# 高数精讲（16）

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# 第二节　偏导数与全微分的计算

## 本节内容要点

### 一. 考试内容要点精讲

（一）复合函数求导法

（二）隐函数求导法

# 二. 常考题型方法与技巧

题型一 求一点处的偏导数与全微分

题型二 求已给出具体表达式函数的偏导数与全微分

题型三 含有抽象函数的复合函数的偏导数与全微分

题型四 隐函数的偏导数与全微分

# 一．考试内容要点精讲

$dy = y'_x dx = y'_u u'_x dx$

## （一）复合函数求导法

$y=f(u),\ u=\varphi(x),\Rightarrow y=f[\varphi(x)],\ y'_x=y'_u u'_x = y'_u du$

设 $u=u(x,y), v=v(x,y)$ 可导，$z=f(u,v)$ 在相应点有连续一阶偏导数，则

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

$z$

$u \quad v$

$x \quad y \quad x \quad y$

**全微分形式不变性**

设 $z=f(u,v), u=u(x,y), v=v(x,y)$ 都有连续一阶偏导数

则 $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \quad = \quad dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$

# （二） 隐函数求导法

$F_y \neq 0 \Rightarrow y = y(x, z)$

## 1. 由一个方程所确定的隐函数

设 $F(x,y,z)$ 有连续一阶偏导数，$F_z \neq 0$，$z = z(x,y)$ 由 $F(x,y,z) = 0$ 所确定.

方法：（1） 公式 $\boxed{\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}}$， $\boxed{\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}}$；

（2） 等式两边求导

$$F_x + F_z \frac{\partial z}{\partial x} = 0, \quad F_y + F_z \frac{\partial z}{\partial y} = 0,$$

（3） 利用微分形式不变性

$$F_x dx + F_y dy + F_z dz = 0$$

$dz = \left(\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial z}{\partial y}\right)dy$

## 2. 由方程组所确定的隐函数（仅数一要求）

设 $u=u(x,y), v=v(x,y)$ 由 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 所确定.

方法:

（1）等式两边求导 $\begin{cases} F_x+F_u\dfrac{\partial u}{\partial x}+F_v\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\dfrac{\partial u}{\partial x}+G_v\dfrac{\partial v}{\partial x}=0 \end{cases}$

（2）微分形式不变性 $\begin{cases} F_x dx+F_y dy+F_u du+F_v dv=0 \\ G_x dx+G_y dy+G_u du+G_v dv=0 \end{cases}$

$\Rightarrow$ $du=(\overset{u_x}{\quad})dx+(\overset{u_y}{\quad})dy$

$dv=(\underset{v_x}{\quad})dx+(\underset{v_y}{\quad})dy$

# 题型一 求一点处的偏导数与全微分

$f'_x(x_0,y_0)=\frac{d}{dx}f(x,y_0)\Big|_{x=x_0}$

【例1】设 $f(x,y)=\begin{cases}\dfrac{\sqrt{|x|}}{x^2+y^2}\sin(x^2+y^2) & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

求 $f_x(0,0)$ 和 $f_y(0,0)$.

$\varphi'(0)=\lim\limits_{x\to0}\dfrac{\frac{\sqrt{|x|}}{x^2}\sin x^2}{x}=\lim\limits_{x\to0}\dfrac{\sqrt{|x|}}{x}=\infty$

[补2] 先代后求. $f(x,0)=\begin{cases}\frac{\sqrt{|x|}}{x^2}\sin x^2, & x\neq0\\ 0, & x=0\end{cases}=\varphi(x)$

【解】由于 $\lim\limits_{\Delta x\to0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}$

$$=\lim_{\Delta x\to0}\frac{\dfrac{\sqrt{|\Delta x|}}{(\Delta x)^2}\sin(\Delta x)^2}{\Delta x}=\lim_{\Delta x\to0}\frac{\sqrt{|\Delta x|}}{\Delta x}=\infty.$$

$f(0,y)=\begin{cases}0, & y\neq0\\ 0, & y=0\end{cases}$

$f'_y(0,0)=0$

则 $f_x(0,0)$ 不存在；而

$$f_y(0,0)=\lim_{\Delta y\to0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to0}\frac{0-0}{\Delta y}=0.$$

【例2】设 $f(x,y)=\dfrac{2x+3y}{1+xy\sqrt{x^2+y^2}}$，求 $f_x(0,0)$ 和 $f_y(0,0)$.

