# Basic Category theory

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October 2, 2022

**Remark 1.** *This note is based on:*`https://www.bilibili.com/video/BV1id4y1N7e1/?spm_id_from=333.1007.top_right_bar_window_custom_collection.content.click&vd_source=a5a8460500e2a3b8256e6de946ff552f`*,which is an introductory lecture to basic notations to category theory.*

## Contents



# 1 Category,Functor,Natural transformation

## 1.1 Category

**Definition 1** (**Category**).
*A category* $C = (obj(C), hom(C))$ *consists of:*

- *$obj(C) = \{A, B, ...\}$ is a class of objects.*

- *$hom(C)$ is a class of morphisms/arrows between objects.The morphisms between two objects is a class $Hom_C(A, B) = C(A, B) = \{f : A \to B | A = dom(F), B = cod(f)\}$*

- *There's a binary operation on $hom(C)$ known as composition of morphism:*[1]

$$\circ : C(A, B) \times C(B, C) \to C(A, C)$$
$$(f, g) \to g \circ f$$

  *The composition is associative:*

$$f \circ (g \circ h) = (f \circ g) \circ h$$

- *There a special morphism in each $C(A, A)$,the identity morphism $id_A$,defined by:*

$$id_A \circ f = f, g \circ id_A = g$$

- *A morphism $f \in C(A, B)$ is known as an isomorphism if $\exists\ g \in C(B, A)$ such that*

$$f \circ g = id_{cod(f)}, g \circ f = id_{cod(g)}$$

  *$g$ is known as the inverse of f,denoted $g = f^{-1}$.*

*A category is known as:*

- *Locally small: $C(A, B) \in$ **Set**,thus $hom(C) = \cup_{A,B} C(A, B) \in$ **Set***

- *Small: $obj(C), hom(C) \in$ **Set***

The oppesite/dual category is denoted $\mathbf{C}^{op}$, defined by reversing all arrows.

**Lemma 1.**

- *The identity morphisms are unique by definition.*

- *The inverse of an isomorphism is unique by definition.*[2]

**Remark 2.** *There are more specific definitions of morphisms, like monomorphism and epimorphism. There are also more definitions of special categories.*

[1]For convenience,the presciption $\forall$ is implicit in definitions.The composition is allowed for morphisms with matched domain and codomain,they are implicit in the composition.

[2]The notation $f^{-1}$ is well defined.

### Macro v.s Micro

A category is defined by the axioms, as long as the appropriate objects and morphisms are defined explicitly, and accept well defined associative composition and identities, it is identified as a category.

In category theory, we care about the morphisms between objects, only when we have to define objects and morphisms explicitly will we have to infer to the 'microscopic construction' of the objects. The microscopic definitions must be make sure well-defined: there's no contradiction and ambiguity.[3]

The uniqueness of identities implies that we can focus only on the morphisms, with identities representing the objects. The morphism $A \overset{id_A}{\to} A$ may be shrink to a point identified with (co)domains of other arrows.

By relation between objects, we mean relations on the class obj(C). When the objects do have explicit microscopic constructions over sets, there can be further relations on the set, i.e, inside the objects. Actually, by constructing appropriate category, any relations on a set can be expressed as relations/morphisms in certain category.

One important example are equivalence relations on set-constructed objects, in certain category they are described as isomorphisms between objects. The isomorphism is a generalization of identitcal equivalent, it defines an equivalence relation on the class $obj(C)$.

The requirement of associativity is not met by some set-theoretical structures/operations, but it is met by every category. Thus it's important not to confuse with macro and micro constructions. In explicit constructions we have to make sure the morphisms are well-defined with respect to the definition of objects. This means the morphisms are defined to be compatible with the algebraic/topological structures of the objects.

Since the macro details matter, in category theories we usually commutative diagrams extensively. The requirements of well-defined/compatibility/natural-definition can usually be translated into certain commutative diagrams. Conversely, many definitions/constructions are implied by commutative diagrams, the requirement of commutivity of compositions are usually strong enough to give unique(up to isomorphism) definitions. An important class of examples are limits.

