F_e^{fcd})_{(g,\beta,\gamma),(l,\delta,\nu)} (F_e^{abl})_{(f,\alpha,\delta),(k,\lambda,\mu)}$$

$$= \sum_{h,\sigma,\psi,\rho} (F_g^{abc})_{(f,\alpha,\beta),(h,\sigma,\psi)} (F_e^{ahd})_{(g,\sigma,\gamma),(k,\lambda,\rho)} (F_k^{bcd})_{(h,\psi,\rho),(l,\mu,\nu)}$$

# *Explainations

- R-symbol: $R_{ab}|a,b;c,\mu\rangle = \sum_\nu (R^{ab}_c)_{\mu\nu}|b,a;c,\nu\rangle$.With more particles,the action of R involve changing basis(F-symbol),and consistency lead to H-eqn:

$$\sum_{\lambda,\gamma}(R^{ca}_e)_{\alpha\lambda}(F^{acb}_d)_{(e,\lambda,\beta),(g,\mu,\gamma)}(R^{cb}_g)_{\gamma\nu}$$
$$=\sum_{f,\sigma,\delta,\psi}(F^{cab}_d)_{(e,\alpha,\beta),(f,\sigma,\delta)}(R^{cf}_d)_{\sigma\psi}(F^{abc}_d)_{(f,\delta,\psi),(g,\mu,\nu)}$$
$$\sum_{\lambda,\gamma}(R^{ac-1}_e)_{\alpha\lambda}(F^{acb}_d)_{(e,\lambda,\beta),(g,\mu,\gamma)}(R^{bc-1}_g)_{\gamma\nu}$$
$$=\sum_{f,\sigma,\delta,\psi}(F^{cab}_d)_{(e,\alpha,\beta),(f,\sigma,\delta)}(R^{fc-1}_d)_{\sigma\psi}(F^{abc}_d)_{(f,\delta,\psi),(g,\mu,\nu)}$$

- Rigidty:Given Fusion ring,the P-eqn have only finite number of gauge[4] equivalent classes of solutions,picking a solution of P-eqn,the H-eqns have only finite number of gauge equivalent classes of solutions.

[4]Here the gauge freedom is the basis transformation of split/fusion space

# (Quasi)-quantum double of finite groups

# Quantum double: $D(H)$

- Given finite group H,we can form two dual Hopf algebras: (1)$\mathbb{C}[H]$ (2)$\mathcal{F}(H)$.With the help of the duality we construct:

**Definition**

$$D(H) := \mathcal{F}(H) \otimes \mathbb{C}[H] = Span\{\delta_g \otimes x \equiv {}^{g}\underset{x}{\llcorner} | g, h \in H\} \tag{6}$$

**Hopf algebra structure**

- Product:

$$ {}^{g}\underset{x}{\llcorner} \cdot {}^{h}\underset{y}{\llcorner} = \delta_{g,xhx^{-1}} \ {}^{g}\underset{xy}{\llcorner} \tag{7}$$

- Unit[a]:

$$ {}^{1}\underset{e}{\llcorner} = \sum_g {}^{g}\underset{e}{\llcorner} \tag{8}$$

where ${}^{1}\underset{x}{\llcorner} = \sum_g {}^{g}\underset{x}{\llcorner}, \forall x \in H$

[a] Unital associative algebra

## Hopf algebra structure

- Coproduct:

$$\Delta({}^{g}\llcorner_{x}) = \sum_{hk=g} {}^{h}\llcorner_{x} \otimes {}^{k}\llcorner_{x} \tag{9}$$

- Counit[a]:

$$\epsilon({}^{g}\llcorner_{x}) = \delta_{g,e}\, {}^{1}\llcorner_{e} \tag{10}$$

- Antipode[b]:

$$S({}^{g}\llcorner_{x}) = {}^{x^{-1}g^{-1}x}\llcorner_{x^{-1}} \tag{11}$$

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[a] Counital coassociative coalgebra
[b] Bialgebra have at most one antipode

## Quasitriangularity

$D(H)$ is actually a quasitriangular Hopf algebra:exist a R-matrix element:

$$R = \sum_{g} {}^{g}\llcorner_{e} \otimes {}^{1}\llcorner_{g} = \sum_{g,h} {}^{g}\llcorner_{e} \otimes {}^{h}\llcorner_{g} \tag{12}$$

satisfying three conditions:

$$\begin{aligned} \tau \circ \Delta(h) &= R\Delta(h)R^{-1} \\ (\Delta \otimes id)(R) &= R_{13}R_{23} \\ (id \otimes \Delta)(R) &= R_{13}R_{12} \end{aligned} \tag{13}$$

where $R = R^{(1)} \otimes R^{(2)}$,$R_{ij} = 1 \otimes ... \otimes R^{(1)} \otimes ... \otimes R^{(2)} \otimes ... \otimes 1$.These conditions lead to the Yang-Baxter equation:

$$\mathcal{R}_1\mathcal{R}_2\mathcal{R}_1 = \mathcal{R}_2\mathcal{R}_1\mathcal{R}_2 \tag{14}$$

where $\mathcal{R} = \tau(R)$

# Remark

- The most wanted element in the quasitriangular Hopf algebra is the R-matrix,which is directly related to physical quantities.Given a Hopf algebra,by constructing its representations we find matrix solutions to the Yang-Baxter equation.The converse is also true,given solution of the Y-B eqn,we can construct corresponding Hopf algebra.
- The bialgebra structure of a Hopf algebra lead to good representation properties,we can form tensor product representations with the help of coproduct,which then lead to a tensor category which is needed for a fusion category

