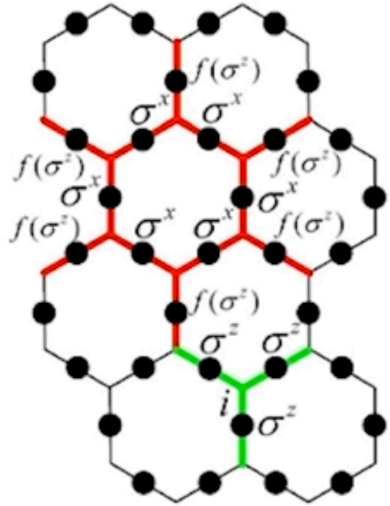
# An simple example of topological order beyond toric code: the double semion model

$$H_{\text{dsemion}} = -U \sum_v \prod_{i \in v} \sigma_i^z - \sum_p \left( \prod_{i \in p} \sigma_i^x \prod_{\text{legs of} p} i^{\frac{1+\sigma_l^z}{2}} \right)$$




$$|\Psi_{\text{dsemion}}\rangle = \sum_{X \text{closed}} (-)^{n(X)} |X\rangle$$

$$f(x) = \frac{1+x}{2}$$

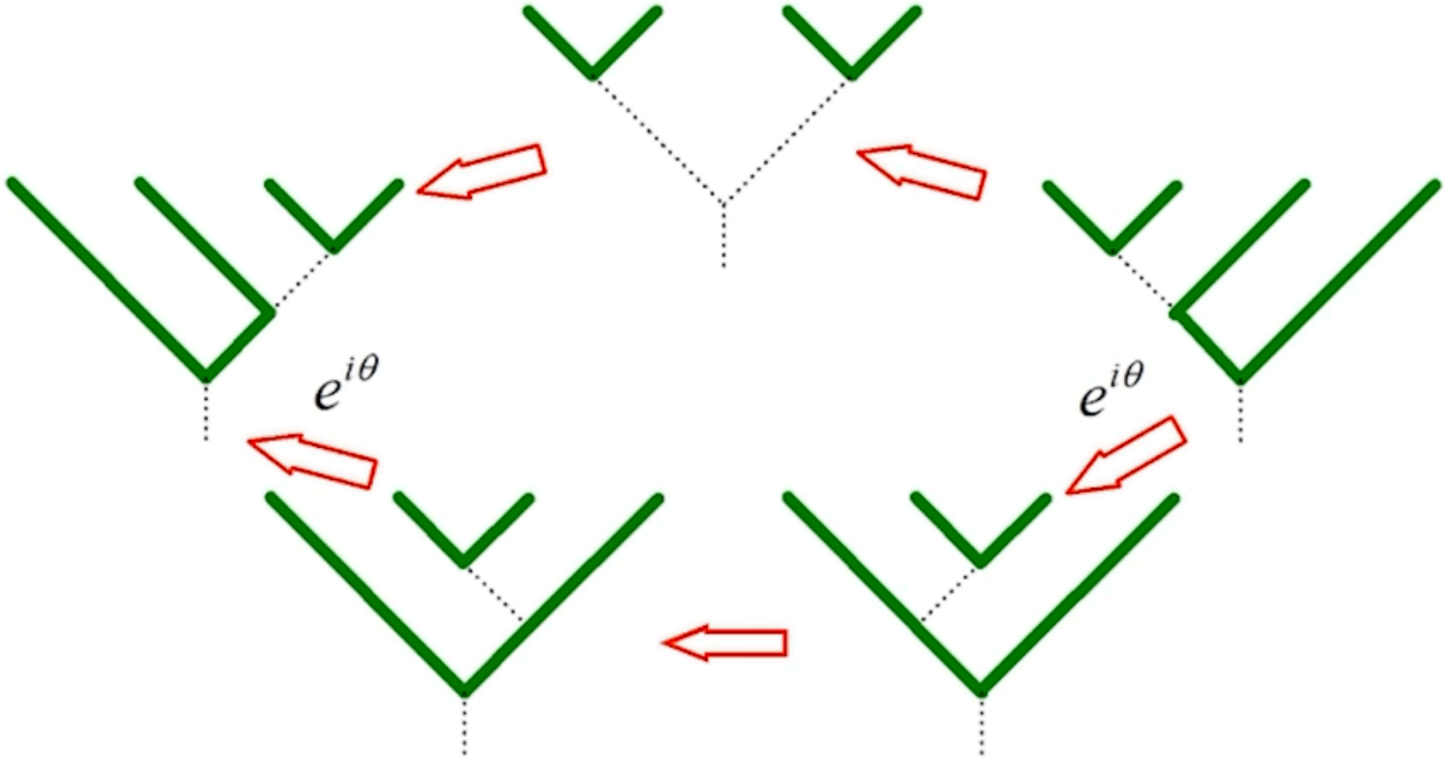
**Quasi-particle in double semion model:**