### How to obtain categories

The categories are ussually constructed by starting with defining objects, then finding the appropriate definitions of morphisms[4].The objects may be defined explicitly with sets and the morphisms are then defined by certain commutative diagrams.

[3]For example, to check if a morphism is an isomorphism, we have to construct the inverse morphism explicitly, aside from the two composition condition, we have to make sure the inverse is indeed a morphism.

[4]The definition of morphisms are not unique, they give rise to different categories.

Another way to obtain categoris is use the exsisting categories. The categories are presented as graphs: points and arrows, we can then extract certain kind of subgraphs and define it to be the object of new categories. The morphism between them are defined to the connection between two subgraphs using appropriate morphisms in old categories such that the conneted diagram is commutative. This process is actually a functor. The construction of limits is of this kind.

#### Why bother with categories

The construction of category shows the similarity between different mathmatical entities. Thus the results proved in categorical forms implies to all these special cases once the corresponding constructions are identified. Conversely, many results in different special cases are proved with high similarity, though the explicit constructions of the elements of the construction or theorems are different, this implies that these constructions or theorems can be generalized to other cases with help of category theories.

The language of category is usually more efficient, transparent, illuminating, vivid. It can bring techniques in other mathematical branches into the study. Modern mathematical study all capture this feature.

This of course doesn't mean there's no difference between different categories. Some results are indeed unique in one category, they usually concern with the detailed structure of the objects rather than the relations between them.

Practically, using the language of category, many(not all) constructions are easy to remember, since thery are defined uniquely(up to isomorphism) by commutative diagrams!

## 1.2 Groupoid

For locally small categories with only one object, they are identified with monoids:

$$Hom_C(*,*) = End_C(*) \in \textbf{Monoid}$$

The objects are endomorphisms,the binary operator(multiplication) is identified with composition, the identity element is the identity morphism. The crucial point is that the endomorphisms form a set for locally small categories.The inverse is also true, every momoid is also a locally small category with only one object.

Even further, when $End_C(*) = Aut_C(*)$ consists of only isomorphisms, then there's also inverses. These are known as groupoid with only one element, and they are identified with groups.

Different categories have different names for morphisms and isomorphisms, but for all locally small categories, by picking out only one objects, we always obtain a monoid, further constrainting to automorphisms(unit:invertible elements) we obtain a group.

## 1.3 Functor

**Definition 2 (Functor).**
*The functor F from category* $\boldsymbol{C}$,$\boldsymbol{D}$,*denoted* $F \in [\mathbf{C}, \mathbf{D}]$ *is defined by:*

- $\forall A \in \mathbf{C},\ F(A) \in \mathbf{D}$
- $\forall f \in C(A, B), F(f) \in D(F(A), F(B))$,*with compatibility requirements:*
  - $F(id_A) = id_{F(A)}$
  - $F(f \circ g) = F(f) \circ F(g)$

  *The last requirements is known as covariant.There's another version of functors known as contravariant, it is defined as* $F \in [\mathbf{C}^{op}, D]$, *for* $f, g \in hom(C)$, *this is expresees as:*

  $$F(f \circ g) = F(g) \circ F(f) = F(g^{op}) \circ_{op} F(f^{op}) = F(g^{op} \circ_{op} f^{op})$$

  *Actually the 1st requirement follows form the 2nd requirement.It is also clear that the isomorphisms are mapped to isomorphisms and* $F(f^{-1}) = F(f)^{-1}$.

The functor maps[5] the diagram from one category to another category, without changing the "topology": the functor maps the points to points and morphisms to morphisms without "changing" the (co)domain.

To construct a functor explicitly, we have to specify how objects are mapped and how the mapped morphism are defined "microscopicly". We then have to make sure this construction meet the two compatibility conditions.

Two functors can also be composed:

$$F \in [\mathbf{C}, \mathbf{D}], G \in [\mathbf{D}, \mathbf{E}], G \circ F \in [\mathbf{C}, \mathbf{E}] :$$
$$(G \circ F)(A) = G(F(A)), (G \circ F)(f) = G(F(f)) \in E(G(F(A)), G(F(B)))$$

It can be easily verified that this definition of composition indeed satisfy the two conditions.