# UIRs of $D(H)$

## Notations

- Denote conjugacy classes of H as $\{C_A = \{^A g_i\}\}$ and corresponding[a] centralizer subgroups $\{N_A \equiv N_{^A g_1} \cong N_{^A g_i}\}$.
- Denote the left-cosets as $G/N_A = \{[^A x_i] | ^A x_i ^A g_1 ^A x_i^{-1} =^A g_i\}$.Fixing the representatives we have 1-1 correspondence:$^A g_i \leftrightarrow^A x_i \leftrightarrow [^A x_i]$。

[a] Elements in same conjugacy class have isomorphic centralizer subgroup

The UIRs of $D(H)$ are obtained through induced representation:

## UIRs of $D(H)$

$$\pi^A_\alpha(\underset{x}{\overset{g}{\llcorner}}) |^A g_i, ^\alpha e_j\rangle = \delta_{g, ^A g_k} |^A g_k, \alpha(\tilde{x})^\alpha e_j\rangle, ^A g_k = x ^A g_i x^{-1}, \tilde{x} =^A x_k^{-1} x ^A x_i \quad (15)$$

where $\alpha$ is the UIR of $N_A$,the UIRs of $D(H)$ are labeled by two indices:

i=(conjugacy class A,UIR $\alpha$ of $N_A$)

# $H^3(H, U(1))$ and quasi-quantum double:$D^\omega(H)$

- The algebraic structure of quantum double can be twisted by the 3rd cohomology group $H^3(H, U(1))$ leading to a quasi-Hopf algebra known as quasi/twisted quantum double $D^\omega(H)$.

### quasi-coassociativity

3-cocycles $\omega \in H^3(H, U(1))$ lead to the associator:

$$\phi = \sum_{g,h,k} \omega^{-1}(g,h,k)^{g}\llcorner_{e} \otimes^{h} \llcorner_{e} \otimes^{k} \llcorner_{e} \tag{16}$$

which twist coassociativity to quasi-coassociativity:

$$(\Delta \otimes id)\Delta(a) = \phi(id \otimes \Delta)\Delta(a)\phi^{-1} \tag{17}$$

which implies the isomorphism:

$$(\pi_1 \otimes \pi_2) \otimes \pi_3 \cong \pi_1 \otimes (\pi_2 \otimes \pi_3) \Leftrightarrow \Phi : (V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3) \tag{18}$$

with intertwiner given by:$\Phi = \pi_1 \otimes \pi_2 \otimes \pi_3(\phi)$

## quasi-coassociativity

$$\begin{tikzcd}
((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \arrow[r, "\Phi \otimes id"] \arrow[d, "\Delta \otimes id \otimes id(\phi)"] & (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 \arrow[r, "id \otimes \Delta \otimes id(\phi)"] & V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) \arrow[d, "id \otimes \Phi"] \\
(V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \arrow[rr, "id \otimes id \otimes \Delta(\phi)"] & & V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\end{tikzcd}$$

Figure: Pentagonal condition

Pentagonal equation:

$$(id \otimes id \otimes \Delta)(\phi)(\Delta \otimes id \otimes id)(\phi) = (1 \otimes \phi)(id \otimes \Delta \otimes id)(\phi)(\phi \otimes 1) \tag{19}$$

Here it's equivalent to the 3-cocycle condition:

$$\omega(g, h, k)\omega(g, hk, l)\omega(h, k, l) = \omega(gh, k, l)\omega(g, l, kl) \tag{20}$$

- With the associator constructed from $\omega$,the Hopf algebra structure is twisted by two $U(1)$ phases related to $\omega$[5].

## Quai-Hopf algebra structure

- Product:

$$ {}^{g}\underset{x}{\llcorner} \cdot {}^{h}\underset{y}{\llcorner} = \delta_{g,xhx^{-1}}\ {}^{g}\underset{xy}{\llcorner}\ \theta_g(x,y) \tag{21} $$

- Coproduct:

$$ \Delta({}^{g}\underset{x}{\llcorner}) = \sum_{hk=g} {}^{h}\underset{x}{\llcorner} \otimes {}^{k}\underset{x}{\llcorner}\ \gamma_x(h,k) \tag{22} $$

With definition:

$$ \begin{aligned} \theta_g(x,y) = i_g\omega(x,y) &= \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}g(xy))}{\omega(x,x^{-1}gx,y)} \\ \gamma_x(g,h) = \tilde{i}_x\omega(g,h) &= \frac{\omega(g,h,x)\omega(x,x^{-1}gx,x^{-1}hx)}{\omega(g,x,x^{-1}yx)} \end{aligned} \tag{23} $$

Here operator:$i_g, \tilde{i}_x$ are known as slant product.

[5] The antipode is also twisted but it have no physical intepretation yet

# Remark

- According to definition:

$$\tilde{\delta}\theta_g(x,y) = 1 \Leftrightarrow \theta_g(x,y)\theta_g(x,y) = \theta_g(x,yz)\theta_{x^{-1}gx}(yz)$$

  which implies $\theta_g$ is a conjugate 2-cocycle.Here $\tilde{\delta}$ is the conjugate 2-coboundary operator.[6]

- It can be verified that 3-cocycles differ by a 3-coboundary $\delta\beta$ give rise to isomorphic $D^{\omega}(H)$

[6] Simply view the lower index as a varible and H act on it by conjugation.

