Def a differentiable Fiber bundle.

$$(E, \pi, M, F, G)$$

total space, projection, base space, Fiber, structure grp

$$0 \to F \xhookrightarrow{i} E \underset{s \in \Gamma(M,F)}{\overset{\pi}{\rightleftarrows}} M \longrightarrow 0$$

sections.

① $\pi^{-1}(p) = F_p \cong \{p\} \times F \cong F$, $\pi(u) = p$.

② Local trivialization: $\{U_i\}$ chart of M

diffeomorphism: $\phi_i : U_i \times F \to \pi^{-1}(U_i) = \bigcup_{p \in U_i} \pi^{-1}(p)$

F_p is some differential structure over $p \in M$.

$$= \bigcup_{p \in U_i} \{p\} \times F$$

$$= \bigcup_{p \in U_i} F_p.$$

$$\bigcup_{p \in U_i} \{p\} \times F$$

$$\|$$

$$\Longrightarrow \underbrace{U_i \times F}_{\text{direct product}} \cong \pi^{-1}(U_i) = \{u \in \pi^{-1}(U_i)\}$$

Manifold.

$$\|$$

$$\bigcup_{p \in U_i} F_p.$$

direct product

~ trivial

$\sim$ trivial extension.

$$\Phi_i^{-1}(u \in \pi^{-1}(p)) = (p, f_i^{(u)})$$

$$\Phi_{i,p}(f) = \phi_i(p, f), \quad \phi_{i,p} : F \to F_p$$

③ transition function:

On $U_i \cap U_j \neq \phi$, $\quad t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$

$\uparrow$ ϕ_i $\quad \uparrow$ ϕ_j

$\in G$

Left act on F

$\rho(t_{ij}(p)) \in \mathrm{Aut}(F)$.

ϕ_i, ϕ_j are related by smooth map:

$$t_{ij} : U_i \cap U_j \to \underline{\rho(G) \subset \mathrm{Aut}(F)}.$$

$t_{ij}(p) \overset{\rho}{\leftrightarrow} t \in G$

$\rho(G) \cong G$

Local pieces $\pi^{-1}(U_i) \cong U_i \times F$ are glued/pasted together

by t_{ij}:

$$\Phi_j(\overset{\pi(u \in E)}{\overset{\shortparallel}{p}}, f_j) = \overset{(p, f^{(u)})}{\overset{\wr}{u \in \pi^{-1}(p)}} = \Phi_i(p, f_i).$$

$t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p} \in \rho(G) \subset \mathrm{Aut}(F)$.

$$\underbrace{\rho(g_\cdot) \in \mathrm{Aut}(F)}$$

$$t_{ij}(p) f_j = \phi_{i,p}^{-1} \circ \phi_{j,p}(f_j)$$

$$= \phi_{i,p}^{-1}(u \in F_p \sim (p, f^{(u)}).)$$

$$= f_i$$

$\Longrightarrow$ $\underline{f_i = t_{ij}(p) f_j}$.

consistency conditions: 1) $t_{ii}(p) = id_F$

2) $t_{ij}(p) = t_{ji}^{-1}(p)$

3) $t_{ij}(p) \circ t_{jk}(p) = t_{ik}(p)$.

$(E \xrightarrow{\pi} M, F \cong \pi^{-1}(p), G \xrightarrow{t} \mathrm{Aut} F; \{(U_i, \phi_i)\})$

define a coordinate bundle.