① $f_x$ $f_y$

② 代入

"先代后求"

【解】 $f_x(0,0)=\dfrac{d}{dx}f(x,0)\Big|_{x=0}=\dfrac{d}{dx}(2x)\Big|_{x=0}=2;$

$f_y(0,0)=\dfrac{d}{dy}f(0,y)\Big|_{y=0}=\dfrac{d}{dy}(3y)\Big|_{y=0}=3;$

【例3】设 $z=\ln(1+xy^2)$，则 $\left.\dfrac{\partial^2 z}{\partial x\partial y}\right|_{(0,1)}=$ ______.

【解】 $\dfrac{\partial z}{\partial x}=\dfrac{y^2}{1+xy^2}$

$x=0$

$$\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(0,1)}=\left.\frac{d}{dy}\left(\frac{\partial z(0,y)}{\partial x}\right)\right|_{y=1}=\left.\frac{d}{dy}(y^2)\right|_{y=1}=2$$

“先代后求”

【例4】设 $f(x,y,z)=\sqrt[z]{\dfrac{x}{y}}$，则 $df(1,1,1)=$ ______.

【解】 $f_x(1,1,1)=\dfrac{d}{dx}f(x,1,1)\Big|_{x=1}=\dfrac{d}{dx}(x)\Big|_{x=1}=1$

$$f_y(1,1,1)=\frac{d}{dy}f(1,y,1)\Big|_{y=1}=\frac{d}{dy}\left(\frac{1}{y}\right)\Bigg|_{y=1}=-\frac{1}{y^2}\Bigg|_{y=1}=-1$$

$$f_z(1,1,1)=\frac{d}{dz}f(1,1,z)\Bigg|_{z=1}=\frac{d}{dz}(1)\Big|_{z=1}=0$$

故 $df(1,1,1)=dx-dy$

# 题型二 求已给出具体表达式函数的偏导数与全微分

【例1】设 $z=(x^2+y^2)e^{-\arctan\frac{y}{x}}$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$ 及 $dz$.

【例2】设 $z=(1+x^2+y^2)^{\boxed{xy}}$，求 $\dfrac{\partial z}{\partial x}$ 及 $\dfrac{\partial z}{\partial y}$.

【解1】 $z=e^{xy\ln(1+x^2+y^2)}$

【解2】 $\ln z = xy\ln(1+x^2+y^2)$

【解3】令 $u=1+x^2+y^2, v=xy$，则 $\boxed{z=u^v}$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=vu^{v-1}2x+u^v\ln u\cdot y$$

$$=(1+x^2+y^2)^{xy}\left[\frac{2x^2y}{1+x^2+y^2}+y\ln(1+x^2+y^2)\right]$$

同理可得 $\dfrac{\partial z}{\partial y}$.

$(x^y)'=yx^{y-1}$

$(a^x)'=a^x\ln a$

$y=(1+x^2)^{\sin x}=u^v$

$u=1+x^2$

$v=\sin x$

$$\frac{dy}{dx}=vu^{v-1}2x+u^v\ln u\cdot\cos x$$

【例3】若函数 $z=f(x,y)$ 满足 $\frac{\partial^2 z}{\partial y^2}=2$，且 $f(x,1)=x+2$，又 $f_y(x,1)=x+1$，则 $f(x,y)$ 等于

A) $y^2+(x-1)y-2$;　　B) $y^2+(x+1)y+2$;

C) $y^2+(x-1)y+2$;　　D) $y^2+(x+1)y-2$;

【解1】 容易验证（C）正确.

