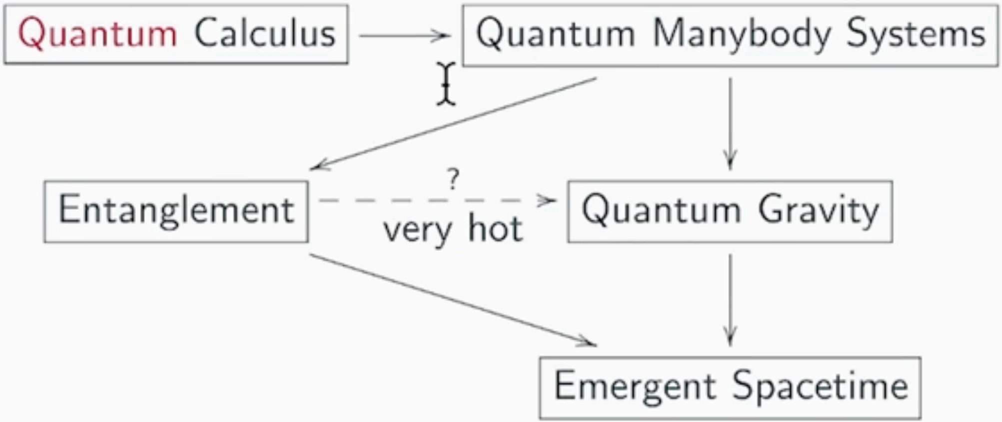
In summary, what I have proposed is that there is a transcendental solution to quantum many body problems, which further leads to quantum gravity, the origin of quantum entanglement and the emergence of spacetime.



"Quantum": discrete/algebraic/categorical

40 years later, among many other things (e.g. dualities, ...), we know that there are many new infinite dimensional mathematical structures, which are emergent only in the infinite size limit (or thermodynamic limit):

1. effective field theories: conformal field theories, topological field theories, etc.
2. category theory, E_n-algebras or higher algebras (or higher symmetries).

The term emergent means that they are not really a limit of any finite things. It is a transcendental structure only exists in the infinite dimensional worlds.

For example,

1. A 2d CFT has a substructure of a so-called chiral algebra, which is called vertex operator algebra (VOA) in mathematics. It is an infinite dimensional structure that does not exists in finite dimensions. For example, it has infinite number of multiplication rules, which are parameterized by the moduli space of Riemann surfaces.

2. It is not a limit of any finite dimensional mathematical structures. (Finite dimensional 2d CFTs are all trivial.)

Geometry: Calabi-Yau manifolds, Mirror Symmetry, Gromov-Witten theory, elliptic cohomology, Fukaya categories, Donaldson-Thomas Invariants, non-commutative geometry, derived algebraic geometry, ...

Topology: Jones polynomial, Donaldson theory, Chern-Simons theory, Seiberg-Witten theory, Khovanov homology, topological field theories, factorization homology, ...

Algebra: chiral algebras, quantum groups, vertex operator algebras, modular tensor categories, subfactors, fusion categories, algebras in a tensor category, A-infinity (L-infinity, G-infinity, ...) algebras, geometric Langlands correspondence, ...

Probability: Stochastic Loewner evolution

For example, the first mathematically (and independently) discovered quantum field field is 2d monster moonshine CFT:

1. Its underlying mathematical structure is a VOA (no one have seen it before!).
2. Its partition function is the famous j-function, which is the generator of all modular functions.
3. Its automorphism group is the famous Monster Group G, the largest sporadic simple group!

$$\begin{aligned}|G| = {} & 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \\ & \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \\ \approx {} & 8 \cdot 10^{53}\end{aligned}$$

This is a mathematical miracle!

The modern lesson in mathematics is that

Infinitely more *is fundamentally different from* **finitely more**.

I think that infinity holds the key to understand

1. non-locality (quantum entanglement);
2. holographic phenomena: one part hold the information of the whole. For example, the boundary of a topological order determines the bulk uniquely.

The mathematical reason for a part encoding the information of the whole system is that they are both infinite dimensional. Example: $\{\text{even integers}\} \simeq \{\text{all integers}\}$.