Compatibility (equivalence relation) : $\{(U_i, \phi_i)\} \cup \{(U_i, \tilde{\phi}_i)\}$

also define a $(\cdots)$

$$\Longrightarrow \quad (E \xrightarrow{\pi} M, F, G) = \left[(E \xrightarrow{\pi} M, F, G; \{(U_i, \phi_i)\}\right]_{\{(U_i, \phi_i)\}}$$

Rmk

① local trivialization, transition functions are not unique :

$$\{U_i\} \to \{\phi_i\} \qquad g_i(p) = \phi_{i,p}^{-1} \circ \tilde{\phi}_{i,p} : F \to F$$
$$\searrow \{\tilde{\phi}_i\} \qquad \in \rho(G) \cong G \quad \text{homeomorphism}$$

$$\Longrightarrow \quad \tilde{t}_{ij}(p) = g_i^{-1}(p) \circ t_{ij}(p) \circ g_j(p).$$

$$\begin{cases} t_{ij} : \text{Gauge transformation.} \\ g_i : \text{Gauge D.O.F in } U_i \end{cases}$$

② section : $E \underset{s \in \Gamma(M,F)}{\overset{\pi}{\rightleftarrows}} M$. (smooth).

$\pi \circ s = id_M$.
$\uparrow$
injective.

$$s|_p \leftrightarrow f^{(s)} \in F$$

$$s(p) = u_s^{(p)} \sim (p, f^{(s)}) \in F_p = \pi^{-1}(p)$$

S : smooth choice of elements of fiber.

In general, we have only

local section: $U_i \to S|_{U_i}$

<u>not</u> all E have <u>global</u> section.

(depend on <u>topological properties</u> of E).

③ trivial bundle :

$$t_{ij}(p) \equiv id_F$$

$$\Rightarrow \quad E = \bigcup_p \pi^{-1}(p) \cong U_i \times F$$

$$= M \times F \quad \text{(trivial extension).}$$

As <u>set</u> $E \underset{\text{set}}{\cong} M \times F$. $\quad E \underset{\text{bundle}}{\cong} M \times F \Longleftrightarrow$ trivial.

Theorem

Reconstruction of Fiber bundle : (More practical definition)

$$(M, \{U_i\}; F, G; \{t_{ij}\}).$$

Set of extension : $X = \bigcup_i U_i \times F \cong M \times F$

gluing equivalence :

$$(p, f) \in U_i \times F \sim (q, f') \in U_j \times F$$

$$\iff p = q, \quad f' = t_{ij}(p) f$$

gluing $\quad \phi_i, \phi_j$

$$\Rightarrow \quad E = X/\sim_t, \quad u \in E = [(\pi(u) = p, f)] \in F_p$$

$$\begin{tikzcd} & u \arrow[dl, "\phi_{i,p}"'] \arrow[dr, "\phi_{j,p}"] & \\ f_i \arrow[rr, "t_{ji}(p)"'] & & f_j \end{tikzcd}$$

unique E

Same fiber u in F_p $\longrightarrow$ gauge transformation

① $\pi : u = [(p, f)] \longmapsto p$

② $\phi_i : U_i \times F \longrightarrow \pi^{-1}(U_i)$. <u>diffeomorphism</u>.

$(p, f_i) \longmapsto [(p, f_i)]_t$

given a choice of ϕ_i, $(p, f_i) \xleftrightarrow{1\text{-}1} [(p, f_i)]_t$

<u>Def</u>. Bundle map

$$\begin{tikzcd} E \arrow[r, "\bar{f} \text{ smooth}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

$\forall p$. $\bar{f}$ maps $\pi^{-1}(p) = F_p$

$= \{u \in E \mid \pi(u) = p\}$.

<u>onto</u>: $\pi'^{-1}(q \in M') = F'_q$

this induce. <u>$f(p) = q$</u>

$\Longrightarrow$ Bundle map: base to base.

<u>corresponding</u> fiber to fiber.

Def. Equivalent bundle (isomorphism).

bundle map $\quad \bar{f} : E' \xrightarrow{\sim} E \quad \bar{f}$ induce $f = id_M$.

$\qquad\qquad\qquad \uparrow$

$\qquad\qquad$ diffeomorphism.