**1, s, $\underline{s}$, b=s$\underline{s}$**

**Local deformation rule of $Z_2$ string**

But why not?

# Topologically consistent condition for fixed point wavefunction



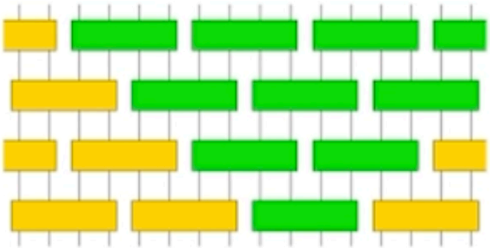

$$(e^{i\theta})^2 = 1 \Rightarrow e^{i\theta} = \pm 1$$

# Finite depth local unitary transformation

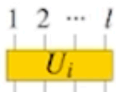
- Two gapped ground states belong to the same phase iff they are related by a local unitary evolution.

- In quantum information theory, it is known that finite time unitary evolution with local Hamiltonian can be simulated with constant depth quantum circuit and vice-verse.

- Thus, two states describe the same topological phase iff they are connected by finite depth local unitary(LU) transformation.



(a)




(b)

$$|\Phi(1)\rangle \sim |\Phi(0)\rangle \text{ iff } |\Phi(1)\rangle = U^M_{circ}|\Phi(0)\rangle$$

$$U_{pwl} = \prod_i U_i$$

$$U^M_{circ} = U^{(1)}_{pwl} U^{(2)}_{pwl} \cdots U^{(M)}_{pwl}$$

- $U_i$ is l-local unitary operator

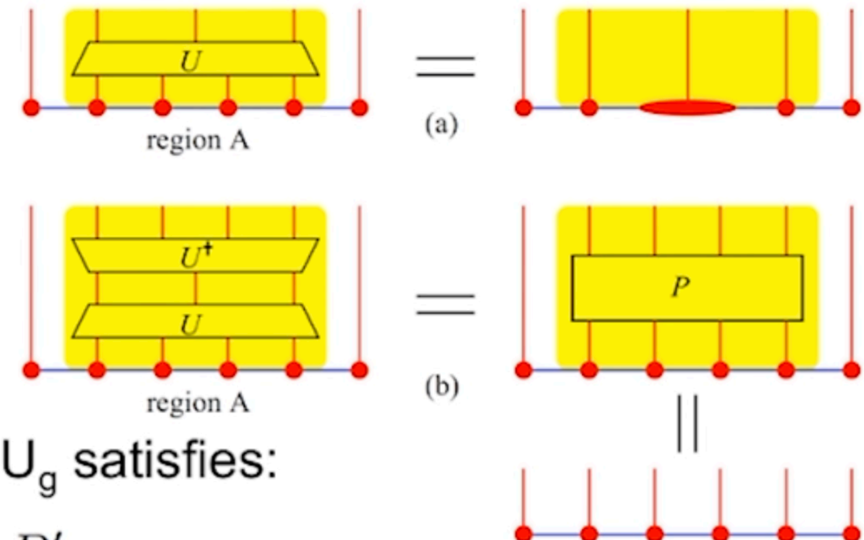
# Generalized local unitary transformation