【解2】由 $\frac{\partial^2 z}{\partial y^2}=2$ 知 $\frac{\partial z}{\partial y}=\int 2dy=2y+\varphi(x)$

由 $f_y(x,1)=1+x$ 知

$$1+x=2+\varphi(x)\Rightarrow\varphi(x)=x-1\Rightarrow\frac{\partial z}{\partial y}=2y+x-1$$

$$z=\int(2y+x-1)dy=y^2+y(x-1)+\psi(x)$$

由 $f(x,1)=x+2$ 知

$$x+2=1+(x-1)+\psi(x)\Rightarrow\psi(x)=2$$

$f'(x)=2x$

$f(x)=\int 2x\,dx=x^2+C$

【例4】已知 $\frac{\partial^2 z}{\partial x\partial y}=1$，且当 $x=0$ 时，$z=\sin y$；当 $y=0$ 时，$z=\sin x$ 则 $z(x,y)=$ ______.

【解1】由 $\frac{\partial^2 z}{\partial x\partial y}=1$ 知，$\frac{\partial z}{\partial x}=\int 1dy=y+\varphi(x)$

$$z=\int[y+\varphi(x)]dx=xy+\int\varphi(x)dx+\psi(y)=xy+g(x)+\psi(y)$$

由 $x=0$ 时，$z=\sin y$ 知，$\sin y=g(0)+\psi(y)$; $\rightarrow 0=g(0)+\psi(0)$

由 $y=0$ 时，$z=\sin x$ 知，$\sin x=g(x)+\psi(0)$

$$z=xy+\sin x+\sin y-g(0)-\psi(0)$$

$$g(0)+\psi(0)=0$$

故 $z(x,y)=xy+\sin x+\sin y$

*【解2】$z(x,0)=\sin x$

$z_x(x,0)=\cos x$

$\cos x=\varphi(x)$

$\frac{\partial z}{\partial x}=y+\cos x$

$z=xy+\sin x+\psi(y)$

$\sin y=\psi(y)$

【例5】已知 $(axy^3-y^2\cos x)dx+(1+by\sin x+3x^2y^2)dy$ $= df(x,y)$

是某一函数的全微分，则 $a,b$ 取值分别为：

（A） − 2 和 2；　（B） 2 和 − 2；　（C） − 3 和 3；　（D）3 和 − 3；

【解】$df(x,y)=(axy^3-y^2\cos x)dx+(1+by\sin x+3x^2y^2)dy$

则 $\dfrac{\partial f}{\partial x}=axy^3-y^2\cos x$，　$\dfrac{\partial f}{\partial y}=1+by\sin x+3x^2y^2$

从而有 $\dfrac{\partial^2 f}{\partial x\partial y}=3axy^2-2y\cos x$，　$\dfrac{\partial^2 f}{\partial y\partial x}=by\cos x+6xy^2$

由于 $\dfrac{\partial^2 f}{\partial x\partial y}$ 和 $\dfrac{\partial^2 f}{\partial y\partial x}$ 都连续，从而有 $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}$ 即

$$\boxed{3axy^2-2y\cos x=by\cos x+6xy^2}$$

则 $\begin{cases}3a=6,\\ b=-2,\end{cases}$ 即 $\begin{cases}a=2\\ b=-2\end{cases}$，故应选（B）.

【注】若 $P(x,y)$ 和 $Q(x,y)$ 有一阶连续偏导数，且

$y$　　　　$x$

$P(x,y)dx+Q(x,y)dy$ 是某一函数的全微分，则

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$ x

✓

$Pdx+Qdy=df$

$P=\frac{\partial f}{\partial x}$　　$Q=\frac{\partial f}{\partial y}$

$\frac{\partial P}{\partial y}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial Q}{\partial x}=\frac{\partial^2 f}{\partial y\partial x}$

【例6】设 $f(x)$ 有连续一阶导数，

$$(xy - yf(x))dx + (f(x) + y^2)dy = du(x, y),$$

求 $f(x)$ 及 $u(x, y)$, 其中 $f(0) = -1$.

