Recap

- $\mathrm{Conf}(\mathbb{R}^{p,q})$ ← conformal diffeomorphism of the pesudo-Riemannian manifold

  ↑ connected componet

- Infinitesimal conformal transformations

  → useless in higher d

  → Witt algebra (1,1)d, (2,0)d

(continuous)
More symmetry, more conserved quantities, less independent D.O.F, more probability to be solved exactly

- Conformal field theory (working def.)

  v1) relevant D.o.F furnish PUIR of $\mathrm{Conf}(\mathbb{R}^{p,q})$

  Not all PUIR are compatible with "locality"

(classical / quantum)
observables : $\Phi_a(x)$ $\xleftrightarrow[\text{related}]{\text{Not necessarily}}$ Representation

↑ "Local"

- Finite dim:

  $G \to M_n(\mathbb{C})$

- Infinite dim:

  $G \to B(\mathcal{H})$ or $L(\mathcal{H})$

  ↑ bounded operators ↑ linear operators

hard

It's likely $\overset{\text{(Local)}}{\pi} \in \mathrm{Rep}(G) = \bigoplus \pi^{(ir)}$

forced reducible easy irreducible ones.

More assumptions. More constraint than simply rep-theory of Lie algebras

§ Classical field rep of conformal group

easy to furnish Locality

— Assumptions: (simplification)

① $S = \int d^d x \, \mathcal{L}(\phi, \underline{\partial_\mu \phi})$ Local.

ϕ_a

tensor components, particle types.

② classical transformation:

$$x \longmapsto x'$$

$$\phi(x) \longmapsto \phi'(x') = F[\phi(x)]$$ active view

$\Rightarrow$ observe at the old position Finite rep (reducible)

$$\phi'(x) = (\underline{O \cdot \phi})(O^{-1} \cdot x)$$

③ $S' = \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}'\left(F(\phi'(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu F(\phi'(x))\right)$

$$\left|\frac{\partial x'}{\partial x}\right| = J = \det \frac{\partial x'}{\partial x}$$

Symmetry : equation of motion left invariant

— Infinitesimal (generators) Conf($\mathbb{R}^{p,q}$)

Vector fields ($T_x M$) :

- translation : $P_\mu = -i\partial_\mu$
- dilation : $D = -i x^\mu \partial_\mu$
- rotations : $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
- SCTs : $K_\mu = -i\left(2 x_\mu (x^\nu \partial_\nu) - x^2 \partial_\mu\right)$

$[\ ,\ ] \longrightarrow$ all infinitesimal ones.

$\Longrightarrow$ Lie algebra of $Conf(\mathbb{R}^{p,q})$

* Using LIH rep of generators is the most efficient way to obtain commutators.

- $[D, P_\mu] = i P_\mu$, $[D_\mu, K_\mu] = -i K_\mu$.
- $[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu})$
- $[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu)$
- $[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)$
- $[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + (\text{permutations}))$

Poincaré algebra

* With even supersymmetry include the list is even longer.

* $[\cdot,\cdot] \sim \underline{eeee}$ physical meaning.

- other's commute

$\rightarrow$ promote to fields.

Irreps of $Lie(G) \longleftrightarrow \Phi_a$ - tensors.

Start with Poincare subalgebra $\longrightarrow$ (mass spin). study transformation properties of these under extra generators

- Finding Irreps.

Focus on subgroup of transformations leaving origin fixed. ($\underbrace{J_{\mu\nu}, D, K_\mu}_{G_a}$, not P_μ)

$$\phi_a(0) \to \sum_b \underline{\pi(e^{i\omega^a G_a})_{ab}}\, \phi_b(0)$$

Form a little group! (subgroup).

Further Use the rep of $J_{\mu\nu}, P_\mu$.

scaler dimension

$$\pi(G_a) = \begin{cases} \pi(D) \equiv \tilde{\Delta} \\ \pi(K_\mu) \equiv \kappa_\mu \\ \pi(L_{\mu\nu}) \equiv S_{\mu\nu} \end{cases}$$

$$\Rightarrow \quad [\tilde{\Delta}, S_{\mu\nu}] = 0, \quad [\tilde{\Delta}, \kappa_\mu] = -i\kappa_\mu, \quad [\kappa_\mu, \kappa_\nu] = 0$$

suppose $S_{\mu\nu}$ irreducible (as in Poincare group)
(labelled by spin/helicity)

By Schur's lemma $\Longrightarrow$ $\begin{cases} \tilde{\Delta} \propto I \\ -i\kappa_\mu \equiv 0 \end{cases}$ $\quad \tilde{\Delta} = i\Delta I$

(both spin(er)/hel)

Action of dilations on (spinless) field (for simplicity).

