

Motivation

I. provide good effective description of the system at or near criticality (2^{nd} phase transition)

e.g. Long-range correlations

Scaling invariance

II. the only true QFTs (Cut-off independent)

"All" QFTs are perturbations of CFT

(the perspective from RG)

CFT correspond to RG fixed point

III. CFT can be made by d large rigorous

at least at $1+d$ (Quantum fluctuations very drastic)

* Note, there're now 3 mathematical formulations
haven't unified yet!

{ What's CFT

- It's quantum field theory (local)
- Invariant under the symmetry group:

conformal group

→ { classical:
leave S invariant
(equ)
Quantum:

Wigner's theorem:

Transformation \sim (anti)unitary operators

the \hat{Q} commute with \hat{H}

A Quantum theory is symmetric under G



Furnish projective representation of G

↳ { QM: projective H

Group cohomology & extension.

§ Conformal transformations in d dimensions.

Def $M = \mathbb{R}^{p+q}$, $\mathbb{R}^{d=p+q}$ with metric $g_{\mu\nu} = \text{diag}(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q}) = \eta_{\mu\nu}$

Flat



* AdS/CFT have something to do with $g=2$.

- Conformal transf. leave the metric invariant up to a

Scale factor \sim angles.

$M \ni x \xrightarrow[\text{smooth change of coordinates}]{\substack{\text{conformal} \\ \text{transf}}} x'(x)$ note this is a global coordinate system.

such that, $g_{\mu\nu} \xrightarrow[\text{conformal}]{\substack{\text{transf.}}} g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^u} \frac{\partial x^\beta}{\partial x'^v} g_{\alpha\beta}(x)$

$\propto \eta_{\mu\nu} (= g_{\mu\nu})$

$\implies g'_{\mu\nu}(x) = \Omega(x) g_{\mu\nu}(x), \Omega(x) > 0$

\uparrow
avoid singularity.

though destroying the coordinate system, the angles are preserved by conformal transf.

$$\angle \theta = \frac{u \cdot v}{|u||v|} = \frac{u \cdot v}{\sqrt{u^2 v^2}} \quad u \cdot v = g(u, v) \\ = g_{uv} u^{uv}$$

Def. the conformal grp $\text{Conf}(M)$

the connected component containing 1 (identity)

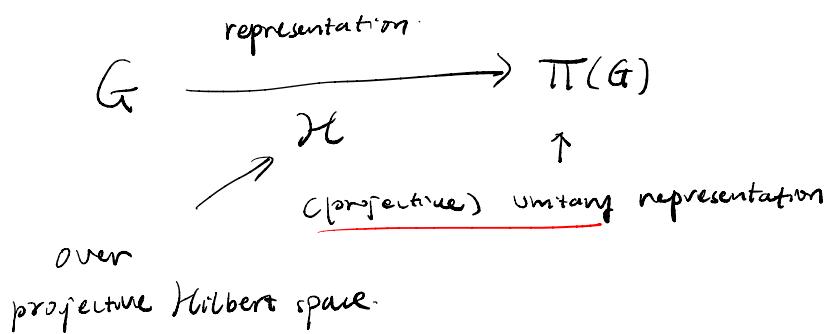
allow infinitesimal version
not including ones linked by
P.T etc. (Lie algebra)

is the grp of all conformal
transfs. of M .

* Note, connectedness of group of transfs. are usually
in compact-open cases.

* Note the definition of general M , i.e. including (pseudo-)Riemannian
manifold is straight forward. $g_{uv} \rightarrow$ a varying tensor field.

QM:



The interesting part of CFT is finding these PUIRs in

local QFTs

↓

making things complicated.

Note, most of (may be all) PUIRs of Conf(M) have been found and classified by mathematician at least for specific M.

Nontrivial physics:

The hard part is to encode the VIRs using local constructions, like what happens to relativistic field theory and Lorentz group / Poincaré group.

tension between {
Locality
Unitarity}

Classification of $G = \text{Conf}(M = \mathbb{R}^{p,q})$
 \hookrightarrow flat here.

For large p, q , the G usually turns out to be ordinary Lie groups, but for small p, q like 2d case, things are complicated and interesting.

Not all will be Lie algebra.