Functors relate two categories, thus they are even more important than the categories themselves. Famous functors include:

- Free functors give constructions of free objects.
- Forgetful functors remove certain structures.
- Group action is the functor: $F \in [\mathbf{G}, \mathbf{D}]$, specificly, identifying the group as a groupoid with only one object, then: $F(*) = S \in [D], F(f \in G = Aut_C(*)) \in Aut_D(S)$, the two requirements implies F induce a group homomorphism: $\rho : G \to Aut_D(S)$.
- Fundamental group, homotopy, (co)homology in algebraic topology are functors from **Tops** to **Grp**

[5] since obj(-),hom(-) are classes

## 1.4 Natural transformations

**Definition 3** (**Natural transformation**).
*Natural transformations are morphisms of the category* $[\mathbf{C}, \mathbf{D}]$ *of all functors.The natural tranformation* $\eta \in [\mathbf{C}, \mathbf{D}](F, G)$ *is defined pointwisely:*

- $\forall A \in \mathbf{C}, \eta_A \in C(F(A), G(A))$
- *There's compatibility condition:*

$$\eta_{cod(f)} \circ F(f) = G(f) \circ \eta_{dom(f)}$$

*This condition is natural defined by commutative diagram.*

Famous natural transformation include:

- $det \in [\mathbf{CRing}, \mathbf{Grp}](GL_n, *)$

### Composition of Natural transformations

There are two way to compose natural transformations:

- $F, G, H \in [\mathbf{C}, \mathbf{D}], \eta \in [\mathbf{C}, \mathbf{D}](F, G), \eta' \in [\mathbf{D}, \mathbf{E}]$, then $\eta' \circ \eta \in [\mathbf{C}, \mathbf{D}](F, H)$, the definition is trivially by composing the corresponding morphisms at each $A \in \mathbf{C}$.
- $F, F' \in [\mathbf{C}, \mathbf{D}], G, G' \in \mathbf{D}, \mathbf{E}, \eta \in [\mathbf{C}, \mathbf{D}](F, F'), \eta' \in [\mathbf{D}, \mathbf{E}](G, G')$,then $\eta' * \eta \in [\mathbf{C}, \mathbf{E}](G \circ F, G' \circ F')$, the definition is indicated by $(\eta' * \eta)_A = \eta'_A * \eta_A \in E(G(F(A)), G'(F'(A)))$[6]:

$$(\eta' * \eta)_A = \eta'_{F'(A)} \circ G(\eta_A) = G'(\eta_A) \circ \eta'_{F(A)}$$

  It is easily shown that these two kinds of definition both satisfy the compatibility condition, thus well-defined.

### Natural isomorphism

Two functors $F, G \in [\mathbf{C}, \mathbf{D}]$ are natural isomorphic if there exsist natural transformations: $\eta \in [\mathbf{C}, \mathbf{D}](F, G), \eta' \in [\mathbf{C}, \mathbf{D}](G, F), \eta \circ \eta' = id_G \in [\mathbf{C}, \mathbf{D}](G, G), \eta' \circ \eta = id_F$, the $id_*$ is the identity natural transformation whose definition is obvious.

Since the natural transformations are defined pointwisely, the isomorphism condition is equivalent to $\exists \eta \in [\mathbf{C}, \mathbf{D}](F, G), \forall A \in \mathbf{C}, \exists!\ \eta_A^{-1}$.

[6]Simply use the fact that $\eta_A \in D(F(A), F'(A)), G(\eta_A) \in E(G(F(A)), G(F'(A))), \eta'_B \in E(G(B), G'(B))$

**Isomorphism and Equivalence between categories**

**Definition 4** (Isomorphism and Equivalence between categories).

- *Two categories* $\mathbf{C}, \mathbf{D}$ *are isomorphic if* $\exists F \in [\mathbf{C}, \mathbf{D}], G \in [D, C], G \circ F = id_C, F \circ G = id_D$
- *Two categories* $\mathbf{C}, \mathbf{D}$ *are equivalent if* $\exists F \in [\mathbf{C}, \mathbf{D}], G \in [D, C], G \circ F \cong id_C, F \circ G \cong id_D$

The 1st requirement is rarely met while the 2nd appears more often.

