Contents

Chapter 1

Motivation and general aspects

1.1 Motivation

Classical field theory:

$$F : Spacetime\ manifold\ M \to Tensor\ space\ T \tag{1.1}$$

- Spacetime manifold M is a manifold in general,in many cases it will be equipped with a **metric**,indicating **the structure of the base space—the spacetime symmetry**:the metric is invariant under
  - Reletivistic field theory: ISO(1,D),with metric given by Lorentz type: $diag(-,+,+,...)$
  - Non-reletivistic field theory: E(1+D=d)=ISO(d),with metric given by Euclidean type: $diag(+,+,...)$
  - Conformal field theory: ???
  - Background irrelevant theory:General reletivity:???
  - Topological field theory: only the topology of M matters

  Usually we will decompose M into $Time \times Space$, with T 1D.
  - In particle physics,T is taken as $(-\infty, +\infty)$
  - In condensed matter physics,usually the T is taken as a circle/ring(compactified):S^1

  It's usually helpful to **Wick rotatation** the reletivistic M to Euclidean ones,this amounts to a **analytic continuation**:
  - To show the convergence of the system(path-integral).which turns the phase in to decaying exponentials
  - To do calculations in regularization.
  - To discuss the vacuum/ground state.
- F is a section,at each point,$F(x)$ is given as a tensor,which transform under certain group G:
  - In reletivistic theory, the tensor is a **finite-dim** irreducible representation of the homogeneous group O(1,D)
  - In condensd matter physics, many other group G can appear: $SO(3), SU(2), ...$

  This **encode the symmetry of the system** effectively
- $F(x)$ is a finit degree of freedom at point x,thus the theory have **infinite degree of freedom**,this have several consequnces:
  - it lead to two types of infinity: UV divergence and IR divergence,to deal with these infinity we can discretize M into *lattice*:functional integrals turn into finite-dim integrals,fields turn into on-site/edge tensors.
  - infinite D.O.F is useful to describe system with **unfixed** point-particle numbers,which is quantum mechanically described by a Fock space.
    - CMP: number of quasiparticles are not conserved

  * Particle physics: number of reletivistic point particles are not conserved due to combination of SR(Energy-Mass equivalence) and QM(uncertainty principle:quantum flunctuation of ΔE and the vacuum) and interactions(lead directly to annihilation and creation)

Phenomelogy: The quantization of classical fields phenomelogically describe the **creation and annihilation** of (quasi)particles and **identical particle principle** follows directly:

- Free field theory have Gaussian-like structure which give rise to **linear** equations,the expansion of the solution into linear combination of modes lead to expansion of fields, promoting the coefficients to operators,the 2nd quantization scheme is restored.
- H_{free} expanded into linear combination of **harmonic modes** in 2nd quantization form,which have the same picture as **harmonic oscillator**,this is common for free theory.
- Free theory is solved by **2-point Green's function**,which turns out to be various propagators with various boundary conditions which have various causality and physical meaning.
- The interactions can also be formed into 2nd quantization forms,solved purturbatively,this lead to **tree-level** Feynman diagrams:the force/potential/interaction results from exchange of particles.
- The classical solutions have their own applications,like electromagenetic fields,mechanical waves in various medium.Special **topological solutions** are of great importance:instanton,monopoles,kinks/solitons,domain walls,...
- The quantization of classical field theory should restore classical theory in the limit of $\hbar \to 0$,the map between quantum theory and classical theory is not 1-1.Classical theory usually have their **effectiveness** at certain scale,like electrodynamics.In this way,they are effective theories in classical limits.
- Not all quantum theory have their classical couterpart,they disappear in classical limit.
- Semi-classical picture: the quantum effect(loops/flunctuation/renormalization) give rise to **quantum correction** of classical results give rise to effective potentials,this is best described by path-integral formalism.

Formalism: Classical field theory is just a generalization of classical mechanics into infinite D.O.F(upon lattice regularization):

- Lagrangian formalsim:
  - The Lagrangian is constrained by certain principle:symmetry(both global and local),stability(Hamiltonian bounded below). Quantum theory have more constrains which makes the theory **inevitible**,this need the idea of quantization.
  - The field equation is the Euler-Lagrange equation,the equation can be used to determine the Lagrangian/action.
  - Continuous symmetry lead to conserved currents and charges,upon quantization,they give rise to symmetry group **generators and Lie algebras** naturally.
- Hamiltonian formalism:
  - The Hamiltonian is constrained by stability,its connection to Lagrangian is through Legendre transformation,which may not exsist.Both Hamiltonian and Lagrangian should give rise to same action.
  - The Hamiltonian formalism manifest the sympletic structure in terms of canonical variables.The phase space determines the true **independent D.O.F and auxiliary D.O.F**.
  - Conserved charges(kinetic constants) as generator lead to symmetry transformations in terms of Possion brackets.The dynamical equation is just time-translation with Hamiltonian the generator.
- The equations and L/H can both be used to determine the transformation behavior of fields under some transformation groups,especially the discrete ones.

Quantization Two equivalent way of quantization of classical field theory:

- canonical/operator/algebraic quantization
- path/functional integral
- both L/H formalism are used in these two quantization scheme

1.2 General aspects of relativistic QFT

Relativistic quantum field theories are **effective** field theories about **relativistic** point-particles and their **interactions**.The basic principles are:

- Spacetime symmetry is given as the Poincare symmetry ISO(1,3)
- Quantum theory: the D.O.F are given as **states** in Hilbert space,in general the Hilbert space is given as the **Fock space**,and operators on the Fock space.The basic aspects of Hilbert space:
  - It's spanned by some **complete** basis,the metric is **Euclidean**.It's natural to choose a **orthogonal** basis which is given as **eigenstates** of complete set of commutative operators.The CSCO are usually given by:
    * Various **commutative generators** of symmetries,especially the 4-momentum for single-particle subspace.
    * Certain **discrete** symmetry operators are used to manifest the **degeneracies**.
  - The spectrum of point-particles are also described by single/multi-particle states(may or may not include interactions) in the Fock space.These states also form a complete basis.The physical D.O.F associated with particles what physically measured.

  For operators:

  - operators in a group will be represented as **unitary/antiunitary** operators,which conserve probability.
  - The whole group is represented **projectively**,usually we will extend the group to it's **universal covering** with **central charges** cancelled by including certain operators.
  - If the operators commute with H,then they are symmetry transformations.
- Fock space must be invariant under spacetime symmetries which lead to constraints on definitions of the states.

1.2.1 Particle and Field

The combination of SR and QM lead to certain results:

- The superselection sectors/building blocks of Hilbert space:Single particle states are **projective irreducible unitary representations of ISO(1,3)=**$T_4 \rtimes O(1,3)$,or equivalently **ordinary unitary representations of extension of ISO(1,3)**.the transformation propertis of multiparticle state is induced from these single-particle states.This is exact:non-purterbative and scale independent.
- There are conflict between SR and QM,which makes the explicit construction of single-particle states hard:the metric of Hilbert space and the metric of Minkowski space are different.Constructing single-particle states simply out of Lorentz-tensors will break **unitarity**:probability will not conserve.This is due to the fact that $ISO(1,3)$ is **not compact**,which have no finite-dim PUIRs.Thus we have to seek for infinite-dim ones.
- The infinite-dim PUIRs can be constructed induced from it's little groups.The general result is that the infinite-dim PUIRs are classified by two basic **Lorentz invariant** quantum numbers:mass and spin/helicity.Mass is given as the Casimir constant,which is related to the Pauli vector.The little groups are:
  - massive:SO(3) with covering SU(2),giving another quantum number as spin
  - massless:$ISO(2) = E(2) = T_2 \rtimes O(2)$,giving another quantum number as helicity.The fact that T_2 should be unobservable(unphysical) is related to gauge invaraince in field theoretical constructions which manifest this unphysical D.O.F as redundancy.
- To construct single-particle states in 2nd quantization scheme,the fields naturally arise.This is the process of embedd particles into fields.In general we construct fields using **finite**-dim PUIRs of $SO(1,3)$ or ordinary UIRs of $SL(2,\mathbb{C})$.The infinite-dim rep arise when we promote these tensors to fields,this is manifested in p-space,using polarization tensors,they are promoted to spacetime-dependet.Fourier transform and form certain linear combinations we ended up with fields.

- There are **antiparticles**:having same mass and spin/helicity but all global symmetry charges.This is a direct result of **causality** from SR:spacelike seperated events can have reversed orders in different frames(upon spacetime transformation).To preserve unitarity(right amplitude) antiparticle must exsist.

- spin-statistic connection:Lorentz invariance lead to causality which lead to exchange properties between reletivistic identical particles.In field theory,these may be manifested from stability of H,causality/locality of (anti)commutators,analytic behavior of propagators,projetive/ordinary representation of SO(3).There are complexity from anyons in 2D space.

- Particles(D.O.F) that drop out of amplitude are unphysical.There are also unphysical states called **ghost** states having minus norm,which break unitarity.The origin of ghost may be:

  - wrong sign of kinetic term:UV ghost

  - wrong spin-statistic connection:F-P ghost

  - wrong behavior of propagators,which may arise from non-local theories:longitude mode of photon

1.2.2 Interaction:S-matrix and observables

Experiments of Particle Physics: According to Lorentz invariance,the interactions must be **local**,which happens only in finite volumn of spacetime.In particle physics,the observables:**cross section and decay** rates are related to the S-matrix,which describe the transition between the initial and final state consists of **effectively/asymptoticly non-interacting/free** multiparticles,which transform properly under ISO(1,3).

The locality can be understood as particles colliding at finite volunm of spacetime,which is effectively a point.This actually lead to the **perturbative expansion** of the process into some Feynman diagrams,where particles are free except at the interacting point,where the particle is annihilated and created or scattered.

The difference of these diagrams with the classical perturbation expansions are that there are loops.

CMP: Other observables are measured in CMP,including:**thermodynamic properties and various response functions**

OFPT,time-dependent perturbation theory,Lorentz covariance

1.2.3 Canonical quantization and path-integral formalism

Axioms

Noether

S-D

Green's functions and exact vertices

1.2.4 Unitarity and Some non-purterbative results

Opetical theorem and cutting rules

Propagators and polarization sums

Spectral decomposition

Poles in Green's functions and LSZ,renormalization

Gauge invarance and Ward identities

Soft particle theorems

Dispersion relations

1.2.5 Gauging,uniqueness of nonabelian gauge theory,F-P,BRST

1.2.6 Renormalization and renormalization group

regularization,counterterm,renormalized purturbation theory

loop behavior:divergence index,logs,polynomials,dimensional analysis

renormalizability,role of gauge invariance

non-renormalizability,prediction,UV completion

scale and renormalization group

IR divergence

1.2.7 quantum effective potential,SSB,anomaly

Part I

Poincare group and particles

Chapter 2

Spectrum:Particles=PUIR of $ISO(1,3)$

2.1 Axioms of quantum mechanics

We are studying effective theories at the scale where quantum effect matters[1],thus the theory must be build on the ground of the axioms of quantum mechanics[2]:

Axioms of Quantum mechanics

- **Physical states** are rays in the projective Hilbert space $\mathbb{P}\mathcal{H} = \mathcal{H}/(\mathbb{C}^* - \{0\})$:
  - Ray $\Psi := [|\Psi\rangle] = \{\alpha|\Psi\rangle, \alpha \in \mathbb{C}\}$
  - Hilbert space $\mathcal{H}$: complex linear space[a] with inner product.The inner product give rise to **positive** norm:$\langle\psi|\psi\rangle \geq 0$.
  - Always adopt unit normalization for physical states:$\langle\psi|\psi\rangle = 1$,thus ray Ψ is $|\Psi\rangle$ up to U(1) phases.$\mathcal{H}$ may be extended to include states with δ-normalization,these are limit/approximation to physical states.
- **Operators:** O is defined as $\hat{O}: \mathcal{H} \to \mathcal{H}$:
  - Linear operators: $A(a|\psi\rangle + b|\phi\rangle) = aA|\psi\rangle + bA|\phi\rangle$,with adjiont defined as:
  $$\langle\psi|A^\dagger|\phi\rangle = \langle A\psi|\phi\rangle = \langle\phi|A|\psi\rangle^*$$
  - Antilinear operators: $A(a|\psi\rangle + b|\phi\rangle) = a^*A|\psi\rangle + b^*A|\phi\rangle$,with adjiont defined as
  $$\langle\psi|A^\dagger|\phi\rangle = \langle A\psi|\phi\rangle^* = \langle\phi|A|\psi\rangle$$
  - Eigenvalue equation:$A|\psi\rangle = \alpha|\psi\rangle$,$|\psi\rangle$s span an eigen/invariant subspace,the dimension/degeneracy can be denoted by other indices which may be eigenvalues of some set of commutative/compatitable operators[b].We usually use CSCO[c],where degeneracy are all manifested,and the eigenstate is denoted by eigenvalues of CSCO.
- **Observables** are represented by **Hermitian** operators,which is **linear** and self-adjiont:$H = H^\dagger$,with **real** eigenvalues.The eigenstate of CSCO may be taken as complete and orthonormal:
  $$\langle\alpha|\beta\rangle = \tilde{\delta}(\alpha - \beta), \tilde{\int} d\alpha|\alpha\rangle\langle\alpha| = I$$
  .The tilde represent product of subindices,for discrete indices:δ refers to symbol,$\int d\alpha$ refers sum.
- **Unitarity**:The probability $P(\mathcal{R} \to \mathcal{R}_\alpha) = |\langle R|R_\alpha\rangle|^2$ is **conserved**:
  $$\tilde{\int} d\alpha P(\mathcal{R} \to \mathcal{R}_n) = \tilde{\int} d\alpha\langle R|R_\alpha\rangle\langle R_\alpha|R\rangle = \langle R|R\rangle = 1$$
  .

[1] $\hbar \neq 0$,the resolution of action and angular momentum is of scale $\hbar$

[2] In this notes,the **Linear-algebra notation and Dirac-notation** are both used for convenience

- **Identical particles:** identical particles are assumed to be indistinguishable,thus exchanging states of identical particles will give no physical consequnces.

[a]Usually infinite dimensional
[b]Usually observables/generators of continuous symmetry or discrete symmetry transformation itself
[c]complete set of **observables**

Remark 1 • *These axioms gives the* ***kinematics****:how to describe a quantum system.The dynamics will be given as equations of motions satisfied by* ***time-evolution*** *operator* $U = \mathcal{T}exp[-\frac{i}{\hbar}\int d\tau H(\tau)]$ *in different pictures.In closed/conserved systems,time-evolution is equivalent to time-translation,there's no need to specify it as an axiom.*

- *It's easier to work with $\mathcal{H}$,the true physics is contained in $\mathbb{P}\mathcal{H}$,thus $|\psi\rangle$ is always accompanied with* ***U(1)-phase equivalence relation****.In theories with particle number unfixed,the Hilbert space is defined as the Fock space:*

$$\mathcal{H} \equiv \mathcal{F} = \oplus_i \mathcal{F}_i$$

,with $\mathcal{F}_0$ the vacuum/ground subspace[3]*,$\mathcal{F}_1 = \mathcal{H}_{non-relativistic}$ the single-particle state which lead to non-relativistic QM.*

- *The observables in most cases will be given as* ***generators*** *of symmetry Lie groups of the system.In relativistic QFT,* ***S-matrix*** *is the only observable to be related to measurements.*

- *Unitarity is the most important constraint,it will lead to many* ***non-perturbative*** *results,here the unitarity means probability is conserved,thus states with* ***negative norms called ghosts*** *are unphysical:*
  - *not the D.O.F of the system,not shown up in spectrum of* ***reletivistic****-particles.*
  - *drop out of all physical quantities*

- *The completeness relation of basis will be vastly used,which may be considered as the equivalent concept of unitarity,which lead to many physical interpretation of the nature of quantum-particles.Besides the completeness of physical*

2.2 Group is projectively represented

Tranformations are the most important operations we use to study physical sysetems,they will also be represented as **operators on** $\mathcal{H}$.These transformations form groups.These groups will always be introduced as the **symmetry group** of certain system,but whether the quantum system is invariant under this group is determined by its Hamiltonian.Different H have different symmetry group.

Convention of transformations

- The group of transformations when H is not specified will be called symmetry group.
- When H is specified,the symmetry group refer to transformations leaving H invariant.
- The transformations will be applied defaultly in **passive view:apply transformations to observables**—observer/coordinates changes,while the system unchanged.The **active view:apply transformations to states**—the system changes while observer/coordinates unchanged is easier to illustrate,especially for discrete transformations.
- For time-translations,passive view correspond to **Heisenberg picture**:the time dependence is from observables,the state will be time-dependent and contain the whole history of the system,not only one particular time.
- In time-dependent perturbation methods,the **Interaction-picture** is also used.

[3]may exsis degeneracy

2.2.1 Wigner's theorem

Physical symmetry transformations[4] refer to $Aut(\mathbb{P}\mathcal{H})$,we will pull back these transformations to $Aut(\mathcal{H})$,this lead to the exact short sequnce:

$$1 \to U(1) \to Aut(\mathcal{H}) \to Aut(\mathbb{P}\mathcal{H}) \to 1$$

One of the most profound theorem in quantum physics is:

Wigner's theorem

Symmetry transformations are defined to preserve probability:

$$|\langle U\alpha|U\beta\rangle| = |\langle\alpha|\beta\rangle|$$

Wigner's theorem implies that U as an **operator in** $Aut(\mathcal{H})$[a] must be either **unitary or antiunitary**:

$$1 \to Unitary(\mathcal{H}) \to Aut(\mathcal{H}) \to Z_2 \to 1$$

- Unitary:linear and $\langle U\alpha|U\beta\rangle = \langle\alpha|\beta\rangle$
- Antiunitary: antilinar and $\langle U\alpha|U\beta\rangle = \langle\alpha|\beta\rangle^*$
- both cases satisfying:$UU^\dagger = 1, U^\dagger = U^{-1}$

[a] act on the linear space $\mathcal{H}$,this lead to group representation

Remark 2
- *Trivial transformation U=I is unitary and linear,**continuity** implies the transformations generated by **Lie algbra**: $U = I + i\epsilon \cdot T$ are unitary thus the generators of Lie groups are represented as **Hermitian operators** when acting on $\mathcal{H}$.*
- *The generators of symmetry Lie algebras will be conserved,thus may be included in CSCO.*
- *In relativistic QFT,the only independent antiunitary and antilinear operator will be **time-reversal**.Which can be decomposed as $T = UK$,where K refers to **complex conjugation**,U is unitary.*

2.2.2 Projective representation and enlarging

In quantum theories,the group act on $\mathbf{P}\mathcal{H}$,the pull back to the action on $\mathcal{H}$,i.e,in terms of (anti)unitary operators lead to **U(1) extension and projevtive representations**:

$$1 \to U(1) \to \tilde{G} \to G \subset Aut(\mathbb{P}\mathcal{H}) \to 1$$

In QM,we talk about **projective unitary (irreducible) representations**.Though conceptionally unavoidable,the complexity of PUIR can be dealed with certain strategies:

Dealing with projective representation

- For connected Lie groups the complexity of projevtive representation is avoided by enlarging the Lie algebra and consider ordinary representations of this Lie algebra.We then ended up with only ordinary representations of the **enlarged Lie group**,without physical difference.
- For discrete symmetries,we can't avoid the possibility of projective representations.In relativistic QFT,we almost need to consider the special case of **time-reversal**.

2.3 Poincare group and Lorentz group

The reletivistic QFT take the spacetime symmetry to be the Poincare group,which arise from the idea of special reletivity.The quantum version of Poincare group is it's projective representation over $\mathcal{H}$.

[4] also known as group action on $\mathbb{P}\mathcal{H}$

2.3.1 Defining/4-vector representation

The principle of special relativity states the equivalence between **inertial** frames.The core axiom is:

Principle of special reletivity

The flat spacetime take the structure of **Minkowski space**:$M_{1,3}$ = ($\mathbb{R}^4$ equipped with a Lorentz type metric):

$$\eta^{\mu\nu} = \eta_{\mu\nu} = diag(-,+,+,+), \eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu_\nu = diag(+,+,+,+) = \eta_{\nu\beta}\eta^{\beta\mu}, (\eta^{-1})_{\mu\nu} \equiv \eta_{\mu\nu}$$

Poincare group consists of transformations of spacetime-coordinates/intertial-frames leaving the **metric** invariant:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = ds^{'2} = \eta_{\mu\nu}dx^{'\mu}dx^{'\nu} \Leftrightarrow \eta_{\mu\nu}\frac{\partial x^{'\mu}}{\partial x^\alpha}\frac{\partial x^{'\nu}}{\partial x^\beta} = \eta_{\alpha\beta}$$

This is solved with **linear** transformations[a]:

$$x^{'\mu} = \Lambda^\mu{}_\nu x^\nu + a^\mu, \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\mu{}_\beta = \eta_{\alpha\beta} \Leftrightarrow \Lambda^T\eta\Lambda = \eta$$

This gives the **defining representation**(representation over $M_{1,3}$) of the Poincare group ISO(1,3),with $\{\Lambda^\mu{}_\nu\}$ form the subgroup called the Lorentz group $O(1,3)$,the inverse of Lorentz transformation is defined as:[b] $(\Lambda^{-1})^\alpha{}_\beta = \Lambda_\beta{}^\alpha \equiv \eta_{\beta\mu}\eta^{\alpha\nu}\Lambda^\mu{}_\nu$.We can infer the group structure from this representation and take it as the **abstract definition**(independent of group action):

$$x^{''\mu} = (\Lambda^{'\mu}{}_\alpha\Lambda^\alpha{}_\nu)x^\nu + (\Lambda^{'\mu}_\alpha a^\alpha + a^{'\mu}) \Rightarrow T(\Lambda', a')T(\Lambda, a) = T(\Lambda'\Lambda, \Lambda' \cdot a + a')$$

The inverse of Poincare transformation is then: $T^{-1}(\Lambda, a) = T(\Lambda^{-1}, -\Lambda^{-1} \cdot a)$.These define Poincare group as a **semi-direct product:**

$$ISO(1,3) = T_4 \rtimes O(1,3), T_4 = \{Translation\} \cong \mathbb{R}^4$$

Using the identities:

$$det(\Lambda)^2 = 1, (\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0\Lambda^i{}_0 \Rightarrow |\Lambda^0{}_0| \geq 1$$

The Lorentz group can be decomposed into 4 pieces,related[c] by **Space-inversion(parity),Time-reversal(T-parity)**,in defining representation:

$$\mathcal{P}^\mu{}_\nu = diag(+,-,-,-), \mathcal{T}^\mu_\nu = diag(-,+,+,+)$$

The one connected to 1 is called the **proper orhochronous Lorentz group:**$SO^+_\uparrow(1,3)$ **abbrevited as** $SO(1,3)$ the corresponding part of ISO(1,3) is $T_4 \rtimes SO(1,3)$

[a]Martix form will only be used for tensors with no more than 2-indices for notational convenience
[b]Λ is not a tensor
[c]Parity reverse sign of determinant,T-parity reverse sign of 00-component

2.3.2 Algebraic and topological properties

In quantum theories,both $T_4 \rtimes SO(1,3)$ and $SO(1,3)$ must take form of projective unitary representations.The algebraic and topological properties of these groups are:

Algebraic and topological properties

- For operational convenience,we will work with exlarged groups,where the central charges are removed and extended to universal coverings.It can be shown that though the Lie algebra of Poincare group is not semi-simple[a] **the central charge in Poincare group is indeed trivial.**The universal covering of $SO(1,3)$ is $SL(2,\mathbb{C})$ with $\pi_1(SO(1,3)) = \mathbb{Z}_2, SL(2,\mathbb{C})/\mathbb{Z}_2 = SO(1,3)$.Thus we will work with $T_4 \rtimes SL(2,\mathbb{C})$ and consider their ordinary representations[b].
- In reletivistic QFT,the projective representation always have relation to **spins of fermions**.The result is related can be traced back to the topologcial properties:$\pi_1(SO(1,3)) = \mathbb{Z}_2 = \pi_1(SO(3))$

thus the identity hold:

$$[U(\Lambda)U(\Lambda')U^{-1}(\Lambda\Lambda')]^2 = 1 \Rightarrow \omega_2 = \pm 1, U(\Lambda, a)U(\Lambda', a') = \pm U(\Lambda\Lambda', \Lambda \cdot a' + a)$$

Minus sign projective representation arise only when the state contains half-integer spins and the path $1 - \Lambda - \Lambda\Lambda' - 1$ not contractible.

These two signs also correspond to **two 1D irreducible representations of $\pi_1 = \mathbb{Z}_2$.**

- **Time-reversal** is projevtively represented when the state contain half-integer spin:$T \cdot T = T^2 = -1$,this can understood from the fact that time-reversal reverse spin,T^2 which amounts to a 2π rotation.
- This topological fact leads to the topological understanding of the **spin-statistic connection**.Exchange particles also amount to a 2π rotation.
- $SO(1,3)$,actually $SL(2,\mathbb{C})$ is not compact[c],thus both group have no finite-dim unitary representations.We have to construct **infinite-dim** unitary representations.**Quantum theories of classical fields provide certain set of infinite-dim unitary representations.**

[a]Lorentz group is indeed semi-simple
[b]conceptionally we still talk about projective representation,to emphasize the quantum nature.
[c]$SO(3) \cong S^3/\mathbb{Z}_2, SL(2,\mathbb{C}) \cong \mathbb{R}^3 \times S^3$

Remark 3
- *Proofs of the above assertions can be found in [WeinbergI,2.7]*
- *One direct extension of Poincare symmetry is the **Conformal symmetry** which also leave the light speed unchanged.The 2d conformal invariance have vast application in string theory and statistical mechanics.The conformal field theory will show up in **fixed point of renormalization group where scale invariance is manifested**The conformal group will have **nontrivial central charges***

2.3.3 Lie algebra of enlarged Poincare group

Consider infinitesimal transformations[5]:$\Lambda^\mu{}_\nu = \omega^\mu{}_\nu, a^\mu = \epsilon^\mu$,Lorentz invariance then lead to:

$$\omega_{\mu\nu} \equiv \eta_{\mu\alpha}\omega^\alpha{}_\nu = -\omega_{\nu\mu}$$

Thus there's totally 4+6=10 independent generators.Expanding near identity[6]:

$$T(1+\omega, \epsilon) = exp[i\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu], J^{\mu\nu} = -J^{\nu\mu}$$

Remark 4
- ***Convention**:since the definition of $\omega_{\mu\nu}$ and $a_\mu$involve metric,a different choice of metric will have sign difference for coefficients for J and P.*
- *Different representation of Lie algebra will give different representation of T.Since the group is non-compact,**finite-dim representation will not be unitary.***
- *Quantum mechanics need unitary representations,where T is represented as U(T) and J,P are **Hermitian operators rather than simply matrices.***
- *The sign covention of coefficients of J,P can be determined according to **passive view convention**:*

$$\hat{U}(g)\hat{O}(x)\hat{U}^{-1}(g) = (R^{-1}(\hat{g}) \cdot O)(V(g) \cdot x)$$

*R correspond to **finite-dim** rep of O as a classical tensor,V is the 4-vector representation acting on $M_{1,3}$.LHS act as operators.The sign is determined to restore the conventional choice of*

$$e^{i\vec{\theta}\vec{J}}Oe^{-i\vec{\theta}\vec{J}}, e^{iHt}Oe^{-iHt}$$

[5]Without mentioning,we consider the connected part and discrete transformations seperately,and **Lie group** always refers to the connected part
[6]not compact

on LHS.[7]

The Lie algebra is obtained from commutation properties of infinitesimal transformations:

$$e^{i\theta A}Be^{-i\theta A} = \sum_n \frac{(i\theta)^n}{n!}A^{[n]}[B] \sim_{O(\theta)} I + i\theta[A,B], e^{i\theta A}e^{i\beta B}e^{-i\theta A} \sim_{O(\beta)} I + i\beta e^{i\theta A}Be^{-i\theta A}$$

LHS can be gotten from group structure/multiplication.For ISO(1,3) this amounts to:

Lie algebra of connected part of enlarged ISO(1,3): $T_4 \rtimes SL(2,\mathbb{C})$

The Lie algebra can be gotten from **any representation**,the simplest way to obtain the Lie algebra is adopt the **field/differential-operator representation** or the **defining/4-vector representation** where the **explicit form** of generators are known.We can also use the **quantum version**,which is more meaningful,since the LHS is just the **action on operators**:

$$U(\Lambda,a)U(1+\omega,\epsilon)U^{-1}(\Lambda,a) =_{group} U(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)$$

Up to $O(\omega), O(\epsilon)$,we obtain the **Poincare transformation properties of generators as Hermitian operators**[a]:

$$\begin{aligned} U(\Lambda,a)J^{\alpha\beta}U^{-1}(\Lambda,a) &= (\Lambda^{-1})^\alpha{}_\mu(\Lambda^{-1})^\beta{}_\nu(J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu) \\ U(\Lambda,a)P^\alpha U^{-1}(\Lambda,a) &= (\Lambda^{-1})^\alpha{}_\mu P^\mu \end{aligned} \tag{2.1}$$

- With a=0,this implies that $J^{\mu\nu}, P^\mu$ are Lorentz tensors: **the indices are indeed Lorentz indices for all representation even when J,P are not hermitian matrices.**
- With $\Lambda = 1$,this implies that the definition of P is independent of the choice of origin(affine),while the definition of J relies on the choice of origin.This is more clear in **field/differential-operator representation**:$J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$

Choose Λ, a infinitesimal and expand up to $O(\theta)$,we obtain the Lie algebra[b]:

$$\begin{aligned} [J^{\mu\nu}, J^{\alpha\beta}] &= i[\eta^{\mu\alpha}J^{\nu\beta} - (\mu \leftrightarrow \nu)] - [\alpha \leftrightarrow \beta] \\ [P^\mu, J^{\alpha\beta}] &= i[\eta^{\mu\beta}J^\alpha - (\alpha \leftrightarrow \beta)] \\ [P^\mu, P^\nu] &= 0 \end{aligned} \tag{2.2}$$

- The Js expand the subalgebra of $SL(2,\mathbb{C})$ and is equivalent to the Lie algebra(complexified) of Lorentz group:$sl(2,\mathbb{C}) = so(1,3)_\mathbb{C} = su(2) \oplus su(2), su(2) = so(3) = sl(2,\mathbb{R}) = so(1,1)$
- The Ps expand the subalgebra of T_4 which is an invariant Abelian subalgebra.The whole Lie algebra is not semi-simple
- It's conventional to choose P^0=H,thus 2nd,3rd implies there are **only 7 conserved operators:**$H, \vec{P} = \{P^i\}, \vec{J} = \{\frac{1}{2}\epsilon_{ijk}J^{jk}\} = \{J^{23}, J^{31}, J^{12}\}$,the remaining 3 generators are not conserved[c]: $\vec{K} = \{J^{i0} = J_{0i}\}$.
- The conserved operators may be included into CSCO to label states,thus we never use eigenvalues of K to label states.

With these definition, the Lie algebra factors into:

$$\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk}J_k,\ [J_i, K_j] = i\epsilon_{ijk}K_k,\ [K_i, K_J] = -i\epsilon_{ijk}J_k \\ &[J_i, P_j] = i\epsilon_{ijk}P_k,\ [K_i, P_j] = -iH\delta_{ij},\ [K_i, H] = -iP_i \\ &[J_i, H] = [P_i, H] = [H, H] = 0 \end{aligned} \tag{2.3}$$

- The 1st line gives $sl(2,\mathbb{C}) = so(1,3)_\mathbb{C}$,the minus sign in the last one is crucial:

[7]Some text define $\tilde{U}' = U^{-1}$.Here active view act as: $U^{-1}|\psi\rangle$,comsistent with the transformation between H-picture and S-picture.The signs really doesn't matter,we can change the sign of the parameters but the meaning of the transformation may be different.Thus the signs should be defined to furnish the meaning of the action itself.**To make things simple,usually choose the sign of parameters to be positive and determine the sign of generators by result of the action.**

- The group is not compact.
- The Lie algebra is a analytic continuation of $so(4)$,which indicate the notation $so(1,3)_{\mathbb{C}}$

Since Lie algebra comes from infinitesimal loops in the group manifold.The commutators implies that: boost around x1 then Bosst around x2 then back will lead to some kinds of rotation,this lead to **Thomas Precession**.

- Defining $\vec{L} = \frac{1}{2}(\vec{J} - i\vec{K}), \vec{R} = \frac{1}{2}(\vec{J} + i\vec{K})$,the $sl(2,\mathbb{C})$ factors into two copies of $su(2)$:

$$[L_i, L_j] = i\epsilon_{ijk}L_k, [R_i, R_j] = i\epsilon_{ijk}R_k, [L_i, R_j] = 0 \tag{2.4}$$

Thus the representation of $SL(2,\mathbb{C})$ is determined by representations of $SU(2)$ completely,these will give rise to various **Lorentz tensors to be promoted to relativistic classical fields.**

- The last line implies the 7 conserved operators.The 2nd line implies $\vec{K}$ is not conserved.
- The commutators with J implies: $\vec{J}, \vec{P}, \vec{K}$ are **SO(3)-vectors**,this is related to the facts:
  - $SO(3) \subset SO(1,3)$,and is the **little group of massive particles.**
  - upton compactification,the Lie algebra will reduce to $su(2) = so(3)$,**the whole theory of massive particles is constructed using** $SU(2)$.
- The J_i's not commutative,only J^2, J_3 are independet D.O.F,will be included in CSCO.
- There are subgroups[d]:
  - Spacetime-translation: $U(1,a) = exp[-ia \cdot P] = exp[iHa^0 - i\vec{a} \cdot \vec{P}]$.We act $U^{-1} = e^{-iHa^0}e^{i\vec{a}\cdot\vec{P}}$ on $|\psi\rangle$ to apply picture-shift($H \to S$)/time-evolution and space-translation($\vec{x}-basis$):$\psi(\vec{x}) \to \psi(\vec{x} + \vec{a})$
  - $SO(3)/SU(2)$-ratation: $U(R_{\vec{\theta}}, 0) = exp[i\vec{\theta} \cdot \vec{J}]$.We act **passively**[e] on spacetime point:
    - act $U = exp[i\vec{\theta} \cdot \vec{J}]$ on spacetime coordinate:$\vec{x}$
    - act $U^{-1} = exp[-i\vec{\theta} \cdot \vec{J}]$ on the tensor O.
  - Lorentz-boost: $U = exp[i\vec{\beta} \cdot \vec{K}]$,similarly:
    - act $U = exp[i\vec{\beta} \cdot \vec{K}]$ on spacetime-coordinate of O
    - act $U^{-1} = exp[-i\vec{\beta} \cdot \vec{K}]$ on the tensor O.

[a] $(\Lambda^{-1})^{\alpha}{}_{\mu} = \Lambda_{\mu}{}^{\alpha}$
[b] As mentioned before,the central charge is only trivial,this is indeed the Lie algebra of ISO(1,3)
[c] Some text choose K=-K,which doesn't change Lie algebra but will change some sign in definitions involving K
[d] Action on different entities implicitly take different representation,here we all denote as U for simplicity,though not all representation is unitary
[e] in some texts on quantum mechanic,active view is adopted

Remark 5 • *Note the action on O and its spacetime-coordinates are different,when we act on x with $T(\Lambda, a)$,we act on O with $T^{-1} = T(\Lambda^{-1}, -\Lambda \cdot a)$.When we study how O as a tensor transform under $T(\Lambda, a)$,we should act on it with $R(T(\Lambda, a)) = exp[i\frac{1}{2}\omega_{\mu\nu}J_R^{\mu\nu} - ia_\mu P_R^\mu] = exp[i\vec{\theta}\cdot\vec{J}_R + i\vec{\beta}\cdot\vec{K}_R + ia^0 H_R - i\vec{a}\cdot\vec{P}_R]$.*

- *When O is just a Lorentz tensor,only $exp[i\frac{1}{2}\omega_{\mu\nu}J_R^{\mu\nu}] = exp[i\vec{\theta} \cdot \vec{J}_R + i\vec{\beta} \cdot \vec{K}_R]$ will be needed,where J_R, K_R are finite-matrices(not unitary-rep);When O is a field operator,the H_R, P_R are field-reps(infinite-dim),the J_R, K_R will be* ***combine both finite-rep and field-rep.***

2.3.4 Lie algebra of Galiliean group

We can obtain the Galiliean group as well as its Lie algebra in the non-relativistic limit[8]:

$$J \sim 1, P \sim mv; H = M + W, M \sim m, W \sim mv^2; K \sim 1/v$$

[8] Known as **Inonu-Wigner contraction**

Galiliean algebra

$$
\begin{aligned}
[J_i, J_j] = i\epsilon_{ijk}J_k,\ [J_i, K_j] = i\epsilon_{ijk}K_k,\ [K_i, K_j] = 0 \\
[J_i, P_j] = i\epsilon_{ijk}P_k,\ [K_i, P_j] = +iM\delta_{ij},\ [K_i, W] = +iP_i \\
[J_i, M] = [J_i, W] = [P_i, M] = [P_i, W] = [K_i, M] = [W, M] = 0
\end{aligned} \tag{2.5}
$$

The Galiliean algebra[a] have nontrivial central charge,known as mass,indicated by:$[K_i, P_j] = iM\delta_{ij}$.Since classically,we have the well-known representation of Galiliean transformation:

- Boost:$exp[i\vec{K}\cdot\vec{v}] : \vec{x} \to \vec{x} + \vec{v}t$,the multiplication is $exp[i\vec{K}\cdot\vec{v}]exp[-i\vec{P}\cdot\vec{a}] : \vec{x} \to \vec{x} + \vec{v}t + \vec{a} \sim exp[i(\vec{K}\cdot\vec{v} - \vec{P}\cdot\vec{a})]$ This implies,the Galiliean algebra should have:

$$[K_i, P_j] = 0$$

- But quantum version gives:

$$exp[i\vec{K}\cdot\vec{v}]exp[-i\vec{P}\cdot\vec{a}] = exp[i\frac{M}{2}\vec{v}\cdot\vec{a}]exp[i(\vec{K}\cdot\vec{v} - \vec{P}\cdot\vec{a})]$$

  The Galiliean group is **intrinsically projectively represented**,with central charge known as Mass.Again,we can enlarge the group by **including a 'mass'-operator** to deal with only ordinary representations.

[a]In some text,$K \to -K$,commutators involve signle K will have a sign difference

Remark 6 • *The fact mass is the nontrivial central charge is closely related to non-relativistic quantume mechanics:*

$$[K_i, \frac{P^2}{2M}] = \frac{1}{2M}(2iM\delta_{ij})P_j = iP_i = [K_i, H] \Rightarrow H = \frac{P^2}{2M} + c$$

When we take non-relativistic limit of some relativistic Lagrangian and equations,the mass as a central charge will shown up to modify the H expansion in $O(\frac{v}{c})$ to the right non-relativistic counterpart.

2.4 Single-particle state

2.4.1 Classification of single-particle state

The irreducible representation of the Lie algebra can be **labelled** by its **Casimir invariants:operators commute with all others**.For enlarged Poincare algebra[9]:

Single-particle states provide subset of UIRs of enlarged Poincare algebra

Since the translation algebra form an invariant Abelian subalgebra,in quantum mechanics they can be simultaneously diagonalized,thus it's natural to include them into the CSCO and use their eigenvalues to label physical states:

$$P^\mu|p\rangle = p^\mu|p\rangle$$

The p^μ is the **total energy and momentum** of the **physical** state.This state in general describe **multiparticle state with or without interactions**,thus there's certain degeneracies,labeled compactly by σ:

- For multiparticle states,σ contain continuous indices like relative momentum,etc
- For **single particle** state,σ is defined to contain **only discrete** indices.[a]
- For the rest,we will focus on only single-particle states,the result is that they form PUIRs of ISO(1,3),but since Poincare symmetry is exact,**any physical state must furnish (projective) unitary representations**.Multiparticle state may or may not be reducible,thus single-particle state only provide as subset of PUIRs,**not all**.

[9]irreducible projective unitary representations of ISO(1,3)

For physical particle,they are **on-shell**:

$$p^2 = p_\mu p^\mu = -m^2 = -E^2 + |\vec{p}|^2$$

Single-particle naturally provide the 1st Casimir invariant[b]:

$$P^\mu P_\mu = P^2, P^2|p\rangle = -m^2|p\rangle$$

This implies the representation furnished by single-particle state will be classified by **Lorentz-invariant label: mass**.To organize the degeneracies,other Casimir invariants are needed.
The other Casimir invariant is provided by the **Pauli-Lubanski 4-vector**:

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\alpha\beta\rho}J^{\alpha\beta}P^\rho$$

Cumbersome calculation implies $W_\mu W^\mu$ commute with all other operators,thus is the 2n Casimir invariant.Since $W_\mu W^\mu$ commute with J,it's **Lorentz invariant**,thus we can calculate its value and obtain the physical meaning of the label using **standard 4-momentum**:

- **Massive particle:rest frame.**:

  $$p^\mu_r = (m,0,0,0) \Rightarrow W_0 = 0, W_i = \frac{1}{2}\epsilon_{ijk}J^{jk} = mJ_i \Rightarrow \frac{1}{m^2}W^2|p_r,\sigma\rangle = J^2|p_r,\sigma\rangle = j(j+1)|p_r,(j,j_3)\rangle$$

  Thus the 2nd label of massive particle has the meaning of **spin**.**The commutator in rest frame reduce to SO(3) subalgebra:**

  $$J_i = \frac{W_i}{m}, W_i[J_i,J_j] = i\epsilon_{ijk}J_k \tag{2.6}$$

- **Massless particle:**

  $$p^\mu_* = (p,0,0,p) \Rightarrow W_0 = pJ_3 = W_3, W_1 = +p(K_2+J_1) = pA, W_2 = -p(K_1-J_2) = -pB$$

  These operators form the **ISO(2)=E(2) algebra**:

  $$\begin{aligned} J_3 = \frac{W_3}{p}, A = \frac{W_1}{p}, B = \frac{-W_2}{p} \\ [J_3,A] = iB, [J_3,B] = -iA, [A,B] = 0 \end{aligned} \tag{2.7}$$

  J_3 generate $O(2) = U(1)$;A,B generate T_2.$ISO(2) = T_2 \rtimes O(2)$.The result is that to get **discrete** labels,**these translations are not observable**:

  $$A|p_*,\sigma\rangle = B|p_*,\sigma\rangle = 0 \Rightarrow \frac{1}{|p|^2}W_\mu W^\mu|p_*,\sigma\rangle = 0$$

  In this case,we may choose:

  $$W^\mu|p_*,j_3\rangle = j_3P^\mu|p_*,j_3\rangle$$

  ,and the j_3 for massless particles is known as **helicity**:

  $$j_3 = h = \frac{\vec{S}\cdot\vec{p}}{|p|}$$

  The helicity must be either integer or half-integer due to **topological reasons**,thus we may still choose the 2nd Lorentz invariant label as **spin**.

[a]discretized D.O.F is the meaning of quanta
[b]this will also hold for any state,since P^2 is a Lorentz scalar,it will commute with J,but the meaning of P^2 is not that clear for multiparticle state as for single-particle state

In summary:**Single particle state gives the subset of PUIRs classified by two Lorentz invariant labels,which physically is (mass,spin).**

2.4.2 Induced representation

The use of W-vector implies the construction of specific single-particle state as PUIR.The choice of the **standard 4-momentum** lead to the definition of **little groups**,as shown by W-vector algebra,the little group is $SO(3)/ISO(2)$ for massive/massless particles.These two groups are **compact** thus have finite-dim PUIRs.As shown above,the finite-dim representation is indicated by label (spin,spin-3/helicity).The PUIR of ISO(1,3) can then be induced by them:

Induced representation

The action[a] of translations:

$$U(1,a)|p,\sigma\rangle = e^{-ia\cdot p}|p,\sigma\rangle$$

Since $U(\Lambda)^{-1}P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu$,we have $U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)|\Lambda p,\sigma\rangle$ .The index convention is fixed by considering $|p,\sigma\rangle$ as basis.
By basis transformation,C-matrix[b] can be made block-diagonal,then it's natural to identify the irreducible PURs as the single-particle states.These transformations will be constructed using the method of **induced representation**.
choose the standard k,with the standard transformation given as:

$$p^\mu = L^\mu{}_\nu k^\nu$$

Define[c] single-particle state:

$$|p,\sigma\rangle \equiv N(p)U(L(p))|k,\sigma\rangle, U(\Lambda)|p,\sigma\rangle = N(p)U(L(\Lambda p))U(L^{-1}(\Lambda p)\Lambda L(p))|k,\sigma\rangle$$

The little group transformation[d] appear,known as **Wigner rotation**:

$$W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p)$$

For any little group transformation:

$$U(W)|k,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W)|k,\sigma'\rangle$$

these standard-momentum basis furnish unitary representations of the little group,then the definition of particles **reduce to PUIRs of little groups**.
The PUIRs now takes form of induced representation[e]:

$$U(\Lambda)|p,\sigma\rangle = N(p)\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))U(L(\Lambda p))|k,\sigma'\rangle = \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|\Lambda p,\sigma'\rangle$$

Since p-orbit contain infinite number of points,these[f] **induced PUIRs is indeed infinite-dim**.
Note that since Lorentz transformation leaves p^2 invariant,as indicated above:
Such induced PUIRs are classified by two indices:spin for little group PUIR,mass for p-oribit

[a]For active view,we should act with U^{-1},but we adopt the passive view,here we change the operator and state(with U) both to keep the matrix element unchanged,the states considered here are **basis/eigenstates**,rather than general state $|\psi\rangle$,the situation is similar to H-picture: eigenstates transform inversely as the general states
[b]σ is discrete by definition
[c]Different definition differs up to a unitary transformation,we then can always choose such a 'diagonal'-definition
[d]leaving standard 4-momentum invariant
[e]Lorentz transform standard-momentum to extend representation of little groups to the whole group
[f]may not be all possible PUIR

Normalization

As basis,we may set them orthonormal:$\langle k,\sigma|k',\sigma'\rangle = \delta_{\sigma\sigma'}\delta^3(\vec{k}-\vec{k}')$. This normalization makes representation-matrix D of little group **unitary**.Generally:

$$\langle p',\sigma'|p,\sigma\rangle = N(p)N^*(p)D(W(L^{-1}(p),p'))^*_{\sigma\sigma'}\delta^3(k-k'), k' \equiv L^{-1}(p)p'$$

Since $k = L^{-1}(p)p$,$\delta^3(k-k') \propto \delta^3(p-p'), W(L^{-1}(p),p) = 1$,thus:

$$\langle p',\sigma'|p,\sigma\rangle = |N(p)|^2\delta_{\sigma\sigma'}\delta^3(k-k')$$

From $\omega_p\delta^3(p-p') = \omega_k\delta^3(k-k')$,we have:

$$\langle p',\sigma'|p,\sigma\rangle = |N(p)|^2(\frac{\omega_p}{\omega_k})\delta_{\sigma\sigma'}\delta^3(p-p')$$

- Non-relativistic normalization: $N(p) = \sqrt{\omega_k/\omega_p}$,then:

  $$\langle p',\sigma'|p,\sigma\rangle = \delta_{\sigma\sigma'}\delta^3(p-p')$$

  which is also useful in CMP.Where the completeness relation have simple form:$\int d^3p|p\rangle\langle p| = 1$

- In some texts,the reletivistic normalization is also used[10]:

  $$\langle p',\sigma'|p,\sigma\rangle = (2\pi)^3(2\omega_p)\delta_{\sigma\sigma'}\delta^3(p-p')$$

Remark 7
- *Different convention of normalization and Fourier transformation only differ by factors of $\sqrt{(2\pi)^3}$, $\sqrt{2\omega_p}$. They only matter in the expansion of fields into modes.The relativistic normalization have better looking expansion and Fourier transformation,but the completeness relation:$\int \tilde{dp}|p\rangle\langle p| = 1, \tilde{dp} = \frac{d^3p}{(2\pi)^3 2\omega}$ is not convention for formal derivation.There's also different normalization conventions for 2nd quantization operators,the same arguments hold.*

- *Non-relativistic normalizations will be convention for formal derivation,while relativistic normalizations may be adopted in field expansions and calculations for convenience,using invariant-measure:$\tilde{dp} = \frac{d^3p}{(2\pi)^3 2\omega}$.*

2.4.3 Little groups

We are interested in 3 cases:

- Massive:$p^2 = -M^2 < 0, p^0 > 0, k^\mu = (M,0,0,0) : SO(3)$
- Massless:$p^2 = 0, p^0 > 0, k^\mu = (1,0,0,1) : ISO(2)$
- Vacuum/Ground:$p^\mu = 0, k^\mu = 0 : SO(1,3)$

These results are rather obvious,which is also indicated by the W-vector algebra.The rest is contructing **PUIRs of the little group**:which is obtained from UIRs of the enlarged Lie algebra.

Massive

The Lie algebra $so(3)$,Ladder operators and Casimir invariants are:

$$[J_i, J_j] = i\epsilon_{ijk}J_k; J_\pm = (J_1 \pm iJ_2); J^2$$

The PUIRs are labelled by $J^{2(j)} = j(j+1)$ with basis chosen as eigenvectors of J_3:

$$(J_3^{(j)})_{\sigma'\sigma} = \sigma\delta_{\sigma\sigma'}, (J_\pm^{(j)})_{\sigma'\sigma} = \delta_{\sigma',\sigma\pm1}\sqrt{(j\mp\sigma)(j\pm\sigma+1)}, \sigma = \{j, j-1, ..., -j\}$$

The group elements can be obtained from infinitesimal ones[11]:

$$D^{(j)}_{\sigma'\sigma}(1+\Theta) = \delta_{\sigma\sigma'} + \frac{1}{2}i\Theta_{ik}(J^{(j)}_{ik})_{\sigma\sigma'}$$

[10] also with a different choice of 2π convention of Fourier transformation

[11] $SO(3)$ compact

Then:

$$U(\Lambda)|p,\sigma^{(j)}\rangle = \sqrt{\omega_{\Lambda p}/\omega_p}\sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))|\Lambda p,\sigma'^{(j)}\rangle \tag{2.8}$$

Wigner rotation is given by:$W(\Lambda,p)=L^{-1}(\Lambda p)\Lambda L(p)$,the standard boost here is:

$$L^i{}_k=\delta_{ik}+(\gamma-1)\hat{p}_i\hat{p}_k, L^i{}_0=L^0{}_i=\hat{p}_i\sqrt{\gamma^2-1}, L^0{}_0=\gamma;\ \gamma=\omega_p/M=1/\sqrt{1-v^2}$$

It's useful to denote:

$$mL^\mu{}_0(p)=p^\mu \tag{2.9}$$

The more physical form of standard boost is:

$$L(p)=R(\hat{p})B_3(|p|)R^{-1}(\hat{p}) \tag{2.10}$$

With rotation R takes the 3-axis to $\hat{p}$ and B_3 is the 3-axis boost.Then for $\Lambda=\mathcal{R}$:

$$W(\mathcal{R},p)=R(\mathcal{R}\hat{p})B^{-1}(|p|)[R^{-1}(\mathcal{R}\hat{p})\mathcal{R}R(\hat{p})]B(|p|)R^{-1}(\hat{p}), R^{-1}(\mathcal{R}\hat{p})\mathcal{R}R(\hat{p})=R_3(\theta),[R_3,B_3]=0$$

$$W(\mathcal{R},p)=R(\mathcal{R}\hat{p})R(\theta)R^{-1}(\hat{p})\equiv\mathcal{R}$$

This means the Wigner rotation for ratation is just the rotation itself,this may seem trivial,since rotations form the little group $SO(3)$.The nontrivial implications are:

- Massive particles have same transformation under rotations as in non-relativistic QM
- The $SU(2)$ apparatus: **Clebsch-Gordon coefficients** will have same meaning as in QM.As we will see, **the little group $SO(3)$ will play crucial role in determine the polarizations in field expansion for massive particles.**

Massless

As indicated by the W-vector algebra,the little group for massless particles is $ISO(2)$,we can also work out the little group by finding transformations leaving $k=p_*$ specificly.

Define $t^\mu=(1,0,0,0)\propto p^\mu_r$,since $(Wt)^2=t^2=-1, Wt\cdot k\equiv t\cdot k=-1$,we have $(Wt)^\mu=(1+c,a,b,c), c=(a^2+b^2)/2$.We then have:$Wt=St, Sk=Wk=k$,for infinitesimal transformations the S-matrix is[12]:

$$S=I-i\alpha B-i\beta A$$

The observation is:

$$Wt=St\Rightarrow S^{-1}Wt=t\Rightarrow S^{-1}W=exp[i\theta J_3]$$

Thus the little group transformation is just $ISO(2)=T_2\rtimes O(2)$ indicated by its Lie algebra expanded by $\{A,B,J_3\}$,the translations form the invariant Abelian subalgebra,thus ISO(2) is **not semi-simple**,while $SO(3)$ is semi-simple.

This means:

$$A|k,a,b\rangle=a|k,a,b\rangle, B|k,a,b\rangle=b|k,a,b\rangle$$

since (A,B) is a $O(2)$-vector,it will be another θ indicating degeneracies in σ:

$$\psi k,a,b,\theta=U^{-1}(R(\theta))\psi k,a,b$$

having same k but rotated (a,b),this contracted our definition of single-particles: σ **must be discrete.**

Physical massless particle is then **defined** by:

$$A|k,\sigma\rangle=B|k,\sigma\rangle=0$$

σ is obtained from J_3:

$$J_3|k,\sigma\rangle=\sigma|k,\sigma\rangle, J_3=\hat{h}=\vec{S}\cdot\vec{p}/|p|$$

The transformation of massless particle is then:

$$U(\Lambda)|p,\sigma\rangle=\sqrt{\omega_{\Lambda p}/\omega_p}exp(i\sigma\theta)|\Lambda p,\sigma\rangle \tag{2.11}$$

The θ-angle is determined by solving: $W(\Lambda,p)=L^{-1}(\Lambda p)\Lambda L(p)=S(\alpha(\Lambda,p),\beta(\Lambda,p))R(\theta(\Lambda,p))$,the standard Lorentz transformation is: $L(p)=R(\hat{p})B_3(|p|)$.

[12]A,B is defined above,here we use the **defining-representation** for them

Remark 8
- *Helicity is **Lorentz invariant**[13],it will be shown that helicity change sign under P-parity,and is invariant under T-parity,thus for process respect P-parity,particles are defined by $ISO(1,3)$ including P,thus $\pm h$ are considered as single-particle state.For process breaking P-parity,$\pm h$ are considered as different particles.*
- *The helicity is **integer or half-integer** due to topological properties of $SO(3)/SO(1,3)$:4π-rotation around momentum is homotopic to 1: $exp[4\pi i\sigma] = 1$.Thus we can also talk about spin,as how helicity-operator is defined.*
- *Physical massless particles doesn't change under gauge-translations,but in reletivistic QFT,redundancy of unphysical D.O.F is introduced,which change under these gauge-translation.This is the origin of **local gauge invariance**.*

2.5 P-inversion and T-reversal

2.5.1 Lie algbera with P,T

The group multiplication invloving P/T-parties are[14]:

$$\begin{aligned} PU(\Lambda, a)P^{-1} &= U(\mathcal{P}\Lambda\mathcal{P}^{-1}, \mathcal{P}a) \\ TU(\Lambda, a)T^{-1} &= U(\mathcal{T}\Lambda\mathcal{T}^{-1}, \mathcal{T}a) \end{aligned} \tag{2.12}$$

Expand to $O(\omega), O(\epsilon)$,we obtain how generators transform[15]:

Lie algebra involving P,T

$$\begin{aligned} P(iJ^{\mu\nu})P^{-1} &= i(\mathcal{P}^{-1})^{\mu}{}_{\alpha}(\mathcal{P}^{-1})^{\nu}{}_{\beta}J^{\alpha\beta} \\ P(iP^{\mu})P^{-1} &= i(\mathcal{P}^{-1})^{\mu}{}_{\alpha}P^{\alpha} \\ T(iJ^{\mu\nu})T^{-1} &= i(\mathcal{T}^{-1})^{\mu}{}_{\alpha}(\mathcal{T}^{-1})^{\nu}{}_{\beta}J^{\alpha\beta} \\ T(iP^{\mu})T^{-1} &= i(\mathcal{T}^{-1})^{\mu}{}_{\alpha}P^{\alpha} \end{aligned} \tag{2.13}$$

The essential to determine (anti)unitarity is by observing that $\mathcal{P}^{0}{}_{0} = 1, \mathcal{T}^{0}{}_{0} = -1$.Thus:

$$PiHP^{-1} = iH, TiHT^{-1} = -iH$$

To avoid negative-energy state(below ground),and to get the symmetry condition:$[P/T, H] = 0$:**P-parity is unitary,T-parity is antiunitary**.Written in J,P,K,the Lie algebra is then:

$$\begin{aligned} [P, H] = [T, H] &= 0 \\ PJP^{-1} = +J, PKP^{-1} &= -K, PPP^{-1} = -P \\ TJT^{-1} = -J, PKP^{-1} &= +K, TPT^{-1} = -P \end{aligned} \tag{2.14}$$

- The 1st line implies **if** P,T is the symmetry,then we can construct coorespoding operators.This is same for all transformations: **only when coorespoding operator can be constructed,thus the Lie algebra satisfied,is the transformation a symmetry,indicated also by conservation/commutation relation with H.**
- For $SO(3)$-vectors(j=1 rep of $SO(3)$):vector refers to those change sign under P-parity, denoted 1^-;pesudovector refers to those invariant under P-parity,denoted 1^+.
- J change sign under T-reversal,this gives the special case of $B = \nabla \times A, L = x \times p$.P change sign under T-reversal can also be understood classically from orbits.
- The fact that T is antiunitary can also be infered from QM: with $T\psi(x,t) = \psi(x,-t)$,from Schrodinger equation we directly get: $TiT^{-1} = -i$ This is just the fact that the above relations hold also in **Galiliean algebra**.

[13]same for spin and mass
[14]P,T are (anti)unitary operators,it's believed that **some process respect these symmetry thus we can find such operators**
[15]LHS are all operatos,generators Hermitian,RHS involve defining-representation

2.5.2 P,T transformation of single-particles

Single particle states are defined using (H, P, J_3),their transformation properties can be refered from algebras above:

Massive

- P-parity:
$$P|k,\sigma\rangle = \eta_\sigma|k,\sigma\rangle, PJ_\pm P^{-1} = J_\pm \Rightarrow \eta_\sigma = \eta$$
$$P|p,\sigma\rangle = \sqrt{M/\omega_p}U(L(\mathcal{P}p))\eta|k,\sigma\rangle \equiv \eta|\mathcal{P}p,\sigma\rangle \tag{2.15}$$

- T-parity:
$$T|k,\sigma\rangle = \zeta_\sigma|k,-\sigma\rangle, TJ_\pm T^{-1} = -J_\mp \Rightarrow -\zeta_\sigma = \zeta_{\sigma\pm1} \Rightarrow \zeta_\sigma = \zeta(-1)^{j-\sigma},$$
Note T reverse J and leave K invariant,thus for $L(p) = R(\hat{p})B(|p|)R(\hat{p})^{-1}$,$\mathcal{T}L(p)\mathcal{T}^{-1} = L(\mathcal{P}p)$[a].We have:
$$T|p,\sigma\rangle = \zeta(-1)^{j-\sigma}|\mathcal{P}p,-\sigma\rangle \tag{2.16}$$
The ζ unlike η has no physical implication,can be abosorbed by U(1)-redefinition,this is due to **antiunitarity** of T: $T|k,\sigma\rangle' = \zeta^{*1/2}T|k,\sigma\rangle = |\zeta|^{1/2}\zeta^{1/2}(-1)^{j-\sigma}|k,-\sigma\rangle = (-1)^{j-\sigma}|k,-\sigma\rangle'$.

[a] $\mathcal{P}: \hat{p} \to -\hat{p}, \omega_p \to \omega_p$

Massive

- P-parity: In this case,P change k^μ,since it contain $\vec{p}$-part.Consider $U(R_2(\pi))^{-1}P, U(R_2(\pi)) = exp[i\pi J_2]$.This gives the wanted transformation:
$$U(R_2(\pi))^{-1}P|k,\sigma\rangle = \eta_\sigma|k,-\sigma\rangle$$
The helicity is reversed under P by definition[a]
$R_2(\pi)^{-1}\mathcal{P}$ commute with $B_3(|p|)$,$\mathcal{P}$ commute with $R(\hat{p})$.Thus:
$$P|p,\sigma\rangle = \sqrt{1/\omega_p}U(R(\hat{p})R_2(\pi)B(|p|))U(R_2(\pi)^{-1})P|k,\sigma\rangle = \sqrt{1/\omega_p}\eta_\sigma U(R(\hat{p})R_2(\pi)B(|p|))|k,-\sigma\rangle$$
There's relation:$U(R(\hat{p})R_2(\pi)) = U(R(-\hat{p}))exp[\pm i\pi J_3]$,and $R(-\hat{p})B(|p|) = L(\mathcal{P}p)$ thus:
$$P|p,\sigma\rangle = \eta_\sigma exp(\mp i\pi\sigma)|\mathcal{P}p,-\sigma\rangle, \mp = -sign(p_2) \tag{2.17}$$

- T-parity: Similarly,T change reverse the 3-components(P) of k and leave the 0-component(H) invariant,T also reverse J thus the helicity is invariant:
$$U(R_2^{-1})T|k,\sigma\rangle = \zeta_\sigma|k,\sigma\rangle$$
$$T|p,\sigma\rangle = \sqrt{1/\omega_p}U(R(\hat{p})R_2B(|p|))\zeta_\sigma|\mathcal{P}p,\sigma\rangle$$
Similarly,
$$T|p,\sigma\rangle = \zeta_\sigma exp(\pm i\pi\sigma)|\mathcal{P}p,\sigma\rangle, \pm = sign(p_2) \tag{2.18}$$

Remark 9
- *The p under P,T both change to $\mathcal{P}p$.For massive case,the spin-3 component:invariant under P,reverse sign under T.For massless case,the helicity :reverse sign under P,invariant under T.*
- *If we assume the P,T are conserved without any interaction,then the particles all carry some intrinsic P/T-phases,the **relative phase is to be determined by experiment**.In QFT,absolute intrinsic phases are defined for convenience: they are **defined to make sure the interactions respect P/T symmetry.***

[a] U as a rotation is also a Lorentz transformation,leaving h invariant

2.5.3 Projective representation of Z_2^T,Kramers degeneracy

- Massive:$T^2|p,\sigma\rangle = T\zeta(-1)^{j-\sigma}|\mathcal{P}p,-\sigma\rangle = (-1)^{2j}|p,\sigma\rangle$
- Massless:$T^2|p,\sigma\rangle = T\zeta_\sigma exp[i\pi\sigma]|\mathcal{P}p,\sigma\rangle = exp(\mp 2i\pi\sigma)|p,\sigma\rangle = (-1)^{2|\sigma|}|p,\sigma\rangle = (-1)^{2j}|p,\sigma\rangle$

Both cases we have:

$$T^2 = (-1)^{2j} \tag{2.19}$$

This result hold generally **as long as the interaction respect T-reversal symmtry**[16]:

$$T^2 = (-1)^{2J}$$

Then states with half-integer spin lead to **Kramers degeneracy**,since $[H,T]=0$.There's **at least** 2-fold degeneracy.

This fact is just the projective representation of Z_2^T: $T\cdot T=-1$.As mentioned before this fact origins from $SO(3)$ **is projectively representated** of spin-$n+1/2$.