Using infinitesimal argument (local algebra) :

$$x \mapsto x + \varepsilon(x) \rightarrow O(1)$$

condition for $\varepsilon(x)$: $g_{\mu\nu}$ independent of x (flat).



$$\dot{g}_{\mu\nu} = g_{\mu\nu} + (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) + O(\varepsilon^2).$$

$$= \Omega(x) g_{\mu\nu} \propto \eta_{\mu\nu} \text{ (flat).}$$

$$\Rightarrow \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \propto \eta_{\mu\nu}$$

take trace \rightarrow

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d} (\delta^\nu_\mu) \eta_{\mu\nu} \quad (*)$$

$$\Rightarrow \Omega(x) = 1 + \frac{2}{d}(\partial \cdot \varepsilon) + O(\varepsilon^2).$$

\downarrow

$d=2$ is special.

It can be verified that from $(*)$

$(**)$

$$(\eta_{\mu\nu} \square + \underbrace{(d-2)\partial_\mu \partial_\nu}_{\uparrow}) (\partial \cdot \varepsilon) = 0$$

$$\square = \partial^2$$

$$= \eta_{\mu\nu} \partial^\mu \partial^\nu$$

$d=2$ less constraining.

Note, $\varepsilon(x)$ is C^∞ (smooth)

T Q: Are these two conditions sufficient to classify all ε ?

(Classification due to $(**)$ & $(*)$ =
(of $\varepsilon(x)$)).

$d > 2$:

$(*)$, $(**)$ \Rightarrow all 3rd derivatives vanishes

$\Rightarrow \varepsilon(x)$ is at most quadratic in x

(Global)

Poincaré

a. $\xi^\mu = a^\mu$ constants.

spacetime translations.

My including
bursts

linear in x { b. $\xi^\mu = \omega^\mu_\nu x^\nu$

$\omega^{\mu\nu}$ antisymmetric

"rotations"

c. $\xi^\mu = \lambda x^\mu$

$(\lambda > 0)$

scale transformations.

constant (linear in x)

quadratic in x d.

$$\xi^\mu = b^\mu x^2 - 2x^\mu (b \cdot x)$$

special conformal transformations. (SCT)

(rigid/global)

Theorem. A conformal transf. of acts on connected subspace

$U \subset \mathbb{R}^{p+q}$, $d=p+q \geq 2$, is a composition of exponentials.

of a. b. c. d. types.

(generators)

(Finite opposed to infinitesimal)

— translation: $x \rightarrow x + c$ $c \in \mathbb{R}^d$

— orthogonal transf: $x \rightarrow \Lambda x$ $\Lambda \in O(p, q)$ ($\stackrel{\text{so}(p, q)}{\text{actually}}$)

— dilations: $x \rightarrow \lambda x$. $\lambda \in \mathbb{R}^+$

- special conformal transf:

$$x^\mu \rightarrow \frac{x^\mu - b^\mu x^2}{1 - 2b^\nu x^\nu + b^2 x^2} \quad b \in \mathbb{R}^d$$

\downarrow

\circ

these SCT bring ∞ to some finite point and some finite point to ∞ (according to b^μ) .

Rmk

By compactification $\mathbb{R}^{p,q} \cup \{\infty\}$, we get a group, else some

SCT's not allowed (ill-defined)

∞ as an extra point

* Note, conformally infinity is interesting topic of relativity.
(general)
and AdS/CFT correspondence.

d=2

$\text{diag}(+1, +1)$

||

For $g_{\mu\nu} = \delta_{\mu\nu}$:

analytic functions
↑

(*) became the Cauchy - Riemann eqns :

$$\partial_1 \mathcal{E}_1 = \partial_2 \mathcal{E}_2$$

$$\partial_1 \mathcal{E}_2 = - \partial_2 \mathcal{E}_1$$

Complex coordinates : $Z = x^1 + i x^2$

$$\mathcal{E}(Z) = \mathcal{E}^1 + i \mathcal{E}^2$$

looks like infinitesimal
analytic functions

! not the same meaning
in gauge theory

(much more possibilities
than d>2)

(finite, no singularity when exponentiated)

⇒ 2d global conformal transformations. (after exponentiating)
(extend to finite distance from id, not infinitesimal).

corresponds to entire holomorphic functions.

(differentiable functions without singularity)

$Z \mapsto f(Z)$ with holomorphic inverse $f^{-1}(Z)$

they have form :

$$f(Z) = \alpha Z + \beta$$

↑
to form a group.

(near id)

There're more transformations called locally (can't be exponentiated)

I)

Q: singularity in finite transf
related to anomaly?

singular transf.

(having poles when exponentiated)

The 2d theories having global conformal symmetry is richer than locally conformal invariant ones since there're less symmetries thus less constraints.