- Let us define generalized local unitary(GLU) transformation $U_g$ as the projection from the full Hilbert space to the support space. So up to some unitary transformations, $U_g$ is a hermitian projection operator:

$$U_g = U_1 P_g U_2$$

$$P_g^2 = P_g, \quad P_g^\dagger = P_g$$

$$U_1^\dagger U_1 = 1, \quad U_2^\dagger U_2 = 1.$$




- It is easy to verify that $U_g$ satisfies:

$$U_g^\dagger U_g = P \qquad U_g U_g^\dagger = P'$$

**Topological order as equivalent class of finite depth (generalized) local unitary transformation!**

# Two basic classes of intrinsic topological phases in 2D

## Chiral topological phases

- Topological phases with protected edge modes, e.g., all FQHE states

## Non-chiral topological phases

- Topological phases with gapped boundary, e.g., the toric code model and double semion model.

- Starting from a chiral topological phases, it is always possible to stacking another layer of topological phase with opposite chirality to make the whole system to be a non-chiral topological phase.

- It is easy to construct exact solvable lattice model to realize all non-chiral topological phases.

# Fixed point wavefunction for 2D non-chiral topological order for bosonic systems:

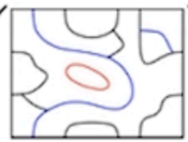
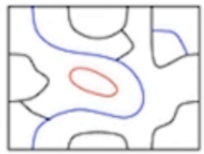
## A trivalent graph with a branching structure


- Physical degree of freedoms for a trivalent graph

$$i = 0, \ldots, N \qquad \alpha = 1, \ldots, N_k^{ij}$$

The branching structure is a set of arrows on edges of a trivalent graph, such that there is no vertex with three in going(out going) arrows.

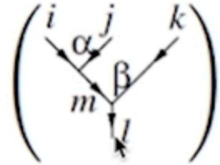
## A fixed point wavefunction

$$|\psi_{\text{fix}}\rangle = \sum_{\text{all conf.}} \psi_{\text{fix}} \Bigg($$

$$\Bigg) \Bigg|$$

$$\Bigg\rangle$$

A fixed point wavefunction can be defined on arbitrary triangulation with a branching structure

# Some basic assumptions for the fixed point wavefunction:

- Wavefunction on a local patch:

$$\phi_{ijkl;\Gamma}(\alpha,\beta,m) = \psi_{\text{fix}}$$ 


- The support space $D_k^{ij}$ surround a vertex in general satisfies:

$$D_k^{ij} \leqslant N_k^{ij}$$

- We assume a saturate condition(this additional condition might limit our solution for topological phase in general, but for bosonic systems we can always impose this condition):

$$D_k^{ij} = N_k^{ij}$$

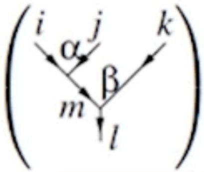
- The support space $D_l^{ijk}$ surround two vertices in satisfies:

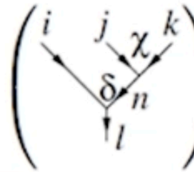
$$D_l^{ijk} = N_l^{ijk} \equiv \sum_{m=0}^{N} N_m^{ij} N_l^{mk}$$

# First type of wavefunction renormalization: The F-move

- Local deformation for a small patch(wavefunction can only be defined up to a overall U(1) phase):

$$\phi_{ijkl;\Gamma}(\alpha,\beta,m) \simeq \sum_{n=0}^{N} \sum_{\chi=1}^{N_n^{jk}} \sum_{\delta=1}^{N_l^{in}} F^{ijm,\alpha\beta}_{kln,\chi\delta}\, \tilde{\phi}_{ijkl;\Gamma}(\chi,\delta,n)$$

