What is anomalous?

- The continuum field theory $H = -iv \int dx \psi^\dagger \partial_x \psi$ has a $U_{em}(1)$ symmetry:

$$U_{\mathrm{em}}(1): \ \psi \to e^{i\alpha}\psi, \quad \psi^\dagger \to e^{-i\alpha}\psi^\dagger$$

According to Noether's theorem, this leads to particle number conservation

$$\partial_\mu j^\mu = 0$$

where $j^0 = \psi^\dagger \psi$ and $j^1 = -iv[\psi^\dagger \partial_x \psi - (\partial_x \psi^\dagger)\psi]$.

- Inconsistency occurs when we turn on electromagnetic field $-i\partial_\mu \to -i\partial_\mu - A_\mu$ $(e = \hbar = c = 1)$:

$$\partial_\mu j^\mu = -\frac{1}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength.

Spectrum flow

More specifically, consider adiabatically increasing the gauge field

$$A_x = -\frac{2\pi}{L}\frac{t}{T}, \quad t \in [0, T]$$

The adiabatic energy spectrum becomes

$$E = vk - A_x = vk + \frac{2\pi}{L}\frac{t}{T}$$

Note that $k = 2\pi m/L$. At the end of the adiabatic process at $t = T$, the total charge increases by $\Delta N = 1$. That is, particle number is not conserved!

$$\int A_x \, dx = -\frac{2\pi}{L} \cdot L \cdot \frac{t}{T} = -2\pi \cdot \frac{t}{T} = -\frac{h}{e} \cdot \frac{t}{T}$$

Remarks

1. This famous **fermion doubling** problem was originally studied by Nielsen and Ninomiya [Nucl. Phys. B 193, 173; Phys. Lett. B 105, 219 (1981)]. They showed that the fermion doubling occurs in all dimensions for chiral fermions (under the assumption of no interaction and with lattice translation).

2. Our analysis is based on $U(1)$ charge conservation symmetry. However, 1D chiral fermion can never be regularized on lattice, regardless of symmetry, as long as it is a *time-independent* local Hamiltonian system. It actually carries **gravitational anomaly**. Experimentally speaking, it has non-vanishing thermal Hall conductance

$$\kappa_{xy} = \kappa \frac{\pi^2 \Delta T}{3h}$$

In conformal field theory, $\kappa = c_-$, the chiral central charge $c_- = c_r - c_l$.

Axial anomaly

Again inconsistency occurs when we turn on electromagnetic gauge field associated with $U_{em}(1)$. One can show that

$$\begin{aligned} \partial_\mu j_L^\mu &= \frac{1}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu} \\ \partial_\mu j_R^\mu &= -\frac{1}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \partial_\mu j_{em}^\mu &= \partial_\mu(j_L^\mu + j_R^\mu) = 0 \\ \partial_\mu j_a^\mu &= \partial_\mu(j_L^\mu - j_R^\mu) = \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} \end{aligned}$$

That is, turning on $U_{em}(1)$ gauge field breaks the conservation law of $U_a(1)$ symmetry.

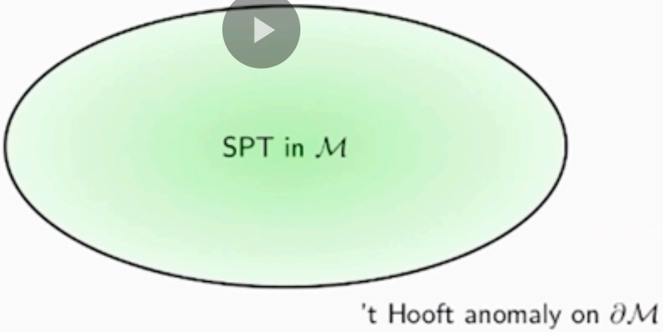
Remarks:

1. If we think of the left-mover as spin $\uparrow$ and right-mover as spin $\downarrow$, this 1D theory is actually the edge of 2D quantum spin Hall effect (a.k.a. topological insulator with spin conservation). So, 2D topological insulator can cure the axial anomaly.
2. Now there is no gravitational anomaly. It is an anomaly purely due to symmetry.
3. The axial anomaly is a mixed anomaly between the two $U(1)$ symmetries, while each $U(1)$ has its own IQHE-like anomaly.

't Hooft anomaly

Definition: global symmetry in a quantum field theory is said to carry 't Hooft anomaly if there exists obstruction to "gauge the symmetry". That is, turning on the corresponding gauge field will lead to some kind of inconsistency.

"Theorem": 't Hooft anomalies of G in $(n-1)$ dimensions have a one-to-one correspondence to n dimensional SPT phases of G.