## Quasitriangularity

There's still a R-matrix,defined still as:

$$R = \sum_{g,h} {}^{g}\llcorner_{e} \otimes {}^{h}\llcorner_{g} \tag{24}$$

But the relations satisfied by R-matrix is twisted:

$$\begin{aligned} \mathcal{R}\Delta(a) &= \Delta(a)\mathcal{R} \\ (\Delta \otimes id)(R) &= \Phi\mathcal{R}_1\Phi^{-1}\mathcal{R}_2\Phi \\ (id \otimes \Delta)(R) &= \Phi^{-1}\mathcal{R}_2\Phi\mathcal{R}_1\Phi^{-1} \end{aligned} \tag{25}$$

with $\mathcal{R} = \tau(R), \phi_{s_1 s_2 s_3} = \phi^{(s^{-1}(1))} \otimes \phi^{(s^{-1}(2))} \otimes \phi^{(s^{-1}(3))}$

# Quasitriangularity

This definition satisfies the Hexagonal conditon:

$$\begin{tikzcd}
V_1(\otimes V_2 \otimes V_3) \arrow[r, "\Phi^{-1}"] \arrow[d, "id\otimes\Delta(\mathcal{R})"] & (V_1 \otimes V_2) \otimes V_3 \arrow[r, "\mathcal{R}_1"] & (V_2 \otimes V_1) \otimes V_3 \arrow[d, "\Phi"] \\
(V_2 \otimes V_3) \otimes V_1 & V_2(\otimes V_3 \otimes V_1) \arrow[l, "\Phi^{-1}"] & V_2(\otimes V_1 \otimes V_3) \arrow[l, "\mathcal{R}_2"]
\end{tikzcd}$$

Figure: Hexagonal condition.The other diagram is similar

and the quasi Yang-Baxter equation:

$$\mathcal{R}_1\Phi^{-1}\mathcal{R}_2\Phi\mathcal{R}_1 = \Phi^{-1}\mathcal{R}_2\Phi\mathcal{R}_1\Phi^{-1}\mathcal{R}_2\Phi \tag{26}$$

The $\Phi$ here implies $\tilde{\mathcal{R}}_2 = \Phi^{-1}\mathcal{R}_2\mathcal{R}$ act on $((V_i \otimes V_j) \otimes V_k)$ same as $\mathcal{R}_1$

# UIRs of $D^{\omega}(H)$

Similarly,using induced representation we obtain the UIRs of $D^{\omega}(H)$ as the following:

**UIRs of $D^{\omega}(H)$**

$$\pi_{\alpha}^{A}(\underset{x}{{}^{g}\llcorner})|^{A}x_i,^{\alpha}e_j\rangle=\theta_g(x,^{A}x_i)\theta_g(^{A}x_k,h)^{-1}\delta_{g,^{A}g_k}|^{A}x_k,\alpha(\tilde{x})(^{\alpha}e_j)\rangle \tag{27}$$

For trivial conjugate 2-cocycle$\theta_g=\tilde{\delta}\epsilon_g$,notice here involve a basis transformation:

$$\pi_{\alpha}^{A}(\underset{x}{{}^{g}\llcorner})|^{A}g_i,^{\alpha}e_j\rangle=\delta_{g,^{A}g_k}\ \epsilon_g(x)|^{A}g_k,\alpha(\tilde{x})^{\alpha}e_j\rangle \tag{28}$$

For this case,the OUIRs of $N_A$ are twisted by $\omega\in H^3(H,U(1))$ to trivial PUIRs:

$$\tilde{\alpha}_{g_1}(x)=\epsilon_{g_1}(x)\alpha(\tilde{x})\Rightarrow\tilde{\alpha}_{g_1}(x_1)\tilde{\alpha}_{g_2}(x_2)=\theta_{g_1}(x_1,x_2)\tilde{\alpha}_{g_1}(x_1,x_2),g_1=x_1g_2x_1^{-1} \tag{29}$$

# Remark

- The reduction of this form happens for trivial conjugate 2-cocyle(including the trivial case:$\omega = 1, D(H)$),it's not clear yet if the basis transformation $|^{A}x_i,^{\alpha}e_j\rangle \rightarrow |^{A}g_i,^{\alpha}e_j\rangle$ is valid.[7]
- The trivial[8] PUIRs here are classified by the conjugate 2-cocycle,not 2-cocyles that encountered more frequently,for abelian H,the two coincide.
- Here the UIRs of $D^{\omega}(H)$ are again labeled by conjugacy classes and ordinary,except some phase difference.In some papers,another form of UIRs is also adopted,they are labeled by $(C_A, \tilde{\rho}_{\alpha}^{^{A}g})$.$\tilde{\rho}_{\alpha}^{^{A}g}$ is the $\theta_{^{A}g}$-regular projective representations,which are indeed 'ordinary' PUIRs.But the isomorphism between this two representations haven't been constructed yet.

[7] I think this is related to the fact that many SSB models have only trivial conjugate 2-cocycle
[8] since conjugate 2-cocyle is trivial

# R-matrix and truncated braid group

- The most physically interested element is $\mathcal{R}$,for both $D(H)$ and $D^{\omega}(H)$, the R-matrix is defined as $R = \sum_g \; {}^{g}\llcorner_{e} \otimes^{1} \llcorner_{g}$,with help of $\Phi$,we can construct representation(due to the (quasi) Yang-Baxter equation) of braid group[9]:

$$\tau_i \rightarrow \Phi_i^{-1} \mathcal{R}_i \Phi_i \tag{30}$$

Note these operators act on $(((V_{\alpha_1}^{A_1} \otimes V_{\alpha_2}^{A_2}) \otimes V_{\alpha_3}^{A_3}) \otimes ...)$(standard form).