① 偏积分

10:11

$\frac{\partial u}{\partial x} = y \Rightarrow u = \int y\,dx = xy + \varphi(y)$

$\frac{\partial u}{\partial y} = x + \varphi'(y)$

【解】由题设知 $x - f(x) = f'(x)$

$\frac{\partial u}{\partial y} = (x-1) + y^2 = x + \varphi'(y)$

$$f'(x) + f(x) = x$$

$u = (x-1)y + \frac{1}{3}y^3 + C$

$\varphi(y) = \frac{1}{3}y^3 - y + C$

$$f(x) = (x-1) + Ce^{-x}$$

$u = xy + \frac{1}{3}y^3 - y + C$

② 凑微分　分组凑

由 $f(0) = -1$ 知, $C = 0$, $f(x) = x - 1$

$$du(x, y) = y\,dx + [(x-1) + y^2]dy$$

$du = y\,dx + \overset{d(x-1)}{(x-1)}dy + d\frac{1}{3}y^3$

$$u(x, y) = y(x-1) + \frac{1}{3}y^3 + C$$

$\Rightarrow u = (\quad) + C$

$= d(x-1)y + d\frac{1}{3}y^3$

## 题型三 含有抽象函数的复合函数偏导数与全微分

【例1】设函数 $f(u,v)$ 由关系式 $f[xg(y),y]=x+g(y)$ 所确定，

其中函数 $g(y)$ 可微，且 $g(y)\neq 0$，则 $\dfrac{\partial^2 f}{\partial u\partial v}=$ ______

【解】令 $xg(y)=u, y=v$，则 $x=\dfrac{u}{g(y)}=\dfrac{u}{g(v)}$ 于是

$$f(u,v)=\frac{u}{g(v)}+g(v)$$

$$\frac{\partial f}{\partial u}=\frac{1}{g(v)},\qquad \frac{\partial^2 f}{\partial u\partial v}=-\frac{g'(v)}{[g(v)]^2}$$

【例3】设 $z=f(xy,x^2+y^2)$，求 $\frac{\partial z}{\partial x},\frac{\partial^2 z}{\partial x\partial y}$. 其中 $f(u,v)$ 有二阶连续偏导数.

①　②

3分

$f_2(xy,x^2+y^2)$　$+yf_{12}\cdot 2y$

【解】$\frac{\partial z}{\partial x}=yf_1+2xf_2$

$\frac{\partial^2 z}{\partial x\partial y}=f_1+yf_{11}x+2xf_{22}2y$

$+2xf_{21}x$

$$\frac{\partial^2 z}{\partial x\partial y}=f_1+y[xf_{11}+2yf_{12}]+2x[f_{21}x+f_{22}\cdot 2y]$$

$$=f_1+xy[f_{11}+4f_{22}]+2(x^2+y^2)f_{12}$$

【例4】设 $f(x,y)$ 可微，又 $f(0,0)=0$, $f_x(0,0)=a$, $f_y(0,0)=b$ 且

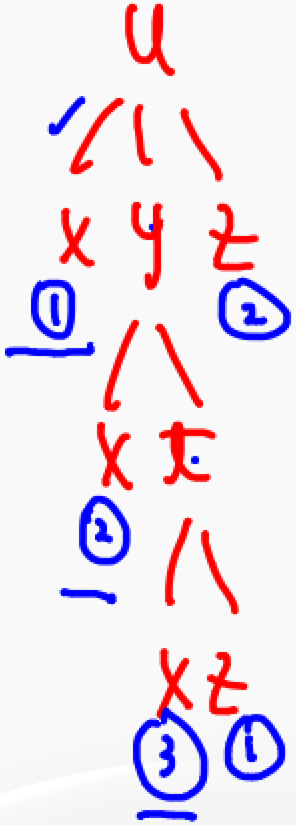
$g(t)=f[t,f(t,t^2)]$，求 $g'(0)$.