Remark 10
Intrinsic electric/gravitational dipole momentum *will break the 2j+1 degeneracy in external fields.They are forbidden by T-reversal for these states.*

2.6 Appendix

2.6.1 $SL(2,\mathbb{C})$ double cover $SO(1,3)$

The fact that $SL(2,\mathbb{C})$ double-covers $SO(1,3)$:

$$1 \to \pi_1(SO(1,3)) = \mathbb{Z}_2 \to SL(2,\mathbb{C}) \to SO(1,3) \to 1$$

can be manifested using the fact in representation theory of Lorentz group:

$$(1/2,1/2) = (1/2,0)\otimes(0,1/2)$$

This fact implies it's equivalent to label a 4-vector with a pair of (L,R) spinor indices. The projective tensor is provided by Pauli-matrices: $\sigma^\mu_{a\dot{a}} = (I,\vec{\sigma})$.We then have:

$$v_{a\dot{a}} = V_\mu\sigma^\mu_{a\dot{a}}, V^2 = -det(v), v\in GL_{2\times 2}(\mathbb{C}), v = v^\dagger$$

Hermiticity and det(v) is perserved under unitary transformation by $\lambda\in SL(2,\mathbb{C}) : det(\lambda) = 1$.Then surjective $\pi : SL(2,\mathbb{C}) \to SO(1,3)$ is given by:

$$\lambda(v)\lambda^\dagger = \lambda(V\cdot\sigma)\lambda^\dagger = (\Lambda(\lambda)V)\cdot\sigma \tag{2.20}$$

It's easy to see that $\pm\lambda$ defines same $\Lambda(\lambda)$,thus the kernal of π is $\mathbb{Z}_2$.

The topology of $SL(2,\mathbb{C})$ can be obtained from the polar decomposition:

$$\lambda = ue^h, u^\dagger u = 1, h = h^\dagger;\ det(\lambda) = 1 \Rightarrow det(u) = 1, tr(h) = 0 \tag{2.21}$$

$u\in SU(2)$,the h matrices can be further written in Pauli matrices[17]:

$$h = \vec{h}\cdot\vec{\sigma}, h \leftrightarrow \vec{h}\in\mathbb{R}^3$$

.

Similarly the $SU(2)$ matrices can be decomposed with Pauli matrice:

$$u = u^0 1 + u^i(i\sigma^i), det(u) = (u^0)^2 + u^i u^i = 1$$

Thus we have:$SU(2)\cong S^3, SL(2,\mathbb{C})\cong S^3\times\mathbb{R}^3, SO(1,3)\cong S^3/\mathbb{Z}_2\times\mathbb{R}^3, ISO(1,3)\cong S^3/\mathbb{Z}_2\times\mathbb{R}^3\times\mathbb{R}^4$

This decomposition is physical:

- u generate rotations: $tr(uvu^\dagger) = tr(u)$,$V^0 = \frac{1}{2}trv$ is invariant under $\Lambda(u)$

[16]even when SO(3)-symmetry is not presented
[17]any hermitian traceless 2by2 matrices

2.6.2 $SU(2)$ double cover $SO(3)$

The above discussion is applicable to:

$$1 \to \pi_1(SO(3)) \to SU(2) \to SO(3) \to 1$$

we have the similar arguments:

$$v = \vec{v} \cdot \sigma, tr(v) = 0, det(v) = |v|^2, v = v^\dagger$$

any $u \in SU(2)$ by unitary transformation leaves properties of v unchanged,thus amounts to a $SO(3)$ rotation:

$$u(v)u^\dagger = u(\vec{v} \cdot \vec{\sigma})u^\dagger = (R(u)v) \cdot \vec{\sigma} \tag{2.22}$$

The same arugument hold:$\pm u \leftrightarrow R(u)$,the kernal is $\mathbb{Z}_2$

2.6.3 $SO(4)$

Since $SO(1,3)$ may be considered as the analytic continuation of the compact group $SO(4)$,we have similar relations:

$$u = u^0 1 + \vec{u} \cdot (i\vec{\sigma}), u \in SU(2) \Leftrightarrow det(u) = 1$$

For any two elements $U, W \in SU(2)$ the map:

$$u \to UuW^\dagger = u'$$

gives another element $u' \in SU(2)$ thus this induces a transformation in Euclidean 4-space:

$$UuW^\dagger = U(u_E \cdot (1, i\vec{\sigma}))W^\dagger = (R(U,W)u_E) \cdot (1, i\vec{\sigma})$$

Actually,this map is only local,and to the same reasoning we have:

$$SO(4) \cong_{locally} (SU(2) \otimes SU(2))/\mathbb{Z}_2$$

This also implies:$so(4) = su(2) \otimes su(2)$,similar for $sl(2, \mathbb{C})$.

Chapter 3

Physical observables and interactions between particles:S-matrix

3.1 Definition of S-matrix

In relativistic theories,the observables are associated with scattering process,where the multiparticle states are free[1] at asymptotic times:it transform under direct product of PUIRs of ISO(1,3)[2]:

$$|\{p_i,\sigma_i,n_i\}\rangle_0 = \otimes_i|p_i,\sigma_i,n_i\rangle$$
$$U(\Lambda,a)_0|\{p_i,\sigma_i,n_i\}\rangle_0 = e^{-ia\cdot(\sum_i \Lambda p_i)}(\prod_i \frac{\omega_{\Lambda p}}{\omega_p})^{1/2}\sum_{\sigma_i'}\prod D^{(j_i)}_{\sigma_i'\sigma_i}(W(\Lambda,p_i))|\{\Lambda p_i,\sigma_i',n_i\}\rangle_0$$

For both massive/massless particles,we denote the little group transformation:$W(\Lambda,p)$,which is determined by standard Lorentz transformations respectively,the same is true for D.Explicit form differs for these two cases,the explicit form is derived before.

Compactify the notation:$|\{p_i,\sigma_i,n_i\}\rangle \to |\alpha\rangle$[3]The normalization and completeness relations can be compactly written as:

$$\langle\alpha|\alpha'\rangle = \delta(\alpha-\alpha'), \int d\alpha|\alpha\rangle\langle\alpha| = 1$$

The $\delta(\alpha-\alpha')$ is the sum over permutations[4] of products of δ-functions/symbols,and $\int d\alpha$ refers to products of sums/integrals respectively.

The transformation implies:$H|\alpha\rangle = E_\alpha|\alpha\rangle, E_\alpha = \sum_i \omega_{p_i}$ as expected: **the energy contain no interactions**.For interacting multiparticle states,it still furnish representations,but the energy contain interaction and have no such decomposition,this will be more clear in **2nd quantizations.**

3.1.1 In/Out states

Scattering process at $t \leftarrow \pm\infty$[5]

The in/out **asymptotic-free** states:

$$|\alpha\rangle_{in/out} = |\alpha\rangle_\pm := \lim_{t\to\mp\infty} e^{-iHt}\Psi$$

- The Ψ refers to the state observed by an observer at standard time $t_0 = 0$,**different observers get different information from** Ψ,by acting on **time-dependent operators**,including the particles α and their interactions.We only specify α for in/out states which is transform in tensor product representations by assumption.
- The physical states are actually wave-packets:$\int d\alpha g(\alpha)|\alpha\rangle$

[1] usually use 0 to denote free,n_i refer to the species of particles,including its global/gauge charges,etc

[2] states are basis of carrier space of PUIR,thus U rather than U^{-1}

[3] 0 is omitted if no ambigurity arise

[4] The convention is:- for exchanging odd number of fermions,actually,if we fix the order of how we state the particles,the permutation is not needed.**With this sum over permutation,orders in the tensror product no longer matters**

[5] In passive view, the time is related to observable/observers.The state contain no time-dependence and decribe the whole history of scattering.**Different time means different observers see different states,up to time translation,we are in H-picture**

There's a more convenient definition of in/out state.Decompose the full H into free parts and interactions:

$$H = H_0 + V, H_0|\alpha\rangle_0 = E_\alpha|\alpha\rangle_0$$

- H_0 is assumed to have same spectrum α as the full H: they contain **same particles**,this need the particles in spectrum of H is defined with **physical mass**:combining interactions and bare mass.
- in/out states are defined to be eigenstates of the full H with same energy[6] as free state :

  $$H|\alpha\rangle_\pm = E_\alpha|\alpha\rangle_\pm$$

  and statisfy the **asymptotic condition**:

  $$\tau \to \mp\infty : e^{-iH\tau}\int d\alpha g(\alpha)|\alpha\rangle_\pm \to e^{-iH_0\tau}\int d\alpha g(\alpha)|\alpha\rangle_0 \Leftrightarrow |\alpha\rangle_\pm = \Omega(\mp\infty)|\alpha\rangle_0, \Omega(\tau) = e^{iH\tau}e^{-iH_0\tau} \tag{3.1}$$

- We may think of the **free states lying in the I-picture,while the in/out states are defined in H-picture.**

According to definition,we have same normalization:

$$\langle\beta;\pm|\alpha;\pm\rangle = \langle\beta;0|\alpha;0\rangle = \delta(\beta-\alpha)$$

3.1.2 Lippmann-Schwinger equation and OFPT

The full eigenequation can be written as:

$$(E_\alpha - H_0)|\alpha;\pm\rangle = V|\alpha;\pm\rangle$$

The formal solution of in/out state is then:

$$|\alpha;\pm\rangle = |\alpha;0\rangle + (E_\alpha - H_0 \pm i\epsilon)^{-1}V|\alpha;\pm\rangle, \Delta_{LS} = (E_\alpha - H_0 \pm i\epsilon)^{-1} \tag{3.2}$$

Δ_{LS} is known as the Lippmann-Schwinger kernal.Define the **transition-matrix operator**:

$$T^\pm|\alpha;0\rangle \equiv V|\alpha;\pm\rangle = V|\alpha;0\rangle + V\Delta_{LS}T|\alpha;0\rangle$$

Constrain to the free Hamiltonian states/subspace,the above equation can be solved purterbationly,this is known as **old-fashioned purterbation theory**[7]:

$$T = V + V\Delta_{LS}T = V + V\Delta_{LS}V + V\Delta_{LS}V\Delta_{LS}V + ...; \Delta_{LS} = \frac{1}{E - H_0 \pm i\epsilon}, E = E_i = E_f \tag{3.3}$$

The in/out state is then[8]:

$$\begin{aligned} |\alpha;\pm\rangle &= |\alpha;0\rangle + \int d\beta \frac{T^\pm_{\beta\alpha}}{E_\alpha - E_\beta \pm i\epsilon}|\beta;0\rangle \\ T^\pm_{\beta\alpha} &= V_{\beta\alpha} + \int d\gamma \frac{V_{\beta\gamma}V_{\gamma\alpha}}{E_\alpha - E_\gamma \pm i\epsilon} + \int d\gamma d\tau \frac{V_{\beta\gamma}V_{\gamma\tau}V_{\tau\alpha}}{(E_\alpha - E_\gamma \pm i\epsilon)(E_\alpha - E_\tau \pm i\epsilon)} + ... \end{aligned} \tag{3.4}$$

- **Causality**:the $\pm i\epsilon$ in Δ_{LS} indicate the definition of in/out state.Consider the wave-packet in S-picture:

  $$\Psi_g(t)^\pm = \Phi_g(t) + \int d\alpha d\beta \frac{e^{-E_\alpha t}g(\alpha)T^\pm_{\beta\alpha}}{(E_\alpha - E_\beta \pm i\epsilon)}\Phi_\beta$$

  The kernal:$K^\pm_\beta = \int d\alpha \frac{e^{-E_\alpha t}g(\alpha)T^\pm_{\beta\alpha}}{(E_\alpha - E_\beta \pm i\epsilon)}$ vanishes[9] for $t \to \mp\infty$ giving the right asymptotic behabior.

[6] same spectrum

[7] $\pm$ omitted

[8] when constraint to free states,we will only use the completeness relation of free states

[9] according to the discribution of the pole

Physical meaning of OFPT

The physical meaning of OPFT is obvious:

- the states are all **free**,except at the localtion where interaction happens.The interaction is indicated by the insertion of intermediate states.The order of the purterbation is indicated by the number of Δ_{LS} kernals.
- the intermediate states are all **free states:can be decomposed as tensor products of free particles**,especially,these free particles are all **on-shell**.
- The intermediate states can have different energy,this is due to **uncertianty relation:energy conservation can be violated in finite times** or the fact that we should consider the states as **wave-packets**.At each interaction vertex,**3-momentum is still conserved**,due to space-translation symmetry.The energy is not conserved during vertices can also be thought that interation at vertices break the time-translation symmetry of free-particles.
- The Δ_{LS} kernal can be considered as **on-shell propagator** with $\pm i\epsilon$ indicating the casulity,for purterbation calculations we have to include all possible states,and all possible causality/order to get Lorentz invariant results.
- In general different causality/order are calculate separately and sum up at the end.Since the intermediate states are all on-shell.OFPT manifest unitarity[a] rather than Lorentz invariance.
- The T-matrix is related to the S-matrix,thus OFPT clarifies the way that **singularities in S-matrix** arise from **intermediate states.**

[a]unitarity here means insertion of completeness relation of physical particle states

Remark 11

- *There's another purterbation scheme manifest Lorentz invariance but not manifest unitarity called* ***the Feynman scheme or time-dependent purterbation theory.*** *The Feynman scheme combine the two causalities,leading to off-shell intermediate states and Lorentz-covariant propagators:* ***the Feynman propagator****.*
- *OFPT will be used in some formal derivations but not for calculations in relativistic QFT.It is useful for calculations in non-relativistic QM.*

3.1.3 S-matrix and scattering amplitude

S-matrix element is defined as the transition amplitude between in/out states,it encode the interactions between the particles in the **scattering process.**:

$$S_{\beta\alpha} = \langle\beta; out|\alpha; in\rangle_H = \langle\beta; -|\alpha; +\rangle = \langle\beta; 0|S|\alpha; 0\rangle_I \tag{3.5}$$

The S-matrix operator is defined by:

$$S = \Omega(+\infty)^\dagger\Omega(-\infty) = U(+\infty, -\infty), U(t, t_0) \equiv \Omega(t)^\dagger\Omega(t_0) = e^{iH_0 t}e^{-iH(t-t_0)}e^{-iH_0 t_0} \tag{3.6}$$

The U operator is the time-evolution operator in I-picture.[10]It will be used in time-dependent purterbation theory.

3.1.4 Relation between S-matrix element and T-matrix element

Using the method of residues:

$$t \to +\infty : K_\beta^+ \to -2i\pi e^{-iE_\beta t}\int d\alpha\delta(E_\alpha - E_\beta)g(\alpha)T_{\beta\alpha}^+$$

By definition,as t goes to infinity:

$$\Psi_g^+(t) = \int d\alpha e^{-iE_\alpha t}g(\alpha)\int d\beta\Psi_\beta^- S_{\beta\alpha} = \int d\beta\Psi_\beta^- e^{-iE_\beta t}\int d\alpha g(\alpha)S_{\beta\alpha}$$

[10]States in I-picture are considered as free.

Then according to definition and Lippmann-Schwinger equation,equating the asymptotic Ψ_g^t:

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2i\pi\delta(E_\beta - E_\alpha)T^+_{\beta\alpha} \tag{3.7}$$

Then the OFPT of T gives the purterbation theory of S.The lowest order gives the **Born-approximation**[11]:

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2i\pi\delta(E_\beta - E_\alpha)V_{\beta\alpha}$$

Remark 12
- *It's more used in relativistic QFT to define:*

$$S = 1 + i\mathcal{T} = 1 + (2\pi)^4\delta^4(p_\beta - p_\alpha)(i\mathcal{M}) \tag{3.8}$$

*In this way,we have a rather simple Feynman rules to calculate $\mathcal{M}$,which is refered to as the **scattering amplitude**.Note in some text there's sign difference and factors of 2π in $\mathcal{M}$,but since observables are related to $|\mathcal{M}|^2$,this sign is a matter of convenience.*

3.2 Properties of S-matrix

3.2.1 Unitarity

In/Out states form two set of **complete orthonoral basis**,since S matrix connect these two basis,**it must be unitary**,specificly:

$$\int d\beta S^*_{\beta\gamma}S_{\gamma\alpha} = \int d\beta\langle\gamma;+|\beta;-\rangle\langle\beta;-|\alpha;+\rangle \equiv \langle\gamma;+|\alpha;+\rangle = \delta(\gamma-\alpha) \Leftrightarrow S^\dagger S = 1$$

Similarly,the completeness relation of in-states implies:

$$\int d\beta S_{\gamma\beta}S^*_{\alpha\beta} = \delta(\gamma - \alpha) \Leftrightarrow SS^\dagger = 1$$

The **unitarity of S-matrix** is a very strong constrains,it lead to many non-purturbative relations.As we have seen,the unitarity of S-matrix is equivalent to the **unitarity(completeness relation) of in/out states**.

Remark 13
- *The unitarity of S-matrix,the completeness relation and orthonormality of in/out states,the relation between S-matrix and T-matrix can be proved using Lippmann-Schwinger equation,without using the asymptotic definition of in/out state and the asymptotic behavior of the K-kernal.For detail,refer to [WeinbergI,p115]*

3.2.2 Spacetime symmetry:Poincare invariance

The S-matrix operator respect Poincare invariance **if** we can **define/construct** the unitary transformation $U(\Lambda, a)$ to act on **both in/out** state,in general,this $U(\Lambda, a) \neq U_0(\Lambda, a)$ acting on true free state.[12]The requirement of the Poincare invariance is **maintained at quantum level** is a strong constraint on theories: **the possible choice of Hamiltonian is constrained.**

The S-matrix element is **covariant**:

$$S_{\beta\alpha} = \langle\beta;-|\alpha;+\rangle \equiv \langle U(\beta;-)|U(\alpha;+)\rangle = (U(\Lambda,a)\Psi^-_\beta, U(\Lambda,a)\Psi^+_\alpha)$$

This relate the S-matrix **elements** as:

$$\begin{aligned} S_{\{p'_i,\sigma'_i,n'_i\},\{p_j,\sigma_j,n_j\}} =& e^{ia\cdot(\sum_{out}\Lambda p'_i - \sum_{in}\Lambda p_j)}\sqrt{\prod_{in,out}\frac{\omega_{\Lambda p_k}}{\omega_{p_k}}}(\prod_{out}\sum_{\bar\sigma'_i}D^{(j_i)*}_{\bar\sigma'_i\sigma'_i}(W(\Lambda,p'_i)))(\prod_{in}\sum_{\bar\sigma_j,in}D^{(j_j)}_{\bar\sigma_j\sigma_j}(W(\Lambda,p_j))) \\ &\times S_{\{\Lambda p'_i,\bar\sigma'_i,n'_i\},\{\Lambda p_j,\bar\sigma_j,n_j\}} \end{aligned}$$

Remark 14
- ***4-momentum conservation**:LHS is independent of a,thus must be the RHS:*

$$\sum_{out} p^\mu - \sum_{in} p^\mu = p_\beta - p_\alpha = 0$$

[11] States constraint to free states

[12] in/out state is asymptotic free,but transform as a free state

- *This suggest the form of S-matrix element*[13]*:*

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2\pi i \delta^4(p_\beta - p_\alpha)\mathcal{M}_{\beta\alpha} \tag{3.9}$$

The $\mathcal{M}$ elements may still contain further δ-functions.

The S-matrix **operator** is **invariant**:

$$S_{\beta\alpha} \equiv \langle \beta; 0|S|\alpha; 0\rangle$$

An important difference is that U_0 **always exsist** since it's the tensor product representation.Then S-matrix respect Poincare invariance **if**:

$$U_0(\Lambda, a)^{-1} S U_0(\Lambda, a) = S$$

Poincare invariance of S-matrix

The above relation implies we **must** have:

$$[H_0, S] = [\vec{P}_0, S] = [\vec{J}_0, S] = [\vec{K}_0, S] = 0, S = U(\infty, -\infty) = \Omega(\infty)^\dagger \Omega(-\infty)$$

Since S matrix have implicit connection to the full Hamiltonian H,thus this gives constraint on the form of H.

- These 'free' generators satisfy the Lie algebra so that U_0 is well-defined.Similarly,**the 'exact'/'full' generators satisfy the Lie algebra** so that U is well-defined.
- Another requirement is that this U must transform both in/out state.

In most reletivistic QFT theories,the effect of interactions is to add an interaction term to free Hamiltonian[a]:

$$H = H_0 + V, P = P_0, J = J_0$$

This can also be obtained from the fact that in/out states are related to free state as:

$$\Psi^\pm_\alpha = \Omega(\mp\infty)\Phi_\alpha$$

Only H, H_0 involve,thus the exact generators for space-translations and rotations are the same as the free ones.The exact Lie algebra gives constraints:

$$[V, P_0] = [V, J_0] = 0$$

Thus P_0, J_0 commute with H, H_0 thus Ω, U and S.Further,S-matrix element involve energy conservation δ-function, S commute with H_0.What left is:

$$[K_0, S] = 0$$

Though J_0, P_0 remain unchanged in exact Lie algebra,the $K_0 \neq K = K_0 + W$,else Lie algebra will imply $H_0 = H$.
The exact Lie algebra then gives constraint:

$$[K_0, V] = -[W, H]$$

The W is not just determined by this[b] alone,there's also requirement these **exact operators act the same way on in/out state.**It can be shown that if W is accompanied with appropriate **smoothness condition**,then $[K_0, V] = -[W, H] \Rightarrow [K_0, S] = 0$ as wanted.This smooth condition on $W(V, H_0, K_0)$ then constraint possible V,thus the full H.To prove this,using the Lie algebra:

$$[K_0, e^{iH_0 t}] = tP_0 e^{iH_0 t}, [K, e^{iHt}] = tPe^{iHt} = tP_0 e^{iHt}$$

This gives,with a little algebra:

$$[K_0, U(\tau, \tau_0)] = -W(\tau)U(\tau, \tau_0) + U(\tau, \tau_0)W(\tau_0), W(\tau) = W_I = e^{iH_0 t} W e^{-iH_0 t} \tag{3.10}$$

If the matrix elements of W between H_0-eigenstates are sufficiently smooth functions of energy,then matrix element of $W(t)$ between smooth superpositions of energy eigenstates(free state as wave-packets) vanish for $t \to \pm\infty$ thus **constraint to free state subspace**[c]:$0 = [K_0, U(\infty, -\infty)] = [K_0, S]$.

[a] **topological twisted fields/defects will add terms to the angular momentum**
[b] by constructing matrix elements of W using K_0, V
[c] We define S-matrix operator and T-matrix operator using free states

[13] As remarked before,there may be sign difference and factors of $(2\pi)^2$in definition of $\mathcal{M}$

Remark 15 • ***Poof of the vanishing:***
*$W_{\beta\alpha;g}(t) = \int d\alpha d\beta g(\alpha)g(\beta)e^{-i(E_\alpha - E_\beta)t}W_{\beta\alpha}(E_\alpha - E_\beta)$ as t goes to infinity,due to **Riemann-Lebsegue lemma**,only the $E_\alpha = E_\beta, W(0) = W_{0,0}$ survive,where 0 refers to H_0 vacuum.Since W is a smooth function of E,we take W(0)=0.This technique of **using energy eigenstate and use R-L lemma to obtain only ground state** will be also be used in path-integral formalism.*

- *The smooth condition of W lead to the vanishing of $W(\pm\infty)$,this is a natural constraint due to $V(\pm\infty) = 0, V(t) = V_I$ since the **interaction is local**.*

Summary

To maintain Lorentz invariance in interaction theories,the full exact Poincare algebra must be furnished/constructed to define U-operator.The condition that this U act the same way on in/out state is the condition that U_0 commute with S,this finnaly reduce to $[V, J_0] = [V, P_0] = 0, [K_0, S] = 0$ the last condition is fulfilled if the correction W to K_0 due to interaction is 'smooth' in free-state energy,this implicitly constrain the form of V.

We also have specific relations between exact generators and free generators:

$$t\Omega(\mp\infty) = \omega(\mp\infty)t_0, t = \{K, P, J, H\} \tag{3.11}$$

As expected,only the relation between H, H_0 and K, K_0 is non-trivial.Since Ω relate free state and in/out state,these implies **in/out state indeed transform like a free state,with U generated by the exact generators.**

3.2.3 Internal symmetry and C,P,T

The internal transformation[14] act on particle species,labelled by n,which includes all kinds of **global/gauge charges,multiplet-representation indices,etc**:[15]

$$U(T)|\{p_i, \sigma_i, n_i\}\rangle = \sum_{n'_i}\prod_i D_{n'_i n_i}(T)|\{p_i, \sigma_i, n'_i\}\rangle$$

Here D is the corresponding PUR[16],generally reducible with each block coorespoding to different multiplets.It's more convenient to group the states into tensor products[17] of multiplets(components are corresponding particle states) so that the transformation is just tensor products of separate irreducible transformations on each multiplet.

The definition of internal **symmetry** of S-matrix element(covariance) is[18]:

$$S_{\beta\alpha} = (\Psi^-_\beta, \Psi^+_\beta) = (U(T)\Psi^-_\beta, U(T)\Psi^+_\beta)$$

The requirements again is that we **if** can construct $U(T)$ acting the same way on **both** in/out state.
Again,this reduce to the requirements:

$$U_0(T)^{-1}SU_0(T) = S, U_0(T)^{-1}H_0U_0(T) = H_0 \Rightarrow [U_0(T), H_0] = 0, [U_0(T), V] = 0 \Leftrightarrow [U_0(T), H] = 0$$

Once these conditions are fulfilled,the relation:$\Psi^\pm = \Omega(\mp\infty)\Phi$ will implies the in/out states transform as a free state,with **exact U**.The situation of internal symmetry is much simplier than spacetime symmetry,since the interaction generally won't change the form of generators.

- **One-parameter Lie group:U(1)-charges**$U(T(\theta)) = e^{iQ\theta}, D_{n'n} = \delta_{n'n}e^{iq_n\theta}$.Similarly,the LHS contain no θ-dependence,the RHS implies conservation of U(1)-charge:

$$\sum_{in} q = \sum_{out} q$$

[14]Usually,**the discrete spacetime symmetry:P,T are also included in internal symmetry**,this is because the intrinsic P/T-phases may be include into n as internal.another example is charge-conjugation.

[15]**Gauge charges for non-Abelian gauge group are also a multiplet-representation indices**.However,except the global gauge symmetry related to fundamental interactions,all internal symmetry are **approximate**.For example:isospin for hadrons,origining from approximate symmetry(SU(2)/SU(3)-flavor,etc) of quarks.

[16]The unitarity is again due to completeness relations of in/out/free states

[17]without interaction between multiplets

[18]expand into components to get transformation rules of S-matrix elements

- Electric charge origin from global U(1)-gauge invariance,other examples include:**baryon,lepton numbers**,**strangeness**,**charmness**

Remark 16 • *The proof here is irrelevent of Poincare symmetry,as long as the symmetry conditions are met.But the V is constrained by Poincare symmetry.Actually,the conservation of U(1)-gauge charge have deep relation to Lorentz invariance and the fact that gauge bosons mediating interactions are spin-1 massless particles,proved using **soft particles** non-purturbatively.*

- There's also examples of non-Abelian internal symmetries: **SU(2)/SU(3)-isospin** for hadrons[19],

The internal symmetry can be used to constrain the form of S-matrix elements,this use the fact that representations is 'multiplies/tensored' at the interaction vertices and factors into different irreducible representaions as possible intermediate states:

- The relative ratio of different process is constraint by group representation.
- The form of S-matrix element can be factored into projection tensors for different tensor product reductions.

  For example[20]:

$$SU(2): S_{t_{C3}t_{D3},t_{A3}t_{B3}} = \sum_{T,t_3} C_{T_C T_D}(Tt_3; t_{C3}t_{D3}) C_{T_A T_B}(Tt_3; t_{A3}t_{B3}) S_T$$

The idea of factorization:the scheme of 'point-particle'

The idea of obtain relative relations using group representation theory and the idea of factor the physical quantities using irreducible representaions and coorespoding projective tensors is very commom:

It's the idea of 'point-particles':different 'particles' are different sectors,which have some completeness(simple objects are building blocks)[a] and orthogonality(the idea of 'species')[b]
Other examples include:

- classical mass-points as points in manifolds.
- relativistic particles as PUIRs of ISO(1,3).(representation theory is also a tensor category)
- conformal blocks in CFT.
- anyons described by Braid tensor categories.examples will include twisted quantum doubles models,string-net models,etc.

The question of **condensation** is another topic.

[a] for group theory,this is given by **representation ring**
[b] for group theory **irreduciblility gives orthogonality relations of representations**

C,P,T

Similarly,when the transformation is a symmetry,and especially,**manifest by S-matrix**,we must be able to construct coorespoding operators **and** they acting the same way on in/out state.The states tranform as a **tensor-product representation**.The S-matrix element transform covariantly,S-matrix operator is invariant(commute with transformation operators).Again,using the relation between in/out-free states,the symmetry condition can be reduced to commutivity between $\{T_0, (H_0, V) = H\}$,and that in/out states truely transform like free-states with exact T.

P

The transformations can be refered from last chapter.The symmetry again lead to conservation of parity:

$$\prod_{in} \eta = \prod_{out} \eta$$

[19] $SU(3): Q = t_3 + (B+S)/2, Y = B + S, SU(2)$ include no S-quark
[20] C is the C-G coefficients

We can redefine P with help of other conserved internal symmetry operators,to set convention for the phases: $P'^2 = 1 \Rightarrow \eta = \pm 1$.The core observation is that P^2 amouts to a phase transition,if this coincide with some U(1)-internal symmetry,then P can be redifined.If P^2 equals to some internal symmetry not connected by U(1)-internal symmetries,then we can't redefine the phases to be ± 1.The situation of T,C is similar.

T

The transformations on particle states can be refered from last chapter,the crucial point is that since T is antiunitary:$S_{\beta\alpha} \to S^*_{\beta\alpha} = S_{\alpha\beta}$ this has the physical meaning of **reversing causality:the time ordering of scattering process is reversed**,especially:$T : \Psi^{\pm} \to \Psi^{\mp}$

This can be seen from(T reverse sign of i):

- T:$\Omega(\mp\infty) \to \Omega(\pm\infty)$.
- T:$\Delta^{\pm}_{LS} \to \Delta^{\mp}_{LS}$,$\pm$ refers to $\pm i\epsilon$

The covariance is then,for massive case(massless case is similar):

$$S_{\beta\alpha} = S_{\mathcal{T}\alpha,\mathcal{T}\beta}, \mathcal{T} : p \to \mathcal{P}p, \sigma \to -\sigma, \times\zeta_n(-1)^{j-\sigma}$$

C

There's a transformation interchange particle with it's antiparticle:

- **Antiparticle:** same mass and spin but reversed all other intrinsic properties included in label n,including charges,P/T-phases,$SU(2) - t_3$,etc.
- The exsistence of antiparticles is a **consequnce of Poincare invariance.**For process having space-like separation,the **causality/time-ordering may be reversed** in different inertial frames.Since physics doesn't depend on coordinates,the transition between 'charges' must be the same,this lead to the picture of **antiparticle carrying oppesite charges**.It's common to illustrate antiparticles as **particles moving backwards in time**.The particle flow will be drawn oppesite to the time-flow,indicating the reversed causality.

The transformation[21] is defined as[22]:

$$C\Psi^{\pm}_{\{p_i,\sigma_i,n_i\}} = \prod_i \xi_{n_i} \Psi^{\pm}_{\{p_i,\sigma_i,n_i^c\}}$$

The covariance of S-matrix is then[23] is obvious,and similar to other symmetries.

The ξ-phases are known as **charge-conjugation parity** which is measurable for **neutral** particles:no conserved quantum numbers(like charge) and is their own antiparticle: $n = n^c$.For process invloving only neutral particles,the C-parity is conserved,similar to P-parity.The explicit phase convention can be obtained using field theories.

Physical implications of C,P,T symmetry

Remark 17

P,T,C is the symmetry for process invloving only strong and electromagenetic interactions and is violated in weak interactions.

If the phases in transformation of S-matrix elements multiply to 1,this says the process related by parity have same amplitude.Similar for C.

The physical implication of T symmetry is not as clear as P.An example would be ***phase shift****,the specific discussion of this and* ***Waston's theorem*** *can be found in [WeinbergI,P129]*

There's also violation of CP,PT symmetry in weak interactions.

[21] in relativistic QFT,**C is unitary**

[22] $n^c = \bar{n}$

[23] if C is a symmetry

The product CPT is exact symmetry in all interactions, at least in theories involving quantum fields.CPT commuting with H,implies the particles and antiparticles have same mass.The consequnce of CPT(antiunitary): particles replaced by antiparticles and J_3 reversed,causality reversed.This then implies ***C reverse helicity:antiparticle have minus helicity of the particle.For massless particles this implies reversed handedness.***[24]

3.3 Physical observables in scattering experiments

3.3.1 Differential transition rate

With box normalization:

$$\delta_V^3(p'-p) = \frac{1}{(2\pi)^3}\int_V d^3x e^{i(p'-p)\cdot\vec{x}} = \frac{V}{(2\pi)^3}\delta_{\vec{p}',\vec{p}}, \delta_T(E_\alpha - E_\beta) = \frac{T}{2\pi}\delta_{E_\alpha,E_\beta}$$

The boxed normalized state is defined as:

$$\Psi_\alpha^B = (\frac{(2\pi)^2}{V})^{\frac{N_\alpha}{2}}\Psi_\alpha, (\Psi_\alpha^B, \Psi_\beta^B) = \delta_{\beta,\alpha} \Rightarrow S_{\beta\alpha} = (\frac{V}{(2\pi)^3})^{\frac{N_\alpha+N_\beta}{2}} S_{\beta\alpha}^B$$

The probability is then[25]:

$$P(\alpha\to\beta) = |S_{\beta\alpha}^B|^2 = (\frac{(2\pi)^3}{V})^{N_\alpha+N_\beta}, d\mathcal{N}_\beta = (\frac{V}{(2\pi)^3})^{N_\beta} d\beta$$

$$dP(\alpha\to\beta) = (\frac{(2\pi)^3}{V})^{N_\alpha}|S_{\beta\alpha}|^2 d\beta$$

- For scattering experiments,we focus on **connected part of S:**

$$S_{\beta\alpha}^C = -2\pi i\delta_V^3(p_\beta - p_\alpha)\delta_T(E_\beta - E_\alpha)\mathcal{M}_{\beta\alpha}^C$$

  - The V,T normalization implies the interaction is **local**.We will finally take V,T to infinity,thus giving only one $\delta^4(p_\beta - p_\alpha)$
  - In general $\mathcal{M}_{\beta\alpha}$ may still contain δ-functions,but for connected part,due to **cluster-decomposition principle**:$\mathcal{M}^C$ is free of δ-function.

- $$dP(\alpha\to\beta) = (2\pi)^2(\frac{(2\pi)^3}{V})^{N_\alpha-1}(\frac{T}{2\pi})|\mathcal{M}_{\beta\alpha}|\delta_V^3\delta_T d\beta$$

Finally,the **differential transition rate:** is

$$d\Gamma(\alpha\to\beta) = \frac{dP}{T} = (2\pi)^{3N_\alpha-2}V^{1-N_\alpha}|\mathcal{M}_{\beta\alpha}|^2\delta^4(p_\beta - p_\alpha)d\beta, d\beta = \prod_{j\in out} d^3p_j \tag{3.12}$$

Remark 18 *For final states containing identical particles,we have to divide the* ***symmetry factor:*** $\prod_j n_j!$ *to divide the redundancy in* $d\beta$

About normalization

We are using non-relativistic normalization,with orthonormal states.For relativistic normalization,the states are normalized by reletivistic δ-function:

$$NR: \langle p'|p\rangle = \delta^4(p'-p), LR: \langle p'|p\rangle = (2\pi)^3(2\omega_p)\delta^4(p'-p)$$

[24] ν is L-handed and h=-1,$\bar{\nu}$ is R-handed and h=+1

[25] Note the normalization is non-reletivistic thus the states is orthonormal,for reletivistic normalization,we will have to divide the normalization of in/out states

Thus the absolute scales of S-matrix element are not the same,since $S_{\beta\alpha} = (\Phi_\beta, S\Phi_\alpha)$:[a]

$$S^{NR}_{\beta\alpha} = \frac{S^{LR}_{\beta\alpha}}{\prod_{in,out}[(2\pi)^3 2\omega_j]^{1/2}}, M^{NR}_{\beta\alpha} = -(2\pi)^3 \frac{M^{LR}_{\beta\alpha}}{\prod_{in,out}[(2\pi)^3 2\omega_j]^{1/2}}$$

The absolute scales of $d\Gamma$ are the same,since it's physically measured,independent of our normalization:

$$\begin{aligned} d\Gamma(\alpha\to\beta) &= (2\pi)^{3N_\alpha-2} V^{1-N_\alpha} |\mathcal{M}^{NR}_{\beta\alpha}|^2 \delta^4(p_\beta - p_\alpha) \prod_{j\in out} d^3p_j \\ &= \frac{1}{V^{N_\alpha-1}\prod_{in}(2\omega_{p_i})} |\mathcal{M}^{LR}_{\beta\alpha}|^2 (2\pi)^4 \delta^4(p_{out} - p_{in}) \prod_{out} \frac{d^3p_j}{(2\pi)^3(2\omega_{p_j})} \\ &= \frac{1}{V^{N_\alpha-1}\prod_{in}(2\omega_{p_i})} dLIPS_{N_\beta} \end{aligned} \tag{3.13}$$

The notation:

$$dLIPS_{N_\beta} = (2\pi)^4\delta^4(p_{out} - p_{in}) \prod_{out} \tilde{dp_j}, \tilde{dp_j} = \frac{d^3p_j}{(2\pi)^3(2\omega_{p_j})}$$

is clearly more compact and especially manisfest Lorentz invariance.But will not be useful in other field theories.

Different normalizations of $\mathcal{M}$ will lead to different Feynman rules,with LR normalization,the factors $2\pi, 2\omega$ will be attached to certain terms.With orthonormal(NR) normalization,the rules will be accompanied with all kinds of (2π),$2\omega_i$ factors.Conversely,the expansion of fields will not have a compact notation.

[a] $-(2\pi)^3$ origin from definition of M in terms of S

Remark 19 • *The $d\Gamma$ defined above is for process with* ***spin specified****,in many experiments,we only set up the initial momentum for each particles,and doesn't clear about the final spin-distribution.For these special theories,we* ***average initial spins and sum final spins****.*

- *It's obvious that $\mathcal{M}$ is not Lorentz-invariant,but covariant.This is easy to understand since spin-components will be different in different inertial frames.According to the Lorentz-covariance of S-matrix elements:*

$$\prod \sqrt{\omega_p} S = (D(spin)) \prod \sqrt{\omega_{\lambda p}} S_\Lambda$$

The phases can be dropped with spin-sumed,since D is unitary:

$$\sum_{spins} |\mathcal{M}^{NR}_{\beta\alpha}| \prod_\beta E \prod_\alpha E \equiv R_{\beta\alpha} \propto \sum_{spins} |\mathcal{M}^{LR}_{\beta\alpha}|^2$$

This quantity is then ***Lorentz-invariant.***

3.3.2 Decay rate

$N_\alpha = 1$:

$$d\Gamma = 2\pi |\mathcal{M}^{NR}_{\beta\alpha}|^2 \delta^4(p_\beta - p_\alpha) d\beta = \frac{1}{2\omega_\alpha} |\mathcal{M}^{LR}_{\beta\alpha}|^2 dLIPS_{N_\beta}$$

The crucial thing is that,for the definition of the intial **unstable** particle to be well-defined as a asymptotic free particle,the T of interactions should be much less than the mean lifetime.Or equivalently,the characteristic energy of the process is much higher than the total decay rate:$s = \Delta E >> \Gamma = \frac{1}{\tau}$.

Remark 20 • *Note the lifetime is not Lorentz invariant,even with spin-summed,shown by LR normalization.*

- *The particle can decay due to unavoidable intractions,the requirement that $E >> \Gamma$ means that the* ***interactions are effectively presented to change the definition of a particle****:*

Effective theories

This also implies what a true particles is: since interactions always exsists,there's nothing known as a free/bare particle:

- It may be bound state of more 'elementary' particles bounded by interactions.For example:hardrons
- It will produce virtual self-interactions.For example:the electron with electromagenetic mass.
- At certain scale,we consider the particle as physical,which means this particle are stable relatively,and **behave just like a true free particle**,but with intrinsic properties being those **physically measured.**
- These particles:bound state or 'elementary' will all be in the spectrum of the system(unitary in sense of as basis of Hilbert space),**with interactions included into the definition(integrated out).**
- This lead to the idea of **renormalization and effecive theory**.Nature present itself with different phenomelogy thus different effective descriptions at different scale.All physical theories are effecive theories,beyond certain scale will fail and need to be replaced by more 'fundamental' ones.When the scale goes higher and the distance goes shorter,the building blocks and their interaction may change.
  - Classical physics:

    mass points and Newtonian mechanics,electromagenetic fields,phenomelogical forces like friction,phenomelogical energies like chemical energy,Newtonian gravity.
  - High-speed:

    special relativiy;Heavy-mass:Einstein's gravity(GR)
  - Quantum physics:

    At short distance,quantum physics are effective.mass points will be presented as all kinds of **subatom particles**.With the distance become shorter and energy become higher,the subatom particles will finally be more 'fundamental'.
  - A phenomelogical model works fine is the **Standard Model**:there's definition of 'fundamental' particles(quark,lepton,Higgs) and their interactions(gluon,...).
  - **Effective field theories:**

    As energy scales goes down,effectively we can use the concepts like mesons,baryons to study strong interaction.The electroweak theories factors into two parts,with the weak theory described by W,Z bosons.At even lower energy,the Fermi theory is effective to study weak interactions.
  - With scale goes down,the theoy will become less symmetric,and phenomelogically richer.The mechanics of **spontaneous symmetry breaking** is important to understand these effective theories.
  - When energy goes even higher than Planck energy(distance shorter than Planck lenghth) the **quantum effect of gravity** is no longer ignored,and certainly the Standard model breaks down.New physical theories needed,may or may not be the final theory.

In summary,particles are not bare/free.To study them,we use effective (field) theories,which need the concept of renormalization to make sure the particles are well-defined as a 'free'/'stable' particle,this incorporate the interactions into the definition of particles.

3.3.3 Cross-section

$N_\alpha = 2$,the cross-section is defined with 'flux' divided: **rate per flux.**

$$\Phi_\alpha = \frac{u_\alpha}{V}, d\sigma(\alpha \to \beta) = \frac{d\Gamma(\alpha \to \beta)}{\Phi_\alpha} = (2\pi)^4 u_\alpha^{-1} |\mathcal{M}_{\beta\alpha}^{NR}|^2 \delta^4(p_\beta - p_\alpha) d\beta = \frac{1}{u_\alpha 4\omega_1\omega_2} |\mathcal{M}_{\beta\alpha}^{LR}|^2 dLIPS_{N_\beta}$$

The $u_\alpha E_1 E_2$ term is a Lorentz scalar[26]:

$$u_\alpha = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}; u_\alpha^{Lab} = \frac{|p_2|}{E_2}, u_\alpha^{CM} = |\vec{v_1} - \vec{v_2}|, \vec{v} = \frac{\vec{p}}{E}$$
$$u_\alpha E_1 E_2 = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$$

Remark 21 *The phase factor $\delta^4(p_\beta - p_\alpha)\prod_j d^3p_j$ will usually be partially integrated to get $d\Omega$ terms,etc.Since measurements will not concern all final distributions,in most cases we consider the solid angle distribution.The process of partial integration will depend on specific frames,and the core step is to identify $\delta^4(f(p))$.These calculations are rather straight,refer to [WeinbergI,p139],[Schwartz,p62]*

3.4 Time-dependent perturbertion theory

3.4.1 Dyson series

As mentioned before,the S-matrix element is related to T^+-matrix element.The T-matrix element can be calculated using OFPT.

The starting point of time-dependent perturbation theory is the definition:$S = U(\infty, -\infty)$,the U operator satisfy equation:

$$i\frac{d}{dt}U(\tau, \tau_0) = V(\tau)U(\tau, \tau_0) = V(\tau)U(\tau, \tau_0), V(\tau) = V_I = e^{iH_0\tau}Ve^{-iH_0\tau}$$

This is solved by the perturbation series:

$$U(\tau, \tau_0) = 1 - i\int_{\tau_0}^{\tau} V(t)U(t, \tau_0) = \sum_{n=0}^{\infty}(-i)^n \int_{\tau_0}^{\tau} dt_1 \int_{\tau_0}^{t_1} dt_2 ... \int_{\tau_0}^{t_{n-1}} dt_n V(t_1)...V(t_n)$$

With the help of **Time-ordering**[27]:

$$T\{V(t_1)V(t_2)\} = \theta(t_1 - t_2)V(t_1)V(t_2) + \theta(t_2 - t_2)V(t_2)V(t_1)$$

thus[28]:

$$U(\tau, \tau_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\tau_0}^{\tau} dt_1 ... \int_{\tau_0}^{\tau} dt_n T\{V(t_1)...V(t_n)\} \equiv Texp(-i\int_{\tau_0}^{\tau} dtV_I(t)), S = Texp(-i\int_{-\infty}^{\infty} dtV_I(t)) \tag{3.14}$$

3.4.2 Equivalence between OFPT and time-dependent perturbation theory

using the identity:

$$\frac{1}{E_\alpha - E_\gamma + i\epsilon} = (-i)\int_0^\infty d\tau e^{i(E_\alpha - E_\gamma)\tau}, (2\pi)\delta(E_\beta - E_\alpha) = \int dte^{i(E_\beta - E_\alpha)t}$$

the equivalence is obvious,changing the integral variable τ to $t - \tau$,then the phases will give the right transformation to I-picture operators.

3.5 Implication of Poincare invariance

3.5.1 Causality condition

The time-dependent perturbation theory has the advantage to manifest Poincare invariance.Expand V into density to manisfet the spacetime structure[29]:

$$V_I(t) \equiv \int d^3x \mathcal{H}_I(\vec{x}, t)$$

[26]u is not physical velocity however

[27]The V here is a **bosonic** operator,consists of even number of fermion operators,same for H and S.This is due to **Hermiticity**,time-ordering for more operators can be defined with iteration

[28]the sum generally won't converge,this is just a compact notation for the perturbation theories,the whole theory is **totally perturbative**,this means some non-perturbative properties can't be derived from this series

[29]This is an important **assumption in relativistic QFT**

Remark 22 *Here 'I' means both for interaction and interaction/free-field picture.*

Then the S-matrix operator is:

$$S = Texp(-i\int d^4x\mathcal{H}_I(x)) = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1...\int d^4x_nT\{\mathcal{H}_I(x_1)...\mathcal{H}_I(x_n)\} \tag{3.15}$$

To preserve Poincare invariance at quantum level,there's constraint:$[U_0(\Lambda, a), S] = 0$,this gives a strong **constraint on the intraction density** $\mathcal{H}_I$:

Constraint on interaction density,Highway to field theory

- The interaction density is a **scalar-field**[a]:

$$[U_0(\Lambda, a), S] = 0 \Rightarrow U_0(\Lambda, a)\mathcal{H}_I(x)U_0^{-1}(\Lambda, a) = \mathcal{H}_I(\Lambda x + a) \tag{3.16}$$

  The U_0 appearing here is illuminating: the interaction density/field in I-picture will be constructed out of **I-picture field-operators**,which have the same form as free field-operators.**The free particle state are related to free-field operators.**

- **Causality:**for Time-ordering to be well-defined for space-like separations,there must be **causality condition:**

$$[\mathcal{H}_I(x), \mathcal{H}_I(x')] = 0, \forall(x - x')^2 > 0 \tag{3.17}$$

- For these constraints to be consistent with the conditions on $K = K_0 + W$.[b]

$$[K_0, \mathcal{H}_I(x)] = it\nabla\mathcal{H}_I + i\vec{x}\frac{\partial}{\partial t}\mathcal{H}_I \Rightarrow [K_0, V] \equiv [K_0, \int d^3x\mathcal{H}_I(\vec{x}, 0)] = [H_0, W]$$

$$W \equiv -\int d^3x\vec{x}\ \mathcal{H}_I(\vec{x}, 0)$$

  The smooth condition of W implies: $\mathcal{H}_I(\vec{x}, 0)$ **between eigenstate of** H_0 **are smooth functions of energies.**This further suggest **constructing** H_0 **also out of free-field operators**.

- Finally, for the exact Lie algebra to hold:$[K_0, V] = -[W, H]$,we need:

$$0 = [W, V] = \int d^3xd^3y\vec{x}[\mathcal{H}_I(\vec{x}, 0), \mathcal{H}_I(\vec{y}, 0)]$$

  this is fulfilled by **causality condition**,since the speration between $(\vec{x}, \vec{y})$ is space-like.

These constraints all comes from the combination of SR+QM.It's important to note that there's no **causality condition** in non-relativistic theories,since **time-ordiering is Galiliean-invariant**.The causality condition from SR is the most resctictive constraint for SR+QM,as we have seen this natrually suggest the framework of field theories: H_0, H_I **all constructed out of free-Relativistic field-operators.**

[a] As an operator,must be **hermitian**,also note the convention of transformation here,we act on components actively
[b] the first is just the **differential operator representation or field-representation**

3.6 Implication of unitarity of S-matrix

3.6.1 The optical theorem

Unitarity of S-matrix is a result of unitarity of quantum theories: the basis must be complete.For S-matrix,this is equivalent to **completeness of in/out state.**

In general:

$$S = 1 + i\mathcal{T}, \mathcal{T}_{\beta\alpha} = -2\pi\delta^4(p_\beta - p_\alpha)\mathcal{M}^{NR}_{\beta\alpha} = (2\pi)^4\delta^4(p_\beta - p_\alpha)\mathcal{M}^{LR}_{\beta\alpha}$$

The uniratrity of S-matrix implies:

$$1 = S^\dagger S = (1 - i\mathcal{T}^\dagger)(1 + i\mathcal{T}) \Rightarrow i(\mathcal{T}^\dagger - \mathcal{T}) = \mathcal{T}^\dagger\mathcal{T}$$

Sandwithing between in/out sates and insert one in/out state in $\mathcal{T}^\dagger\mathcal{T}$,we obtain the **general optical theorem**:

The Optical theorem

- General optical theorem,NR[a]:

$$i(\mathcal{M}^{NR}_{\beta\alpha} - \mathcal{M}^{NR*}_{\alpha\beta}) = \int d\gamma 2\pi\delta^4(p_\beta - p_\alpha)\mathcal{M}^{NR*}_{\gamma\alpha}\mathcal{M}^{NR}_{\beta\gamma} \tag{3.18}$$

The usually optical theorem relate the **forward scattering amplitude to total rate for all reactions**:

- NR:

$$Im\mathcal{M}_{\beta\alpha} = -\pi\int d\beta\delta^4(p_\beta - p_\alpha)|\mathcal{M}_{\beta\alpha}|^2$$

The total rate for all reactions of initial state is:

$$\begin{aligned}\Gamma_{\alpha,total} = \int d\beta\frac{d\Gamma(\alpha\to\beta)}{d\beta} &= (2\pi)^{3N_\alpha-2}V^{1-N_\alpha}\int d\beta\delta^4(p_\beta - p_\alpha)|\beta\alpha|^2 \\ &= -\frac{1}{\pi}(2\pi)^{3N_\alpha-2}V^{1-N_\alpha}Im\mathcal{M}^{NR}_{\alpha\alpha}\end{aligned} \tag{3.19}$$

  - Decay:

$$Im\mathcal{M}^{NR}_{\alpha\alpha} = \frac{-1}{2}\Gamma_{\alpha,total}, Im\mathcal{M}^{LR}_{\alpha\alpha} = E_\alpha\Gamma_{\alpha,total}$$

  - Cross section:

$$Im\mathcal{M}^{NR}_{\alpha\alpha} = \frac{-u_\alpha\sigma_\alpha}{16\pi^3}, Im\mathcal{M}^{LR}_{\alpha\alpha} = 2(u_\alpha E_1E_2)\sigma_{\alpha,total}$$

[a] LR: $\mathcal{M}^{LR}_{\beta\alpha} - \mathcal{M}^{LR*}_{\alpha\beta} = i\int d\gamma(2\pi)^4\delta^4(p_\beta - p_\alpha)\mathcal{M}^{LR*}_{\gamma\alpha}\mathcal{M}^{LR}_{\beta\gamma}$

Remark 23
- *For physical implications like **diffraction peak** can refer to [WeinbergI,p148].*
- ***CPT**:since CPT is an exact antiunitary symmetry:*

$$\mathcal{M}_{\{p_i,\sigma_i,n_i\},\{p_i,\sigma_i,n_i\}} = \mathcal{M}_{\{p_i,-\sigma_i,n_i^c\},\{p_i,-\sigma_i,n_i^c\}}$$

the optical theorem implies,exactly:

$$\Gamma_{\{p_i,\sigma_i,n_i\},total} = \Gamma_{\{p_i,-\sigma_i,n_i^c\},total}$$

*For decaying,spin-rotation symmetry implies σ-indepence,thus **particle and antiparticle have same decay rate,as well as mass,spin.***

- *In field theory,the optical theorem will have more implications.Since this theorem is non-perturbative,and the **both side have different order** in perturbation theory,thus **the optical theorem relate the low-order calculations to higher order calculations.**Especially the classical result(tree level) is related to results in 1-loop level.*
- *In field theory,there's a more efficient method to calculate $Im\mathcal{M}$,known as **cutting rules**,it will give rise many physical implications like **the polarization-sum which is measured must be related to the form of propagators.** and will finally require field theory to**have gauge invariance to preserve unitarity.**For more detail,refer to [Schwartz,24.1]*

3.6.2 Kinetic theory

Similarly,we have:

$$\int d\beta\delta^4(p_\beta - p_\alpha)|\mathcal{M}_{\beta\alpha}|^2 = -\frac{1}{\pi}Im\mathcal{M}_\alpha\alpha = \int d\beta\delta^4(p_\beta - p_\alpha)|\mathcal{M}_{\beta\alpha}|^2$$

This implies:

$$\int d\beta c_\alpha\frac{d\Gamma(\alpha\to\beta)}{d\beta} = \int d\beta c_\beta\frac{d\Gamma(\beta\to\alpha)}{d\alpha}, c_\alpha = (\frac{V}{(2\pi)^3})^{N_\alpha}$$

The change of probability of finding system at state α is:

$$\frac{dP_\alpha}{dt} = \int d\beta P_\beta \frac{d\Gamma(\beta \to \alpha)}{d\alpha} - P_\alpha \int d\beta \frac{d\Gamma(\alpha \to \beta)}{d\beta} \equiv 0$$

This is **thermal equilibrium.** The change of entropy is:

$$\begin{aligned}
-\frac{d}{dt}\int d\alpha P_\alpha \log(P_\alpha/c_\alpha) &= -\int d\alpha d\beta (\log(P_\alpha/c_\alpha)+1)[P_\beta \frac{d\Gamma(\beta \to \alpha)}{d\alpha} - P_\alpha \frac{d\Gamma(\alpha \to \beta)}{d\beta}] \\
&= \int d\alpha d\beta P_\beta \log(\frac{P_\beta c_\alpha}{P_\alpha c_\beta})\frac{d\Gamma(\beta \to \alpha)}{d\alpha} \geq \int d\alpha d\beta [\frac{P_\beta}{c_\beta} - \frac{P_\alpha}{c_\alpha}] c_\beta \frac{d\Gamma(\beta \to \alpha)}{d\alpha} \\
&= \int d\alpha d\beta \frac{P_\beta}{c_\beta}(c_\beta \frac{d\Gamma(\beta \to \alpha)}{d\alpha} - c_\alpha \frac{d\Gamma(\alpha \to \beta)}{d\beta}) = 0
\end{aligned}$$

This is the **Boltzmann H-theorem**,this is the result of unitarity of S-matrix.Note in relativistic QFT,S-matrix describe transition between states or scattering between particles.

Remark 24 *These implies we are working at some fixed temperature,in reletivistic QFT,since particles are allowed to be scattered to infinity,the temperature is effectively T=0K,but for condensed matter physics:*$T \neq 0K$ *the formalism of field theory will have minor difference,this is known as* ***finite-temperature field theory***

Part II

Quantum theory of fields

Chapter 4

Local operator formalism:2nd quantization

The detailed structure of H is not clear by now,in general it can be **defined by its matrix elements** between states in the Hilbert space.For theories with particle number not fixed,the Hilbert space will be the **Fock space**,and the states consists of complete[1] multi-physical-particle[2] states.

It can be shown that **any** such state can be expressed as a function of certain **local** operators that **create/annihilate single-particles.**This is known as the **2nd quantization formalism.**

Advantage of 2nd quantization formalism:Cluster decomposition principle

When Hamiltonian operator H is expressed as a sum of products of creation and annihilation operators with non-singular coefficients,then the S-matrix will automatically satisfy the **cluster decomposition principle:distant experiment decouples(uncorrlated)**.
This is also true for non-relativistic statistical mechanics,where particle number are fixed,or with unfixed number of quasiparticles.
The cluster decomposition is a kind of **locality-axiom** in relativistic QFT,which is actually not experimentally qualified.
Nonlocal relativistic QFT have trouble with obtaining **Poincare invariant S-matrix**.The **combination of SR+QM+Locality** constrain the possible form of local relativistic theories,it lead to **effective field theories.**

4.1 Identical particles and statistics

The Hilbert space is spanned by states containing 0,1,2,... **free particles**.These can be **free/in/out** states,in general,they have different set of creation/annihilation operators.

In quantum theories,**identical particles are indistinguishable**,this implies exchanging/braiding states between identical particles will give no physical consequnces.This means identical-multiplet(tensor product of single particle state with same species) will furnish **unitary representation of the operator exchanging**[3] **states** in the subspace of identical particles.

Remark 25 *In general,these operators form the* ***braid group****,being the fundamental group of the configuration space of states of identical particles in different dimensions of state-space(usually taken as x-space of p-space,forming manifold of certain dimension).The representation of this group gives a common definition of anyons.*

However,in reletivistic (3+1)d QFT,the only physical different types of anyons are fermions and bosons,as 1D representation of the permutation group.

Even in (2+1)d/(1+1)d theories,many theories can be described completely by bosons and fermions with certain redundancy included.In fact many effective theories in these cases are CFT/TQFT,may or may not describing the anyon-quasiparticles.

[1] unitarity

[2] When interaction is included,H, H_0 have same spectrum.

[3] change the **ordering in the tensor product**,these differently ordered tensor-product state expand the identical particle subspace for each species

In (3+1)d,we consider only fermion and boson.There's **spin-statistic connection**:bosons/fermions have (half-)integer-spin.This will be **manifested**[4] in relativistic QFT constructions.

Boson/Fermi-statistics

Consider **basis of the Fock space** as a tensor-products(free/in/out) of single-particles-state(as PUIR).The discussion is valid in **any choice of basis.**
The ordering in tensor products matter,the exchange operator change the ordering in tensor products:

$$|a\rangle \otimes |b\rangle = |a,b\rangle \neq |b\rangle \otimes |a\rangle = |b,a\rangle, \tau|a,b\rangle \equiv |b,a\rangle$$

The principle of identical particle implies that exchange two state in tensor product with same particle type labelled by n[a]

$$\begin{aligned}(...) \otimes |\vec{p}', \sigma', n\rangle \otimes (...) \otimes |\vec{p}, \sigma, n\rangle \otimes (...) &= \tau^{(n)}(...) \otimes |\vec{p}, \sigma, n\rangle \otimes (...) \otimes |\vec{p}', \sigma', n\rangle \otimes (...) \\ &= \alpha_n[(...) \otimes |\vec{p}, \sigma, n\rangle \otimes (...) \otimes |\vec{p}', \sigma', n\rangle \otimes (...)], \alpha_n \in U(1)\end{aligned}$$

- **Cluster decomposition principle** implies α_n is independent of other particle types n' and total particle numbers in the state.
- α_n is independent of spin-components,since the **1D** unitary representation of $SO(3)$ is the trivial one.
- α_n might depent on $p_1^\mu p_{2\mu}$ which is symmetric thus doesn't manifest.
- Thus α_n can only depend manifestly on the the particle type.
- α_n is just the representation of $\tau^{(n)}$:
$$\alpha_n \equiv U_{1D}(\tau^{(n)})$$

The crucial point is that:

$$\begin{aligned}\tau^{(n)2}(...) \otimes |\vec{p}, \sigma, n\rangle \otimes (...) \otimes |\vec{p}', \sigma', n\rangle \otimes (...) &= \alpha^2[(...) \otimes |\vec{p}, \sigma, n\rangle \otimes (...) \otimes |\vec{p}', \sigma', n\rangle \otimes (...)] \\ &\equiv (...) \otimes |\vec{p}, \sigma, n\rangle \otimes (...) \otimes |\vec{p}', \sigma', n\rangle \otimes (...)\end{aligned}$$

The last line follows from the **topology of configuration space of dimension higher than 2**:

$$\tau^2 \equiv 1$$

Physically,we consider hard-core particles,thus they can be considered as singularities in the configuration space.In dimension higher than 2,the pathes can go around these singularities with the help of extra dimension,thus these singularities are implicit: **all loops are homotopic to the identity.** or the fundamental group generated by τ alone is trivial.[b]:

- The fundamental group of configuration space in dimension higher than 2D is $S_n = B_n/\{\tau_i^2 = 1\}$,the **1D ordinary**[c] unitary representation of S_n just correspond to bosons($\alpha_n = 1$),and fermions($\alpha_n = -1$),this follows from $\alpha_n^2 = U(\tau^2) = U(1) = 1$.
- The tensor product state itself will not depend on the path generate it from the identity.[d]

[a]same mass,spin,charge,all that.Except that the particles can have different position,momentum,spin orientation
[b]The generators will satisfy extra relations like Yang-Baxter relation,thus the whole fundamental group of configuration space is nontrivial.
[c]higher-dim representation of S_n is known as **parastatistics** which is **equivalent to 1D cases in categorical sense**
[d]This will not be the case of anyons,where the pathes are not homotopic due to singularities on the configuration space,thus the state will depend on the path,this is consistent with representation of braid group

[4]This connection is generally a topological property,the connection between spin-statistic can be altered by topological defects whic h contribute to angular momentum

Convention of ordering,statistics,and normalization

- For exchange of different particle type,in principle we can separate the particle types to group states with same type to form a braid-multiplet,and tensor product the multiplet by certain fixed order[a].Then there's no problem of exchanging of different particle type.
- In order to deal with **internal symmetry** where different particle type may form a larger multiplet.We adopt the convention: the exchange of **any** particle is symmetric unless the exchange involve **odd** number of fermions.This means even number of fermions may be considered as a boson,
- This will be **manifested in QFT constructions**,where operators with even number of fermion-operators are considered as a bosonic operator.

This lead to the convention of normalization,let q denote all quantum numbers of a single particle:

$$\begin{aligned}(\Phi_0,\Phi_0)=1,(\Phi_{q'},\Phi_q)=\delta(q'-q)=\delta^3(p'-p)\delta_{\sigma'\sigma}\delta_{n'n}\\ (\Phi_{q_1'\dots q_M'},\Phi_{q_1\dots q_N})=\delta_{NM}\sum_{\sigma\in S_n}\epsilon(\sigma)\prod_i\delta(q_i-q'_{\sigma(i)})\end{aligned}\tag{4.1}$$

- The fisrt line gives normalization of single-particle state,we are assuming **the vacuum doesn't have degeneracy**[b],the normalization of single-particle state is non-relativistic,which is more formal.These single-particle state form the **complete orthonormal basis** of $\mathcal{F}_1$-subspace,this gives the single-particle part of the spectrum of H/H_0.[c]
- The second line gives normalization of multi-free-particle state,including in/out/free states.δ_{NM} implies states with fixed particle number form complete orthonormal basis of $\mathcal{F}_n$-subspace,these gives **only part of** the multi-particle spectrum.Note the sum comes from our convention of ordering,$\epsilon(\sigma)=-1$ only when the permutation involve odd number of fermions.If we fix ordering of particle-type Hilbert space,then there won't be any sum of permutations.
- The whole spectrum will also include multi-particles states with **interactions**,since the free-states form a basis,these multi-particle-interacting states will be linear combination of these states and generally **can't be expressed as a tensor product.**

[a]For example: list the states like $\mathcal{H}_{photon}\otimes\mathcal{H}_{electron}\otimes\dots$ and just restrict the exchange-operator to each subspace
[b]If it does,for example when symmetry is SSB,**different vacuum are taken as orthonormal** as usually
[c]again,this need the **renormalization**

Remark 26 ***Possible ambigurity:can the braid group be intrinsically projectively represented?***,*more specificly for S_n,will the $\tau^2=1$ always hold in quantum level?*

This may also be seen from $H^2(B_n,U(1))$ or more specificly here $H^2(S_n,U(1))$.Will the projective representation of B_n lead to more general concept of statistics?