- Actually these operators are of finite order:$\tau_i^m = id$,this can be traced back to the fact that H is finite.More explicitly take $D(H)$ for example,the explicit actions are of the form:

$$\mathcal{R}_{\alpha\beta}^{AB} |{}^{A}g_i, {}^{\alpha} e_j\rangle |{}^{B}g_k, {}^{\beta} e_l\rangle = |{}^{A}g_i^{B} g_k^{A} g_i^{-1}, \beta({}^{B}x_m^{-1} \; {}^{A}g_i^{B} x_k)^{\beta} e_l\rangle |{}^{A}g_i, {}^{\alpha} e_j\rangle \tag{31}$$

By sucessive action for some kind of lcm times,both the elements in the conjugacy classes and the elements in $V_\alpha$ carrier space of $N_A$ may be the initial one.It's better to check directly using definition $\mathcal{R} = \sigma \circ (\pi_1 \otimes \pi_2)(R)$

[9]the definition of $\Phi_i$ take care of the places of the '()'

- This means UIRs of $D^\omega(H)$ actually produce representations of quotient groups of $B_n$:

### Truncated braid group

$$B(n,m) = B_n/\{\tau^m = 1\} \tag{32}$$

Here n specify # of particles and m a constant determind by H.Note that the explicit formula for m is still not clear.As quotient group,the form of B(n,m) are much simpler.
Example:$B(n,2) = S_n$,$B(2,m) \cong \mathbb{Z}_m$
We can find B(n,m) first then using construct their UIRs as UIRs of the braid group produced by UIRs of $D^\omega(H)$.

# Fusion algebra and modular S-matrix

## Fusion algebra

The tensor product representation is generally reducible:

$$\pi_\alpha^A \otimes \pi_\beta^B \cong \oplus_{C\gamma} N_{\alpha\beta\gamma}^{ABC} \pi_\gamma^C \tag{33}$$

Due to (quasi) coassociativity of $D^{(\omega)}(H)$ and quasitriangularity,this algebra is associative and commutative respectively.This implies that by viewing $(N_i)_{jk}$ as matrix,they can be simultaneously diagonalized by modular S-matrix.

## S-matrix

- $D(H)$:$S_{\alpha\beta}^{AB} = \sum_{[{}^A g_i, {}^B g_j]=e} tr\alpha^*({}^A x_i^{-1}\ {}^B g_j^A x_i) tr\beta^*({}^B x_j^{-1}\ {}^A g_i^B x_j)$
- $D^\omega(H)$(trivial $\theta$):
  $S_{\alpha\beta}^{AB} = \sum_{[{}^A g_i, {}^B g_j]=e} tr\alpha^*({}^A x_i^{-1}\ {}^B g_j^A x_i) tr\beta^*({}^B x_j^{-1}\ {}^A g_i^B x_j)\sigma({}^A g_i|{}^B g_j)$
  here $\sigma(g|h) = \epsilon_g(h)\epsilon_h(g)$

# Relation between $\mathcal{R}$,S,N

- The multiplicity $N_{ijk}$ can be obtained using orthogonal relations,which implies the relation:

$$N^{ABC}_{\alpha\beta\gamma} = \sum_{D,\delta} \frac{S^{AD}_{\alpha\delta} S^{BD}_{\beta\delta} S^{CD*}_{\gamma\delta}}{S^{eD}_{0\delta}} \tag{34}$$

known as Verlinde's formula.[10]

- Using explict formula for $\mathcal{R}$ and S,we can verify the following relation:

$$S_{ij} = \frac{1}{|H|} tr \mathcal{R}^{-2}_{ij} \tag{35}$$

This implies that we can find $\mathcal{R}_{ij}$ from explict action then find $S_{ij}$,and get the multiplicity $N_{ijk}$.

[10] This is first derived in study of CFT,there's an elegent proof using anyon diagrams

# Modular group

- Except S-matrix,there's a T-matrix:

$$T^{AB}_{\alpha\beta} = \delta_{\alpha\beta}\delta^{AB}\frac{1}{d_\alpha}tr(\alpha(^A g_1)) \tag{36}$$

- S,T-matrix span the quotient group of modular group:

$$\begin{aligned} SL(2,\mathbb{Z})/\{S^2 = (ST)^3 = C, C^2 = 1 \\ S^* = CS = S^{-1}, S^t = S, T^* = T^{-1}, T^t = T, [C,T] = 0\} \end{aligned} \tag{37}$$

# Remark

- In physical contexts,the T-matrix will be related to topological spins or conformal weights.The C-matrix will be related to 'chagre' conjugation,and [C,T]$=$0 means 'particle-antiparticle' pairs have same 'spin'
- Topologically,the modular group have something to do with the diffeomorphisms of $T^2$.
- In anyon diagrammatics,we can define these matrices and find their relations using diagrams,and the definition of T-matrix will involve the central charge.

# Rep$[D^\omega(H)]$ as anyon theory

- The good representation properties of $D^\omega(H)$ suggest it can produce a tensor category.In short the anyon will be labeled by representations/carrier spaces,the fusion algebra is exactly the fusion ring.
- Quasi-coassociativity gives the isomorphism between tensor product spaces(representations),the F-symbol can be derived using representation techniques.Note the $\Phi$ given here is not the F-symbol since it involve also the explict basis,while F-symbol only involve the labels of UIRs.But they still have deep connection.The Pentagonal condition is related to the pentagon equation.
- Quasitriangularity gives the crucial $\mathcal{R}$ and that the braid process commute with the action of $D^\omega(H)$,thus the anyons can be labeled by

$$(\pi_\alpha^A, UIRs\ of\ B(n,m)) \tag{38}$$

  the first gives the particle type,the second gives its statistic.The UIRs of B(n,m) can be obtained by decomposing reducible $\mathcal{R}$,which are R-symbols.Similarly,the Hexagonal condition coincide with the Hexagon equation.