$\psi_{\text{fix}}$ 

$\simeq \sum_{n\chi\delta} F^{ijm,\alpha\beta}_{kln,\chi\delta}\, \psi_{\text{fix}}$ 

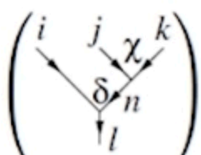

**Saturate condition:**

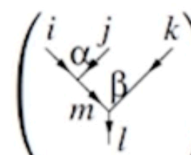
$$\boxed{\sum_{m=0}^{N} N^{ij}_m N^{mk}_l = \sum_{n=0}^{N} N^{jk}_n N^{in}_l}$$

**Unitary condition:**

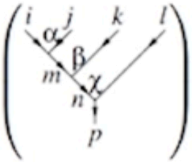
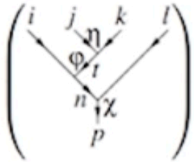
$$\boxed{\sum_{n\chi\delta} F^{ijm',\alpha'\beta'}_{kln,\chi\delta} \left(F^{ijm,\alpha\beta}_{kln,\chi\delta}\right)^* = \delta_{m,m'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}}$$

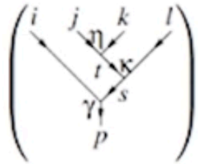
**Inverse transformation:**

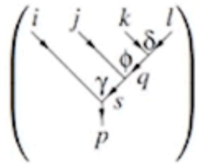
$\Psi_{\text{fix}}$ 

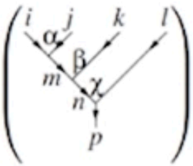
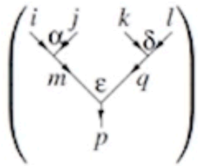
$\simeq \sum_{m\alpha\beta} \left(F^{ijm,\alpha\beta}_{kln,\chi\delta}\right)^* \Psi_{\text{fix}}$ 

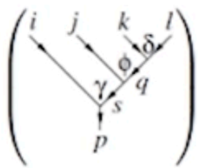
# Coherent condition(the Pentagon equation):

$\psi_{\text{fix}}$  $\simeq \sum_{t\eta\varphi} F^{ijm,\alpha\beta}_{knt,\eta\varphi}\,\psi_{\text{fix}}$ 

$\simeq \sum_{t\eta\varphi;s\kappa\gamma} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma}\,\psi_{\text{fix}}$ 

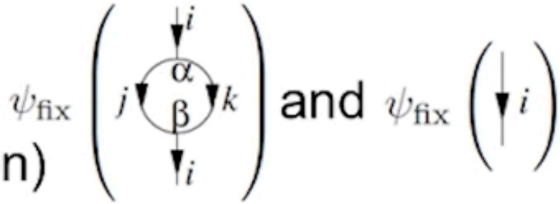
$\simeq \sum_{t\eta\kappa;\varphi;s\kappa\gamma;q\delta\phi} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi}\,\psi_{\text{fix}}$  .

$\psi_{\text{fix}}$  $\simeq \sum_{q\delta\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon}\,\psi_{\text{fix}}$ 

$\simeq \sum_{q\delta\epsilon;s\phi\gamma} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}\,\psi_{\text{fix}}$ 

$$\boxed{\sum_{t}\sum_{\eta,\varphi,\kappa} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi} \simeq \sum_{\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}}$$

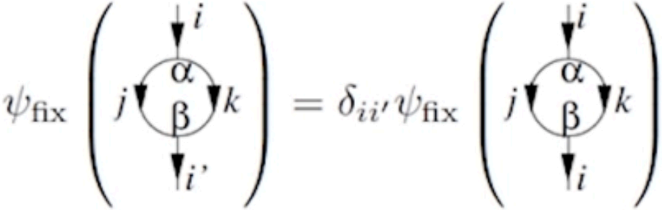
# Second type wavefunction renormalization: The O-move