Back to very original definition of quantum anomaly in QFT

In path integral formulation of quantum field theory,

$$\mathcal{Z} = \int D\Phi \; e^{iS[\Phi]} \xrightarrow{\text{turn on } A} \mathcal{Z}[A] = \int D\Phi \; e^{iS[\Phi, A]}$$

A symmetry transformation $\Phi(x) \to \Phi'(x)$ is anomalous if $S[\Phi'] = S[\Phi]$ but $D\Phi' \neq D\Phi$, i.e. a **classical** symmetry is violated **quantum mechanically**.[1]

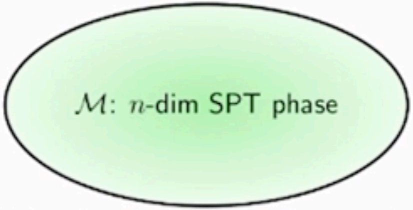
If we turn on gauge field, the functional $\mathcal{Z}[A]$ will be cancelled by an SPT bulk

$$\mathcal{Z}_{\text{bulk SPT}}[A] \cdot \mathcal{Z}[A] = 1$$

which is the so-called **anomaly inflow**.

[1] See textbook by e.g., Peskin and Schroeder

$(n-1)$-dim 't Hooft anomalies $=$ n-dim SPT phases



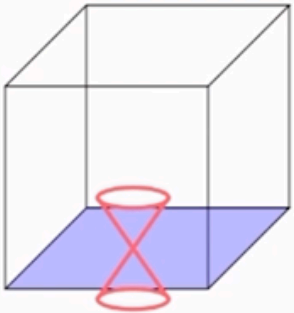
$\partial\mathcal{M}$: QFT has a t' Hooft anomaly

$n-1$	symmetry	SPT/Anomaly	comment
1	$\mathcal{T}^2=-1$ fermion	$\mathbb{Z}_2$	2D TSC.
1	$U_{em}(1)\times U(1)$	$\mathbb{Z}^3$ $(\mathbb{Z}^4)$	IQHE, axial anomaly
2	$\mathbb{Z}_2^T$ boson	$\mathbb{Z}_2^2$	3D bosonic TSC
2	$\mathbb{Z}_N\times\mathbb{Z}_N$ boson	$\mathbb{Z}_N^2$	
2	$U(1)\rtimes\mathbb{Z}_2^T$ boson	$\mathbb{Z}_3^2$	3D bosonic TI
2	$\mathcal{T}^2=-1$ fermion	$\mathbb{Z}_{16}$	3D fermionic TSC
2	$\mathcal{M}^2=1$ fermion	$\mathbb{Z}_{16}$	3D fermionic TCSC
2	$U(1)\rtimes T, T^2=-1$ fermion	$\mathbb{Z}_2^3$	3D fermionic TSC

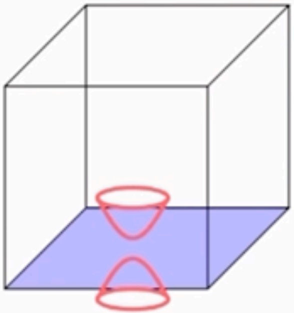
Remarks:

1. identifying 't Hooft anomaly in $(n-1)$ dimensional QFT is equivalent to identifying its n dimensional bulk, i.e., establishing a bulk-boundary correspondence.
2. Question: How to expose anomaly for $\mathcal{T}$ and $\mathcal{M}$, as there is no corresponding gauge field?

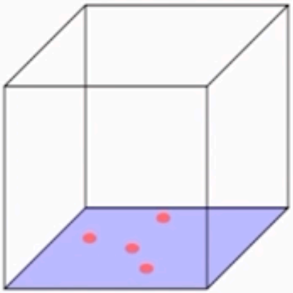
Possible surface QFTs of 3D SPT



Gapless (CFT)



Symmetry breaking

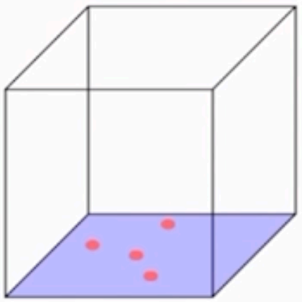


Topological order (TQFT)

All surfaces respect the symmetries of the bulk (i.e., not explicitly broken). Different surfaces may share the same bulk SPT.

To establish bulk-boundary correspondence (i.e., compute anomaly), we need

1. to characterize the surface QFT, in particular its symmetry properties
2. to characterize the bulk SPT phase
3. to make the connection between boundary and bulk



In this lecture, I will focus on the case the the surface is a TQFT equipped with certain symmetries, i.e., **symmetry-enriched topological** (SET) phases.

