Rmk In this chapter, $P(M,G)$ denote principle bundle.

Def. vertical subspace.

R-action on P

$$\downarrow$$

Given $P(M,G)$. $R_{e^{tA}} \cdot u = u \cdot e^{tA}$ $A \in \mathfrak{g}$ (Lie algebra of G).

then $\pi(u) = \pi(ue)$

$\Longrightarrow$ $u(t) = u \cdot e^{tA}$ is a curve in $G_p = G_{\pi(u)}$.

$V_u P$ is vector space tangent to G_p at $u \in P$ $\subset T_u P$

$A^{\#} \in T_u P$ defined by. ($A^{\#}$: fundamental vector field generated by A)

$$A^{\#}[f(u)] = \frac{d}{dt} f(u e^{tA})\Big|_{t=0} = \frac{d}{dt} f(u(t))\Big|_{t=0}$$

Lemma:

① $V_u P \underset{\text{Vect}}{\cong} \mathfrak{g} \cong T_e G$

$A^{\#} \underset{1\text{-}1}{\longleftrightarrow} A$

② $\pi_* X = 0$. $X \in V_uP$.

$$\pi_* X[g \in \mathcal{F}(M)] = X[g \circ \pi(u(t))]$$
$$= X[g(p)] = \frac{d}{dt} g(p) = 0.$$

Def. Connection on principle bundle

A connection on $P(M, G)$ is the unique seperation of

$$T_uP = H_uP \oplus V_uP$$
$$X \in \mathfrak{X}(M) = X^H + X^V$$

the horizontal subspace H_uP satisfies:

right invariance: $R_{g*} H_uP = H_{ug}P$

$$\begin{tikzcd} u \in P \arrow[r, "R_g"] \arrow[d] & u \cdot g \in P \arrow[d] \\ T_uP \arrow[r, "R_{g*}"] & T_{u \cdot g}P \end{tikzcd}$$

Since $\pi(ug) = \pi(u) = p$.

H_uP at u generat all $H_{u'}P$ on the same fiber $\pi^{-1}(p)$. (R-action on G is transitive and free).

Def. Connection one-form / Ehresmann connection.

$\omega \in \mathfrak{g} \otimes T^*P : TP \to \mathfrak{g}$ is Lie-algebra valued one-form.

It's a <u>projection</u> :

$$\begin{array}{c} \mathfrak{g} \\ \text{\rotatebox{90}{$\cong$}} \\ \omega_u : T_uP \twoheadrightarrow V_uP \subset T_uP \end{array}$$

satisfying : (1) $\omega(A^\# \in V_uP) = A \in \mathfrak{g}$.

(2) $R_g^* \omega = Ad_{g^{-1}} \circ \omega$

$R_g^* : T_{ug}^{*(g)} P \to T_g^{*(g)} P$

$$R_g^* \omega_{ug}(X \in T_uP) \equiv \omega_{ug}(\underbrace{R_{g*}X}_{\in T_{ug}P})$$

$$\in T_u^{*(g)} P \qquad = g^{-1} \omega_u(X) g$$

$\Longrightarrow$ $H_uP \equiv \ker \omega = \{ X \in T_uP \mid \omega(X) = 0 \}$

$$\omega(R_{g*}X) = R_g^* \omega(X) = g^{-1} \underbrace{\omega(X)}_{0} g \equiv 0$$

$\Longrightarrow$ $R_{g*} H_uP \equiv H_{ug}P$

(R_{g*} invertible)

Thus ω defines the seperation :

$$\omega \Rightarrow T_u P = \ker \omega \oplus \operatorname{im} \omega = H_u P \oplus V_u P$$

Def. Local connection form / gauge potential (connection)

given chart and local section $\sigma_i : M \to P$;

local connection. g valued.

$$A_i = A|_{U_i} = \sigma_i^* \omega \in \mathfrak{g} \otimes T^*M|_{U_i} = \mathfrak{g} \otimes \Omega^1(U_i) : TM|_{U_i} \to \mathfrak{g}$$

$\pi : P \to M$ $\quad s : M \to P$

$\pi_* : TP \to TM$ $\quad s_* : TM \to TP$

Theorem : $\pi^* : T^*M \to T^*P$ $\quad s^* : T^*P \to T^*M$.

Given (A_i, U_i, σ_i) then there's unique local $\mathfrak{g}$-connection.

$$\omega_i = \omega|_{U_i} = g_i^{-1} \pi^* A_i g_i + g_i^{-1}(d_P g_i)$$

① canonical local trivialization :

$u = \sigma_i(p) g_i(u)$. $\quad \phi_i^{-1}(u) = (p, g_i(u))$ $\quad g_i(u) \equiv g_{i,u}$.