$$x \longmapsto \lambda x = (\lambda^{\varepsilon} \cdots \lambda^{\varepsilon} \cdots) x$$

$$\phi_a(0) \longmapsto \left[(I + i\varepsilon \tilde{\Delta})(I + i\varepsilon \tilde{\Delta}) \cdots \right] \phi_a(0)$$

$$= \lambda^{i\tilde{\Delta}_a} \phi_a(0)$$

$$= \lambda^{-\Delta_a} \phi_a(0)$$

Scaling dimension of conformal fields

$$\left| \frac{\partial x'}{\partial x} \right| = \Lambda^{-\frac{d}{2}}, \quad g'_{\mu\nu} = \Lambda g_{\mu\nu} \implies \Lambda = \lambda^{-2}$$

$$\lambda^{-1} = \left| \frac{\partial x'}{\partial x} \right|^{-\frac{1}{d}}$$

$\implies$ Under dilation: (use $e^{iP_\mu x^\mu}$ to translate to x, $[\,,\,]$ involved)

$$\phi(x) \longmapsto \phi'_a(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\frac{\Delta_a}{d}} \phi_a(\lambda x).$$

Use $e^{ix\cdot P} D e^{-ix\cdot P} \equiv D + x\cdot P$

$$D \cdot \phi(x) = (-ix^\nu \partial_\nu + \tilde{\Delta}) \phi(x)$$

$$\implies \phi'_a(x) = \lambda^{-\Delta_a} \phi_a(\lambda^{-1} x)$$

$$= \left| \frac{\partial x'}{\partial x} \right|^{-\frac{\Delta_a}{d}} \phi_a(\lambda^{-1} x)$$

Quantum: self-adjoint (unitary) operators correspond to generators

the Δ (scaling dimension) also appear, along with spin, central charge (projective)

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§ Constraints of conformal invariance on quantum field theories

assume we've constructed one.

What's QFT? (CFT) No rigorous def. yet!

(1+1)d : 3 approaches (not directly equivalent)

- Vertex operator algebras
- local algebraic quantum field theory
- Functional : n-Category $\longrightarrow$ Hilb

Working definition :

① $\mathcal{H}$: kinematic space (quantum)

② projective unitary representation of conformal group $U(g)$, $g \in \mathrm{Conf}(\mathbb{R}^{p,q})$
(or larger)

not SSB

③ Vacuum: $|0\rangle \in \mathcal{H}$ invariant under global (finite) conformal symmetries

$U(g)|0\rangle = e^{i\varphi(g)}|0\rangle$

④ Observables: $A_{j,x}$ self-adjoint
x ↑ spacetime.

What makes $A_{j,x}$ Local is commutation relations (causality):

$A_{j,x}, A_{k,y}$ are jointly measurable if $x-y$ spacelike.

$$[A_{j,x}, A_{k,y}] = 0 \quad \text{space-like}$$

Rmk. if $p+q>2$ then local = global (finite ones = exp(infinitesimal ones))

for $p=1, q=1$, the global $\mathrm{Conf}(\mathbb{R}^{1,1}) \subset \underbrace{\mathrm{Diff}_+(S^1)}_{\text{bigger}} \times \ldots$

Quasi-primary : Local observables transform under conformal transformations properly.

(assume they exists). (global for now).

Rmk notation changed !

$$\hat{\phi}_k(x) \xrightarrow[x'=g\cdot x]{U(g)} \left|\frac{\partial x'}{\partial x}\right|^{+\Delta_k/d} \hat{\phi}'_k(x')$$

$$= U^{-1}(g)\,\hat{\phi}_k(x)\,U(g)$$

consequences of conformal symmetries $\Longrightarrow$ Constraint on Correlation functions (covariance).

$$\langle 0|\,\hat{\phi}_{k_1}(x_1)\cdots\hat{\phi}_{k_n}(x_n)\,|0\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{\Delta_{k_1}/d}\cdots\left|\frac{\partial x_n'}{\partial x_n}\right|^{\Delta_{k_n}/d} \qquad (*)$$

$$\times\ \langle 0|\,\hat{\phi}'_{k_1}(x_1')\cdots\hat{\phi}'_{k_n}(x_n')\,|0\rangle$$

$$\| \qquad\qquad\qquad\qquad \left(=\langle 0|\,\hat{\phi}_{k_1}(x_1')\cdots\hat{\phi}_{k_n}(x_n')\,|0\rangle\right)$$

$$G(x_1,\dots,x_n)$$

observables in experiments

invariant under conf

Rmk.

① This hold for global conformal transf.

② These n-pt functions are contructed out of invariants of global $Conf(\mathbb{R}^{p,q})$.

Invariants : ($G(x_1, \dots, x_n)$ depend only on $x_1, \dots, x_n$)

translation $\Rightarrow$ $x_j - x_k$
$\uparrow$
n

Rotation $\Rightarrow$ For spinless object & large enough d
(finite Conf(...))

$$r_{jk} = |x_j - x_k|$$

scale $\Rightarrow$ r_{jk}/r_{em}.