A clue may follow from ***twisted quantum double*** *where the Yang-Baxter relation is modified to quasi-Yang-Baxter equation.With* ***3-cocycle and conjugated 2-cocycles*** *presented.*

4.2 2nd quantization formalism

The particle scheme(spectrum) of relativistic QFT is described by single/multi-particle states,these states can be **equivalently** described by **the algebra of creation and annhilation operators**,which allows us to **construct the basis of Fock space thus the whole unitarity-theory.**

4.2.1 Creation and annihilation operator

Define the creation operator[5] $a^\dagger(q)=a^\dagger(\vec{p},\sigma,n)=a_q^\dagger$ by[6]:

$$a^\dagger(q)\Phi_{q_1\dots q_N}\equiv\Phi_{qq_1\dots q_N}$$

[5]the particle is on-shell,as needed by unitarity
[6]Take non-interacting state as an example,same for in/out states.In general the 3 set of $a,a^\dagger$ will be distinguished

Conversely:

$$\Phi_{q_1...q_N} = a^\dagger(q_1)...a^\dagger(q_N)\Phi_0$$

The annihilation operator is defined as Hermitian conjugate,with $\pm = \pm 1$ for boson/ferimon respectively[7] :

$$(\Phi_{q'_1...q'_M}, a(q)\Phi_{q_1...q_N}) \equiv (a^\dagger(q)\Phi_{q'_1...q'_M}, \Phi_{q_1...q_N}) = (\Phi_{qq'_1...q'_M}, \Phi_{q_1...q_N})$$
$$\Rightarrow a(q)\Phi_{q_1...q_N} = \sum_{r=1}^{N}(\pm)^{r+1}\delta(q-q_r)\Phi_{q_1...q_{r-1}q_{r+1}...q_N}$$

This vacuum then can be defined as[8]:

$$a(q)\Phi_0 = 0, \forall n$$

The above defines creation/annihilation operators with their **matrix element** between free-multiparticle-state basis.The more fundamental/algebraic definition can be obtained from these matrix-elements:

Algebraic definition of creation/annhilation operators

From the matrix elements[a]:

$$a(q')a^\dagger(q)\Phi_{q_1...q_N} \equiv \delta(q'-q)\Phi_{q_1...q_N} + \sum_{r=1}^{N}(\pm)^{r+2}\delta(q'-q_r)\Phi_{qq_1...\not{q}...q_N}$$
$$a^\dagger(q)a(q')\Phi_{q_1...q_N} \equiv \sum_{r}^{N}(\pm)^{r+1}\delta(q'-q_r)\Phi_{qq_1...\not{q}...q_N}$$

This lead to[b]:

$$[a(q'), a^\dagger(q)]_\mp = \delta(q'-q), [a^\dagger(q'), a^\dagger(q)]_\mp = 0, [a(q'), a(q)]_\mp = 0 \tag{4.2}$$

This is the **algebraic definition of creation/annihilation operators.**

- The operators manifest the spectrum of the theory,they **defines** the free-particle basis for Fock space.
- The $\mp$ indicate the boson/fermion statistic.For fermions,we have $a^2(q) = a^{\dagger 2}(q) \equiv 0$,this is the **Pauli exclusion principle**.
- Since the basis are constructed out of these operators,matrix element of **any** element can be expressed into functions of products of these operators.This is known as **2nd quantization**.[c]
- These algebra hold for any quantum theory with Hilbert space defined as **Fock space**: expanded by bases being tensor products of certain states,**independent** of any pre-exsisting classical field theory.
- These operators are **local**,in sense of **cluster decomposition**.In general,the 3-copies of **in/out/free** algebra would be **distinguished**.But with **same vacuum.**[d]
- The canonical/operator quantization of classical field will have deep connection to 2nd quantization formalism,we will have to **construct the algebra of creation/annihilation** to get **unitary** theory.

[a]a slash on components means removed

[b]$[,]_\mp$ stands for commutator and anticommutator.The 2nd comes from definition of $a^\dagger$ directly,the 3rd follows from hermitian conjugation

[c]The name 2nd quantization means 2nd type of quantization,it's the generalization of 1st quantization,where the Hilbert space are those with fixed number of D.O.F,in 2nd quantization the **D.O.F will be infinite**.Thus the 2nd quantization usually means quantization of classical fields,which have infinite D.O.F.**The 1nd quantization(usually non-relativistic QM) can be restored by restrict to subspaces of Fock space**

[d]**Caution:in principle there's difference between vacuum with interaction and free-vacuum which is for in/out state and free state resctictively.But since in/out state are related to free state by $\Omega(\mp\infty)$,there's only a phase difference for these two kinds of vacuum,thus annihilated by all a.We do distinguish them in quantization procedure**

[7]Notations like this will always chooes the **upper sign for boson**,proof of this definition is rather trivial,by dividing the permutation into two parts:pick r that is permuted to 1,then and the permutation reduce to permutations with one-state less.This involve:$\epsilon(\sigma) = (\pm)^{r-1}\epsilon(\bar{\sigma})$,then show the definition gives right orthonormal condition for both sides

[8]This doesn't rule out degeneracy

4.2.2 2nd quantization of operators

Any operator can be put into the form:

$$\hat{O} = \sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\int dq_1'...dq_N'dq_1...dq_M\ C_{NM}(q_1'...q_N', q_1...q_M)\ \hat{a}^\dagger(q_1')...\hat{a}^\dagger(q_N')\hat{a}(q_M)...\hat{a}(q_1) \tag{4.3}$$

- The coefficients are chosen[9] such that the matrix element of O coincide.This idea also lead to **renormalization:** when interaction is included,we have to find the **true/renormalized** operator that contain the particles in **physical spectrum**,which have the **same matrix element as operators that create free particle.**
- The convention of the ordering of $a^\dagger, a$ is **normal ordering**:with all creation operator to the left.The ordering of q_i is for convenience[10],since ordering of annihilation operators gives only sign difference,which can be abosorbed into definition of coefficients.

Examples would include:

- **Addictive** operators,a class of this kind of operators would be addictive conserved operators which follows from Noether's theorem,also being generators of the coorespoding Lie algebra: (H, J, P):

  $$F\Phi_{q_1...q_N} = (f(q_1) + ... + f(q_N))\Phi_{q_1...q_N} \Rightarrow F = \int dqf(q)a^\dagger(q)a(q)$$

  We may identify $N_q = a^\dagger(q)a(q)$,as the operator counting number of q-states,this follows from the fact:

  $$[N_q, a(q)] = -a(q), [N_q, a(q)] = +a_q$$

  For fermions,$N^2 = N \Rightarrow E(N) = 0, 1$,this is just **Pauli exclusion principle**.
  Especially $H_0 = \int dq\omega_k a^\dagger(q)a(q), \omega_k = \sqrt{k^2 + m_n^{Bare2}}$.is the usually on-shell/relativistic dispersion relation,for free particle,$m_0 = m_{Bare} = m_{phy} = m_{pole}$.
- Interactions:
- Creation/Annihilation field-operators $\psi^{\mp} \propto a^{\pm}e^{-\pm ip\cdot x}$ will be used to construct field theory.

4.3 Transformation properties of creation/annihilation operators

The transformation properties of $a, a^\dagger$ can be obtained from the **transformation of single-particle state.**

Transformation properties of creation/annihilation operators

- Poincare group[a]:

  $$U(\Lambda, \alpha)a^\dagger(\vec{p}\sigma n)U^{-1}(\Lambda, \alpha) = e^{-i(\Lambda p)\cdot\alpha}\sqrt{\omega_{\Lambda p}/\omega_p}\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))a^\dagger(\vec{p}_\Lambda\sigma' n) \tag{4.4}$$

  - For massive case,D is the spin-j PUIR of $SO(3)$ of Wigner rotation.
  - For massless case,D is $\exp(i\sigma\theta(\Lambda, p))$,with θ given by Wigner rotation implicitly.
- C,P,T:
  - Massive:

  $$\begin{aligned} Ca^\dagger(\vec{p}\sigma n)C^{-1} &= \xi_n a^\dagger(\vec{p}\sigma n^c) \\ Pa^\dagger(\vec{p}\sigma n)P^{-1} &= \eta_n a^\dagger(-\vec{p}\sigma n) \\ Ta^\dagger(\vec{p}\sigma n)T^{-1} &= \zeta_n(-1)^{j-\sigma}a^\dagger(-\vec{p} - \sigma n) \end{aligned} \tag{4.5}$$

[9] This can be proven by induction on N,M.The proof is obvious.
[10] A hint for this ordering: **hermitian conjugation reverse the ordering**

- Massless: σ now means **helicity**,and it reverse sign under P, C,is invariant under T.Just oppesite to the transformation of σ in massive case.for P,T,**extra sign** determined by $sign(p_2)$ are included.

These transformation properties are important,they constrain the possible form of relativistic field-operators.

[a]Note that $U(\Lambda)\Phi_0 = 0$

Remark 27 *Different basis are also used in CMP,they are related by the* ***unitary basis transformation.***

4.4 Locality:cluster decomposition principle

It may be considered as a principle of physics that **distant** experiments[11] are **uncorrlated**.This makes our experiments 'valid',and our theories local,the generality of our theories then follow from spacetime symmetry,which may or may not be exact.

For S-matrix and relativistic QFT,this principle implies:

$$S_{\sum_i \beta_i, \sum_i \alpha_i} \to \prod_{cluster-i} S_{\beta_i \alpha_i}$$

The $\sum$ here means tensor product,**cluster**-i means the particles in each cluster are considered in one **'local'**-process.Particles in different cluster are considered as participating in different local-processes.The factorization means **process in different clusters are uncorrlated.**

A more useful form of cluster decomposition of S-matrix would be:

Connected part of S-matrix

Define the **connected part** of the S-matrix $S^C_{\beta\alpha}$ [a] recursively[b]:

$$S_{\beta\alpha} = \sum_{PART} (\pm) S^C_{\beta_1\alpha_1} S^C_{\beta_2\alpha_2} ... = S^C_{\beta\alpha} + \tilde{\sum_{PAR}} (\pm) S^C_{\beta_1\alpha_1} S^C_{\beta_2\alpha_2} ...$$

The sum is over all possible partition of particles into uncorrlated clusters.The sign is minus only when the partition need to interchange odd number of fermions.
The recursion may start with α, β containing only single-particle:

$$S^C_{q'q} = S_{q'q} = \delta(q' - q)$$

No scattering happen for the single particles.The physical particle is 'free'/**stable.**[c] If the particle is not stable,we can consider more terms like $S_{1\to 3}$.
For 2-2 reaction:

$$S_{q'_1 q'_2, q_1 q_2} = S^C_{q'_1 q'_2, q_1 q_2} + \delta(q'_1 - q_1)\delta(q'_2 - q_2) \pm \delta(q'_1 - q_2)\delta(q'_2 - q_1)$$

The last two term is the normalization:$\delta(\alpha - \beta)$ thus the connected part is $(S-1)_{\beta\alpha}$.
The general case is more complicated,$(S-1)_{\beta\alpha} \propto \delta^4(p_\beta - p_\alpha)\mathcal{M}_{\beta\alpha}$ will contain more terms:

$$|\mathcal{M}^C_{\beta\alpha}|^2 < |\mathcal{M}_{\beta\alpha}|^2$$

With $S^C_{\beta\alpha}$,the cluster principle is interpreted as:$S^C_{\beta\alpha} = 0$ if any particle is not in the cluster of the rest(far away):

$$S_{\beta\alpha} = \sum^{1} (\pm) S^C_{\beta_{11}\alpha_{11}} S^C_{\beta_{12}\alpha_{12}} ... \times \sum^{2} (\pm) S^C_{\beta_{21}\alpha_{21}} S^C_{\beta_{22}\alpha_{22}} ... \tag{4.6}$$

The $\sum^{(j)}$ means partition in i-cluster alone.Using this factorization and the definition of $S^C_{\beta\alpha}$ this implies the wanted cluster decomposition:$S_{\sum_{cluster-i}(\beta_i,\alpha_i)} = \prod_{cluster-i} S_{\beta_i,\alpha_i}$.

[11]actually the physical experiments involve noise,which will lead to **decoherence**,it's then natural to think that distant experiments are uncorrlated

In one cluster the partition can have many different ways,thus **process in same cluster do correlate/couple.But different clusters uncorrelate/decouple.**

[a]As noted before,experiments care only the connected process,and will be calculated perturbatively using **connected Feynman diagrams**

[b]The first term contain more particles and it is determined by S and S^C with less particles,we start with $S^C_{0,0} = 1, S^C_{1,1'} = S_{1,1'}$

[c]**caution:the free particle must be the normalized one,it give rise to exact propagator,including interaction-corrections.**

4.4.1 Implication of cluster decomposition

When considered in momentum space:

$$S^C_{x'_1x'_2...,x_1x_2...} = \int d^3p'_1d^3p'_2...d^3p_1d^3p'_2...S^C_{p'_1p'_2...,p_1p_2...}e^{ip'_1x'_1+ip'_2x'_2+...-ip_1x_1-ip_2x_2-...}$$

According to translational invariance,$S^C_{\beta\alpha} = \delta^4(p_\beta - p_\alpha)C_{\beta\alpha}$ cluster decomposition implies that when the difference between some x, x' goes to infinity,this term must vanish.However,if $C_{\beta\alpha}$ contain extra δ-functions,this condition won't be satisfied.

Thus cluster decomposition gives the constraint that:

$$S^C_{\beta\alpha} = -2\pi i\delta^4(p_\beta - p_\alpha)\mathcal{M}^C_{\beta\alpha}, no\ \delta - function\ in\ \mathcal{M}^C_{\beta\alpha}$$

The connected part of S-matrix,unlike S-matrix itself,contains only one δ-function.

Note that,in general,S^C will contain **various poles and branch-cuts representing the spectrum** but only one singulary δ comes from translational invariance

Remark 28
- *The technique of* ***decomposing quantities into connected part*** *have been vastly used.Like the cluster decomposition methods in statistical mechanics.*
- *The purpose of isolating the connected part of Green's functions,partition functions,resolvents is to* ***deal with objects with simple volumn dependence.***
- *For example,in box normalization,the δ^4 in S^C becomes VT*
- *In theory of noise,cluster decomposition amouts to* ***decompose the correlation function of several random variables into its cumulants.*** *The cumulants will be propotional to N which is the number of independent flunctuations.*

4.4.2 Constraint on the interactions

The cluster decomposition of S-matrix then naturally lead to constraints on the structure of the Hamiltonian,equivalently on $h_{NM}(q'_1...q'_N, q_1...q_M)$:

$$h_{NM} = \delta^3(p'_1 + ... + p'_N - p_1 - ... - p_M)\tilde{h}_{NM}$$

and $\tilde{h}_{NM}$ contains no delta-function.

As we will see,this constraint lead directly to local field-operators.Loosely speaking is the x-space version(not changing of basis,some thing like **Fourier transformation**) of $a, a^\dagger$.

Sketch of the proof

The matrix element can be factored as:

$$(\Phi_\beta, T\{V(t_1...V(t_n))\}\Phi_\alpha) = \sum_{clusterings}(\pm)\prod_{j=1}^{v}(\Phi_{\beta_j,T\{V(t_{j1}...V(t_{jn_j}))\}\Phi_{\alpha_j}})_C$$

The sum is over all ways of splitting up the in/out particles and V operators into v clusters(v=1,...,n),with n_j V opeartors and (α_j, β_j) all in the jth cluster.C means connected,represented by **connected Feynman diagrams**:

With a little algebra,the S-matrix element is:

$$S_{\beta\alpha} = \sum_{PART} (\pm) \prod_{j=1}^{v} \sum_{n_j=0}^{\infty} \frac{(-1)^{n_j}}{n_j!} \int_{-\infty}^{\infty} dt_{j_1}...dt_{j_n} (\Phi_{\beta_j}, T\{V(t_{j_1})...V(t_{j_{n_j}})\}\Phi_{\alpha_j})_C$$

Thus:

$$S^C_{\beta\alpha} \equiv \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dt_{j_1}...dt_{j_n} (\Phi_\beta, T\{V(t_1...V(t_n))\}\Phi_\alpha)_C \tag{4.7}$$

This greatly simplifies perturbation calculation:**the wanted connected part of S-matrix is calculated from connected part of Green's functions**.We will derive the formal LSZ formula to obtain S-matrix from Green's functions.

We should note that the Green's functions are richer:

- It's pole and branch-cut structure gives the whole spetrum of the theory.
- The whole procedure of renormalization will be dealing with renormalization of Green's functions.

The assumption:h_{NM} contain only one delta-function is equivalent to that V_{NM} contain only one delta-function,since H_0 meets the requirement automatically.

The technique of Feynman diagram implies the V(t) at each vertices contribute **only one delta-function to assure momentum-conservation.**Then the total number of delta-functions remaining in the integral will be:

$$V - I + L = C \equiv 1$$

Thus there's only one delta-function in S^C required by cluster decomposition if we assume V_{NM} contain only one delta-function as required by translational invariance.

This implies,**when we use Feynamn rules to calculate $\mathcal{M}^C$ we will contain no delta-functions.**

2-body scattering implies 3-body scattering

For simplicity,drop other indices,the exsisting of 2-body scattering:

$$v_{2,2}(p_1'p_2', p_1p_2) = V_{p_1'p_2'p_1p_2}$$

$v_{2,2}$ contain one delta-function,the 3-body scattering is then:

$$V_{p_1'p_2'p_3',p_1p_2p_3} = v_{3,3}(p_1'p_2'p_3', p_1p_2p_3) + v_{2,2}(p_1'p_2', p_1p_2)\delta^3(p_3' - p_3) \pm permutations$$

If $V_{3\to3} = 0$ then $v_{3,3} = -v_{2,2}\delta^3 + ...$ will contain 2 delta-function thus $V_{2\to2} \neq 0 \Rightarrow V_{3\to3} \neq 0$

Implications of this kind can also been manifested using Feynman diagrams,since the cluster decomposition is manifested by Feynman diagrams.

Chapter 5

Local field operators:Free fields

5.1 Highway to fields:Creation/Annhilation fields

As shown before[1]:

- SR+QM lead to constraints[2]:

$$
\begin{aligned}
H = H_0 + V,\ V_I(t) = \int d^3x \mathcal{H}_I(\vec{x}, t) \\
U_0(\Lambda, a)\mathcal{H}_I(x)U_0^{-1}(\Lambda, a) = \mathcal{H}_I(\Lambda x + a) \\
[\mathcal{H}_I(x), \mathcal{H}_i(x')] = 0, \forall (x - x')^2 > 0
\end{aligned}
\tag{5.1}
$$

- The 2nd quantization formalism combining cluster decomposition principle lead to constraint on $\mathcal{H}_I$:it a **function $\mathcal{H}_I(a^\dagger, a)$ of products of $a^\dagger, a$ with coefficients contain only one delta-function from translational invariance.**

The combination of these two set of constraint lead naturally to **fields**:

> **Creation/Annihilation local-field-operators**
>
> The definition of creation/annihilation local-field-operators are[a]:
>
> $$
> \begin{aligned}
> \psi_l^-(x) = \sum_{\sigma_n n} \int d^3p v_l(x; \vec{p}\sigma n) a^\dagger(\vec{p}\sigma n) \\
> \psi_l^+(x) = \sum_{\sigma_n n} \int d^3p u_l(x; \vec{p}\sigma n) a(\vec{p}\sigma n)
> \end{aligned}
> \tag{5.2}
> $$
>
> - These may be considered as some x-space version of $a^\dagger, a$.As we will see these field operators can be used to construct $|x\rangle$ rather than $|p\rangle$
> - We then use field operators instead of $a, a^\dagger$ to construct $\mathcal{H}^b$,to make sure $\mathcal{H}[\psi^-(x), \psi^+(x)]$ is indeed a **Poincare-scalar-field/density**,we demand these field-operators also transform properly under Poincare group:
>
> $$U_0(\Lambda, a)\psi^\mp U_0^{-1}(\Lambda, a) = [\mathfrak{D}(\Lambda^{-1})\psi^\mp](\Lambda x + a) \tag{5.3}$$
>
> Where $(\mathfrak{D}(\Lambda^{-1})\psi^\pm)_l \equiv \sum_{\bar{l}} \mathfrak{D}_{l\bar{l}}(\Lambda^{-1})\psi^\pm_{\bar{l}}$.Here the $\psi^\pm$ are not basis but rather **Lorentz-tensors**,it's a multiplet labelled by l,they furnish **finite-dim**[c] **ordinary** representation of $SL(2, \mathbb{C})$.Note that by certain similar transformation,the representation is the same for both $\pm$.
>
>   - This then gives **transformation properties of the coefficients**,which will be crucial in construction of specific field-operators.

[1]**Caution:the fields are defined in I-picture just as V_I indicated,in principle with interactions the H-picture field operators have different form and in general can't be expansioned into $\psi^\pm$ for all time.But in I-picture,the field operator have the same form of field(non-interacting) operators in H-picture.Since we assume interaction for most of time,the 'free' is somewhat 'I-picture',similar for ground state:**$|0\rangle = |Vacuum\rangle_I, |\Omega\rangle = |Vacuum\rangle_H$

[2]Here Λ may also include $\mathcal{P}$,which interchange **chirality**

  - The reasion for restricting to finite-dim representation is related to **renormalizability**

- Since the particles are PUIR of $ISO(1,3)$,$D_{l\bar{l}}$ in general is reducible,with each block being the corresponding irreducible representation of $SL(2,\mathbb{C})$ for each particle type.**Thus the field-operator for specific particle type is indeed IR.**
  - An important caution is that different particle types $n=(m,s;q;(t,t_3),...)$ can correspond to same irreducible representation of $SL(2,\mathbb{C})$,determined by only **spin**[d].This is to be manifested by relation between $SL(2,\mathbb{C})$ and the little groups of particles.
  - Especially,particles and antiparticles are embedded into same IR,since they have same PUIR.

[a]the $\mp$ refer to latter $e^{\mp ip\cdot x}$

[b]include $H_0=\int d^3x\mathcal{H}_0$,since $[U_0,H_0]=0$ automatically,again,we work in I-picture,where the ψ_I are recognized as free fields ψ_0

[c]Since the group is not compact,the matrix D will **not be unitary**

[d]**embedd particle into field**

The reason to define $\psi^\pm$ as Lorentz tensors is to construct scalar-$\mathcal{H}_I$ as:

$$\mathcal{H}_I(x)=\sum_{NM}\sum_{l'_1...l'_N}\sum_{l_1...l_M}g_{l'_1...l'_N,l_1...l_M}\psi^-_{l'_1}(x)...\psi^-_{l'_N}(x)\psi^+_{l_1}(x)...\psi^+_{l_M}(x)$$

such that:

$$\int d^3x\sum_{NM}\sum_{l'_1...l'_N}\sum_{l_1...l_M}g_{l'_1...l'_N,l_1...l_M}\psi^-_{l'_1}...\psi^-_{l'_N}\psi^+_{l_1}...\psi^+_{l_M}=\sum_{NM}\sum_{\sigma_n n}\prod d^3p_ih_{INM}a^\dagger...a \tag{5.4}$$

- This relation gives the connection between 2nd quantization formalism and local-field formalism.The $\int d^3x$ will give rise to translational-invariance delta-fucntion in h_{INM},not the only dependece of x in $\psi^\pm$ is from coefficients thus by translational invariance it will give factors of $e^{\pm ip\cdot x}$.The sum over σ_n,n,d^3p_n is absorbed into definition of ψ.
- The **constant** g-coefficients are Lorentz **covariant**[3],such that the $\mathcal{H}_I(x)$ is a scalar-field.In essence,this g-coefficients are known as **projective-tensors/invariant-symbols**,which is used to**project out the trivial representation out of tensor products of representations of $SL(2,\mathbb{C})$.**

5.1.1 Poincare transformation of the coefficient functions

Using unitarity of representations of compact little groups[4],the transformation rule of $a,a^\dagger$ can be put into the form[5]:

$$U_0(\Lambda,\alpha)a^{(\dagger)}(\vec{p}\sigma n)U_0^{-1}(\Lambda,\alpha)=e^{-i(\Lambda p)\cdot\alpha}\sqrt{\omega_{\Lambda p}/\omega_p}\sum_{\bar{\sigma}}D^{(*)}_{\sigma\bar{\sigma}}(W^{-1}(\Lambda,p))a^{(\dagger)}(\vec{p}_\Lambda\bar{\sigma}n)$$

Again,the D differs for massive/massless cases,we may denote D with j_n of the particle n.Using the identity:

$$\frac{d^3p}{\omega_p}=\frac{d^3p_\Lambda}{\omega_{\Lambda p}}\Rightarrow d^3p=d^3p_\Lambda\frac{\omega_p}{\omega_{\Lambda p}}$$

We have:

$$\begin{aligned}U_0(\Lambda,\alpha)\psi^\mp_l(x)U_0^{-1}(\Lambda,\alpha)&=\sum_{\sigma n,\bar{\sigma}}\int d^3p_\Lambda\begin{pmatrix}v_l(x;\vec{p}\sigma n)\\u_l(x;\vec{p}\sigma n)\end{pmatrix}e^{\mp i(\Lambda p)\cdot\alpha}D^{(*)}_{\sigma\bar{\sigma}}(W^{-1}(\Lambda,p))\sqrt{\omega_p/\omega_{\Lambda p}}a^{(\dagger)}(\vec{p}_\Lambda\bar{\sigma}n)\\&=\sum_{\sigma n,\bar{l}}\mathfrak{D}_{l\bar{l}}(\Lambda^{-1})\int d^3p_\Lambda\begin{pmatrix}v_{\bar{l}}(\Lambda x+\alpha;\vec{p}_\Lambda\sigma n)\\u_{\bar{l}}(\Lambda x+\alpha;\vec{p}_\Lambda\sigma n)\end{pmatrix}a^{(\dagger)}(\vec{p}_\Lambda\sigma n)\end{aligned}$$

This gives the transformation of coefficients:

[3] $gD...D=g'$

[4] $D_{\bar{\sigma}\sigma}(W)=D^{-1}_{\bar{\sigma}\sigma}(W^{-1})=D^\dagger_{\bar{\sigma}\sigma}(W^{-1})=D^*_{\sigma\bar{\sigma}}(W^{-1})$

[5] the combination is $(\psi^-,v_l,D^*,a^\dagger)$

Poincare transformation of coefficient functions

$$\begin{aligned}\sum_{\bar{l}}\mathfrak{D}_{l\bar{l}}(\Lambda^{-1})u_{\bar{l}}(\Lambda x+\alpha;\vec{p}_\Lambda\sigma n)&=\sqrt{\omega_p/\omega_{\Lambda p}}e^{+i(\Lambda p)\cdot\alpha}\sum_{\bar{\sigma}}D_{\sigma\bar{\sigma}}(W^{-1}(\Lambda,p))u_l(x;\vec{p}\bar{\sigma}n)\\ \sum_{\bar{l}}\mathfrak{D}_{l\bar{l}}(\Lambda^{-1})v_{\bar{l}}(\Lambda x+\alpha;\vec{p}_\Lambda\sigma n)&=\sqrt{\omega_p/\omega_{\Lambda p}}e^{-i(\Lambda p)\cdot\alpha}\sum_{\bar{\sigma}}D^*_{\sigma\bar{\sigma}}(W^{-1}(\Lambda,p))v_l(x;\vec{p}\bar{\sigma}n)\end{aligned}\tag{5.5}$$

- $\mathfrak{D}$ with label l denote ordinary representation of $SL(2,\mathbb{C})$,D with label σ denote projective representation of little groups of $ISO(1,3)/SO(1,3)$
- The transformation of coefficients $f_l(x;q)$ consists two parts:
  - RHS:The poincare transformation of the **particles embedded(represented by $a,a^\dagger$)** is induced by corresponding little groups.This part is indicated by q labels.
  - LHS:The Lorentz transformation is induced by it's **tensor** nature,the Poincare transformation on spacetime coordinates is induced by it's **field** nature.The field operator representation may be considered as **combining finite-dim tensor representation and differential-operator representation to form a infinite-dim unitary representation.**This part is indicated by (l,x) labels.
  - These two parts together induce the wanted Poincare transformation of the local-field-operator.Just send $\mathfrak{D}$ to RHS,with LHS identified as coefficients after transformation.
  - The sum over n is absent,this is promising,since $\sum_n$ in definition of ψ_l is the **direct sum** of representations for each particles:$\sum_n\sim\oplus\psi_l^{IR}$,we will actually focus on irreducible representations,except when $\mathcal{P}$ is included where the representation need to combine two chiral part.

A more convenient form is:

$$\begin{aligned}\sum_{\bar{\sigma}}D_{\bar{\sigma}\sigma}(W(\Lambda,p))u_{\bar{l}}(\Lambda x+\alpha;\vec{p}_\Lambda\sigma n)&=\sqrt{\omega_p/\omega_{\Lambda p}}e^{+i(\Lambda p)\cdot\alpha}\sum_{\bar{l}}\mathfrak{D}_{\bar{l}l}(\Lambda)u_l(x;\vec{p}\bar{\sigma}n)\\ \sum_{\bar{\sigma}}D^*_{\bar{\sigma}\sigma}(W(\Lambda,p))v_{\bar{l}}(\Lambda x+\alpha;\vec{p}_\Lambda\sigma n)&=\sqrt{\omega_p/\omega_{\Lambda p}}e^{-i(\Lambda p)\cdot\alpha}\sum_{\bar{l}}\mathfrak{D}_{\bar{l}l}(\Lambda)v_l(x;\vec{p}\bar{\sigma}n)\end{aligned}\tag{5.6}$$

5.1.2 Specific form of coefficient functions

More specificly,we can simplify the expression of $f_l(x;q)$:

Translation

Translation:feed in $\Lambda=1$,we obtain the factorization[6]:

$$\begin{cases}u_l(x;\vec{p}\sigma n)=\frac{1}{(2\pi)^{3/2}}e^{ip\cdot x}u_l(\vec{p}\sigma n)\\ v_l(x;\vec{p}\sigma n)=\frac{1}{(2\pi)^{3/2}}e^{-ip\cdot x}v_l(\vec{p}\sigma n)\end{cases}\tag{5.7}$$

Thus:

$$\psi_l^{\mp}=\sum_{\sigma n}(2\pi)^{-3/2}\int d^3p\begin{pmatrix}v_l(\vec{p}\sigma n)\\ u_l(\vec{p}\sigma n)\end{pmatrix}e^{\mp ip\cdot x}a^{\pm}(\vec{p}\sigma n)\tag{5.8}$$

- In this form we can see more clearly that we **embedded single-particle state(in p-space) into p-space-field** u/v,upon **Fourier transformation**,we get the x-space local field operators representing the particles.
- The d^3p implies the particle is **on-shell**,$s^{ip\cdot x}=e^{-i\omega_pt+i\vec{p}\cdot\vec{x}}$,the NR-normalization of Fourier transformation takes the more formal measure:$\int d^3p|\vec{p}\rangle\langle\vec{p}|=1,\langle\vec{x}|\vec{p}\rangle=(\frac{1}{\sqrt{2\pi}})^3e^{i\vec{p}\cdot\vec{x}}$.

[6]The 2π factor is a convention,which is different from **LR-normalization,which is actually more compact**

- We will use LR-normalization for calculation of $\mathcal{M}^{LR}$,most of time we will use more formal/symmetric NR-normalization
- Meaning of creation field operator can be manifested by:

$$\langle 0|\psi^+(x)|\vec{p}\sigma n\rangle = u(\vec{p}\sigma n)e^{-i\omega_p t}\langle\vec{x}|\vec{p}\rangle$$

  We may consider **the 'spectrum' of** $\psi^\pm(x)$ **contain single-particles**,and can create/annihilate such particles at x.[7] This equation gives the meaning of the spectum of operators and will be crucial to understand **renormalization of operators**:

  - For free field,the mass is taken as the bare mass
  - For interacting field,the renormalized field operator have **same residue** but with mass taken as the **physical mass.**

Remark 29 *This expansion can be understood in canonical quantization of classical* ***free*** *fields,with* ***relativistic linear equations.*** *These equations can be solved by linear combinations of* ***plane-waves with relativistic dispersion relation*** *multiplying* ***polarization functions.*** *Upon canonical quantization,the coefficients will be identified as creation/annihilation operators and the general solution will be identified as free-field-operators,the plane-wave becomes particle.*

The transformation is reduced to:

$$\begin{aligned}\sum_{\bar\sigma} D_{\bar\sigma\sigma}(W(\Lambda,p))u_{\bar l}(\vec{p}_\Lambda\sigma n) &= \sqrt{\omega_p/\omega_{\Lambda p}}\sum_{\bar l}\mathfrak{D}_{\bar l l}(\Lambda)u_l(\vec{p}\bar\sigma n)\\ \sum_{\bar\sigma} D^*_{\bar\sigma\sigma}(W(\Lambda,p))v_{\bar l}(\vec{p}_\Lambda\sigma n) &= \sqrt{\omega_p/\omega_{\Lambda p}}\sum_{\bar l}\mathfrak{D}_{\bar l l}(\Lambda)v_l(\vec{p}\bar\sigma n)\end{aligned} \tag{5.9}$$

Boost

Set $\vec{p} = \vec{k}$ the standard momentum,$\Lambda = L(q), \Lambda p \equiv q$,note for $\vec{k}, L(q)$ are different for massive/massless case.For both cases,$L(p) = L(k) = 1, W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(q)L(q) = 1$.Thus:

- Massive:

$$\begin{aligned}u_{\bar l}(\vec{p}\sigma n) &= \sqrt{m/\omega_p}\sum_l \mathfrak{D}_{\bar l l}(L(p))u_l(0\sigma n)\\ v_{\bar l}(\vec{p}\sigma n) &= \sqrt{m/\omega_p}\sum_l \mathfrak{D}_{\bar l l}(L(p))v_l(0\sigma n)\end{aligned} \tag{5.10}$$

- Massless:

$$\begin{aligned}u_{\bar l}(\vec{p}\sigma n) &= \sqrt{|\vec{k}|/\omega_p}\sum_l \mathfrak{D}_{\bar l l}(L(p))u_l(\vec{k}\sigma n)\\ v_{\bar l}(\vec{p}\sigma n) &= \sqrt{|\vec{k}|/\omega_p}\sum_l \mathfrak{D}_{\bar l l}(L(p))v_l(\vec{k}\sigma n)\end{aligned} \tag{5.11}$$

These equations imply:

- once we know how to construct u in standard frame,we can construct u at any $\vec{p}$ using the corresponding representation of standard Lorentz transformation.This is **similar to the induced representation.**We then reduce the construction of field to **constructing representations of Lorentz group and coefficient functions in standard frame.**Similarly,we will relate these standard-frame-coefficient-functions to **representation theory of little groups.**

[7]**Caution**:any operators with the **particles in the spectrum** can do the same thing,the spectrum of operators will be more clear in **Green's functions**

Embedding particles(little groups) into fields($SO(3) \subset SO(1,3)$)

Massive case

Set $\vec{p}=0$,$\vec{p}_\Lambda = 0, \Lambda = R, W(\Lambda,p) = R$:

$$\sum_{\bar{\sigma}} u_{\bar{l}}(0\bar{\sigma}n) D^{(j_n)}_{\bar{\sigma}\sigma}(R) = \sum_{l} \mathfrak{D}_{\bar{l}l}(R) u_l(0\sigma n)$$
$$\sum_{\bar{\sigma}} v_{\bar{l}}(0\bar{\sigma}n) D^{(j_n)*}_{\bar{\sigma}\sigma}(R) = \sum_{l} \mathfrak{D}_{\bar{l}l}(R) v_l(0\sigma n)$$

Infinitesimally,this is:

$$\begin{aligned} \sum_{\bar{\sigma}} u_{\bar{l}}(0\bar{\sigma}n) J^{(j_n)}_{\bar{\sigma}\sigma} &= \sum_{l} \mathcal{J}_{\bar{l}l} u_l(0\sigma n) \\ \sum_{\bar{\sigma}} v_{\bar{l}}(0\bar{\sigma}n) (-J)^{(j_n)*}_{\bar{\sigma}\sigma} &= \sum_{l} \mathcal{J}_{\bar{l}l} v_l(0\sigma n) \end{aligned} \tag{5.12}$$

Here $J^{j_n}, \mathcal{J}$ are unitary[8] representation of $SO(3)$-rotation generator J:

- For LHS,we have spin-j_n representation of the **little group** of massive particles.
- For RHS,we have in-general-**reducible** projective unitary representation of $SO(3) \subset SO(1,3)$.
- These equations means the coefficient functions specify how $D^{j_n}(R)$ is embedded as the irreducible block $\mathfrak{D}^{j_n}(R)$ into $\mathfrak{D}(R)$.More physically,they specify **how particle of spin-j is embedded into the field.**The Lorentz-tensor ψ as a representation of $SL(2,\mathbb{C})$ will decompose into irreducible representations of $SO(3)$ subgroup,with **each irreducible representation representing particle in the spectrum with corresponding spin.**

Massless case

The massless case is more complex,due to the peculiar form of $ISO(2) = T_2 \rtimes O(2)$.Similarly,we arrive at:

$$u_{\bar{l}}(\vec{k},\sigma)\exp(+i\sigma\theta(W,k)) = \sum_{l} \mathfrak{D}_{\bar{l}l}(W) u_l(\vec{k},\sigma)$$
$$v_{\bar{l}}(\vec{k},\sigma)\exp(-i\sigma\theta(W,k)) = \sum_{l} \mathfrak{D}_{\bar{l}l}(W) v_l(\vec{k},\sigma)$$

The little group element can be divided into two classes:**Rotation** $R(\theta)$ and **gauge translation** $S(\alpha,\beta)$:

$$\begin{aligned} \begin{pmatrix} u_{\bar{l}}(\vec{k},\sigma) \\ v_{\bar{l}}(\vec{k},\sigma) \end{pmatrix} e^{\pm i\sigma\theta} &= \sum_{l} \mathfrak{D}_{\bar{l}l}(R(\theta)) \begin{pmatrix} u_{\bar{l}}(\vec{k},\sigma) \\ v_{\bar{l}}(\vec{k},\sigma) \end{pmatrix} \\ \begin{pmatrix} u_{\bar{l}}(\vec{k},\sigma) \\ v_{\bar{l}}(\vec{k},\sigma) \end{pmatrix} &= \sum_{l} \mathfrak{D}_{\bar{l}l}(S(\alpha,\beta)) \begin{pmatrix} u_{\bar{l}}(\vec{k},\sigma) \\ v_{\bar{l}}(\vec{k},\sigma) \end{pmatrix} \end{aligned} \tag{5.13}$$

- The equations for v is just the complex conjugation of equations for u,thus we can set:$v = u^*$
- In constructing field of massless particles,the problem lies at that we can't find a u_l satisfying the equations for general representations of $SL(2,\mathbb{C})$.
- The field-theortical(using representation of $SL(2,\mathbb{C})$) will introduce redundant D.O.F,since for massless particles,the helicity takes only 2 value:$\pm j_n$($\mathcal{P}$ included), while the SO(3) embedding will give $2j_n + 1$ values.
- It seems for massless spin-0/$\frac{1}{2}$ particle,we have D.O.F mathched.However,the construction of these fields actually use the $m \to 0$ limit,we are still using $SO(3)$ little group.The $m \to 0$ limit of higher spin fields will be problematic due to unphysical D.O.F.
- These redundancy D.O.F manifest the unphysical gauge translation.Poincare transformation of field for massless particles with $j_n \geq 1$ will be accompanied by gauge transformation.Defining local gauge invariance a exact symmetry constraint.

[8] J is represented as a hermitian matrix,this will not be true for K.

5.1.3 Implementing the cluster decomposition principle

In 2nd quantization form:

$$V = \sum_{NM} \int dq'_1 ... dq'_N dq_1 ... dq_M \mathcal{V}_{NM} a^\dagger_{1'} ... a^\dagger_{N'} a_M ... a_1$$

Feed into the specific form of field operators,we obtain the **explicit** form of the interaction density:

$$\mathcal{V}_{NM}(q'_1 ... q'_N, q_1 ... q_M) = \delta^3(p'_1 + ... + p'_N - p_1 - ... - p_M)\tilde{\mathcal{V}}_{NM}(q'_1 ..., q_1 ...)$$
$$\tilde{\mathcal{V}}_{NM} \equiv (2\pi)^{3-3N/2-3M/2} \sum_{l'_1 ... l'_N} \sum_{l_1 ... l_M} g_{l'_1 ... l'_N, l_1 ... l_M} v_{l'_1}(q_1) ... v_{l'_N}(q_N) u_{l_1}(q_1) ... u_{l_M}(q_M)$$

Thus by constructing the scalar interaction density out of local-relativistic-field-operators automatically meet the requirement of cluster decomposition.[9]

The logic may be truned around:

Inevitable appearence of relativistic field operators

The combination of SR+QM+Locality(cluster decomposition) lead to constraints:

- The interaction density must be a Poincare scalar density.
- The interaction density must satisfy causality conditions,implied by Poincare invariance(of S-matrix).
- The interaction density can be put into 2nd quantization form,with coefficients $\mathcal{V}_{NM}$.
- The coefficients contain only one delta-function from translational invariance.The rest factor is smooth.
- Any smooth functions without further delta-functions can be expressed as a linear combinations of products of polarization functions which transform properly and can be used to construct reletivistic field operators.
- Finally,both $H_0, H_I = V(t)$ can be constructed out of creation/annihilation relativistic field operators using projective tensors.The final form will be constraint by causality condition.

5.1.4 Implication of causality condition

The causality condition is not satisfied by arbitary coupling of $\psi^\pm$,it will constraint the form of projective tenrsor $g_{l'...,l}$.This follows from the fact:

$$[\psi^+_l(x), \psi^-_{\bar{l}}(y)]_\mp = (2\pi)^{-3} \sum_{\sigma n} \int d^3p u_l(\vec{p}\sigma n) v_{\bar{l}}(\vec{p}\sigma n) e^{ip\cdot(x-y)} \neq 0$$

We then try the linear combination[10]:

$$\psi_l(x) = \kappa_l \psi^+_l(x) + \lambda_l \psi^-_l(x)$$

then adjust κ_l, λ_l to satisfy the condition,which may be stronger:

$$[\psi_l(x), \psi_{l'}(y)]_\mp = [\psi_l(x), \psi^\dagger_{l'}(y)]_\mp = 0, \forall (x-y)^2 \neq 0 \tag{5.14}$$

- This is the **causality condition for field operators**.The causality condition for Hamiltonian density is satisfied if it is constructed out of these operators[11].Since H is hermitian,there's always **even number** of field operators,this means the H is always a **bosonic operator**.Similar for other hermitian operators.

[9]For example,$\tilde{\mathcal{V}}_{NM}$ contain at most branch point singularities at zero particle(massive) momentum

[10]Note since ψ is not hermitian,we can't measure them.

[11]To respect cluster decomposition we should using the **normal ordering**:$V = \int d^3x : \mathcal{H}_I(\psi, \psi^\dagger) :$

- The meaning of $\psi_l^{(\dagger)}$ is the same as $\psi^{\mp}$,since $\psi|0\rangle \propto \psi^-|0\rangle$
- This causality condition will lead to many implications:
  - There's field theoretical spin-statistic connection
  - There exsist antiparticle.

Remark 30 *The overall* ***dimensionless*** *scales of κ, λ are finnaly fixed by procedure of* ***renormalization:the field operator should contain the physical spectrum****,this manifest the notation of* ***effectiveness: we are describing physics at certain energy/length scale.***

Antiparticles

For U(1)-charges,like electric(U(1)-gauge) charge,Lepton/Baryon numbers,there exsist commutation relations[12]:

$$[Q, a^{\pm}(\vec{p}\sigma n)] = (\pm q_n)a^{\pm}(\vec{p}\sigma n) \tag{5.15}$$

Since the generator comes from symmetry,it is conserved[13].:

$$[H, Q] = 0 \Rightarrow [\mathcal{H}_I(x), Q] = 0$$

We must have[14]:

$$[Q, \psi_l(x)] = -q_l\psi_l(x) \tag{5.16}$$

This agrees with the meaning of $\psi(x)$.This is satisfied with condition:

- $q(n) = q_l$ thus we will use **irreducible** representations,one field for one particle.
- the ψ^- component should carry charge $q(\bar{n}) = -q_l = q(n^c)$.Thus $a^\dagger$ actually stands for antiparticle[15]:

  $$\psi \sim \kappa a e^{ipx} + \lambda a_c^\dagger e^{-ipx}$$

  and that[16]:$[Q, a_c^{\pm}] = [Q, a^{\pm}(\vec{p}\sigma n^c)] = (\mp q_n)a^{\pm}(\vec{p}\sigma n^c)$
- As noted,the causality condition follows form SR,and lead to such expansions and finally lead to exsistence of $a_c^\dagger = a^\dagger(n^c)$,thus **the exsistence of antiparticle is a implication of SR**.
- Only for completely neutral particles, can we have in expansion $a = a_c$,but this is not necessary.

The charge operator can be writtne as:

$$N = a^\dagger a + a_c^\dagger a_c, Q = |q_e|[a^\dagger a - a_c^\dagger a_c]$$

where the charge is discrete,quantized in unit of $|q_e|$

5.1.5 Free field equations

As mentioned,the field expansions can be considered as quantization of solutions to free field equations.The operators constructed above indeed satisfy the **operator-equation**:

$$(\Box - m^2)\psi_l(x) = 0, \Box \equiv \partial^2 = \partial_\mu\partial^\mu = -\partial_0^2 + \nabla^2 \tag{5.17}$$

This is known as **Klein-Gordon equation**,satisfied by all reletivistic field operators since they contain relativistic plane waves factors from translational covariance.

Some fields satisfy other field equations as well,**depending on whether or not there are more field components than independent particle states.**

Since we want to describe the whole history of the system,we must constraint all free D.O.F.This is the meaning of field equations.This is actually more transparent in p-space and boost to standard frame: **the field equations are just the Fourier transformation of the Lorentz-covariant constraint on fields**

From the logical construction of the field operators,we see that **free field equations are equivalent to Poincare transformation properties of coefficient functions**.

[12]Generators are hermitian,thus bosonic.**An exception may be BRST generator?**
[13]Else it will not be a good quantum number to be included in CSCO to label states
[14]The commutation with H follows from $\sum_\psi q - \sum_{\psi^\dagger} q = 0$
[15]Implicitly,the particle and antiparticle have **same mass**
[16]This is for massive case,for massless case,charge conjugation also inverse the helicity,as well as chirality

5.2 Representation of $SL(2,\mathbb{C})$

5.2.1 $sl(2,\mathbb{C}) = so(1,3) = su(2) \oplus su(2)$

Defineing $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}, K_i = J^{i0} = J_{0i}, L_i = \frac{1}{2}(J_i - iK_i), R_i = L_i^\dagger = \frac{1}{2}(J_i + iK_i)$ the Lie algebra of $sl(2,\mathbb{C}) = so(1,3)$ factors into two copies of $su(2)$:

$$\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk}J_k, [J_i, K_j] = i\epsilon_{ijk}K_k, [K_i, K_j] = -i\epsilon_{ijk}K_k \\ \Leftrightarrow &[L_i, L_j] = i\epsilon_{ijk}L_k, [R_i, R_j] = i\epsilon_{ijk}R_k, [L_i, R_j] = 0 \end{aligned} \tag{5.18}$$

- The two copies of $su(2)$ algebra is generated by $\vec{L}, \vec{R}$ separately,they interchange under P-parity:$P: J \to J, K \to -K, L \leftrightarrow R$,hence **L means Left-chirality;R meas Right-chirality.**
- The J algebra form the subalgebra of $SO(3) \subset SO(1,3)$.As generator $J_i = L_i + R_i$;The K algebra generate boosts,but the Lie braket is not closed,thus not a subalgebra[17].As generators,$K_i = i(L_i - R_i)$
- The Lorentz transformation will be denoted locally/infinitesimally[18] as $T(\Lambda) = e^{i\vec{\theta}\cdot\vec{J} + i\vec{\beta}\cdot\vec{K}}$.When we consider the tensor/finite-dim representations,we will act **actively**,with representation matrix:$\mathfrak{D} = e^{i\vec{\theta}\cdot\vec{\mathcal{J}} + i\vec{\beta}\cdot\vec{\mathcal{K}}}$
- The **minus-sign** is crucial,it can be traced back to the metric of $M_{1,3}$:$\eta_{\mu\nu} = diag(-,+,+,+)$.This minus sign manisfest the i in L/R generator:
  - Leading to the interchange between L/R under parity.
  - Manisfest the fact that $SO(1,3)/SL(2,\mathbb{C})$ are not compact,may be considered as the analytic continuation of $SO(4)$,this implies finite-dim representations are **not unitary.**
- Representation of the algebra between L/R gives the representation of $SL(2,\mathbb{C})$.While in **finite** representations L/R are represented as **hermitian**[19] matrices $\mathcal{L}, \mathcal{R}$.Thus in finite-dim representation $\mathcal{J} = \mathcal{L} + \mathcal{R}$ is **hermitian**,since SO(3) subgroup have finite-dim unitary representation.$\mathcal{K} = i(\mathcal{L} - \mathcal{R})$ is **antiunitary**[20]
- The action of P-parity on Lie algebra is equivalent to hermitian-conjugation in finite-dim representations

Remark 31 *As noted before,these tensor representations are used in construction of field-operators,there's no problem with non-unitarity.The Lorentz invariant positive norm is provided by single-particle state as PUIR,not directly by fields.These* ***tensor representations are combined with field/differential representations to obtain infinite-dim representation.***

$$U\Psi(x)U^{-1} = (R\cdot\Psi)(\Lambda x + a) \sim \exp(i\theta T_{PDE})(R\cdot\Psi)(x)$$

The U is PUIR of ISO(1,3) furnished by single-particles these single-particle state do have Poincare invariant positive norm;The R is the tensor/finite-dim representation we construct.The transformation on spacetime component is furnished by T_{PDE}.

This kinds of representation will be used in canonical formalism:

- *Construction of Noether current in classical field theory,which will be promoted to generators.*
- *Commutators of generators between other operators.*

5.2.2 Finite-dim representations as direct product of $SU(2)_{L/R}$-representations

The decomposition:$su(2) \oplus su(2)$[21] implies the representations can be written as a tensor/direct product:

$$\begin{aligned} \mathfrak{D}(g) = e^{i\vec{\alpha}\cdot\vec{\mathcal{L}}} \otimes e^{i\vec{\eta}\cdot\vec{\mathcal{R}}} &\equiv \exp(i\vec{\alpha}\cdot(\vec{\mathcal{L}}\otimes 1) + i\vec{\eta}\cdot(1\otimes\vec{\mathcal{R}})) \equiv \exp(i\vec{\theta}\cdot(\vec{\mathcal{L}}\otimes 1 + 1\otimes\vec{\mathcal{R}}) + i\vec{\beta}\cdot i(\vec{\mathcal{L}}\otimes 1 - 1\otimes\vec{\mathcal{R}})) \\ &= \exp(i\vec{\theta}\cdot\vec{\mathcal{J}} + i\vec{\beta}\cdot\vec{\mathcal{K}}) = \exp(i\frac{1}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}) \end{aligned}$$

[17] This is related to the fact that K is not independently conserved,will not be included in CSCO

[18] $SL(2,\mathbb{C})$ is not compact

[19] SO(3) is compact

[20] As generator in quantum level,they are all hermitian operator.This motivate the fields.

[21] this are also used in addition of angular momentum problem,which is indeed a problem of **decompose the tensor product of representations of $SO(3)$ into irreducible representations,the Clebsch-Gordon coefficients are just projective tensors**

The tensor product representations will then be labelled by:

$$\mathfrak{D}^{(A,B)}_{a'b',ab} = \mathfrak{D}^{A}_{a'a} \otimes \mathfrak{D}^{B}_{b'b}, \mathcal{L}^{(A,B)}_{a'a,b'b} = \mathcal{L}^{A}_{a'a} \otimes 1_{b'b} = \mathcal{L}^{A}_{a'a}\delta_{b'b}, \mathcal{R}^{(A,B)}_{a'a,b'b} = 1_{a'a} \otimes \mathcal{R}^{B}_{b'b} = \delta_{a'a}\mathcal{R}^{B}_{b'b}$$

- The representation of L/R generated $SO(3)$ algebras are denoted separately as A,B.With $(\mathcal{L}/\mathcal{R})^{j}_{\sigma'\sigma} = J^{j}_{\sigma'\sigma}, \sigma = j, ..., -j$ are just the usually **spin-A/B representation.**
- The representation of $SL(2,\mathbb{C})$ is the tensor/direct product of these chiral $SU(2)$-representation:(A,B).With dimension $dim = (2A+1)(2B+1)$.This gives the number of **unconstaint D.O.F** in the tensor.
- If $\mathcal{P}$ is included then there must be representation matrix β reversing sign of tensors with odd numbers of space indices,in particular:
$$\beta\mathcal{L}/\mathcal{R}\beta^{-1} = \mathcal{R}/\mathcal{L}$$
thus **only irreducible representation with** $A = B$ **or reducible representation like** $(A,B) \oplus (B,A), A \neq B$ **can provide representation of** $\mathcal{P}$($\mathcal{P}$ well-defined[22])

Finite dim representation:matrix notation and tensor notation

The finite-dim representation represent group elements in matrices,with the carrier/represenation space expanded by orthonormal basis:$\{e_i\}$ and the matices act as:

$$e'_i = e_{i'} = \sum_i D_{i'i}e_i, D_{i'i} = (e_{i'}, De_i) = \langle e_{i'}|D|e_i\rangle$$

This is the case of the Hilbert space,which is a linear space(usually infinite-dim) spanned by single-particle space,which furnish the PUIR of ISO(1,3).
To have compact expression of transformations,we can use the **matrice notation**,denote multiplets as **column vectors**:

$$\psi = (\psi_a), \psi' = D\psi$$

In discussion of fields,we promote tensors to fields,the tensors also furnish finite-dim representation of groups,but the entities are defined as a multiplet,this comes from the definition of a tensor:$\psi = \psi_a e_a$,where the e_a are the true basis expanding the finite-dim carrier space.Thus the $\psi_a = \langle e_a|\psi\rangle$ transform actively:$\psi'_a = \psi_{a'} = \sum_a D_{a'a}\psi_a$.For these representations,we usually have projevtive tensors to have invariants(trivial-representations)$\psi \cdot \chi = m^{ab}\psi_b\psi_a$.It's then better to introduce the **tensor notation:index and dual index** to have more compact expressions:

$$\psi = \psi_a e^a, \psi^a = m^{ab}\psi_b, \psi = \psi^a e_a = \psi_b m^{ab} e_a, e^b = e_a m^{ab}$$

In many cases,the **matrix**:$(M)_{ab} \equiv m_{ab}$ is invertible,$(M^{-1})_{ab} \equiv m^{ab}$ thus we can consider m as the **metric** to lift/lower indices(switch between index and dual index).In terms of matrix,M is the similar transformation,thus **the representation and its dual representation are equivlaent**,with basis transformation given by the constant metric components.With non-Euclidean metric presented,we should distinguish between these two type of indices.
In tensor notation:

$$\psi_a = D_a{}^b\psi_b, (D)_{ab} = D_a{}^b$$

these **transformation matrix** specify the what representation of the index represent(spinor or 4/Lorentz-vector).It's important to remember $D_a{}^b$ are just **constants**,and they are not tensors.The have relation:v

$$m^{ab}\psi_b\chi_a = m^{ab}D_b{}^cD_a{}^d\psi_d\chi_c = m^{cd}\psi_c\chi_d \Rightarrow m^{ab}D_b{}^cD_a{}^d = m^{dc}$$

In martix form,this is:$(M)_{ab} = m^{ab}, D_{ab} = D_a{}^b, D_{ba} = D^T_{ab} \Rightarrow D^tMD = M$.Note that in general $D^t \neq D^{-1}$,the inverse is defined by $m_{ab}m^{bc} = m^{cb}m_{ba} = \delta^c_a \Leftrightarrow (M^{-1})_{ab} = m_{ab}, M^{-1}M = MM^{-1} = 1$,$M^{-1}D^TMD = 1 \Rightarrow D^{-1} = M^{-1}D^TM \Rightarrow (D^{-1})_{ab} \equiv (D^{-1})_a{}^b = M^{-1}_{ac}D^T_{cd}M_{db} = m_{ac}D_d{}^cm^{db}$
If $m_{ab} = \pm m_{ba}$,then $(D^{-1})_{ab} \equiv (D^{-1})_a{}^b = m^{bd}D_d{}^cm_{ca} \equiv D^b{}_a$
This gives the **definition** of lifting/lowering indices of transformation:

$$m^{bd}D_d{}^cm_{ca} \equiv D^b{}_a$$

[22]In quantum theory,$\mathcal{P}$ is conserved only when P is constructed as a unitary operator and commute with H,here we are considering representaion over some finite-dim linear space,there's no H,etc.

This relate to dual transformations:

$$\psi^a = m^{ab}\psi_b = (M\psi)_a \to MDM^{-1}M\psi \equiv D_{dual}(M\psi) \Rightarrow D_{dual} = MDM^{-1}$$

$$(D_{dual})_{ab} = (D_{dual})^a{}_b = M_{ac}D_{cd}(M^{-1})_{db} = m^{ac}D_c{}^d m_{db} \equiv D^a{}_b = (D^{-1})_a{}^b$$

This is a similar transformation as noted.Note that $D_{dual} \equiv MDM^{-1} \equiv (D^t)^{-1}$.The generator is:

$$(J_{dual})^a_b = m^{ac}J_c{}^d m_{db} = J^a{}_b \equiv -J_a{}^b$$

The invariant of contraction is:

$$T^a{}_a = D^a{}_b D_a{}^b T^a{}_a = (D^{-1})_b{}^a D_a{}^b T^b{}_b = T^b{}_b$$

The specific example include **inner product**:

$$m^{ab}\psi_b\chi_a = \psi^a\chi_a = \psi\cdot\chi$$

In matix form,inner product is:$\psi^{dual}\chi = \mp\chi^T M\psi = \mp\psi^T M\chi \equiv (M\psi)^T\chi$

- **The tensor from and matrix form are both used in scalar densities,when discussing transformations,matrix form are often used for its compactness.**
- **We will not use generators of dual representations,the dual transformations are defined via transformations** by $D^a{}_b = (D^{-1})_a{}^b = m^{ac}D_c{}^d m_{db}$
- For other symbols like Pauli matrix,the definition of lifting/lowering may be different.

Rotation subgroup $SO(3) \subset SO(1,3)$

The generator J is given by the usually addtion of 'angular momentum':

$$\mathcal{J} = \mathcal{L}\otimes 1 + 1\otimes\mathcal{R}$$

Since SO(3) is a subalgebra,the (A,B) irreducible representation of $SL(2,\mathbb{C})$ gives the reducible representation of $SU(2)$ which can be decomposed into irreducible representation,with the decomposition given by the **representation ring**:

$$(A,B)|_{su(2)} = A\otimes_{su(2)} B = |A+B| \oplus_{su(2)} |A+B| - 1 \oplus_{su(2)} ... \oplus_{su(2)} |A-B|$$

This is equivalent to addtion of spin-A and spin-B angular momentum.This means the corresponding tensor:ψ_l can also be decomposed $\psi_l = \phi_l^{j=A+B} + ... + \phi_l^{j=|A-B|}$ with ψ_l a irreducible $SL(2,\mathbb{C})$-tensor, ϕ_l^j irreducible $SU(2)$-tensor with spin-j.In many cases,the decomposition is recognized as $\psi_l = \phi_{l_1}^{j_1} \oplus \phi_{l_2}^{j_2} \oplus ...$

Remark 32
- *An important feature is that spin-j is included* ***at most once****,this makes the embedding of particles unambiguous.*
- *For one particle,we only use the spin-j part only,other D.O.F of the field coorespoding to other spins will have to be removed using* ***constraints****,these contraints are automatically fulfilled by the construction of coefficient functions,and they turn out to be* ***extra field equations and gauge fixing conditions.***

5.2.3 Important examples:Weyl spinor,Lorentz vector and others

$\triangleright(0,0)$**:trivial representaion**

This is the trivial representation,with dimension 1.All transformations are represented by a complex number,which is equivalent to $\mathfrak{D} = 1 \in \mathbb{C}$.

Examples include:

- **Hamiltonian density** $\mathcal{H}(x) = \mathcal{H}_0(x) + \mathcal{H}_I(x)$
- In quantization of classical field,we will be interested in scalar **action** I, and scalar **Lagrangian density**:$\mathcal{L}(x) = \mathcal{L}_0 + \mathcal{L}_I$

The introduction of these reletivistic tensors and fields is to construct these scalar densities.Trvial representation is also trivial for its rotation subgroup,thus trivial representation contain only **spin-0**.