② $d_p g_i(u) \in \Omega^1(P)$

Check :

③ $\sigma_i^* \omega_i(X) = \omega_i(\sigma_{i*} X)$

$\sigma_i(p) = \sigma_i(p) g_i \Rightarrow g_i = e$ $\longrightarrow$ $g_i = e$

$$\equiv A_i(\underline{\pi_* \sigma_{i*}} X) + (d_p g_{iu})(\sigma_{i*} X)$$

$(\pi\sigma)_* = id_{T_p(M)}$ $\qquad$ $\sigma_{i*} X [g_{iu}] = 0$, $g_{iu} = e$

$$= \underline{A_i}$$

④ $\omega(A^\#) = g_{iu}^{-1}(d_p g_{iu})(A^\#)$

$\pi_* A^\# = 0$

$$= g_{iu}^{-1} A^\# [g_{iu}]$$

$$\equiv g_{iu}^{-1} \frac{d}{dt} g_i(u e^{tA}) \Big|_{t=0}$$

$$= (g_{iu}^{-1} g_{iu}) \frac{d}{dt} e^{tA} \Big|_{t=0} = A$$

⑤ given $X \in T_u P$, $h \in G$

$$R_h^* \omega_i(X) = \omega_i(R_{h*} X)$$

$\pi R_h = \pi$

$$= g_{iuh}^{-1} A_i(\underbrace{\pi_* R_{h*}}_{= \pi_* X} X) g_{iuh}$$

$$+ \underline{g_{iuh}^{-1}(d_p g_{iuh})(R_{h*} X)}.$$

$$= g_{iuh}^{-1} R_{h*} X [g_{iuh}]$$

$\underline{\gamma(0) = u,\ \dot{\gamma}(0) = X}$ $\qquad = g_{iu h}^{-1} \frac{d}{dt} g_{i\gamma(t)h}\Big|_{t=0}$

$\underline{g_{iuh} = g_{iu} \cdot h}$ $\qquad = h^{-1} g_{iu}^{-1} \frac{d}{dt} g_{i\gamma(t)}\Big|_{t=0} h.$

$$= h^{-1}\left[g_{iu}^{-1}\, d_p g_{iu}(X)\right] h.$$

$$\Longrightarrow \quad R_h^* \omega_i(X) = h^{-1}\omega_i(X) h = Ad_h \circ \omega_i(X).$$

<u>Lemma</u>. $X \in T_p M$

$$\sigma_{j*} X = R_{t_{ij}*}(\sigma_{i*} X) + \left(t_{ij}^{-1}\, d t_{ij}(X)\right)^{\#}$$

$$\underbrace{\sigma_{j*} X}_{\in T_{\sigma_j(p)}P} \equiv \frac{d}{dt} \overbrace{\sigma_j(\underbrace{\gamma(t)}_{\in M})}^{\in P}\Big|_{t=0} = \frac{d}{dt}\left(\overset{\sigma_i(\gamma(t))}{\sigma_i(t)}\ \overset{t_{ij}(\gamma(t))}{t_{ij}(t)}\right)\Big|_{t=0}$$

$$= \underbrace{\left(\frac{d}{dt}\sigma_i(t)\Big|_{t=0}\right) \cdot t_{ij}(p)}_{\sigma_{j*}X} + \underbrace{\sigma_i(p)\left(\frac{d}{dt} t_{ij}(t)\Big|_{t=0}\right)}_{\underline{\sigma_j t_{ij}^{-1}(p) \frac{d}{dt} t_{ij}(t)\big|_{t=0}}}$$

$(*)$.

$$= \frac{d}{dt}\left(\sigma_i(\gamma(t))\Big|_{t=0} \cdot t_{ij}(p)\right)$$

$$\equiv (R_{t_{ij}} \circ \sigma_j)_* X = R_{t_{ij}*}(\sigma_{j*} X).$$

$$(\ast) = t_{ij}^{-1}(p)\,\frac{d}{dt}t_{ij}(\gamma(t))\Big|_{t=0} = \underline{t_{ij}^{-1}(p)(dt_{ij})(X)}$$

$$= \frac{d}{dt}\Big(\underbrace{t_{ij}^{-1}(p)\,t_{ij}(\gamma(t))}_{=e}\Big|_{t=0}\Big) \in T_eG \cong \mathfrak{g}$$

$$\Longrightarrow \quad \sigma_j(p)(\ast) = (t_{ij}^{-1}\,dt_{ij}(X))^{\#}$$

$$(u = \sigma_j(p)\,g_i)$$

Theorem Uniqueness of $\omega \iff$ consistency condition.