# Discrete gauge theories obtained from SSB

# Lagrangian

## Lagrangian for the C-S gauge theory

$$
\begin{aligned}
\mathcal{L} = \{-\frac{1}{4}F^a_{\alpha\beta}F^{a\alpha\beta} - j^\mu_a A^a_\mu\} \\
+\{(D_\alpha\Phi)^\dagger D^\alpha\Phi - V(\Phi)\} \\
+\{\frac{\mu}{4}\epsilon^{\alpha\beta\gamma}[F^a_{\alpha\beta}A^a_\gamma + \frac{1}{3}e\epsilon^{abc}A^a_\alpha A^b_\beta A^c_\gamma]\}
\end{aligned}
\tag{39}
$$

The three parts are known as:

1. gauge boson& matter 'charge'
2. Gauge 'charged' Higgs field
3. Chern-Simons term[a]

[a] here use the form of $SU(2)$ or $SO(3)$

# Classification and discrete gauge theory

## Topological aspects of $\mathcal{F}(M \to G)$

$$S_{C-S} = \frac{8\pi^2\mu}{g^2} W[A], W[A] = \int dx^3 \mathcal{L}_{C-S} \tag{40}$$

For W[A] to behave properly(no divergence),the local gauge transformations need to satisfy asympotic compactification:$\Omega(x \to \infty) \to I$.Thus the local gauge transformations are classified by $\pi_3(G) = \mathbb{Z}$[a].Variation of W[A] is given by the winding number:

$$\delta W[A] = \omega(U) = \frac{1}{24\pi^2} \int d^3x \epsilon^{\alpha\beta\gamma} tr a_\alpha a_\beta a_\gamma \in \mathbb{Z}, a_\alpha = U^{-1}\partial_\alpha U \tag{41}$$

Thus gauge invariance implies the quantization of $\mu$:

$$\frac{8\pi^2\mu}{g^2} = 2\pi p \Rightarrow \mu = \frac{g^2}{4\pi} p, p \in \mathbb{Z} \tag{42}$$

[a] compact,connected G

### Classification of Chern-Simons actions

This result coincide with the general result that Chern-Simons actions are classified by $H^4(BG,\mathbb{Z})$,here $H^4(BG,\mathbb{Z})=\mathbb{Z}=\pi_3(G)$.For discrete Chern-Simons theories,the classification reduce to $H^4(BH,\mathbb{Z})\cong H^3(H,U(1))$ for finite gauge group.

- Discrete Chern-Simons gauge theories can be obtained with spontaneous symmetry breaking:

$$H\subseteq G\Rightarrow \varphi : H^4(BG,\mathbb{Z})\rightarrow H^3(BH,\mathbb{Z})\cong H^3(H,U(1)) \tag{43}$$

- The possible physically different models are classified by

$$\omega\in Im\phi\subseteq H^3(H,U(1))$$

# $\mu = 0$

- Gauge bosons all acquire mass by absorbing the Goldstone bosons created by SSB of continuous symmetry via Higgs mechanism.Consider the low-energy effective theory where all gauge bosons and higgs particles are gapped.Thus there's no gauge interations.But there can still be topological interactions due to topological defects(excitation)

## Topological defects:Vortices(flux)

Topological defects here are vortices,they are semiclassical ground states.Quantum mechanically,we view them as point particles carrying flux degrees of freedom(Hilbert space),with the flux carried identified with the Wilson loop operator:

$$D\Phi = 0 \Rightarrow W = \mathcal{P}exp(\oint_c A^a_\alpha T^a ds^\alpha) = \Gamma(h), h \in H \tag{44}$$

Here $h \in H$ makes $\Phi$ single-valued.

### Topological defects:Vortices(flux)

'topological' comes from the fact that the origin of this nonvanishing Wilson loop comes from the nontrivial winding of the Goldstone mode around the gound state manifold $G/H$.The flux are then classified by

$$\pi_1(G/H) = \pi_1(\bar{G}/\bar{H}) \cong \bar{H} \tag{45}$$

By including the instantons,which are monopoles in 3d Euclidean spacetime and describe tunneling effect in 2+1d Minkowski spacetime,classified by $\pi_1(G)$,the spectrum can be reduced to the above result:

$$\bar{H}/\pi_1(G) = H \tag{46}$$

# Quasiparticle:Dyon

- Wilson loop is not gauge invariant:

$$W \mapsto U(x_0)WU(x_0)^{-1} \tag{47}$$

  thus the flux Hilbert spaces are spanned by conjugacy classes.
- The pure charge are identified with the UIRs of H upon SSB making UIRs of G reducible.
- The full spectrum are dyons combining flux and charge,the dyon charges are identified with the UIRs of centralizer subgroup $N_A$.A explanation is given below.

Two operations can be defined for dyons:

1. Flux measurement:$P_h^2 = P_h$
2. Gauge transformation:$g \in H$

They satisfy relation:$P_h g = g P_{g^{-1}hg}$,thus the combined operators $D_{(h,g)} = P_h g$ satisfy the algebra:

$$D_{(h,g)} D_{(h',g')} = \delta_{h,gh'g} D_{(h,gg')} \tag{48}$$

Further,the action of $P_h g$ on the tensor product space is exactly given by coproduct:

$$\Delta(P_h g) = \sum_{h'h''=h} (P_{h'} g) \otimes (P_{h''} g) \tag{49}$$

which means gauge transform the two dyons seperately by g and keeping the total flux conserved.

### Dyons are UIRs of $D(H)$

This physical model can be summarized by $D(H)$.Especially the dyons are identified with UIRs of $D(H)$,labeled by $(A, \alpha)$ with internal Hilbert space spanned by $|^A g_i, ^\alpha e_j\rangle$

# Topological interactions

- The topological interactions are just generalized A-B effect.When one dyon passes through the Dirac string of the other dyon,it will be gauge transformed by the other dyon's flux element.
- This action is just the R-matrix:

$$R = \sum_{h,g} P_g \otimes P_h g \tag{50}$$

act the first dyon's flux on the second dyon while keeping the total flux fixed.