$\triangleright(\frac{1}{2},0),(0,\frac{1}{2})$**:L/R Wely spinors**

The representations $(\frac{1}{2},0)/(0,\frac{1}{2})$ are furnished by Left-handed/Right-handed Wely spinors:

- $(\frac{1}{2},0)$:$\vec{\mathcal{L}}=\frac{1}{2}\vec{\sigma},\vec{\mathcal{R}}=1$
$$\vec{\mathcal{J}}=\frac{\vec{\sigma}}{2},\vec{\mathcal{K}}=i\frac{\vec{\sigma}}{2};\mathcal{J}_L^{ij}=\epsilon^{ijk}\frac{\sigma_k}{2},\mathcal{J}_L^{i0}=i\frac{\sigma_i}{2}$$
- $(0,\frac{1}{2})$:$\vec{\mathcal{L}}=1,\vec{\mathcal{R}}=\frac{1}{2}\vec{\sigma}$
$$\vec{\mathcal{J}}=\frac{\vec{\sigma}}{2},\vec{\mathcal{K}}=-i\frac{\vec{\sigma}}{2};\mathcal{J}_R^{ij}=\epsilon^{ijk}\frac{\sigma_k}{2},\mathcal{J}_R^{i0}=-i\frac{\sigma_i}{2}$$
- For both cases,dim=2,the Weyl spinors are 2-component,with entities in $\mathbb{C}$,thus totally 4-independent-real-components.They transform like(infinitesimally):
$$\psi_{L/R}\to\exp(i\frac{1}{2}\omega_{\mu\nu}\mathcal{J}_{L/R}^{\mu\mu})\psi=\exp(i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\mp\vec{\beta}\cdot\frac{\sigma}{2})\psi_{L/R}$$
$\mathfrak{D}=\exp(i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\mp\vec{\beta}\cdot\frac{\sigma}{2})$ is clearly not unitary.
- There's a more compact notation,define:
$$\begin{aligned}\sigma^\mu=(I,\vec{\sigma}),\bar{\sigma}^\mu=(I,-\vec{\sigma})\\ \sigma^{\mu\nu}=\frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu-\sigma^\nu\bar{\sigma}^\mu),\bar{\sigma}^{\mu\nu}=\frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu-\bar{\sigma}^\nu\sigma^\mu)\\ \sigma^{ij}=\bar{\sigma}^{ij}=-i\frac{1}{2}\epsilon_{ijk}\sigma_k,\sigma^{k0}=-\bar{\sigma}^{k0}=\frac{1}{2}\sigma_k\end{aligned}\tag{5.19}$$
Thus we have:
$$\mathcal{J}_L^{\mu\nu}\equiv i\sigma^{\mu\nu},\mathcal{J}_R^{\mu\nu}\equiv i\bar{\sigma}^{\mu\nu};\mathfrak{D}_L=\exp(-\frac{\omega_{\mu\nu}}{2}\sigma^{\mu\nu}),\mathfrak{D}_R=\exp(-\frac{\omega_{\mu\nu}}{2}\bar{\sigma}^{\mu\nu})\tag{5.20}$$
- Both L/R-spinors contain **only spin-1/2.**

Relation between L/R spinors

For representation,infinitesimally:$D=1+i\theta^aT_R^a$,the complex comjugation doesn't change the order of matrix multiplication,thus representation R is always accompanied with a **complex-conjugate representation** $\bar{R}$:
$$\psi'=D(\theta)\psi,D(\theta')D(\theta)=D(\theta'\theta)\Rightarrow\psi^{*'}=D^*(\theta)\psi^*,D(\theta')^*D(\theta)^*=D(\theta'\theta)^*$$
$$D_{\bar{R}}=D_R^*,\psi_{\bar{R}}=\psi_R^*$$
The generator of complex-conjugate representation is:
$$D_{\bar{R}}=1+i\theta^aT_{\bar{R}}^a=D_R^*=1-i\theta^aT_R^{a*}\Rightarrow T_{\bar{R}}=-T_R^*$$
Note that if $D_{\bar{R}}=D_R^*=SD_RS^{-1}$,then these two representations are equivalent,and the representation R is (pesudo)real.
The relation between L/R spinors is manifested by the relation:
$$-\sigma_i^*=\sigma_2\sigma_i\sigma_2=(i\sigma_2)\sigma_i(-i\sigma_2)=\epsilon\sigma_i\epsilon^{-1},\epsilon=i\sigma_2=-\epsilon^{-1}=-\epsilon^t$$
This can be compactly written as:
$$\bar{\sigma}^{\mu\nu}=\epsilon(\sigma^{\mu\nu})^*\epsilon^{-1}$$
Then:
$$\mathcal{J}_{\bar{L}/\bar{R}}^{\mu\nu}=-(\mathcal{J}_{L/R}^{\mu\nu})^*=\epsilon\mathcal{J}_{R/L}^{\mu\nu}\epsilon^{-1}$$
$$\Rightarrow D_{\bar{L}/\bar{R}}=D_{L/R}^*=\epsilon D_{R/L}\epsilon^{-1}=\epsilon^{-1}D_{R/L}\epsilon,D_{R/L}=\epsilon^{-1}D_{L/R}^*\epsilon=\epsilon D_{L/R}^*\epsilon^{-1}$$

This means the **complex conjugation representation** of L/R spinor representation is **equivalent** to the R/L spinor representation.Since there's no similar representation between L/R spinor representations:

- the L/R spinor representations are indeed complex.Complex conjugation will interchange two representations,this fact comes from:**complex comjugatioon interchange $\vec{L}/\vec{R}$ as generators**
- the R/L are not identified with the $\bar{L}/\bar{R}$ representation but equivalent up to a similar transformation provided by $S = \epsilon = i\sigma_2$ or $S = \epsilon^{-1} = -i\sigma_2$.
- It's then natural to consider the similar transformed L/R representations,which are known as the **dual L/R representations**,these dual spinor representations are **equivalent** representations,just a changing of basis,but helps going between L/R-spinor representations.[a]

The dual representations are motivated by the fact:

$$(\epsilon\psi_L^*)' = \epsilon D_L^* \epsilon^{-1}\epsilon\psi_L^* = D_R(\epsilon\psi_L^*), (\epsilon^{-1}\psi_R^*)' = \epsilon^{-1}(\epsilon D_L \epsilon^{-1})\epsilon\epsilon^{-1}\psi_R^* = D_L(\epsilon\psi_R^*)$$

These implies that:

- $\epsilon\psi_L^* = i\sigma_2\psi_L^*$ is a R-spinor,$\epsilon^{-1}\psi_R^* = -i\sigma_2\psi_R^*$ is a L-spinor.Note that the sign is for convenience,since ϵ is **antisymmetric**[b]

[a]We will not focus on transformations of L/R-spinors,define the generator $J^{\mu\nu}_{L/R,dual}$ in terms of $J^{L/R}$

[b]In contrast the metric $\eta_{\mu\nu}$ is symmetric

The above discusstion promote the **dotted-undotted** notation of spinors,which is useful for going between (dual) L/R:

Dotted-Undotted notation of L/R spinors

We start by defining:$\psi_L = (\psi_a), \psi_R = \bar{\psi} = (\psi_{\bar{B}})$.The antisymmetric nature of $\epsilon = i\sigma_2 = -\epsilon^T = -\epsilon^{-1}$ motivate the notation[23]:

$$(\epsilon)_{ab} = \varepsilon^{ab}, (\epsilon^{-1})_{ab} = \varepsilon_{ab}, \varepsilon^{ab}\varepsilon_{bc} = \varepsilon_{cb}\varepsilon^{ba} = \delta^a_c, (\epsilon)_{\bar{A}\bar{B}} = \varepsilon^{\bar{A}\bar{B}} \equiv \varepsilon_{\dot{a}\dot{b}}, (\epsilon^{-1})_{\bar{A}\bar{B}} = \varepsilon^{\bar{A}\bar{B}} = \varepsilon_{\dot{a}\dot{b}} \tag{5.21}$$

- Spinors:$\psi_a, \bar{\psi}^{\dot{a}}$
- Metric:
$$\varepsilon_{AB}\varepsilon^{BC} = \varepsilon^{CB}\varepsilon_{BA} = \delta^C_A, A = a/\dot{a}$$
- Dual-spinors:
$$\epsilon\psi \sim \epsilon^{ab}\psi_b = \varepsilon^{ab}\psi_b \equiv \psi^a, \epsilon^{-1}\bar{\psi} \sim \varepsilon^{\bar{A}\bar{B}}\bar{\psi}_{\bar{B}} = \varepsilon_{\dot{a}\dot{b}}\bar{\psi}^{\dot{b}} \equiv \bar{\psi}_{\dot{a}}$$
- The contraction of two indices should be taken with care,since ϵ is antisymmetric.We adopt the **convention** that contraction goes like:${}^a{}_a$ and ${}_{\dot{a}}{}^{\dot{a}}$.
Then:
$$\psi \cdot \chi \equiv \psi^a\chi_a = \varepsilon^{ab}\psi_b\chi_a = -\varepsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b = \chi^b\psi_b = \chi \cdot \psi$$
same for ψ_R.
  - Here we define **components of L/R spinors are Grassman variables,which anticommute rather than commute.**The reason of this choosing Grassman variables for half-integer spin fields is a consequnce of **spin-statistic connection.**
  - This construction is needed in field theories,in principle,group theory doesn't infer to the nature of the entities.
  - The different contraction orientation between $a, \dot{a}$ comes from definition $\bar{\psi}_{\bar{A}} = \bar{\psi}^{\dot{a}}$,origin form the antisymmetric nature of $\epsilon = \varepsilon_{ab}$.

[23]This convention is known as **van der Waerden** convention.There's a reversed convention,with a up-stair and $\dot{a}$ down-stair.This convention is useful in **spin-helicity formalism**

The transformation rules are then:

$$\psi'_a = \exp(i\frac{1}{2}\omega_{\mu\nu}\mathcal{J}_L^{\mu\nu})_a{}^b\psi_b = \exp(i\frac{1}{2}\omega_{\mu\nu}(i\sigma^{\mu\nu}))_a{}^b\psi_b, \psi^{\dot{a}'} = \exp(i\frac{1}{2}\omega_{\mu\nu}\mathcal{J}_R^{\mu\nu})^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}} = \exp(i\frac{1}{2}\omega_{\mu\nu}(i\bar{\sigma}^{\mu\nu}))^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}}$$

Thus we have the indices convention for generators,and $\sigma^\mu, \bar{\sigma}^\mu$:

$$(\mathcal{J}_L^{\mu\nu})_a{}^b = i(\sigma^{\mu\nu})_a{}^b, (\mathcal{J}_R^{\mu\nu})^{\dot{a}}{}_{\dot{b}} = i(\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}}$$

These notations are suggestive:

$$(\sigma^{\mu\nu})_a{}^b \sim \sigma^\mu_{a\dot{c}}\bar{\sigma}^{\nu\dot{c}b}, (\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}} \sim \bar{\sigma}^{\mu\dot{a}c}\sigma^\nu_{c\dot{b}} \Rightarrow \sigma^\mu_{a\dot{a}}, \bar{\sigma}^{\mu\dot{a}a}$$

The relations between L/R representation then is more clear,since the they are related by complex conjugation,we have:

$$(\psi_a)^* = \bar{\psi}_{\dot{a}}, (\bar{\psi}^{\dot{a}})^* = \psi^a; (i\sigma_2)\psi_L^* \to \varepsilon^{\dot{a}\dot{b}}\psi_{\dot{b}} = \psi^{\dot{a}}, (-i\sigma_2)\psi_R^* \to \varepsilon_{ab}\psi^b = \psi_a \tag{5.22}$$

The core connection is:

$$(\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}} \equiv \varepsilon^{\dot{a}\dot{c}}[(\sigma^{\mu\nu})_c{}^d]^*\varepsilon_{\dot{d}\dot{b}} \sim \varepsilon^{\dot{a}\dot{c}}[(\bar{\sigma}^{\mu\nu})_{\dot{c}}{}^{\dot{d}}]\varepsilon_{\dot{d}\dot{b}}$$

suggest that **the bar on the entities can be considered as some operator denoting the representation we are in**,similar to the dot on indices,they are manipulated by the complex conjugation.Note that in last step,the convention of lifting/lowering transformation/generator indices are adopted,the 2nd index is manipulated differently form the 1st which coincide with definition of dual spinors.

Summary of L/R-spinor representation

- Spinors are denoted as $\psi_a, \bar{\psi}^{\dot{a}}$,the dual spinors are related by the spinor metric:$\psi^a = \varepsilon^{ab}\psi_b, \bar{\psi}_{\dot{a}} = \varepsilon_{\dot{a}\dot{b}}\bar{\psi}^{\dot{b}}$.This also defines the how indices are lifted/lowered.
- Transformations and generators are defined by the rule:

$$(\mathfrak{D}_{dual})^a{}_b = \varepsilon^{ac}\mathcal{D}_c{}^d\varepsilon_{db} \equiv \mathfrak{D}^a{}_b = (\mathfrak{D}^{-1})_b{}^a$$

  same for R-spinor.
- The contraction is given by ${}^a{}_a$ and ${}_{\dot{a}}{}^{\dot{a}}$ and gives minus sign when changed to ${}_a{}^a$ and ${}^{\dot{a}}{}_{\dot{a}}$
- The generators are represented as:

$$(\mathcal{J}_L^{\mu\nu})_a{}^b = i(\sigma^{\mu\nu})_a{}^b, (\mathcal{J}_R^{\mu\nu})^{\dot{a}}{}_{\dot{b}} = i(\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}}$$
$$(\mathcal{J}_{L,dual}^{\mu\nu})^a{}_b = i(\sigma^{\mu\nu})^a{}_b \equiv -i(\sigma^{\mu\nu})_a{}^b, (\mathcal{J}_{R,dual}^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = i(\bar{\sigma}^{\mu\nu})_{\dot{a}}{}^{\dot{b}} \equiv -i(\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}}$$

  These agree with:

$$\mathcal{J}^{\mu\nu}_{L/R,dual} = \epsilon\mathcal{J}^{\mu\nu}_{L/R,dual}\epsilon^{-1} = -(\mathcal{J}^{\mu\nu}_{R/L})^* = \mathcal{J}^{\mu\nu}_{\bar{R}/\bar{L}}$$

- With definition:

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \bar{\sigma}^{\mu\nu} = \overline{\frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)} = \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$$

- The core identity promoting the index convention is:

$$-\sigma_i = \varepsilon\sigma_i^*\varepsilon^{-1} \Rightarrow \bar{\sigma}^\mu = \varepsilon(\sigma^\mu)^*\varepsilon^{-1}, \bar{\sigma}^{\mu\nu} = \varepsilon(\sigma^{\mu\nu})^*\varepsilon^{-1}$$

  This gives the index convention:$\sigma^\mu_{a\dot{a}}, \bar{\sigma}^{\mu\dot{a}a}$ and the action of **complex conjugation:add/move the bar and bar simultaneously.**
- The definition of the **value** of Pauli-matrices is:

$$(\bar{\sigma}^\mu)^{\dot{a}a} \equiv \varepsilon^{\dot{a}\dot{c}}\varepsilon^{ac}(\sigma^\mu)^{c\dot{c}}$$

  The tensor notation distinguish upper/lower indices,Since these are also **projective tensors** like metric,they are lifted/lowered in the same way,**not the same as transformation/generator**:

$$(\bar{\sigma}^{\mu\dot{a}a})^* = \sigma^{\mu a\dot{a}} = -\varepsilon^{ac}\sigma^\mu_{c\dot{c}}\varepsilon^{\dot{c}c} = \varepsilon^{ac}\varepsilon^{\dot{a}\dot{c}}\sigma^\mu_{c\dot{c}}, \sigma^{\mu c\dot{c}} \neq \sigma^\mu_{c\dot{c}}$$

- Explicitly:

$$\sigma^{\mu} = \sigma^{\mu}_{a\dot{a}} e^a \otimes e^{\dot{a}}, \psi = \psi_a e^a, \bar{\psi} = \bar{\psi}^{\dot{a}} e_{\dot{a}}$$

$\triangleright(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$**:Dirac spinor**

When the theory respect parity(the action of $\mathcal{P}$ is well-defined),we should combine the L/R part into a single one.This defines the Dirac spinor as the direct sum:

$$\Psi = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix} = \Psi_L + \Psi_R = \Psi_l, l = 1, 2, \dot{1}, \dot{2} = 1, 2, 3, 4$$

This is 4-dim,with 8 real-unconstaint-D.O.F.And only contain spin-1/2.
The transformation is $\mathfrak{D} = \exp(i\frac{\omega_{\mu\nu}}{2}\mathcal{J}^{\mu\nu}_D)$,the generator are represented as:

$$\mathcal{J}^{\mu\nu}_D = \begin{pmatrix} \mathcal{J}^{\mu\nu}_L & 0 \\ 0 & \mathcal{J}^{\mu\nu}_R \end{pmatrix} = \begin{pmatrix} i(\sigma^{\mu\nu})_a{}^b & 0 \\ 0 & i(\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}} \end{pmatrix}$$

These can be put into a compact notation,using Clifford algebra:

Clifford algebra and Gamma-matrices

The Clifford algebra used in (3+1)d[24] relativistic QFT:

$$[\gamma^\mu, \gamma^\nu]_+ = \{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$$

- The conventional choice of $-2\eta^{\mu\nu}$,makes explicit expression of the gamma matrices free of i.
- There's another choice,,with $+2\eta^{\mu\nu}$,denoted as $\hat{\gamma}^\mu$,is more related to SO(4),with $2\delta_{\mu\nu}$.The connection is $i\hat{\gamma}^\mu = \gamma^\mu$[25]
- These different choice differ by important i,in expressions involving even numbers of γ there may be sign differences;in expressions involving $^{-1}$,* or $\dagger$ of γ there may be sign differences.
- Explicitly,we have:

$$(\gamma^0)^2 = +1, (\gamma^i)^2 = -1, (\gamma^i)^{-1} = -\gamma^i; \gamma^\alpha\gamma^\beta = -\gamma^\beta\gamma^\alpha, \alpha \neq \beta; \gamma_\mu = \eta_{\mu\nu}\gamma^\nu$$

define $\Sigma^{\mu\nu} = +\frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and[26]

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = -\eta^{\mu\nu} - i\Sigma^{\mu\nu}$$

the generator in Dirac representation is:

$$\mathcal{J}^{\mu\nu}_D = \frac{1}{2}\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

This comes form:

$$[\mathcal{J}^{\mu\nu}_D, \gamma^\alpha] = +i\gamma^\mu\eta^{\nu\alpha} - i\gamma^\nu\eta^{\mu\alpha}$$

- By some algebra,we show this indeed satisfies the Lie algebra of $SL(2, \mathbb{C})$
- This implies,since $\mathcal{J}^{\mu\nu}_D$ generate Lorentz transformations:

$$\mathfrak{D}(\Lambda)\gamma^\mu\mathfrak{D}^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\alpha\gamma^\alpha$$

Thus the μ index is indeed a Lorentz-index.This expression also implies the **gamma matrices are also the projective tensors in this representation**.

[24] In general Clifford algebra is associated with universal covering group of SO(d) to find its reducible representation.Since SO(1,3) is analytic continuation of SO(4),so is the Clifford algebra.

[25] The gamma matrices including gamma5 used in CMP are more close to the convention in [WeinbergI],where $\hat{\gamma}^\mu = \frac{1}{i}\gamma^\mu$ used in this note,and $\bar{\gamma}^5 = -\gamma^5$.This is just some sign difference in definition,when expressed all in Pauli matrices,they will coincide.

[26] The choice of $+\eta^{\mu\nu}$ makes expressions like this free of signs

The parity is defined as $\beta = \gamma^0 = \beta^{-1}$,with the action being $\mathcal{P} : M \to \beta M \beta^{-1}$ this comes form the fact:

$$\beta\gamma^0\beta^{-1} = \gamma^0, \beta\gamma^i\beta^{-1} = -\gamma^i$$

This agree with μ is a Lorentz-index.Thus[27]

$$\beta\mathcal{J}_D^{ij}\beta^{-1} = \mathcal{J}_D^{ij}, \beta\mathcal{J}_D^{k0}\beta^{-1} = -\mathcal{J}_D^{k0}$$

Since the product of γ^μ can be factored into antisymmetrized product times products of $\eta^{\mu\nu}$,the totally antisymmetric tensors(Lorentz-tensors) form a complete basis for the set of matrices can be constructed from the Dirac matrices.In 4d,the sequnce is:

$$\{1, \gamma^\mu, \gamma^{[\mu}\gamma^{\nu]}, \gamma^{[\mu}\gamma^\nu\gamma^{\alpha]}, \gamma^{[\mu}\gamma^\nu\gamma^\alpha\gamma^{\beta]}\}$$

- These projective tensors all transform differently under $\Lambda^\mu{}_\nu$ and/or $\mathbb{P}$.They are linearly independent,with scalar product defined as trace of their products.
- These **projective tensors** gives all possible **Dirac bilinears**.Number of independent components is 1+4+6+4+1=16.Since for $\nu - dim$ matrices,there's ν^2 independent matrices,the least dimension of γ^μ **in 4d** is $\sqrt{16} = 4$,these γ^μ are irreducible,they can't be put to block-diagonal form simultaneously.**Thus we take γ^μ as 4×4 matrices**[28]

The decomposition of Dirac representation into two copies of Weyl spinors is revealed by the **Weyl basis**,the basis here refer to the choice of $e^l, \Psi = \Psi_l e^l$,in different basis Ψ, γ^μ have different form,but the**abstract expressions hold in all basis,without mention we will work with Weyl basis:**

- Weyl basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \mathcal{J}_D^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \equiv \mathcal{J}_L^{\mu\nu} \oplus \mathcal{J}_R^{\mu\nu}$$

  - **Gamma-5**[29]:The L/R part is given by $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$,in Weyl basis:

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

    Thus $\gamma^5\Psi_{L/R} = \mp\Psi_{L/R}$,this defines **chirality**.[30]In general,**chirality≠helicity**.More abstractly,in all basis[31]:

$$(\gamma^5)^2 = 1, \{\gamma^5, \gamma^\mu = 0\}$$

    We then define the $P_{L/R} = \frac{1\mp\gamma^5}{2}, P_{L/R}^2 = P_{L/R}$,then $\Psi_{L/R} = P_{L/R}\Psi_{L/R} = P_{L/R}\Psi$.
  - These expressions hold in any basis,thus γ^5 may be called the chirality-operator.
  - The parity in Weyl basis is

$$\beta = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

    the action of parity interchange L/R part as wanted:

$$\mathcal{P}\Psi = \gamma^0\Psi = \begin{pmatrix} \bar{\chi} \\ \psi \end{pmatrix}$$

    This fact can also seen from $\beta P_{L/R}\beta^{-1} = P_{R/L}$ due to γ^5 containing odd γ^i.

[27]entities contain odd γ^i are odd under parity

[28]Generally,in even-d dimension,antisymmetric tensors can have 0,...,d indices.Thus the liearly independent tensor-components are $\sum_{n=0}^d \binom{d}{n} = 2^d$,thus irreducible $\gamma^a, a = 0, ..., d-1$ are taken 2^{d-1}-dim.For odd-d,there's connection between rank-n,rank-d-n Lorentz-tensor are ralated by the **Levi-Civita** symbol:$\epsilon^{\mu_1,\ldots,\mu_d}$,thus there are only 2^{d-1} linearly independent tensor-components,the γ^μ is also $2^{\frac{d-1}{2}} = 2^{2k/2} = 2^k$-dim

[29]Some text difine $\gamma^5 \to -\gamma^5$

[30]Chiral fermions usually are taken to be massless,where chirality=helicity,thus γ^5 is also known as helicity operator,this can be seen from H

[31]This means in 5d,this is the needed extra gamma-matrice.Also note that gamma-5 is defined in even-dim,in odd-dim,γ^5 is equivalent to the trivial one:1,as noted before.Also note that gamma-5 have only one independent Lorentz-tensor component

- Dirac basis:this basis is useful in non-relativistic limit,the difference with Weyl basis is:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

  The Dirac spinor in this basis decompose into particle and antiparticle 'wave-function'.In non-relativistic limit,only the first is factored out,leaving equations for particles alone.

- Majorana basis:this is useful for **Majorana spinors**:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}$$

  all these are manifestly imaginay.

More properties of γ^5:

- $$[\mathcal{J}_D^{\mu\nu}, \gamma^5] = 0, \beta\gamma^5\beta^{-1} = -\gamma^5$$

  thus γ^5 is a **pseudoscalar**

- There's relation:

$$\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]} \propto \epsilon^{\mu\nu\alpha\beta}\gamma^5\gamma_\beta, \gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta]} \propto \epsilon^{\mu\nu\alpha\beta}\gamma^5$$

  thus the last two set of independent Lorentz-tensors are just $\{\gamma^5\gamma^\mu, \gamma^5\}$

The action of parity can lead to:[32]:

$$\beta M^\dagger \beta \equiv \bar{M} : \ \bar{\gamma^\mu} = \gamma^\mu, \bar{\mathcal{J}_D^{\mu\nu}} = \mathcal{J}_D^{\mu\nu}, \bar{\mathfrak{D}} = \mathfrak{D}^{-1}, \bar{\gamma}^5 = -\gamma^5, \overline{\gamma^5\gamma^\mu} = (\gamma^5\gamma^\mu) \tag{5.23}$$

$\triangleright(\frac{1}{2}, \frac{1}{2})$ **:Lorentz/four vector,defining representation**

This representation is just the **defining representation**,acting on $\psi^\mu, \mu = 0, 1, 2, 3$,with transformation:

$$\mathfrak{D}(\Lambda) \equiv \Lambda^\mu{}_\nu = \exp(i\frac{\omega_{\mu\nu}}{2}\mathcal{J}_V^{\mu\nu})$$

The generator can be obtained directly from infinitesimal transformation:

$$\omega^\alpha{}_\beta = i\frac{\omega_{\mu\nu}}{2}(\mathcal{J}_V^{\mu\nu})^\alpha{}_\beta \Rightarrow (\mathcal{J}_V^{\mu\nu})^\alpha{}_\beta = -i(\eta^{\alpha\mu}\delta_\beta^\nu - \eta^{\alpha\nu}\delta_\beta^\mu)$$

This can also be adopted form $\vec{\mathcal{L}} = \frac{\vec{\sigma}}{2} = \vec{\mathcal{R}} \Rightarrow \vec{\mathcal{J}} = \frac{\vec{\sigma}}{2} \otimes 1 + 1 \otimes \frac{\vec{\sigma}}{2}, \vec{\mathcal{K}} = i(\frac{\vec{\sigma}}{2} \otimes 1 - 1 \otimes \frac{\vec{\sigma}}{2})$

Here we usd the tensor product form,this is not directly the defining representation,but **up to unitary transformation**,the equivalence can be shown using **projective tensor** $\sigma^\mu_{a\dot{a}}$,as we did in proving $SL(2,\mathbb{C})$ double-covering SO(1,3).We will work with defining representation,where the index is known as a **Lorentz-index**,and using multiplet ψ^μ rather than tensor[33] $\psi_{a\dot{a}}$.

This representation contain spin-0,spin-1,and written in Lorentz index,the 0-components is the spin-0 part,the 1-3 components are spin-1 part,which reverse sign under parity.In non-chiral representation,parity is well defined.

Similarly,the metric is given by $\eta^{\mu\nu} = \eta^{\nu\mu} = \eta_{\mu\nu} = \eta_{\nu\mu}$,the dual vector is $\eta_{\mu\nu}\psi^\mu = \psi_\mu$.The dual transform under $\Lambda_\mu{}^\nu \equiv (\Lambda^{-1})^\mu{}_\nu$ as noted.

[32]There're 3 bar: bar on Dirac spinor,bar on R-spinor,bar on matrices.All have different meaning;Again,**note possible sign difference in different conventions**

[33]this tensor have mixed type of indices

5.2.4 Projective tensors and Lorentz-invariants

Representaion ring of $SL(2,\mathbb{C})$

One of the main reasons for introducing these representations of $SL(2,\mathbb{C})$ is to construct trivial representations.One way to construct higher representations is to form tensor product of lower representations and factor out higher representations using corresponding projective tensors.For $SL(2,\mathbb{C})$ the most fundamental representation is $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ all other irreducible representations can be obtained by corresponding projective tensors.

The representation ring of $SL(2,\mathbb{C})$ can be obtained directly form representation ring of $SU(2)$,in general:

$$\begin{aligned}(A',B')\otimes(A,B) =&[A'\tilde{\otimes}B']\otimes[A\tilde{\otimes}B] = (A'\tilde{\otimes}A)\otimes(B'\tilde{\otimes}B) = [|A'+A|\oplus...\oplus|A'-A|]\otimes[|B'+B|\oplus...\oplus|B'-B|]\\ =&(|A'+A|,|B'+B|)\oplus...\oplus(|A'-A|,|B'-B|)\end{aligned}$$

The projective tensors form $A\otimes B\to C$ are defined as:

$$\begin{aligned}C^{ab}_c\psi_a\chi_b \to C^{ab}_c(\mathfrak{D}^{(A)})_a{}^{a'}(\mathfrak{D}^{(B)})_b{}^{b'}\psi_{a'}\chi_{b'} \equiv (\mathfrak{D}^{(C)})_c{}^{c'}(C^{a'b'}_{c'}\psi_{a'}\chi_{b'})\\ \Rightarrow C^{ab}_c(\mathcal{J}^{\mu\mu}_A)_a{}^{\alpha}(\mathcal{J}^{\mu\mu}_B)_b{}^{\beta} = C^{\alpha\beta}_{\gamma}(\mathcal{J}^{\mu\mu}_C)_c{}^{\gamma}\end{aligned}$$

- These C are constants rather than tensors,their value remain unchanged upon transformation,though the indices are changed.
- These projective tensors can be obtained from the projective tensors of $SU(2)$:the Clebsch-Gordon coefficients $\mathcal{C}^m_{m_1m_2} = \langle j,m|j_1,m_1;j_2,m_2\rangle = \langle j,m|(|j_1,m_1\rangle\otimes|j_2,m_2\rangle)$ but to note that these are the basis.
- The general form of projective tensors are rather complex,we will work out only some examples.

Examples of invariants and projective tensors

$$\triangleright(\frac{1}{2},0)\otimes(\frac{1}{2},0) = (1,0)\oplus(0,0),(0,\frac{1}{2})\otimes(0,\frac{1}{2}) = (0,1)\oplus(0,0)$$

The projection tensor to trvial tensor is given by[34]: ε^{ab},this is just the metric.The invariant is given by inner product between two L-spinor,with indices contracted:

$$\psi\cdot\chi = \psi^a\chi_a = \varepsilon^{ab}\psi_b\chi_a$$

The Lorentz invariance is trivial:

$$\psi^a\chi_a \to \mathfrak{D}^{a'}{}_a\mathfrak{D}_{a'}^{\ a}\psi^{c'}\chi_{c'} = (\mathfrak{D}^{-1})_a{}^{a'}\mathfrak{D}_{a'}^{\ a}\psi^{a'}\psi^{c'}\chi_{c'} \equiv \psi^{c'}\chi_{c'}$$

Note the fact $\mathfrak{D}^a{}_b = \mathfrak{D}^{-1}{}_b{}^a$ is equivalent to the definition of metric:$D^tMD = M, MDM^{-1} = (D^{-1})^t$.This can also be verified using explicit expression for $\mathcal{J}^{\mu\nu}_L$ but again,the vanishing can be traced back to metric.

The same hold for R-spinor:$\varepsilon_{\dot{a}\dot{b}}$ is the projective tensor to (0,0),the invariant is:

$$\bar{\psi}\cdot\bar{\chi} = \bar{\psi}_{\dot{a}}\bar{\chi}^{\dot{a}} = \varepsilon_{\dot{a}\dot{b}}\bar{\psi}^{\dot{b}}\bar{\chi}^{\dot{a}}$$

Since L/R are connected,denote for convenience:$\psi^*\equiv\psi^\dagger$,this gives the **matrice form**:

$$\psi^\dagger\bar{\chi} \equiv (\psi_a)^*\bar{\chi}^{\dot{a}} = \bar{\psi}_{\dot{a}}\bar{\chi}^{\dot{a}} = \bar{\psi}\cdot\bar{\chi}, \bar{\chi}^\dagger\psi \equiv \chi\cdot\psi \tag{5.24}$$

These also gives the **invariants formed by both L/R spinors.**

The projection tensors to $(1,0)/(0,1)$ are also given by the Clebsch-Gordon coefficients,and this is equivalent to:$|1,1/0/-1\rangle = ...$,the coefficients can't be written as a single matrice,but rather 3-matrice[35]:

$$C^1_{m_1,m_2} = J^+_{s=\frac{1}{2}}, C^2_{m_1,m_2} = \frac{1}{\sqrt{2}}\sigma_1, C^3_{m_1,m_2} = J^-_{s=\frac{1}{2}}$$

- Actually,this tensor product labelling is not convenient,the (1,0)/(0,1),a more compact notation is obtained using $(\frac{1}{2},\frac{1}{2})$ representation,where the 2×2 components turn into $\mu = 1,2,3,4$ multiplet.

[34] This is trivial:$|0,0\rangle = |+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle$

[35] For L-spinor the index is on the upper floor

$$\triangleright(\frac{1}{2},0)\otimes(0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2})=(0,\frac{1}{2})\otimes(\frac{1}{2},0)$$

This fact implies that the Lorentz-vector representation is equivalent to product of L and R spinors:**One Lorentz-index equals one (dual)L-spinor index plus one R-spinor index**.

The projective tensor is given by Pauli-matices[36]:

$$\sigma^\mu=(I,\vec{\sigma})=\sigma^\mu_{a\dot{a}}$$

This fact can be obtained from the transformation of the combination,in **matrix form**,using the Lie algebra of σ^i:

$$\delta(\bar{\psi}^\dagger\sigma^0\bar{\chi})=\beta_i\bar{\psi}^\dagger\sigma_i\bar{\chi},\delta(\bar{\psi}^\dagger\sigma^i\bar{\chi})=\beta_i\bar{\psi}^\dagger\bar{\chi}-\theta_i\epsilon_{ijk}\bar{\psi}^\dagger\sigma_k\bar{\chi}$$

This is exactly:$\delta(V^0,V^i)=(\beta_iV_i,\beta_iV_0-\epsilon_{ijk}\theta_jV_k)$.This fact is equivalent to:

$$\sigma^\mu_{a\dot{a}}(\mathcal{J}^{\alpha\beta}_L)^a{}_{a'}+\sigma^\mu_{a\dot{a}}(\mathcal{J}^{\alpha\beta}_R)^{\dot{a}}{}_{\dot{a}'}=(\mathcal{J}^{\alpha\beta}_V)^\mu{}_\nu\sigma^\nu_{a'\dot{a}'}$$

Similarly,there's projective tensor:$\bar{\sigma}^{\mu\dot{a}a}$,and $\psi^\dagger\bar{\sigma}^\mu\chi=\bar{\sigma}^{\mu\dot{a}a}\bar{\psi}_{\dot{a}}\chi_a$ also transform as a Lorentz tensor of rank-1.The difference between $\sigma^\mu,\bar{\sigma}^\mu$ is up to a basis transformation between dual representation.These two Pauli matrice all gives the basis transformation between Lorentz vector representation and direct product of L and R spinor representations.Conversely,any Lorentz vector can be expressed like:

$$v_{a\dot{a}}=V_\mu\sigma^\mu_{a\dot{a}},v^{\dot{a}a}=V_\mu\bar{\sigma}^{\mu\dot{a}a}$$

- This fact implies that the most fundamental representation is the L/R spinor representations,any other representation can be constructed out of these representations.
- Especially,the vector representation and the Dirac representation are mostly used,they can be decomposed into L/R spinor representations,and higher representations are usually constructed out of these two representations in practice.

$$\triangleright(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2})=(1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0)$$

The projective tensor of trivial representation is just the metric$\eta_{\mu\nu}$,thus the invariants are given by:

$$A\cdot B=A^\mu B_\mu$$

more importantly,the dynamic of 3+1d reletivistic QFT are given by ∂_μ.Which is considered as a **dual** Lorentz-tensor of rank-1.[37]

The $(1,0)\oplus(0,1)$ is just the antisymmetric tensor:

$$A^{[\mu}B^{\nu]}$$

which have 6 independent D.O.F,and is invariant under parity.A special case is $F^{\mu\nu}=\partial^{[\mu}A^{\nu]}$,which can be factored into $\vec{E},\vec{B}$ parts in some basis[38].The $(1,0)/(0,1)$ part are defined as the antisymmetrized tensor accompanied with extra 'duality' conditions[39]:

$$K^{\mu\nu}=A^{[\mu}B^{\nu]}=\pm\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}K_{\alpha\beta}=\pm\frac{i}{2}*K^{\mu\nu}$$

There's another trivial representation occuring in $(\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{1}{2})$,with projective tensor:$\epsilon^{\mu\nu\alpha\beta}$,thus we also have invariants:

$$\epsilon^{\mu\nu\alpha\beta}A_\mu B_\nu C_\alpha D_\beta,...$$

A special example is $\epsilon^{\mu\mu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}=*F^{\mu\nu}F_{\mu\nu}$

$$\triangleright[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]\otimes[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]=(0,0)\oplus(0,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{1}{2})\oplus(1,0)\oplus(0,1)$$

[36]the L-spinor is taken as dual representation,which is equivalent and makes the projective tensor better looking:they have same value of Pauli matices

[37]the contraction between ∂ and A^μ have **no refer** to $\eta^{\mu\nu}$

[38]$\vec{F}=\vec{E}+i\vec{B}$

[39]A special case will be **instanton solutions**.

This decomposition gives the all **Dirac bilinears**[40].The coorespoding projective tensors are given by[41]:

$$\Gamma^i = \{1, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \gamma^{[\mu}\gamma^{\nu]} \propto \Sigma^{\mu\nu}\}$$

These projective tensors to trivial representation comes from linear combination of :

$$\begin{pmatrix} \varepsilon^{ab} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} \Rightarrow \mathcal{C}^{-1} = \begin{pmatrix} i\sigma^2 = \varepsilon^{ab} & 0 \\ 0 & -i\sigma^2 = \varepsilon_{\dot{a}\dot{b}} \end{pmatrix} = i\gamma^2\beta = i\gamma^2\gamma^0 = -\mathcal{C}, \mathcal{C}^{-1}\gamma^5 = \begin{pmatrix} -\varepsilon^{ab} & 0 \\ 0 & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix}$$

The invariants are $\Psi^T\mathcal{C}\Phi'$, $\Psi^T\mathcal{C}\gamma^5\Phi$,define the bar-Dirac spinor,as a matix:

$$\bar{\Psi} = \Psi^\dagger\beta = (\chi^a, \bar{\psi}_{\dot{a}}), \Psi = \begin{pmatrix} \psi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix}$$

The invariants are now,in **matrix form**:

$$\bar{\Psi}\Psi = \chi \cdot \psi + \bar{\psi} \cdot \bar{\chi} = \bar{\chi}^\dagger\psi + \psi^\dagger\bar{\chi} \tag{5.25}$$

The invariance can also be shown by:

$$\beta\mathfrak{D}^\dagger\beta = \mathfrak{D}^{-1}$$

The fact $\bar{\Psi}\gamma^\mu\Phi, \bar{\Psi}\gamma^5\gamma^\mu\Phi$ transform is a Lorentz-tensor is clear.The $\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ clearly gives projection to $(1,0) \oplus (0,1)$

These Γ are orthogonal thus independet as noted before,this can be clearly seen by[42]:$tr(\Gamma^i\Gamma^j) = 4\delta^{ij}$

More symmetry properties of gamma-matrices are,these will be used in **Charge-conjugation**,note the different definition may have different sign due to i difference[43]:

- $\beta M^\dagger\beta = \pm\bar{M}$:
  - $\bar{M} = M$:$\gamma^\mu, \mathcal{J}_D^{\mu\nu}, \Sigma^{\mu\nu}, \gamma^5\gamma^\mu$
  - $\bar{M} = -M$:γ^5
- $-\gamma_\mu^t = \mathcal{C}\gamma_\mu\mathcal{C}^{-1} \Rightarrow \mathcal{C}M\mathcal{C}^{-1} = \pm M^T$:
  - +:$\gamma^5, \gamma^5\gamma^\mu$
  - -:$\gamma^\mu, \mathcal{J}_D^{\mu\nu}$
- $-\beta\mathcal{C} = -i\gamma^2 = \mathcal{C}\beta = (\beta\mathcal{C})^{-1}, \beta\mathcal{C}M(\beta\mathcal{C})^{-1} = \beta\mathcal{C}M\mathcal{C}^{-1}\beta = \pm M^*$:
  - +:$\gamma^5\gamma^\mu$
  - -:$\gamma^\mu, \gamma^5, \mathcal{J}_D^{\mu\nu}$

We see that the meaning of $\mathcal{C}$ is related to complex conjugation,which will also be involved in **time-reversal**

5.2.5 Some higher representations

- The (A,A) representation correspond to traceless symmetric tensor of rank 2A.[44]Examples would include (1,1) representation,which contain spin-2,used for **gravition**.The extra D.O.F should be removed with constraints.
- The $(\frac{1}{2}, 1) \oplus (1, \frac{1}{2})$ contain spin-3/2,used for the **Rarita-Schwinger field**.This representation can be obtained from:

  $$(\frac{1}{2}, \frac{1}{2}) \otimes [(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})] = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2}, 1) \oplus (1, \frac{1}{2})$$

  The quantity $\gamma_\mu\Psi^\mu$ transform as an ordinary Dirac field,thus the rest part can be obtained by the constraint:

  $$\gamma_\mu\Psi^\mu = 0$$

  The extra D.O.F are removed further by equations.[45]

[40] All in matrix form
[41] Weyl basis
[42] Any 4×4 matrix,$\Gamma = \sum_{i=1}^{16} \frac{1}{4}tr(\Gamma\Gamma^i)\Gamma^i$
[43] **Caution:different difinition involve i difference,thus in 1st and 3rd cases,there are sign difference in** $\gamma^\mu, \gamma^5\gamma^\mu$
[44] symmetric tensor of rank 2A in 4d have independent components: $[2A] = \frac{(3+2A)!}{3!(2A)!}, [k]_D = \frac{(D-1+k)!}{(D-1)!k!}$,traceless condition gives $[k-2]_D$ constraints,thus totally $(2A+1)^2$ as wanted.
[45] ordinary Dirac field and $\partial_\mu\Psi^\mu = 0$

5.3 General causal free fields

The expansion of the free field operator have the form[46]:

$$\psi_{ab}(x) = \sum_{\sigma} \int \frac{d^3p}{(2\pi)^3}[\kappa a(\vec{p},\sigma)e^{ip\cdot x}u_{ab}(\vec{p},\sigma) + \lambda a_c^\dagger(\vec{p},\sigma)e^{-ip\cdot x}v_{ab}(\vec{p},\sigma)] \tag{5.26}$$

- The κ,λ are constants,known as scale of field operator,the overall value will be fixed by **renormalization** at certain physical scale.The free field operators contain the particle in the sense $\langle 0|\psi|\vec{p}\rangle \sim ue^{ip\cdot x}$,u/v are residuals.
- field operator combine annhilation field of particle and creation field of **antiparticle**,this is due to **causality condition** of fields.Only when the particle is neutral can $a = a_c$.
- In specific cases,we usually rearrange the indices,so that $(a,b) \equiv l$,we have multiplet rather than tensors,and **matrix form** can be used.
- The d^3p implies the particle is on-shell.And this is just a Fourier transformation.[47]
- There's still a sum over **independent** polarizations(j_3-component or helicity),the polarization/coefficient functions will be fixed by group theoretical methods.[48]

5.3.1 Constructing coefficient functions:massive case

For massive case,the littel group is $SO(3)$.The transformation properties of coefficient functions in standard(rest) frame are:

$$\begin{aligned} \sum_{\bar{\sigma}} u_{\bar{a}\bar{b}}(0,\bar{\sigma})\vec{J}^{(j)}_{\bar{\sigma}\sigma} &= \sum_{ab} \vec{\mathcal{J}}^{(A,B)}_{\bar{a}\bar{b},ab} u_{ab}(0,\sigma) \\ -\sum_{\bar{\sigma}} v_{\bar{a}\bar{b}}(0,\bar{\sigma})\vec{J}^{(j)*}_{\bar{\sigma}\sigma} &= \sum_{ab} \vec{\mathcal{J}}^{(A,B)}_{\bar{a}\bar{b},ab} v_{ab}(0,\sigma) \end{aligned}$$

These can be put into the form,using $\vec{\mathcal{J}}^{(A,B)} = \vec{\mathcal{L}}^{(A)} \otimes 1 + 1 \otimes \vec{\mathcal{R}}^{(B)}$:

$$\begin{aligned} \sum_{\bar{\sigma}} u_{\bar{a}\bar{b}}(0,\bar{\sigma})\vec{J}^{(j)}_{\bar{\sigma}\sigma} &= \sum_{a} \vec{\mathcal{L}}^{(A)}_{\bar{a}a} u_{a\bar{b}}(0,\sigma) + \sum_{b} \vec{\mathcal{R}}^{(B)}_{\bar{b}b} u_{\bar{a}b}(0,\sigma) \\ -\sum_{\bar{\sigma}} v_{\bar{a}\bar{b}}(0,\bar{\sigma})\vec{J}^{(j)*}_{\bar{\sigma}\sigma} &= \sum_{a} \vec{\mathcal{L}}^{(A)}_{\bar{a}a} v_{a\bar{b}}(0,\sigma) + \sum_{b} \vec{\mathcal{R}}^{(B)}_{\bar{b}b} v_{\bar{a}b}(0,\sigma) \end{aligned} \tag{5.27}$$

where $\vec{\mathcal{L}}/\vec{\mathcal{R}}$ are generators of $SU(2)$ in spin-A,spin-B representation.This form indicate that the coefficient functions in standard frame are just the **projective tensors of little group** $SU(2)$,they specify how spin-A and spin-B are combined to produce spin-j[49]:

$$u_{ab}(0,\sigma) \propto C_{AB}(j\sigma;ab) = C^{\sigma}_{ab}$$

The **conventional** choice is:

$$u_{ab}(0,\sigma) = \frac{1}{\sqrt{2m}} C_{AB}(j\sigma;ab) \tag{5.28}$$

- This is exact/unambiguous since representation (A,B) contain spin-j **at most once**

[46] The field is for single particle type,thus the representation of Lorentz group is **irreducible.**And the constants are index independent,we are using the more formal non-relativistic normalization.For the case of reducible representation being the direct sum of two chiral counterpart,we simply **direct sum the field indices**.It should be remember that the expressions may differ for different direct-sum-part,the derivation below is for **single chiral part**

[47] As noted above,in canonical formalism,this operator is the quantization of solutions to the free field equations,with reletivistic dispersion.

[48] In canonical formalsim,these comes from solution of field equations.Which is equivalent to group theoretical methods.

[49] ψ_{ab} transform under rotation as $e^{i\theta\cdot\mathcal{L}^{(A)}} \otimes e^{i\theta\cdot\mathcal{R}^{(B)}}$,the combination $\sum_{ab} C_{AB}(j\sigma;ab)\psi_{ab}$ transform under rotation as $e^{i\theta\cdot\mathcal{J}^{(j)}}$,the combination can be understood as:

$$\psi_{ab} = \langle A,B;a,b|\psi\rangle, \psi^j_\sigma = \langle j;\sigma|\psi\rangle = \sum_{ab}\langle j;\sigma|A,B;a,b\rangle\langle A,B;a,b|\psi\rangle, C_{AB}(j\sigma;ab) = \langle j;\sigma|A,B;a,b\rangle$$

- This gives the embedding of massive particle of spin-j into field promoted from (A,B)-representation/tensor containing[50] spin-j irreducible[51] part.
- Group theoretically,this is projection from $A \otimes B$ representation of SO(3)-rotation(tensor) subgroup into the spin-j representation of the little group(particle).By induced representaion,we obtain infinite-dim irreducible projective unitary representations both for $SO(1,3)$ and $ISO(1,3)$.

From the relation[52]:

$$-\vec{J}^{(J)*}_{\bar{\sigma}\sigma} = (-1)^{\bar{\sigma}-\sigma}\vec{J}^{(j)}_{-\bar{\sigma},-\sigma} \Rightarrow (-1)^{j-\sigma}v_{ab}(0,-\sigma) \propto u_{ab}(0,\sigma)$$

This comes form the fact that $(-1)^{j-\sigma}v_{ab}(0,-\sigma)$ satisfies the same equation as u_{ab},thus we have the **conventional unique** solution:

$$v_{ab}(0,\sigma) = (-1)^{j+\sigma}u_{ab}(0,-\sigma) \tag{5.29}$$

The general coefficient functions are obtained by **standard boost**: $L(p)^0{}_0 = cosh\theta = \gamma, L^i{}_0 = L^0{}_i = \hat{p}_i sinh\theta = \gamma\vec{v}, L^i{}_j = \delta_{ij} + (cosh\theta - 1)\hat{p}_i\hat{p}_j$.

The boost transformation in (A,B) representation is:

$$\vec{\mathcal{K}}^{(A,B)} = i(\vec{\mathcal{L}}^{(A)} \otimes 1 - 1 \otimes \vec{\mathcal{R}}^{(B)}), \mathfrak{D}(L(p)) = \exp(+i\hat{p}\cdot\vec{\mathcal{K}}^{(A,B)}\theta) \equiv \exp(-\hat{p}\cdot\vec{\mathcal{L}}^{(A)}\theta)\exp(+\hat{p}\cdot\vec{\mathcal{R}}^{(B)}\theta)$$

Then from the transformation properties of coefficient functions:

Coefficient functions of (A,B)-irreducible-tensor field operators of massive (m,j) single-particle

$$\begin{aligned} u_{ab}(\vec{p},\sigma) &= \sqrt{m/\omega_p}\sum_{a,b}\mathfrak{D}(L(p))_{ab,a'b'}u_{a'b'}(0,\sigma) \\ &= \frac{1}{\sqrt{2\omega_p}}\sum_{a'b'}(\exp(-\hat{p}\cdot\vec{\mathcal{L}}^{(A)}\theta))_{aa'}(\exp(+\hat{p}\cdot\vec{\mathcal{R}}^{(B)}\theta))_{bb'}C_{AB}(j\sigma;a'b') \\ v_{ab}(\vec{p},\sigma) &= (-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma) \end{aligned} \tag{5.30}$$

- In specific representations,the (a,b) is replaced with l and we usually use matrice form for compactness.
- For representations being a direct sum of chiral counterparts,these relations are not identical for each counterparts,and we should include in l implicitly an index manifesting irreducible representations,and these expressions depend on the index implicitly.
- C_{AB} appearing here manifest the little group $SO(3)$.
- The fact that $v(\sigma) \leftrightarrow u(-\sigma)$ is that **Charge conjugation reverse j_3.**

Remark 33 ***Qusetion:****Is it possible to construct coefficient functions similarly using purely group theoretical methods? The starting point is that we wish to embed the superselection sector(which may be classified by certain representation,especially the little groups) into the coefficient functions,whose transformation properties can be figured out by transformation properties of superselection sectors(a,a†).Finaly we reduce the embedding in some standard frame related to the little group.The embedding of relativistic particle is rather simple,since the representation of $SL(2,\mathbb{C})$ and the little group* ***both*** *reduce to $SU(2)$ for massive case,thus the coefficient functions are constructed out of projective tensors,purely group theoretically.Is it possible for this procedure to be true for any case?It is known that when the little group is ISO(2),there would be extra gauge transformations due to the extra structure of ISO(2).*

Remark 34
- *In* ***canonical formalism****,the u/v are considered as the polarization vector in p-space,then the u/v are obtained from the independent solutions of the classical equaitons.In many cases,this is actually more practical,since we don't need to further use projective tensors to change to another index basis.*

[50] reducible,decomposed into direct sums
[51] As SU(2)-rep
[52] This can be varified from explicit representation

5.3.2 Constructing scalar H

The field operators for **massive** operator transform as[53]:

$$U_0(\Lambda)\psi_{ab}(x)U_0^{-1}(\Lambda) = \sum_{a'b'} \mathfrak{D}^{(A,0)}_{aa'}(\Lambda^{-1})\mathfrak{D}^{(0,B)}_{bb'}(\Lambda^{-1})\psi_{a'b'}(\Lambda x)$$

Then the projective tensor $g_{...,...}$ needed to construct the scalar H-density is constructed out of projective tensors of $SO(3)$ as noted before,the scalars would take form as[54]:

$$\sum_{a_1...a_n} \sum_{b_1...b_n} g_{a_1...a_n,b_1...b_n} \psi^{(1)}_{a_1 b_1}(x)...\psi^{(n)}_{a_n b_n}(x)$$

The g are constructed out of projective tensors like **Clebsch-Gordon coefficients** and **Wigner 3-j symbols**,etc.But in specific representations,we will work with some basis transformed projective tensors,like **Pauli matrices,Gamma-matrices**, which are more convenient.

5.3.3 Causality condition and spin-statistic connection

In general,for $(A,B)-\psi, (\tilde{A},\tilde{B})-\tilde{\psi}$ of the **same** particle with $(m\neq 0, j)$:[55]

$$\begin{aligned}
[\psi_{ab}(x), \psi^{\dagger}_{a'b'}(y)]_{\mp} &= \int \frac{d^3p}{(2\pi)^3(2\omega_p)} \pi_{ab,\tilde{a}\tilde{b}}(\vec{p})[\kappa\tilde{\kappa}^* e^{ip\cdot(x-y)} \mp \lambda\tilde{\lambda}^* e^{-ip\cdot(x-y)}] \\
(2\omega_p)^{-1}\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) &\equiv \sum_{\sigma} u_{ab}(\vec{p},\sigma)\tilde{u}^*_{\tilde{a},\tilde{b}}(\vec{p},\sigma) = \sum_{\sigma} v_{ab}(\vec{p},\sigma)\tilde{v}^*_{\tilde{a},\tilde{b}}(\vec{p},\sigma)
\end{aligned} \tag{5.31}$$

- Due to $[a,\tilde{a}]_{\mp}\equiv 0$,$\psi^{+/-}$ (anti)commute with $\tilde{\psi}^{+/-}$ automatically.Thus what left in $[\psi,\tilde{\psi}^{\dagger}]_{\mp}$ is $[\psi^{\pm},\tilde{\psi}^{\mp}]_{\mp}$.The need to use $\psi,\psi^{\dagger}$ is from hermiticity of $\mathcal{H}_I$[56],this fact will constraint the way $\psi^{\pm}$ are combined:**the combination must form a field operator.**[57]
- The π[58] is known as spin-sum,the 2nd equality follows form the relation $v(\sigma)=(-1)^{j+\sigma}u(-\sigma)$,$\sum_{\sigma} vv^* = \sum_{\sigma}(-1)^{2(j-\sigma)}uu^*(-\sigma)$,and note that the summation over σ is symmetric,and $(-1)^{2(j-\sigma)}=1$
- The spin-sum is for irreducible representations,for reducible representations we should consider the direct sum couterparts independently.
- The physical meaning of spin-sum of coefficient functions can be understood as some projection operator as having the form:

$$spin-sum \sim \sum_{\sigma} |u\rangle\langle u|, normalization \sim \langle u|u\rangle =_{matrix} u^{\dagger}u$$

- These spin-sum will appear in propagators in p-space[59].

The explicit form of π is:

$$\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) = \sum_{a'b'}\sum_{\tilde{a}'\tilde{b}'}\sum_{\sigma} C_{AB}(j\sigma;a'b')C_{\tilde{A}\tilde{B}}(j\sigma;\tilde{a}'\tilde{b}')\exp(-\theta\hat{p}\cdot\vec{\mathcal{L}}^{(A)})_{aa'}\exp(+\theta\hat{p}\cdot\vec{\mathcal{R}}^{(B)})_{bb'}\exp(-\theta\hat{p}\cdot\vec{\mathcal{L}}^{(\tilde{A})})^*_{\tilde{a}\tilde{a}'}\exp(+\theta\hat{p}\cdot\vec{\mathcal{R}}^{(\tilde{B})})^*_{\tilde{b}\tilde{b}'}$$

This gives the property:

$$\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) = P_{ab,\tilde{a}\tilde{b}}(\vec{p},\omega_p), P(-\vec{p},-\omega_p) = (-1)^{2A+2\tilde{B}}P(\vec{p},\omega_p)$$

[53]The translational invariance is absorbed into the Fourier transformation factors.

[54]**The derivative of a field can always be decomposed into fields of other types without derivatives,thus only field is needed**

[55]We consider the (anti)commutator between field and adjiont of field of various types,but as noted,in specific representaiton,it would be more convenient to use some other basis,like $\bar{\Psi}$ for Dirac representation

[56]**Normal ordering**

[57]This fact is used in discussion of discrete transformations

[58]There are various normalization convention,in specific representations

[59]the appearence can be considered comes form residue of spin/polarization in experiments.This fact is revealed by the cutting rule,a result from unitarity.

since $\omega_p^2 = \vec{p}^2 + m^2$:

$$\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) = P_{ab,\tilde{a}\tilde{b}}(\vec{p}) + 2\omega_p Q(ab,\tilde{a}\tilde{b})(\vec{p}), P(-\vec{p}) = (-1)^{2A+2\tilde{B}}P(\vec{p}), Q(-\vec{p}) = -(-1)^{2A+2\tilde{B}}Q(\vec{p})$$

Using P,Q-function,the **space-like** (anti)commutator is(take $x^0 - y^0 = 0$)[60]:

$$\begin{aligned}[\psi_{ab}(x), \tilde{\psi}^\dagger_{\tilde{a},\tilde{b}}(y)]_\mp &= [\kappa\tilde{\kappa}^* \mp (-1)^{2A+2\tilde{(B)}}\lambda\tilde{\lambda}^*]P_{ab,\tilde{a}\tilde{b}}(-i\nabla)\Delta_+(0,\vec{x}-\vec{y}) \\ &+ [\kappa\tilde{\kappa}^* \pm (-1)^{2A+2\tilde{B}}\lambda\tilde{\lambda}^*]Q_{ab,\tilde{a}\tilde{b}}(-i\nabla)\delta^3(\vec{x}-\vec{y})\end{aligned} \tag{5.32}$$

- The $(-1)^{2A+2\tilde{(B)}}$ factor comes from $\vec{p} \to -\vec{p}$ in integral of $e^{-ip\cdot(x-y)}$,the sign form properties of Q is crucial in the 2nd braket.The $2\omega_p$ in front of Q cancels the $2\omega_p$ in the measure and leading to δ^3
- The P,Q are pulled out of the integral and is a fucntion of differential operators.
- The Δ_+ is known as the **on-shell forward-propagator**[61]:

$$\Delta_+(x) = \int \tilde{dp} e^{ip\cdot x} = \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{ip\cdot x} \equiv (2\pi)^{-3} \int d^4p \delta(p^2+m^2)\theta(p^0)e^{ip\cdot x}$$

  - This is manifestly **Lorentz-invariant**,depending only on x^2,thus **even in x**.
  - For space-like x,choose $x^0 = 0$,then

$$\Delta_+(x) = (2\pi)^{-3}\int \frac{d^3p}{2\sqrt{|p|^2+m^2}} e^{i\vec{p}\cdot\vec{x}} = \frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{p^2dp}{2\sqrt{|p|^2+m^2}} \frac{\sin(p\sqrt{x^2})}{p\sqrt{x^2}} \equiv \frac{m}{4\pi^2\sqrt{x^2}}K_1(m\sqrt{x^2}) \neq 0$$

  K_1 is the **Hankel function**

The causality condition,following form Poincare invariance of S-matrix and consequntly the causality condition of scalar density $\mathcal{H}$,requires:

$$\kappa\tilde{\kappa}^* = \pm(-1)^{2A+2\tilde{B}}\lambda\tilde{\lambda}^*$$

- For $(A,B) = (\tilde{A},\tilde{B})$,this case always appear due to causality condition and hermiticity of $\mathcal{H}$:

$$|\kappa|^2 = \pm(-1)^{2A+2B}|\lambda|^2 \equiv \pm(-1)^{2j}|\lambda|^2 \Rightarrow |\kappa|^2 = |\lambda^2|, \pm\ for\ 2j\ even/odd$$

  This is the **spin-statistic connection:the particle is a boson/fermion thus the commutator taken as commutator/anticommutator,the field-components considered as ordinary/Grassman number for spin is integer/half-integer** in relativistic field theory.Note this theory also **hold for massless particles.**[62]

- For general case[63]:

$$\frac{\kappa}{\tilde{\kappa}} = (-1)^{2B+2\tilde{B}}\frac{\lambda}{\tilde{\lambda}} \Rightarrow \frac{\lambda}{\kappa} = (-1)^{2B}c, |c| = 1$$

  The constant phase c can be eliminated by redifinition of the relative phase of $a, a^\dagger$:

$$\lambda = (-1)^{2B}\kappa \tag{5.33}$$

  This implies that **up to scale**[64],any (A,B)-irreducible-field for a given particle j is **unique**[65]:

$$\psi_{ab}^{(A,B)} \propto \sum_{\sigma(j)}\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}[u_{ab}(\vec{p},\sigma)a(\vec{p},\sigma)e^{ip\cdot x} + (-1)^{2B}v_{ab}(\vec{p},\sigma)a_c^\dagger(\vec{p},\sigma)e^{-ip\cdot x}]$$

[60] If this vanishes,$[\psi,\psi']$ will vanish automatically,or,think ψ' annhilates antiparticle

[61] This comes form:

$$[\psi^+,\psi^-]_\mp = \pi(-i\partial_\mu)\Delta_+$$

,the commutator can be understood as as a the propagation(create and annihilate) of a on-shell particle,not that this include**no antiparticle,thus the causality is fiexed**,the Feynman propagator will combine propagation of both particle and antiparticle:$[\psi,\psi^\dagger]_\mp$,thus a virtual particle,and the causality combine both situation.**The causality is manifested by the pole structure of the propagators**

[62] For massless case,the commutator can be put into a similar form,with certain spin-sum over **physical** polarizations

[63] dividing $|\tilde{\kappa}|^2 = |\tilde{\lambda}|^2$,and use $\pm(-1)^{2A+2B} = 1$

[64] The relative phase between λ,κ is fixed for any field,thus causality condition will constraint all (A,B)-field of the same particle to have the same relative phase,this will be used in discussion of discrete symmetries.

[65] For massless field this is not true,since the field is determined up to gauge transformation

Remark 35

- *In **canonical formalism**:the field operators will be considered as promoted from classical solution of the field equations following form the classical Lagrangian in **I-piture**.The (anti)commutator $[\psi, \psi^\dagger]$ follows from the **Possion braket of canonical variables**,and is taken as an assumption,this assumption is equivalent to the (anti)commutation $[a, a^\dagger]$,this restore the whole 2nd quantization.*

- ***Free vs. Interacting**:As mentioned before,the free fields refer to the **I-picture** field operators,where the 'free' particle is on-shell,and that $a, a^\dagger$ transform under time-translation as:*

$$e^{iHt}a(\vec{p})_I e^{-iHt} = e^{-i\omega_p t}a(\vec{p})$$

In S-picture,this is equivalent to:

$$[H_0, a_S] = 0$$

and we may define the S-piture field operator as:

$$\psi(\vec{x}) \sim a(\vec{p})_S e^{+i\vec{p}\cdot\vec{x}}$$

When there's no interaction,the H-picture operator agree with the I-picture operaor,and they are called the free fields.

*When there's interaction,the I-piture still have the same form as free fields,where the particles are **on-shell**.However the H-piture field operator will not have such a form,this is due to:*

$$[H, a_S] \neq 0 \Rightarrow e^{iHt}a_S e^{-iHt} \neq e^{-iHt}a_S$$

In general,the $a_H(t)$ will mix $a_S = a_I = a_{free}$,but since asymptoticly,the field is free,we have $a_H(t \to \mp\infty) = a_{in/out}$ and the asymptotic field can be expanded like a free field.

Note that in H-piture the free (anti)commutator $[\psi, \psi^\dagger]$ is also modified by interaction.

5.3.4 Different field operators of same particle are physically equivalent

The fact that (A,B)-field of massive spin-j particle is unique implies that different field of the same particle are related by derivatives.

In general,any (A,B)-field of spin-j particle can be expressed as a differential operator of rank 2B/2A acting on (j,0)/(0,j) type field.This fact follows from the fact that $\{\partial_{\mu_1}...\partial_{\mu_{2B}}\}$ being a symmetric traceless tensor of rank 2B,is a (B,B) representation.Thus:

$$\{\partial_{\mu_1}...\partial_{\mu_{2B}}\}\phi_\sigma^{(j,0)}$$

can be decomposed into irreducible representations from $|j - B| \leq A \leq j + B \Leftrightarrow |A - B| \leq j \leq A + B$ thus the unique(up to scale) (A,B) representation can always be obtained in this way.