$\omega_i = \omega_j$ on $U_i \cap U_j$

$\iff$ gauge transformation of A_i

Consistency condition:

ω is uniquely defined on $P \iff \omega_i = \omega_j$ on $U_i \cap U_j$

$$\iff \quad A_j(X) = \sigma_j^{\ast}\omega(X) = \omega(\sigma_j^{\ast}X)$$

$$= \omega\big(R_{t_{ij}\ast}\sigma_{i\ast}X + (t_{ij}^{-1}dt_{ij}(X))^{\#}\big)$$

$$= R_{t_{ij}}^{\ast}\omega(\sigma_{i\ast}X) + t_{ij}^{-1}dt_{ij}(X)$$

$$= t_{ij}^{-1}\underbrace{\omega(\sigma_{i\ast}X)}_{\sigma_i^{\ast}\omega(X)}t_{ij} + t_{ij}^{-1}dt_{ij}(X)$$

$$\iff \quad A_j = t_{ij}^{-1}A_i t_{ij} + t_{ij}^{-1}dt_{ij}$$

Choose two section (two choice of fiber elements).

$$\sigma_2(p) = \sigma_1(p)\,\underline{g(p)}$$ local gauge transformation.

$$\Longrightarrow \quad A_2 = g^{-1} A_1 g + g^{-1} d g$$

Rmk.

① the notation here is mathematical. differ from physical ones by possible $I i$

② ω is defined uniquely globally over P

while A_i is defined only locally.

~ trivial bundle $U_i \times G$

$\omega \Leftrightarrow \{A_i, t_{ij}\}$ carry global information of $P(M,G)$

pasting the bundle.

gauge transform/ corresponding connection

connecting

③ There're different definition of connections,

but they're equivalent.

the one used here relies on unique seperation

practically we have to find suitable ω, (σ_i, A_i).

Def. Horizontal lift

Given curve in M: $\gamma : [0,1] \to M \quad \in \gamma(M) \subset M$.

the horizontal lift of γ: $\tilde{\gamma} : [0,1] \to P \in \gamma(P)$. satisfying

① $\pi \circ \tilde{\gamma} = \gamma$

② $\tilde{X} \in T\tilde{\gamma} \in H_{\tilde{\gamma}(t)} P$ tagent vector to $\tilde{\gamma}(t) \subset P$ is horizontal.

constructing $\tilde{\gamma}$ and $\tilde{X}$:

given (U_i, σ_i): $\tilde{\gamma}(t) = \sigma_i(\gamma(t))\, g_i(\gamma(t))$

$\tilde{\gamma} : \gamma(M) \to P$

$\tilde{\gamma}_* : T\gamma(M) \to TP$

$\sigma_{i*} \to \tilde{\gamma}_*$

$t_{ij} \to g_i$

Let $X \in T_{\gamma(t)} \gamma \big|_{t=0}$ then the lift of X is

$$\tilde{X} = \tilde{\gamma}_* X \in T_{\tilde{\gamma}(t)} \tilde{\gamma} \big|_{t=0} \subset H_{\tilde{\gamma}(t)} P$$

$$\Rightarrow \quad \tilde{X} = g_i^{-1}(t) (\sigma_{i*} X)\, g_i(t) + [g_i^{-1}(t)\, d g_i(X)]^{\#}$$

$$\Rightarrow \quad 0 = \omega(\tilde{X}) = g_i(t)^{-1} \underbrace{\omega(\sigma_{i*}X)}_{A_i(X)} g_i(t) + g_i(t)^{-1} \frac{d}{dt} g_i(t)$$

$$\Rightarrow \quad \frac{dg_i(t)}{dt} = -A_i(X)\, g_i(t)$$

$\xRightarrow{\text{ODE}}$ unique solution: $\quad (X = \dot{\gamma}(0))$.

path-ordering

$$g_i(\gamma(t)) = \mathcal{P}\exp\left(-\int_0^t A_{i\mu} \frac{dx^\mu}{dt}\, dt\right) g_i(\gamma(0))$$

$$= \mathcal{P}\exp\left(-\int_{\gamma(0)}^{\gamma(t)} A_{i\mu}(\gamma(t))\, dx^\mu\right) g_i(\gamma(0))$$

the expansion may not convergent $\longrightarrow$

$$= \underline{\mathcal{P}\exp\left(-\int_{\gamma(0)}^{\gamma(t)} A_i(\gamma(t))\right)}\, g_i(\gamma(0)).$$

the unique horizontal lift of $\gamma(t) \subset M$ is:

$$P \supset \tilde{\gamma}(t) = \sigma_i(\gamma(t))\, g_i(\gamma(t))$$

depend on $\underline{\gamma(t)\ \&\ A_i}$

Lemma

① $\tilde{\gamma}'$ is another H-lift, $\tilde{\gamma}'(0) = \gamma(0)\, g$ then.

$$\left\{ \begin{array}{l} R_{g*} H_u P = H_{ug} P \ \text{(right action invariant)} \\ \text{uniqueness of solution to ODE} \end{array} \right. \Rightarrow \quad \tilde{\gamma}'(t) = \gamma(t) \cdot \underline{g}$$

<u>Def</u>. Parallel transport of elements $\in P$

given $\tilde{\gamma}(t)$, $\Gamma(\tilde{\gamma}(t)) : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1))$

$$u_0 = \tilde{\gamma}(0) \longmapsto u_1 = \tilde{\gamma}(1)$$

$$u_1 = \sigma_i(1)\left(P e^{-\int_{\gamma(0)}^{\gamma(1)} A_i(\gamma(t))} g_i(\gamma(0))\right), \quad u_0 = \sigma_i(\gamma(0))\, g_i(\gamma(0))$$

Lemma

$$\Longrightarrow \quad \underline{R_g \Gamma(\tilde{\gamma}) = \Gamma(\tilde{\gamma}) R_g}.$$

Further: ① $\Gamma(\tilde{\gamma}^{-1}) = \Gamma(\tilde{\gamma})^{-1}$, $\tilde{\gamma}^{-1} = \tilde{\gamma}(1-t)$.