【解】 $g'(t)=f_1[t,f(t,t^2)]+f_2[t,f(t,t^2)]\cdot[f_1(t,t^2)+f_2(t,t^2)\cdot 2t]$

$f_1(0,0)$ $f_2(0,0)$

$g'(0)=a+b[a+0\times b]=a(1+b)$

【例5】设 $u=f(x,y,z),y=\varphi(x,t),t=\psi(x,z)$，其中 $f,\varphi,\psi$ 可微，求 $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial z}$.




【解】
$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial x}$$

$$\frac{\partial u}{\partial z}=\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial z}+\frac{\partial f}{\partial z}$$

$u=u(\xi,\eta)$

【例7】设函数 $u=f(x,y)$ 具有二阶连续偏导数，且满足 $4\dfrac{\partial^2 u}{\partial x^2}+12\dfrac{\partial^2 u}{\partial x\partial y}+5\dfrac{\partial^2 u}{\partial y^2}=0.$

确定 $a,b$ 的值，使等式在变换 $\xi=x+ay,\eta=x+by$ 下简化为 $\dfrac{\partial^2 u}{\partial\xi\partial\eta}=0.$

$=\dfrac{\partial^2 u}{\partial\xi^2}+\dfrac{\partial^2 u}{\partial\xi\partial\eta}+\dfrac{\partial^2 u}{\partial\eta\partial\xi}+\dfrac{\partial^2 u}{\partial\eta^2}$

【解1】

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta}\qquad \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial\xi^2}+2\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\partial^2 u}{\partial\eta^2}$$

$$\frac{\partial u}{\partial y}=a\frac{\partial u}{\partial\xi}+b\frac{\partial u}{\partial\eta}\qquad \frac{\partial^2 u}{\partial y^2}=a^2\frac{\partial^2 u}{\partial\xi^2}+2ab\frac{\partial^2 u}{\partial\xi\partial\eta}+b^2\frac{\partial^2 u}{\partial\eta^2}$$

$$\frac{\partial^2 u}{\partial x\partial y}=a\frac{\partial^2 u}{\partial\xi^2}+(a+b)\frac{\partial^2 u}{\partial\xi\partial\eta}+b\frac{\partial^2 u}{\partial\eta^2}$$

u
ξ η
x y x y

u
x y
ξ η ξ η

$$\underbrace{(5a^2+12a+4)}_{=0}\frac{\partial^2 u}{\partial\xi^2}+\underbrace{[10ab+12(a+b)+8]}_{\neq 0}\frac{\partial^2 u}{\partial\xi\partial\eta}+\underbrace{(5b^2+12b+4)}_{=0}\frac{\partial^2 u}{\partial\eta^2}=0$$

$$\begin{cases}5a^2+12a+4=0\\5b^2+12b+4=0\end{cases}\qquad 10ab+12(a+b)+8\neq 0$$

【例7】设函数 $u=f(x,y)$ 具有二阶连续偏导数，且满足 $4\dfrac{\partial^2 u}{\partial x^2}+12\dfrac{\partial^2 u}{\partial x\partial y}+5\dfrac{\partial^2 u}{\partial y^2}=0.$

确定 $a,b$ 的值，使等式在变换 $\xi=x+ay,\eta=x+by$ 下简化为 $\dfrac{\partial^2 u}{\partial\xi\partial\eta}=0.$

【解2】由 $\xi=x+ay,\eta=x+by$ 解得 $\begin{cases} x=\dfrac{a\eta-b\xi}{a-b}, \\ y=\dfrac{\xi-\eta}{a-b}, \end{cases}$

$$\frac{\partial u}{\partial\xi}=\frac{-b}{a-b}\frac{\partial u}{\partial x}+\frac{1}{a-b}\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=\frac{-ab}{(a-b)^2}\frac{\partial^2 u}{\partial x^2}+\frac{a+b}{(a-b)^2}\frac{\partial^2 u}{\partial x\partial y}+\frac{-1}{(a-b)^2}\frac{\partial^2 u}{\partial y^2}=0$$


欲使 $\dfrac{\partial^2 u}{\partial\xi\partial\eta}=0$， 即 $-ab\dfrac{\partial^2 u}{\partial x^2}+(a+b)\dfrac{\partial^2 u}{\partial x\partial y}-\dfrac{\partial^2 u}{\partial y^2}=0$

$$\frac{-ab}{4}=\frac{a+b}{12}=\frac{-1}{5}$$

【例8】设 $f(u)$ 具有二阶连续导数，而 $z=f(e^x\sin y)$ 满足方程

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=ze^{2x}.$$ 求 $\boxed{f(x)}$.