We then focus on cases $(j, 0), (0, j), (j, 0) \oplus (0, j)$ fields for **massive** spin-j particles.Note that for massless fields,this argument is not true,we have to consider (A,A) fields,etc.[66]

5.3.5 Discrete symmetries of field operators:massive case

P-parity

The transformation of $a, a^\dagger$ for massive(σ is spin-3) particles transform under P[67] as:

$$Pa(\vec{p}, \sigma)P^{-1} = \eta^* a(-\vec{p}, \sigma), Pa_c^\dagger(\vec{p}, \sigma)P^{-1} = \eta_c a_c^\dagger(-\vec{p}, \sigma)_c^\dagger$$

Using properties of Clebsch-Gordon coefficients,which helps to change the variable in u/v from $\vec{p}$ to $-\vec{p}$:

$$C_{AB}(j\sigma; ab) = (-1)^{A+B-j}C_{BA}(j\sigma; ba) \Rightarrow$$
$$u_{ab}^{(A,B)}(-\vec{p}, \sigma) = (-1)^{A+B-j}u_{ba}^{(B,A)}(\vec{p}, \sigma), v_{ab}^{(A,B)}(-\vec{p}, \sigma) = (-1)^{A+B-j}v_{ba}^{(B,A)}(\vec{p}, \sigma)$$

[66]The flaw lies in the transformation of u/v or equivalently,gauge transformations of ψ;The fact that we have to consider (A,A) fields are nontrivial,since there's $(1, 0) \oplus (0, 1)$ also containing a massless spin-1 particle

[67]We would assume all particles not only participating in weak interactions having well defined unitary P operator.This means for particles like ν_i the intrinsic P/T phases are ill-defined

Thus the transformation of massive field operators under Parity is:

$$
\begin{aligned}
P\psi_{ab}^{(A,B)}(x)P^{-1} &= \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}[\eta^* a(-\vec{p},\sigma)e^{ip\cdot x}u_{ab}^{(A,B)}(\vec{p},\sigma) + \eta_c(-1)^{2B}a_c^\dagger(-\vec{p},\sigma)e^{-ip\cdot x}v_{ab}^{(A,B)}(\vec{p},\sigma)] \\
&= \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}(-1)^{A+B-j}[\eta^* a(\vec{p},\sigma)e^{ip\cdot \mathcal{P}x}u_{ba}^{(B,A)}(\vec{p},\sigma) + \eta_c(-1)^{2B}a_c^\dagger(-\vec{p},\sigma)e^{-ip\cdot \mathcal{P}x}v_{ba}^{(B,A)}(\vec{p},\sigma)]
\end{aligned}
$$

For the RHS to be the unique (B,A) field up to scale[68]:

$$
\eta_c(-1)^{2B}/\eta^* = (-1)^{2A} \Rightarrow \eta_c = \eta^*(-1)^{2j}
$$

- The intrinsic parity of **particle-antiparticle pair** is ± 1 for boson/fermion.
- Under parity,the field transform like:

$$
P\psi_{ab}^{(A,B)}(x)P^{-1} = \eta^*(-1)^{A+B-j}\psi_{ba}^{BA}(\mathcal{P}x) \tag{5.34}
$$

  - The action of P on field operators contain two part:act on spacetime coordinates as $\mathcal{P}$ in $(\frac{1}{2},\frac{1}{2})$ representation,interchange the representation to (B,A) by interchanging the chiral generators.
  - The phase η comes from the particle ψ describe.
  - There's extra signs due to parity of coefficient functions.
  - For reducible representation $(A,B)\oplus(B,A)$,this interchange implies that the two counterpart in $\psi_{l=l_L\oplus l_R}$ are changed to $\psi_{l=l_R\oplus l_L}$.This is usually down by specific representation of $\mathcal{P}$ in this representation.

C-parity

The effect of charge conjugation on $a, a^\dagger$:

$$
Ca(\vec{p},\sigma)C^{-1} = \xi^* a_c(\vec{p},\sigma), Ca_c^\dagger(\vec{p},\sigma)C^{-1} = \xi_c a^\dagger(\vec{p},\sigma)
$$

using the fact $v_{ab}(\vec{p},\sigma) = (-1)^{j+\sigma}u_{ab}(\vec{p},-\sigma)$,and use the fact that sum over σ is symmetric,the transformation of ψ is:

$$
C\psi_{ab}^{(A,B)}(x)C^{-1} = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}u_{ab}^{(A,B)}(\vec{p},\sigma)[\xi^* a_c(\vec{p},\sigma)e^{ip\cdot x} + \xi_c(-1)^{2B}a^\dagger(\vec{p},-\sigma)(-1)^{j-\sigma}]
$$

The adjiont of the (B,A) field of the **same** particle:

$$
\psi_{ba}^{(B,A)\dagger} = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}u_{ba}^{(B,A)*}(\vec{p},\sigma)[(-1)^{2A}(-1)^{j-\sigma}a_c(\vec{p},-\sigma)e^{ip\cdot x} + a^\dagger(\vec{p},\sigma)e^{-ip\cdot x}]
$$

To calculate u^*,use the result[69]:

$$
-\vec{\mathcal{J}}^{(j)*} = -\vec{\mathcal{J}}^{(j)t} = \mathcal{C}\vec{\mathcal{J}}\mathcal{C}^{-1} \equiv (-1)^{\sigma-\sigma'}\vec{\mathcal{J}}^{(j)}_{-\sigma,-\sigma'} \Rightarrow \mathcal{C}_{\sigma\sigma'} \propto (-1)^{j-\sigma}\delta_{\sigma',-\sigma}
$$

The Clebsch-Gordon coefficients are real:

$$
u_{ba}^{(B,A)}(\vec{p},\sigma)^* = \frac{1}{\sqrt{2\omega_p}}\sum_{a'b'}\exp(-\theta\hat{p}\cdot\vec{\mathcal{L}}^{(A)})_{-a,-a'}\exp(+\theta\hat{p}\cdot\vec{\mathcal{R}}^{(B)})_{-b,-b'}(-1)^{a'-a}(-1)^{b'-b}C_{BA}(j\sigma;b'a')
$$

using the reflection properties of C_{AB}:

$$
C_{BA}(j,-\sigma;-a,-b) = C_{AB}(j\sigma;a,b)
$$

we have,with $a+b = a'+b' = \sigma$:

$$
u_{-b,-a}^{(B,A)*}(\vec{p},-\sigma) = (-1)^{a+b-\sigma}u_{ab}^{(A,B)}(\vec{p},\sigma)
$$

[68]This is the requirement of **causality condition**:if the system have coorespoding discrete symmetry,then there exsist the transformed fields,and these field must also **(anti)commute with all other fields**,thus the tranformed RHS must also be a field operator

[69]note that in any representation,J is unitary

Thus,also using $(-1)^{-2A-j}=(-1)^{2B+j}, a+b=\sigma$:

$$(-1)^{2A-a-b-j}\psi^{(B,A)\dagger}_{-b,-a}(x)=\sum_\sigma\int\frac{d^3p}{(2\pi)^{3/2}}(-1)^{a+b-\sigma}u^{(A,B)}_{ab}(\vec{p},\sigma)[a_c(\vec{p},\sigma)e^{ip\cdot x}+(-1)^{j-\sigma+2B}a^\dagger(\vec{p},\sigma)e^{-ip\cdot x}]$$

Thus for the RHS of transformation of ψ to be the unique up to scale $\psi^{(B,A)\dagger}$,we need:

$$\xi^*=\xi_c \tag{5.35}$$

and we have:

$$C\psi^{(A,B)}_{ab}C^{-1}=\xi^*(-1)^{-2A-a-b-j}\psi^{(B,A)\dagger}_{-b-a}(x) \tag{5.36}$$

- For particles being their own particles[70],we have **reality condtions**:

$$\psi^{(A,B)}_{ab}=(-1)^{-2A-a-b-j}\psi^{(B,A)^\dagger}_{-b,-a}(x) \tag{5.37}$$

  This will reduce the number of real free D.O.F(components)

- The action of charge conjugation on field of massive particles have several effects:
  - the chirality(thus helicity for massless particles) is interchanged,since p is not reversed, the J_3 is reversed.
  - There's dagger acting on ψ,for field operator,the dagger implies $a\to a_c$,the particle is replaced by its antiparticle.For tensor of the representation,the dagger simply means we go to the **complex conjugate representation**
  - When we express the field operator as a multiplet(column vector),then $\dagger\to *$ and note that the $*$ act on $a, a^\dagger$ as $\dagger$
  - These actions will have more compact expressions using projective tensors in certain representations.
  - There's phase and sign factors.

T-reversal

The action of time reversal on $a, a^\dagger$:

$$Ta(\vec{p},\sigma)T^{-1}=\zeta^*(-1)^{j-\sigma}a(-\vec{p},-\sigma), Ta^\dagger_c(\vec{p},\sigma)T^{-1}=\zeta_c(-1)^{j-\sigma}a^\dagger_c(-\vec{p},-\sigma)$$

Thus,note that T is **antiunitary**:

$$T\psi^{(A,B)}_{ab}(x)T^{-1}=\sum_\sigma\int\frac{d^3p}{(2\pi)^{3/2}}u^{(A,B)*}_{ab}(\vec{p},\sigma)(-1)^{j-\sigma}[\zeta^*a(-\vec{p},-\sigma)e^{-ip\cdot x}+\zeta_c(-1)^{2B}a^\dagger_c(-\vec{p},-\sigma)e^{ip\cdot x}]$$

From:

$$C_{AB}(j\sigma;ab)=(-1)^{A+B-j}C_{AB}(j,-\sigma;-a;-b)$$

Then[71]:

$$u^{(A,B)*}_{ab}(-\vec{p},-\sigma)=(-1)^{A+B-j+a+b+\sigma}u^{(A,B)}_{-a,-b}(\vec{p},\sigma)$$

For the transfomed field to be another field operator up to scale:

$$\zeta_c=\zeta^* \tag{5.38}$$

and the field transform as:

$$T\psi^{(A,B)}_{ab}(x)T^{-1}=(-1)^{a+b+A+B-2j}\zeta^*\psi^{(A,B)}_{-a,-b}(-\mathcal{P}x) \tag{5.39}$$

The action of T:

- Reverse j_3.
- There are phase and sign factors.
- T act on the spacetime coordinates in $(\frac{1}{2},\frac{1}{2})$ representation.

[70] must be neutral
[71] Also use the result in C-parity

5.3.6 CPT theorem

There's a presice/exact relation between particle and antiparticle,known as the **CPT theorem:**For an appropriate choice of discrete symmetry phases,the product CPT is conversed at least in relativistic field theories.

Putting together the results[72]:

$$(CPT)\psi^{(A,B)}_{ab}(x)(CPT)^{-1} = (-1)^{2B}\psi^{(A,B)\dagger}_{ab}(-x)$$

To project out scalar hermitian density $\mathcal{H}_I$,we need:$\sum_i B_i, \sum_i A_i = integers$,thus:

$$(CPT)\mathcal{H}_I(x)(CPT)^{-1} = \mathcal{H}_I(-x) \Rightarrow [CPT, V] = 0, [CPT, H] = 0$$

Remark 36 *The discrete transformation of the fields can also be discussed in terms of field equations,where the fields are considered as solution,and the transformed fields must also be a solution.* ***This methods is more practical,but not essential.***

Summary of inversion phases in relativistic field theory

For 3 cases we have:

$$\eta_c = \eta^*(-1)^{2j}, \xi_c = \xi^*, \zeta_c = \zeta^*$$

for particle-antiparticle pair,the intrinsic phases are $(-1)^{2j}, +1, +1$,for particle being its own particle:$\xi = \pm 1, \zeta = \pm 1$,

$$\eta = \sqrt{(-1)^{2j}}$$

For boson,$\eta = \pm 1$,for fermion,$\eta = \pm 1, \pm i$,this is the case for Majorana fermion.
The definition of the phases have some ambigurity ($\alpha^2 = e^{i\theta}$)as noted before,if there's continuous phase transformation.We will adjust for all particles:

$$\zeta\xi\eta \equiv 1$$

5.4 Examples of massive fields and the problem of massless limit

Remark 37 *This section is rather redundantly written,and should be rewritten with more clearly*

5.4.1 About normalization

Different normalization differ in the convention of the measure of momentums,and correspondingly the normalization of $a, a^\dagger$,normalization of other indices are the same,so we focus on the p-index.

The difference in measure of p-state is equivalent to the differ in normalization of 2nd quantization algerbas,this comes from the **definition** of p-state of single-particle[73]:

$$|\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle$$

Non-relativistic normalization of field operators

$$\int d^3p|\vec{p}\rangle\langle\vec{p}| = 1, \langle\vec{x}|\vec{p}\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\vec{p}\cdot\vec{x}}, [a(\vec{p}), a^\dagger(\vec{p}')] = \delta^3(p-p') = \langle\vec{p}|\vec{p}'\rangle$$

The Fourier transformation is:

$$f(\vec{x}) = \int d^3p\langle\vec{x}|\vec{p}\rangle\langle\vec{p}|f\rangle = \int \frac{d^3p}{(2\pi)^{3/2}}e^{i\vec{p}\cdot\vec{x}}f(\vec{p})$$

Relativistic normalization

$$\langle\vec{p}|\vec{p}'\rangle = (2\pi)^3(2\omega_p)\delta^3(\vec{p}-\vec{p}') = [a(\vec{p}), a^\dagger(\vec{p}')], \langle\vec{x}|\vec{p}\rangle = \sqrt{2\omega_p}e^{i\vec{p}\cdot\vec{x}}$$

Thus:

$$\int \frac{d^3p}{(2\pi)^3(2\omega_p)}|\vec{p}\rangle\langle\vec{p}| = \int \tilde{dp}|\vec{p}\rangle\langle\vec{p}| = 1, f(\vec{x}) = \int \tilde{dp}\langle\vec{x}|\vec{p}\rangle\langle\vec{p}|f\rangle = \int \frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot\vec{x}}f(\vec{p})$$

[72]The phases of all particles are chosen such that $\zeta\xi\eta \equiv 1$

[73]Some text would define $|\vec{p}\rangle = \sqrt{2\omega_p}a^\dagger(\vec{p})|0\rangle$,this is actually not convenient

Changing between two normalization

The normalization is related by:

$$|\vec{p}\rangle_{LR} = \sqrt{(2\pi)^3(2\omega_p)}|\vec{p}\rangle_{NR}, a^\dagger_{LR} = \sqrt{(2\pi)^3(2\omega_p)}a^\dagger_{NR}$$

Thus the field operator is:

$$\psi_{ab} = \int \frac{d^3p}{(2\pi)^{3/2}} u_{ab} a e^{ip\cdot x} + v_{ab} a_c^\dagger e^{-ip\cdot x}$$

in LR-normalization:

$$\psi_{ab} = \int \frac{d^3p}{(2\pi)^3(2\omega_p)} u_{ab} a e^{ip\cdot x} + ... = \int \tilde{d}p u_{ab} a e^{ip\cdot x} + ...$$

Note that in LR-noremalization,the normalization of u_{ab} is also different,same for v:

- NR:
$$u_{ab}(\vec{p},\sigma) = \frac{1}{\sqrt{2\omega_p}} C_{AB}(j\sigma; ab)$$
- LR:
$$u_{ab}(\vec{p},\sigma) \sim \langle \vec{p},\sigma|u_{ab}\rangle = \sqrt{2\omega_p}\langle \vec{p}; NR,\sigma|u_{ab}\rangle = C_{AB}(j\sigma; ab)$$
  And thus:
$$\pi_{ab,\tilde{a}\tilde{b}}(\vec{p}) = \sum_\sigma u_{ab}(\vec{p},\sigma) u^*_{\tilde{a}\tilde{b}}(\vec{p},\sigma)$$

It's clear that NR is more formal,while LR is manifestly Lorentz-invariant in measure and the coefficient functions are directly C_{AB}.The spin-sum is also free of $\sqrt{2\omega_p}$.

5.4.2 Scalar field

For spin-0 particles,the most simple field is the $(0,0)$-field,using scalar tensor,which is dim-1.For this case,$C_{00}(00;00) = 1$,thus $u_{ab} = \frac{1}{\sqrt{2\omega_p}}$.The spin-sum is trivial.

The field is then[74]:

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2\omega_p)}}[a(\vec{p})e^{ip\cdot x} + a_c^\dagger(\vec{p})e^{-ip\cdot x}]$$

For **any** x,y,the **commutator** is:

$$[\phi(x), \phi^\dagger(y)]_- = \Delta_+(x-y) - \Delta_+(y-x) \tag{5.40}$$

This will vanish at spacelike separation.The transformation under discrete symmetry is(RHS must still be a field operator):

$$\begin{aligned}
\eta_c &= \eta_*, ...\\
P\phi(x)P^{-1} &= \eta^*\phi(\mathcal{P}x)\\
C\phi(x)C^{-1} &= \xi^*\phi^\dagger(x) = \xi^*\phi^*(x)\\
T\phi(x)T^{-1} &= \zeta^*\phi(-\mathcal{P}x)\\
(CPT)\phi(x)(CPT)^{-1} &= (\zeta\xi\eta)^*\phi^*(-x) \equiv \phi^*(-x)
\end{aligned} \tag{5.41}$$

- If $\eta = +1$ the particle is known as a scalar,if $\eta = -1$ the particle is known as a pesudoscalar.
- For $n = n^c$ we have the reality condition:$\xi_c = \xi^* = \xi$,$\phi^*(x) = \phi(x)$
- The complex scalar field then can be decomposed as:
$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

[74]B=0,no σ index,this is NR normalization,for the LR normalization is just a change of measure

Field equation and Lagrangian

The field satisfies the **Klein-Gordon equation**:

$$(-\partial^2 + m^2) = (\Box + m^2)\phi(x) = 0 \tag{5.42}$$

with Lagrangian:

$$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi^* - m^2|\phi|^2 = \sum_i -\frac{1}{2}(\partial\phi_i)^2 - \frac{1}{2}m^2\phi_i^2 \tag{5.43}$$

Non-relativistic limit:Schrodinger equation as a field theory

The 1st quantization theories just take the single particle subspace as the full Hilbert space,thus by limiting on single particle states,we can restore 1st quantizations:

$$\langle 0|\phi(x)|\vec{p}\rangle = e^{ip\cdot x} = \langle x|p, on - shell\rangle$$

The Schrodinger picture wave function is:

$$\Psi(x) = \langle x|\psi\rangle, |x\rangle = \phi(x)|0\rangle$$

then the **Schrodinger equation of wave functions** can be restored by:

$$i\partial_t\Psi(x) = i\langle 0|\partial_t\phi(x)|\Psi\rangle = \langle 0|\sqrt{m^2 - \nabla^2}\phi(x)|\Psi\rangle = \sqrt{m^2 - \nabla^2}\Psi(x)$$

Take the non-relativistic limit,we restore the Schrodinger equation of non-relativistic spinless particle:

$$\sqrt{m^2 - \nabla^2} = m - \frac{\nabla^2}{2m} + ...$$

The m term is to be removed by defining:

$$\Psi(x) = e^{-imt}\Phi(x), i\partial(\Psi) = m\Phi + i\partial_t\Phi$$

Then:

$$i\partial_t\Phi(\vec{x}, t) = -\frac{\nabla^2}{2m}\phi(\vec{x}, t) \tag{5.44}$$

The Lagrangian of Φ,as a classical field is:

$$\mathcal{L} = i\Phi^\dagger\partial_0\Phi^\dagger - \frac{1}{2m}\partial_i\Phi\partial_i\Phi^\dagger$$

This can be reversed,if we consider in **1st quantization limit,the field becomes the wave-fucntion**.Then we can directly take the non-relativistic limit of the $\mathcal{L}$,the conserved current origin form this will have the meaning of **probability density and probability flow.**The integral of the j^0 is propotional to the **conserved** total number of particles.

It is useful to write the non-relativistic limit in $\Phi = \sqrt{\rho}e^{i\theta}$:

$$\mathcal{L} = \frac{i}{2}\partial_0\rho - \rho\partial_0\theta - \frac{1}{2m}[\rho(\partial_i\theta)^2 + \frac{1}{4\rho}(\partial_i\rho)^2]$$

Then:

$$[\rho(x), \theta(x')] = i\delta^{(D)}(x - x') \Rightarrow [N, \theta] = i, N = \int d^D x\rho(x)$$

Remark 38 *This fact is important in CMP.For more application,refer to [Zee.III.5, V, VI]*

5.4.3 Massive vector field

The equations of the $(\frac{1}{2},\frac{1}{2})$ coefficient functions are not convenient written in $u/v_{a\dot{a}}$,we use Pauli matrices to transforme directly to u/v^{μ} and the equations are now:

$$\sum_{\bar{\sigma}} u^{\mu}(0,\bar{\sigma})\vec{\mathcal{J}}^{(j)}_{\bar{\sigma}\sigma} = (\vec{\mathcal{J}}_V)^{\mu}{}_{\nu}u^{\nu}(0,\sigma), -\sum_{\bar{\sigma}} v^{\mu}\vec{\mathcal{J}}^{(j)*}_{\bar{\sigma}\sigma} = (\mathcal{J}_V)^{\mu}{}_{\nu}\nu^{\nu}(0,\sigma)$$

The representation $\vec{\mathcal{J}}_V$ is:

$$(\mathcal{J}_{Vk})^{i}{}_{j} = -i\epsilon_{ijk}$$

thus we have:

$$\sum_{\bar{\sigma}}\begin{pmatrix}u^0(0,\bar{\sigma})\\ v^0(0,\bar{\sigma})\end{pmatrix}\begin{pmatrix}(\vec{\mathcal{J}}^{(j)})^2_{\bar{\sigma}\sigma}\\ (\vec{\mathcal{J}}^{(j)*})^2_{\bar{\sigma}\sigma}\end{pmatrix} = 0, \sum_{\bar{\sigma}}\begin{pmatrix}u^i(0,\bar{\sigma})\\ v^i(0,\bar{\sigma})\end{pmatrix}\begin{pmatrix}(\vec{\mathcal{J}}^{(j)})^2_{\bar{\sigma}\sigma}\\ (\vec{\mathcal{J}}^{(j)*})^2_{\bar{\sigma}\sigma}\end{pmatrix} = 2\begin{pmatrix}u^i(0,\sigma)\\ v^i(0,\sigma)\end{pmatrix}$$

This has two possibility:

Spin-0

For the $(\frac{1}{2},\frac{1}{2})$ field of spin-0 field,it must be constructed from the complex scalar field using ∂^{μ},this can be verified that the solution of u/v in this situation is[75]:

$$u^0 = i\sqrt{m/2}, v^0 = -i\sqrt{m/2}, u^i = v^i = 0 \Rightarrow u^{\mu} = ip^{\mu}(2\omega_p)^{-1/2} = -v^{\mu}(\vec{p})$$

The RHS used the transformation of standard boost $L^{\mu}{}_{\nu}(p)$.This is clear then:

$$\phi^{\mu}(x) \equiv \partial^{\mu}\phi(x)$$

Spin-1

The spin-1 solution is:

$$\begin{aligned} u^{\mu}(0,\sigma=0) = v^{\mu}(0,\sigma=0) &= (2m)^{-1}(0,0,0,1)^t \\ u^{\mu}(0,+1) = -v^{\mu}(0,-1) &= -\frac{1}{\sqrt{2}}(2m)^{-1}(0,1,+i,0)^t \\ u^{\mu}(0,-1) = -\nu^{\mu}(0,+1) &= \frac{1}{\sqrt{2}}(2m)^{-1}(0,1,-i,0)^t \end{aligned} \tag{5.45}$$

- This shows explicitly that charge conjugation(particle-antiparticle) have reversed j_3:$u(+1) \leftrightarrow v(-1), u(-1) \leftrightarrow v(+1)$

The general solution is:

$$u^{\mu}(\vec{p},\sigma) = v^{\mu *}(\vec{p},\sigma) = (2\omega_p)^{-1/2}e^{\mu}(\vec{p},\sigma), e^{\mu}(\vec{p},\sigma) = L^{\mu}{}_{\nu}e^{\nu}(0,\sigma)$$

The $e^{\mu}(0,\sigma) = \epsilon^{\mu}(0,\sigma)$ are known as standard polarization[76]:

$$e^{\mu}(0,0) = (0,0,0,1), e^{\mu}(0,+1) = -\frac{1}{\sqrt{2}}(0,1,+i,0)^t, e^{\mu}(0,-1) = \frac{1}{\sqrt{2}}(0,1,-i,0)^t$$

- The fact that $e^{\mu} = 0$ implies that v^0 is unphysical,or projective out.This implies extra field equations.
- The reason that this solution is not manifestly $v(\sigma) = (-1)^{j+\sigma}u(-\sigma)$ is that we have **implicit used the projective tensor σ^{μ} which contain imaginary constants.**
- The $e^{\mu}(\sigma = 0)$ is also known as the **longitude mode**,for massive case,this mode is indeed **physical**.This is not the case for massless case,and the solution will be propotional to p^{μ}
- The $u^{\mu}(\sigma = \pm 1)$ are transverse modes,these modes will be helicity eigenstate in **massless case**.

[75]by certian normalization,what ever normalization we pick,the core is that this field is equivalent to acting partial on scalar field,up to scale/noremalization

[76]Some text will use other normalization,the e^{μ} are determined up to some constants

- The normalization is[77]:
$$e^{\mu *}(\sigma)e_{\mu}(\sigma) = \langle e(\sigma)|e(\sigma)\rangle = +1$$
- The spin-sum is:
$$(2\omega_p)^{-1}\pi^{\mu\nu} = \Pi^{\mu\nu}(\vec{p}) = \sum_{\sigma} e^{\mu}(\vec{p},\sigma)e^{\nu *}(\vec{p},\sigma) = \eta^{\mu\nu} + \frac{p^{\mu}p^{\nu}}{m^2} \tag{5.46}$$
$\Pi^{\mu\nu}(0)$ is the projection matrix on the space orthogonal to the time-direction,thus $\Pi^{\mu\nu}(\vec{p})$ is the projection matrix on the orthogonal space orthogonal to p^{μ}:
$$\Pi^{\mu\nu}(p)p_{\nu} = 0, e^{\mu}(\vec{p},\sigma)p_{\mu} = 0 \tag{5.47}$$
Thus the spin-sum have the meaning of measurement spin-sum over $M(\sigma)$,the final result of measurement will have factor of spin-sums,and the measurement see the polarization as orthogonal to momentum.[78]

The commutator(spin-1 must be a boson):

$$[\phi^{+\mu}(x), \phi^{-\nu}(y)] = \Pi(-i\partial_{\mu})\Delta_{+}(x-y) = [\eta^{\mu\nu} - \frac{\partial^{\mu}\partial^{\nu}}{m^2}]\Delta_{+}(x-y)$$

The field is then[79]:

$$v^{\mu} = \sum_{\sigma=\{+1,0,-1\}} \int \frac{d^3p}{\sqrt{(2p)^3 2\omega_p}}[e^{\mu}(\vec{p},\sigma)a(\vec{p},\sigma)e^{ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a_c^{\dagger}(\vec{p},\sigma)e^{-ip\cdot x}] \tag{5.48}$$

and the commutator is then:

$$[v^{\mu}(x), v^{\nu\dagger}(y)] = [\eta^{\mu\nu} - \frac{\partial^{\mu}\partial^{\nu}}{m^2}]\Delta(x-y), \Delta(x-y) = \Delta_{+}(x-y) - \Delta_{+}(y-x)$$

Field equation and Lagrangian

The field equations are:
$$(\Box - m^2)v^{\mu}(x) = 0, e^{\mu}(\vec{p},\sigma)p_{\mu} = 0 \Rightarrow \partial_{\mu}v^{\mu}(x) = 0$$

The 2nd comes from properties of e^{μ} thus is **equivalent to equations giving e^{μ}.**This project out the spin-0 D.O.F.

These two equations can be combined together to give:

$$(\eta^{\mu\nu}(-\partial^2 + m^2) + \partial^{\mu}\partial^{\nu})v_{\nu} = 0$$

This implies the Lagrangian is:

$$\mathcal{L} = -\frac{1}{2}A_{\mu}[(-\partial^2 + m^2)\eta^{\mu\nu} + \partial^{\mu}\partial^{\nu}]A_{\nu} \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_{\mu}A^{\mu}, F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$$

This is known as the **Proca Lagrangian.**

Discrete transformations

Since we have changed basis,the discrete transformation is obtained from the transformation properties of $e^{\mu}(\vec{p},\sigma)$[80]

- The space inversion have effect:
$$e^{\mu}(-\vec{p},\sigma) = L^{\mu}{}_{\nu}(-\vec{p})e^{\mu}(0,\sigma), L^{\mu}{}_{\nu}(-\vec{p}) = \mathcal{P}^{\mu}{}_{\alpha}L^{\alpha}{}_{\beta}(\vec{p})\mathcal{P}^{\beta}{}_{\nu} \Rightarrow e^{\mu}(-\vec{p},\sigma) = -\mathcal{P}^{\mu}{}_{\nu}e^{\nu}(\vec{p},\sigma)$$
The RHS used the fact that in spin-1 case,$e^0(0,\sigma) \equiv 0 \Rightarrow \mathcal{P}^{\mu}{}_{\tau}e^{\tau}(0,\sigma) \equiv -e^{\mu}(0,\sigma)$

[77]Some text use the normalization $\langle e|e\rangle = -1$ they differ only by some sign constants
[78]This can be shown generally by cutting rules.
[79]LR noremalization is just a change of measure
[80]Note that σ here is j_3,this will be different from the massless case,thus we **can't just take the massless limit.**

- The effect of time-reversal is:

$$(-1)^{1+\sigma} e^{\mu *}(0,-\sigma) \equiv -e^{\mu}(0,\sigma) \Rightarrow (-1)^{1+\sigma} e^{\mu *}(-\vec{p},-\sigma) = \mathcal{P}^{\mu}{}_{\nu} e^{\nu}(\vec{p},\sigma)$$

Finally,using transformation rule of $a, a^{\dagger}$ and the requirement that the RHS is still a field operator,we have:

$$\begin{aligned} \eta_c &= \eta_*, ... \\ P v^{\mu} P^{-1} &= -\eta^* \mathcal{P}^{\mu}{}_{\nu} v^{\nu}(\mathcal{P}x) \\ C v^{\mu} C^{-1} &= \xi^* v^{\mu\dagger}(x) \\ T v^{\mu} T^{-1} &= \zeta^* \mathcal{P}^{\mu}{}_{\nu} v^{\nu}(-\mathcal{P}x) = -\zeta^* \mathcal{T}^{\mu}_{\nu} v^{\nu}(\mathcal{T}x) \\ (CPT) v^{\mu} (CPT)^{-1} &= -(\zeta\xi\eta)^* v^{\mu\dagger}(-x) = -v^{\mu\dagger} \end{aligned} \tag{5.49}$$

- Since we have transformed basis,the various signs in the transformations are aborbed into representation of $\mathcal{P}, \mathcal{T}, \mathcal{C}$
- The sign in P,T transformation indicate that $\eta = +1 : v^{\mu}$ is a **pesudovector**:$v^i \to -v^i$,$\eta = -1 : v^{\mu}$ is a scalar.Similar for T.
- Spin-1 field have CPT phase:-1.

Remark 39 • *There do exsist massive spin-1 field,the $W^{\pm}, Z$ particles in weak theory.*

- *The treatment of massless spin-1 field can somehow take $m \to 0$ limit,this works fine for $\mathcal{L}$ and field equations,but the polartizations will be ill-define in massless limit,this is because one of the polarizaiton will become unphysical(ghost,with negative norm)*

Massless limit is ill-defined

To see that the massless limit is ill-defined for spin-1 case:consider the rate:

$$\sum_{\sigma} |\langle J_{\mu} \rangle e^{\mu *}(\vec{p},\sigma)|^2 = \langle J_{\mu} \rangle \langle J_{\nu} \rangle^* \Pi^{\mu\nu}(\vec{p})$$

this implies that the observable will indeed depend on $\frac{1}{m^2}$ and blows up as $m \to 0$,this means we can't directly take the $m \to 0$ limit for spin-1 case.Also,to avoid this blow up,we must have at $m \to 0$,but **not exactly zero**,thus $\langle J_{\mu} \rangle p^{\mu} = 0$,this means the current must be conserved even when coupled to massive spin-1 field:

$$\partial_{\mu} j^{\mu} = 0$$

- This is just the **implication of unitarity**,there's no ghost v^0 state is excited,the spin-0 component of v^{μ} is completely ruled out,**if the current is not conserved,the ghost v^0 part could be excited**.We the need to include further a **physical spin-0 particle in the spectrum to be embedded into the spin-0 component to make the ghost physical**.
- This is exactly what happens to weak interaciton theory:when energy goes beyond the effectiveness of W,Z bosons,a UV completion theory is need which would include further a Higgs boson,and finally unifying weak interaction and EM interaction.

To see more clearly the origin,note that,take $\vec{p} = p\hat{p}_3$:

$$e^{\mu}(\vec{p},0) = \epsilon^L = L^{\mu}{}_{\nu} e^{\nu}(0,0) = \frac{1}{m}(\omega_p, 0, 0, p)^t$$

This polarization becomes unphysical(ghost) as $m \to 0 : e^{\mu}(\vec{p},0) \to \epsilon^L_{\mu} \propto p_{\mu}$

As we have seen the blowing up of rates due to this unphysical polarizaiton is cured by **conservation of currents**,for this time,the conservation of currents assure both unphysical spin-0 component and unphysical Longitude polarization is ruled out.This will is exact the consequnce of gauge invariance,in p-space the gauge invariance is **the Ward identity**:

$$p_{\mu} M^{\mu}(p) = 0$$

Gauge invariance will automatically force the current to be conserved.

Thus as $m \to 0$,the unphysical polarizaiton drops out.The drop out of spin-0 component and polarization is different,since the spin-0 component becomes physical when UV completed,the unphysical polarization of Longitude mode is due to **the nature of massless spin-1 particle:it has only two helicity state.**

Massless spin-1 vector field$\neq$ Massless limit of massive spin-1 vector field

The conclusion is that:

- The massless limit of massive spin-1 vector field works fine **only** in Lagrangian/field-equation,and perturbative calculations(assured by current conservation and **gauge invariance** of **massless** spin-1 vector field)
- There's essential difference in construciton of massive/massless spin-1 vector field,we should **construct massless spin-1 field independently,this procedure will introduce new feature like gauge transformation.**

5.4.4 Weyl field and Dirac field

The Dirac field is denoted as a multiplet(column vector) known as Dirac spinor:

$$\Psi = \Psi_l = \begin{pmatrix} \psi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix} = \Psi_L + \Psi_R, \Psi_{L/R} = P_{L/R}\Psi, P_{L/R} = \frac{1 \mp \gamma^5}{2}$$

We first consider Dirac field only,denote the index as:$(a)\oplus(\dot{a}) = l \equiv (m, \pm)$,index m denote rows and columns of submatrices,and $\pm$ denote rows and columns of supermatrices:$\pm \leftrightarrow L/R$,with m denoting the indices of L/R spinors.

Since the Dirac spinor is direct sum of $\pm = L/R$ spinors,the equations for coefficient functions are:

$$\begin{aligned} \sum_{\bar{\sigma}} u_{\bar{m}\pm}(0,\bar{\sigma})\vec{\mathcal{J}}^{(j)}_{\bar{\sigma}\sigma} &= \sum_m \frac{1}{2}\vec{\sigma}_{\bar{m}m} u_{m\pm}(0,\sigma) \\ -\sum_{\bar{\sigma}} v_{\bar{m}\pm}(0,\bar{\sigma})\vec{\mathcal{J}}^{(j)*}_{\bar{\sigma}\sigma} &= \sum_m \frac{1}{2}\vec{\sigma}_{\bar{m}m} v_{m\pm}(0,\sigma) \end{aligned}$$

When we seperate $\pm$,this means the equations of L/R spinors are equal,this is trivial,since $\vec{\mathcal{J}}_L = \vec{\mathcal{J}}_R$,the only difference is the convention of indices,here we use the compact index for both L/R:$m = a/\dot{a}$.

The coefficient functions can also be obtained by solving the equations directly,use **matrice form**:

$$U_\pm \vec{\mathcal{J}}^{(j)} = \frac{\vec{\sigma}}{2} U_\pm, -V_\pm \vec{\mathcal{J}}^{(j)*} = \frac{\vec{\sigma}}{2} V_\pm$$

Since $\vec{\mathcal{J}}^{(j)}$ also furnish irreducible representations,U/V must vanish or be square and non-singular[81].Thus this representation only have solution $\frac{1}{2}$,this agree with the result from representation of $SL(2,\mathbb{C})$

Thus,we have:$\vec{\mathcal{J}}^{\frac{1}{2}} = \frac{\vec{\sigma}}{2}, -\vec{\mathcal{J}}^{\frac{1}{2}*} = \sigma_2 \frac{\vec{\sigma}}{2} \sigma_2$,and $U_\pm, V_\pm\sigma_2$ commute with $\vec{\sigma}$,thus:

$$u_{m\pm}(0,\sigma) = c_\pm \delta_{m\sigma}, v_{m\pm}(0,\sigma) = -id_\pm \sigma^2_{m\sigma}$$

Thus[82]:

$$\begin{aligned} u(0,\frac{1}{2}) &= (c_+, 0; c_-, 0)^t, u(0,-\frac{1}{2}) = (0, c_+; 0, c_-)^t \\ v(0,\frac{1}{2}) &= (0, d_+; 0, d_-)^t, v(0,-\frac{1}{2}) = -(d_+, 0; d_-, 0) \end{aligned} \tag{5.50}$$

The general solution is obtained using standard boost.The relation between $c_\pm, d_\pm$ are obtained from discrete transformations,note that in this representation,$\mathcal{P} = \beta$.Upon parity,for the transformed field to be a field operator,we need[83]:

$$\beta u(0,\sigma) = b_u u(0,\sigma), \beta v(0,\sigma) = b_v(0,\sigma), b_i^2 = 1$$

[81] **Due to Schur's lemma**
[82] $V = (V_{m+}; V_{m-})$
[83] For spin-1 case b=-1

This means,by rescaling:

$$c_- = b_u c_+, d_- = -b_v d_+, c_+ = d_+ = 1$$

The spin-sum is:

$$N_0 = \sum_\sigma uu^* = \frac{1 + b_u\beta}{2}, M(0) = \sum_\sigma uu^* = \frac{1 + b_v\beta}{2}$$

generally,$N/M(\vec{p}) = \frac{m}{2\omega_p} D(L(p))(1 + b_{u/v}\beta)D^\dagger(L(p))$,using the relation $\beta D^\dagger \beta = D^{-1}, DD^\dagger = D\beta D^{-1}\beta$,and:

$$D(L(p))\beta D^{-1}(L(p)) = L_\mu{}^0\gamma^\mu = -\frac{p_\mu}{m}\gamma^\mu$$

The spin-sum is finally(up to normalization):

$$N/M(\vec{p}) = \frac{1}{2\omega_p}[-p_\mu\gamma^\mu + b_{u/v}m]\beta$$

The final relation between b_u, b_v is provided by causality condition:

$$[\Psi_l(x), \Psi^\dagger_{l'}(y)]_\mp = (|\kappa|^2[i\gamma^\mu\partial_\mu + b_u m]\beta\Delta_+(x-y) \mp |\lambda^2|[i\gamma^\mu\partial_\mu + b_v m]\beta\Delta_+(y-x))_{ll'}$$

This implies:

$$\kappa = \lambda = 1, b_u = -b_v = +1$$

Thus finaly:

$$\begin{aligned} u(0, \frac{1}{2}) = \frac{1}{\sqrt{2}}(1,0,1,0)^t, u(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}}(0,1,0,1)^t \\ v(0, \frac{1}{2}) = \frac{1}{\sqrt{2}}(0,1,0,-1)^t, v(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}}(-1,0,1,0)^t \end{aligned} \tag{5.51}$$

The spin-sum is,up to normalization:

$$\sum_\sigma u(\vec{p},\sigma)u^\dagger(\vec{p},\sigma) = [-p_\mu\gamma^\mu + m]\beta, \sum_\sigma v(\vec{p},\sigma)v^\dagger(\vec{p},\sigma) = [-p_\mu\gamma^\mu - m]\beta$$

More conveniently,we use,in LR-normalization:

$$\sum_\sigma u\bar{u} = -p_\mu\gamma^\mu + m, \sum_\sigma v\bar{v} = -p_\mu\gamma^\mu - m$$

Explicitly,the solutions are,in LR-normalization:

$$u(\vec{p}, \pm\frac{1}{2}) = \begin{pmatrix} \sqrt{-p\cdot\sigma}\xi_\pm \\ \sqrt{-p\cdot\bar{\sigma}}\xi_\pm \end{pmatrix}, \xi_\pm = \begin{pmatrix} 1 \\ 0 \end{pmatrix} / \begin{pmatrix} 0 \\ 1 \end{pmatrix}; v(\vec{p}, \pm\frac{1}{2}) = \begin{pmatrix} \sqrt{-p\cdot\sigma}\eta_\pm \\ -\sqrt{-p\cdot\bar{\sigma}}\eta_\pm \end{pmatrix}, \eta_\pm = \begin{pmatrix} 0 \\ 1 \end{pmatrix} / \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

with LR-normalization:

$$\bar{u}u = 2m\delta_{ll'} = \bar{v}v$$

in NR normalization:

$$\begin{aligned} \Psi_l = \sum_{\sigma=\pm\frac{1}{2}} \int \frac{d^3p}{\sqrt{(2\pi)^2 2\omega_p}} [u(\vec{p},\sigma)a(\vec{p},\sigma)e^{ip\cdot x} + v(\vec{p},\sigma)a_c^\dagger e^{-ip\cdot x}] \\ \sum_\sigma u(\vec{p},\sigma)\bar{u}(\vec{p},\sigma) = -\not{p} + m, \sum_\sigma v(\vec{p},\sigma)\bar{v}(\vec{p},\sigma) = -\not{p} - m \end{aligned} \tag{5.52}$$

- We hadly concern about the specific form of u,v,but their spin-sum.
- In LR normalization,the measure is squared,and the u,v have same normalization.
- The u/v are also obtained from the equations in canonical formalism this is way more convenient.

The anticommutator is:

$$[\Psi_l(x), \Psi^\dagger_{l'}(y)]_+ = \{[i\gamma^\mu\partial_\mu + m]\beta\}_{ll'}\Delta(x-y)$$

The β may be obsorbed into $[\Psi, \bar{\Psi}]_+$.

Field equations and Lagrangian

The transformation properties of u/v implies they satisfy the following equations:

$$(\not{p}+m)u(\vec{p},\sigma)=0,(-\not{p}+m)v(\vec{p},\sigma)=0$$

These two equations are known as Weyl equations,in p-space.They imply the **Dirac equation**:

$$(-i\gamma^\mu\partial_\mu+m)=(-i\not{\partial}+m)\Psi(x)=0 \tag{5.53}$$

Explictly:

$$\begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix}\psi \\ \bar{\chi}\end{pmatrix}=0\Leftrightarrow\begin{cases} i\sigma^\mu\partial_\mu\bar{\chi}=m\psi \\ i\bar{\sigma}^\mu\partial_\mu\psi=m\bar{\chi}\end{cases} \tag{5.54}$$

The Lagrangian is:

$$\mathcal{L}=\bar{\Psi}(i\not{\partial}-m)\Psi=i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi+i\bar{\chi}^\dagger\sigma^\mu\bar{\chi}-m(\psi^\dagger\bar{\chi}+\bar{\chi}^\dagger\psi)$$

Thus the **Dirac mass** couple L/R hand and is invariant unde Parity.

Massless limit

In massless limit,the L/R spinors decouple,this can be seen both in Lagrangian and the field equations.The Lagrangians are known as Weyl Lagrangian,and they are massless.Since the D.O.F matches in $m\to 0$ limit,it's fine to go to massless limit of $j=\frac{1}{2}$ Dirac field.

Also we see that in massless limit,the chiral spinors are just the eigenstates of helicity operator.Thus **In massless limit,chirality=helicity,L/R=**$\mp j$

Discrete transformations

Similarly,the the parity transformed field to be a field operator,we have:

$$\eta_c=\eta_*$$

The complex conjugate of u/v can be obtained from[84]:

$$\exp(\frac{\omega_{\mu\nu}}{2}i\mathcal{J}_D^{\mu\nu})^*=\beta\mathcal{C}\exp(\frac{\omega_{\mu\nu}}{2})\mathcal{C}^{-1}\beta$$

and the fact:

$$\mathcal{C}\beta u/v(0,\sigma)=-v/u(0,\sigma)$$

thus:

$$u^*(\vec{p},\sigma)=-\beta\mathcal{C}v(\vec{p},\sigma),v^*(\vec{p},\sigma)=-\beta\mathcal{C}u(\vec{p},\sigma)$$

Thus we have:

$$C\Psi(x)C^{-1}=-\xi^*\beta\mathcal{C}\psi^*=\xi^*\mathcal{C}\bar{\Psi}^t=\begin{pmatrix}\chi_a \\ \bar{\psi}^{\dot{a}}\end{pmatrix},\beta\mathcal{C}=-i\gamma^2$$

The reality condition is,known as **Majorana fermion**:

$$\Psi(x)=\Psi_c=-i\gamma^2\psi^*=-\beta\mathcal{C}\Psi^*(x)=\mathcal{C}\bar{\Psi} \tag{5.55}$$

This means the Majorana fermion is composed with single Weyl spinor:

$$\Psi=\Psi_c=\begin{pmatrix}\psi \\ i\sigma_2\psi^*\end{pmatrix}=\begin{pmatrix}\psi_a \\ \bar{\psi}^{\dot{a}}\end{pmatrix} \tag{5.56}$$

For Majorana fermions,$\eta=\pm i,\xi=\pm 1$.

The Lagrangian of Majorana fermion is:

$$\mathcal{L}=i\psi^\dagger\sigma^\mu\partial_\mu\psi+\frac{m}{2}(\psi^\dagger i\sigma_2\psi^*+h.c.) \tag{5.57}$$

[84] using properties of the generators

For time-reversal,using the identity:

$$D^*(L(-\vec{p})) = \gamma^5\beta D^*(L(\vec{p}))\beta\gamma^5 = \gamma^5\mathcal{C}D(L(p))\mathcal{C}^{-1}\gamma^5$$

and:

$$-\gamma^5\mathcal{C}^{-1}u(0,-\sigma) = (-1)^{\frac{1}{2}-\sigma}u(0,\sigma), -\gamma^5\mathcal{C}^{-1}v(0,-\sigma) = (-1)^{\frac{1}{2}-\sigma}v(0,\sigma)$$

thus

$$(-1)^{\frac{1}{2}-\sigma}u^*(-\vec{p},-\sigma) = \gamma^5\mathcal{C}u(\vec{p},\sigma), (-1)^{\frac{1}{2}-\sigma}v^*(-\vec{p},-\sigma) = \gamma^5\mathcal{C}v(\vec{p},\sigma)$$

and finally:

$$T\Psi(x)T^{-1} = \zeta^*\gamma^5\mathcal{C}\Psi(-\mathcal{P}x) \tag{5.58}$$

The charge conjugation of bilinears are:

$$C(\bar{\psi}M\psi)C^{-1} = (\beta\mathcal{C}\psi)^t M(\beta\mathcal{C}\psi^*) = -(\beta\mathcal{C}\psi^*)^t M^t\mathcal{C}\psi = \bar{\psi}\mathcal{C}^{-1}M^t\mathcal{C}\psi = \pm\bar{\psi}M\psi$$

The conjugation of bilinears are:

$$(\bar{\psi}M\phi)^\dagger = \bar{\phi}(\beta M^\dagger M\beta)\psi = \bar{\phi}\bar{M}\psi$$

The CPT transformation is:

$$(CPT)\psi(x)(CPT)^{-1} = \gamma^5\psi^*(-x)$$

C,P,T transformations in Dirac representation

In Weyl basis,using relations of γ^μ,we have:$\gamma^5\mathcal{C} = \gamma^1\gamma^3, \beta\mathcal{C} = -\gamma^2, \beta = \gamma^0$

$$\begin{aligned} C &: \psi \to -i\gamma^2\psi^* \\ T &: \psi \to \gamma^1\gamma^3\psi^* \\ P &: \psi \to \gamma^0\psi \\ CPT &: \psi \to \gamma^5\psi^* \end{aligned} \tag{5.59}$$

Using these tranformations,we can deduce the transformation of other field coupling to spinor by demanding the **Lagrangian is invariant under these transformations**[a]

[a]For detail,refer to [Schwartz.11]

Remark 40 • *The discrete transformations will be rarely used.They will appear in reality condition,discussion of the C,P,T symmetry of the Lagrangian,etc.*

Meaning of Dirac equation

The Dirac equation can be obtained using group theoretiacl methods above,it's meaning is clear with identification $P \leftrightarrow \gamma^0$.

We wish theory involving the Dirac field to be invariant under parity,thus in rest frame we may have:

$$(\gamma^0 - 1)\Psi(p_r) = 0$$

this means we **treat L/R hand equivalently**,boost this to arbitary momentum:

$$(\frac{\gamma^\mu p_\mu}{m} - 1)\psi = 0$$

This is just the p-space Dirac equation.It is a constraint on u and v as noted before.

5.5 Massless field

5.5.1 Transformation properties

As mentioned there's no problem for scalar field and Dirac field go to massless limit,the massless Dirac field will decouple to two Weyl spinor field describing massless particles.

The core is that:

- the massless particle of spin-0,spin-$\frac{1}{2}$ have 0/2 physical D.O.F,and the scalar/Weyl-spinor field have the right number of independent tensor/field component to encode these,there's no reduandancy.Then the massless particle can be safely embedded into massive particle field at massless limit.
- For particles with spin$\geq$ 1,however,the physical D.O.F are always 2,but the massive field have $2j+1$ independent component,in massless limit,the redundant ones will become unphysical and will need further treatment.
- New feature like gauge invariance must be taken as a exact symmetry,this originated form this redundancy.

Similarly,we construct:

$$\psi_l = \sum_{\sigma=\pm j} \int \frac{d^3p}{(2\pi)^{3/2}} [\kappa u_l(\vec{p},\sigma)a(\vec{p},\sigma)e^{ip\cdot x} + \lambda v_l(\vec{p},\sigma)a_c^\dagger(\vec{p},\sigma)e^{-ip\cdot x}]$$

the creation/annihilation operators for **massless** particles transform under Lorentz group as:

$$U_0(\Lambda)a^\dagger U_0(\Lambda)^{-1} = \sqrt{\omega_{\Lambda p}/\omega_p}\exp(i\sigma\theta(\Lambda,p))a^\dagger(\vec{p}_\Lambda,\sigma) \tag{5.60}$$

If we wish the field transform under Lorentz group properly:

$$U(\Lambda)\psi_l(x)U(\Lambda)^{-1} = \mathfrak{D}_{ll'}(\Lambda^{-1})\psi_{l'}(\Lambda x)$$

we must force the equation of coefficient functions:

$$\begin{aligned} u_{l'}(\vec{p}_\Lambda,\sigma)\exp(i\sigma\theta(\Lambda,p)) &= \sqrt{\omega_p/\omega_{\Lambda p}}\sum_l \mathfrak{D}_{l'l}(\Lambda)u_l(\vec{p},\sigma) \\ v_{l'}(\vec{p}_\Lambda,\sigma)\exp(-i\sigma\theta(\Lambda,p)) &= \sqrt{\omega_p/\omega_{\Lambda p}}\sum_l \mathfrak{D}_{l'l}(\Lambda)v_l(\vec{p},\sigma) \end{aligned} \tag{5.61}$$

We can satisfy this by going to standard frame which lead to little group:

$$\begin{aligned} u_{l'}(\vec{p},\sigma) &= \sqrt{|k|/\omega_p}\sum_l \mathfrak{D}_{l'l}(L(p))u_l(\vec{k},\sigma) \\ v_{l'}(\vec{p},\sigma) &= \sqrt{|k|/\omega_p}\sum_l \mathfrak{D}_{l'l}(L(p))v_l(\vec{k},\sigma) \end{aligned}$$

The equation then can be reduced to equations involving little group ISO(2),this is just the precedure to embed the little group into the rotation subgroup of Lorentz group:

$$\begin{aligned} u_{l'}(\vec{k},\sigma)\exp(i\sigma\theta(k,W)) &= \sum_l \mathfrak{D}_{l'l}(W)u_l(\vec{k},\sigma) \\ v_{l'}(\vec{k},\sigma)\exp(-i\sigma\theta(k,W)) &= \sum_l \mathfrak{D}_{l'l}(W)u_l(\vec{k},\sigma) \end{aligned} \tag{5.62}$$

There're two kinds of transformations in little group ISO(2),the rotation subgroup $SO(2)\cong U(1)$ and the gauge translation,which is unphysical.

For rotation:

$$\begin{aligned} u_{l'}(\vec{k},\sigma)\exp(i\sigma\theta) &= \sum_l \mathfrak{D}_{l'l}(R(\theta))u_l(\vec{k},\sigma) \\ v_{l'}(\vec{k},\sigma)\exp(-i\sigma\theta) &= \sum_l \mathfrak{D}_{l'l}(R(\theta))u_l(\vec{k},\sigma) \end{aligned}$$

For gauge translation:

$$u_{l'}(\vec{k},\sigma) = \sum_l \mathfrak{D}_{l'l}(S(\alpha,\beta))u_l(\vec{k},\sigma)$$
$$v_{l'}(\vec{k},\sigma) = \sum_l \mathfrak{D}_{l'l}(S(\alpha,\beta))u_l(\vec{k},\sigma)$$

from the equations we can take $v_l(\vec{p},\sigma) = u_l^*(\vec{p},\sigma)$.The problem is that **we can't find u statisfying transform under gauge translation properly**.

5.5.2 Massless $(\frac{1}{2},\frac{1}{2})$ field

Conventionally:$u^\mu(\vec{p},\sigma) = \frac{1}{\sqrt{2\omega_p}}e^\mu(\vec{p},\sigma)$.Thus:

$$e^\mu(\vec{p},\sigma) = L(p)^\mu{}_\nu e^\nu(\vec{k},\sigma)$$
$$e^\mu(\vec{k},\sigma)e^{i\sigma\theta} = R(\theta)^\mu{}_\nu e^\nu(\vec{k},\sigma)$$
$$e^\mu(\vec{k},\sigma) = S(\alpha,\beta)^\mu{}_\nu e^\nu(\vec{k},\sigma)$$

up to normalization,we can choose the **physical transverse polarization** as[85]:

$$e^\mu(\vec{k},\sigma=\pm1) = \frac{1}{\sqrt{2}}(1,\pm i,0,0)^t$$

The equation of gauge translation implies that in general:$\alpha \pm i\beta \equiv 0$,this **can't be satisfied.**

Instead this construction will lead to redundant unphysical transformation:

$$D^\mu_\nu(W(\theta,\alpha,\beta))e^\nu(\vec{k},\pm1) = S^\mu{}_\lambda R^\lambda{}_\nu e^\nu(\vec{k},\pm1) = \exp(\pm i\theta)\{e^\mu(\vec{k},\pm1) + \frac{\alpha \pm i\beta}{\sqrt{2}|k|}k^\mu\}$$

The physical rotation will mix the physical polarization,but the unphysical gauge translation will **induce in field theory** the unphysical Longitude mode:

$$\epsilon^L_\mu \propto p_\mu$$

The little group transformation leading to:$\epsilon^\mu \to \epsilon^\mu + p^\mu$ is the **p-space gauge transformation.**This gauge transformation is redundant,from combination of the redundant field components and unphysical gauge translation in little group of massless particles.As noted before,this lead to **Ward identity**:

$$p_\mu M^\mu(p) = 0$$

This identity ensure the unphysical polarization drop out of physical quantities,implied by **unitarity**.

If we still use redundant field to embedd massless particle,we will have **more constraint**,known as **gauge fixing conditions.**

The **full constraint** which remove all unphysical D.O.F is known as the **Coulomb gauge**,there are other gauges,which will accept residual gauge transformations.

5.5.3 Coulomb gauge

The Coulomb gauge follows form:

$$e^\mu(\vec{p},\pm1) = R(\vec{p})^\mu{}_\nu e^\nu(\vec{k},\pm1), e^0(\vec{p},\pm1) \equiv 0, \vec{k}\cdot\vec{e}(\vec{k},\pm1) = 0$$

Thus we have:

$$e^0(\vec{p},\pm1) = 0, p_i e^i(\vec{p},\pm1) = 0 \Leftrightarrow a^0(x) = 0, \nabla\cdot\vec{a}(x) = 0$$

The fact that a^0 vanishes implies that a^μ is **not a massive vector field in massless limit**.

the polarization will in general satisfy:

$$e^\mu(\vec{p}_\Lambda,\pm1)e^{\pm i\theta(\vec{p},\Lambda)} = \mathfrak{D}^\mu{}_\nu(\Lambda)e^\nu(\vec{p},\pm1) + p^\mu\Omega_\pm(\vec{p},\Lambda)$$

This manifest the failure of constructing massless spin-1 field using vector fields.In x-space,there's extra gauge transformation:

$$U(\Lambda)a^\mu(x)U(\Lambda)^{-1} = (\Lambda^{-1})^\mu{}_\nu a^\nu(x) + \partial_\mu\Omega(x,\Lambda) \quad (5.63)$$

Thus **gauge invarince follows form redundancy in field theory**,and **in any theory involve massless spin-1 vector field,there must be exact gauge invariance holding at quantum level**.

[85]The normalization here is different form the massive case,but these are equivalent just a change of basis in polarization space

Spin-sum in Coulomb gauge

Using explicit expression of polarizations,the spin-sum is:

$$\sum_{\sigma=\pm 1} e^i(\vec{p},\sigma)e^{j*}(\vec{p},\sigma) = \delta_{ij} - \frac{p^i p^j}{|\vec{p}|^2}$$

It should be note that since polarizations will change gauge condition upon gauge transformation,**different gauge fixing condition will have different spin-sums.**

5.5.4 $(1,0)\oplus(0,1)$ massless spin-1 field is well-defined

It is fine to embed massless spin-1 particle into $f_{\mu\nu} = \partial_{[\mu}a_{\nu]}$,this field is clearly invariant under gauge transformations,with the coefficient functions defined by:

$$u^{\mu\nu}(\vec{p},\pm 1) = \frac{i}{\sqrt{(2\pi)^3(2\omega_p)^3}}[p^\mu e^\nu(\vec{p},\pm 1) - p^\nu e^\mu(\vec{p},\pm 1)]$$

It can be shown that this coefficient function indeed satisfy the wanted equations,without extra gauge transformations.

The field equations are:

$$\partial_\mu f^{\mu\nu} = 0, \partial_\sigma * f^{\sigma\rho} = 0, *f^{\sigma\rho} = \epsilon^{\sigma\rho\alpha\beta} f_{\alpha\beta}$$

5.5.5 The need of redundancy in field theory:gauge theories

The reason for using a^μ for massless spin-1 particles rather than $f_{\mu\nu}$ is that:

- There's derivative in $f_{\mu\nu}$,the interaction density constructed out of $f_{\mu\nu}$ and its derivative will have give matrix elements vanish too fast.Then using $f_{\mu\nu}$ alone will give rise to interactions vanishing faster than **observed** square-inverse law.
- The gauge-invariant theory(known as **gauge theory**) using a^μ for massless spin-1 particles represent a more general[86] class of theories.
  - The **Standard model** of strong,electroweak interactions all use massless **spin-1** vector fields,and with the mechanics of spontaneous symmetry breaking helping to explain the origin of the mass of massive vetor bosons.
  - These theory are build with **redundancies**,this redundancy form the theory in a more **symmetry-based** form,and use certain mechanics to obtain the more physical theories.This is the idea of using auxiliary Hilbert space plus conditions defining physical states.
- It's important to note that a^μ are not only auxiliary,it has physical consequnces.The a^μ can be used to construct the Wilson loop operator,which give rise to **general A-B effect** which are observable.Further,this is nonlocal.
- There are many **nonlocal topological defects** can be constructed using a^μ,including monopole,instanton,etc.

The similar argument hold for massless spin-2 particles,we can construct the $(1,1)$ symmetric traceless field $h_{\mu\nu}$,the creation/annihilate operators of massless spin-2 particles will make $h_{\mu\nu}$ ill-defined,it will transform under **Poincare group**[87] properly upon gauge transformation known as **general coordinate transformations**.We can also construct the Reimann-Christroffel curvature tensor $R_{\mu\nu\alpha\beta}$ which is invariant under gauge transformations.But to explain the square-inverse law of gravity,in field theory,we must use $h_{\mu\nu}$ and introduce redundancy to form a gauge theory.

Gauge theory are constructed using (A,A) fields

Different to massive spin-j particles are embedded into $(j,0)/(0,j)$ fields[a],gauge theories use $(A,A) = j(\frac{1}{2},\frac{1}{2})$ field(Lorentz tensors) for **massless** spin-j particle,these field are used to describe fundamental interactions which at high enough scale,are long-ranged,in a reduandant way.

[a]Others are physical equivalent upon $\{\partial_\mu ... \partial_\nu\}$

[86]since $f_{\mu\nu}$ is constructed out of $\partial^\nu a^\mu$,they give only a subset of theories.

[87]This is the **backgroup dependent** formalism,it's **perturbative.**

Coupling to conserved current

To assure the theory is gauge invariant,we must couple the gauge field $a^\mu, h_{\mu\nu}$ to conserved currents:

- For spin-1 case:J^μ from corresponding **global G symmetry**,this is a internal symmetry between several particles/fields.
- For spin-2 case:$\theta^{\mu\nu}$ the **symmetric energy-momentum tensor**,this follows from global coordinate transformation:**spacetime translation**.
- Generally,for $\partial_{\mu_1}\theta^{\mu_1\cdots\mu_N} = 0, \int d^3x \theta^{0\mu_2\cdots\mu_N}$ is a conserved quantity of rank-(N-1) Lorentz-tensor.However,the only such tensors are scalar charges from continuous global symmetry,and energy-momentum vector.Any higher tensors will lead to bad behaved interactions.This fact is related to the fact that there's **no gauge-interactions could be form by spin>2 fields.**

5.6 Discussion on higher spin-fields:Massive and Massless

There's no promblem for massive spin≥ 3 fields.In general,these fields are constructed **tensoring Lorentz indices and Dirac-Spinor indices.** and then by project out reduandant D.O.F.We can form the interactions(not gauge theory) using projective tensors.

For general massless particles to be embedded into free (A, B) field,we have:

$$\sigma u_{ab}(\vec{k}, \sigma) = (a+b) u_{ab}(\vec{k}, \sigma)$$
$$-\sigma v_{ab}(\vec{k}, \sigma) = (a+b) v_{ab}(\vec{k}, \sigma)$$

This means $\sigma = a + b/(-a-b)$,The equation of u:$u_{l'}(\vec{k}, \sigma) = \sum_l \mathfrak{D}_{l'l}(S(\alpha, \beta)) u_l(\vec{k}, \sigma)$ then equivalent to:

$$(\mathcal{L}_1^{(A)} - i\mathcal{R}_2^{(A)})_{aa'} u_{a'b}(\vec{k}, \sigma) = 0$$
$$(\mathcal{L}_1^{(B)} + i\mathcal{R}_2^{(B)})_{bb'} u_{ab'}(\vec{k}, \sigma) = 0$$

this implies:

$$a = -A, b = +B$$

Thus the (A,B) field can be formed from annhilation operator of massless particle of helicity σ,and creation operator for its antiparticle of helicity $-\sigma$,$\sigma = B - A$

- This is the statement that there's no problem of $(0,0), (\frac{1}{2}, 0)/(0, \frac{1}{2}), (1,0)/(0,1), (2,0)/(0,2)$ field,and particles/antiparticles have oppesite helicity.The right D.O.F are formed using **direct sums**.
- There's problem with (A, A) Lorentz-tensor fields,these field are equvalent to $(0,0)$ field and describe helicity-0 particles.
- To use (A, A) field to describe spin-j particle,we would have to introduce gauge transformations and gauge invariance.

5.7 Appendix:Other views of spin-statistic

Remark 41 *This section comes form [Schwartz.12],more details should be included in quantization of classical fields.*

5.7.1 Topological arguments

The essence of spin-statistic connection is that the path of **exchanging two identical particles are equivalent to a 2π rotation of a single particle**Then for simplicity abelian anyon case:

$$e^{i\Theta} = \mathfrak{D}(R(2\pi)) = e^{2\pi i s}$$

- In 3D,$\pi_1(SO(3)) = \mathbb{Z}_2$,thus $e^{i\Theta} = \pm 1$ this gives only (half)-integer spins,corresponding to fermion/boson.
- In 2D,$\pi_1(SO(2)) = \pi_1(U(1)) = \mathbb{Z}$,thus $e^{i\Theta} \in \mathbb{C}$,this give rise to fractional spins.The origin of complex braing phase $e^{i\Theta}$ may comes from topological interactions like A-B effects,which need the combination of flux and charge(dyons).