# 2 $Hom$ functor and Yoneda lemma

## 2.1 $Hom$ functor

**Definition 5** ($Hom$ **functor**)**.** *For locally small category* $\mathbf{C}, \forall A \in \mathbf{C}$*,we can define a Hom functor* $h_A \in [\mathbf{C}, \mathbf{Set}]$ *as:*

- $h_A(X) := C(A, X) \in \mathbf{Set}$*, then it's also indicative to denote* $h_A \equiv C(A, *)$
- $h_A(f \in C(X, Y)) \in Set(h_A(X), h_A(B)) = Set(C(A, X), C(A, Y))$*, for* $g \in C(A, X), f \in C(X, Y)$ *it's natural to define* $h_A(f)(g) \equiv C(A, *)(f)(g) = f \circ g \in C(A, Y)$*. It can be shown that this definition indeed satisfies the two compatibility condition.*

The h in $h_A$ itself is indicative:

**Definition 6.**
*For locally small category, we can define a contravariant functor* $h \in [\mathbf{C^{op}}, [\mathbf{C}, \mathbf{Set}]]$ *by:*

- $h(A) = h_A \in [\mathbf{C}, \mathbf{Set}]$ *is the Hom functor.*
- $h(f \in C(A, B)) \in [\mathbf{C}, \mathbf{Set}](h_B, h_A)$*, since* $h(f)_{X \in C} \in Set(C(B, X)), C(A, X)$*, for* $g \in C(B, X)$*, it's natural to define:*

$$h(f)_X(g) = g \circ f$$

  *It can be easily shown that the two compatibility requirement are satisfied and for* $\mathbf{C}$*,* $h$ *is indeed contravariant.*

It also indicative to denote $h = C(*, -)$ the 1st argument is filled by $A \in C^{op}$ to obtain the Hom functor, this functor is contravariant and $h(f)_A(g)$ simply compose the f right to g. While the Hom functor use $A \in C$ to fill the 2nd argument to obtain a set, $h_A(f)(g)$ simply compose f left to g.

This motivate a more efficient notation:

- $h = C(*, -)$,$h_A = C(A, -)$,$h^A = C(*, A)$ the former is covariant while the latter is controvariant.

- The definition of $h_A(f), h^A(f)$ is obvious, the former compose f left to g,while the latter compose f right to g.

## 2.2 Yoneda lemma

**Lemma 2** (**Yoneda lemma**).
*For locally small category* $\mathbf{C}$*,* $\forall A \in \mathbf{C}, \forall F \in [\mathbf{C}, \mathbf{Set}]$*, we have the isomorphism:*

$$[\mathbf{C}, \mathbf{Set}](h_A, F) \equiv [h_A, F] \underset{Set}{\cong} F(A)$$

*Expressed in functors this implies:*

- $\forall F \in [\mathbf{C}, \mathbf{Set}], [h_*, F] \underset{[C,Set]}{\cong} F(*) = F$
- $\forall A \in \mathbf{C}, [h_A, *] \underset{[[C,Set],Set]}{\cong} *(A)$

*The isomorphism is a natural isomorphism.Similarly, we have:*

$$[h^A, G \in [C^{op}, Set]] \cong G(A)$$

***Sketch of the proof.*** 
- To prove the isomorphism as sets, define two maps whose definition are natural:$\hat{*}(\eta \in [h_A, F]) = \hat{\eta} := \eta_A(id_A)$,since $\eta_A \in Set(C(A,A), F(A))$; $\bar{*}(x) \in [h_A, F], \bar{x}_Y \in Set(C(A,Y), F(Y)), \bar{x}_Y(f \in C(A,Y)) = F(f)(x)$. It can be easily shown that these two maps are inverse of each other. For functors the map of objects are obvious.