- At long wave length, the wavefunction


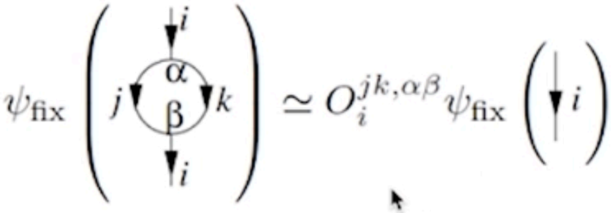
make no difference (up to normalization)

- Therefore, the support space dimension is 1 for i=i' and 0 otherwise

$$D_{ii'} = \delta_{ii'}$$




$$D_{ii} = 1$$

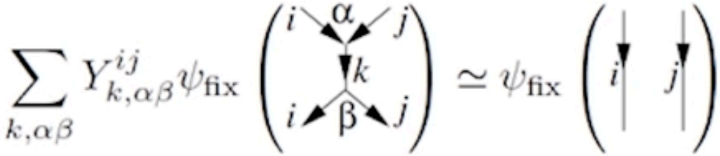



**Generalized unitary condition:**

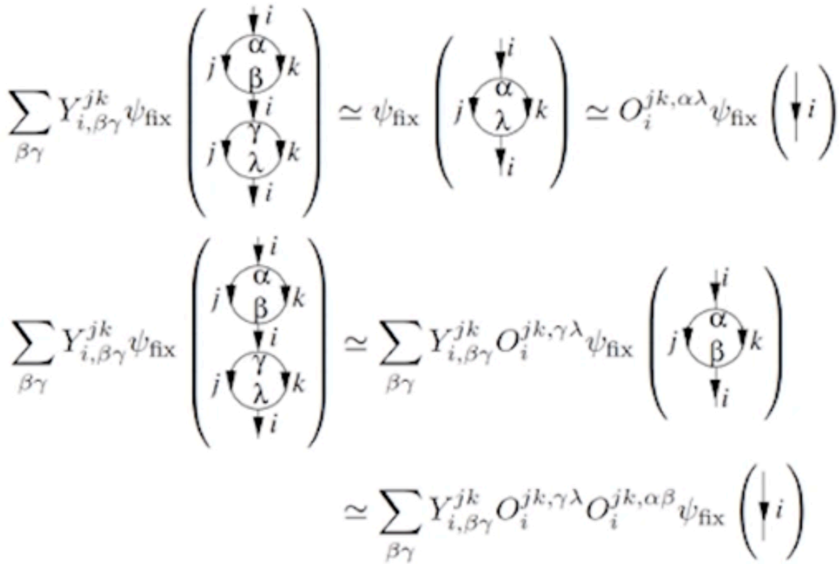
$$\boxed{\sum_{k,j} \sum_{\alpha\beta} O_i^{jk,\alpha\beta} (O_i^{jk,\alpha\beta})^* = 1}$$

# Third type of wavefunction renormalization: The Y move

- We also need a move to increase the number of vertices




**A relation between O and Y moves:**




$$\boxed{O^{jk,\alpha\lambda}_{i} \simeq \sum_{\beta\gamma} Y^{jk}_{i,\beta\gamma} O^{jk,\gamma\lambda}_{i} O^{jk,\alpha\beta}_{i}}$$

# A gauge choice:

$$\Psi_{\text{fix}}\begin{pmatrix} i & \alpha & j \\ & k & \end{pmatrix} \to \sum_{\beta} f^{ij,\alpha}_{k,\beta}\Psi_{\text{fix}}\begin{pmatrix} i & \beta & j \\ & k & \end{pmatrix} \qquad \sum_{\beta} f^{ij,\alpha}_{k,\beta}\left(f^{ij,\alpha'}_{k,\beta}\right)^* = \delta_{\alpha\alpha'}$$

$$O_i^{jk,\alpha\beta} \to f^{i,\alpha}_{jk,\alpha'} f^{jk,\beta}_{i,\beta'} O_i^{jk,\alpha'\beta'},$$