Why topological order?

Let us focus on the so-called quantum phases where quantum features are dominant.

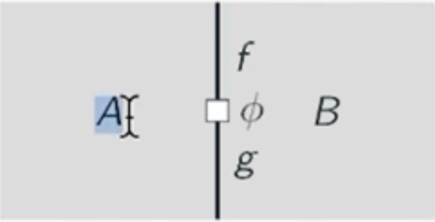
1. A quantum matter is a state of matter at zero temperature.
2. A quantum phase is the universal class of quantum matters. Quantum phases are distinguished from each other by quantum phase transitions.
3. There are two types of quantum phases: gapped and gapless. Gapped quantum phases are simpler than the gapless ones. We are interested in a nice family (finite-type) of gapped quantum phases called gapped quantum liquids:
   - 3.1 A gapped quantum liquid without symmetry is called a topological order.
   - 3.2 Gapped quantum liquids with symmetries include gapped spontaneous symmetry breaking orders, symmetry enriched topological (SET) orders and symmetry protected trivial (SPT) orders (including topological insulators).

Conclusion: topological orders are the simplest quantum phases!

The study of gapped quantum phases, however, provides us a chance to study the notation of a quantum phase from the first principle. Indeed, it has already motivated many attempts to define the notion of a gapped quantum liquid precisely from both the microscopic perspective and the macroscopic perspective.

1. Microscopic definition: The "gapped" condition implies that only the ground state matters. This leads to the definition of a gapped quantum liquid as a equivalence class of (ground) states with the equivalence relation defined by local unitary transformations and the stacking of the product states Chen-Gu-Wen:10, Zeng-Wen:15.
   A product state: $\otimes_i |i\rangle \in \mathcal{H}_{tot} = \otimes_i \mathcal{H}_i$, where $|i\rangle \in \mathcal{H}_i$.
2. Macroscopic definition: the collection of all observables in the long wave length limit form a higher category Kitaev:06, K.-Wen:14, K.-Wen-Zheng:15, Johnson-Freyd:20, K.-Lan-Wen-Zhang-Zheng:20.

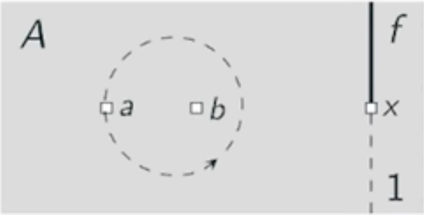
A topological order is gapped. All correlation functions exponentially decay. It seems that there is no "observable" in the long wave length limit. It turns out that a topological order allow non-trivial topological defects.



$$\rightsquigarrow \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \phi"] & B \end{tikzcd} = \hom(A, B).$$

All these defects form a higher category of nd topological orders $\mathbf{TO}_n$:
$\mathbf{TO}_n =$ a set of 0-morphisms or objects $A, B, C, \cdots$ (labels of topological orders) $+$ 1-morphisms between 0-morphisms $f, g, \cdots$ $+$ 2-morphisms between 1-morphisms ϕ, so on and so forth.

For a 2d (spatial dimension) topological order A:



$$\rightsquigarrow \quad \begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow \phi"] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow \psi"] & f \end{tikzcd}$$

1 labels the trivial 1d wall; ϕ, ψ are defects in time axis (i.e. instantons).

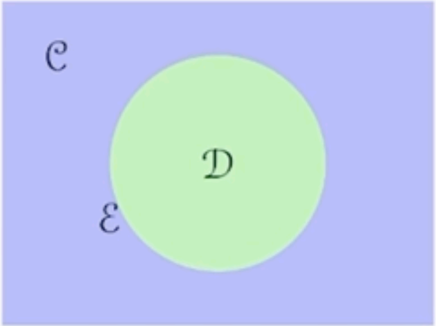
1. $a \xrightarrow{\phi} b$ for a 1-category + fusion + braiding = unitary modular tensor 1-category (UMTC) $\mathcal{M}$. Kitaev:06, K.-Zhang: An invitation to topological orders and category theory, in preparation
2. 1d domain walls $1, f, g, h, \cdots$ + 0d walls $x, y, \cdots$ + instantons = unitary fusion 2-category $\Sigma\mathcal{M}$, which is the delooping of $\mathcal{M}$. Douglas-Reutter:18, Johnson-Freyd:20, K.-Lan-Wen-Zhang-Zheng:20. Note that $\mathcal{M} = \hom_{\Sigma\mathcal{M}}(1, 1)$.