(Pull back)

Def. Pull back bundles (Fiber bundle on Fiber product)

$$f^*E \cong \begin{tikzcd} N \times_{(f,\pi)} E \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

$(m, f^{(u)})$.

$$N \times_{(f,\pi)} E = \{ (p, u) \in N \times E \mid \underline{f(p) = \pi(u)} \}$$

$$\pi_1 : (p, u) \longmapsto p \implies \pi_1^{-1}(p) = \{ u \in E \mid \pi(u) = f(p) \}$$
$$\cong F_{\underline{f(p)}} \cong F$$

$\pi_2 : (p, u) \longmapsto u$. $\underline{\text{bundle map}}$

$$F_{f(p)} \xrightarrow{\text{onto}} F_{f(p)}$$

over p over $f(p)$

Local trivialization :

$$\Phi_i(u) = (\underline{f(p)}, f_i^{(u)})$$

$$\Phi_i^*((p, u^*)) = (\underline{p}, f_i^{(u)})$$

$$= f^*[\Phi_i]$$

$$\Longrightarrow$$

$$f_i = t_{ij}(p) f_j$$

$\downarrow$ pull back

$$f^*[t_{ij}(p)] = t^*_{ij}(\underline{p}) = t_{ij}(\underline{f(p)})$$

<u>Theorem</u> . Homotopy axiom

$$\begin{tikzcd}
f^*E \arrow[dd, phantom, "\cong"] & & & E \arrow[d, "\pi"] \\
 & & N \arrow[ull] \arrow[dll] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & M \\
g^*E & & &
\end{tikzcd}$$

$$f \underset{\text{homotopic}}{\sim} g \Longrightarrow f^*E \cong g^*E$$

trivial pull back : $M \xrightarrow{f_0} M$

$$m \longmapsto m_0$$

$$f_0^* E \cong \bigcup \{p_0\} \times F = M \times F$$

then $\forall$ $\pi_1(M) = 0$ contractible to a point.

$\forall$ E over M is trivial ($\cong M \times F = f_0^* E$).

Def. Vector bundle:

$F \in \mathrm{Vect}$, $F \cong \mathbb{K}^k$. $\mathbb{K}$ usually $= \mathbb{R}, \mathbb{C}$ (or $\mathbb{H}$).

$\dim E := \dim F = k$. $\dim M = m$, $\dim E = k+m$.
(locally $\cong \mathbb{R}^m$) (manifold) (locally $\mathbb{R}^m \times \mathbb{K}^k$)

Structure grp: $G = GL(n, \mathbb{K})$

$t_{ij}(p) = \rho(GL(n,\mathbb{K}))$ is a n-dim rep.

Linear map $\xrightarrow{\text{basis}}$ matrix

Example:

(1) Tangent bundle: $F_p \cong T_pM \cong F$ $\quad \dim F = \dim M$

$$E \equiv TM = \bigcup_p T_pM.$$

given coordinate functions: $\varphi_i(p) = x^\mu$
$\varphi_j(p) = y^\mu$.

$u \leftrightarrow (p, V \in T_pM) \leftrightarrow \begin{cases} (p, V^\mu) \\ (p, \tilde{V}^\mu) \end{cases}$

local trivialization :

$$\begin{cases} \phi_i^{-1}(u) = (p, \underset{\in \mathbb{R}^m}{V^\mu}) = (p, (\underset{\in T_pM}{V}, \varphi_i)) \\ \phi_j^{-1}(u) = (p, \tilde{V}^\mu) = (p, (V, \varphi_j)) \end{cases}$$

the $t_{ij}(p) = G^\mu{}_\nu(p) = \left(\frac{\partial x^\mu}{\partial y^\nu}\right)\Big|_p \in GL(m, \mathbb{R})$

changing of the basis/frame of the Fiber

Section = Vector fields

$\Gamma(M, TM) = \mathfrak{X}(M)$

(2) Normal bundle

$M \underset{\text{embed}}{\subset} \mathbb{R}^{m+k}$

$N_pM \in$ Vect normal to T_pM

vanishing inner product in $\mathbb{R}^{m+k}$

$$NM = \bigcup_p N_p M.$$

(3) Cotangent bundle.