Fractional charge: field-theoretical description

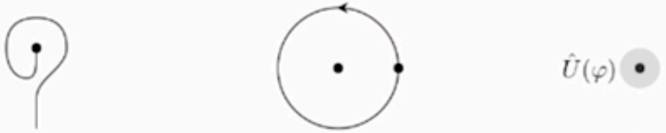
Recall the Chern-Simon theory for Abelian FQHE

$$\mathcal{L} = \frac{m}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{\tau}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda$$

where a is dynamical $U(1)$ gauge field and A is an external probing $U(1)$ field. The coupling constant τ between a and A specifies **symmetry properties of the theory**. Anyons, labeled by l with $l = 0, 1, \ldots, (m-1)$, have the following properties

$$\theta_l = \frac{\pi}{m}l^2, \quad \theta_{l,l'} = \frac{2\pi}{m}ll', \quad q_l = \frac{\tau}{m}l$$

Note that θ_l and $\theta_{l,l'}$ are **intrinsic** topological properties and q_l is $U(1)$ **symmetry** property. Different values of τ denote different $U(1)$ Symmetry-Enriched Topological (SET) phases.[2]



$e^{i\theta_l}$ $\quad$ $e^{i\theta_{l,l'}}$ $\quad$ $e^{iq_l\varphi}$

[2] It requires $m = \tau$ mod 2 if m is odd. Bosonic and fermionic FQHEs must have m even and odd, respectively.

More generally, Abelian topological orders can be described by the K-matrix theory:

$$\mathcal{L} = \frac{\mathcal{K}_{IJ}}{4\pi}\epsilon^{\mu\nu\lambda} a^I_\mu \partial_\nu a^J_\lambda + \frac{\tau_I}{2\pi}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a^I_\lambda$$

where $\mathcal{K}$ is a non-singular integer matrix, and τ is an integer vector. Number of anyons equals $\det(\mathcal{K})$. Anyons, labeled by integer vector l, have the following properties

$$\theta_l = \pi l^T \mathcal{K}^{-1} l, \quad \theta_{l,l'} = 2\pi l^T \mathcal{K}^{-1} l', \quad q_l = \tau^T \mathcal{K}^{-1} l$$

Any vector of the form $\mathcal{K}\Lambda$ is a trivial (local) excitation.

Example: Toric-code topological order

$$\mathcal{K} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Fractional charge of anyons $q_e = \tau_2/2, \quad q_m = \tau_1/2, \quad q_\epsilon = (\tau_1 + \tau_2)/2$. Depending on τ, we can have four SET phases of toric-code topological order with $U(1)$ symmetry.

Fractional charge: algebraic description

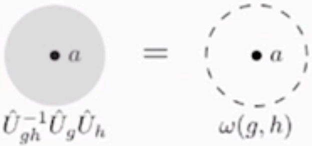
We know there are purely algebraic description of topological properties (Tensor Category), so are symmetry properties. Consider a symmetry group G and its **local** action on an anyon. Let $|a,\mu\rangle$ be the local quantum state around anyon a, where $\mu = 1,\dots,D$ with D equal to the local degeneracy. Then, local symmetry action is

$$\hat{U}(g)|a,\mu\rangle = \sum_{\nu}(U_g^a)_{\nu\mu}|a,\nu\rangle$$

where $g \in G$. The matrices $\{U_g^a\}_{g\in G}$ form a **projective** representation of G:

$$U_g^a U_h^a = e^{i\alpha_a(g,h)} U_{gh}^a \tag{1}$$

where $\alpha_a(g,h) = \theta_{a,\omega(g,h)}$ with $\omega(g,h)$ being anyon [we assume topological order is Abelian for simplicity]. Physically, one may imagine



Fractional charge: algebraic description

Associativity of group multiplication leads to

$$e^{i\alpha_a(g,h)}e^{i\alpha_a(gh,k)} = e^{i\alpha_a(h,k)}e^{i\alpha_a(g,hk)}$$

which further leads to the **2-cocycle condition**:

$$\omega(g,h)\omega(gh,k) = \omega(h,k)\omega(g,hk)$$

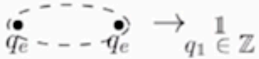
where multiplication of ω corresponds to anyon fusion. The anyon $\omega(g,h)$ is ambiguous up to redefinition $U_g^a \to U_g^a e^{i\beta_a(g)}$, which leads to

$$\omega(g,h) \to \omega(g,h)\frac{\nu(gh)}{\nu(g)\nu(h)}$$

This is a co-boundary transformation. Hence, fractional charge, or more generally **symmetry fractionalization** is classified by the second cohomonogy group $H^2(G,A)$.