② $\Gamma(\widetilde{\alpha * \beta}) = \Gamma(\tilde{\beta}) \circ \Gamma(\tilde{\alpha})$.

③ $u \underset{\tilde{\gamma}(t)}{\sim} v \iff u, v \in \text{same } \tilde{\gamma}(t)$.

$\iff \exists$ unique invertible

$$u \underset{\Gamma^{-1}(\tilde{c}(\tilde{\gamma}(t)))}{\overset{\Gamma(\tilde{c}(\tilde{\gamma}(t)))}{\rightleftarrows}} v$$

Def. Holonomy & Holonomy group

In general, $\alpha(0) = \beta(0) = p_0$, $\alpha(1) = \beta(1) = p_1$

$\Rightarrow$ $\tilde{\alpha}(0) = \tilde{\beta}(0) = u_0$, $\tilde{\alpha}(1) \neq \tilde{\beta}(1) \in P$

$\Rightarrow$ consider loops $\gamma \in L(M) = \{ \gamma(t) \in \gamma(M) \mid \gamma(0) = \gamma(1) = p \}$.

$\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$, $\pi(\tilde{\gamma}(0)) = \pi(\tilde{\gamma}(1)) \equiv p$.

different in fiber $F_p \cong G$

$\Rightarrow$ $\gamma \in L(M) \Leftrightarrow \tau_\gamma : \underset{G}{\underset{\shortparallel}{\pi^{-1}(p)}} \to \underset{G}{\underset{\shortparallel}{\pi^{-1}(p)}}$ (depend on both γ and A)

$\tau_\gamma(ug) = \tau_\gamma(u) g$. ($\Gamma(\gamma)$ commute with R_g)

holonomy group:

$$\Phi_u = \{ g \in G \mid \tau_\gamma(u) \equiv u g_\gamma,\ \gamma \in L_{p=\pi(u)}(M) \}.$$

group structure: ① $\gamma = \alpha * \beta \Rightarrow \tau_\gamma = \tau_\beta \circ \tau_\alpha$.

$$\tau_\gamma(u) = \tau_\beta(u g_\alpha) \equiv \tau_\beta(u) g_\alpha \equiv u g_\beta g_\alpha$$
$$= u g_\gamma.$$

$$\Longrightarrow \quad \tau_\gamma = \tau_\beta \circ \tau_\alpha \iff g_\gamma = g_\beta g_\alpha$$

② $\tau_{\gamma^{-1}} = \tau_\gamma \iff g_{\gamma^{-1}} = g_\gamma^{-1}$

③ $\tau_\alpha(u) = u g_\alpha$.

$$\tau_\alpha(ug) = \tau_\alpha(u) g = ug(g^{-1} g_\alpha g)$$

$$= ug(\mathrm{ad}_g(g_\alpha)).$$

$$\Longrightarrow \quad \Phi_{ug} \cong g^{-1} \Phi_u g.$$

④ Φ_* is a Functor:

$$\Phi_* : u \to \Phi_u.$$

$$u \underset{\gamma(t)}{\sim} v \longrightarrow \Phi_u \cong \Phi_v.$$

$$\Longrightarrow \left[M \text{ connected} \Rightarrow \Phi_u \cong \Phi_v, \ \forall u, v \in P \right]$$

⑤ $\tau_\gamma(u) = u g_\gamma$

$$g_\gamma = g_i(\gamma(1)) = \underline{P\exp\left(-\oint_\gamma A_i(\gamma(t))\right)} \cdot g_i(\gamma(0))$$

Wilson loop operator.

⑥ Restricted holonomy group:

$\underline{\gamma \sim \text{constant loop}}$, denoted Φ_u^0.

Def. Covariant derivatives in $P(M,G)$

given $\phi = \sum_{\alpha=1}^{k} \underbrace{\phi^\alpha}_{\in \Omega^r(P)} \otimes e_\alpha \in \Omega^r(P) \otimes V$, $X_i \in T_u P$

the covariant derivative of ϕ (in $P(M,G)$):

$$D : \Omega^r(P) \otimes V \longrightarrow \Omega^{r+1}(P) \otimes V$$

separation. depend on connection

$$D\phi(X_1, \dots, X_{r+1}) = d_P\phi(X_1^H, \dots, X_{r+1}^H)$$

$$= \underbrace{(d_P\phi^\alpha)}_{\in \Omega^{r+1}(P)}(X_1^H, \dots, X_{r+1}^H) \otimes e_\alpha$$

Def. Curvature 2-form

since $\omega \in \Omega^1(P) \otimes \mathfrak{g}$, $\Omega \equiv D\omega \in \Omega^2(P) \otimes \underline{\mathfrak{g}}$

Lemma.