$$T^*M = \bigcup_p T_p^* M = \bigcup_p \Omega^1(M)\Big|_p.$$

$\phi \sim$ choice of basis : $\omega \to \omega^\mu.$

$t_{ij} \sim$ changing basis : $G_\mu{}^\nu(p) = \dfrac{\partial x^\nu}{\partial y^\mu}\Big|_p.$

Def. Line bundle

$$F \cong \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C} \ (\text{or } \mathbb{H}\ ).$$

Canonical line bundle. :

$$M = \mathbb{C}P^n = \{ [x] \,/\, v = \alpha x,\ \alpha \in \mathbb{C},\ x \in \mathbb{C}^{n+1} \neq 0 \}$$

Lines through. 0 - x.

$$\underline{\pi^{-1}(p) \equiv [p] \cong \mathbb{C}}$$

Def. Frames

$E \xrightarrow{\pi} M$, $\dim E = k$,

k linearly independent sections $\{e_1(p), \dots, e_k(p)\}$ over U_i

local frame (basis of fiber).

$$F_p \ni V = V^\alpha e_\alpha(p) \longmapsto \{V^\alpha\} \in F \cong \mathbb{K}^k.$$

$$\Phi_i^{-1}(\underset{\underset{(p, V)}{\wr}}{u}) \equiv \Phi_i^{-1}(V) = (p, V^\alpha(p))$$

$$\Longrightarrow \quad \Phi_i(p, (0, \dots, \underset{\alpha}{\underset{\uparrow}{1}}, \dots, 0)) = e_\alpha(p).$$

$$\tilde{e}_\beta(p) = e_\alpha(p) \underbrace{G(p)^\alpha{}_\beta}_{t_{ij}}.$$ basis/frame transformation

$G \in GL(k, \mathbb{K})$.

$$\tilde{V}^\beta = G^{-1}(p)^\beta{}_\alpha V^\alpha.$$

For TM, T^*M, the frame is given by φ coordinate functions.

Def. Dual bundle.

vector bundle $E \xrightarrow{\pi} M$.

dual bundle $E^* \xrightarrow{\pi} M$.

$$F^* = \mathrm{Hom}(F, \mathbb{K}).$$

frame $\{e_\alpha(p)\} \longrightarrow \{\theta^\alpha(p)\}$ dual basis/frame:

$$\langle \theta^\alpha(p), e_\beta(p) \rangle = \delta^\alpha{}_\beta .$$

---

local section = local vector fields:

$$\left\{ \begin{aligned} (s+s')(p) &= s(p) + s'(p) \\ (fs)(p) &= f(p)\, s(p) \end{aligned} \right.$$

global null section: $\phi_i^{-1}(s_0(p)) = (p, 0)$.

Fiber metric: $h_{\mu\nu}(p)$.

$$\Rightarrow \quad (s, s')_p = h_{\mu\nu}\, \overline{s^\mu(p)}\, s'^\nu(p).$$

inner product　　　　by choosing a frame (basis).

Def Product bundle, Whitney sum bundle.

$E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} M'$ be Vector bundles

$$\begin{array}{c} E \times E' \\ \downarrow \pi\times\pi' \\ M \times M' \end{array}$$

$$(\pi\times\pi')^{-1}(p \in M\times M') \quad \text{Vect}$$

$$= F_p \oplus F'_p \cong F \oplus F'$$

$$\pi\times\pi'(u, u') = (p, p').$$

$$\begin{tikzcd} f^*(E\times E') = E\oplus E' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \times E' \arrow[d, "\pi\times\pi'"] \\ M \arrow[r, "f = \mathrm{id}_M \times \mathrm{id}_M"'] & M \times M \end{tikzcd}$$

$$E\oplus E' = f^*(E\times E') = \{ (p, (u, u')) \in M \times (E\times E') \mid$$