- Since the only interactions are topological interaction,the charge is observed through A-B scattering:

$$\begin{aligned}\langle h, v|\langle h'|\mathcal{R}^2|h'\rangle|h, v\rangle &\equiv \langle h, v|h'hh'^{-1}, \alpha(h')\nu\rangle\langle h'|(h'h)h'(h'h)^{-1}\rangle \\ &= \langle v|\alpha(h')\rangle\delta_{h,h'hh'^{-1}}\end{aligned} \tag{51}$$

which implies only centralizer charges are observable,thus dyon charges are UIRs of $N_A$.

# Topological spin

- Dyons have topological spin,due to spin-statistic connection,the braiding of two identical particle is captured by the spin of the particle.While the braiding process is captured by:

$$\pi_\alpha^A(\sum_h P_h h)|h,v\rangle = |h_i, \alpha(^A g_1)v\rangle, \alpha(^A g_1) = e^{2\pi i s_{A,\alpha}} I_{d_\alpha} \tag{52}$$

  which act the flux of the dyon on the flux itself thus the origin of this topological spin is the topological interaction.

- This factor is captured by T-matrix:

$$T_{ij} = \delta_{ij}\frac{1}{d_\alpha} tr\alpha(^A g_1) = \delta_{ij}\frac{tr\alpha(^A g_1)}{tr\alpha(e)} = e^{2\pi i s_i} \tag{53}$$

# $\mu \neq 0$

- The physical influence of Chern-Simons remains after SSB in the theory through $\omega \in H^3(H, U(1))$,i.e,the resulting discrete Chern-Simon gauge theory is classified by 3-cocycles.
- An interesting feature of simple examples like $U(1) \to \mathbb{Z}_N, U(1)^k \to \mathbb{Z}_N^k$ suggest that only trivial conjugate 2-cocyle appear after SSB.If this is true for general case is unknown,since there's no unified result of $\phi : H^4(G, \mathbb{Z}) \to H^3(H, U(1))$ for general $H \subseteq G$.The rest part will be limited to trivial $\theta$.

## Noether charge

The most direct consequnce of the remaining discrete Chern-Simons term is the modification of the Noether charge,take $U(1) \to \mathbb{Z}_N$ for example:

$$U(l)|m,n\rangle = exp(i\frac{4\pi}{Ne}l\tilde{Q})|m,n\rangle = exp(i\frac{2\pi}{N}l(n+p\frac{m}{n}))|m,n\rangle \tag{54}$$

here $exp(i\frac{2\pi}{N}\frac{pm}{n})$ is the extra phase giving the trvial $\theta_g = \tilde{\delta}\epsilon_g$.More generally,the change of the Noether charge,thus extra topological interactions can be absorbed into the trivial PUIRs of $N_A$,i.e,the discrete Chern-Simons term change OUIRs of $N_A$ to trivial PUIRs:

$$\pi^A_\alpha({}^g_{\llcorner\atop x})|^A g_i,^\alpha e_j\rangle = \delta_{g,^A x_k}\ \epsilon_g(x)|^A g_k, \alpha(\tilde{x})^\alpha e_j\rangle$$

### $D^\omega(H)$

Simple examples suggest the influence of the discrete Chern-Simons term,classified by $H^3(H,U(1))$ introduce a 'twist' to the original $D(H)$-model,by inducing extra topological interactions between dyons via $\theta_g = i_g\omega = \tilde{\delta}\epsilon_g$.This will also introduce extra phase modification to T-matrix,which implies dyons have extra spin due to Chern-Simons term.

However,a general theory of how general Chern-Simons term lead to algebra and coalgebra of $D^\omega(H)$ is still lacking.

# Summary

- Without Chern-Simons term,the physics of the effective low-energy discrete gauge theory with topological excitation and interactions are characterized by the algebra $D(H)$,with dyons classified as UIRs of $D(H)$,and the topological interaction,summarized in the operaotr R-matrix,implies that these dyons are $Rep[D(H)]$ anyons.
- By including extra Chern-Simons term,SSB introduce extra physical consequnces characterized by discrete Chern-Simons actions $\omega \in H^3(H, U(1))$,which twist the original $D(H)$ algebra to $D^\omega(H)$ algebra,resulting in $Rep[D^\omega(H)]$ anyons.
- There are several questions remaining:
  1. Can these effective models arise in real world?More fundamentally,what's the mechanism of the combination of charge and flux?
  2. Is it possible for nontrivial conjugate 2-cocyle arise via SSB?

# Exactly solvable lattice models

# Introduction to TQFT

- Give space manifold[11] $\Sigma$,we can associate it with a Hilbert space(kets) V($\Sigma$).Inverting the orientation gives the dual space:$V(\Sigma^*) = V^*(\Sigma)$(bra).If we don't attach particles(by punctuation[12]),then this V stands for ground states,else it's the fusion/split space,coincide with anyon diagrammatics.
- Locality:$V(\Sigma_1 \cup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2), V(\emptyset) = \mathbb{C}$。
- Define:$Z(M) \in V(\partial M = \Sigma)$ which can be considered as the wavefunction(ket/bra) in the boundary Hilbert space of spacetime.$Z(M)$ depends only on the topology of M.Using above definitions,the sewing of two M via common boundary $\Sigma$ correspond to inner product:

$$Z(M_1 U_\Sigma M_2) = \langle Z(M)_1 | Z(M_2) \rangle \tag{55}$$

- For two $\Sigma_1, \Sigma_2$ with cobordism:$\partial M = \Sigma_1^* \ U \ \Sigma_2$,$Z(M)$ describe the time evolution:

$$Z(M) \in V^*(\Sigma_1) \otimes V(\Sigma_2) \tag{56}$$

---

[11] only introduce 2+1d,$\Sigma$ orientable

[12] anyons introduce nontrivial braiding which can't be ignored

# Remark

## Implications

- For $\partial M = 0$(closed),$Z(M) \in \mathbb{C}$ descrbie the evolution from vacuum to vacuum,i.e,the ground state amplitude $\langle vacuum|O|vacuum\rangle$.By definition $Z(M)$ is a topological invariant of M.If we include anyons the worldlines of anyons may form knots/links,which then lead to knot/link invariants.
- $Z(\Sigma \times I) = 1$ given same basis.
- $$Z(\Sigma \times S^1) = tr(Z(\Sigma \times I)) = dimV(\Sigma) \tag{57}$$

  $S^1$ implies the sewing of the two faces of $\Sigma$,which implies the trace.When there're no anyons,$dimV(\Sigma)$ is the ground state degeneracy.Especially,we have:

  $$dimV(S^2_*) = 1, dimV(T^2_*) = \#(anyon\ types) \tag{58}$$

  * stands for ground state of the anyon model.The second can be understood using $\pi_1$,where nontrivial braiding are introduced by nontrivial loops in 2D manifolds or by sewing punctures on $S^2$

# Dijkgraaf-Witten TQFT

- Dijkgraaf-Witten gives a concrete construction of discrete Chern-Simons gauge theories classified by $H^3(H, U(1))$.The effective $D^\omega(H)$ model can be considered as some kind of DW TQFT,with dyons being anyons. Especially,$Z(T^2 \times S^1)$ gives the number of UIRs of $D^\omega(H)$ which can be verified in simple cases.
- DWTQFT also describe the ground state of some exactly solvable lattice models with excitations identified as $Rep[D^\omega(H)]$ anyons.

## Branched simplex

The definition of $Z_{DW}$ involve the following triangulation and orientation:

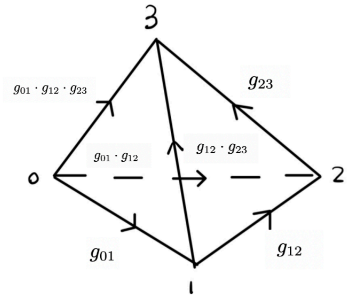



Figure: The orientation goes form smaller to larger number,each edge is associated with a gauge connection $g_{ij} \in H, i < j$The whole configuration satisfy flat-connection/zero-flux principle(trival Wilson loop).There's only 3 degrees of freedom

## Definition of $Z_M$

$$Z_{DW}(T(M)) = \frac{1}{|H|} \prod_{3-B.S.\in T(M)} \Omega(3-B.S.) \tag{59}$$
$$\Omega(3-B.S.) = \omega(g_{i_0 i_1}, g_{i_1 i_2}, g_{i_2 i_3})^{s_{i_0 i_1 i_2 i_3}}, \omega \in H^3(H, U(1))$$

s stands for the chirality of the brached simplex,can be determind by the chirality of the brached 2-simplex oppesite to the smallest vertex.The 3-cocyle is fully determined by the 3 degrees of freedom in branched simplex.

- Topological(P-move/retriangulation) invariance:

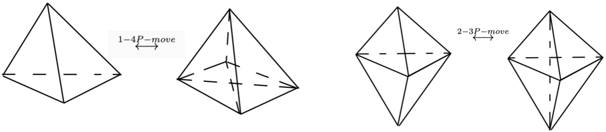



Figure: P-move。The core of this invariance is the 3-cocycle condition and the definition of 3-cocycles in terms of branched simplexes

## Gauge invariance

The local gauge transformation of gauge connections:$g_{ij} \to u_i g_{ij} u_j^{-1}, u_i \in H$ can be defined by on-site transformation.

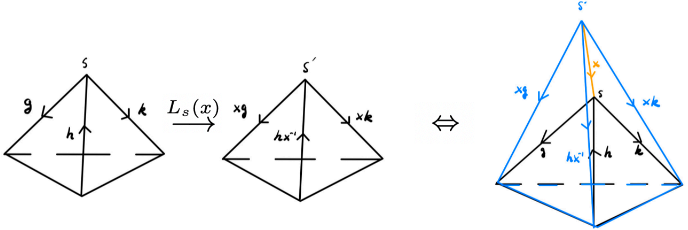



Figure: Gauge transformation act on vertex s by acting on the associated edges with $L^+(g)(L^-(g))$:$g_{s*}(g_{*s})$left multiply g(right multiply $g^{-1}$),This form is just another kind of 1-4 P-move.

# Boundary configurations

- For M with closed boundary $\Sigma$,we can associate a physical model(lattice discrete gauge theory) on $\Sigma$,and $Z_{DW}(M) = \Psi \in V((\partial M = \Sigma))$ is just the wavefunction the ground state.For physical models,the wavefunction must be gauge invariant,thus is the 3-cocyle-weighted linear combination of flat-connetion configurations.
- Generally,for 2D lattice,the gauge connection on each edge turned into a local physical degrees of freedom:$\mathcal{H}_e \cong \mathbb{C}[H]$,$\mathcal{N} = \otimes_e \mathcal{H}_e$。 Note that the excitation states break gauge invariance or flat-connection.In this model,gauge theory is dynamically generated!
- The ground state degeneracy equals $Z_{DW}(\Sigma \times S^1)$.

# Defining Hamiltonian

- The Hamiltonian is suggested by the ground state:flat-connections suggest flux-measurement operators;gauge invariance suggests gauge-transformation operators.