- Fix F, we define the functor $[h_*, F] \in [C, Set]$, the map of morphisms are naturally defined as $[h_*, F](f)(\eta) = \eta \circ h(f)$,note $h$ is the contravariant functor giving Hom functors. It's easy to shown this indeed a well-defined functor. The natural isomorphism is defined by:$\hat{*}, \hat{*}_A \in Set([h_A, F], F(A))$ which is just the bijection above, this gives the natural isomorphism between functors.

- Similarly, the functor maps morphism as: $[h_A, *](f)(\eta) = f \circ \eta$. It's easy to shown this is indeed as covariant functor. The natural isomorphism is given by $\hat{*}, \hat{*}_F \in Set([h_A, F], F(A))$ which is just the $\hat{*}$ above.[7]

- The rest is to make sure these $\hat{*}$ are well-defined natural transformations, the commutative diagram follows directly from definition of $\hat{*}_{A/F}$

□

[7] Note that the notation is suggestive: for unfilled 1st/2nd argument the functor is a contravariant/covariant functor. The definition is also simple: contravariant/covariant one compose the f right/left to g.

# 3 Adjiont functor,Universal properties,Limit

## 3.1 Adjiont functor

**Definition 7** (**Adjiont functor**).

*For two locally small categories* $\mathbf{C}, \mathbf{D}$,*given* $F \in [\mathbf{C}, \mathbf{D}], G \in [\mathbf{D}, \mathbf{C}]$, *F is the left adjiont of G or G is the right adjiont of F if for all* $X \in \mathbf{C}, Y \in \mathbf{D}$, $C(X, G(Y)) \underset{Set}{\cong} D(F(X), Y)$. *Equivalently:*

- $C(-, G(Y)) \underset{[\mathbf{C^{op}}, \mathbf{Set}]}{\cong} D(F(-), Y)$
- $C(X, G(-)) \underset{[\mathbf{D}, \mathbf{Set}]}{\cong} D(F(X), -)$

*The functors map the morphism as:*

- $C(-, G(Y))(f)(g) = g \circ f, D(F(-), Y)(f)(g) = g \circ F(f)$
- $C(X, G(-))(f)(g) = G(f) \circ g, D(F(X), -)(f)(g) = f \circ g$

The proof of adjiont functors are similar to the proof of Yoneda lemma: we first construct set isomorphism then promote this isomorphism pointwisely to natural isomorphism and finally show that the functors are natural isomorphic.

Famous adjiont functors include:

- Forgetful functor $G \in [\mathbf{Set}, \mathbf{Grp}]$ is the left adjiont of free functor $F \in [\mathbf{Grp}, \mathbf{Set}]$.
- $\Delta \in [\mathbf{Grp}, \mathbf{Grp^2}]$ is the left adjiont of $\pi \in [\mathbf{Grp^2}, \mathbf{Grp}]$
- $\cdot^{ab} \in [\mathbf{Grp}, \mathbf{AB}]$ is the left adjiont of $i \in [\mathbf{Ab}, \mathbf{Grp}]$

## 3.2 Universal morphism

**Definition 8** (**Universal morphism**).

- *Given* $F \in \mathbf{C}, \mathbf{D}, X \in D, u_A \in D(X, F(A))$ *is a universal morphism from X to F if* $\forall A' \in C, \forall f \in D(X, F(A'))$,$\exists!\ h \in C(A, A')$ *such that* $f = F(h) \circ u$.*This means any morphism* $f \in D(X, F(A'))$ *factor through uniquely the universal morphism* $u_A \in D(X, F(A))$ *from X to F.*
- *By reversing the arrows we have the definition of a universal morphism from F to X.*

**Universal property of universal morphism**

It can be easily shown that if the universal morphism from X to F exsist then it is unique up to unique isomorphism.This means if $u_A, u'_{A'}$ are two universal morphism then they are unique and the isomorphism is unique.

This follows directly from unique factorization in definition of universal morphisms,and the fact that there's already an isomorphism in $id_A \in C(A,A)$.Thus the compositions of the arrows between the $F(A), F(A')$ are $id_{F(A)}, id_{F(A')}$.