① $(R_{a*}X)^H = R_{a*}(X^H)$ (HuP right action invariant)

② $d_P R_a^* = R_a^* d_P$

Lemma

$$R_a^* \Omega = a^{-1} \Omega a$$

$$R_a^* \Omega(X, Y) = \Omega(R_{a*}X, R_{a*}Y) = d_p\omega(R_{a*}X^H, R_{a*}Y^H)$$

$$= R_a^*(d_p\omega(X^H, Y^H)) = d_p(R_a^*\omega)(X^H, Y^H)$$

$$= d_p(a^{-1}\omega a^{-1})(X^H, Y^H)$$

$$= a^{-1}(d_p\omega)(X^H, Y^H)a = a^{-1}\Omega(X, Y)a$$

Def. Commutator between $\mathfrak{g}$-valued differential forms:

$$\zeta \in \Omega^p(P) \otimes \mathfrak{g}, \quad \eta \in \Omega^q(P) \otimes \mathfrak{g}$$

$$[\zeta, \eta] \equiv \zeta \wedge \eta - (-1)^{pq} \eta \wedge \zeta$$

$$= T_\alpha T_\beta \otimes \zeta^\alpha \wedge \eta^\beta - (-1)^{pq} T_\beta T_\alpha \otimes \eta^\beta \wedge \zeta^\alpha$$

$$= [T_\alpha, T_\beta] \otimes (\zeta^\alpha \wedge \eta^\beta) = f_{\alpha\beta}{}^\gamma T_\gamma \otimes (\zeta^\alpha \wedge \eta^\beta) \in \Omega^{p+q}(P) \otimes \mathfrak{g}$$

$$\zeta = \eta \implies \underline{[\zeta, \zeta] = 2\zeta \wedge \zeta} = f_{\alpha\beta}{}^\gamma T_\gamma \otimes (\zeta^\alpha \wedge \zeta^\beta)$$

Lemma

$$X \in H_uP, \quad Y \in V_uP \implies \underline{[X, Y] \in H_uP}$$

$$L_Y(X) = [Y, X] = \lim_{t \to 0} \frac{1}{t} [\underbrace{R_{g(t)*}X}_{\in H_{ug(t)}P} - X) \in \underline{H_uP}$$

Y generated by $g(t)$
(tangent to G_p) $\longrightarrow$ $Y \in V_uP$.
the path generated by Y lies in $\underline{G_p}$

Lemma $\quad d_p\omega(X,Y) = X\cdot\omega(Y) - Y\cdot\omega(X) - \omega([X,Y])$

Theorem $\quad$ Cartan's structure equation :

$$\Omega(X,Y) = d_p\omega(X,Y) + [\omega(X),\omega(Y)]$$

3-cases: ① $X,Y \in H_uP$: $\underbrace{\Omega(X,Y)}_{\overset{\shortparallel}{d_p\omega(X^H,Y^H)}} = d_p\omega(X,Y) + \underbrace{[\omega(X),\omega(Y)]}_{0}$

② $X \in H_uP$, $Y \in V_uP$, $(Y^H = 0)$

$$\underbrace{\Omega(X,Y)}_{0} = d_p\omega(X,Y) + [\underbrace{\omega(X)}_{0},\omega(Y)] \qquad d_p\omega(X,Y) \equiv X\cdot\underbrace{\overset{\in V_uP}{\omega(Y)}}_{=0} - \underbrace{Y\cdot\omega(X)}_{0} - \omega(\underbrace{[X,Y]}_{\in H_uP}) = 0$$

③ $X,Y \in V_uP$

$$\underbrace{\Omega(X,Y)}_{0} = \underbrace{d_p\omega(X,Y) + [\omega(X),\omega(Y)]}_{0} \qquad d_p\omega(X,Y) \equiv \underbrace{X\cdot\omega(Y) - Y\cdot\omega(X)}_{0} - \omega(\underbrace{[X,Y]}_{\in V_uP})$$

$$\omega([X,Y]) = [X,Y] = [\omega(X),\omega(Y)] \quad ([X^\#,Y^\#] = [X,Y]^\#)$$

From $\quad [\omega,\omega](X,Y) = [T_\alpha,T_\beta]\,\omega^\alpha\wedge\omega^\beta(X,Y)$

$$= [T_\alpha,T_\beta]\left(\omega^\alpha(X)\omega^\beta(Y) - \omega^\beta(X)\omega^\alpha(Y)\right)$$

$$= [\omega(X),\omega(Y)] - [\omega(Y),\omega(X)]$$

$$= 2[\omega(X),\omega(Y)]$$

$$= 2\,\omega\wedge\omega(X,Y).$$

$$\Longrightarrow \qquad \Omega = d_p\omega + \omega\wedge\omega$$

Rmk

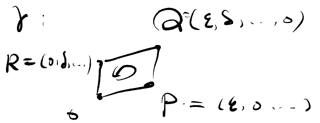
1) Geometrical meaning of Ω:

For $X, Y \in H_u P$ $\quad \Omega(X,Y) = d_P\omega(X,Y) = -\omega([X,Y])$

Note $\begin{cases} X \in H_u P,\ Y \in V_u P \Rightarrow [X,Y] \in H_u P \\ X \in V_u P,\ Y \in V_u P \Rightarrow [X,Y] \in V_u P \end{cases}$ $\quad X \in H_u P,\ Y \in H_u P \not\Rightarrow [X,Y] \in H_u P$

Let $X, Y \in H_u P$, $\pi_* X = \varepsilon V$, $\pi_* Y = \delta W$. $\quad V = \frac{\partial}{\partial x^1}, W = \frac{\partial}{\partial x^2}$ (φ on U).

take γ:



$$\Rightarrow \pi_*([X,Y]^H) = \varepsilon\delta[V,W] = \varepsilon\delta\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right] = 0$$

$$\Rightarrow [X,Y] \in V_u P \quad (\text{not changing } \pi(u) = p).$$

$\Longrightarrow$ $\tilde{\gamma}$ of closed loop γ fail to close. (the fiber not coincide).

holonomy group $\longleftrightarrow$ curvature in $P(M,G)$.

Explicity:

Theorem. Lie algebra $\mathfrak{h}$ of Φ_{u_0} of $u_0 \in P(M,G)$

$$\cong \operatorname{Span}\{\Omega_u(X,Y) \mid X, Y \in H_u P,\ \Omega_u(X,Y) \in \mathfrak{g}\} \subset \mathfrak{g}$$
$$u \sim_{\tilde{\gamma}(t)} u_0$$

Def. Local form of curvature 2-form.

$A_i = \sigma^* \Omega$, $F_i = \sigma_i^* \Omega = \sigma^* (d_p \omega + \omega \wedge \omega)$

$= d(\sigma^* \omega) + (\sigma^* \omega) \wedge (\sigma^* \omega)$

$\underline{\sigma^* d_p = d \sigma^*}$ $= d A_i + A_i \wedge A_i$

$\Rightarrow \quad F(X, Y) = dA(X, Y) + [A(X), A(Y)]$

given φ_i on $U_i \Rightarrow$ $A = A_\mu dx^\mu$ gauge potential.

$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$

$\Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ curvature tensor.

$= F_{\mu\nu}{}^\alpha T_\alpha$

$F_{\mu\nu}{}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + f_{\beta\gamma}{}^\alpha A_\mu^\beta A_\nu^\gamma$

Compatibility condition (gauge transformation):

$A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1} d t_{ij}$

$\Rightarrow F_j = [\underline{\cdots} + t_{ij}^{-1} dA_i t_{ij} + \underline{\cdots} + t_{ij}^{-1} (A_i \wedge A_i) t_{ij}$

$+ \underline{\cdots}] = \underline{t_{ij}^{-1} F_i t_{ij}}$

cancel

$d(t^{-1}) = -t^{-1} (dt) t^{-1}$

$(d(t t^{-1}) = 0)$.

Lemma. Pure gauge $A = g^{-1} dg \underset{\text{Locally}}{\Longleftrightarrow} F = 0$

(A not defined globally).

Theorem. Bianchi identity :

$$\forall\, X, Y, Z \in T_u P . \quad D\Omega(X, Y, Z) = d_p\omega(X^H, Y^H, Z^H) = 0$$

$$T^\alpha : d_p\Omega^\alpha = f^\alpha_{\beta\gamma}\, d\omega^\beta \wedge \omega^\gamma + f^\alpha_{\gamma\beta}\, \omega^\beta \wedge d\omega^\gamma$$

$$\Longrightarrow \qquad D\Omega \equiv 0$$

Local form :

$$\sigma^* d_p\Omega = d\,\sigma^*\Omega = dF .$$

$$= \sigma^*(d_p\omega \wedge \omega - \omega \wedge d_p\omega)$$

$$= d(\sigma^*\omega) \wedge \omega - \omega \wedge d(\sigma^*\omega)$$

$$= dA \wedge A - A \wedge dA$$

$$= F \wedge A - A \wedge F$$

$$\Longrightarrow DF \equiv dF + A \wedge F - F \wedge A = dF + [A, F] = 0 .$$

Def. $\eta \in \Omega^3(M) \otimes \mathfrak{g}$. $\quad D\eta = d\eta + [A, \eta]$

Def Covariant derivative on associated bundle (of sections).