$z=f(u)$

【解】令 $u=e^x\sin y$，则

$$\frac{\partial z}{\partial x}=f'(u)e^x\sin y,\quad \frac{\partial^2 z}{\partial x^2}=f''(u)e^{2x}\sin^2 y+f'(u)e^x\sin y$$

$$\frac{\partial z}{\partial y}=f'(u)e^x\cos y,\quad \frac{\partial^2 z}{\partial y^2}=f''(u)e^{2x}\cos^2 y-f'(u)e^x\sin y$$

$$\boxed{f''(u)=f(u)}$$

$r^2-1=0$

$$f''(u)-f(u)=0$$

$$f(u)=c_1e^u+c_2e^{-u}$$

【例9】设 $(r,\theta)$ 为极坐标，$u=u(r,\theta)$ 具有二阶连续偏导数，

并满足 $\frac{\partial u}{\partial \theta}\equiv 0$，且 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$，求 $u(r,\theta)$.

$\varphi = c_1\ln r + c_2$

【解】由 $\frac{\partial u}{\partial \theta}=0$ 知，$u$ 仅为 $r$ 的函数，令 $u=\varphi(r)$，$r=\sqrt{x^2+y^2}$

$u=\varphi(\sqrt{x^2+y^2})$

$$\frac{\partial u}{\partial x}=\varphi'(r)\frac{x}{\sqrt{x^2+y^2}}=\varphi'(r)\frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2}=\varphi''(r)\frac{x^2}{r^2}+\varphi'(r)\frac{r-\frac{x^2}{r}}{r^2}=\varphi''(r)\frac{x^2}{r^2}+\varphi'(r)\left(\frac{1}{r}-\frac{x^2}{r^3}\right)$$

$$\frac{\partial^2 u}{\partial y^2}=\varphi''(r)\frac{y^2}{r^2}+\varphi'(r)\left(\frac{1}{r}-\frac{y^2}{r^3}\right)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\varphi''(r)+\varphi'(r)\frac{1}{r}=0 \qquad \varphi(r)=C_1\ln r+C_2$$

$\boxed{r\varphi''+\varphi'=0}$

$(r\varphi')'=0$

$r\varphi'=c_1 \qquad \varphi'=\frac{c_1}{r}$

$f(x,y)=x^4+y^4$　　$f(x,y)=x^2+y^2$ 2次

**【例10】** 若对任意 $t>0$ 有 $f(tx,ty)=t^n f(x,y)$，则称函数 $f(x,y)$ 是 $n$ 次齐次函数，试证：若 $f(x,y)$ 可微，则 $f(x,y)$ 是 $n$ 次齐次函数 $\Leftrightarrow x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf(x,y)$

$f\left(\frac{y}{x}\right)$ 0次

**【证】必要性**　由 $f(tx,ty)=t^n f(x,y)$ 得

$$xf_1(tx,ty)+yf_2(tx,ty)=nt^{n-1}f(x,y)$$

$f(x,y)=x^2y^3$

$f(tx,ty)=t^2xy$

令 $t=1$ 得　$xf_1(x,y)+yf_2(x,y)=nf(x,y)$

即　$x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf(x,y)$

**充分性**　令 $F(t)=f(tx,ty)\,(t>0)$　则 $\dfrac{dF}{dt}=xf_1(tx,ty)+yf_2(tx,ty)$

$$t\frac{dF}{dt}=txf_1(tx,ty)+tyf_2(tx,ty)\overset{*}{=}nf(tx,ty)=nF(t)$$

$f(tx,ty)=t^n f(x,y)$

$$\frac{dF}{F}=\frac{n}{t}dt,\quad \boxed{F(t)=Ct^n},\ F(1)=C,\ F(1)=f(x,y)$$

2次 0 2次

$=2z$

【例】设 $z=xyf(\frac{y}{x})$，且 $f(u)$ 可导，若 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy(\ln y-\ln x)$，则（ ）