Adding a missing data: chiral central charge c, we obtain $A = (\mathcal{M}, c)$ or $(\Sigma\mathcal{M}, c)$.

If the set of "observables" is complete, then miracles happen.

1. **Boundary-bulk relation**: "bulk = the center of the boundary": K.-Wen-Zheng:15,17
   $Z_2(\mathcal{M})$ and $Z_1(\Sigma\mathcal{M})$ are trivial. Johnson-Freyd:20
2. **Condensation theory**: a boson condensation from $(\mathcal{C}, c)$ to $(\mathcal{D}, c)$ (with a gapped domain wall $\mathcal{E}$) is determined by a condensable algebra A in $\mathcal{C}$. K:13

- $\mathcal{D}$ = the category $\mathcal{C}_A^{loc}$ of local A-modules in $\mathcal{C}$;
  the trivial anyon $\mathbf{1} \in \mathcal{D}$ is $A \in \mathcal{C}$; $\otimes_{\mathcal{D}} = \otimes_A$.
- $\mathcal{E}$ = the category of confined particles
  = the category $\mathcal{C}_A$ of A-modules in $\mathcal{C}$.
- anyons in $\mathcal{C}$ move onto $\mathcal{E}$ according to
  $x \mapsto x \otimes A$;
- Boundary-bulk relation: $Z_1(\mathcal{E}) = \mathcal{C} \boxtimes \overline{\mathcal{D}}$.



Category theory as a new language

The mathematical theory of symmetries (in the the classical sense) is that of groups. The new exotic phases of matter challenge us to find radically new mathematical language and structures to characterize gapped quantum phases. The enormous effort has been made to meet this challenge. As a result, up to invertible gapped quantum phases, the mathematical structure that characterizes a gapped quantum phase is found to be a higher category, which describes topological excitations or defects in the phase. Kitaev:06, Levin-Wen:05, Kitaev-K.:12, K.-Wen:14, Barkeshli-Bonderson-Cheng-Wang:19, K.-Wen-Zheng:15, Johnson-Freyd:20, K.-Lan-Wen-Zhang-Zheng:20

Moreover, it unifies "continuous" with "discrete", "algebraic" with "analytic or geometric" and "finite" with "infinite": 1+1 vs. open-closed duality in string theory:

$$
\begin{tikzcd}
1 \arrow[r] \arrow[dr] & 1+1 \arrow[d, "\exists!"] & 1 \arrow[l] \arrow[dl] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Z(A)\boxtimes A \arrow[ddr] & \\
& X\boxtimes A \arrow[u, "\exists!"] \arrow[dr] & \\
A \arrow[uur] \arrow[ur] \arrow[rr] & & A
\end{tikzcd}
$$

1 as a limit of a sequence of numbers:

$$
\begin{tikzcd}
0.9 \arrow[r] \arrow[dr] \arrow[drr] & 0.99 \arrow[r] \arrow[d] \arrow[dr] & 0.999 \arrow[r] \arrow[dl] \arrow[d] & 0.9999 \arrow[r] \arrow[dll] \arrow[dl] & \cdots \\
& 1 \arrow[r] & x & &
\end{tikzcd}
$$

Category theory simplifies all mathematical notions into only two types of notions: "Limits" and "Colimits", e.g. $1+1$ is a colimit and 1×1 is a limit.

There has been a new wave in mathematics of replacing the set theory by the category theory as the new foundation of mathematics since the rise of the new paradigm of algebraic geometry developed in 1960's by Alexandre Grothendieck and his schools.

Therefore, the real question is: can the category theory become the language of a new calculus for many body quantum physics?