$$P(M,G) \xrightarrow{\text{associated}} E = P \times_{\rho} V/\sim_{R_G}, \quad X \in T_pM \text{ is } T_p\gamma|_{p_0}$$

representation of G

$s \in \Gamma(M,E)$.

gauge invariance

gauge D.o.F physical D.o.F

$$s(p) = [(\sigma_i(p), \xi(p))]$$

local

$\in G_p \subset P$, $\cong G$

$\in V \cong F_E$ (vector fields)

choice of representative = fixing the gauge

$$[(u,v)] = \{(ug, g^{-1}v) \mid u \in P, v \in V, g \in G\}$$

gauge transformation

Parallel transport of vector in E along $\gamma \subset M$:

constant with respect to $\tilde{\gamma}$ of γ in P

$$s(\gamma(t)) = [(\tilde{\gamma}(t) = \sigma_i(\gamma(t))\, g_i(\gamma(t)), \eta(\gamma(t)))]$$

is parallel transported if η is constant along $\gamma(t)$

Covariant derivative of $s(\gamma(t))$ along $\gamma(t) \subset M$ at $p_0 = \gamma(0)$:

$X = \dot{\gamma}(0)$.

$$\nabla_X s = \left[\left(\tilde{\gamma}(0), \frac{d}{dt}\eta(\gamma(t))\Big|_{t=0}\right)\right]$$

Rmk

① $\nabla_X s$ independent of choice of lift $\tilde{\gamma}$:

$$\check{\gamma}(t) = \tilde{\gamma}(t)a \qquad [(\tilde{\gamma}(t), \eta(t))] = [(\check{\gamma}(t)a^{-1}, \eta(t))]$$
$$= [(\check{\gamma}(t), a^{-1}\eta(t))]$$

(Another choice of gauge)

$$[(\check{\gamma}(0), \frac{d}{dt}(a^{-1}\eta(t))\Big|_{t=0})] \equiv [(\check{\gamma}(0)a^{-1}, \frac{d}{dt}\eta(t)\Big|_{t=0})]$$
$$= [(\tilde{\gamma}(0), \frac{d}{dt}\eta(t)\Big|_{t=0})]$$

$\nabla_X s$ depend on only $(\gamma(t), X)$, section s, connection on $P(M, G)$

② $\nabla_X : \Gamma(M, E) \longrightarrow \Gamma(M, E)$.

$$\nabla_X s\Big|_p = \nabla_{X_p} s \qquad X_p = T_p\gamma \in T_p M.$$

$$\nabla_* : \Gamma(M, E) \to \Gamma(M, E) \otimes \Omega^1(M)$$

$$\nabla s(X) = \nabla_X s$$

Lemma

Local expression for ∇ on $P \times_\rho V / \sim_{R_G}$

$(e_\alpha^\circ)^\beta = \delta_\alpha{}^\beta$

take frame section : $e_\alpha(p) = [(\sigma_i(p), e_\alpha^\circ)]$

α-th e_α° is the basis of V.

Canonical local trivialization

$\tilde{\gamma}(t) = \sigma_i(\gamma(t))\, g_i(\gamma(t))$

$$e_\alpha(\gamma(t)) \equiv e_\alpha(t) = [(\sigma_i(t), e_\alpha^\circ)]$$

$$= [(\tilde{\gamma}(t) \circ g_i^{-1}(\gamma(t)), e_\alpha^\circ)]$$

$$= [(\tilde{\gamma}(t), g_i^{-1}(t) \cdot e_\alpha^\circ)]$$

compensate for the change of basis along $\gamma(t)$

$$\nabla_X e_\alpha = \left[\left(\tilde{\gamma}(0), \frac{d}{dt}(g_i^{-1}(t) e_\alpha^\circ)\Big|_{t=0}\right)\right] = \left[\left(\tilde{\gamma}(0), -g_i^{-1}(t)\left(\frac{d}{dt} g_i(t)\right) g_i^{-1}(t)\, e_\alpha^\circ\Big|_{t=0}\right)\right]$$

$\frac{d}{dt}(g g^{-1}) = 0$

$$= \left[\left(\tilde{\gamma}(0) g_i^{-1}(0), \left(\frac{d}{dt} g_i(t)\right) g_i(t)\, e_\alpha^\circ\Big|_{t=0}\right)\right]$$

$$= [(\tilde{\gamma}(0) g_i^{-1}(0), \underline{A_i(X)}\, e_\alpha^\circ)] \leftarrow \frac{d g_i(\gamma(t))}{dt} = -A_i(X)\, g_i(t)$$

$$A_{i\mu}{}^\alpha{}_\beta = A_{i\mu}{}^\gamma (T_\gamma)^\alpha{}_\beta$$

$$A_i(X) e_\alpha^\circ = \frac{dx^\mu}{dt} e_\beta^\circ A_{i\mu}{}^\beta{}_\gamma \delta_\alpha{}^\gamma = \frac{dx^\mu}{dt} A_{i\mu}{}^\beta{}_\alpha e_\beta^\circ$$

$$\Rightarrow \quad \nabla_X e_\alpha = \left[\left(\sigma_i(\gamma(0)),\ \frac{dx^\mu}{dt} A_{i\mu}{}^\beta{}_\alpha e_\beta{}^\circ \right) \right]$$

$$= \frac{dx^\mu}{dt} A_{i\mu}{}^\beta{}_\alpha\, e_\beta(\gamma(0))$$

how basis of U are transformed along $\gamma(t)$.