The universal morphism is a special case of limit,which is defined similarly, but with more general categories.

**Universal property of adjiont functors**

Given $F \in [\mathbf{C}, \mathbf{D}], G \in [\mathbf{D}, \mathbf{C}]$, F is the left adjiont of G, if for $\forall Y \in \mathbf{D}$,$\exists$ universal morphism from F to Y.That is $\forall Y \in \mathbf{D}$, $\exists G(Y) \in \mathbf{C}, u_Y \in D(G(Y), Y)$, such that $\forall X \in \mathbf{D}, \forall f \in D(F(X), Y)$, $\exists!\ h \in C(X, G(Y))$ such that $f = u_Y \circ F(h)$.The unique factorization through universal morphism gives bijection between the two sets of morphisms.

***Sketch of the proof.*** The oneway proof is rather obvious, just using definition of the natural isomorphism and its relation between functors. The other way need the definition of the natural transformation, the definition is provided by the universal morphism:

$$\hat{h} = u_Y \circ F(h), f = u_Y \circ F(\bar{f})$$

It can be shown that these two map are indeed isomorphisms and promoted to natural transformations they are well-defined with respect to the functors, this also defines the functor G naturally. $\square$

## 3.3 Limit

**Definition 9** (**Graph;Cone**).

- *A* $\mathbf{J}$ *graph in* $\mathbf{C}$ *is a functor* $F \in [\mathbf{J}, \mathbf{C}]$
- *A cone of J-graph F consist of* $(N, \{f_A \in C(N, F(A))\}_{A \in \mathbf{J}})$ *such that* $\forall f \in J(A,B), f_B = F(f) \circ f_A$.

**Definition 10** (**Limit**).
*Given J-graph F in* $\mathbf{C}$*, the limit of F is the special cone* $(L, \{l_A \in C(L, F(A))\}_{A \in J})$ *such that for* $\forall$ *other cone* $(N, \{f_A \in C(N, F(A))\}_{A \in \mathbf{J}})$, $\exists!\ u \in C(N, L), \forall A \in \mathbf{J}, f_A = \phi_A \circ u$

*That is the cone factor through limit uniquely.Similarly it can be shown that if limit exsist then it's unique up to unique isomorphism.*

*By reversing the arrows we obtain the dual definitions of cocone,colimit.*

### 3.3.1 Examples of (co)limit

The most elementary examples are:

- Final objects & Initial objects. With $J = \varnothing$

Other examples can all be considered as the Final/Initial object in appropriate categories consists of subgraphs((co)cones):

- Product & Coproduct. With $J = \{A, B\}$
- Pull back & Push forward,these are also known as Fibered projuct and fibered coproduct. With $J = \{C, A, B; f : A \to C, g : A \to C\}, J^{op}$
- Equilizer & Coequilizer $J = \{A, B; f : A \to B, g : A \to B\}, J^{op}$
- The notion of product and equilizer can be generalized to include more objects.

Explicit definitions of these objects are natural from the commutative diagrams defining the unique (co)limit. Usually the colimits have more complex construction, but it is easy to understood in category theory as the dual construction of the corresponding limit.

Two useful criteria[8]:

- If **C** accept definition of product and fibered product then the equilizer can be constructed from the universal properties of these two construction.
- If **C** accept definition of all arbitary products and all equilizers then any J-graph F in C have limit.

The proof is implicit in the diagrams:

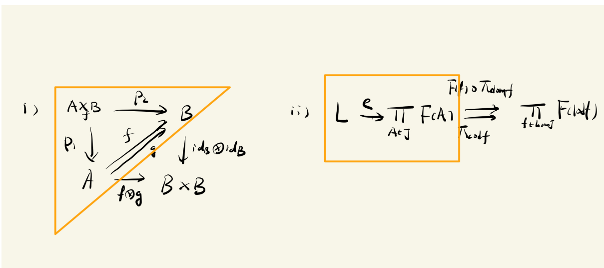


Figure 1: Construction of equilizer and limit

[8]Same for dual statements