5.7.2 Relativistic field theory

In 4d relativistic field theory,without topological defects,the fermion/boson statistics are encoded into $[A, B]_{\mp}$.The essential argument lead to spin-statistic connection,is the Lorentz invariance.The spin-statistic connection:**(Anti)commutato for Grassman/Ordinary field** are manifested at many places:

Lorentz invariance of S-matrix

Lorentz invariance of S-matrix lead to Lorentz invariance of time-ordered operators.It can be shown that for boson/fermion,we must define $T\{A(x)B(y)\} = \theta(x^0 - y^0)A(x)B(y) \pm \theta(y^0 - x^0)B(y)A(x)$ this will lead to **Lorentz invariant matrix elements**,for example the manifestly Lorentz invariant propagators.If we adopt the inverse convention,then the matrix element will break Lorentz invariance.

For example,consider spin-0 field:

$$\langle 0|\phi(x)\phi(y)^\dagger|0\rangle \equiv \Delta_+(x-y)$$

Then if we define T bosonicly:

$$\langle T_+\{\phi(x)\phi(y)^\dagger\}\rangle \equiv \Delta_F(x-y)$$

RHS is the Lorentz invariant Feynman propagator.If we define T fermionicly:

$$\langle T_-\{\phi(x)\phi(y)^\dagger\}\rangle \equiv \int \frac{d^4p}{(2\pi)^4}\frac{p^0}{\omega_p}\frac{i}{p^2+m^2-i\epsilon}e^{i\vec{p}\cdot x}$$

This is not Lorentz invariant.The same hold for others.Mathematically,the core of picking T_- for fermions is that there's odd number of p_μ in the expressions to contract γ^μ and this is odd under rotation that $p \to -p$,for two spinors,the rotation gives $i \times i = -1$,thus we need to define T with T_-.

Stability

We need to have 2nd quantized Hamiltonian to be**bounded below**.This will hold directly form Lorentz invariance and causality,this stability argument is a necessay condition,and hold also in CMP,where SR is not manifested.

In this argument,what matters is $[a, a^\dagger]_{\mp}$,it can be shown that only we take + for fermion can the H be bounded below.The mathmatical essence is that:There's only odd number of time derivatives for fermions,and this will lead to:

$$H_f \sim a^\dagger a - a_c a_c^\dagger$$

for fermions,but bosons have even number of time-derivatives thus:

$$H_b \sim a^\dagger a + a_c a_c^\dagger$$

the stability requires:

$$H \sim a^\dagger a + a_c^\dagger a_c$$

Causality

This involve the analytic properties for special functions,consider first the spin-0 case:

$$[\phi(x), \phi(y)]_- = \Delta_+(x-y) = iD(t,r) = i\frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases}\pm\mathcal{J}_0(m\sqrt{t^2-r^2}) & time-like \\ 0 & space-like\end{cases}$$

Here the Δ_+ or $D(t,r)$ is the Green function with properties:

$$D(t,\vec{r}) = -D(-t,\vec{r}); D(t,\vec{r}) = D(t,-\vec{r}) \Rightarrow D(t,\vec{r}) = -D(-t,-\vec{r})$$

For anticommutator:

$$[\phi(x), \phi(y)]_+ = iD_1(t,r) = i\frac{-1}{4\pi r}\frac{\partial}{\partial r}\begin{cases}i\mathcal{Y}_0(m\sqrt{t^2-r^2}) & time-like \\ \mathcal{H}_0(im\sqrt{r^2-t^2}) & space-like\end{cases}$$

The $D_1(t,r)$ satisfies:

$$D_1(t,\vec{r}) = D_1(t,\vec{r})$$

The mathmatical essence is that,for fermions there'are odd derivatives in Lagrangian and thus odd p in spin-sum while:

$$[,]_- = \pi(\partial)D, [,]_+ = \pi(\partial)D_1$$

LHS is odd/even under PT transformation:$x \to y$ thus the RHS must be odd/even under PT,while ∂ is odd,thus for odd partial in spin-sum for fermions,we must choose D_1 which is even under PT.For bosons with even number of partials,we must choose D which is odd under PT.

Mathematical essence of spin-statistic in field theory

In field theory,fermions are contracted with gamma matrices,and this lead to odd number of derivatives in Lagrangian,while for bosons,there's even number of derivatives.This fact manifest in time-ordered operators as odd/even number of p,and the interchange of two operator will lead to parity transformation of p this lead to π rotation of particles.For fermions this will give further minus sign.For (anticommutators) this manifest in spin-sum,which have odd/even number of derivatives acting on special functions on RHS.In stability of H,the odd/even derivative will manifest in signs before $a_c a_c^\dagger$.
These signs and PT transformation properties of partials will force the choice of (anti)commutator and Grassman/ordinary components and thus the definiton of time-ordering.

5.8 Brief Summary

5.8.1 What is reletivistic quantum field theory

The principles of relativistic quantum field theory include:

- The physical D.O.F are states in the physical Hilbert space:the projective Hilbert space $\mathbb{P}\mathcal{H}$,more conveniently, the complex linear space $\mathcal{H}$ is used.
  - In any system with unfixed particle numbers,states in Hilbert space being a Fock space include vacuum subspace,single-particle/1st-quantization subspace,multiparticle(interacting or free or in/out) subsapce.
  - The non-interacting(free or in/out) multiparticle-states as tensor product states are the basis of the Fock space,interacting multiparticle states can't be decomposed as a tensor product.
  - The interaction between these states are described by some operator,in relativistic QFT,the operator is the S-matrix between free multiparticle states,which is the transition amplitude between in/out states.The in/out states are asymptoticly free due to locality of interactions.
  - The completeness and orthonormality of the bases are equivalent to the 2nd quantization operator formalism,where identical single particle states are given by creation and annhililation operators of bosons/fermions.This works in 3+1d QFT.The operators can always be expressed in functions of products of these operators in normal ordering.With coefficient functions related to the matrix elements.
- The system respect certain symmetry,these symmetries are represented as (anti)unitary operators on $\mathcal{H}$,and the symmetry group of the system is determind by the Hamiltonian thus the S-matrix.The quantum level transformation groups are projectively represented over the Hilbert space.
  - The most fundamental symmetry is the spacetime symmetry,in relativistic QFT,it's the Poincare group.This means there exsist hermitian operators expanding the enlarged Poincare algebra.
  - Single particle states furnish certain projective irreducible unitary representations of Poincare group,which are infinite-dimensional,labelled by mass and spin.These single particle states are constructed inducing the representation of little groups perserving the standard momentums:
    - For vacuums,it's the Lorentz group.
    - For massive particles,it's $SO(3)$,with finite dimensional unitary representations labelled by spin and spin-3.
    - For massless particles,it's $ISO(2)$,with finite dimensional unitary representations labelled by spin and helicity.The gauge translation are unphysical.
  - The Poincare invariance must also be preserved by S-matrix,this will lead to certain constraints on interactions:

    * The generator of boost are modified by interacitons,in order for the Poincare algebra to be satisfied and the S-matrix is Poincare invariant,the interaction can be constructed from scalar densities in I-piture.
    * The scalar interaction density must satisfy the causality condition.

- There's another principle known as cluster decomposition of S-matrix.This will constraint the 2nd quantization form of interaction densities:the coefficient functions of products of creation/annhilation operators must contain only one delta-function from translational invariance.

5.8.2 Construction of free field operators

Combination of these principles lead natrually to the introduction of relativistic creation/annhiliation field operators which are x-space version of the p-space creation/annihilation operators:

- The combination of the principles suggest the construction of scalar interaction density in I-piture using free(I-piture) creation/annihilation field operators.To satisfy the causality condition and conserving U(1) charges ,the annihilation field of particles and the creation field of antiparticles are coupled together to a relativistic field operator:
  - The statistic and the algberaic nature of the fields follow from causality condition manifesting spin-statistic connection,a result from topological properties of rotation subgroup of spacetime symmetry group.
  - The field operator combine finit dimensional non-unitary projective representation of the Lorentz group and differential operator representation to furnish infinite dimensional representation of the Poincare group.
  - The scalar interaction density is constructing using projective tensors of Lorentz group.
- Since the creation/annhilation operators contain identical informations about single particle states,the properties of these partiles are tranlated into properties of creation/annhilation operators.Most fundamentally,the transformation properties of these operators under Poincare group plus $\mathbb{Z}_2 : C, P, T$ discrete transformations.Then to have well-defined field operator,there're symmetry transformation properties of the coefficient functions of creation/annhilation operators:
  - These serves equations for the construction of these coefficient functions thus the field operators.
  - These equations with the help of standard boost related to little groups,are reduced to equations in standard frame,which relate the construction to representation of little groups.The physical explaination of this equation is to embedd little group(particle) into $SO(3)$ subgroup of the Lorentz group,or use coefficient functions to encode the informations of the particles.

Thus the final job is reduced to constructing relativistic field operators from finite dimensional representations of Lorentz group.The Lie algebra of Lorentz group is special,it's a direct sum of two chiral $SU(2)$ algebra,thus the represenions are obtained by tensoring representations of $SU(2)$.

The final result of construction of free field has some features:

- For massive case,the construction implies the coefficient functions are just Clebsch-Gordon coefficients of $SU(2)$,modified by certain projective tensors in order to be expressed as a multiplet/column,special examples include:
  - $(0,0)$ scalar field
  - $(\frac{1}{2},0)/(0,\frac{1}{2})$ Weyl-spinor field,$(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ Dirac-spinor field.
  - $(\frac{1}{2},\frac{1}{2})$ massive Lorentz-vector field,$(1,0) \oplus (0,1)$
  - Higher representations can be obtained form tensoring these fields and project out unphysical components.For example $(\frac{1}{2},1) \oplus (1,\frac{1}{2})$ Rita-Schwinger field
- The field operator for each representation is unique up to scale.The scale is fixed by renomalization at certain energe/length scale.This implies:
  - The field operators for certain particles are all physically equivalent,related by tensoring with Lorentz-tensors.

  - Under discrete transformations,the field operator trun into some other field operator.The discrete transformation make use of finite dimensional representations of the C,P,T operators in each representations and the transformation properties of the explicit solutions of coefficient functions.

- For massless particle and corresponding creation/annihilation operators,there's some difference:
  - The equations only accept solutoins for $(A, B), \sigma = A = -B$ solutions thus massless particles can be embedded into fields like $(0,0), (\frac{1}{2}, 0)/(0, \frac{1}{2}), (1,0)/(0,1), (2,0)/(0,2)$ correspondingly.
    * This implies massless particles and antiparticles have inverse helicity and chirality,helicity=chirality in massless limit.
  - If we construct $(A, A) = j(\frac{1}{2}, \frac{1}{2})$ field for massless spin-j particles,the equations of gauge translations won't be satisfies,instead there will be gauge transformations of coefficent functions,due to gauge translation in little group.Equivalently,there will be inevitable gauge transformations of these fields due to redundancy in field components.
    * The theories using these field operators must have gauge invariance at quantum level.
    * To model the observed interactions,these gauge fields are used.
    * Further,there are topological properties of these fields that are observable,thus they are physical.
    * Gauge invariance is a feature of field theories.

5.8.3 Field equations and Lagrangian:the path to quantization of classical field theory

The transformation properties/equations of the coefficient functions translate into partial differential equations for the field operators.

- These field equations project out redundant field components to make sure the field describe only single type of particles.
- For massless case,the gauge transformation will trun some equations into gauge fixing conditions.

These field equations are equivalent to the Lagrangian if we consider reduce the field operators into classical fields.This manifest the validity of quantizing a classical field.This reversed approach of obtain the quantum theory(Constructing states expanding Hilbert/Fock space,Constructing symmetry generators satisfying right algebra.) from classical Lagangian and equations of classical fields is known as canonical formalism:

- The free Lagrangian is constructed using representation of Lorentz group and projective tensors.The interaction is included by coupling different fields into gauge invariant scalars.
- Starting with the linear relativistic field equation,it accept solutions of linear combination of Fourier mode solutions:in p-space,the solutions are expanded by independent physical polarization multiplets,they are just the coefficient fucntions obtained using group theory.
- Fourier transform these solutions and compose them in a Lorentz covariant way,we then obtain classical field solutions.
- Canonical quantization promote the classical field into field operators.
  - The classical equation will then become an operator equation.
  - The coefficient before the polarization will be promoted into operators,the canonical (anti)commutation relation between fields is equivalent to these operators being creation/annihilation operators,we then obtain the Fock space.
- Symmetry of the Lagrangian will lead to conserved charges constructing from the field and differential operator representations of the Lorentz group,canonical quantization promote these charges into generators of the Poincare group.

There's another equivalent approach known as path-integral formalism,which is more compact and formal.The equivalence between the two approach are shown by Schwinger-Dyson equations.The path-integral approach goes directly to quantities like Green's functions,which give rise to S-matrix elements.

In doing specific calculations,two approach both give rise to same perturbative calculation scheme known as Feynman rules.

It should be note that

- Not all classical results will be presented in quantum level,the examples include spontaneous symmetry breaking,classical symmetry anomaly.
- The classical theory should be considered as the classical limit of quantum theories,not an equivalent description of the quantum physics.There's ambigurity in the concept ”quantization”.
- Starting form classical field theory and do the quantization is of great convenience:
  - Solving the equation to get the polarizations are simpler that solving the equations of coefficient functions.Thus easier to construct and represent different fields using equations and Lagrangians.
  - The 1st quantization is easy to obtain considering the classical field as a wave-function.
  - Discussion of discrete transformations of field are simpler using directly the field equations.
  - It's easier to discuss interactions,and make models using Lagrangian
  - It's easier to easier to manifest symmetries using Lagrangian,and make use of the SSB mechanics and implications of anomalies to explain observed quantum physics.
  - It's easier to dicuss topological properties and other nonperturbative results using path-integral formalism.
  - Methods like auxiliary field,auxiliary Hilbert space are more easy to apply with path-integral formalism.
  - It's more easy to do perturbative calculations starting form Lagrangian and derive the Feynman rules.

SR + QM + Locality (cluster decomposition)

↓ Unitarity

$\mathcal{H} = \mathcal{F} = \mathrm{Span}\{ a^\dagger_{\sigma/\mp} \cdots a^\dagger |0\rangle \}$

$a^\dagger|0\rangle = |p, \sigma = j/\lambda\rangle$

(PUIR of ISO(1,3))

↓

Poincaré algebra (generators)

local Interaction $S = \exp\left(-\int d^4x\, \mathcal{H}_I(x)\right)$

causality condition $[\mathcal{H}_I(x), \mathcal{H}_I(y)] \overset{\text{space like}}{=} 0$

↓

2nd quantization $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$,

$h_0 = \omega_p$ (on-shell)

$h^{(I)}_{NM} = \delta^3(\sum \vec{p})\, \tilde{h}^{(I)}_{NM}$

↓

Field operators $\tilde{h}_{NM} = C^0_{l_1 \cdots l_{N'}} \psi^{(+)}_{l_1} \cdots \psi^{(+)}_{l_N} \psi^{\dagger(+)}_{l_1'} \cdots \psi^{\dagger(+)}_{l_{M'}}$

$[\psi^{(+)}_l, \psi^{\dagger(+)}_l]_\mp \overset{\text{space like}}{=} 0$

$\psi_l(x) = \sum_{\sigma = \pm 1/2} \int d\tilde{p}\; u_l a e^{ipx} + v_l a^\dagger_c e^{-ipx} \xrightarrow{\text{Wick theorem}}$ Feynman rules. propagators. Green's functions

Rep(SL(2,C))

canonical quantization

↓

eqns of u, v

↓ polarization

eqns of ψ $\longrightarrow$ canonical structure $\longrightarrow$ $H(\psi, \Pi)$ $\longrightarrow$ $\mathcal{L}(\psi, \partial_\mu \psi)$

symmetry principle

↓

Path-integral

$Z[J]$

Part III

Quantization of classical field and perturbative calcualtions

Chapter 6

Quantization of classical field theory

6.1 Canonical structure is common but not always

All of the simplest examples of relativistic quantum field theory furnish canonical systems **both in classical limit and at operator level**,these theories can then be put in a Lagrangian form[1].But,there's **no proof** that all possible relativistic quantum theory can be formulated in this way,there may be a quantum field theory that produce the right Hilbert space and algebraic identities and produce right observables but do not have a Lagrangian,the lack of a 'quantized' Lagrangian implies there's **no classical limit.**

The advantage of Lagrangians is it makes it easy to **satisfy symmetry principles** at least at classical level:

- The Lagriangian in field operators will have same symmetry of the classical version constructed out of classical fields.The form of the Lagriangian will differ form the classical version only by ordering.

- Classical symmetries are not necessarily quantum symmetries,the phenomena that gauge invariance and some classical symmetry can't be simultaneously preserved at quantum level is known as **anomaly**:

  - The failure of classical symmetry can be seen for example that the **renormalized current operator** doesn't satisfy current conservation **operator** equations.
  - The anomaly is better manifested using path-integrals,where the measure is not invariant under this symmetry transformation.

- There's **no anomaly in Poincare symmetry and gauge invariance**,thus classical Poincare symmetry of the classical Lagrangian will become quantum Poincare symmetry after 'quantization':

  - Such theories will allow the construction of operators satisfying the **Poincare algebra** and the single **particle states.**
  - These operators will satisfy the current conservation operator equations.Or their commutator between Hamiltonian vanishes.

There's another advantage of canonical formalism:

- It will provide appropriate non-covariant terms to cancel the non-covariant terms in propagator derived from Wick's theorem in canonical formalism.

- However,in path-integral formalism,the propagator is manifestly Lorentz covariant.

6.1.1 Practical approach:Starting from geussing the classical Lagrangian/action

In constrution of specific models/theories,it is more convenient to:

[1]Note that the Lagrangian here is made out of field **operators**,this is canonical quantized version of Lagrangian for classical fields

Canonical quantize a classical field theory

- Start with **geusssing** a Lorentz invariant and gauge invariant **classical** Lagrangian then derive the classical Hamitonian.
- Canonical quantization the canonical classical fields into canonical operators.And obtain the possible Hamiltonian **operator**.
- If the Hilbert space and the Poincare algebra can be restored,then this Hamiltonian operator is **the right one to compute S-matrix.**For free field theories,the construction start by expanding the classical solutions to classical field equations and quantize the solution to operators,the canonical quantization conditions then **reproduce the 2nd quantization formalism.**
- There's ambigurity in quantization:the order of operators matters,thus the form of Hamiltonian operator may differ from classical Hamiltonian.Also note that various quantum theories may have same classical limit,thus the quantization is ambigurous.
- To obtain **right Hamiltonian operator**,we need trial and error.
- As noted,some quantum theories may not even have a Lagrangian(but still have a Hamiltonian operator).Thus this approach is valid for certain set of theories,they work well for simple free/interacting relativistic field theories.

The **validity** of canonical quantization comes from the fact:

- The field operators,the appropriate 2nd quantized operators(including H,$\vec{P}$,$\vec{J}$,$\vec{K}$,...)and the equations they satisfy have been contructed already **as operators.**
- It can be shown that these simple free/interacting theories do have caonical structure.The Hamiltonian can then be put into the form of canonical field operators,using Legendre transformation,we obtain the **right** Lagrangian **as a operator.**
- In **classical limit**,the classical field theories are restored.

Remark 42

- *The canonical quantization formalism best manifest the validity of quantization procedure,but it is not the best way to derive nonperturbative results and do perturbative calculations.The equivalent **path-integral quantization formalism,**which start with the same **classical** Lagrangian are more convenient to use in practice.*

- *The basic ingredients in theories starting with a **classical** Lagrangian are the **free classical fields** and their Lagrangians,the **interactions are included as couplings between different fields**.*

- *These free fields have already been derived,we simply shown that they satisty the canonical structure and derive the classical Lagrangian.*

- *A appendix on histroical canonical quantization of these classical Lagrangians are included to display techniques related to field equations.*

6.2 Canonical formalism

Convention of variables

Since we are discussing the quantization of classical field theories,there are two types of variables presented:

- For classical field theory,where the field and the Hamiltonian,Lagrangian formed out them,are taken as c-numbers,according to the nature of the field,the c-number have two types:
  - Commuting ordinary numbers for bosonic fields
  - Anticommuting Grassman numbers for fermionic fields
  - For both cases,the order matters only up to signs.The Lagrangian and Hamiltonian are constructed to be bosonic thus the order doesn't matter.
  - The order do matter in path-integral formalism.
  - The field and conjugate field are considered as **infinite-dim**generalization of canonical coordinates and momentum.Thus classical field theory are generalization of classical mechanics,both are classical caonical systems,with **sympletic structure**.
  - As a **classical** canonical system,these field and conjugate field are assumed to satisfy the canonical relations expressed in **Possion braket**:$[\ ,\ ]_P$.The infinitesimal tranformations and generators,classical equations are all formed in Possion braket.
- For canonical quantized systems,the field are quantized to be operators,and the Hamiltonian,Lagrangian formed out of them are also operators:
  - The Possion braket are assumed[a] to be replaced by (anti)commutators
  - For bosonic operators,commutators are involved.
  - For fermionic operators,anticommutators are involved.
  - In general the order in Hamiltonian and Lagrangian matters,different order will involve various signs and (anti)commutators.Usually we will choose a common ordering principle in formal derivation
  - Note that not all ordering are physially the same.We need trial and error to obtain the true Hamiltonian operator.

  In classical system,the $\vec{x}$,t are taken symmetricly,in quantum theory however,the space and time are treated differently:

  - In caonical quantization,the quantization is down implicitly in H-piture for free/interacting fields.Even with presentation of interactions,the free field operators can be obtained in I-piture.
  - In H-picture,all operators are constructed out of interacting field operators.In I-picture,all operators are constructed out of free field operators.Especially the $\mathcal{H}_I(x)$ and S-matrix.
  - Since the free fields are the building blocks of both free and interacting theories,we work in I-pitrure to do perturbative calculations.[b]
  - The H-piture is used to discuss canonical structure and in path integral formalism.
  - Ψ are used for H-piture implicitly,ψ are used for free fields and I-piture implicitly.

  Implicitly we will work with operators directly in canonical quantization,and classical fields in path integral formalism.

[a] The replacement need some condition,see [WeinbergI] on Dirac brakets

[b] Only in I-piture can we expand the field operators in free on-shell modes.

Remark 43 *As in the case of classical mechanics,there are two equivalent way to discuss canonical systems:*

- *The Lagrangian formalism:which involve canonical(general) coordinates and its **time** derivative.*
- *The Hamiltonian formalism:which involve canonical coordinates and their conjugate momentum.*

Both two formalism have their advantages:

- *The Lagrangian formalism manifest symmetries more clearly,easy to derive Noether currents and the equations are eaiser to solve.*
- *The Hamiltonian formalism manifest the true independent D.O.F(field components) in terms of canonical relations.The Possion braket manifest the generator of symmetries and their relation between Hamiltonian.*
- *The Hamiltonian formalism is more closely related to canonical formalism.*
- *The Lagrangian formalism is more closely related to path integral formalism.*
- *Both formalism are used in derivation.*

6.2.1 Action and field equation

The fields are generalization[2] of general coordinates,we may denote fields in H-piture with interaction generally,as:

$$Q_a = \Psi_l(x)$$

- The 'a' label include both discrete l label and continuous x label.The l label denote representation of Lorentz group,in general block diagonal,thus l may include different field types.
- Note all field components are independent,there are axuliary fields,which is determined by independent variables.**These axuliary fields are not canonical variables**,but we still include them for compactness.
- Indepent D.O.F is manifested in Hamiltonian formalism.

The Lagrangian is a **functional** of the set of fields and their **time derivatives**:

$$L(t) = L[\Psi(t), \dot{\Psi}(t)]$$

- The L is a functional as there's infinite number of Q,classically the configuration space is infinite dimensional.
- L depend only on t,thus RHS is implies in expression of L,the tensor indices are contracted so that L is a **scalar**,the $\vec{x}$ are integrated,thus the L can be expressed as a scalar density:

$$L = \int d^3x\mathcal{L}$$

The **conjugate** field is defined as:

$$P_a = \Pi^l(\vec{x}, t) \equiv \frac{\delta L[\Psi(t), \dot{\Psi}(t)]}{\delta \dot{\Psi}_l(\vec{x}, t)}$$

- Since in general,$\Psi, \dot{\Psi}$ do not satisfy the canonical relations,the functional derivative is ill-defined for operators,we then define it by the form when they are classical field.[3]
- We put the index on Π oppesite as on Ψ,since these indices are tensors,two indices are contracted implicitly with a metric.

The field equation is:

$$\dot{\Pi}^l(x) = \frac{\delta L[\Psi(t), \dot{\Psi}(t)]}{\delta \Psi_l(\vec{x}, t)}$$

These can be formulated in **variational principle**,the action as a scalar is defined to be:

$$I[\Psi] \equiv \int_{-\infty}^{\infty} dt L[\Psi(t), \dot{\Psi}(t)] = \int d^4x\mathcal{L}$$

[2] **Caution**:In principle we should work with lattice regularized spacetime,and take continuum limit only at final calcualtions.This perspective may be used to understand the presentation of delta-functions and all kinds of divergence

[3] In principle,it need be amended with signs and (anti)commutators.

under variation $\delta\Psi_l(x) \to 0, t \to \pm\infty$:[4]

$$\begin{aligned}\delta I[\Psi] &= \int dt \sum_a [\frac{\delta L}{\delta Q_a}\delta Q_a + \frac{\delta L}{\delta \dot{Q}_a}\delta \dot{Q}_a] \equiv \int dt \int d^3x [\frac{\delta L}{\delta \Psi_l(x)}\delta\Psi_l(x) + \frac{\delta L}{\delta \dot{\Psi}_l(x)}\delta\dot{\Psi}_l(x)] \\ &= \int d^4x [\frac{\delta L}{\delta \Psi_l(x)} - \frac{d}{dt}\frac{\delta L}{\delta \dot{\Psi}_l(x)}]\delta\Psi_l(x) + [0]\end{aligned}$$

The final equality use the vanishing of variation at boundary of time.The **operator/classical** field equation equivalent to:

$$\delta I[\Psi] = 0 = -\dot{\Pi}_l(x) + \frac{\delta L[\Psi(t), \dot{\Psi}(t)]}{\delta \Psi_l(\vec{x}, t)} = \frac{\delta L}{\delta \Psi_l(x)} - \frac{d}{dt}\frac{\delta L}{\delta \dot{\Psi}_l(x)}$$

- The action is assumed to real,then for complex field,we can break them into real and imaginary complex and think I as functional of real field components[5]
- This make sure that the number of equations coincide with the number of fields.
- The reality of action also assure the generators of continuous symmetries are hermitian operators.

To manifest special relativity,the scalar density is defined to be[6]:

$$\mathcal{L} = \mathcal{L}(\Psi(x), \nabla\Psi(x), \dot{\Psi}(x)) = \mathcal{L}(\Psi(x), \partial_\mu\Psi(x))$$

The variational is now[7]:

$$\delta L = \int d^3x [\frac{\partial \mathcal{L}}{\partial \Psi}\delta\Psi + \frac{\partial \mathcal{L}}{\partial \nabla\Psi}\nabla\delta\Psi + \frac{\partial \mathcal{L}}{\partial \dot{\Psi}}\delta\dot{\Psi}] = \int d^3x [(\frac{\partial \mathcal{L}}{\partial \Psi_l} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla\Psi_l})\delta\Psi_l + \frac{\partial \mathcal{L}}{\partial \dot{\Psi}_l}\delta\dot{\Psi}_l]$$

This gives:

$$\begin{aligned}\Pi^l &= \frac{\delta L}{\delta \dot{\Psi}_l} = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}_l} \\ \dot{\Pi}^l &= \frac{\delta L}{\delta \Psi_l} = \frac{\partial \mathcal{L}}{\partial \Psi_l} - \nabla \cdot \frac{\partial \mathcal{L}}{\nabla \Psi_l}\end{aligned} \tag{6.1}$$

The field equation can be written more compactly as:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Psi}_l} = \frac{\delta L}{\delta \Psi_l} = \frac{\partial \mathcal{L}}{\partial \Psi_l} - \nabla \cdot \frac{\partial \mathcal{L}}{\nabla \Psi_l} \Leftrightarrow \frac{\partial \mathcal{L}}{\partial \Psi_l} - \partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu \Psi_l} = 0 \tag{6.2}$$

This is known as **Euler-Lagrange equation**,and is Lorentz invairant.

How to geuss the Lagrangian

The symmetry principle constraint the possible form of the action,the most fundamental symmetries are:

- Spacetime symmetry:
  - CMP:Galiliean group
  - Relativistic QFT:Poincare group
  - CFT:conformal group

  We will consider Poincare group,the action is a Poincare scalar,this need the Lagrangian density being constructed out of representations of Lorentz group,the translation symmetry is satisfied if we integral over the whole spacetime:

  $$I = \int d^4(x+a)\mathcal{L}(x+a) \equiv \int d^4x\mathcal{L}(x)$$

  The $\mathcal{L}$ is constructed by contracting the tensor indices,more generally use projective tensors to project out scalars.

[4] The sum over l is implicit

[5] Dirac field is assumed to be complex,we always think of real independent field components/D.O.F

[6] The space derivative are seperated from the time derivative,since CMP doesn't use ∂_μ,this form is more general.

[7] ∇,∂_t commute with δ that does not involve spacetime variation,**only internal variation** on field itself,not that here we assume variation of Ψ vanish at space-infinity

- Gauge invariance:when the theory involve (A, A) field for massless particles,the theory must have gauge invariance,else the quantum theory would not be unitary.

Other classical symmetries may also be used for specific models,like C,P,T.But these are not general symmetries.Another important feature is that classical symmetries are not always preserved at quantum level.

The Poincare invariance and gauge invariance will always be preserved at quantum level,thus these constraints are exact.

The symmetry principle constraint possible couplings,but there's still too many terms.There's another criterion,known as **renormalizability**:

- Renormalizability can only be satisfied by certain terms.Thus practically,**renormalizability is simplicity**.We start with simple theories to extract informations first,then add more complex nonrenormalizable terms.

- It should be note that nonrenormalizable terms are also important.Only at low enough energy,is the renormalizable theories effective.When the energy goes high enough,the theories will generally be modified by effective theories,which contain all terms allowed by symmtries.

In view of renormalizability:

- Only include terms with no more than 1nd order derivatives of fields.[8]

- Only include local terms,with no nonlocal coupling.

- There will be **accident symmetry** due to the constraint form renormalizability:even though we don't force these symmetries at first hand,they will emerge when we consider only renormalizable terms.These accidental symmetries will be break when nonrenormalizable terms are included.

6.2.2 Hamiltonian and canonical quantization conditions

The Hamiltonian is defined by **Legendre transformation**[9]:[10]

$$H(t) = H[\Pi(t), \Psi(t)] = \sum_a P_a Q_a - L[Q, \dot{Q}] = \int d^3x[\Pi^l(x)\Psi_l(x) - \mathcal{L}(\Psi, \partial_\mu \Psi)] = \int d^3x \mathcal{H}(x)$$

- The Hamiltonian is a functional of both canonical variables:the field and the conjugate field.

- In general H can depend on t,but for conserved system with no background/external dynamical field,the H is independent of t and is conserved.Without mentioning we will **take H as conserved.**

The variational derivatives are[11]:

$$\begin{aligned}
\frac{\delta H}{\delta \Psi_l(x)} &= \int d^3y[\Pi^{l'}(y)\frac{\delta \dot{\Psi}_{l'}(x)}{\delta \Psi_l(x)} - \frac{\delta L}{\delta \Psi_l(x)} - \frac{\delta L}{\delta \dot{\Psi}_{l'}(y)}\frac{\delta \dot{\Psi}_{l'}(y)}{\delta \Psi_l(x)}] \equiv -\frac{\delta L}{\delta \Psi_l(x)} = -\dot{\Pi}^l(x) \\
\frac{\delta H}{\delta \Pi^l(x)} &= \int d^3y[\frac{\delta \Pi^{l'}(y)}{\delta \Pi^l(x)}\dot{\Psi}_{l'}(x) + \Pi^{l'}(y)\frac{\delta \dot{\Psi}_{l'}(y)}{\delta \Pi^l(x)} - \frac{\delta L}{\delta \dot{\Psi}_{l'}(y)}\frac{\delta \dot{\Psi}_{l'}(y)}{\delta \Pi^l(x)}] \equiv \dot{\Psi}_l(x)
\end{aligned} \tag{6.3}$$

- We have implicitly used the facts that Π, Ψ are independent canonical variables,both classically and as operator,and the property of variational derivative[12]:

$$\frac{\delta A_l(x)}{\delta A_{l'}(y)} = \delta(x-y)\delta_{ll'} \Leftrightarrow \frac{\delta A_a}{\delta A_b} = \delta_{ab}$$

- These two eqution s are known as the **Hamilton equations**.Similarly we have:

$$\frac{\delta H}{\delta A} = \frac{\partial \mathcal{H}}{\partial A}$$

[8] Upon integral by parts,2nd order can appear.

[9] This hold only when the Jacobi matrix of changing viables from $\dot{\Psi} \to \Pi$ is not singular.

[10] The summation over l is implicit.

[11] The variational derivative of some variable is valued when other variables hold fixed correspondingly

[12] δ_{ab} contain products of delta-functions and symbols

For **canonical operators**,they are **assumed** to satisfy the canonical relation in **equal time (anti)commutators**.[13]:

$$[Q_n(\vec{x},t),P^{n'}(\vec{y},t)]_{\mp} \equiv i\delta^3(x-y)\delta_n^{n'}, [Q_n(\vec{x},t),Q_{n'}(\vec{y},t)] \equiv 0, [P^n(\vec{x},t),P^{n'}(\vec{y},t)] \equiv 0 \tag{6.4}$$

- The (anti)commutator are used for fermionic/bosonic operators,which classically are Grassman/Ordinary variables.
- The operators are in H-piture,and they are generally interacting:

  $$A(\vec{x},t)_H = e^{iHt}a(\vec{x},0)_S e^{-iHt}$$

  We may also use the I-piture,where the fields have the same form of **free fields**:

  $$a(\vec{x},t)_I = e^{iH_0t}a(\vec{x},0)_S e^{-iH_0t}$$

- The canonical relations hold **with or without interaction.**
- These will be taken as axioms of canoncial quantization,together with the exsistence of a Hamiltonian/Lagrangian.Note there's condition for replacing the classical Possion braket with (anti)commutators.
  - Axiom means upon renormalization,this will also be true,not all relations will be preserved upon quantization/renormalization.

For operators,we can define the functional derivative of bosonic[14] operators:

$$\begin{aligned}
\frac{\delta F[Q(t),P(t)]}{\delta Q_n(x)} &\equiv i[P^n(x),F[Q(t),P(t)]]_- \\
\frac{\delta F[Q(t),P(t)]}{\delta P^n(x)} &\equiv i[F[Q(t),P(t)],Q_n(x)]_- \\
\delta F[Q(t),P(t)] &= \sum_n \int d^3x[\delta Q_n(x)\frac{\delta F[Q,P]}{\delta Q_n(x)} + \frac{\delta F[Q,P]}{\delta P^n(x)}\delta P^n(x)]
\end{aligned}$$

- We have assumed that F is expressed with **all Q to the left of all P** for operators,thus these two equations are just left/right-derivatives with respect to classical Q/P.Implicitly shown in the expression of δF
- This definition express the equations and other transformations in a compact form[15],note that H is bosonic:

$$\begin{aligned}
\dot{Q}_n(x) &= i[H,Q_n(x)] = \frac{\delta H}{\delta P^n(x)} \\
\dot{P}^n(x) &= -i[P_n(x),H] = -\frac{\delta H}{\delta Q_n(x)}
\end{aligned} \tag{6.5}$$

- For free fields:

$$\begin{aligned}
\dot{q}_n(x) &= i[H_0,q_n(x)] = \frac{\delta H_0}{\delta p^n(x)} \\
\dot{p}^n(x) &= -i[p_n(x),H_0] = -\frac{\delta H_0}{\delta q_n(x)}
\end{aligned} \tag{6.6}$$

There's difference between free conjugate field and interacting conjugate field,by definition:

$$P^n = \frac{\partial \mathcal{L}}{\partial \dot{Q}_n}, p^n = \frac{\partial \mathcal{L}_0}{\partial \dot{q}_n}$$

If $\mathcal{L}_I = \mathcal{L} - \mathcal{L}_0$ depend on $\dot{Q}/\partial_\mu Q$ then $P(x) \neq p(x)$.When interactions are involved,$P(x)$ is always used for derivation.This is the origin of the **non-covariant**[16] terms in H of interacting fields.

[13]For classical case,$[,]_P \leftrightarrow -i[,]_{\mp}$
[14]Contain even number of fermionic field,involving commutators
[15]This may be obtained from classical form by replacing Possion braket,but note the convention of ordering is different
[16]Since this only involve ∂_0,SR is not hidden

Independent fields and axuliary fields

Not all fields appearing are independent canonical variables P,Q,some fields have no conjugate field this means their time-derivative are absent in $\mathcal{L}$,these fields are known as axuliary fields,denoted as $C_r(x)$:

- The fields can be seperated into two set:$\Psi_l(t) = \{Q_a, C_r\}$,where Q are canonical variables and will satisfy the equations and are **independet** variables.The number of real independent component(D.O.F) are the number of independent real Qs.And is half the dimension of phase space.
- The Qs still will satisfy the canonical relations while Cs don't
- The Lagrangian is now $L[Q(t), \dot{Q}(t), C(t)]$.
- The axuliary field are presented due to Lorentz invairance.

The Hamiltonian is:

$$H = \sum_n \int d^3x[P^n(x)\dot{Q}_n(x)] - L[Q(t), \dot{Q}(t), C(t)]$$

,to express $H = H[Q, P]$ we need to solve $C, \dot{Q}$ in terms of Q,P.

The equations satisfied by C involve only fields and their first derivatives:

$$0 = \frac{\delta L}{\delta C_r(x)}$$

These equations together with definition of P,can be solved to obtain$C_r[Q, P], \dot{Q}_n[Q, P]$ for simple theories.

- For gauge theories,we must **fix the gauge** to obtain the H in terms of P, Q only.This can be down by either choose a gauge or use Fadeev-Popov methods.

6.2.3 Time-dependent perturbation

Once we express $H \equiv H[Q, P]$,the transition to I-piture is easy:for time-independent H,$H = [Q(0), P(0)] = [q(0), p(0)]$,thus:

$$H_I = e^{iH_0t}H[q(0), p(0)]e^{-iH_0t} = H_I[q(t), p(t)] = H_0[q(t), p(t)] + V_I[q(t), p(t)] \tag{6.7}$$

The Hamiltonian is split into two parts,where H_0 in terms of **free** fields are exactly solvable.The interaction in I-picture is also constructed out of **free** fields,and is considered in perturbation.

We usually have the relation:

$$V[q(t), p(t)] = \int d^3x\mathcal{H}_I(q(t), p(t)) \sim -\int d^3x\mathcal{L}_I$$

Thus the S-matrix is:

$$S = \exp(-i\int d^4x\mathcal{H}_I(x)) = \exp(i\int d^4x\mathcal{L}_I(q(x), \partial_\mu q(x), c(x), \nabla c(x)))$$

Since only free fields are invloved,the two axioms are more implicative

- Hamilton/Lagrange operator equations of free fields can be solved with linear combinations of free modes,the coefficent are operators.
- Canonical relations then implies that these operators satisfy the 2nd quantization algebra,thus they are creation/annhilation operators.
- The V_I then is expressed in $V_I(a, a^\dagger)$,and we can express the perturbative calculations in Feynman diagrams.

Remark 44

- *This have restore the free field operators,thus this* ***successively quantized the classical free field theory in particular****.We can show that the generators are put into 2nd quantization form and the Poincare algebra is indeed statisfied.*
- *Note that we need the* ***right*** *free field Lagrangian to obtain the expansion,not all Lagrangian created out of free fields accept the solution of free field operators.The canonical quantization of free fields is indeed historical,since if we change the Lagrangian a little[17],the quantization procedure will fail,the validity of the quantization procedure comes form the free field operator itself,which is yet to be found,the most deep insight is that it follows naturally form the axioms of relativistic QFT.*

[17]Renormalizability/simplicity is never a reasonsing but a strategy.

6.2.4 Total derivatives,topological terms,implicit variables and extension

The total derivatives in Lagrangian give contribution of action as:

$$\Delta I = \int_M d^4x \partial_\mu \mathcal{F}^\mu(\Psi, \partial_\mu \Psi) = \int_{\partial M} \mathcal{F}^\mu dS_\mu$$

- For **trivial** topology field:$\Psi \to 0$,$\Delta I = 0$ and these total derivatives can be ignored.This means we can integrel by parts for these topological trivial fields in Lagrangian directly,with no physical consequnce.

- For topological nontrivial fields,some of the total derivatives known as topological terms matters,and in general the integration gives the topological constants of the manifold.

  - For example,the Chern-Simons term in gauge theories,the total derivative of this term gives the θ-term[18] as a total derivative.The integration gives the Pontryagin index.

The total derivative terms will never have effect on classical level,they do not contribute to the field equations,and will thus never appear in perturbation theories.

The total derivatives in Lagrangian density may still contribute to the Lagrangian however,it's obvious that $\nabla \cdot \vec{\mathcal{F}}$ terms drop out safely for topological trivial fields.This is not clear for $\partial_0 \mathcal{F}^0$.

Consider the contribution:

$$\Delta L(t) = \int d^3x D_{n,\vec{x}}[Q(t)]\dot{Q}_n(x), \Delta P^n(x) = D_{n,\vec{x}}[Q(t)]$$

If we express $H = H[Q, \dot{Q}]$ then there's no difference,due to:

$$\int d^3x \Delta P^n Q_n - \Delta L = 0$$

there's also no change in H expressed as $H = H[Q, P]$,but these old variables will not satisfy the canonical relations,the canonical relations will be **extended**:

$$\begin{aligned} [P^n(x), P^m(y)] =& [P^n(x) + \Delta P_n(x), P^m(y) + \Delta P^m(y)] \\ & - [\Delta P^n(x), P^m(y) + \Delta P^m(y)] - [P^n(x) + \Delta P_n(x), \Delta P^m(y)] \\ & + [\Delta P^n(x), \Delta P^m(y)] \\ =& - i\frac{\delta D_{n\vec{x}}[Q(t)]}{\delta Q_m(y)} + i\frac{\delta D_{m\vec{y}}[Q(t)]}{\delta Q_n(y)} \end{aligned} \tag{6.8}$$

in general this will not vanish,but if the added term is a total time-derivative,then this will vanish:

$$\Delta L = \frac{d}{dt}G = \int d^3x \frac{\delta G[Q(t)]}{\delta Q_n(x)}\dot{Q}_n(x)$$

then:

$$D_{n\vec{x}}[Q(t)] \equiv \frac{\delta G[Q(t)]}{\delta Q_n(x)}$$

For general case,the extension of the canonical relations thus also the Lie algebras will not be trivial,and when H is expressed in true canonical variables,the there will be **extra terms**.These terms are implicit in H when expressed in old/traditional variables.

Remark 45
- *This modification of algebraic relations due to implicit terms in H is highly related to projective representations and group cohomology.For more details of topological terms,refer to [Jackiw],[Altland],[Nakahara],[Ze*

- *A classical example will be systems with external magnetic field added,the conjugate momentum of the particle will be modified to include the momentum of the electromagenetic field.The usually Heisenberg group is extended,for example the magnetic translation group and magnetic BZ.*

[18]Related to CP violation

6.2.5 Global symmetry and Noether's theorem

The most important feature of canonical formalism is that it provide a natural framework for the **quantum** mechanical implementation of **symmetry principles**.

The definition of a quantum level symmetry may be:

- Quantum mechanical:
  - The (antiunitary) operator of transformations must be constructed.
  - For continuous symmetries,the we must have to construct the operators explicitly to satisfy the quantum/enlarged **Lie algebra**.
  - Especially,there must be some system with H commuting with the generators we constructed.For discrete symmetries,all transformation operators should commute with certain H.
  - This means the operators for the united group of spacetime transformation and the other possible symmetry group[19] should all be constructed with appropriate operators and Lie algebras.
- Classical:
  - The action is left invariant by the transformations of the fields.
  - The generators of continuous transformations can be constructed out of these fields and these transformations can be expressed in $[\ ,\ ]_P$
- Canonical formalism:
  - When canonical quantized,the action as a functional of field operators is still invariant under transformations of the field operators.
  - The generator operator of continuous transformations can be constructed out of field operators and the transformations are expressed in $[\ ,\ ]_\mp$
  - These generators satisfy the quantum level Lie algebra.
- Path integral formalism:
  - The measure is also invariant under classical transformations.
  - The quantum effective action is invariant under transformations.

We will focus on continuous symmetries,where the enlarged group of **all** infinitesimal transformations and Lie algebra are,:

$$T = \exp(i\epsilon^a t_a), [t_a, t_b] = if_{ab}^{\ c} t_c$$

- Note,here ϵ are constants,the group is just a group G.
- There's anther kind of symmetry known as local symmetries where $\epsilon(x)$ depend on spacetime,this group is denoted as functions from spacetime manifold to group manifold:$\mathcal{F}[\mathcal{M} \to G] = G^{\mathcal{M}}$
- Gauge theories involve promoting global (internal)-G symmetry to local G symmetry and introduction of gauge boson fields consists of (A, A) fields invariant under this local G symmetry.The introduction gives the coupling between gauge fields and matter fields.

Quantum mechanically,the generators are constructed as $T[\Psi, \partial_\mu \Psi]$[20],and the transformations are,for any bosonic operator furnishing some representation $O[\Psi, \partial_\mu \Psi]$:[21]

$$O(x) + \delta O(x) = e^{i\epsilon^a T_a} O(x) e^{-i\epsilon^a T_a} = \sum_n \frac{(i\epsilon^n)}{n!} \cdot T^{[n]}[O(x)]_- = e^{+i\epsilon^a t_a^{(R)}} O(e^{-i\epsilon^a t_a^{(V)}} x) \tag{6.9}$$

- For compactness,we omitted the index on T,in principle there's also summations over $a_1, ..., a_n$ for each n.
- The generators are assumed to be bosonic[22],$T^{[n]}[O]_-$ means n-th commutator with right order.

[19] May be semiproducts or simply products
[20] Actually we it will depend on only P,Q
[21] δO is bosonic/fermionic if O is bosonic/fermionic
[22] Exception is that generator of supersymmetry is fermionic

- A special case is that $O_n = \Psi_n(x)$,n include both field components and particle types related by internal symmetries.

Matching both side to $O(\epsilon)$,we obtain the commutator of generators with O,note that RHS may also include the tranformation of spacetime,giving rise to $\partial_\mu O(x)\delta x$.

Noether's theorem

Consider the **global/physical** continous infinitesimal transformation[23]:

$$\delta\Psi_l(x) = i\epsilon\mathcal{F}_l(x)$$

Classical symmetry means that the action is left invariant,but when the field is classical and a operator,when the field equation is **not** satisfied[24]:

$$0 = \delta I = i\epsilon\int d^4x \frac{\delta I[\Psi]}{\delta\Psi_l(x)}\mathcal{F}_l(x) \tag{6.10}$$

In general the action is not invariant under the local transformations,but it must have the form:

$$\delta I = -\int d^4x J^\mu(x)\partial_\mu\epsilon(x)$$

- As $\epsilon \to \epsilon$,$\delta I \to 0$
- If the field is stationay,statisfying the equation of motion,and that $\delta\Psi_l = i\epsilon(x)\mathcal{F}_l(x) = 0, x \in \partial M$ $\delta I = 0$,this means the field have trivial topology,then we upon **integral by part**:

$$\Psi stationary \Leftrightarrow 0 = \delta I \Leftrightarrow \partial_\mu J^\mu = 0$$

This is known as the **Noether's theorem**:

Noether's theorem

Continuous classical global symmetry that leaves the **action** invariant will give rise to conserved Noether currents:

- The current is conserved **when E.O.M is satisfied**,there exsist conserved quantity:

$$(\partial_\mu J^\mu)_{E.O.M} = 0 \Rightarrow F = \int d^3 J^0, \frac{d}{dt}F = 0$$

- The explicit expression of $J^\mu[\Psi, \partial_\mu\Psi]$ can be obtained by promoting the global symmetry to local symmetry in variation of the action:

$$\delta I = -\int d^4x J^\mu(x)\partial_\mu\epsilon(x) \tag{6.11}$$

- This argument is classical,when quantized and renormalized,the classical symmetry not be preserved and the renomalized operator J^μ may not be conserved.However,the Poincare symmetry and Gauge invariance are always preserved at quantum level.

If the transformation leaves the Lagrangian invariant,then we can obtain the explicit expression of F using the Lagrangian itself:

$$0 = \delta L = \int d^3[\frac{\delta L}{\delta\Psi_l(x)}\mathcal{F}_l(x) + \frac{\delta L}{\delta\dot{\Psi}_l(x)}\frac{d}{dt}\mathcal{F}_l(x)]$$

Then,promote $\epsilon \to \epsilon(t)$:

$$\delta I = i\int dt\int d^3x[\frac{\delta L}{\delta\Psi_l(x)}\epsilon(t)\mathcal{F}_l(x) + \frac{\delta L}{\delta\dot{\Psi}_l(x)}\frac{d}{dt}(\epsilon(t)\mathcal{F}_l(x))] \equiv i\int dt\int d^3x\frac{\delta L}{\delta\dot{\Psi}_l(x)}\dot{\epsilon}(t)\mathcal{F}_l(x)$$

[23]For simplicity,consider one generator

[24]for stationary fields(solutions to E.O.M),the variation vanishes identically.

Thus the invariance of L produce the explicit form of F as:

$$F = -i\int d^3x \frac{\delta L}{\delta\dot{\Psi}_l(x)}\mathcal{F}_l(x) = \int d^3x J^0$$

- This will be the case of spactial translation and rotation,but general Lorentz transformation will leave only action invariant.

If the transformation leaves further the Lagrangian density invariant,then we can obtain the explicit expression of J^μ using the Lagrangian density itself:

$$0 = \frac{\partial\mathcal{L}}{\partial\Psi_l(x)}\mathcal{F}_l(x) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l(x)}\partial_\mu\mathcal{F}_l(x)$$

then,promoting $\epsilon \to \epsilon(x)$:

$$\delta I = i\int d^4x[\frac{\partial\mathcal{L}}{\partial\Psi_l(x)\epsilon(x)}\mathcal{F}_l(x) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l(x)}\partial_\mu(\mathcal{F}_l(x)\epsilon(x))] \equiv i\int d^4x\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l(x)}\mathcal{F}_l(x)\partial_\mu\epsilon(x)$$

Thus the invariance of $\mathcal{L}$ produce the explicit form of J^μ as:

$$J^\mu = -i\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l(x)}\mathcal{F}_l(x) = J^\mu[\Psi, \partial_\mu\Psi]$$

The general form of J^μ for symmetry transformations leaves action invariant but change Lagrangian density by a total derivative(trivial topology),can be obtained like this[a]:

$$\begin{aligned}
\delta\mathcal{L} =& \partial_\mu\mathcal{L}\delta x^\mu + \bar{\delta}\mathcal{L} = \partial_\mu\mathcal{L}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\Psi_l(x)}\bar{\delta}\Psi_l(x) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l(x)}\partial_\mu\bar{\delta}\Psi_l(x) \\
=& \partial_\mu\mathcal{L}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\Psi_l(x)}(\delta\Psi_l(x) - \partial_\nu\Psi_l(x)\delta x^\nu) + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l(x)}\partial_\mu(\delta\Psi_l(x) - \partial_\nu\Psi_l(x)\delta x^\nu) \\
=& \partial_\mu[(\eta^\mu_\nu\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\partial_\nu\Psi_l)\delta x^\nu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\delta\Psi_l] - [\frac{\partial\mathcal{L}}{\partial\Psi_l} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}][\partial_\nu\Psi_l\delta x^\mu - \delta\Psi_l] \\
=& \partial_\mu[(\eta^\mu_\nu\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\partial_\nu\Psi_l)\delta x^\nu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\delta\Psi_l] + [\frac{\partial\mathcal{L}}{\partial\Psi_l} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}]\bar{\delta}\Psi_l \\
=& \partial_\mu\bar{J}^\mu + \frac{\delta I}{\delta\Psi_l}\bar{\delta}\Psi_l
\end{aligned} \tag{6.12}$$

Thus the general expression of Noether current in terms of $\mathcal{L}$ is:

$$J^\mu = (\eta^\mu_\nu\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\partial_\nu\Psi_l)\delta x^\nu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\delta\Psi_l$$

- **Caution**:the use of $\bar{\delta}$ is a little confusing,here with specification of $\bar{\delta}$,the meaning of δ is truely the internal variation,this is different from the meaning before,in terms of $\bar{\delta}$ notation here,$\delta\mathcal{L} = 0$
- To manifest the difference: we better redenote the current as:

$$J^\mu = (\eta^\mu_\nu\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\partial_\nu\Psi_l)\delta x^\nu + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l(x)}\Delta\Psi_l \tag{6.13}$$

  here the Δ means variation in tensors,not fields.

- The current is conserved only when E.O.M is statisfied:

$$\partial_\mu J^\mu = -\frac{\delta I}{\delta\Psi_l}\bar{\delta}\Psi_l$$

  here we have the definition only when the field is topologically trivial:

$$\delta I = \int d^4x[\frac{\partial\mathcal{L}}{\partial\Psi_l}\delta\Psi_l + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\partial_\mu\delta\Psi_l] = \int d^4x[\frac{\partial\mathcal{L}}{\partial\Psi_l} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}]\delta\Psi_l + \int d^4x\partial_\mu(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\delta\Psi_l)$$

These hold in classical level and for bare operators,these also hold as operator expressions.Note that J^μ, F are **bosonic.**

- These conserved currents will be the ones coupling to the (A, A) gauge fields.

[a]it's better to distinguish the contribution from internal transformations and spacetime transformations:

$$O \to \tilde{O}(\tilde{x}), \bar{\delta}O = \tilde{O}(x) - O(x), \delta O(x) = \bar{\delta}O(x) + \partial_\mu O(x)\delta x^\mu, \bar{\delta}\partial_\mu = \partial_\mu\bar{\delta}$$

,note that $\delta \equiv \Delta$ is the viation of tensors,$\bar{\delta} \equiv \delta$ used before.

For quantum case,we may express $C[Q, P]$,thus $F[Q, P]$ and Nother's theorem implies that the conserved quantities are **generators.**Expressed in terms of canonical variables:$\mathcal{F}_n(x) = \mathcal{F}_n[Q(t); \vec{x}]$.

For transformations leaving the **Lagrangian** invariant,the conserved quantities are **expressed in terms of P,Q**,and generate the transformation:

$$F = -\int d^3x P^n(x)\mathcal{F}_n[Q(t); \vec{x}] \equiv F[Q, P]$$

Then:

$$-i\frac{\delta F}{\delta P^n(x)} = [F, Q_n(x)]_- = -\mathcal{F}_x(x)$$

This coincide with the definition of generator:

$$\delta Q_n = i\epsilon\mathcal{F}_n = -i\epsilon[F, Q_n] = e^{-i\epsilon F}Q_n e^{i\epsilon F} - Q_n$$

Similarly,we have:

$$[F, P^n(x)]_- = \int d^3y P^m(y)\frac{\delta\mathcal{F}_m[Q; \vec{y}]}{\delta Q_n(x)}$$

Important examples are the following:

Translational invariance

The infinitesimal global spacetime translation is:

$$\delta x^\nu = \epsilon^\nu, \Delta\Psi_l = 0$$

thus we have:

$$J^\mu = T^\mu_{\ \nu}\epsilon^\nu = [\eta^\mu_\nu\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\partial_\nu\Psi_l]\epsilon^\nu$$

Since the ϵ^ν are constants this defines for 4 conserved current:

$$\partial_\mu T^\mu_{\ \nu} = 0, T^{\mu\nu} = \eta^{\nu\alpha}T^\mu_{\ \alpha} = \eta^{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\partial^\nu\Psi_l$$

these can also be derived from promoting $\epsilon \to \epsilon(x)$ in $\delta I, \delta\Psi(x) = \epsilon^\mu(x)\partial_\mu\Psi_l(x)$ and identifiy the current multiplying the $-\partial_\mu\epsilon$.These two methods coincide.

This is known as **energy-momentum tensor**,with conserved quantities known as **energy-momentum vector**,note that $T^{\mu\nu}$ is not symmetric in general:

$$P_\nu = \int d^3x T^0_{\ \nu}, \frac{d}{dt}P_\nu = 0$$

specificly:

$$P_\nu = \int d^3x[\delta^0_\nu\mathcal{L} - \Pi^l\partial_\nu\Psi_l], P_i = P^i = -\int d^3x\Pi^l\partial_i\Psi_l, P_0 = -P^0 = \int d^3x[\mathcal{L} - \Pi^l\dot{\Psi}_l] = -H$$
$$\Rightarrow P^\mu = (H, \vec{P}), P_\mu = (-H, \vec{P}) \tag{6.14}$$

- Since space-translation leaves L invariant,we have,for any possible Ψs:

$$\vec{P} = -\int d^3x P^n(x)\nabla Q_n(x) \Rightarrow [\vec{P}, Q_n] = i\nabla Q_n, [\vec{P}, P^n] = i\nabla P^n, [\vec{P}, A[Q, P]] = i\nabla A \tag{6.15}$$

  this shows $\vec{P}$ constructed out of P,Q is indeed the generator operator:

$$e^{i\vec{P}\cdot\epsilon}Ae^{-i\vec{P}\cdot\epsilon} = \epsilon(x - \vec{\epsilon}) = A - \epsilon\cdot\nabla A = A + i\epsilon\cdot[\vec{P}, A] \Rightarrow [\vec{P}, A] = i\nabla A$$

- Similarly for time translation we have,as generator:

$$[H, A] = -i\partial_0 A$$

The explicit form of H in terms of P,Q however is not constructed,we should solve $C, \dot{\Psi}$ in terms of P, Q first then put them into the definition obtained from the Noether's theorem.The result is that P^0 is indeed the generator H.

- The differential operator representation:

$$P_\mu \to -i\partial_\mu = (-i\partial_0, i\nabla) = (-H, \vec{P})$$

Internal symmetries

For internal symmetries,the **global** infinitesimal transformations have form of:

$$\Psi_l(x) \to e^{i\epsilon^a (t_a)_l{}^{l'}} \Psi_{l'}(x) \Leftrightarrow \delta Q_n(x) = i\epsilon^a (t_a)_n{}^m Q_m(x), \delta C_r(x) = i\epsilon^a (\tau_a)_r{}^{r'} C_{r'}(x)$$

- Here t, τ furnish finite representation of the Lie algebra of the internal symmetry group.
- In general,the internal transformation mix different fields,these fields are grouped into multiplet of certain representations.In general since the field index include all particles,the generators can be put into block diagonal forms,each block furnishing some irreducible finite dimensional representation.
- Special examples include:
  - $U(1)$:where τ_a is diagonal and with U(1) charges on the main diagonal.The representation of U(1) is denoted as $e^{iQe\theta}, q = Qe, Q \in \mathbb{Z}$,by seperating complex field components into real components,$U(1) \cong O(2)$
  - $SU(2)$ in weak interactions:fermions participating in weak interactions will furnish weak-multiplet.
  - $SU(3)$ in strong interactions:quarks participating in strong interactions will carry colors,furnishing fundamental representation of $S(3)$
  - In all these cases,the gauge bosons furnish adjiont representation of the **global**-G group.
  - All kinds of approximate symmetries related to masses in strong interactions.

For each generator,there's a conserved current,since the Lagriangian density as well as L,I are invariant:

$$J_a^\mu = -i\frac{\partial \mathcal{L}}{\partial \partial_\mu Q_n}(t)^n{}_m Q_m - i\frac{\partial \mathcal{L}}{\partial \partial_\mu C_r}(\tau)_r{}^s C_s$$

when integraled and with C expressed in P,Q,the generators are only functionals of P,Q.

$$\partial_\mu J_a^\mu = 0, T_a = \int d^3 J_a^0 = -i\int d^3x P^n(x)(t_a)_n{}^m Q_m(x)$$

Note there's no contribution from axuliary fields in conserved generator operators T_a,they are expressed in terms of P,Q.

We have the relation:

$$[T_a, Q_n(x)] = -(t_a)_n{}^m Q_m(x)$$
$$[T_a, P^n(x)] = +(t_a)^n{}_m P^m(x)$$

- For U(1) symmetry,the Q,P are just ladder operators lowering the eignevalue of $T = Q$,this coincide with the definition of field operators.
- These **bare** generators furnish the Lie algebra,using the Lie algebra satisfied by representation t_a:

$$[T_a, T_b] \equiv i\int d^3x[-P^m(t_a)_m{}^n(t_b)_n{}^k Q_k + P^n(t_b)_n{}^k(t_a)_k{}^m Q_m] = if_{ab}^c T_c$$

There're useful equal-time commutators:

$$\begin{aligned}
&J_a^0 = -iP^n(t_a)_n{}^m Q_m \to [J]\\
\Rightarrow &[J_a^0(\vec{x},t), Q_n(\vec{y},t)] = -\delta^3(x-y)(t_a)_n{}^m Q_m(\vec{x},t)\\
&[J_a^0(\vec{x},t), P^n(\vec{y},t)] = \delta^3(x-y)(t_a)^n{}_m P_m(\vec{x},t)\\
&[J_a^0(\vec{x},t), C_t[Q(y),P(y)]] = -\delta^3(x-y)(\tau_a)_r{}^s C_s(\vec{x},t)
\end{aligned}$$

In general:

$$[J_a^0(\vec{x},t), \Psi_l(\vec{y},t)] = -\delta^3(x-y)(t_a)_l{}^{l'} Q_{l'}(\vec{x},t) \tag{6.16}$$

- These may be used to derive relations including Ward identities.

Lorentz invariance

For infinitesimal Lorentz transformations:$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$.

Use the genral formula,with the definition of Δ:

$$\Delta\Psi_l(x) = \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})_l{}^{l'}\Psi_{l'}(x), \Delta\partial_\alpha\Psi_l = \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})_l{}^{l'}\Psi_{l'}(x) + \omega_\alpha{}^\lambda\partial_\lambda\Psi_l$$

- The meaning of Δ is that we consider the variation as tensors,and $\partial_\mu\Psi_l$ is considered as tensor products of Lorentz-vectors and general representation indicated by $\mathcal{J}^{\mu\nu}$
- The differential can also be considered as the differential operator representation.

Having seperated contribution from spacetime transformation,we have:

$$\Delta\mathcal{L} = 0 = \frac{\partial\mathcal{L}}{\partial\Psi_l}\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})_l{}^{l'}\Psi_{l'} + \frac{\partial\mathcal{L}}{\partial\partial_\alpha\Psi_l}\frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})_l{}^{l'}\partial_\alpha\Psi_{l'} + \frac{\partial\mathcal{L}}{\partial\partial_\alpha\Psi_l}\omega_\alpha{}^\lambda\partial_\lambda\Psi_l$$

Setting the coefficient of ω to zero:

$$0 = \frac{i}{2}\frac{\partial\mathcal{L}}{\partial}\Psi_l(\mathcal{J}_{\mu\nu})_l{}^{l'}\Psi_{l'} + \frac{i}{2}\frac{\partial\mathcal{L}}{\partial\partial_\alpha\Psi_l}(\mathcal{J}_{\mu\nu})_l{}^{l'}\partial_\alpha\Psi_{l'} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\partial_\alpha\Psi_l}(\eta_{\alpha\mu}\partial_\nu - \eta_{\alpha\nu}\partial_\mu)\Psi_l$$

Using E.O.M and definition of $T^{\mu\nu}$(not symmetric):

$$0 = \partial_\alpha[\frac{i}{2}\frac{\partial\mathcal{L}}{\partial\partial_\alpha\Psi_l}(\mathcal{J}_{\mu\nu})_l{}^{l'}\Psi_{l'}] - \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$$

This suggest the symmetric version of energy momentum tensor:

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{i}{2}\partial_\alpha[\frac{\partial\mathcal{L}}{\partial\partial_\alpha\Psi_l}(\mathcal{J}_{\mu\nu})_l{}^{l'}\Psi_{l'} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}(\mathcal{J}^{\alpha\nu})_l{}^{l'}\Psi_{l'} - \frac{\partial\mathcal{L}}{\partial\partial_\nu\Psi_l}(\mathcal{J}^{\alpha\mu})_l{}^{l'}\Psi_{l'}] \tag{6.17}$$

- This is known as the **Belinfante tensor** and is symmetric:

  $$\Theta^{\mu\nu} = \Theta^{\nu\mu}$$

  this is used in general relativity and can actually be derived from variational principle in curved spacetime.
- The term in braket is antisymmetric in μ, α thus this term will not effect the conservation:

  $$\partial_\mu\Theta^{\mu\nu} = 0$$

- It can be shown that $\int d^3x\Theta^{0\nu} = \int d^3x T^{0\nu} = P^\nu$.