$$\Rightarrow \quad \nabla e_\alpha = A_i{}^\beta{}_\alpha\, e_\beta = e_\beta \otimes A_i{}^\beta{}_\alpha.$$

$$D_{\partial_\mu} e_\alpha = A_{i\mu}{}^\beta{}_\alpha\, e_\beta$$

Covariant derivative on associated vector bundle is specified by connection A on priciple bundle.

In general, $\quad s(p) = [(\sigma_i(p), \xi_i(p))] = \underline{\xi_i^\alpha(p)\, e_\alpha}.$

$$\Rightarrow \quad \nabla_X s = \nabla_X(\xi_i^\alpha e_\alpha) = X[\xi_i^\alpha]\, e_\alpha + \xi_i^\alpha\, \nabla_X e_\alpha$$

$$= \left[\left(\sigma_i(0),\ \frac{d\xi_i}{dt} + A_i(X)\,\xi_i \Big|_{t=0} \right) \right]$$

$$= \frac{dx^\mu}{dt} \left\{ \frac{\partial \xi^\alpha}{\partial x^\mu} + A_{i\mu}{}^\alpha{}_\beta\, \xi_i{}^\beta \right\} e_\alpha.$$

$$= X^\mu \left\{ \partial_\mu + A_{i\mu}{}^\alpha{}_\beta \right\} \xi_i{}^\beta\, e_\alpha.$$

choice of basis on U

choice of coordinates on M

covariant derivative D_μ

physical D.o.F

choice of connection on $P(M,G)$.

Rmk.

① Local expression of $D_X s$ is independent of local trivialization

$$D_X s = \left[\left(\sigma_i(0) ,\ \frac{d\xi_i}{dt} + A_i(X)\xi_i \Big|_{t=0} \right) \right]$$

$$= \left[\left(\sigma_j(0) t_{ij}^{-1} ,\ \frac{d}{dt}(t_{ij}\xi_j) + A_i(X) t_{ij}\xi_j \Big|_{t=0} \right) \right]$$

$$= \left[\left(\sigma_j(0) ,\ \frac{d}{dt}\xi_j + A_j(X)\xi_j \Big|_{t=0} \right) \right] \qquad \left(t_{ij}^{-1} dt_{ij} \Big|_{t=0} = 0 \right)$$

$D_X s$ is independent of $\tilde{\gamma}$, σ_i, depend only on $\gamma(t)$, s, A_i.

thus is the most natural derivative on an associated vector bundle, compatible with the connection on $P(M,G)$.

Similary, we can define derivative on $E_{\mathfrak{g}} = P \times_{Ad} \mathfrak{g}$.

$$Ad_g V = g^{-1} V g.$$

$$D_X s = \left[\left(\tilde{\gamma}(0) ,\ \frac{d}{dt}\left[Ad_{g(t)} V(t) \right] \Big|_{t=0} \right) \right]$$

$$= \left[\left(\sigma_i(0) ,\ \frac{dV(t)}{dt} + \left[A_i(X), V(t) \right] \Big|_{t=0} \right) \right]$$

$$= X^\mu \left(\frac{\partial V^\alpha}{\partial x^\mu} + \underbrace{f_{\beta\gamma}{}^{\alpha}}_{\text{adjoint representation}} A_{i\mu}{}^{\beta} V^\gamma \right) \left[(\sigma_i(0), T_\alpha) \right]$$

More generally:

$$\eta \in \Omega^p(M), \quad \nabla(s \otimes \eta) = \overbrace{\nabla s \wedge \eta}^{\Omega^{p+1}(M)} + s \otimes d\eta$$

$$\Rightarrow \quad \underline{\nabla\nabla e_\alpha} = \nabla(e_\beta \otimes A_i{}^\beta{}_\alpha) = \nabla e_\beta \wedge A_i{}^\beta{}_\alpha + e_\beta \otimes dA_i{}^\beta{}_\alpha$$

$$= e_\beta \otimes (dA_i{}^\beta{}_\alpha + A_i{}^\beta{}_\gamma \wedge A_i{}^\gamma{}_\alpha) = \underline{e_\beta \otimes F_i{}^\beta{}_\alpha}$$

$$\Rightarrow \quad \nabla\nabla s = \nabla\nabla(\xi^\alpha e_\alpha) = e_\alpha \otimes F_i{}^\alpha{}_\beta \xi^\beta$$