SCT $\Rightarrow$ Cross ratios : $\boxed{\dfrac{r_{jk} \cdot r_{em}}{r_{je} \cdot r_{km}}}$

Example :

① 2-pt functions : $G(x_1, x_2) = f(r_{12})$

under dilation.

$$f(r_{12}) = \lambda^{\Delta_1 + \Delta_2} f(\lambda r_{12})$$

$$\Rightarrow f(r_{12}) = \sum_a f_a r_{12}^a$$

a might be continuous.

$f_a = 0$ unless $a = -\Delta_1 - \Delta_2$.

determined by field normalization

$$f(r_{12}) = r_{12}^{-\Delta_1 - \Delta_2} \cdot C_{12}$$

under SCT:

only solutions: $\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \dfrac{C_{12}}{r_{12}^{2\Delta}} & \Delta_1=\Delta_2=\Delta \\ 0 & \text{otherwise.} \end{cases}$

② 3-pt functions:

$$G(x_1,x_2,x_3) = \sum_{\substack{a,b,c \\ a+b+c=\Delta_1+\Delta_2+\Delta_3}} \frac{C_{abc}}{r_{12}^{a}\, r_{23}^{b}\, r_{13}^{c}}$$

SCT: $\Longrightarrow$ $a=\Delta_1+\Delta_2-\Delta_3,\ b=\Delta_2+\Delta_3-\Delta_1,\ c=\Delta_3+\Delta_1-\Delta_2.$

$$G(x_1,x_2,x_3) = C_{abc}\, r_{12}^{-a}\, r_{23}^{-b}\, r_{13}^{-c}$$

③ For 4-pt:

$$G(x_1,x_2,x_3,x_4) = \underset{\text{arbitrary}}{F}\Big(\underbrace{\frac{r_{12}r_{34}}{r_{13}r_{24}},\ \frac{r_{12}r_{34}}{r_{23}r_{14}}}_{\text{independent}}\Big) \times \prod_{j<k} r_{jk}^{-(\Delta_j+\Delta_k)+\Delta/3}$$

$\Delta = \sum_j \Delta_j$

Rmk. CFT is not trivial, since $n \geq 4$ pt functions are (not too constraint) not determined uniquely

(Generalization of conf transf).

Conformal theories in 2D

$(2 \to) d$.

There's nontrivial Φ's.

Rmk

(infinitesimal ones).

the def of conformal transf are generalized, more constraint!

So does quasi-primary $\Longrightarrow$ primary of type $(h, \bar{h})$

or conformal weight

analytic continuation

$$\phi(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \qquad z' = f(z).$$

Infinitesimally. $z \longmapsto z + \varepsilon(z)$. $\partial = \partial_z \quad \bar{\partial} = \partial_{\bar{z}}$

$$\delta_{\varepsilon, \bar{\varepsilon}} \phi(z, \bar{z}) = \left\{ (h \partial \varepsilon + \varepsilon \partial) + (\bar{h} \bar{\partial} \bar{\varepsilon} + \bar{\varepsilon} \bar{\partial}) \right\} \phi(z, \bar{z})$$

I.

$\Longrightarrow$ 2 p-t function :

$$\delta_{\varepsilon, \bar{\varepsilon}} G(z, \bar{z}) = \langle \delta_{\varepsilon \bar{\varepsilon}} \phi_1, \phi_2 \rangle + \langle \phi_1, \delta_{\varepsilon \bar{\varepsilon}} \phi_2 \rangle$$

vectors ! (z_i).

$O(\varepsilon^2)$ $\overline{}$ 0

consequence of constraints:

$z_1 - z_2$.

1. $\varepsilon(z) = \varepsilon \Rightarrow G^{(2)}$ depend on differences.

2. $\varepsilon(z) = \varepsilon z \Rightarrow G^{(2)}(z, \bar{z}) = \dfrac{C_{12}}{z_{12}^{h_1+h_2}\, \bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}$

3. $\varepsilon(z) = z^2 \Rightarrow G^{(2)}(z, \bar{z}) = \dfrac{C_{12}}{z_{12}^{2h}\, \bar{z}_{12}^{2\bar{h}}}$

Example

suppose given bosonic fields: $h - \bar{h} = 0$. set $\Delta = h + \bar{h}$

$$\Rightarrow G^{(2)}(z, \bar{z}) = \frac{C_{12}}{|z_{12}|^{2\Delta}}$$

Rmk. this best will focus on spinless (bosonic) fields

For fields with spins. $h \neq \bar{h}$

I. 3-pt functions:

$$G^{(3)}(z, \bar{z}) = C_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} \, z_{23}^{h_2+h_3-h_1} \, z_{13}^{h_1+h_3-h_2}}$$

$$\times \, (z \to \bar{z})$$