$$Y^{ij}_{k,\alpha\beta} \to \left(f^{ij,\alpha'}_{k,\alpha}\right)^*\left(f^{k,\beta'}_{ij,\beta}\right)^* Y^{ij}_{k,\alpha'\beta'},$$

$$F^{ijm,\alpha\beta}_{kln,\chi\delta} \to f^{ij,\alpha}_{m,\alpha'} f^{mk,\beta}_{l,\beta'}\left(f^{jk,\chi'}_{n,\chi}\right)^*\left(f^{in,\delta'}_{l,\delta}\right)^* F^{ijm,\alpha'\beta'}_{kln,\chi'\delta'}$$

## Under proper gauge:

$$O_i^{jk,\alpha\beta} = O_i^{jk,\alpha}\delta_{\alpha\beta}, \quad O_i^{jk,\alpha} \geqslant 0$$

## Y becomes O inverse:

$$Y^{ij}_{k,\alpha} \simeq 1/O_k^{ij,\alpha}$$

# Unitary condition and a simple solution

$$\sum_{l\mu\tau} \left(F^{jkl,\mu\tau}_{mip',\chi'\alpha'}\right)^* \frac{O^{km,\chi'}_{p'} O^{jp',\alpha'}_{i}}{O^{lm,\tau}_{i} O^{jk,\mu}_{l}} F^{jkl,\mu\tau}_{mip,\chi\alpha} \frac{O^{km,\chi}_{p} O^{jp,\alpha}_{i}}{O^{lm,\tau}_{i} O^{jk,\mu}_{l}} = \delta_{pp'}\delta_{\chi\chi'}\delta_{\alpha\alpha'}$$

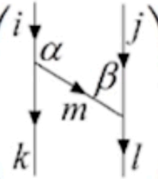
## A simple solution of O move:

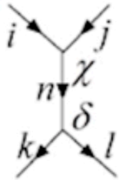
$$\boxed{O^{ij,\alpha}_{k} = \sqrt{\frac{d_i d_j}{D d_k}}\delta^{ij}_{k}, \quad D = \sum_l d_l^2, \quad d_i > 0}$$

## Projective unitary condition on O move becomes:

$$\boxed{\sum_{ij} d_i d_j N^{ij}_{k} = d_k D, \quad D = \sum_l d_l^2}$$

# Forth type of wavefunction renormalization: H move and dual H move

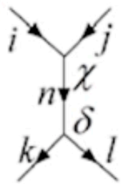
$\psi_{\text{fix}}\Bigg($ 

$\Bigg) \simeq \sum_{n\chi\delta} H^{kim,\alpha\beta}_{jln,\chi\delta}\,\psi_{\text{fix}}\Bigg($ 

$\Bigg).$

$$H^{kim,\alpha\beta}_{jln,\chi\delta} \simeq \sqrt{\frac{d_m d_n}{d_i d_l}}\left(F^{kmi,\alpha\chi}_{jnl,\beta\delta}\right)^*$$

$\Psi_{\text{fix}}\Bigg($ 

$\Bigg) \simeq \sum_{n\chi\delta} \tilde{H}^{kim,\alpha\beta}_{jln,\chi\delta}\,\Psi_{\text{fix}}\Bigg($ 

$\Bigg)$

$$\tilde{H}^{kim,\alpha\beta}_{jln,\chi\delta} \simeq \sqrt{\frac{d_m d_n}{d_j d_k}} F^{imk,\alpha\delta}_{lnj,\beta\chi}$$

**Unitary condition for these moves lead to an additional condition on F symbol:**

$$\boxed{\sum_{n\chi\delta} d_n F^{km'i,\alpha\chi}_{jnl,\beta'\delta'}\left(F^{kmi,\alpha\chi}_{jnl,\beta\delta}\right)^* = \frac{d_i d_l}{d_m}\delta_{mm'}\delta_{\alpha\alpha'}\delta_{\beta\beta'}}$$