In another perspective, locally conformal invariant 2d theories are highly constraint

Riemann sphere.

e.g. $\mathbb{R}^{2,0} \xrightarrow{\sim} \mathbb{C} \xrightarrow{\text{compactification}} \mathbb{C} \cup \{\infty\} = S^2$

global: $\text{Conf}(\mathbb{C})$
 $= \{ \text{linear complex variable functions } f \}$

\downarrow

$\Rightarrow \text{Diff}(S^2)$

$\text{Conf}(\mathbb{C} \cup \{\infty\}) = \{ f(z) \mid f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \alpha, \beta, \gamma, \delta \in \mathbb{C} \}$

\Rightarrow group of Möbius transformations

$\frac{\alpha z + \beta}{\gamma z + \delta}, \alpha \bar{\beta} - \beta \bar{\alpha} \neq 0$

$$\text{Conf}(\mathbb{C} \cup \{\infty\}) = \{ f(z) \mid f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \alpha, \beta, \gamma, \delta \in \mathbb{C} \}$$

\Rightarrow group of Möbius transformations

§ More on $d=2$ case : local conformal transformations
 (infinitesimal) singular finite ones.

(local is less constrained since it only ask it be be conformal
 transfs up to $O(\epsilon)$) . thus there's more "vector field" $X: \mathbb{R}^{2,0} \rightarrow \mathbb{R}^{2,0}$

like generators in Lie
 algebra)

$$\phi = id + \epsilon X : \mathbb{R}^{2,0} \xrightarrow{=} \mathbb{R}^{2,0}$$

↑

need not to be welldefined on

all the manifold of finite transfs. (upon exponentiation)

up to all orders $O(\epsilon)^k$

(there doesn't
 form a group!)
algebra!

usually, we'll have ∞ -dim algebras near \mathbb{I}

For infinitesimal conformal transf. we focus on the algebras rather
 than groups.
 (usually ∞ -dim algebras)

(infinitesimally & only for continuous groups).

conformal algebra.

is just (not always)

like $G = \text{Lie group} \longrightarrow \text{Lie algebra } \mathfrak{g}$ (be exp).

rep of \mathfrak{g} : $(x, \pi: \mathfrak{g} \rightarrow L(x))$

↓

↓ exp

linear operators, not

necessarily bounded especially
when $x \in \mathbb{R}^{\text{dim}}$.

$\exp(iX\theta) \Rightarrow \pi(g \in G) = U(x)$

↑

elements in \mathfrak{g}

↑

unitary, since U is

This picture already assume well-defined G , but for local conformal transfs. there's more in algebra.

(infinitesimally)

\mathfrak{g} ∞ dimensional: $\exp(isX)$ may not be continuous in s at some points.

(infinitesimal transf.)

Work with "Lie" algebra, rather than finite group
(no group of finite transf.).

(not well-defined
for all generators)

§ Local algebra: $(1+ld)$

$$z = x + ix^2$$

holomorphic

Global: $z \xrightarrow{f} f(z)$

$$\varepsilon = \varepsilon_1 + i\varepsilon_2$$

Local: $z \mapsto z + \varepsilon(z)$ $\varepsilon \text{ is holomorphic}$

$\bar{z} \mapsto \bar{z} + \bar{\varepsilon}(z)$ i.e. ε depends only
 ϕ on ε , not $\bar{\varepsilon}$

A convenient basis: $\varepsilon_n(z) = -\varepsilon z^{n+1}$, $n \in \mathbb{Z}$, $\varepsilon \sim O(1)$

\uparrow
∞-dim.

Corresponding vector field:

vector field $\in T_z(M = \mathbb{C})$

$$\phi(z)(\equiv z + \varepsilon_n(z)) = \exp(\epsilon \overset{\downarrow}{X})$$

\nearrow
diffeomorphism

$$\text{of } \mathbb{C} \implies X_{\mathbb{R}} \models l_n = -z^{n+1} \partial_z, \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}$$

\uparrow depend on z

Differential operator/representation of the (Lie) algebra
field (like $p_x \rightarrow -i\partial_x$)

the Lie brackets:

$$[l_m, l_n] = (m-n) l_{m+n}$$

∞-dim

$$\forall m, n \in \mathbb{Z} \quad [\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n}$$

Lie algebra

$$[\tilde{e}_n, \tilde{e}_m] = 0$$

∞ -dim Lie algebra
is more subtle & consequential -

this is called the Witt algebra :

$$\text{Witt} = \mathbb{A} \oplus \bar{\mathbb{A}}$$

$$\text{Span}\{\tilde{e}_n\} \quad \{\tilde{e}_{\bar{n}}\}$$

Though the exp of \tilde{e}_n 's may not be well-defined, Witt is indeed a Lie algebra of some group but not $\text{Conf}(M)$

(Many structures are defined via algebra, like quantum group)

Subalgebra corresponds to global transfs: $\text{Lie}(\text{Conf}(M))$.