The conserved current from Lorentz invariance can be expressed as:

$$\mathcal{M}^{\lambda\mu\nu} \equiv x^\mu\Theta^{\lambda\nu} - x^\nu\Theta^{\lambda\mu}, \partial_\lambda\mathcal{M}^{\lambda\mu\nu} = \Theta^{\mu\nu} - \Theta^{\nu\mu} = 0, \mathcal{M}^{\lambda\mu\nu} = -\mathcal{M}^{\lambda\nu\mu}, J^{\mu\nu} = \int d^3x\mathcal{M}^{0\mu\nu}, \frac{d}{dt}J^{\mu\nu} = 0 \tag{6.18}$$

The compact expression of $\mathcal{M}^{\lambda\mu\nu}$ can be obtained by expanding $\Theta^{\mu\nu}$,there's a quicker way form the general expression of J^μ,putting in $\delta x^\mu = \omega^\mu{}_\nu x^\nu, \Delta\Psi_l(x) = \frac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})_l{}^{l'}\Psi_{l'}(x)$

$$0 = \partial_\mu[T^{\mu\nu}x^\lambda + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi_l}\frac{i}{2}(\mathcal{J}^{\nu\lambda})_l{}^{l'}\Psi_{l'}]\omega_{\nu\lambda}$$

by certain antisymmetrization:

$$\mathcal{M}^{\lambda\mu\nu} = -T^{\lambda\mu}x^\nu + T^{\lambda\nu}x^\mu - i\frac{\partial\mathcal{L}}{\partial\partial_\lambda\Psi_l}(\mathcal{J}^{\mu\nu})_l{}^{l'}\Psi_{l'} \tag{6.19}$$

The specific expression of $J^{\mu\nu}$ is:

$$J^{\mu\nu} = \int d^3x[-T^{0\mu}x^\nu + T^{0\nu}x^\mu - i\Pi^l(\mathcal{J}^{\mu\nu})_l{}^{l'}\Psi_{l'}]$$

For angular momentum:

$$J^{ij} = \int d^3x\Pi^l[-x^i\partial_j\Psi_l + x^j\partial_i\Psi^l - i(\mathcal{J}^{ij})_l{}^{l'}\Psi_{l'}] \tag{6.20}$$

- The fisrt two terms gives the orbital angular momentum.

- The last term gives the spin angular momemtum.Which is related to representation of the rotation subgroup of the Lorentz group furnished by the field.

- The identification of these operators with the generator is again by forming $[J^{ij}, A]$,in general:

$$\begin{aligned} &[P^\mu, Q/P] = i\partial^\mu(Q/P) \\ &[J^{\mu\nu}, Q/P] = \mp[i(-x^\mu\partial^\nu + x^\nu\partial^\mu) + (\mathcal{J}^{\mu\nu})_n^{n'}](Q/P) \end{aligned} \tag{6.21}$$

- Using explicit form of the generators we can show that,with some algebra,these generators indeed furnish the quantum Poincare algebra.[25]It should be note that the generator of boost is not conserved,this is due to it doesn't leave the Lagrangian invariant.

- Using explicit expansion of fields,these operators can be put into **2nd quantization form correctly**,further we can show the single particle states furnish representations of these operators,the states have **right eigenvalues**

Remark 46
- *Almost only the current of internal symmetries will be used,for coupling with gauge bosons.*

- *In practice it's quicker to obtain the currents simply promoting global symmetry to local symmetry,or use the scheme of **gauging/minimal-coupling**,the one coupling to gauge field will be the current.*

[25]A more quicker way is to recognize the RHS as the differential operator representation,and put them back to show Poincare algebra

6.3 Canonical structure of simple free fields

6.3.1 Scalar field

Canonical structure of free scalar field operators

We've obtained the field operator using axioms without any reference to classical Lagrangian.The result is:

$$\begin{aligned}\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2\omega_p)}}[a(\vec{p})e^{ip\cdot x} + a_c^\dagger(\vec{p})e^{-ip\cdot x}] \\ [\phi^+(x), \phi^-(y)]_- =&\Delta_+(x-y) = \int \tilde{dk}e^{ik\cdot(x-y)}, [\phi(x), \phi^\dagger(y)]_- = \Delta(x-y) = \Delta_+(x-y) - \Delta_+(y-x) \\ \Delta(\vec{x},0) = 0,&\dot{\Delta}(\vec{x},0) = (\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}[-ie^{ik\cdot x} - e^{-ik\cdot x}])_{t=0} = -i\delta^3(x)\end{aligned} \tag{6.22}$$

Thus we have the canonical structure:

$$q(x) = \phi(x), p(x) = \dot{\phi}^\dagger(x)$$

This is for free field,when interaction is included $p \neq P$ in general.

Since complex scalar field can be decomposed into two scalar field,we consider the general interacting Lagrangian of scalar fields:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 - \mathcal{H}(\Phi) - J^\mu\partial_\mu\Phi$$

- Here the field operators are in H-pitrure,thus in general $\Pi \neq \pi, P \neq p$.
- The interaction density:$\mathcal{L}_{int} = -\mathcal{H}(\Phi) = -\mathcal{H}_I$ are taken to be independet of $\dot{\Phi}$,this is not the most general case.
- Here we include possible coupling to other fields through the external current.When considered as external field,J^μ is simply a c-number,when considered as operators of other fields,we should include Lagrangian density of them as well to have full Lagrangian.

The conjugate field in **H-piture** is,the J^0 appearing is crucial:

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \dot{\Phi} - J^0 \Rightarrow \dot{\Phi} = \Pi + J^0$$

The Hamiltonian is[26]:

$$H = \int d^3x[\Pi\dot{\Phi} - \mathcal{L}] = \int d^3x[\Pi(\Pi + J^0) + \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{2}(\Pi + J^0)^2 + \frac{1}{2}m^2\Phi^2 + J^0(\Pi + J^0) + \vec{J}\cdot\nabla\Phi + \mathcal{H}(\Phi)]$$

Seperate the H into free part and interaction part:

$$\begin{aligned}H =&H_0 + V \\ H_0 =& \int d^3x[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}m^2\Phi^2] \\ V =& \int d^3x[\Pi J^0 + \vec{J}\cdot\nabla\Phi + \frac{1}{2}(J^0)^2 + \mathcal{H}(\Phi)]\end{aligned}$$

Perturbation theory is applicable in **I-piture**,where the H_0 have exactly the same form of free field H_0 and is exactly solvable,this is down by replacing $\Pi \to \pi, \Phi \to \phi$[27]:

$$\begin{aligned}H_0 =& \int d^3x[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2] \\ V_I =& \int d^3x[\pi J^0 + \vec{J}\cdot\nabla\phi + \frac{1}{2}(J^0)^2 + \mathcal{H}(\phi)]\end{aligned} \tag{6.23}$$

[26]The seperation of ∂_μ into not manifestly Lorentz invariant ∂_0, ∇ is to manifest the canonical structure:which fields are truely independent D.O.F,which fields are only axuliary,this is not important in simple theories but is crucial in more complex theories.

[27]note that $P \neq p$,this is replacing not identification

Note that $\pi = \dot{\phi} = \frac{\partial \mathcal{L}_0}{\partial \dot{\phi}} \neq \Pi \neq \dot{\phi} - J^0$,thus:

$$V_I(t) \equiv \int d^3x [J^\mu \partial_\mu \phi + \frac{1}{2}[J^0]^2 + \mathcal{H}(\phi)]$$

- The extra $\frac{1}{2}[J^0]^2$ term is known as non-covariant term,this is used to cancel non-covariant term in the expression of propagators derived using canonical formalism.
- This term would not be presented if we naively consider the interacting Lagrangian as being constructed out of free field operators as it were when these fields are considered as classical.The crucial point is to identify the quantized operator in H-piture and the H-piture interacting fields have different form of free fields in I-piture.We only trun to I-piture after seperate H-piture H appropriately.

Conversely,we may show that by canonical quantizing the H_0 only,we can restore the quantum results,including:

- The explicit expression of field operators in terms of creation/annihilation operator
- The 2nd quantized form of generators,and all other operators.

Starting form the **right free classical** Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 \equiv -\frac{1}{2}\phi(-\Box + m^2)\phi = -\frac{1}{2}\phi \Delta_{KG}^{-1} \phi$$

here the field is classical,when quantized the H-piture operator $\hat{\phi}$ in free theory is identical to the I-piture operator in interacting theory.

The equation is the **Klein Gordon equation**:

$$(-\partial^2 + m^2)\phi = 0$$

This is a linear second order differential Lorentz invariant equation.The solution can be obtained by free mode in p-space:

$$\phi(x) = u(p)e^{ip\cdot x} \Rightarrow (p^2 + m^2)u(p)e^{i\cdot x} = 0 \Rightarrow p^2 = -E^2 + |p|^2 = -m^2$$
$$\Rightarrow E \equiv \omega_p = \sqrt{|p|^2 + m^2}, u(p) \equiv \frac{1}{\sqrt{2\omega_p}}$$

The general solution may be taken as:

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}}[ae^{ip\cdot x} + a^* a^{-ip\cdot x}], \pi(x) = \frac{\partial \mathcal{L}_0}{\partial \dot{\phi}} = \dot{\phi}$$

- The d^3p indicate the particle is on-shell,the choice of independent polarization u(p) is conventional.
- The normalization here is NR,LR normalization is just a replacement:

$$\frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}} \to \frac{d^3p}{(2\pi)^3(2\omega)}$$

We then canonical quantize the field,and the axiom is that:

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = [\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\delta^3(x-y), [\phi(x), \phi(y)]_t = [\pi(x), \pi(y)]_t = 0$$

This is equivalent to relations:

$$[a(\vec{p}), a^\dagger(\vec{p'})] = \delta^3(p-p'), [a^\dagger, a^\dagger] = 0, [a,a] = 0$$

- The LR normalization will add extra $(2\pi)^3(2\omega_p)$ factors

Also the H expanded into $H(a, a^\dagger)$ will have **right** 2nd quantization form:

$$H_0 = \int d^3k \omega_k \frac{1}{2}[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})] = \int d^3k \omega_k N_k + \frac{1}{2}\int d^3k \delta^3(0)$$

- We always use the normal ordering for 2nd quantized operators:

$$: H :\equiv \int d^3k\omega_k a^\dagger(\vec{k})a(\vec{k})$$

- The infinite term will only matter when gravity is involved,it stands for the quantum **zero energy** of the vacuum,this is due to **uncertainty principle**,which is the origin of the nonconservation of particle numbers.

- This zero energy have observable effect known as **Casimir force.**

It's trivial to show that quantities like P,J also have right 2nd quantization form,and that these generators do satisfy the Poincare algebra.This implies success in 'quantization'.

The quantization of free complex scalar field is trivial,the only difference is to combine two real scalars leading to two set of operators:a, a_c.

6.3.2 Dirac field

Canonical structure of Dirac field

We have:

$$\begin{aligned}\psi_l(x) = \sum_{\sigma=\pm}\int \frac{d^3p}{\sqrt{(2\pi)^3(2\omega_p)}}u_l(\vec{p},\sigma)a(\vec{p},\sigma)e^{ip\cdot x} + v_l(\vec{p},\sigma)a_c^\dagger(\vec{p},\sigma)e^{-ip\cdot x}\\ [\psi_l(x),\psi^\dagger_{l'}(y)]_+ =&[(i\gamma^\mu\partial_\mu + m)\beta]_{ll'}\Delta(x-y)\\ =&\{\int \tilde{dk}[m\beta(e^{ik(x-y)} - e^{-ik(x-y)}) - \gamma^\mu k_\mu\beta(e^{ik(x-y)} + e^{-ik(x-y)})]\}_{ll'}\end{aligned} \tag{6.24}$$

It can be shown that the equal time anticommutator is[a]:

$$[\psi_l(\vec{x},t),\psi^\dagger_{l'}(\vec{y},t)]_+ = \delta^3(x-y)\delta_{ll'}$$

thus we identify:

$$q_n(x) = \psi_n(x), p^n(x) = i\psi_n^\dagger(x)$$

This means $\psi, \psi^\dagger$ are not independent canonical variables,they are indeed conjugate.

- Thus the independent real D.O.F have $4 \times 2 = 8$,with 4 fixed by **Dirac equation**,**there's only 4 independent real D.O.F**,2 for fermion and 2 for antifermion.These 4 independent D.O.F are fixed by Klein Gordon equations seperately.

[a]Simply changing p to -p,and use $(\gamma^0)^2 = +1$ to get the last term

Consider the Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - \mathcal{H}(\bar{\Psi},\Psi)$$

This is not hermitian,but the action is hermitian:

$$(\bar{\Psi}i\gamma^\mu\partial_\mu\Psi) - (\bar{\Psi}i\gamma^\mu\partial_\mu\Psi)^\dagger = (\bar{\Psi}i\gamma^\mu\partial_\mu\Psi) + i\partial_\mu\Psi^\dagger\beta\beta(\gamma^\mu)^\dagger\beta\Psi = i\partial_\mu(\bar{\Psi}\gamma^\mu\Psi)$$

The canonical field is:

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = i\bar{\Psi}\gamma^0 \equiv i\Psi^\dagger, \bar{\Psi} = -i\Pi\gamma^0$$

The Hamiltonian is:

$$H = \int d^3x[\Pi\dot{\Psi} - \mathcal{L}] = \int d^3x(-i\Pi\gamma^0)[\vec{\gamma}\cdot(-i\nabla) + m]\Psi + \mathcal{H} \sim \int d^3x\bar{\Psi}(\vec{\gamma}\cdot(-i\nabla) + m)\Psi + \mathcal{H}$$

Seperate this as:

$$\begin{aligned}H =&H_0 + V\\ H_0 =& \int d^3x(-i\Pi\gamma^0)(\vec{\gamma}\cdot(-i\nabla) + m)\Psi\\ V =& \int d^3x\mathcal{H}\end{aligned} \tag{6.25}$$

- Perturbation theory in I-piture involve the replacement:

$$\Pi \to \pi = i\bar{\psi}\gamma^0 = i\psi^\dagger, \Psi \to \psi$$

- Similarly,inclusion of extra term:$J^\mu \partial_\mu \Psi$ will modify:

$$\Pi = i\Psi^\dagger + J^0, \bar{\Psi} = -i(\Pi - J^0)\gamma^0, V = \int d^{3x}x[\mathcal{H} + J^0\dot{\Psi} + \gamma^0\vec{\gamma}J^0\nabla\Psi + iJ^0\gamma^0 m\Psi]$$

  This is not compact,thus we won't consider this generalization.The term non-covariant term will be talked in coupling to other fields.

The quantization of classial free Dirac Lagrangian is similar:
Start with the right free Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

The Dirac equation is:

$$(i\gamma^\mu\partial_\mu - m)\psi$$

Try the plane wave solutions $u(p)e^{ip\cdot x}, v(p)e^{-ip\cdot x}$,the equations are:

$$(\not{p} + m)u(p) = 0, (\not{p} - m)v(p) = 0$$

This shows they are eigenvectors of $\not{p}$ with eigenvalue $\mp m$,thus the spin-sum must be propotional to the projection matrices:

$$\frac{1}{2m}(\mp\not{p} + m)$$

the normalization gives:

$$\sum u\bar{u} \equiv -\not{p} + m, \sum v\bar{v} = -\not{p} - m$$

Thus we have the expansion:

$$\psi_l(x) = \sum_{\sigma=\pm}\int \frac{d^3p}{\sqrt{(2\pi)^3(2\omega_p)}} u_l(\vec{p},\sigma)a(\vec{p},\sigma)e^{ip\cdot x} + v_l(\vec{p},\sigma)a_c^\dagger(\vec{p},\sigma)e^{-ip\cdot x}$$

The conjugate field is $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger$
similarly the axiom:

$$[\psi(x)_\alpha, \pi(y)_\beta]_{t,+} = i\delta^3(x-y)\delta_{\alpha\beta}, ...$$

is equivalent to the 2nd quantization algebra,with anticommutator.

- The 2nd quantized Dirac Hamiltonian,by expanding the fields into $a, a^\dagger$s,and use the 2nd quantization algebra and Fourier transform,we obtain[28]:

$$H_0 = \sum_\sigma \int d^3k\omega_k(a^\dagger a - a_c a_c^\dagger) = \sum_\sigma \int d^3p\omega_p[N_n + N_{n^c}] - \delta^3(0)$$

- For other operators,for example P, J we can show they are also quantized correctly,also note that J will include spin.Further,these statisfies the right Lie algebra.Using the 2nd quantization alegbra we can shown that**single particle state do furnish representations of these operators**

[28]The minus sign comes from odd number of derivative in H

6.3.3 Massive vector field

Canonical structure of massive vector field

We have:

$$\begin{aligned} v^\mu(x) =& \sum_{\sigma=0,\pm1}\int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}}[e^\mu(\vec{p},\sigma)a(\vec{p},\sigma)e^{ip\cdot x} + e^{\mu *}(\vec{p},\sigma)a_c^\dagger(\vec{p},\sigma)e^{-ip\cdot x}] \\ [v^\mu(x), v^{\nu\dagger}(y)]_- =&[\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\Delta(x-y)] \\ =& \int \tilde{dk}(\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2})[e^{ik(x-y)} - e^{-ik(x-y)}] \end{aligned} \tag{6.26}$$

It can be shown that the canonical variables are:

$$q_i(x) = v^i(x), p^i = \dot{v}^{i\dagger}(x) + \partial_i v^{0\dagger}(x)$$

For **real** vector field:

$$v^\mu = v^{\mu\dagger}$$

To show this:

$$\begin{aligned} [v^i(\vec{x},t), \dot{v}^{j\dagger}(\vec{y},t)] =& i\int \frac{d^3k}{(2\pi)^3}[\delta^{ij} + \frac{k^ik^j}{m^2}]e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \\ [v^i(\vec{x},t), \partial_i v^{0\dagger}(\vec{y},t)] =& -i\int \frac{d^3k}{(2\pi)^3}[\frac{k^ik^j}{m^2}]e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \\ \Rightarrow [v^i(x), \dot{v}^{j\dagger} + \partial^j v^{0\dagger}(y)]_{t,-} =& i\delta^3(x-y)\delta^{ij} \end{aligned}$$

This shows that v^0 is only axuliary,its time-derivative won't appear in the Lagrangian density. we can show that,using field equations[a]:

$$v^0 = \frac{1}{m^2}\nabla\cdot\vec{p} = \frac{1}{m^2}[\partial_i\dot{v}^{i\dagger} + \partial_i\partial^i v^{0\dagger}]$$

[a] $(-\Box + m^2)v^\mu = 0, \partial_\mu v^\mu = 0$

The general possible Lagrangian for **real** massive vector field can be determined like the following:

$$\mathcal{L} = -\frac{1}{2}\alpha\partial_\mu V_\nu\partial^\mu V^\nu - \frac{1}{2}\beta\partial_\mu V^\nu\partial^\nu V^\mu - \frac{1}{2}V^\mu V_\mu + J^\mu V^\mu$$

- To make sure that the V^μ do transform as a vector instead of being a direct sum of 4 scalar fields,we must have $\beta \neq 0$

The equation reads:

$$-\alpha\Box V_\nu - \beta\partial_\nu(\partial_\mu V^\mu) + m^2 V_\nu = J_\nu$$

taking the divergence gives:

$$-(\alpha+\beta)\Box(\partial_\alpha V^\alpha) + m^2(\partial_\alpha V^\alpha) = \partial_\alpha J^\alpha$$

This is exactly the equation of a **scalar** field with mass $\frac{m^2}{\alpha+\beta}$ and source $\frac{\partial_\mu J^\mu}{\alpha+\beta}$ To completely **project out** the scalar D.O.F,we must have:

$$\alpha = -\beta = 1, \partial_\mu J^\mu = 0$$

This gives the wanted equations:

$$(-\Box + m^2)v^\mu = 0, \partial_\mu v^\mu = 0$$

- This gives the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}V_\mu V^\mu + J_\mu V^\mu, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

with field equation:

$$\partial_\mu F^{\nu\mu} = J^\nu - m^2 V^\nu$$

- The fact that external current must be conserved is to project out scalar D.O.F,this also renders the high energy physics of massive vector fields well-behaved,with no infinite amplitude due to $m \to 0$.

The conjugate field is[29]:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{V}_\mu} = -F^{0\mu} = F^{\mu 0}, \Pi^0 \equiv 0, \Pi^i = F^{i0} = \dot{V}^i + \partial_i V^0 \equiv -E^i, \dot{V}^i = \Pi^i - \partial_i V^0$$

Only the spatial part is canonical,as expected,they are just the **electric field** up to sign.
The field equation for $v = 0$ is,notice the crucial J^0:

$$\partial_i F^{i0} = m^2 V^0 - J^0 \Rightarrow V^0 \equiv \frac{1}{m^2}(\nabla \cdot \Pi + J^0)$$

The Hamiltonian is then,using $\dot{V}^i = -\partial^i V^0 + \Pi^i = \Pi^i - \frac{1}{m^2}\partial_i(\nabla \cdot \Pi + J^0)$:[30]

$$\begin{aligned}
H &= \int d^3x[\Pi^i \dot{V}_i - \mathcal{L}] \\
&= \int d^3x[\vec{\Pi}^2 + m^{-2}(\nabla \cdot \vec{\Pi})(\nabla \cdot \vec{\Pi} + J^0) - \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla \times V)^2 + \frac{1}{2}m^2\vec{V}^2 \\
&\quad - \frac{1}{2}m^{-2}(\nabla \cdot \vec{\Pi} + J^0)^2 - \vec{J} \cdot \vec{V} + m^{-2}J^0(\nabla \cdot \vec{\Pi} + J^0)]
\end{aligned}$$

We may seperate as:

$$\begin{aligned}
H_0 &= \int d^3x[\frac{1}{2}\Pi^2 + \frac{1}{2m^2}(\nabla \cdot \vec{\Pi})^2 + \frac{1}{2}(\nabla \times \vec{V})^2 + \frac{m^2}{2}\vec{V}^2] \\
V &= \int d^3x[-\vec{J} \cdot \vec{V} + \frac{1}{m^2}J^0(\nabla \cdot \vec{\Pi}) + \frac{1}{2m^2}(J^0)^2]
\end{aligned} \tag{6.27}$$

The perturbation theory in I-piture is obtained by replacement:

$$\Pi \to \pi = \dot{\vec{v}} + \nabla v^0, V \to v, V$$

- The free fields obtained using Hamilton equation are:

$$\dot{v} = \frac{\delta H_0}{\delta \pi} = \pi - m^{-2}\nabla(\nabla \cdot \pi), \dot{\pi} = -\frac{\delta H_0}{\delta v} = +\nabla^2 v - \nabla(\nabla \cdot v) - m^2 v$$

  this is not manefestly Lorentz invariant,but putting together we restore the two Lorentz invariant field,using

$$v^0 \equiv m^{-2}\nabla \cdot \pi \Rightarrow \pi = \dot{v} + \nabla v^0$$

  The detail is rather tedious but straight.

- The quantization of classical free massive vector field is also straight forward,starting form the classical Lagrangian and equation,we obtain the expansion of v^μ in terms of polarization:$e^\mu, e^{\mu *}$ satisfying:

$$p_\mu e^\mu(\vec{p}, \sigma) = 0$$

  this means the spin-sum is propotional to the projection into space orthogonal to the 4-momentum:

$$\sum e^\mu e^{\nu *} = \eta^{\mu\nu} + \frac{p^\mu p^\mu}{m^2}$$

  Agagin,the axiom of canonical relations between $v_i, \pi^i = F^{i0} = \dot{v}^i + \partial_i v^0 = -e^i$ gives the 2nd quantization algebra of $a, a^\dagger$,we then express the H,J,P obtained from Neother's theorem in 2nd quantization form,and show that the single particle states are the eigenstates.

- Having turned to I-piture,the interaction is:

$$V_I(t) = \int d^3x[-J_\mu v^\mu + \frac{1}{2m^2}[J^0]^2] \tag{6.28}$$

  There is also a non-covariant term arising from interactions.This is used to cancel the non-covariant term in propagator derived using canonical formalism.

[29]Note the distinguish between upper index and lower index
[30]This form break the gauge invariance,thus the non-covariant term can be presented

6.3.4 Massless vector field:Coulomb gauge,Covariant gauge

Gauge transformation and gauge invariance

The canonical structure of massless vector field relies on the choice of gauge:the origin of gauge invariance in field theory relies on the fact that we can't create (A, A) field describing massless particles of helicity $\pm 2A$ using their creation/annihilate operators,there will be redundant gauge transformation.

- This is the **view of 'particles'**,where the exsistence of particle/superselection-sectors with zero-mass and spin-1/2/... comes first.And the gauge invariance emerge as a requirement of build effective field theories,this view is also adopted in String theory.
- There' another view,starting from gauge invariance and **fields**,and deduce the exsistence of (A, A) field describing massless particles.This is the process of gauging a symmetry,which is used to describe fundamental forces,accompanied with techniques like spontaneous symmetry breaking.

For massless vector field,the gauge transformation comes from the gauge translation in the little group,which not only mix physical helicity polarizations,but also induce unphysical longitude modes:

$$U_0(\Lambda)a_\mu(x)U_0(\Lambda)^{-1} = \Lambda_\mu{}^\nu a_\nu(x) + \partial_\mu\Omega(x)$$
$$e_\mu(\vec{p}, \sigma = \pm 1) \to \Lambda_\mu{}^\nu e_\nu(\vec{p}, \sigma = \pm 1) + \alpha e_\mu(\vec{p}, \sigma = 0), e_\mu(\vec{p}, \sigma = 0) = \epsilon^L_\mu \propto p_\mu$$

We can create another field $(1, 0) \oplus (0, 1)$ with no problem of gauge transformation:$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$,but natural of fundamental interactions adopt a_μ as fundamental.Thus gauge invariance becomes a **nontrivial constraint on field theories**,gauge invariance must be a quantum level symmetry,at classical level:

$$\delta I_M = \int d^4x \frac{\delta I_M}{\delta A_\mu}\partial_\mu\epsilon(x)$$

- This is the infinitesimal gauge transformation $\delta A_\mu = \partial_\mu\epsilon(x)$,in view of gauge theory,this is the local transformation promoted form U(1)-global transformations.Here I_M stands for matter fields.
- Gauge invariance implies:
$$\partial_\mu \frac{\delta I_M}{\delta A_\mu} \equiv 0$$
  - If I_M involve only $F_{\mu\nu}$,which is manifestly invariant under gauge transformation,then this is trivial,since $F_{\mu\nu}$ is antisymmetric[31]:
  $$\frac{\delta I_M}{\delta A_\mu} = 2\partial_\nu \frac{\delta I_M}{\delta F_{\mu\nu}}$$
  - For I_M involve A_μ,this will be a nontrivial constraint:the matter fields must tranform according to the U(1)-local transformation:
  $$\delta\Psi_l = i\epsilon(x)q_l\Psi_l(x), q_l = Q_l e, (Q)_l{}^{l'} = q_l\delta_{ll'} \tag{6.29}$$
  this local transformation implies exsistence of global transformation,and if the theory has this global-U(1) invariance,then there will be the wanted conserved current:
  $$\delta I_M = -\int d^4x J^\mu\partial_\mu\epsilon(x), \partial_\mu J^\mu = 0$$
  we then have the identification:
  $$J^\mu = \frac{\delta I_M}{\delta A_\mu} \equiv -i\sum_l \frac{\partial\mathcal{L}_M}{\partial\partial_\mu\Psi_l} q_l\Psi_l = \sum_n q_n J_n$$
    * Since we've proved the charge is the same for all field components of single particles.The current is a sum over current of particles.
    * The conserved charge is the generator of this **global** U(1)-transformations,and the fields carry eigenvalue of this operator:U(1)-charges:$[Q, \Psi_l] = -q_l\Psi_l, Q = \int d^3x J^0$

31 $\frac{\delta F_{\alpha\beta}(y)}{\delta A_\mu(x)} = \partial_\alpha(\delta_{\beta\mu}\delta^3(x-y)) - (\alpha \leftrightarrow \beta)$,RHS is obtained upon integral by parts

Thus gauge invariance gives nontrivial constraint:**the matter action is invariant under local transformations of both matter fields and gauge fields**:

$$\begin{aligned} A_\mu &\to e^{i\epsilon(x)}(A_\mu + i\partial_\mu)e^{-i\epsilon(x)}, \delta A_\mu(x) = \partial_\mu \epsilon(x) \\ \Psi_l &\to e^{i\epsilon(x)q_l}\Psi, \delta\Psi_l(x) = i\epsilon(x)q_l\Psi_l(x) \end{aligned} \tag{6.30}$$

Inversely,we can introduce the A_μ field by starting with gauge invariance,rather than take it as a consequnce of massless particles,this is the strategy adopted by **gauge theories**:

- We start with global symmtery G,and promote this to local symmetry,the need for the action to be invariant will force the introduction of gauge fields.
- For U(1) symmetry:
$$\delta\Psi_l = i\epsilon q_l \Psi_l \to \delta\Psi_l = i\epsilon(x) q_l \Psi_l$$
  This first gives the current J^μ from **global** symmetry,further to make sure the action is invariant,we have to make the derivative of fields also gauge invariant:
$$\delta\partial_\mu\Psi_l(x) = i\epsilon(x)q_l\partial_\mu\Psi_l + iq\Psi_l\partial_\mu\epsilon$$
  Define the **covariant derivative**:
$$D_\mu\Psi_l = \partial_\mu\Psi_l - iq_l A_\mu \Psi_l, \delta D_\mu \Psi_l \equiv i\epsilon(x)q_l D_\mu \Psi_l$$
  The replacement $\partial_\mu \to D_\mu$ makes the action invariant under global symmetries also invariant under local symmetries.This introduce the coupling with gauge field:
$$\mathcal{L}_{int} = +A_\mu J^\mu$$
  This can be seen form:
$$\frac{\delta I_M}{\delta A_\mu} = J^\mu_{Matter} = \sum_l \frac{\partial \mathcal{L}_M}{\partial D_\mu \Psi_l}(-iq_l\Psi_l) = -i\sum_l \frac{\partial \mathcal{L}_M}{\partial\partial_\mu\Psi_l} q_l \Psi_l \equiv J^\mu_{Global}$$
- The masslessness of these fields are from gauge invariance:mass term will break gauge invariance and is forbidden.At quantum level:
  - This will forbid counterterms of massterms in renormalization
  - Force relations between couterterms
  - Surpressed Logarithm divergence in loops amplitudes.
  - Ward identities.
- The masslessness itself will have implications:
  - Using **soft particles**,we can show the conservation of U(1) charges directly form Lorentz invariance.The conservation of U(1) charge is expressed as the global U(1) symmetry in field theory,but this is not essential.**The essence is the exsistence of massless spin-1 particles.**Similarly there's a theorem for gravitons.And can be used to prove that there's no interactions mediated by higher spins.

Summary of (abelian) gauge invariance

- The introduction of vector field of massless particle of spin-1 which have extra gauge transformation,will force the theory to be invariant under gauge invairance,this force the matter fields to transform under local U(1) transformations,implying the matter field have global U(1) symmetry,and the gauge field coupling to the Noether current from global U(1)-invariance of matter fields.
- Conversely,starting with U(1) global invariant matter fields,promoting the global invariance to local invariance will force the replacement of derivatives to covariant derivatives,this procedure is known

as **minimal coupling**,this will lead to the coupling term between gauge field and the Noether current.

- This above procedure can be generalized to **any internal global symmetry G**,at least for compact groups:
  - Matter fields furnish representations of G,this induce Noether currents for each generator.These generator generate the global G group and the fields or single particles states they create will carry eigenvalues of these generator operators,known as global/gauge-G-charge.
  - Promoting the aciton to be invariant under local G transformations will replace the derivatives to **covariant derivatives**,which have **same transformation rule as the fields**
  - This will introduce the gauge fields,and that how gauge fields themselves transform under local G transformations.The conclusion is that the gauge fields furnish **adjiont representations** of the globa G group.
  - This procedure introduce the coupling between gauge fields and Noether currents.
- The nonabelian gauge theories are more complex,one feature is that the global G is nonabelian thus the gauge field also carry global/gauge charge,there will be self-interaction from the free Lagrangian,without coupling to matters.The quantization of nonabelian gauge theories can hardly be done in canonical formalism,it's usually done with path-integral and will involve ghost fields.And after quantization there will be redundant gauge invariance,known as BRST invariance.
- **The gauge invariance are crucial in proving renormalizability of gauge theories.**

The Lagrangian of gauge fields is[32]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

- This is the massless limit of **classical real** massive vector fields.But the massless limit of quantized massive vector field is not well-defined,which need gauge invariance.The problem lies in the possible excitation of ghost longitude mode,which makes the amplitude infinite.
- This is the classical Lagrangian for electromagenetic fields,up to constant this is unique with the requirements:the functional is **quadratic** in $F_{\mu\nu}$,this gives linear equations.

The equation for this is:

$$0 = \frac{\delta}{\delta A_\nu}[I_\gamma + I_M] = \partial_\mu F^{\mu\nu} + J^\nu$$

and the Bianchi relation:

$$\partial_\alpha \epsilon^{\alpha\beta\mu\nu}F_{\mu\nu} = \partial_\alpha * F^{\alpha\beta} = 0 = \partial_{[\alpha}F_{\mu\nu]}$$

Remark 47
- *These can all been put into geometric language,with gauge fields being connections on fiber bundles and the field strength being curvature.The nonlocal properties of gauge field like Wilson loop giving the flux is known as holonomy.*
- *The formulation of these results can all been put into* ***differential forms****,which is more geometric and more compact.*

Examples:interaction between scalar,spinor,vector

- The Dirac field can couple to scalr field as:

  $$\phi\bar{\psi}\psi$$

  This is known as Yukawa coupling,in standard model this is used to generate the mass of fermions.
- The Dirac Lagrangian have U(1) global symmetry,by promoting to local U(1) and introduce the covariant derivative,the coupling is:[33]

$$\begin{aligned}\mathcal{L} =& \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi + q\bar{\psi}\gamma^\mu\psi A_\mu, J^\mu_e = qJ^\mu_V, q = Qe, Q \in \mathbb{Z} \\ J^\mu_V =& \bar{\psi}\gamma^\mu\psi\end{aligned} \tag{6.31}$$

[32] This is taken as real

[33] The definition of the current may have some sign and i difference with the one obtained using variatinal principle,but this is of no importance

- For massless case,there's another **classical** U(1) symmetry known as **chiral symmetry**:

$$\begin{aligned} \psi \to e^{i\theta\gamma^5}\psi, \bar{\psi} \to \bar{\psi}e^{i\theta\gamma^5}, e^{i\theta\gamma^5}\gamma^\mu e^{i\theta\gamma^5} = \gamma^\mu \\ \psi = \begin{pmatrix} \chi \\ \bar{\xi} \end{pmatrix}, \chi \to e^{-i\theta}\chi, \bar{\xi} \to e^{+i\theta}\bar{\xi} \\ \mathcal{L}_{L/R-Weyl} \to \mathcal{L}_{L/R-Weyl} \end{aligned} \tag{6.32}$$

  The corresponding current is the **axial** current:

$$J_A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi = \bar{\xi}^\dagger\bar{\sigma}^\mu\chi - \chi^\dagger\sigma^\mu\bar{\xi} \tag{6.33}$$

  - The vector current implies the conservation of $\#n - \#n^c$:

$$J_V^0 = \psi^\dagger\psi \sim a^\dagger a - a_c^\dagger a_c$$

    since antiparticle carry oppesite charge,this implies the conservation of U(1) global/gauge charges.
  - The axial current implies

$$J_A^0 = \psi^\dagger\gamma^5\psi = \bar{\xi}^\dagger\chi - \chi^\dagger\bar{\xi} \tag{6.34}$$

  - Since γ^5 anticommute with γ^0,we have under parity:J_V is transform as a vector,while J_V transform as a pesudovector.
  - The theory with parity conservation breaking can be created out of chiral components which involve the insertion of $P_{L/R} = \frac{1\pm\gamma^5}{2}$ in the current this will lead to axial currents in the Lagrangian naturally.
  - Chiral/Axial anomaly:the chiral symmetry and gauge invariance can't be preserved simultaneously at quantum level,the anomalous divergence of the current is:

$$\partial_\mu J^\mu_{A,renormalized} \propto *F_{\mu\nu}F^{\mu\nu}$$

    It can be shown that L/R Weyl spinor contribute to this anomaly with oppesite sign,thus the **anomaly cancellation implies the number of L/R Weyl spinors must be equal,as in Dirac spinor with a mass**.

- The complex scalar Lagrangian have U(1) symmetry,with current:

$$\begin{aligned} \mathcal{L} =& - D_\mu\phi D^\mu\phi^* - m^2\phi\phi^* = -|\partial\phi|^2 - m^2|\phi|^2 + q^2A_\mu A^\mu\phi\phi^* + iqA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \\ J^\mu =& i(\phi\partial^\mu\phi^*) + h.c. = \phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1 \end{aligned} \tag{6.35}$$

The need of gauge fixing

The **real** vector field have totally 4 independent components:

- For massive case,one is project out by $\partial_\mu A^\mu$,and the rest three is fixed by Klein-Gordon equation.

- For massless case,however,we must project out one more degrees of freedom,since massless particle have only **two** polartization to be fixed by Klein-Gordon equation.

- Since the gauge fields are determined up to gauge transformations,the equations besides Klein-Gordon equations are known as **gauge fixing constraints**,thus we are quantizing **constraint classical systems.**

- Different gauge constraints have different independent solutions.For Lorentz gauge:$\partial_\mu A^\mu = 0$ and other covariant gauges,we have 3-indepent polarizations,this actually doesn't rule out all unphysical D.O.F,thus quantization in these gauges will have ghost modes,usually the longitude mode.

- The Coulomb gauge:$A^0 = 0, \nabla \cdot A = 0$ have two constraints and project out all unphysical D.O.F.

- There are other gauges,which have different conditions on A as well as different conditions on e^μ,thus different solutions.They are all related by gauge transformations,both in x-space and p-space.

The systematic method of canonical quantizing constraint classical systems use **Dirac brakets**.In principle,the massive classical vector field are also constraint systems,the quantization can also been down using Dirac brakets,or more **simply by treating A_0 as axuliary.**[34]

However,the quantization of massless vector field can't be done using Dirac brakets,to see this,note that the constraints are:

$$\begin{aligned}\Pi_0 = 0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} \\ \partial_i \Pi^i = -\partial_i \frac{\partial L}{\partial F_{i0}} &= \partial_i F^{i0} = -\partial_i F^{0i} = -\partial_i E^i = -\frac{\partial \mathcal{L}}{\partial A_0} = -J^0\end{aligned} \tag{6.36}$$

- The first is known from the structure of Lagrangian and is known as **primary constraint**
- The 2nd is from field equations for the quantity fixed by primary constraints:the 0th Hamiltonian equation with primary constraint implemented.This is known as **secondary constraint**
- These two constraints are all **first class**
- These constraint makes the naive quantization $[A_\mu, \Pi^\nu]_t = i\delta^3(x-y)\delta^\nu_\mu$ fails:this assumption is inconsistent with the operator constraints.[35]
- The Dirac braket method fails due to the fact the constraints:$\Pi_0 = 0, \partial_i \Pi_i + J^0 = 0$ has **vanishing Poisson brakets**

The difference between massive case and massless case is that:for massive case,we can solve $A^0 = [\partial_i \Pi^i + J^0]m^{-2}$ uniquely for all time,but this is not ture for massless case,since A^μ **is determined up to gauge invariance,not uniquely from any equation.**[36]

To apply the canonical quantization of massless vector field,we need first **choose a gauge**.There are several useful gauge:

- Lorentz gauge $\partial_\mu A^\mu = 0$,Coulomb gauge $A^0, \nabla \cdot A = 0$,Temporal gauge $A^0 = 0$,Axial gauge $A^3 = 0$,Unitarity gauge:$\Phi real$,R_ξ covariant gauge.

Some of the gauges are used to simplify perturbative calculations,the unitarity gauge is used to absorb the Goldstone mode into gauge fields in SSB.The R_ξ gauge is used to have covariant propagator and the Lorentz gauge,Feynman gauge are special cases.The temporal gauge is used to discuss topological properties of gauge transformations.[37]

Canonical quantization in Coulomb gauge

Fix the gauge using **weak** Coulomb gauge,this doesn't projector out all unphysical D.O.F,but is useful to discuss non-covariant terms:$\nabla \cdot A = 0$

Thus the constraints are:

$$\Pi^0 = 0, -\partial_i \Pi^i = \nabla \cdot E = J^0$$

From the constraint we can obtain:

$$-\partial_i(\partial^i A^0 - \partial^0 A^i) = J^0 \Leftrightarrow -\nabla^2 A^0 = J^0$$

We can then make **another gauge fixing condition**:

$$A^0 \equiv \int d^3y \frac{J^0(y)}{4\pi|\vec{x} - \vec{y}|}$$

- $\frac{-1}{4\pi|\vec{x}-\vec{y}|}$ is the Green's fucntion:$[-\nabla^2]^{-1}$
- For free theory,$J^0 = 0$,this reduce to the **full** Coulomb gauge.

---

[34]For detail of Dirac braket and the corresponding quantization of massive classical vector field as constraint system,refer to [WeinbergI.7.6]

[35]For massive case,this is avoided by identifying the true canonical variables and axuliary A_0,the axuliary field is project out by field equations,which serve as a constraint:$\partial_i \Pi_i - m^2 A^0 + J^0 = 0$

[36]This fact lead to the vanishing of the determinant of field equation operator in path-integral,this lead to the same conclusion that **gauge fixing in needed in quantization**

[37]refer to [Jackiw]

- This gauge fixing condition is the common **Coulomb potential** in classical electrodynamics.
- Note that A^0 is considered as **axuliary**,and the canonical variables are A^i, Π^i,this is implicit in the constraint $\Pi^0 = 0$

The constraint is now reduced to:

$$\chi_1 = \partial_i A^i(\vec{x}) = 0, \chi_2 = \partial_i \Pi^i(\vec{x}) + J^0 = 0$$

- We can't directly assume $[A_i, \Pi^j]_t = i\delta_{ij}\delta^3(x-y)$
- After gauge fixing,these constraints have non-vanishing Possion brakets and we can use the Dirac braket[38]:

$$C_{1\vec{x},2\vec{y}} = [\chi_{1\vec{x}}, \chi_{2\vec{y}}]_P = -\nabla^2\delta^3(\vec{x}-\vec{y}), C_{i\vec{x},i\vec{y}} = [\chi_{i\vec{x}}, \chi_{i\vec{y}}]_P = 0$$

The Dirac braket method gives[39]:

$$\begin{aligned}[A^i(\vec{x}), \Pi^j(\vec{y})] = i\delta^i_j\delta^3(x-y) + i\partial_i\partial_j(\frac{1}{4\pi|\vec{x}-\vec{y}|}) \\ [A^i(\vec{x}), A^j(\vec{y})] = [\Pi_i(\vec{x}), \Pi_j(\vec{y})] = 0\end{aligned} \tag{6.37}$$

It can be shown that:

$$\Pi = \dot{A} + \nabla A^0$$

To facilitate the transition to the interaction picture,define the **solonoidal** part of Π:

$$\Pi_\perp = \Pi - \nabla A^0 = \dot{A}$$

- It can be shown that $\Pi_\perp$ have same commutation relation of Π,and also the constraint $\nabla \cdot \Pi_\perp = 0$

The general Lagrangian of interacting massless vector field and matter fields is:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J_\mu A^\mu + \mathcal{L}_{matter}$$

The Hamiltonian is given by,H_M is the pure matter Hamiltonian:

$$\begin{aligned}H =& H_M + \int d^3x[\Pi_\perp^2 + \frac{1}{2}(\nabla\times A)^2 - \frac{1}{2}(\Pi_\perp + \nabla A^0)^2 - \vec{J}\cdot\vec{A} + J^0A^0] \\ =& H_M + \int d^3x[\frac{1}{2}\Pi_\perp^2 + \frac{1}{2}(\nabla\times A)^2 - \vec{J}\cdot\vec{A} + \frac{1}{2}J^0A^0]\end{aligned} \tag{6.38}$$

Here the last term is known as Coulomb energy:

$$V_{Coulomb} = \frac{1}{2}\int d^3x J^0A^0 = \frac{1}{2}\int d^3x d^3y \frac{J^0(\vec{x})J^y(\vec{y})}{4\pi|\vec{x}-\vec{y}|}$$

This term can be used to cancel the non-covariant term in propagator.
The transition to interaction picture is similar:

$$\begin{aligned}H_0 &= \int d^3x \frac{1}{2}(\Pi_\perp^2 + (\nabla\times A)^2) + H_{M,0} \\ V &= -\int d^3x \vec{J}\cdot\vec{A} + V_{Coulomb} + V_M\end{aligned}$$

We then replace $A, \Pi_\perp, P, Q \to a, \pi, p, q$,note that a, π have the same commutation relations which is more complex than the massive case:

$$[a^i(\vec{x},t), \pi^j(\vec{y},t)] = i\delta^i_j\delta^3(x-y) + i\partial_i\partial_j(\frac{1}{4\pi|\vec{x}-\vec{y}|}), \nabla\cdot\vec{a} = 0, \nabla\cdot\vec{\pi} = 0$$

[38] The definition of Possion braket is just the generalization of classical mechanics:

$$[A,B]_P = \frac{\delta A}{\delta Q}\frac{\delta B}{\delta P} - \frac{\delta A}{\delta P}\frac{\delta B}{\delta Q}$$

,the first is $\int d^3x \nabla_x\delta^3(x-y)\nabla_y\delta^3(x-y) = -\nabla^2\delta^3(x-y)$

[39] For detail,refer to [WeinbergI,8.3]

- There still constraints on a,π,known as Coulomb gauge and Gauss constraint.The physical states are given by $\nabla \cdot \Pi_\perp |phy\rangle = 0$
- It can be shown that the **Hamilton equation** of free fields gives,using the commutation relation[40]:

$$\dot{\vec{a}} = \vec{\pi}, \Box \vec{a} = 0$$

in interacting picture,we can take $a^0 = 0$,this is just the requirement of Coulomb gauge in free theory.

The most general free real massless vector field can then be expanded in Coulomb gauge as:

$$a^\mu(x) = \sum_{\sigma=\pm 1} \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}} [a(\vec{p},\sigma)e^\mu(\vec{p},\sigma)e^{ip\cdot x} + a^\dagger(\vec{p},\sigma)e^{\mu *}(\vec{p},\sigma)e^{-ip\cdot x}] \tag{6.39}$$

- Since we have full Coulomb gauge in free theory,there's no ghost longitude polarization.
- In Coulomb gauge,the equation of the **two** independent physical polarization/helicity is:

$$\vec{p} \cdot e^\mu(\vec{p},\sigma) = 0, e^0(\vec{p},\sigma) = 0 \tag{6.40}$$

the explicit solution is given before.

- The spin-sum in Coulomb gauge is:

$$\sum_\sigma e^i e^{j*} = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}$$

- The interaction is:

$$V_I(t) = -\int d^3x j_\mu a^\mu + V_{Coulomb} + V_M$$

- Though complex,the commutation relation between a, π indeed lead to the right 2nd quantization algebras.The Hamiltonian is given by:

$$H_0 = \sum_{\sigma=\pm 1} \int d^3k \omega_p \frac{1}{2}[a, a^\dagger]_+ = \sum_\sigma \int d^3k \omega_k N_k + \frac{1}{2}\delta^3(0). \tag{6.41}$$

Similar for other generators.

The commutation relation is modified by constraints,this can be considered as projection into some subspace defined by these constraints:

$$[a^i(x), \pi^j(y)]_t = i(\delta^{ij} - \frac{\partial^i \partial^j}{\nabla^2})\delta^3(x-y) = i\int \frac{d^3p}{(2\pi)^3} e^{ip(x-y)}(\delta_{ij} - \frac{k_j k_j}{|p|^2})$$

The projection is given by the spin-sum in these constraint.For other free theories before,there's no such constraints,thus the commutation relation are already in the constrainted subspace,we don't need to act the spin-sum.

Remark 48
- *The quantization of constraint system is complex,the more efficient and more clear scheme is indeed the path-integral formalism.There's no problem due to non-covariant terms origin form coupling to currents,and the gauge fixing procedure is more clear.*
- *Still in CMP,2nd quantization rooted in canonical formalism this form are still used.*

---

[40] $i\dot{o} = [o, H_0]$,the detail of the simplification can refer to [WeinbergI,8.4]

6.3.5 Free propagators in canonical formalism

In perturbative calculation of S-matrix elements and Green's functions,we will have to evaluate free propagators.[41]

The **free** propagators can be viewed as the pairing(Wick's theorem) between $\psi_l(x)$ in $\mathcal{H}_I$ with a $\psi_m^\dagger$ in $\mathcal{H}_I(y)$:

$$\begin{aligned}
-i\Delta_{lm}(x-y) =&\theta(x^0-y^0)[\psi_l^+(x),\psi_m^{+\dagger}(y)]_\mp \pm \theta(y^0-x^0)[\psi_m^{-\dagger}(y),\psi_l^-(x)]_\mp \\
\equiv&\theta(x^0-y^0)\langle[\psi_l^+(x),\psi_m^{+\dagger}(y)]_\mp\rangle_0 \mp \theta(y^0-x^0)\langle[\psi_m^{-\dagger}(y),\psi_l^-(x)]_\mp\rangle_0 \\
\equiv&\theta(x^0-y^0)\langle\psi_l^+(x),\psi_m^{+\dagger}(y)\rangle_0 \mp \theta(y^0-x^0)\langle\psi_m^{-\dagger}(y),\psi_l^-(x)\rangle_0 \\
\equiv&\theta(x^0-y^0)\langle\psi_l(x),\psi_m^\dagger(y)\rangle_0 \mp \theta(y^0-x^0)\langle\psi_m^\dagger(y),\psi_l(x)\rangle_0 \\
\equiv&\langle T\{\psi_l(x)\psi_m^\dagger(y)\}\rangle_0 = \langle 0|T\{\psi_l(x)\psi_m^\dagger(y)\}|0\rangle
\end{aligned} \tag{6.42}$$

- This is a special case of Green's functions in **free theory**
- Perturbation will expand the exact Green's functions in interacting theory into products of free Green's functions,the final reduction is the 2-point function:free propagator.

The meaning of free propagator(Feynman propagator) is clear:

- The pairing shoule be of the form $[a,a^\dagger]_\mp$ to have nontrivial value.This include two possibility(causality) according to the x^0,y^0 thus there's time ordering.
- Since the fermionic fields anticommute,the time ordering operator is defined with a minus sign.This also renders the final result **Lorentz invariant**
- The two possibility $[a,a^\dagger],[a_c,a_c^\dagger]$ manifest the fact that antiparticles are presented for reversed causality.
- The term $[a,a^\dagger]\sim[\psi^+,\psi^{+\dagger}]$ has the meaning of annhihilate and create thus **propagate** a real(on-shell,with mass coincide with bare mass[42]).
- These two terms are the **on-shell** propagators with different causality:

$$\begin{aligned}
[\psi_l^+,\psi_m^{+\dagger}]_\mp =& \int\frac{d^3pd^3p'}{(2\pi)^3\sqrt{2\omega_p2\omega_{p'}}}\sum_{\sigma,\sigma'}u(\vec{p},\sigma)u^*(\vec{p}',\sigma')e^{ip\cdot x-ip'\cdot x'}[a(\vec{p},\sigma),a^\dagger(\vec{p},\sigma)]_\mp = \\
=&P_{lm}(-i\partial_\mu)\Delta_+(x-y), P_{lm}(-\partial_\mu)=[\sum_\sigma u(\vec{p},\sigma)u^*(\vec{p},\sigma)]_{p=-i\partial} \\
\Delta_+(x-y) =&\Delta_-(y-x)=\Delta_-^*(x-y)=\int\tilde{dk}e^{ik\cdot(x-y)}=\langle 0|\phi(x)\phi^\dagger(y)|0\rangle, \phi=(0,0)-rep
\end{aligned} \tag{6.43}$$

  the Feynman propagators combine these two causality.

The spin-sum generally is:

$$\sum_\sigma u_l u_m^* = P_{lm}(\vec{p},\omega_p), \sum_\sigma v_l v_m^* = \pm P_{lm}(-\vec{p},-\omega_p), \pm\leftrightarrow boson/fermion \tag{6.44}$$

- The mass occuring in spin-sum is **bare mass** for free theory.
- As noted,these spin-sums have meaning as certain **projection operators**.The cutting rule implies they are also the must appearing term from **spin-averaged** measurements.

[41] **Caution**:

- since we do the perturbation in I-picture,the field operator and creation/annihilation operator and the vacuum states are all those in a free theory:$\psi\sim ae^{ipx}+a^\dagger e^{-ipx}, a_0^{(\dagger)}, |0\rangle$.
- The Green's function is defined using interacting fields and vacuum in interaction theories,but the perturbation calculation relate it to free states.
- **Only** in free theory can be expand into $a_0^{(\dagger)}$.
- The S-matrix is related to Green's functions through LSZ reduction formula,which involve asymptotic free fields,which can be expanded into $a_{in/out}^{(\dagger)}$

[42] For interacting field,the mass is physical mass

- Scalar field:$P(p) = 1$
- Dirac field:$P_{lm}(p) = [(-\gamma^\mu p_\mu + m)\beta]_{lm} = [(-\not{p} + m)\beta]_{lm}$,note that we define the propagator by $\psi, \psi^\dagger$,in explicit theories involving Dirac field,we'd better define using $\psi, \bar{\psi}$,then the β is abosorbed.
- Massive vector field:$P_{\mu\nu}(p) = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}$
- Massless vector field in **Coulomb gauge**:$P_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{|p|^2}$
- The general form can be constructed using representations of $\mathcal{L}, \mathcal{R}$ and C_{AB},as in derivation of causality conditions of general free fields.

To simplify the expression,we define the **off-shell** spin-sum,this act on off-shell Feynman propagators instead of on-shell propagators:

$$P^L(p) = P^{(0)}(\vec{p}) + q^0 P^{(1)}(\vec{p}), P^L(p) =_{p^0=\omega_p} P(p)$$

Using $\partial_{x^0}\theta(x^0 - y^0) = \delta(x^0 - y^0)$,we obtain:

$$\begin{aligned}
\Delta_{lm}(x-y) =& P_{lm}(-\partial_\mu)\Delta_F(x-y) + \delta(x^0 - y^0)P^{(1)}_{lm}(-i\nabla)\Delta(x-y) \\
-i\Delta_F(x) \equiv& \theta(x^0)\Delta_+(x) + \theta(-x)\Delta_+(-x) = \theta(x^0)\Delta_+(x) + \theta(-x)\Delta_-(+x) \\
\Delta(x) =& \Delta_+(x) - \Delta_+(-x)
\end{aligned} \tag{6.45}$$

- The 2nd term is dropped due to $\Delta(\vec{x}, 0) \equiv 0$,thus finally we have:

$$\begin{aligned}
-i\Delta_{lm} =& P^{(L)}_{lm}(-i\partial_\mu)[-i\Delta_F(x-y)] = \langle 0|T\{\psi_l(x)\psi^\dagger_m(y)\}|0\rangle \\
-i\Delta_F(x-y) =& \theta(x^0 - y^0)\Delta_+(x-y) + \theta(y^0 - x^0)\Delta(y-x) = \langle 0|T\{\phi(x)\phi^\dagger(y)\}|0\rangle, \\
\phi(x) =& complex - (0,0)
\end{aligned} \tag{6.46}$$

Pole structure:causality

The on-shell propagator can be put into the form:

$$\Delta_\pm(x) = \int \frac{d^3p}{(2\pi)^3(2\omega_p)} e^{\pm ip\cdot x} \equiv \int \frac{d^4p}{(2\pi)^3} e^{ip\cdot x}\delta(p^2 + m^2)\theta(\pm p^0)$$

We then use the Fourier transformation of the step function:

$$\theta(x) = \frac{-1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-isx}}{s + i\epsilon}ds, \theta(-x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-isx}}{s - i\epsilon}ds$$

- $\theta(x)$ has pole on the lower half plane,while $\theta(-x)$ have pole on the upper half plane.

The expression of Feynman propagator is:

$$\begin{aligned}
\vec{q} =& \vec{p}, q^0 = p^0 + s, p^0 = \omega_p \\
-i\Delta_F(x) =& -\frac{1}{2\pi i}\int \frac{d^3q}{(2\pi)^3}\int dq^0 \frac{e^{-iq^0x^0 + i\vec{q}\cdot\vec{x}}}{2\omega_p}[\frac{1}{q^0 - \omega_q + i\epsilon} + \frac{1}{-q^0 - \omega_q + i\epsilon}], \\
=& \int \frac{d^4q}{(2\pi)^4}\frac{e^{iq\cdot x}}{q^2 + m^2 - i\epsilon}
\end{aligned} \tag{6.47}$$

- The q in Δ_F is **off-shell**,this can be understood as the combination of **real particle and antiparticle** result in the picture of **virtual (anti)particles**.
- The result of the combination is that the propagator is manifestly **Lorentz invariant.**
- The propagator have two poles in the complex q^0 plane:$\pm(\omega_q - i\epsilon)$ this this combines the causality of two on-shell propagators,the two on-shell propagators is just the replacement:

$$\frac{-1}{(q^0 - (\omega_q - i\epsilon))(q^0 + (\omega_q - i\epsilon))} \to \frac{-1}{[(q^0 \pm i\epsilon) - \omega_p][(q^0 \pm i\epsilon) + \omega_p]} = \frac{1}{q^2 + m^2 - isign(q^0)\epsilon}$$

These pole structure can be understood as boundary condition of the Green's functions,these propagators are all Green's functions of the Klein-Gordon equation:

$$(-\partial^2+m^2)\Delta_i(x)=\delta^4(x), i=F,\pm$$

the $i\epsilon$ difference lyies on boundary conditions and distinguish forward/backward/Feynman propagators.

The same is true for other Feynman propagators,they are Green's functions of the field equations besides Klein-Gordon equation.In general:

$$\Delta_{lm}\equiv O_{PDE}^{-1}(-i\partial)\delta^4(x)=O_{PDE}^{-1}(-i\partial)\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}=\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{O_{PDE}(p)}=\int\frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{p^2+m^2-i\epsilon}P_{lm}^{(L)}(p)$$

Cancellation of non-covariant terms

- The etended $P^{(L)}$ will in general contain noncovariant terms,this term can be cancelled by the non-covariant terms in interaction Hamiltonian,note that the propagator always comes from pairing two interaction Hamiltonian density,thus the non-covariant term is presented.
- The result is that by cancelling between the non-covariant terms,we can safely **ignore both the non-covariant term in propagator and the interaction terms.** and use the normal P_{lm} and $V_I(t)-V_{NC}$
- In this aspect,there's no need to go first to H-piture,we can directly go to I piture of $\mathcal{H}$ and their no need to strictly distinguish between Π,π,Ψ,ψ.We have $\mathcal{L}_{int}\equiv\mathcal{H}_{int}$

As an example[43],the propagator for massless vector field may be obtained by inversing the field equation operator:

$$\partial_\nu F^{\mu\nu}=0=\partial_\nu(\partial^\mu A^\nu-\partial^\nu A^\mu)=(-\partial^2\eta^{\mu\nu}+\partial^\mu\partial^\nu)A_\nu=0$$

The problem is that $O_\gamma^{\mu\nu}=-\partial^2\eta^{\mu\nu}+\partial^\mu\partial^\nu$ is not invertible,it has zero-eigenvalue with eigenvector:

$$O_\gamma^{\mu\nu}[\partial_\mu\phi(x)]\equiv 0$$

or in p-space:$(p^2\eta^{\mu\nu}-p^\mu p^\nu)p_\nu\equiv 0$.This fact is the manifest of **gauge invariance**,the A^μ is not uniquely determined by inversing the equation operator.Before inverting the operator,we have to first fixing the gauge.

- Coulomb gauge:

$$\Delta_{\mu\nu}(x-y)=\int\frac{d^4q}{(2\pi)^4}\frac{P_{\mu\nu}(q)}{q^2-i\epsilon}e^{iq\cdot(x-y)}, P_{i0}=P_{0i}=P_{00}=0, P_{ij}=\delta_{ij}-\frac{q_iq_j}{|q|^2}$$

  Use the non-covariant term in $V_I=V_{Coulomb}$,we can show that after cancellation,the effective photon propagator is:

$$\Delta_{\mu\nu}\equiv\int\frac{d^4p}{(2\pi)^4}\frac{\eta_{\mu\nu}}{p^2-i\epsilon}e^{ip\cdot(x-y)}$$

  This is also the **Feynman gauge** propagator

Since gauge invariance lead to Ward identity:$p_\mu\mathcal{M}^\mu(p)=0$,we can effectively add terms propto to p_μ freely:

$$P_{\mu\nu}(p)=\eta_{\mu\nu}\to\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}$$

This is known as **R_ξ-covariant gauge**.There is a systematic way of getting covariant gauges like this known as Fadeev-Popov method.The Lagrangian after gauge fixing will be:

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Special cases include:

- $\xi=1$,Feynman gauge;$\xi=0$ Landau gauge or Lorentz gauge.$\xi=\infty$,unitary gauge

[43]For whole detail,refer to [WeinbergI,8.6,6.2]

Perturbation theory of classical field theories

The classical field theory is the classical limit of the quantum theory:$\bar{h} \rightarrow$,the meaning of this can be clarified using **path integral**.

For the interaction theory $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$,the equation of motion with external current is then:

$$O_{PDE}[\psi] = \frac{\partial \mathcal{L}_{int}}{\partial \psi} + J$$

We then solve this perturbatively[44]:

$$\psi = O_{PDE}^{-1}[J + f(\Psi)], \psi^0 = \frac{J}{O_{PDE}}$$

To next order:

$$O_{PDE}[\psi^0 + \psi^1] = J + f(\psi^0 + \psi^1) \Rightarrow O_{PDE}(\psi^1) = f(\frac{J}{O_{PDE}}, \psi^1)$$

in general the f is just a polynomial and this process can be done order by order.This perturbation solution can be drawn as Feynman digrams with only **trees** end up with sources and the vertices indicate the order and stands for interaction/perturbation:

- The difference between classical physics and quantum physics is that perturbatively,quantum particles can create **virtual** propagators to form **loops**.
- Dimensional analysis show that loops are of order $O(\hbar^{n\geq 0})$ while trees are of order $O(\hbar^{-1})$,thus upon classical limit,only the tree level dominate.
- The difference between trees and loops are manifested by **Schwinger-Dyson** equations,where there're extra **contact terms** due to noncommutivity between operators.
- The loops can give quantum correction to the effective $\mathcal{L}_{int}$ known as **quantum effective action.**Calculating the tree level of effective action include calculation of loops in original effect.
  - This quantum effective action is what needed to model experiments,since **experiments talk nothing about loops,**trees are what happens in experiments.
  - This also gives the meaning of renormalization,for interacting theory,the physical particles are defined to contain all quantum effect including self-interaction,this means replacing loops with trees in certain scale.Only when we go to higher scale,can we find the 'internal' structure of the physical particles.