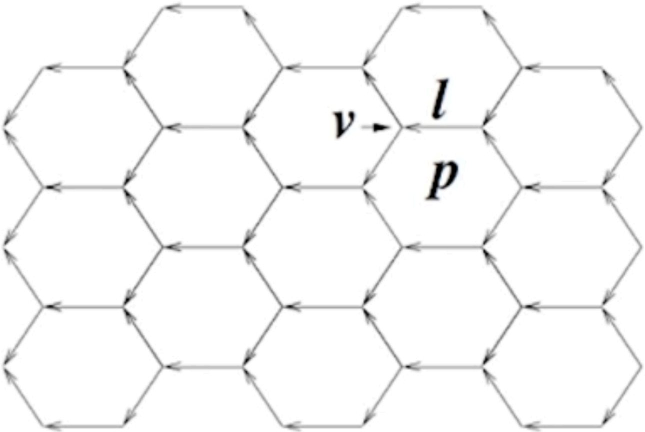
# Summaries of all conditions:

- By applying all four types of wavefunction renomarlization, one can deform the ground state wavefunction from one trivalent graph to any other trivalent graph.

- $$\sum_m N^{ij}_m N^{mk}_l = \sum_n N^{jk}_n N^{in}_l$$
- $$\sum_{n\chi\delta} F^{ijm',\alpha'\beta'}_{kln,\chi\delta} (F^{ijm,\alpha\beta}_{kln,\chi\delta})^* = \delta_{mm'}\delta_{\alpha\alpha'}\delta_{\beta\beta'},$$
- $$\sum_t \sum_{\eta\varphi\kappa} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi} = \sum_\epsilon F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma}$$
- $$\sum_{ij} d_i d_j N^{ij}_k = d_k D, \quad D = \sum_l d_l^2$$
- $$\sum_{n\chi\delta} \frac{d_n d_m}{d_i d_l} F^{km'i,\alpha'\chi}_{jnl,\beta'\delta} (F^{kmi,\alpha\chi}_{jnl,\beta\delta})^* = \delta_{mm'}\delta_{\alpha\alpha'}\delta_{\beta\beta'},$$

**Mathematically, the solutions of these algebra equations define a unitary tensor category(UTC) theory.**

# Parent Hamiltonian:




$$\boxed{\hat{H} = \sum_{\boldsymbol{v}} (1 - \hat{Q}_{\boldsymbol{v}}) + \sum_{\boldsymbol{p}} (1 - \hat{B}_{\boldsymbol{p}})}$$

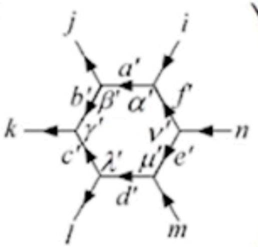
$$\hat{Q}_{\boldsymbol{v}} \left| \begin{pmatrix} i \searrow \alpha \swarrow j \\ \uparrow k \end{pmatrix} \right\rangle = \left| \begin{pmatrix} i \searrow \alpha \swarrow j \\ \uparrow k \end{pmatrix} \right\rangle \quad \text{if } N_k^{ij} > 0,$$

$$\hat{Q}_{\boldsymbol{v}} \left| \begin{pmatrix} i \searrow \alpha \swarrow j \\ \uparrow k \end{pmatrix} \right\rangle = 0 \qquad \text{otherwise.}$$

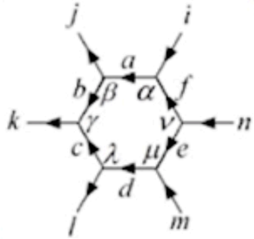
- The projector $B_p$ is more complicated, and we can define its matrix element as:

$$B^{a\alpha,b\beta,c\gamma,d\lambda,e\mu,f\nu}_{a'\alpha',b'\beta',c'\gamma',d'\lambda',e'\mu',f'\nu'}(i,j,k,l,m,n)$$