Vector field : $v(x) = \tilde{v}(z) = - \sum_{n=-\infty}^{+\infty} v_n z^{-n-1} \partial z$

When does $v(x)$ exponentiate to be a holomorphic function.

necessary

\Rightarrow nonsingular as $z \rightarrow 0$

$\Rightarrow \underline{v_n = 0 \quad n < -1}$

inverse exp to be holomorphic \Rightarrow non singular as $z \rightarrow \infty$
on Riemann sphere.

\Rightarrow If we don't compactification, then $v_n = 0 \quad n > 0$

we end up with linear maps $(f(z) = \alpha z + \beta)$

If we compactified, then we demand holomorphism
on Riemann sphere, we have more possibilities

$\Rightarrow v_n = 0 \quad n > 1$ only b_1, b_0, b_1 left
 $\& (\bar{b}_1, \bar{b}_0, \bar{b}_1)$

these 6 generators close to form a subalgebra. under [,]

Note that for $\mathbb{R}^{2,0}$ compactified to $\mathbb{C} \cup \{\infty\}$, the global
conformal transfs. consists of Möbius transf. discussed above.
Thus, this subalgebra (6 generators) of W_{+} generate subgroup of
 $\text{Conf}(\mathbb{C} \cup \{\infty\})$

$\Rightarrow \left\{ \begin{array}{l} \text{the linear fractional transformations : } z \mapsto \frac{az+b}{cz+d} \\ \text{ s.t. } ad - bc = 1 \end{array} \right\}$

$$= \boxed{\text{PSL}(2, \mathbb{C})}$$

$\left\{ \begin{array}{l} \text{Möbius transfs} \\ = \text{Conf}(\mathbb{C} \cup \{\infty\}) \end{array} \right\}$

Matrix form (representation on \mathbb{C})

explicitly :

$$z \mapsto \frac{az+b}{cz+d} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ act on } \mathbb{C}$$

- $\exp(s\text{sl}_1) : z \mapsto z-s$ translation • $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$
- $\exp(s(\text{dil} + \bar{\ell}_0)) : z \mapsto e^{-s}z$ dilation • $\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$
- $\exp(is(\bar{\ell}_0 - \ell_0)) : z \mapsto e^{is}$ rotation
- $\exp(s\text{sl}_1) : z \mapsto \frac{z}{1+sz}$ SCT: • $\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$

Claim . For $\mathbb{R}^{p,0}$, $\boxed{p \geq 2}$.

global ones \longleftrightarrow local ones.

imaginary time = temperature.

Physical interpretation:

($\mathbb{R}^{2,0}$ is good for Statistical

Mechanics. by Wick rotation)

Minkowski space : $d=2$, $\mathbb{R}^{1,1}$

Conf($\mathbb{R}^{1,1}$) is special!

$$(x,y) \mapsto (u(x,y), v(x,y))$$

Theorem: A smooth map $\varphi: M \rightarrow \mathbb{R}^{1,1}$

from a subset $M \subset \mathbb{R}^{1,1}$ is conformal iff

(pull back the metric
to a scalar multiplication
of the metric)