$$(p, p) = f(p) = \pi\times\pi'(p, p') = (p, p') \}$$

$$\cong \{ (u, u') \in E \times E' \mid \pi(u) = \pi(u') \}$$

$$(\pi\times\pi')^{-1}(p) = \pi^{-1}(p) \oplus \pi'^{-1}(p) = F_p \oplus F'_p \cong F \oplus F'$$

$$T_{ij}(p) = \begin{pmatrix} t^{E}_{ij}(p) & \\ & t^{E'}_{ij}(p) \end{pmatrix}$$

Def. Tensor product bundles

$$\begin{array}{c} E \otimes E' \\ \downarrow \\ M \end{array} \qquad \pi^{-1}(p) = F_p \otimes F'_p \cong F \otimes F'$$

$$\Longrightarrow \quad \Lambda^r(E) : \quad \pi^{-1}(p) = \mathcal{A}[\otimes F]$$

expanded by $\{ e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_r} = \mathcal{A}[e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r}] \}$

Fiber consists of antisymmetric tensors.

Example:

(1) $\Lambda^r(T^*M)$, $\Omega^r(M) \cong \Gamma(M, \Lambda^r(T^*M))$.

$\uparrow$ space of r-forms fields over M.

$\uparrow$ sections.

Rmk this is most closely related to group cohomology

Def. Principle bundles

$$P(M,G) = (P \xrightarrow{\pi} M, F \cong G).$$

(G-bundle over M.)

Besides left action on F, we can also define right action.
(transition function)

right action induce curve/orbit in $F_p \cong G \cong F$

$$\phi_i^{-1}(u) = (p, g_i), \quad \phi_i^{-1}(u \cdot a) = (p, \underline{g_i a})$$

$u \in \pi^{-1}(p) = \underline{F_p}$

G right act on F_p

or $\quad u \cdot a = \phi_i(p, g_i a) = \phi_i(p, g_i) \cdot a$

Rmk

① t act on left, G act on right, they commute.

$$ua = \phi_j(p, g_j a) = \phi_j(p, t_{ji} g_i a) = \phi_i(p, g_i a)$$

G - right action on F_p is trivialization independent.

② $\pi(u \cdot a) = \pi(u)$. within same Fiber

Since G act on G transitively and freely :

$\pi^{-1}(p) = F_p \cong G \equiv \{ u \cdot a \mid a \in G \}$

③ given section $\{ s_i$ over $U_i \}$. then we have

canonical local trivilization :

$u = s_i(p) g_u$. $\Longrightarrow$ $\phi_i(p, g_u) = u = s_i(p) g_u$.

$\Longrightarrow$ $s_i(p) = \phi_i(p, e)$.

d. $s_i(p) = \phi_i(p, e) = \phi_j(p, t_{ji}(p) e) = \phi_j(p, e) t_{ji}(p)$

$= s_j(p) t_{ji}(p)$.

$\phi_i \leftrightarrow f_i^{(u)} \leftrightarrow s_i$

Example :

(1) H bundle over G/H.

$$0 \to H \to G \xrightarrow{\pi} G/H \to 0$$

(2). monople, instanton . . . .

$\Rightarrow$ (non)trivial bundle $\longleftrightarrow$ t_{ji} $\longleftrightarrow$ $\pi(G)$

Def. Associated bundle.

given principle bundle $P(M, G)$.

then define L-action on $P \times F$

another manifold.

$(u, f) \longmapsto (ug, g^{-1} \cdot f)$. $\Longrightarrow$ transition function $\in G$.