$= \Big\langle \Psi_{\text{fix}} \Big($



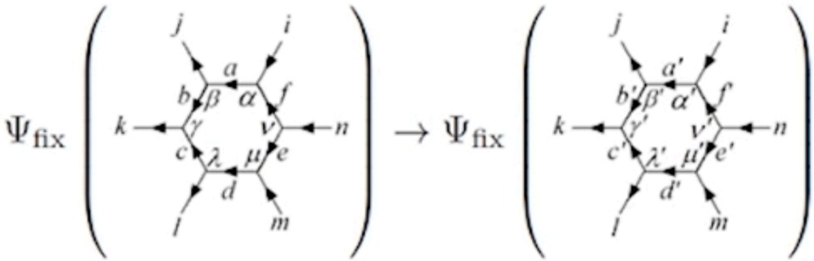

$\Big) \Big| \hat{B}_{\boldsymbol{p}} \Big| \Psi_{\text{fix}} \Big($



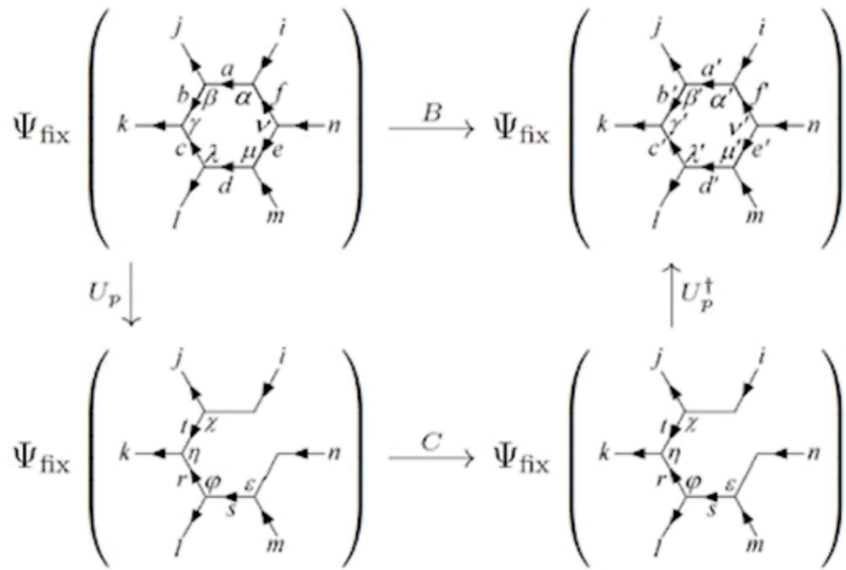

$\Big) \Big\rangle$

# Parent Hamiltonian:

- Apparently, for the fixed point wavefunction, the matrix element of $B_p$ operator can be evaluated from the local deformation:




$$B = U_P^{\dagger} C U_P$$




- Here C is the identity matrix in the support space

# References and extended reading material

- References. (The major contents are based on a coming book with Tian Lan and X G Wen. For relevant papers, see Michael A. Levin and Xiao-Gang Wen, Phys. Rev. B 71, 045110 (2005), X Chen, ZC Gu, XG Wen Physical review B 82, 155138 (2010))

## Extended Study

- Topological entropy for ground state of topological phases. (Alexei Kitaev and John Preskill, Phys. Rev. Lett. 96, 110404, Michael Levin and Xiao-Gang Wen, Phys. Rev. Lett. 96, 110405 (2006))
- Computing braiding statistics of excitations from ground state wavefunctions on torus via modular transformation.(Fangzhou Liu, Zhenghan Wang, Yi-Zhuang You, Xiao-Gang Wen, arXiv:1303.0829)
- Tensor network representation for ground state wavefunction and entanglement renormalization. (ZC Gu, M Levin, B Swingle, XG Wen, Physical Review B 79, 085118 (2009))
- Non-chiral topological phase in 2D interacting fermion systems. (ZC Gu, Z Wang, XG Wen, Physical Review B 90, 085140(2014), ZC Gu, Z Wang, XG Wen, Physical Review B 91, 125149 (2015))