Remark 49 *Not all fields have classical couterpart:*

- *the Dirac field is essentially quantum,with dimension analysis showing the Lagrangian is of order $O(\hbar)$,this means there's actually **no classical matter fields**.*
- *The electromagenetic field and gravitation field do have classical limit.*

6.4 Path integral formalism

6.4.1 Motivation

The canonical formalism have several features:

- Unitarity:The Hilbert space is expanded by multiparticle states generated by 2nd quantization algebra,free fields are obtained form creation/annihilation operators.
- Poincare invariance:The generators of Poincare group as hermitian operators satisfy the quantum Poincare algebra.They are obtained using Nother's theorem as functionals of the interacting fields in H-piture.
- Dynamics:the field operators satisfy Lorentz invariant operator equations.

This is not the unique formalism of quantum theories.There's an equivalent but more compact and useful formalism known as path integral formalism.The path integral formalism have the following features:

[44]This is similar to OFPT,but this is covariant since the field operator is invariant

- Unitarity [45] :the Hilbert space is expanded by complete,orthonormal states describing **field configurations**,the field configuration is just the infinite dimension generalizaiton of general coordinate,with several component(D.O.F) on **each** spacetime point.
  - In canonical formalism,these states are **eigenstates** of the field operators.In path integral formalism,the field operators are defined by these states conversely.
  - These states contain equivalent information:the generate same Hilbert space but with different basis.
- Dynamics:the transition amplitude between these states are given by the summing over all paths with phase determined by **classical action**.
- Poincare invariance:the Lagrangian and the measure will be invariant under Poincare transformations.

These two formalism are equivalent:

- We can construct the basis in path integral formalism using the canonical field operators.The amplitude between these states can be derived from the time evolution in H-piture.
- The canonical formalism can also been restored form the path integral formalism:the field operators are defined by matrice elements between these states,the field equation and Poincare group generators(Noether's theorem) can be obtained form the **Schwinger action principle**.
- **Green's functions** in interacting/free theories are identical in both formalism.Both formalism give rise to the same perturbative calculation scheme(Feynman rules) for Green's functions,thus the observable:S-matrice elements are the same in both formalism.

There are some difference and similarity:

- The canonical formalism construct the states directly and manifest unitarity.But Poincare invariance is not manifested as we construct Hilbert space on time-slices of spacetime manifold.[46],it's not obvious that the amplitude will be Poincare invariant.
- The path integral formalism manifest Poincare invariance using the action and measures,the amplitudes are manifestly Poincare invariant.The unitarity is not manifested since we rarely refer to Hilbert states and field operators.
- The canonical formalism is more related to the multiparticle states in Hilbert space thus manifest the particle aspects of quantum states
- The path integral formalism is more related to the wave aspect of quantum states.
- Both formalism take fields as the fundamental degree of freedom.Since this is a infinite dimensional generalization of canonical system,this will give rise to **infinity in bare physical observables.**To regularize these infinity in formalism,in principle we should **lattice regularize** the spacetime such that the fields have finite D.O.F.
- Specificly the infinity of path integral formalism is hidden in the measure.

The advantages of the path integral formalism:

- The canonical formalism not manifest SR,thus the formalism of interaction theroies will contain non-covariant terms and the propagator will also include non-covariant terms.We need a roundabout procedure to obtain the covariant interaction Hamiltonian and covariant propagators.The path integral however gives directly to covariant propagators.

[45]Unitarity:complete,orthonomal **Basis**:

- Free multiparticle states defined 2nd quantization algebra,including free states and in/out states.
- More generally the full spectrum of physical particles in the Hamiltonian/system
- Eigenstates of generators of symmetry groups like Poincare group,especially the energy eigenstates.
- Eigenstates of field operators.

[46]The equivalence between Hilbert space at different time is indicated by Poincare invariance

- The canonical quantization of constraint systems are very complex,this becomes almost impossible for nonabelian gauge theories and general relativity.The gauge fixing and quantization of nonabelian gauge theories are easily down in path integral formalism.
- The discussion of **spontaneous symmetry breaking and anomaly** are more clear in path integral formalism.
- The **non perturbative aspects** are more easy to discuss using path integrals.

The canonical formalism is still important:

- It implies the path integral formalism is indeed unitary.
- The constraints of constaint system are manifest by canonical formalism.

6.4.2 From canonical formalism to path integral formalism

Bosonic operator

The quantum system is described by canonical conjugate variables[47] denoted as Q_a, P_b.We start with **S-piture**[48]:

- For field operators,$Q_a = Q_m(\vec{x}), P_b = P_n(\vec{x})$.This is an infinite dimensional generalization,in principle we should lattice regularize the spacetime such that the D.O.F is finite:

$$Q_a = Q_{m,\vec{x}_i}, P_b = P_{n,\vec{x}_j}$$

- Assuming that we have elinimate and solved the constraints[49] such that these operators satisfy[50]:

$$[Q_a, P_b] = i\delta_{ab}, [Q_a, Q_b] = [P_a, P_b] = 0$$

The crucial point is that for Q/P **commute** with each other,thus we can find **simultaneous eignestates**:[51]

$$Q_a|q\rangle = q_a|q\rangle$$

These eigenstates form a **complete,orthonormal basis**,similar for P:

$$\langle q'|q\rangle = \prod_a \delta(q_{a'} - q_a) = \delta(q' - q), \int \prod_a dq_a|q\rangle\langle q| = 1$$

From the canonical relation,we have[52][53]:

$$\langle q|p\rangle = \prod_a \frac{1}{\sqrt{2\pi}} e^{iq_a p_a}$$

We then trun to the **H-picture**:

$$|q;t\rangle = e^{iHt}|q\rangle, |p;t\rangle = e^{iHt}|p\rangle$$

- Since we are deriving path integral from canonical formalism,the basis is still defined on time slices,axiomatically,this is not necessary,we can define the states on any **hypersurfaces**.The advantage of this choice is that we can seperate time and space.
- The basis transform conversely as the states.

[47] the axuliary fields are expressed as functional of P,Q.They can be included using the method of axuliary fields generally

[48] In path integral formalism,we use only S/H piture,the I piture is used in canonical formalism in perturbation theories

[49] Else the commutation relation is modified by constraints,this is not crucial

[50] the delta-symbol is the product of delta symbols and delta functions

[51] The construction of eigenstates of fermionic operators are more complex

[52] For field case,$P = -i\frac{\delta}{\delta Q}, \langle Q|P\rangle \sim \exp(\int d^3xQP)$,for usually QM,the core is the canonical relation,thus **any** conjugate pairs having this commuation relation can be put into the path integral form.For example:X,P

[53] This is the quantum Lie algebra(extension of the classical Lie algebra of transitions in phase [q,p]=0) of translation in phase space,this may be obtained using canonical quantization.Form $[q,p] = i$,we have $e^{ip\delta}Q|q\rangle = xe^{iP\delta}|q\rangle = Qe^{iP\delta}|q\rangle - \delta e^{iP\delta}|q\rangle$,thus $e^{iP\delta}|q\rangle \equiv |q+\delta\rangle \Rightarrow P_{qq'} = -i\partial_q\delta(q-q') \Rightarrow p\langle q|p\rangle = \langle q|P|p\rangle = -i\partial_q\langle q|p\rangle$,this is solved by **Plane wave**

- At any time the states are orthonormal and complete.Inner product between states at different time is known as transition amplitude.Since these states form basis,we only need to evaluate transition amplitude between basis at different time.

The dynamic of system is given by the basic transition amplitude:

$$\langle q';t'|q;t\rangle_H$$

Infinitesimally,insert $\int dp|p,t\rangle\langle p,t| = 1$:

$$\begin{aligned}\langle q';t+dt|q;t\rangle \equiv \langle q';t|e^{-iHdt}|q;t\rangle &= \int\prod_a dp_a\langle q';t|e^{-iH[Q(t),P(t)]dt}|p;t\rangle\langle p;t|q;t\rangle \\ &= \int\prod_a \frac{dp_a}{2\pi}\exp(-iH[q'(t),p(t)]dt + i\sum_a(q'_a - q_a)p_a)\end{aligned} \tag{6.48}$$

- We have odopted the convention that H is a functional of Q,P in **Weyl order:all Q to the left of P**.Different ordering of H will lead to (anti)commutators,these will eventially lead to **different measure**.
- For other operators we will use the oppeseite Weyl odering:all P to the left of Q,thus $O[P(t), Q(t)]$

The general transition amplitude is obtained by seperating the time also into lattice:

$$dt = t_{k+1} - t_k = \frac{t'-t}{N+1}, t < t_1 < ... < t_N < t'$$

Then:

$$\begin{aligned}&q_0 = q, q_{N=1} = q' \\ \langle q';t'|q;t\rangle &= \int dq_1...dq_N\langle q';t'|q_N,t_N\rangle\langle q_N;t_N|q_{N-1};t_{N-1}\rangle...\langle q_1;t_1|q;t\rangle \\ &= \int(\prod_{k=1}^N\prod_a q_{k,a})(\prod_{k=0}^N\prod_a\frac{dp_{k,a}}{2\pi})\exp(i\sum_{k=1}^{N+1}\{\sum_a(q_{k,a}-q_{k-1,a})p_{k-1,a} - H(q_k,p_{k-1})dt\}) \\ &= \int(\prod_{k=1}^N\prod_a q_a(t_k))(\prod_{k=0}^N\prod_a\frac{dp_a(t_k)}{2\pi})\exp(i\sum_{k=1}^{N+1}\{\sum_a \dot{q_a}(t_k)p_a(t_k) - H(q(t_k),p(t_{k-1}))\}dt)\end{aligned}$$

Defining the measure:

$$\begin{aligned}&\int\prod_l\mathcal{D}q_l = \lim_{dt\to 0}\int\prod_a\prod_t dq_a(t) = \lim_{dt\to 0}\int\prod_l\prod_{x_i}dq_l(x_i) = \lim_{dt\to 0}\int\prod_a\prod_{k=1}^{N=1}q_a(t_k) \\ &\int(\prod_l\mathcal{D}q_l)_t = \lim_{dt\to 0}\int\prod_l\prod_{\vec{x}_i}dq_l(\vec{x},t), \int\prod_l\mathcal{D}q_l = \lim_{dt\to 0}\int\prod_{k=1}^{N+1}(\mathcal{D}q_l)_{t_k}\end{aligned} \tag{6.49}$$

Thus the final result is:

$$\langle q';t'|q;t\rangle_H = \int_{q_a(t)=q_l(\vec{x},t)=q_l(\vec{x})}^{q_a(t')=q_l(\vec{x},t')=q'_l(\vec{x})}\prod_l\mathcal{D}q_l\prod_{l'}\frac{\mathcal{D}p_l}{2\pi}\ \exp(i\int_t^{t'}d\tau\{\sum_a p_a(\tau)\dot{q}_a(\tau) - H(q(\tau),p(\tau))\}) \tag{6.50}$$

- The measure at fixed time point is define to be product of all variation of all D.O.F,for each space point,there's l field components,and we product the space points
- The measure for the general spacetime is the product of all spacetime point D.O.F,each point have l field component.The total measure is the product of time points at different times.
- The completeness relation of the basis **at time t** can be written as:

$$\int\prod_a q_a|q;t\rangle\langle q;t| = \int\prod_l\prod_{\vec{x}_i}q_l(\vec{x}_i,t)|q;t\rangle\langle q;t| \equiv \int\prod_l(\mathcal{D}q_l)_t|q;t\rangle\langle q;t| = 1$$

  Similar for p.

- The insertion if **ordered by time**,the direction of path fixed from small t to large t.However,written in final path integral form,the time ordering is **hidden.**
- The meaning of the final result in known as **path integral**,since we integrate over all paths that between $q_a(t) \to q_a(t')$,with the final D.O.F fixed.Note that for final point,the D.O.F is an eigenvalue of the **S-piture** field operator: $q_l(\vec{x})$ and $q'_l(\vec{x})$,the $\vec{x}$ should be understood as lattice label for D.O.F.Similar for p.
- An important notation is that there's no axuliary field,the axuliary field can be included by some procedure.

The matrix elements of operators $O[P(t), Q(t)]$ are also easy to express:

$$\langle p;\tau|O[P(t), Q(t)]|q;\tau'\rangle = O(p(t), q(\tau'))\langle p;\tau|q;\tau'\rangle$$

thus:

$$\begin{aligned}
&\langle q';t'|T\{O_1[P(t_1), Q(t_1)]...O_k[P(t_k), Q(t_k)]\}|q;t\rangle \\
= &\int_{q_a(t)=q_a}^{q_a(t')=q'_a} (\prod_l \mathcal{D}q_l \prod_{l'} \frac{\mathcal{D}p_{l'}}{2\pi})\{O_1(p_l(\vec{x}, t_1), q_l(\vec{x}, t_1))...O_k(p_l(\vec{x}, t_k), q_l(\vec{x}, t_k))\} \\
&\times \exp(i\int_t^{t'} d\tau\{\int d^3x \sum_l p_l(\vec{x}, \tau)\dot{q}_l(\vec{x}, \tau) - H(q_l(\vec{x}, \tau), p_l(\vec{x}, \tau))\})
\end{aligned} \tag{6.51}$$

- There's time ordering in matrix element.This comes from the fact that we insert the time slices by the order:**latest to the leftest**.This is due to the fact:$t' > t$
- RHS contain all **classical results**,the p,q are not constraint to be stationary:in RHS,the $p(t_i), q(t_i)$ are independent **variables** at fixed time t_i,they depend on particular path.Thus the **value of classical O_i varies as well as $H(q, p)$ on different pathes**
- The derivation from classical equation of motion and the fact that the pathes are integrated indicate the notion of unitarity,this summation give rise to quantum physics,manifesting the **uncertainty principle.**
- This implies that H in the phase is **not constant in** τ,this is also the manifest of the uncertainty principle:**energy conservation can be broken in finite times**.
- In formulation we take t_i as independent,when t_i approch to some other t_j,there will be a term propotional to $\delta(t_i - t_j)$ and its derivative,this is implicitly the time derivation of $\theta(t_i - t_j)$ in LHS.This fact will be more clear in Schwinger Dyson equations.
- The O, H in RHS are classical,their form is independent of orders up to signs.But in LHS we have operators.Choosing a different order of O, H as operators in LHS will lead to a change of measure in RHS.The difference will be hidden in the infinity in the measure.

6.4.3 S-matrix element and Green's functions

Generating functional of Green's functions

The physical observables in relativistic QFT are S-matric elements,by definition:

$$S_{\beta,\alpha} \equiv \langle\beta, out|\alpha, in\rangle_H = \langle\beta; +\infty|\alpha; -\infty\rangle_H = \langle\beta|S|\alpha\rangle_I$$

In path integral the S-matrix element can be calculated by multiplying the 'wave-functionals'[54]

$$\langle\beta, out|q';t'\rangle, \langle q;t|\alpha;in\rangle$$

and integral over $q_l(\vec{x}, t) = q_l(\vec{x}), q_l(\vec{x}, t') = q'_l(\vec{x})$ to remove $\langle q;t|q;t\rangle, \langle q';t'|q';t'\rangle$ and finally getting the translation amplitude between in/out state expressed in path integrals.It convenient to take $t \to -\infty, t' \to \infty$.The

[54]Since the coordinates are fields,we are considering functionals

integral with constraint at $t = \pm\infty$ can be removed when the field parameter is integrated.More generally:

$$\begin{aligned}
&\langle\beta; out|T\{O_1[P(t_1), Q(t_1)]...O_k[P(t_k), Q(t_k)]\}|\alpha; in\rangle \\
=&\int (\prod_l \mathcal{D}q_l \prod_{l'} \frac{\mathcal{D}p_{l'}}{2\pi})\{O_1(p_l(\vec{x}, t_1), q_l(\vec{x}, t_1))...O_k(p_l(\vec{x}, t_k), q_l(\vec{x}, t_k))\} \\
&\times \exp(i\int_t^{t'} d\tau\{\int d^3x \sum_l p_l(\vec{x}, \tau)\dot{q}_l(\vec{x}, \tau) - H(q_l(\vec{x}, \tau), p_l(\vec{x}, \tau))\}) \\
&\times\langle\beta; out|q(\infty); \infty\rangle\langle q(-\infty); -\infty|\alpha; in\rangle
\end{aligned} \tag{6.52}$$

The S-matrix element is a special case of this,with no operator inserted.There is an important formula for S-matrix when coupled to some **external source/field/operator**:

$$H_\epsilon = H_0 + V(t) + \sum_A \int d^3x \epsilon_A(\vec{x,t})O_A(\vec{x}, t)$$

$$\begin{aligned}
S_{\beta\alpha}[\epsilon] \equiv \langle\beta; out|\alpha; in\rangle_\epsilon = &\int (\prod_l \mathcal{D}q_l \prod_{l'} \frac{\mathcal{D}p_{l'}}{2\pi}) \\
&\times \exp(i\int_t^{t'} d\tau\{\int d^3x \sum_l p_l(\vec{x}, \tau)\dot{q}_l(\vec{x}, \tau) - H(q_l(\vec{x}, \tau), p_l(\vec{x}, \tau)) - \sum_A \int d^3x \epsilon_A O_A\}) \\
&\times\langle\beta; out|q(\infty); \infty\rangle\langle q(-\infty); -\infty|\alpha; in\rangle
\end{aligned}$$

Take derivative of ϵ_A,we obtain the most general form:

$$(\frac{1}{i^k}\frac{\delta S_{\beta\alpha}[\epsilon]}{\delta\epsilon_1(x_1)...\delta\epsilon_k(x_k)})_{\epsilon=0} \equiv (\frac{1}{i^k}\frac{\delta\langle\beta; out|\alpha; in\rangle_\epsilon}{\delta\epsilon_1(x_1)...\delta\epsilon_k(x_k)})_{\epsilon=0} \equiv \langle\beta; out|T\{O_1(x_1)...O_k(x_k)\}|\alpha; in\rangle \tag{6.53}$$

This can also be obtained using definition of S-matrix:[55]

$$\begin{aligned}
S[\epsilon] \equiv& T\{\exp(-i\int dt V_I[\epsilon])\} \\
(\frac{\delta S_{\beta\alpha}[\epsilon]}{\delta\epsilon_1(x_1)...\delta\epsilon_r(x_r)})_{\epsilon=0} =& \sum_{N=0}^{\infty} \frac{(-i)^{N+r}}{N!}\int dt_1...dt_N(\Phi_\beta, T\{V(t_1)...V(t_n)o_1(x_1)...o(r)(x_r)\}\Phi_\alpha)_I \\
=&(-i)^r(\Phi_\beta, U(\infty, x_1^0)o_1(x_1)U(x_1^0, x_2^0)o_2(x_2)...o_r(x_r)U(x_r^0, -\infty)\Phi_\alpha) \\
=&(-i)^r(\Omega(\infty)\Phi_\beta, T\{O_1(x_1)...O_r(x_r)\}\Omega(-\infty)\Phi_\alpha)_H = (-i)^r(\Psi_\beta^-, T\{O_1(x_1)...O_r(x_r)\}\Psi_\alpha^+)_H
\end{aligned}$$

- The S-matrix element may be considered as **on-shell** external particles $\alpha + \beta$ interacting intermediate internal off-shell virtual particles.The meaning of the LHS is perturbatively in Feynman diagram:
  - Adding k addtional vertices attached with $n_a, ..., n_b$ **internal** lines if o_a contain n_a field factors.
  - These vertices contain label $x_1, ..., x_k$ we don't integrate.Each vertices contribute $-i$ times whatever numerical factors appear in the associated **external current/operator** o_a
  - The special case is when $o_a = \psi_l$,then the rule is that we add further vertex for each o,and attach a **off-shell** internal line of field type l that labelled by x-the position of source that we don't integrate.
  - These new off-shell internal lines with sources stripped can be considered as **off-shell external lines**,in p-sapce they give contribution of **off-shell propagators rather than on-shell polarizations**,and $-i$ from the vertex standing for stripped vertex(which may be considered as not exsisting thus the internal line is indeed external)
  - By **replace the propagator with polarization**,we force the off-shell particle **on-shell**,this then gives the S-matrix with more interacting particles.
- The above fact implies:**S-matrix can be generated from vacuum scattering with off-shell particles given by external source by forcing these particles on-shell**:

$$S_{\beta\alpha} = LSZ[\langle\Omega|T\{\Psi(x_1)...\Psi(x_{n_{\alpha+\beta}})\}|\Omega\rangle] \tag{6.54}$$

The 'LSZ' means that we force the external particles on-shell.This is known as LSZ reduction.

[55] For detail,refer to [WeinbergI.6.4]

Thus in relativistic QFT,what we really need is to study the **Green's functions**:

$$\begin{aligned}
\langle\Omega|T\{\Psi_{l_1}(x_1)...\Psi_{l_k}(x_k)\}|\Omega\rangle \equiv&(\frac{1}{i^k}\frac{\delta\langle\Omega;out|\Omega;in\rangle_\epsilon}{\delta J_{l_1}(x_1)...\delta J_{l_k}(x_k)})_{\epsilon=0}\\
\langle\Omega;out|\Omega;in\rangle_J \equiv Z_I[J] =& \int(\prod_l \mathcal{D}q_l\prod_{l'}\frac{\mathcal{D}p_{l'}}{2\pi})\\
\times\exp(i\int_t^{t'}d\tau\{\int d^3x\sum_l p_l(\vec{x},\tau)\dot{q}_l(\vec{x},\tau) - H(q_l(\vec{x},\tau),p_l(\vec{x},\tau)) &- \int d^3x\sum_l J_l(\vec{x},t)\Psi_l(\vec{x},t)\}) \\
&\times\langle\Omega;out|q(\infty);\infty\rangle\langle q(-\infty);-\infty|\Omega;in\rangle
\end{aligned} \tag{6.55}$$

- $\langle\Omega;out|\Omega;in\rangle_J \equiv Z_I[J]$ is known as **generating functional**,having obtained the expression of Z[J],we then take derivative of it to generate the Green's functions.This will be drawn as vacuum scattering with extra off-shell particles.
- Using the LSZ reduction,we can obtain wanted S-matrix elements.
- It is important to note that though the Green's functions are central,they are in general **not observables**,some of them are not gauge invariant.But observables are extracted from Green's functions.
- Green's functions contain more information than observables.
- To show the theory is renormalizable,we show that all Green's functions can be made finite,the Fourier transformed Green's functions are known as **exact vertices**,with 2-point Green's function known as **exact propagator**.
- The Green's functions are expressed in H-picture,thus it's different to free theories.
- Perturbatively,the Green's functions are calculated using diagrams generated by **free generating functional** $Z_0[J]$.For simple free fields,$Z_0[J]$ is exactly obtained.

Wave-functional and $i\epsilon$

In interacting theory,the fields and the ground states are asymptoticly free,this implies:

- The interacting field at asymptotic time can be expanded as a free field with $a_{in/out}$
- The vacuum state in interaction theory are defined to be[56]:

$$a_{in/out}|\Omega;in/out\rangle = 0$$

By expanding $a_{in/out}$ into P,Q and go to the 'Q-basis',we can solve these equations[57]

For real scalar field:

$$\begin{aligned}
a_{in/out} = \lim_{t\to\mp\infty}\frac{e^{iEt}}{(2\pi)^{3/2}}\int d^3xe^{-i\vec{p}\cdot\vec{x}}[\sqrt{\frac{E}{2}}\Phi(\vec{x},t)+i\sqrt{\frac{1}{2E}}\Pi(\vec{x},t)],\Pi(\vec{x},t)\to -i\frac{\delta}{\delta\phi(\vec{x},t)}\\
0 = \int d^3xe^{-i\vec{p}\cdot\vec{x}}[\frac{\delta}{\delta\phi(\vec{x},t)}+E(\vec{p})\phi(\vec{x})]\langle\phi(t\to\mp\infty);\mp\infty|\Omega;in/out\rangle
\end{aligned}$$

This is solved by **Gasussian wave-packet**:

$$\begin{aligned}
\langle\phi(t\to\mp\infty);\mp\infty|\Omega;in/out\rangle = \mathcal{N}\exp(-\frac{1}{2}\int d^3xd^3y\mathcal{E}(\vec{x},\vec{y})\phi(\vec{x})\phi(\vec{y}))\\
0 = \int d^3xe^{-i\vec{p}\cdot\vec{x}}[\int d^3y\mathcal{E}(\vec{x},\vec{y})\phi(\vec{y}) - E(\vec{p})\phi(\vec{x})] \Rightarrow \mathcal{E}(\vec{x},\vec{y}) = \int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}E(\vec{p}),E(\vec{p}) = \omega_p
\end{aligned}$$

[56] It's more convenient to identify $|\Omega;in/out\rangle,|\Omega;0\rangle = |\Omega\rangle_H = |0\rangle$ as unique up to phases,this identification is useful in LSZ reduction

[57] This is similar to the case of Harmonic oscillator

- The $\mathcal{N}$ is obtained from the renormalization condition of vacuum state:

$$\int (\mathcal{D}\phi)_{\mp\infty}\langle\Omega; in/out|\phi(\mp\infty);\mp\infty\rangle\langle\phi(\mp\infty);\mp\infty|\Omega; in/out\rangle = \langle\Omega; in/out|\Omega; in/out\rangle$$
$$= \mathcal{N}^2\int (\mathcal{D}\phi)_{\mp\infty}|\exp(-\frac{1}{2}\int d^3xd^3y\mathcal{E}(\vec{x},\vec{y})\phi(\vec{x})\phi(\vec{y}))|^2 = 1$$

  We need not to solve $\mathcal{N}$,it will drop out when we evaluate connected part of the vacuum expectation values.

- the final result is[58]:

$$\langle\Omega; out|\phi(\infty);\infty\rangle\langle\phi(-\infty);-\infty|\Omega; in\rangle = |\mathcal{N}|^2\exp(-\frac{1}{2}\int d^3xd^3y\mathcal{E}(\vec{x},\vec{y}))[\phi(\vec{x},\infty)\phi(\vec{y},\infty)+\phi(\vec{x},-\infty)\phi(\vec{y},-\infty)]$$
$$= |\mathcal{N}|^2\exp(-\frac{1}{2}\varepsilon\int d^3xd^3y\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\tau\mathcal{E}(\vec{x},\vec{y})\phi(\vec{x},\tau)\phi(\vec{y},\tau)e^{-\varepsilon|\tau|})$$
$$\Rightarrow H \to H - i\varepsilon$$

This is common for **all simple free fields**,the whole effect of the vacuum wavefunctional is to add a $-i\varepsilon$ term to the Hamiltonian,this convert the initial and final states to vacuum states:

$$\langle\Omega|T\{O_1...O_k\}|\Omega\rangle = |\mathcal{N}|^2\int\mathcal{D}q\frac{\mathcal{D}p}{2\pi}(O_1...O_k)\exp(i\int_{-\infty}^{+\infty}dt\{\int d^3x\sum_l p_l(x)\dot{q}_l - (H-i\varepsilon)\}) \tag{6.56}$$

- The $i\epsilon$ term have the effect of putting the right $-i\epsilon$ in the denominators of propagators thus this $i\epsilon$ is related to **causality**

- For more general wavefunctionals,with vacuum replaced by multiparticle states,the wavefunctional can be obtained by acting creation operators in wavefunctional representation.
  - The vacuum state wavefunctinal is a Gasussian wavepacket
  - The general multiparticle state wavefunctional will be **Hermite polynomials in field** times the vacuum Gaussian.
  - Working out general wavefunctionals gives us directly the generalized S-matrix element,but this is not necessary,since we can use LSZ reduction to obtain S-matrix element from vacuum expectation values(Green's functions)

- For S-matrix element,as noted,it factors into connected parts,thus we focus on the connected part of the Green's functions:

$$\langle\Omega|T\{...\}|\Omega\rangle_C = \frac{\langle\Omega|T\{...\}|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \frac{1}{Z_{I,\varepsilon}[0]}[(\frac{\delta}{i\delta\epsilon_i}...)Z_{I,\varepsilon}[\epsilon]]_{\epsilon=0} \tag{6.57}$$

  by LSZ reduction,we obtain **connected S-matrix and connected Feynman amplitude** automatically.
  - $\langle\Omega|\Omega\rangle = \langle\Omega; out|\Omega; in\rangle$ contain **vacuum bubbles/flunctuation**.They drop out in connected parts.
  - In the generating functional:

$$\mathcal{H} = (\mathcal{H}_0 - i\epsilon) + \mathcal{H}_I + i\sum_A\epsilon_A(x)O_A(x)$$

    The ε indicate we are evaluating vacuum expectation values,and the end point D.O.F are also integrated over,not fixed.

- The $i\epsilon$ factor can also be understood in this way,inserting the **energy eigenstate**:

$$\lim_{T\to\infty}\langle q';T|T\{O_1...O_k\}|q;-T\rangle = \sum_n\langle q'|e^{-iE_nT}|n\rangle\langle n|T\{...\}|n'\rangle\langle n'|e^{-iE_{n'}T}|q\rangle$$

[58] $f(+\infty)+f(-\infty) = \lim_{\varepsilon\to 0^+}\varepsilon\int_{-\infty}^{+\infty}d\tau f(\tau)e^{-\varepsilon|\tau|}$

The effect of the $i\epsilon$ term in H lead to $e^{-\epsilon(E_n-E_\Omega)\infty}$ suppression factors,thus by **Riemann-Lebsegue lemma**,only the vacuum term remains:

$$\langle q;+\infty|T\{...\}|q;-\infty\rangle = \langle\Omega|T\{...\}|\Omega\rangle \times (\lim_{T\to\infty} e^{-2iE_\Omega(1-i\epsilon)T}\langle q(+\infty);+\infty|\Omega\rangle\langle\Omega|q(-\infty);-\infty\rangle)$$

While $\langle q';+\infty|q;-\infty\rangle = \lim_{T\to\infty} e^{-2iE_\Omega(1-i\epsilon)T}\langle q(+\infty);+\infty|\Omega\rangle\langle\Omega|q(-\infty);-\infty\rangle$,we finally have:

$$\langle\Omega|T\{...\}|\Omega\rangle = \frac{\langle q';+\infty|T\{...\}|q;-\infty\rangle}{\langle q';+\infty|q;-\infty\rangle} \equiv \langle\Omega|T\{...\}|\Omega\rangle_C \tag{6.58}$$

- Though the RHS fix the final configuration to $q'(\infty) = q'(\vec{x}), q(-\infty) = q(\vec{x})$,the LHS is independet of these configurations,as shown by the explicit derivation.Thus the path integrals in RHS can also be safely generalized to contain all paths,**with final configurations unfixed.**
- Thus this is just the connected part,since $\langle\Omega|\Omega\rangle = \frac{\langle q';+\infty|q;-\infty\rangle}{\langle q';+\infty|q;-\infty\rangle} = 1$

6.4.4 Quadratic theories and Lagrangian form

In the general formula,the integrated variables p, q are **independet**,they are not related as conjugate pairs as the **operators**.The quadratic theories:the classical Hamiltonian is quadratic in the canonical momenta can be simplified,they have **S-picture** Hamiltonian:

$$H[Q,P] = \frac{1}{2}\sum_{nm}\int d^3x d^3y A_{\vec{x}n,\vec{y}m}[Q]P_n(\vec{x})P_m(\vec{y}) + \sum_n \int d^3x B_{\vec{x}n}[Q]P_n(\vec{x}) + C[Q]$$

- The 'matrix'(disrtibution) is **real,symmetric,positive,non-sigular** and denpend **only on Q**

This implies:

$$\begin{aligned}\int d^4x \sum_n p_n(x)\dot{q}_n(x) - H[q(t),p(t)] = &-\frac{1}{2}\sum_{nm}\int d^3x d^3y d\tau d\tau' \mathcal{A}_{\tau\vec{x}n,\tau'\vec{y}m}[q]p_n(\vec{x},\tau)p_m(\vec{y},\tau') \\ &-\sum_n \int d^3x \int d\tau \mathcal{B}_{\tau\vec{x}n}[q]p_n(\vec{x},\tau) - \mathcal{C}[q]\end{aligned} \tag{6.59}$$

$$\mathcal{A}_{\tau\vec{x}n,\tau'\vec{y}m}[q] = A_{\vec{x}n,\vec{y}m}[q(\tau)]\delta(\tau-\tau'), \mathcal{B}_{\tau\vec{x}n}[q] = B_{\vec{x}n}[q(\tau)] - \dot{q}_n(\vec{x},\tau), \mathcal{C}[q] = \int d\tau C[q(\tau)]$$

This is the generalized **Gaussian integral**.With the matrix being the general distribution(usually differential operators),consider the continuous labels also as index,as long as the operators O are independent of P,we can evaluate the Gaussian integrals by setting the varibles(ps) to the **stationary point of the quadratic expression in the argument of the exponential**:

$$\frac{\delta}{\delta p_n(\vec{x},t)}[\int d^4x p_n\dot{q}_n - (H-i\varepsilon)] = \dot{q}_n(\vec{x},t) - \frac{\delta}{\delta p_n(\vec{x},t)}H[p,q] = 0, \dot{q}_n(\vec{x},t) = (\frac{\delta}{\delta p_n(\vec{x},t)}H[p,q])_{p=\bar{p}}$$

this means the integral over $\mathcal{D}p$ is just by setting p being the canonical value,and the argument is set **equal** the ordinary **Lagrangian**:

$$L[q,\dot{q}] = \int d^3x \sum_n \bar{p}_n\dot{q}_n - H$$

Thus:

$$\begin{aligned}\langle\Omega|T\{...\}|\Omega\rangle &= |\mathcal{N}|^2 \int \mathcal{D}q(det(\frac{i\mathcal{A}[q]}{2\pi}))^{-1/2}\frac{1}{2\pi}(...)\exp(i\int d^4x\mathcal{L} - i\varepsilon) \\ &= |\mathcal{N}|^2 \int \mathcal{D}q(det(2\pi i\mathcal{A}[q]))^{-1/2}\exp(iI[q_l(\vec{x},t)])\end{aligned} \tag{6.60}$$

- If $\mathcal{A}$ is independent of q,then it is in general another infinite constant,can be pulled out of the integral.Thus drop out when connected part of the Green's function is evaluated.We may absorb all constant infinities into the measure:
$$\mathcal{D}q = |\mathcal{N}|^2(det(2\pi i\mathcal{A}))^{-1/2}\prod_l\prod_{x_i} q_l(x_i) = n(N)\prod_l\prod_{x_i} q_l(x_i)$$

- When $\mathcal{A}[q]$ is indeed dependent of q,then we can expand:

$$det(\mathcal{A}[q])^{-1/2} = \exp(tr \log \mathcal{A}[q]) = \exp(i \int d^4x \Delta \mathcal{L}_{eff}) \tag{6.61}$$

This gives the effective Lagrangian.The crucial point is:**The effective Lagrangian in Lagrangian form of path integral may differ from the classical Lagrangian**.[59]

Auxiliary fields

If the classical Lagrangian contain axuliary fields,then the Hamiltonian is expressed in terms of canonical variables will contain no axuliary fields,thus the final Lagrangian will have no axuliary fields in it,it is expressed in terms of P,Q already.

For example,for the massive vector field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu + J^\mu A_\mu$$

The Hamiltonian containing non-covariant term is:

$$H = \int d^3x[\frac{1}{2}\vec{\Pi}^2 + \frac{1}{2}(\nabla \times \vec{A})^2 + \frac{1}{2}m^2\vec{A}^2 + \frac{1}{2m^2}(\nabla\cdot\Pi)^2 + \vec{J}\cdot\vec{A} - \frac{1}{m^2}J^0(\nabla\cdot\Pi) + \frac{1}{2m^2}(J^0)^2]$$

we have:

$$\mathcal{A}_{ix,jy} = \delta_{ij}\delta^4(x-y) - \frac{1}{2m^2}\partial_i\partial_j\delta^4(x-y)$$

This is independent of q.The axuliary A^0 can be reintroduced by containing the effective Hamiltonian as well as a new integral over A^0:

$$\Delta H = -\frac{1}{2}\int d^3x[A^0 - \frac{1}{m^2}\nabla\cdot\vec{\Pi} + \frac{1}{m^2}J^0]^2$$

- the stationary point is:

$$\frac{\delta}{\delta A^0}\Delta H = 0 \Rightarrow \bar{A}^0 = \frac{1}{m^2}[\nabla\cdot\vec{\Pi} - J^0]$$

- Note that this is different from the canonical constraint $A^0 = \frac{1}{m^2}(\nabla\cdot\Pi + J^0)$
- The crucial point is:this term have no effect if we integral A^0 first:

$$\int \mathcal{D}A^0 e^{-i\Delta H} = e^{-i\Delta H[\bar{A}^0]} = 1$$

Thus the introduction of axuliary fields have no effect as long as we choose the ΔH appropriately,so that integrate over the axuliary field first this term is identical to 1.

- If we integral over $\vec{\Pi}$ first then the classical Lagrangian may be restored:

$$H + \Delta H = \int d^3x[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\times\vec{A})^2 - \frac{1}{2}m^2(A^0)^2 + \frac{1}{2}m^2\vec{A}^2 + \vec{J}\cdot\vec{A} - J^0\vec{A}^0 + A^0(\nabla\cdot\Pi)]$$

The integral over Π is by replacing it with stationary point:

$$\bar{\Pi} = \dot{A} + \nabla A^0$$

This gives the classical Lagrangian:

$$(\int \Pi\cdot\vec{A} - H - \Delta H[A^0])_{\bar{\Pi}} = L[A^\mu]$$

Effectively,we can always inlude the axuliary classical fields in the path integral so that the Lagrangian obtained by setting the canonical variables to stationary point will be the classical Lagrangian up to effective terms due to determinant:

$$\langle\Omega|T\{O[\Psi]...\}|\Omega\rangle = |\mathcal{N}|^2\int\mathcal{D}\Psi(O[\Psi]...)\exp(i\int d^4x(\mathcal{L}+i\varepsilon) + (\Delta\mathcal{L}_{eff})) \tag{6.62}$$

Another example of the use of axuliary field is the **Harbard**

[59] An example is **non-linear σ-model**:$\mathcal{L} = -\frac{1}{2}\sum_{nm}\partial_\lambda\phi_n\partial^\lambda\phi_m[\delta_{nm} + U_{nm}(\phi)] - \mathcal{V}(\phi)$

Fermionic operator

For fermionic case,the canonical relations are now anticommutation relations:[60]

$$[Q_a, P_b]_+ = i\delta_{ab}, [Q_a, Q_b]_+ = [P_a, P_b]_+ = 0$$

The first step is to construct the complete orthonomal basis **defined by** these operators.The relation $Q_a^2 = P_a^2 = 0$ defines two states:

$$Q_a|0;Q\rangle = 0, \langle 0;P|P_a = 0 \forall a$$

- The normalization may be $\langle 0;P|0;Q\rangle = 1$
- In general $\langle 0;P| \neq (|0;Q\rangle)^\dagger$,else the normalization will vanish.[61]

The complete basis is generated by acting on P:

$$|a, b, ...\rangle = P_a P_b ...|0;Q\rangle, \langle a, b, ...| = \langle 0;P|...(-iQ_b)(-iQ_a)$$

- Labels in the set {a,b,...} are distinct.
- Using the anticommutation relations:

$$\begin{aligned}
&Q_a|b, c, ...\rangle = 0, P_a|b, c, ...\rangle = |a, b, c, ...\rangle \\
&Q_a|a, b, ...\rangle = i|b, c, ...\rangle, P_a|a, b, ...\rangle = 0 \\
&\langle b, c, ...|Q_a = i\langle a, b, ...|, \langle b, c, ...|P_a = 0 \\
&\langle a, b, ...|Q_a = 0, \langle a, b, ...|P_a = -i\langle b, c, ...|
\end{aligned}$$

- The basis is orthonormal:

$$\langle c, d, ...|a, b, ...\rangle = \begin{cases} 0 & \{c, d, ...\} \neq \{a, b, ...\} \\ 1 & c = a, d = b, ... \end{cases}$$

  The normalization is defined to be one if the labels on both sides 'cancel' in order,with a different order,there will be sign differences.

The eigenstates of the operators are:

$$Q_a|q\rangle = q_a|q\rangle, q_a q_b + q_b q_a = 0$$

- The **classical** eigenvalue/field for fermionic operators are **anticommuting Grassmann variables**.These variables act like c-number but satisfy the anticommutation relations:

$$[q_a, q'_b]_+ = [q_a, Q_b]_+ = [a_a, P_b]_+ = 0$$

- These eigenstates can be constructed explicitly as:[62]

$$|q\rangle = \exp(-i\sum_a P_a q_a)|0;Q\rangle = \prod_a (1 - iP_a q_a)|0\rangle$$

$$\langle q| = \langle 0;P|(\prod_a Q_a)\exp(-i\sum_a q_a P_a) = \langle 0;P|(\prod_a Q_a)(1 + \prod_b P_b q_b)$$

- These states have scalar product:[63]

$$\langle q'|q\rangle = \prod_a (q_a - q'_a)$$

  this will effectively act like a **delta-function** for integral of Grassman variables q, q'.

---

[60] S-piture

[61] $\langle 0;Q|[Q,P]_+|0;Q\rangle = 0 = i\langle 0;Q|0;Q\rangle$

[62] This is bosonic since both the operator and the variable anticommute.Varification:

$$[Q_a - q_a]|q\rangle = [Q_a - q_a](1 - iP_a q_q)e^{-i\sum_{b\neq a} P_b q_b}|0;Q\rangle = [-i[Q_a, P_a]_+ q_1 - q_a]e^{-i\sum_{b\neq a} P_b q_b}|0;Q\rangle = 0$$

,similar for $\langle q|$

[63] the sign convention is implicit in $\prod_a Q_a$,thus also in $\prod_a \delta q_a$

Eigenstates of P are defined similarly:[64]

$$|p\rangle = \exp(-i\sum_a Q_a p_a)(\prod_b P_b)|0;Q\rangle$$
$$\langle p| = \langle 0;P|\exp(-i\sum_a p_a Q_a)$$

The scalar product is similar to **wave packet**,since Q,P satisfy the fermionic version of Heisenberg algebra:$[Q,P] = i, [q,p]_P = 1$:

$$\langle q|p\rangle = (\prod_a \exp(1-iq_a p_a))\langle 0;P|(\prod_a Q_a \prod_b P_b)|0;Q\rangle = \chi_N \exp(-i\sum_a q_a p_a) = \chi_N \exp(+i\sum_a p_a q_a)$$
$$\chi_N = \langle 0;P|(\prod_a Q_a)\prod_b (Q_b)|0;Q\rangle = i^N(-1)^{\frac{N(N-1)}{2}}$$

- The phase is independent of the ordering in both products,since the sign have mod 2 algebra in the power.We may order the two products in the same way.
- the expression of the 'complex conjugation' is more simple:

$$\langle p|q\rangle = \prod_a \exp(-ip_a q_a)$$

The two complete basis:$|a,b,...\rangle, |q\rangle/|p\rangle$ are equivalent:

$$|q\rangle = (1-i\prod_a P_a q_a)|0;Q\rangle = |q\rangle_0 + \sum_a q_a|q\rangle_a + ...$$

thus $|a,b,...\rangle$ is up to phases the coefficient of $q_a q_b$ in RHS:$|a_1,a_2,...,a_k\rangle \equiv |q\rangle_{a_1a_2...a_k}$
The general state is:

$$|f\rangle = f_0|q\rangle_0 + \sum_a f_a|q\rangle_a + \sum_{a\neq b} f_{ab}|q\rangle_{ab} + ... = f_0|0;Q\rangle + \sum_a f_a|a\rangle + \sum_{a\neq b} f_{ab}|a,b\rangle + ...$$

6.4.5 Summary:Axiom of path integral formalism

6.4.6 Schwinger Dyson equation

6.4.7 Green's functions from canonical formalism

6.4.8 non-perturbative results of Green's functions

6.4.9 Unitarity:Pole and branch cut in Green's functions

6.4.10 Propagator

6.4.11 Spectral decomposition of exact propagator

6.4.12 LSZ

6.4.13 Gauge invariance and Ward identites

6.4.14 Appendix:Down-up formalism of gauge field

[64]the varification is similar.

Chapter 7

Covariant purturbative calculation:Feynman diagrams and rules

7.0.1 Soft particle and infrared divergence

Chapter 8

General gauge theory

Chapter 9

Effective theory:Renormalization and RG

Part IV

Quantum level symmetry

Chapter 10

Spontaneous symmetry breaking

Chapter 11

Anomaly

Part V

Standard model and effective theories

Chapter 12

Ingredients of standard model

Chapter 13

Electroweak

Chapter 14

QCD

Chapter 15

Examples of effective theories

Chapter 16

Appendix

16.1 Conventions

16.1.1 Dimensional analysis

16.1.2 Fourier transformations,Normalizations,Signs

Fourier transformation

2π convention origin from the choice of measure of p-eigenstate:

$$NR: \int dp|p\rangle\langle p| = 1, \langle p|p'\rangle = \delta(p-p'), \langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipx}$$

$$LR: \int \frac{dp}{2\pi}\langle p|p\rangle = 1, \langle p|p'\rangle = 2\pi\delta(p-p'), \langle x|p\rangle = e^{ipx}$$

Note that the definition of δ-functions:

$$\delta(x) = \int dpe^{2\pi ipx} = \int \frac{dp}{2\pi}e^{ipx}, \langle x|p\rangle = e^{ipx}$$

thus:

$$|p\rangle_{LR} = (2\pi)^{1/2}|p\rangle_{NR}$$

These discussion hold for all other possible factors like 2ω,which will further give factors to $\langle x|p\rangle$ thus the Fourier transformation is uneffected.

The convention of Fourier transformation origin from this choice of measure:

$$LR: f(x) = \int \frac{dp}{2\pi}\langle x|p\rangle\langle p|f\rangle = \int \frac{dp}{2\pi}e^{ipx}f(p), f(p) = \int dx\langle p|x\rangle\langle x|f\rangle = \int dxe^{-ipx}f(x)$$

Since $\delta_x = \int dpe^{\pm 2\pi ipx}$,the Fourier transformation between E,t is chosen as:

$$f(t) = \int \frac{dE}{2\pi}e^{-iEt}f(t), f(E) = \int dte^{iEt}f(t)$$

such that:

$$f(x) = \int \frac{d^4p}{(2\pi)^4}e^{ip\cdot x}f(p), f(p) = \int d^4xe^{-ip\cdot x}f(x)$$

16.2 Gamma matrix identities and conventions

16.2.1 Gorden identity

By computing $\bar{u}(\vec{p}', \sigma')(\not{p}'\gamma^\mu + \gamma^\mu\not{p})u(\vec{p}, \sigma)$,we obtain the Gordon identities:

$$\begin{aligned} 2m\bar{u}(\vec{p}', \sigma')\gamma^\mu u(\vec{p}, \sigma) &= \bar{u}(\vec{p}', \sigma')[(p'+p)^\mu - 2iJ_D^{\mu\nu}(p'-p)_\mu]u(\vec{p}, \sigma) \\ -2m\bar{v}(\vec{p}', \sigma')\gamma^\mu v(\vec{p}, \sigma) &= \bar{v}(\vec{p}', \sigma')[(p'+p)^\mu - 2iJ_D^{\mu\nu}(p'-p)_\mu]v(\vec{p}, \sigma) \end{aligned} \tag{16.1}$$

For special case:

$$\bar{u}(\vec{p},\sigma')\gamma^\mu u(\vec{p},\sigma) = 2p^\mu \delta_{\sigma'\sigma}, \bar{v}(\vec{p},\sigma')\gamma^\mu v(\vec{p},\sigma) = 2p^\mu \delta_{\sigma'\sigma}$$

and:

$$\bar{u}(\vec{p},\sigma')\gamma^0 v(-\vec{p},\sigma) = 0, \bar{v}(\vec{p},\sigma')\gamma^0 u(-\vec{p},\sigma) = 0$$

16.3 Gaussian integrals and Grassmann integrals

16.3.1 Gaussian integrals and central identity in path integral

The basic Gaussian integral is:

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}$$

with **moments**:

$$\int dx e^{-\frac{1}{2}ax^2} x^{2n} = \sqrt{\frac{2\pi}{a}} \frac{1}{a^n}(2n-1)!!$$

with linear term included,the integral is implicitly over **all space**:

$$\int dx e^{-\frac{1}{2}ax^2+Jx} = \int dx e^{-\frac{1}{2}a(x-\frac{J}{a})^2-\frac{1}{2a}J^2} = e^{-\frac{J^2}{2a}} \int dx e^{-\frac{1}{2}} = \sqrt{\frac{2\pi}{a}} e^{-\frac{J^2}{2a}}$$

a generalization to more variables will be:

$$I = \int \prod_r d\xi_r e^{-[\frac{1}{2}\sum_{rs} K_{rs}\xi_r\xi_s + \sum_r L_r\xi_r + M]} = \int d\vec{\xi} e^{-[\frac{1}{2}\vec{\xi}^T K\vec{\xi} + \vec{L}^t\vec{\xi} + M]}$$

- the matrix K is required to be symmetric,non-singular.We start with real K,L,M and that K is positive.The general case is obtained by analytic continuation.

For real symmetric matrix,it can be diagonalized by an orthogonal matrix:

$$\mathcal{Y}^t = \mathcal{Y}^{-1}, (\mathcal{Y}^t K \mathcal{Y})_{rs} = \delta_{rs}\lambda_r, \xi_r = \sum_s \mathcal{Y}_{rs}\xi'_s, |det\mathcal{Y}| = 1$$

thus:

$$\begin{aligned} I =& e^{-M} \prod_r \int d\xi' e^{-\frac{\lambda_r}{2}\xi'^2 - (\lambda^t L)_r \xi'} = e^{-M} \prod_r \sqrt{\frac{2\pi}{\lambda_r}} e^{\frac{1}{2\lambda_r}(\mathcal{Y}^t L)_r^2} = (det(\frac{K}{2\pi}))^{-1/2} \exp(\frac{1}{2}\sum_{rs} L_r L_s K_{rs}^{-1} - M) \\ & det(K) = \prod_r \lambda_r, K_{rs}^{-1} = \sum_l \mathcal{Y}_{rl}\mathcal{Y}_{sl}\lambda_l^{-1} = \sum_l \mathcal{Y}_{rl}\lambda_l^{-1}\mathcal{Y}^t_{ls} \end{aligned} \tag{16.2}$$

More compactly this is[1]:

$$I = \int d\xi e^{-\frac{1}{2}\xi^t K\xi - L^t\xi - M} = \sqrt{\frac{(2\pi)^n}{det(K)}} e^{+\frac{1}{2}L^t K^{-1} L - M}$$

It can be shown that:

$$\bar{\xi} = -K^{-1}\vec{L}, \bar{\xi}_r = -\sum_s K_{rs}^{-1} L_s$$

is the stationary point:

$$(\frac{\partial Q(\xi)}{\partial \xi_r})_{\xi=\bar{\xi}} = 0$$

thus:

$$Q(\bar{\xi}) = \frac{1}{2}\sum_{rs} L_r L_s K_{rs}^{-1} - M$$

[1] $\pm L$ have same result

Finally:

$$I = \int \prod_r d\xi_r e^{-Q(\xi)}, Q(\xi) = (det\frac{K}{2\pi})^{-1/2} e^{-Q(\bar{\xi})}$$

Gaussian integrals can be evaluated up to a determinant factor by setting the integration variable equal to the point where the argument of the exponential is **stationary**.

For ξ complex and K hermtian,the generalization is trivial:

$$I = \int d\xi^\dagger d\xi e^{-\frac{1}{2}\xi^\dagger K\xi - L^\dagger \xi - M} = (det\frac{K}{2\pi})^{-1/2} e^{\frac{1}{2}L^\dagger K^{-1} L - M}$$

when generalized to field theory:

- The integral will become path integral
- The matrix will be differential operators or disrtibutions,for simple free theory,the quadratic term comes from Lagrangian.
- The linear term comes from coupling to external source which is present in generating functional
- For interacting theories,there will be non quadratic terms it can be dealed by perturbation:

$$\int \mathcal{D}\phi e^{-\frac{1}{2}\phi\cdot K\cdot\phi + J\cdot\phi - V(\phi)} = e^{-V(\frac{\delta}{\delta\phi})}(det(\frac{K}{2\pi}))^{-1/2} e^{\frac{1}{2}J\cdot K^{-1}\cdot J}$$

  the generalization to complex case and with some i factor is trivial.

16.3.2 Wick's theorem

The Green's functions are generalization of the **moments**[2]:

$$I_{r_1 \dots r_{2N}} = \int \prod_r d\xi_r \ \xi_{r_1} \dots \xi_{r_{2N}} \ e^{-\frac{1}{2}K_{rs}\xi_r\xi_s}$$

This can be obtained from the Gaussian integral as generating function:

$$\begin{aligned}\sum_{N=0}^{\infty} \sum_{r_1,\dots,r_{2N}} \frac{1}{(2N)!} I_{r_1\dots r_{2N}} L_{r_1}\dots L_{r_{2N}} \equiv & \int \prod_r d\xi_r \ e^{-\sum_r L_r\xi_r - \frac{1}{2}\sum_{rs} K_{rs}\xi_r\xi_s} \\ =&(det\frac{K}{2\pi})^{-1/2} e^{\frac{1}{2}\sum_{rs} K^{-1}_{rs}\xi_r\xi_s} \\ =&(det\frac{K}{2\pi})^{-1/2} \sum_{N=0}^{\infty} \frac{1}{N!2^N} (\sum_{rs} K^{-1}_{rs}\xi_r\xi_s)^N\end{aligned} \tag{16.3}$$

This gives:

$$I_{r_1\dots r_{2N}} = c_N \sum_{r_i - paring} \prod_{pairs} (K^{-1})_{r_i r_j}$$

the number of different ways of pairing is:

$$v_n = \frac{(2N)!}{N!2^N}$$

thus:

$$\sum_{r_1\dots r_{2N}} L_{r_1}\dots l_{r_{2N}} I_{r_1\dots r_{2N}} = v_N c_N (\sum_{rs} L_r L_s K^{-1}_{rs})^N \Rightarrow c_N = (det(\frac{K}{2\pi}))^{-1/2} = I_0 = \int \prod_r d\xi_r e^{-\frac{1}{2}\sum_{rs} K_{rs}\xi_r\xi_s}$$

[2] odd number of variables vanish by parity

16.3.3 Grassmann variables,functions and integrals

The **functions** of classical Grassmann variables q_i:

$$f(q) = f^{(0)} + f_i^{(1)} q_i + f_{ij}^{(2)} q_i q_j + ...$$

- The summation convention is implicit and no repeat summation.
- The indices $f_{ijk...}$ are **antisymmetric**.
- If we consider the q_i as expanding some vector space,then we have a exterior algebra,with $q_i q_j \equiv q_i \wedge q_j$,and $f : \wedge(V) = \oplus \wedge^n (V) \rightarrow \wedge V$.The q_i are considered as **basis of 1-forms.**

We have derivative(exterior derivative):

$$\begin{aligned} \partial : V \times \wedge(V) \rightarrow \wedge(V) \\ \partial_{q_i} q_j = \delta_{ij}, \partial_{q_i}(q_j q_k) = \partial_{q_i}(q_j \wedge q_k) = (\partial_{q_i} q_j) q_k - q_j(\partial_{q_i} q_k), \partial_{q_i} a^{(0)} = 0 \\ \Rightarrow [\partial_{q_i}, \partial_{q_j}]_+ = 0 \end{aligned}$$

This is defined as the **left-derivative**.

- This also implies:$[dq_i, dq_j]_+ = 0$,the dq_i are just Grassmann variables as q_i,they have similar anticommutation relations.
- In canonical formalism,the derivatives in **canonical field,Noether current,Equations of motion right derivative**.It act starting form the right rather than left.Either we ignore the Grassmann nature field as ordinary fields and consider the derivative as ordinary or we evaluate the derivative starting form the right.[3]

The integral over Grassman varibles are rather peculiar,we wish it have the following properties:

- The integral over $d\xi$ have **no range**,the normalization is of $d\xi$ is taken to be $\int d\xi \xi = 1 \in \mathbb{R}$
- We wish the integral have the useful properties:

$$\int d\xi_1 d\xi_2 ... a(\xi_i' \notin \{\xi_1, ...\}) f(\xi) = a(\xi_i') \int d\xi_1 ... f(\xi)$$

The constants can always be pulled out.

$$\int d\xi f(\xi + a) = \int d\xi f(\xi)$$

The integral is linear:

$$\int d\xi f(\xi) + g(\xi) = \int d\xi f(\xi) + \int d\xi g(\xi)$$

These properties suggest the identification:

$$\int d\xi = \frac{\partial}{\partial \xi} \tag{16.4}$$

The derivatives are **derivatives**,thus we make the convention that:

$$\int d\xi_1 d\xi_2 ... d\xi_n = \partial_{\xi_1} ... \partial_{\xi_n}, f(\xi) = a + f_i \xi_i + ... + f_{n,n-1,...,1} \xi_n \xi_{n-1}, ... \Rightarrow \int d\xi_1 d\xi_2 ... d\xi_n f(\xi) = f_{n,n-1,...,1}$$

In brief:

- Differentials of Grassmann variables are considered as Grassmann variables
- The integral is ordered left derivatives of Grassmann variables,the derivative operator is also Grassmannian.

[3] $\frac{\partial(\bar{\Psi}\gamma^\mu \partial_\mu \Psi)}{\partial \partial_0 \Psi} = \Psi^\dagger \neq -\Psi^\dagger$

- To evaluate integrals,we simply expand the functions into products of Grassmann varibles ordered **oppesite** to the differentials,and act on the derivatives by order,this will give no sign ambigurity.
- **All Grassmann entities anticommute,integration is just left derivative.In canonical formalism,we use right derivative.**

The central identity is,for antisymmetric A_{ij}:

$$\int d\bar{\xi}_1...d\bar{\xi}_n d\xi_1...d\xi_n e^{-\bar{\xi}_i A_{ij}\xi_j} = \int d\xi_1...d\xi_n d\bar{\xi}_1...d\bar{\xi}_n e^{+\bar{\xi}_i A_{ij}\xi_j} = det(A) \tag{16.5}$$

and

$$\int d\bar{\xi}_1...d\bar{\xi}_n d\xi_1...d\xi_n e^{-\bar{\xi}_i A_{ij}\xi_j + \bar{\eta}_i \xi_i + \eta_i \bar{\xi}_i} = det(A) e^{\bar{\eta} A^{-1} \eta} \tag{16.6}$$

16.4 Regularizations

16.5 Useful identities

16.6 Basic Lie algebra

16.6.1 Connected Lie algebra

Group of transformations parameterized by **finite** set of **real** continuous parameters.Each element is connected to **1** by a path within the group[4].

Group multiplication is given as:

$$T(\theta')T(\theta) = T(f(\theta', \theta)), f^a(\theta, 0) = f^a(0, \theta) = \theta^a$$

expand near **1**(infinitesimal),the group element can be expressed in Lie algebras **locally**[5]

$$T(\theta) = 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + O(\theta^3), t_{bc} = t_{cb}$$

expand f as:$f^a(\theta', \theta) = \theta^a + \theta'^a + f^a_{\ bc}\theta'^b\theta^c + O(\theta^3)$,matching terms up to $O(\theta^2)$ gives:

$$t_{bc} = -t_b t_c - i f^a_{\ bc} t^a, [t_b, t_c] = iC^a_{\ bc} t_a$$

with $C^a_{\ bc} = -f^a_{\ bc} + f^a_{\ cb}$ the **structure constant** of the Lie algebra.When $C = 0, f(\theta', \theta) = \theta' + \theta$ the group is **abelian**,which is compact: $T(\theta) = exp[i\theta \cdot T]$ for **any** θ.Special cases include one-parameter subgroups

16.6.2 Projective representation

In quantum mechanics,groups are represented projectively:

$$U(T)U(T') = exp(i\phi(T, T'))U(T, T'), \phi(T, 1) = \phi(1, T') = 0$$

The projective representations are classified by $H^2(G, U(1))$,in terms of ϕ,the associativity condition lead to 2-cocycle equation of $\omega(T, T') = exp(i\phi(T, T'))$.

Generally,it's hard to study $H^2(G, U(1))$ directly,for Lie groups we turn to it's algebraic structure and topological properties for help. Expand near I: $\phi(T(\theta), T(\theta')) = f_{ab}\theta^a\theta^{b'}$,up to order $O(\theta^2)$ we have:

$$[t_b, t_c] = iC^a_{\ bc} t_a + iC_{bc} 1, C_{bc} = -f_{bc} + f_{cb}$$

The 2nd term propotional to 1 is known as **central charge**.

Jacobi identity lead to two constraints:

$$C^e_{\ ad} C^a_{\ bc} + (bcd)_{permute} = 0$$
$$C_{ad} C^a_{\ bc} + (bcd)_{permute} = 0$$

The 1st is the usually constraint on structure constants due to Jacobi identity,the 2nd is the constrain on 2-cocycles,

[4] the group itself is a connected manifold

[5] for compact Lie group,finite transformations can also be expressed as exponential maps of Lie algebra,but this is not true for non-compact groups

- There always exsist trivial solutions:$C_{ab} = {C^e}_{ab}\phi_e, \forall\phi_e \in \mathbb{R}$.The 2nd constraint can be removed by redefining the generator:

$$t_a \to \tilde{t}_a := t_a + \phi_a, [\tilde{t}_b, \tilde{t}_c] = i{C^a}_{bc}\tilde{t}_a$$

  This is necessary for trivial cocycle solutions,which is removed by redefining the operators with a U(1)-phase giving rise to the trivial 2-cocycle.These solutions have the ordinary Lie algebra thus equivalent to ordinary representations.

  Generally,the 2-cocycle is always trivial if two condition is met:

  1. Algebraicly,near **1**,the central charges can be removed by certain redefinition of the generators.
  2. The group is **simply connected**:$\pi_1(G) = 1$

  For Lie algebra,though it's hard to determine $H^2(G, U(1))$ directly,the origin of non-trivial 2-cocycles have two origins:

  1. Algebraicly,the Lie algebra is intrinsically extended by central charges:the central charges are not linear combinations of the structure constants.
  2. Topologically,the group is not simply connected.

- **Semi-simple**[6] Lie algebra have no nontivial central charges.

16.6.3 Superselection rule and Enlarging the Lie group

The 2-cocycle may depend on class of states when there's a **superselection rule**:certain states can't be supercomposed.Inversely, if the representation have 2-cocycles varing with certain quantities,then states with different such quantities can't be supercomposed.

Enlarging the Lie group

Physical equivalently,we can always enlarge the Lie group to work with only ordinary groups:

- If there's nontrivial central charges can't be removed by redefinition,we may include operators commuting with all other generators with these central charges being eigenvalues.Then enlarged Lie algebra have no nontrivial central charges.

- If the Lie group is not simply connected,enlarge the group to its **universal covering** given by:

$$1 \to H \to \tilde{G} \to G = \tilde{G}/H \to 1$$

  .This is related to the fact that the representations of Lie algebras[a] give rise to ordinary representations of the universal covering group.

In physical applications,we work with enlarged Lie algebras and their representations. The superselection may or may not exist,which can't be determined from symmetry principles,whatever one thinks the symmetry group of nature may be,there's always another group whose consequnces are identical except for the absence of superselection rules.

[a]no central charge

[6]No invariant Abelian subalgebra:spanned by subset of commutative generators