G

the associated bundle is reconstructed as

$$E = P \times F / G = \{ [(u, f)]_{L\text{-}G\,\text{action}} \}$$

left action

denoted : $\left(E = P \times F / G \xrightarrow{\pi_E} M, F, G \right)$

$$\pi_E([u, f]) = \pi_P(u) = p \in M$$

well-defined since $\pi_P(u \cdot g) = \pi_P(u)$.

$$\pi_E^{-1}(p) = \{ [u, f] \in E \mid \pi_P(u) = p \}$$

$$= F_p^{(E)} \cong F$$

(since u, $u \cdot g$ are identified $[u, f]$ correspond to only F).

transition function is the left action of G on F

induced by L-action on $P \times F$

Example

$F \in Vect$

(1) $P(M, G) \longrightarrow P \times_\rho V / G$ is a vector bundle.

vector bundle $\longrightarrow$ principle bundle.

$V,\ t \in GL(k, \mathbb{K}) \rightarrow F = G \equiv GL(k, \mathbb{K}),\ t \in G.$

$\rightarrow$ reconstruction

$P(M, Aut(V))$.

(2). Frame bundle.

TM $\longrightarrow$ Frame bundle. (associated principle bundle).

Vect bundle.

$$LM = \bigcup_{p \in M} L_p M.$$

set of frames.

$u \in LM = \{X_1, \ldots, X_m\}$ Linearly independent

dim M = dim F

$$X_\alpha = \underline{X^\mu{}_\alpha} \frac{\partial}{\partial x^\mu}\Big|_p \quad \text{given } \varphi_i \text{ on } U_i.$$

$\in GL(m, \mathbb{R})$.

$$\phi_i^{-1}(u) = (p, \underline{(X^\mu{}_\alpha)})$$

$\in GL(m, \mathbb{R})$

$$\underline{G = F = GL(m, \mathbb{R})}.$$

$$= \underline{\mathrm{Aut}(T_pM \cong \mathbb{R}^k)}$$

1) right action : $u \cdot a$

$$\underline{Y_\beta = X_\alpha a^\alpha{}_\beta}.$$

2) left action :
(transition function)

$$X^\mu{}_\alpha = \underbrace{\left(\frac{\partial x^\mu}{\partial y^\nu}\right)_p}_{t^L_{ij}(p)} \tilde{X}^\mu{}_\alpha.$$

(locally flat).

$\Rightarrow$ In GR, right action $\Longleftrightarrow$ $\underline{\text{local}}$ Lorentz transf.

left action $\Longleftrightarrow$ general coordinate transf.

(3) Spin bundle

4-dim Lorentz frame bundle $LM_{1,3}$ $\longrightarrow$ associated. spin bundle.

(principle bundle) (Vector bundle)

$\uparrow$

spinor.

Structure group : $O^{+}_{\uparrow}(1,3)$.

$O^{+}_{\uparrow}(1,3)$ act on V_{spinor} as projective rep

($SL(2,\mathbb{C})$).

$\Longrightarrow$

Weyl spinor field : section of

$(W \xrightarrow{\pi} M,\ F \cong \mathbb{C}^2,\ SL(2,\mathbb{C}))$.

Dirac spinor field : section of

$(D \xrightarrow{\pi} M,\ F \cong \mathbb{C}^4,\ SL(2,\mathbb{C}) \oplus \overline{SL(2,\mathbb{C})})$.

$\uparrow$

complex conjugate rep

C, D are associated vector bundle

of $LM_{1,3}$. (principle bundle, $G = O^{+}_{\uparrow}(1,3)$)

Theorem triviality:

(1) $P(M,G)$ is trivial iff it admit global section.

$$\Phi : P(M,G) \to M \times G \quad \text{global diffeomorphism.}$$

$$u = s(p)g \mapsto (p,g).$$

(2) Vector bundle is trivial. $\iff$ $t \in GL(k,\mathbb{K})$ trivial.

$\iff$ associated principle bundle

$P(M, GL(k,\mathbb{K}))$ trivial.

(same t_{ij})

$\iff$ $P(M, GL(k,\mathbb{K}))$ admit global sections.

$$\cong M \times GL(k,\mathbb{K})$$

$\uparrow$

trivial t_{ij}.