## Hamiltonian

This lead to definition:

$$H = -\sum_s A_s - \sum_f B_f \tag{60}$$

with operators defined as:

- Zero-flux projector:

$$B_f|(s_1, s_2, s_3)\rangle = \delta_{e, g_{s_1 s_2} g_{s_2 s_3} g^{-1}_{s_1 s_3}} |(s_1, s_2, s_3)\rangle \tag{61}$$

- Gauge-invariance projector:

$$A_s = \frac{1}{|H|} \sum_{g \in H} A_s^g \tag{62}$$

## Hamiltonian

The gauge transformation operaotr are defined as:

$$A^g_{s_0}|(s_0, s_1, s_2, s_3)\rangle = |(s_{0'}, s_1, s_2, s_3)\rangle \frac{\omega(g, g_{s_0 s_1}, g_{s_1 s_2})\omega(g, g_{s_0 s_1}, g_{s_1 s_3})}{\omega(g, g_{s_0 s_2}, g_{s_2 s_3})} \tag{63}$$

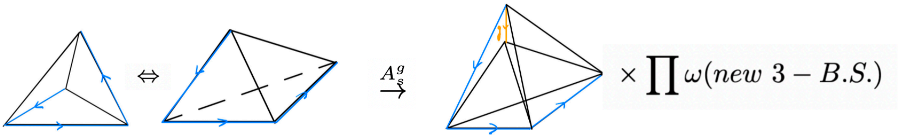



Figure: The triplet vertex can be considered as a branched simplex,then the gauge transformation is the similarly defined as before.Except the Blue lines are considered as physical degrees of freedom and thus the extra U(1) phases are included.

## Hamiltonian

The interesting feature about this model is that the operators are commutative projectors:

- $B_f^2 = B_f$,$[B_f, B_{f'}] = 0$
- $A_s^2 = A_s$,$[A_s, A_{s'}] = 0$
- $[A_s, B_f] = 0$ this can be verified directly.This is actually the fact:$W \to hWh^{-1} = heh^{-1} = e$.

Thus this model is exactly solvable,with eigenvalues differing by 1.The ground state correspond to $B_f = A_s = 1$ state,thus is the gauge invariant,flat-connection state,described by $Z_{DW}$.But the excitation states break some condition of $A_s = 1$,$B_f = 1$ thus is not described by DWTQFT.

# Excitations and local operators

- If we difine the quasiparticle/elementary-excitation as breaking a pair of $A_s = B_f = 1$ at a site(s,f),then the local operators $L(V_{excitation})$[13],describing the exication are spanned by the following operators

## Local operators

Here we leave the site=(s,f) implicit.

- $A^g$ defined as before.
- Flux-measurement:$B_f^h = \sum_{h_1 h_2 h_3 = h} \prod_{m=1}^{3} T^{h_m}(e_m, f)$,T obtain the gauge connection on the edge,the total flux/Wilson-loop is h.The zero-flux projector is just:$B_f = B_f^e$

[13] the properties of the excitation is describe by their behavior under linear operators on the Hilbert space

## Local operators

These operators satisfy the same algebra discussed in SSB models:

- For trivial $\omega$:

$$A^h A^g = A^{hg}; B^h B^g = \delta_{h,g} B^h;$$
$$(B^g A^x)(B^h A^y) = \delta_{g,xhx^{-1}} B^g A^{xy} \tag{64}$$

  Defining:$D_{(h,g)} = B^h A^g$ these operators span algebra $D(H)$.Note that the Ribbon operators generating a pair of particle-antiparticle at the ends of a Ribbon(connecting two sites) span a dual $D(H)$ algebra.

- For nontrivial $\omega$,the definition of local operators is more complex,but the same conclusion holds:$D_{(h,g)}$ span the algebra $D^\omega(H)$,same for the Ribbon operators.

## Elementary excitations are $Rep[D^\omega(H)]$ anyons

The conclusion is:elementary excitations in this exactly solvable model are $Rep[D^\omega(H)]$ anyons.

# Summary

- A exactly solvable model is constructed via DWTQFT:the ground state theory is dynamically generated lattice gauge theory described by DWTQFT as gauge invariant,flat-connection gauge connection configurations.
- The excitations in this model are characterized by local operators,which span algebra $D^{\omega}(H)$ thus are $Rep[D^{\omega}(H)]$ anyons.

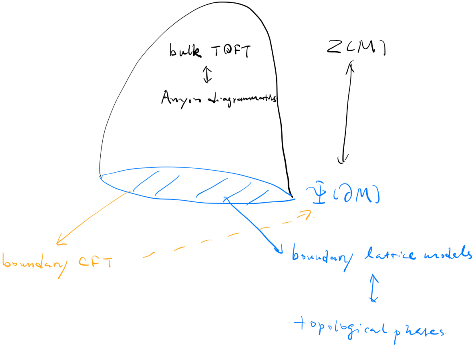
# Main results

## Main results

- Introduced the basic elements of an anyon theory.
- Reviewed an algebraic theory of anyons:$Rep[D^{\omega}(H)]$
- Introduced two discrete gauge theory models that give rise to this kind of anyon:(1) SSB models of discrete Chern-Simon gauge theories (2) Exactly solvable lattice models with discrete gauge group.
- These two models are related via Dijkgraaf-Witten TQFT,which describe the ground states of both models.This is due to DW TQFT have $D^{\omega}(H)$ structure.

# Remark

- The exactly solvable models have been used to construct some topological phases with long-range entanglement.With on-site symmetry included into the model:$H = GG \times SG$,they have been used to classify some symmetry enriched topological phases.
- There's interesting correspondence:



# Thanks for listening!