（A）$f(1)=\frac{1}{2}, f'(1)=0.$ （B）$f(1)=0, f'(1)=\frac{1}{2}.$

（C）$f(1)=\frac{1}{2}, f'(1)=1.$ （D）$f(1)=0, f'(1)=1.$

$f(1)=0 \quad f'(1)=\frac{1}{2}$

【解1】$\frac{\partial z}{\partial x}=yf(\frac{y}{x})-\frac{y^2}{x}f'(\frac{y}{x})$

$\frac{\partial z}{\partial y}=xf(\frac{y}{x})+yf'(\frac{y}{x})$

$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2xyf(\frac{y}{x})=xy\ln\frac{y}{x}$ $\quad f(x)=\frac{1}{2}\ln x$

【解2】$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2xyf(\frac{y}{x})$

$t^2 xyf(\frac{y}{x})$

# 题型四　隐函数的偏导数与全微



【例1】设 $z=z(x,y)$ 是由方程 $z+e^z=xy$ 所确定，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

【解1】公式法 $\boxed{\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},}$ $\boxed{\frac{\partial z}{\partial y}=-\frac{F_y}{F_z};}$

【解2】等式两边求导

$$\frac{\partial z}{\partial x}=-\frac{\frac{1}{z}F_1'}{\left(-\frac{x}{z^2}\right)F_1'+\frac{1}{y}F_2'}$$

【例3】利用微分形式不变性

【例2】设方程 $F(\frac{x}{z},\frac{z}{y})=0$ 可确定函数 $z=z(x,y)$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

**【例4】** 设 $u=f(x,y,z)$, $\varphi(x^2,e^y,z)=0$, $y=\sin x$ 确定了函数 $u=u(x)$, 其中 $f,\varphi$ 都有一阶连续偏导数, 且 $\dfrac{\partial\varphi}{\partial z}\neq 0$, 求 $\dfrac{du}{dx}$.

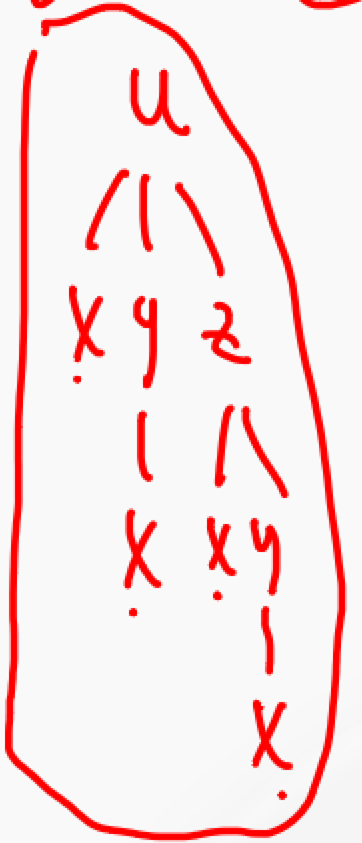
**【解1】** $$\frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cos x+\frac{\partial f}{\partial z}\frac{dz}{dx}$$

$\varphi(x^2,e^y,z)=0$ 两端对 $x$ 求导得

$$\varphi_1 2x+\varphi_2 e^y\cos x+\varphi_3\frac{dz}{dx}=0$$

解得 $$\frac{dz}{dx}=-\frac{1}{\varphi_3}(2x\varphi_1+\varphi_2 e^y\cos x)$$

$$\frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cos x-\frac{\dfrac{\partial f}{\partial z}}{\varphi_3}(2x\varphi_1+\varphi_2 e^y\cos x)$$




【例4】设 $u = f(x,y,z)$, $\varphi(x^2, e^y, z) = 0$, $y = \sin x$ 确定了函数 $u = u(x)$, 其中 $f, \varphi$ 都有一阶连续偏导数, 且 $\dfrac{\partial \varphi}{\partial z} \neq 0$, 求 $\dfrac{du}{dx}$.