Examples: Toric code $\{\mathbb{1}, e, m, \epsilon\}$, $A = \mathbb{Z}_2 \times \mathbb{Z}_2$

- $G = U(1)$: $H^2(U(1), A) = \mathbb{Z}_2^2$, four distinct SETs, characterized by $q_e, q_m = 0, 1/2$. Physical argument on why q_e, q_m must be 0 or $1/2$:



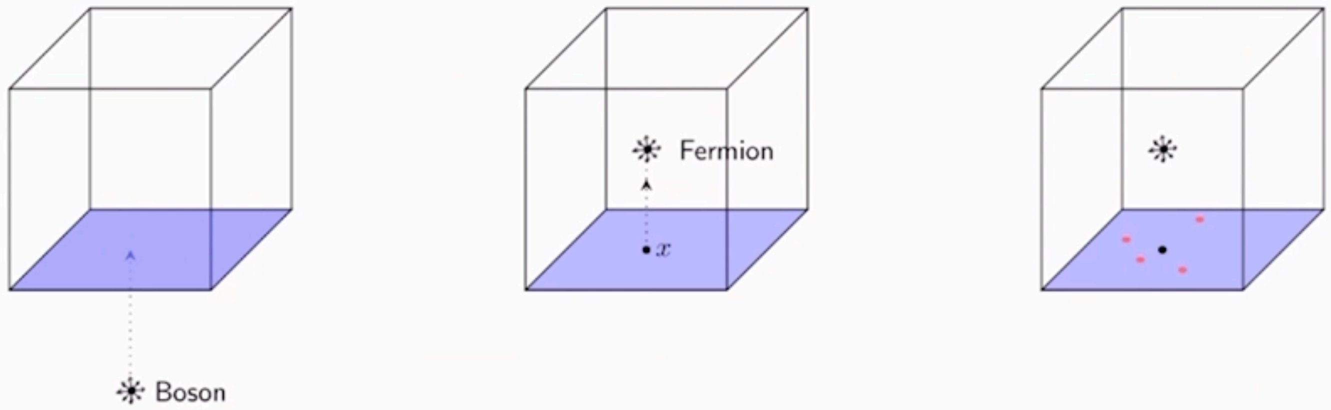
- $G = \mathbb{Z}_2$: the same as $U(1)$. Note that the continuum field theoretical description will be a bit unnatural to deal with discrete gauge theory.

- $G = \mathbb{Z}_2^{\mathcal{T}}$: $H^2(\mathbb{Z}_2^{\mathcal{T}}, A) = \mathbb{Z}_2^2$. The four time-reversal SETs correspond to whether e, m carry Kramers degeneracy, i.e,

$$U_{\mathcal{T}}^{e} = K \text{ or } i\sigma_y K, \quad U_{\mathcal{T}}^{m} = K \text{ or } i\sigma_y K \tag{2}$$

where K is complex conjugation. The 2D representation is projective, $U_{\mathcal{T}} U_{\mathcal{T}} = -1$. Hence, Kramers degeneracy are usually referred to as "$\mathcal{T}_e^2 = \pm 1$", "$\mathcal{T}_m^2 = \pm 1$".

Monopole tunnelling event



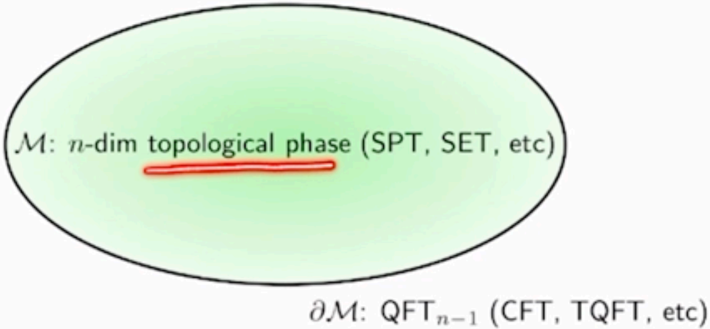
A gapped surface must host a topological order, with a special fermion x.

General picture

- Bulk-boundary correspondence:

$$\text{quantum field theory in } \partial\mathcal{M} \xrightarrow{\text{many-to-one}} \text{topological phase in } \mathcal{M}$$

- The boundary QFT carries a quantum anomaly if the bulk is topologically non-trivial. Such QFT cannot be regularized (e.g., on a lattice) in $(n-1)$ dimension (with or without symmetries).



- Depending on details, we may have **'t Hooft anomaly**, **gravitational anomaly**, **gauge-gravitational mixed anomaly**, **invertible/non-invertible anomaly**, etc.