$$\varphi^* g_{\mu\nu} \propto g_{\mu\nu}$$

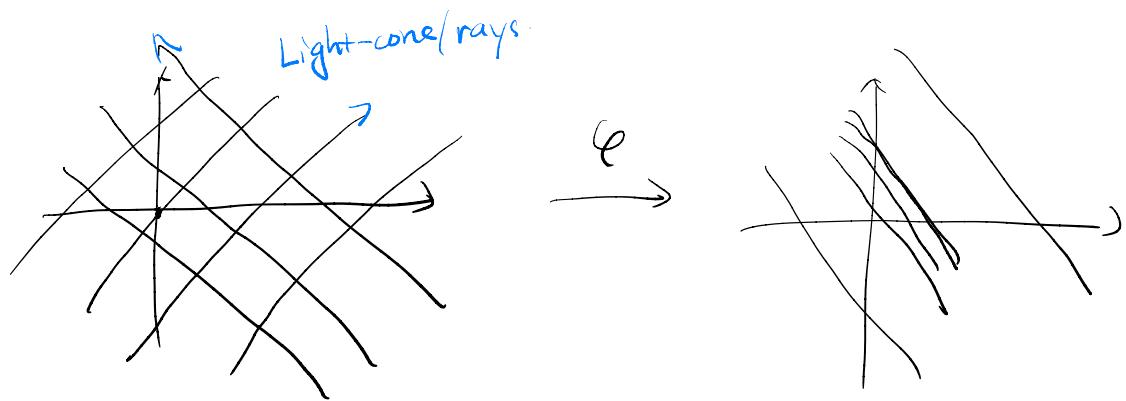
Arbitrary conformal
transformation of $\mathbb{R}^{1,1}$

$$u_x^2 > v_x^2 \quad \& \quad u_x = v_y, \quad u_y = v_x$$

$$\frac{\partial}{\partial x} u$$

$$\text{or } u_x = -v_y, \quad u_y = -v_x$$

Using lightcone coordinates to make the form of these S.T. clear:



$(\mathbb{R}^{1,1})$

All possible conformal transfs. take those form.

dilate-contract of the lightcone coordinates preserve angles.

two independent series.

Theorem: For $f \in C^\infty(\mathbb{R})$ let $f_\pm \in C^\infty(\mathbb{R}^2, \mathbb{R})$

defined by $f_\pm(x, y) = f(x \pm iy)$

\sim
 \mathbb{R}^2 coordinates

lightcone coordinates

(two axes are distinguished)
(independent).

construct dilations.
contractions
of each axis

go to
light-cone
coordinates.

the map $\Phi : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}^2, \mathbb{R})$

$$(f(x), g(y)) \longmapsto \frac{1}{2} + \underbrace{\frac{f_+ + g_-}{2}(x,y)}_{u(x,y)}, \underbrace{\frac{f_+ - g_-}{2i}(x,y)}_{v(x,y)}$$

use f, g , dilate/contract one

axis's correspondingly

has properties: vector functions.

necessary for C.T.

$$(1) \quad \text{Im } \Phi = \left\{ (u, v) \mid u_x = v_y, u_y = -v_x \right\}$$

(2) $\Phi(f, g)$ is conformal transf. iff $f' > 0$ & $g' > 0$

or $f' < 0$ & $g' < 0$

(3) $\Phi(f, g)$ bijective iff f, g bijective.

$$(4) \quad \Phi(f \cdot h, g \cdot k) = \Phi(f, g) \circ \Phi(h, k)$$

(a homomorphism-)

The two theorems

the group of orientation-preserving transformations of
 $M = \mathbb{R}^{1|1}$ is isomorphic to (M hasn't compactified)

together orientation-preserving.

$$(\text{Diff}_+(\mathbb{R}) \times \text{Diff}_+(\mathbb{R})) \sqcup (\text{Diff}_-(\mathbb{R}) \times \text{Diff}_-(\mathbb{R}))$$

↑
orientation-preserving

↑
orientation-reversing

Subgroup in interest

Dropped unreality

Study $\text{Diff}_+(\mathbb{R})$ (finding VIs) and
construct tensor product representations (to obtain $\text{Conf}(\mathbb{R}^{1|1})$)

Compactification: $\mathbb{R}^{1|1} \rightarrow S^{1|1}$
 $\subset \mathbb{R}^{2|0} \times \mathbb{R}^{0|2}$

$\Rightarrow \dots$ is isomorphic to

$$\underbrace{(\text{Diff}_+(S^1) \times \text{Diff}_+(S^1))}_{\text{not interested.}} \cup \text{...} \rightarrow$$

$$\cong \text{Conf}(\mathbb{R}^{1,1}_{\text{compactified}})$$

since not connected to \mathbb{R}

Focus on $\boxed{\text{Diff}_+(S^1)}$ (chiral half)

Q: Why $\text{Diff}_+(S^1)$ have ∞ -dim Lie(G) while $\text{Conf}(S^1)$ have finite dim

Note, $(\text{Diff}_+(S^1))$

VIRs of this group actually won't lead to sensible Quantum theories, some of them don't give H bounded below (unstable Quantum theories)

Note, all PUIRs of this group correspond to the number of central charge, use PUIR to construct sensible QFTs