【解2】

$u = f(x,y,z)$ ✓

$\varphi(x^2, e^y, z) = 0$ ✓

$y = \sin x$ ✓

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\varphi_1 2x dx + \varphi_2 e^y dy + \varphi_3 dz = 0$$

$$dy = \cos x dx$$

$$dz = -\frac{1}{\varphi_3}(\varphi_1 2x + \varphi_2 e^y \cos x)dx$$

$$du = [\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cos x - \frac{\frac{\partial f}{\partial z}}{\varphi_3}(2x\varphi_1 + \varphi_2 e^y \cos x)]dx$$

【例5】设 $y = f(x,t)$, 且方程 $F(x,y,t)=0$ 确定了函数 $t = t(x,y)$, 求 $\dfrac{dy}{dx}$.

$y=y(x)$

$y=f(x,t(x,y))$ 方程.

【解1】由 $y=f(x,t)$ 得

$$\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}\left(\frac{\partial t}{\partial x}+\frac{\partial t}{\partial y}\frac{dy}{dx}\right)$$

由 $F(x,y,t)=0$ 得

$$\frac{\partial t}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial t}},\quad \frac{\partial t}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial t}}$$

$$\frac{dy}{dx}=\frac{\partial f}{\partial x}-\frac{\partial f}{\partial t}\left(\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial t}}+\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial t}}\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}=\frac{\frac{\partial F}{\partial t}\frac{\partial f}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial t}}{\frac{\partial F}{\partial t}+\frac{\partial F}{\partial y}\frac{\partial f}{\partial t}}$$

【例5】设 $y=f(x,t)$，且方程 $F(x,y,t)=0$ 确定了函数 $t=t(x,y)$，求 $\frac{dy}{dx}$.

【解2】$y=f(x,t)$，
$F(x,y,t)=0$

$$\begin{cases} dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial t}dt \\ \frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial t}dt=0 \end{cases}$$

$$dt=-\frac{1}{\frac{\partial F}{\partial t}}\left(\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy\right)$$

$$\frac{dy}{dx}=\frac{\frac{\partial F}{\partial t}\frac{\partial f}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial t}}{\frac{\partial F}{\partial t}+\frac{\partial F}{\partial y}\frac{\partial f}{\partial t}}$$

【例6】设 $f(x,y)$ 有二阶连续偏导数，且 $f_y' \neq 0$，证明：对任意常数 $C$，$f(x,y)=C$ 为一条直线 $\Leftrightarrow f_2^2 f_{11} - 2f_1 f_2 f_{12} + f_1^2 f_{22} = 0.$

$y=y(x) \Leftrightarrow y=ax+b \Leftrightarrow y''=0$

【证】$f_1 + f_2 \dfrac{dy}{dx} = 0.$ $\quad \dfrac{dy}{dx} = -\dfrac{f_1}{f_2} = -\dfrac{f_1(x,y)}{f_2(x,y)}$

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{f_1}{f_2}\right)$$

$$= -\frac{\left(f_{11} + f_{12}\dfrac{dy}{dx}\right)f_2 - \left(f_{21} + f_{22}\dfrac{dy}{dx}\right)f_1}{f_2^2}$$

$$= -\frac{f_2^2 f_{11} - 2f_1 f_2 f_{12} + f_1^2 f_{22}}{f_2^3} = 0$$

$= 0$

$$\frac{\partial z}{\partial x} = x^2 + xy.$$

$$z = \int (x^2 + x\underline{y})\,dx = \frac{1}{3}x^3 + \frac{x^2}{2}y + \varphi(y)$$

/20 例3. $u=f(x,y,z)$ $z(x,y)$

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$

?

u

x y z

还不关注，你就懂了