From Projection to Perception: A Mathematical Exploration of Shadow-based Neural Reconstruction

A research report submitted to the Scientific Committee of the Hang Lung Mathematics Award

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Date
July 4, 2025

Abstract

This paper explores ShadowNeuS [LWX23], a neural network that reconstructs 3D geometry from single-view camera images using shadow and light cues. Unlike traditional 3D reconstruction methods relying on multi-view cameras or sensors, ShadowNeuS leverages a neural signed distance field (SDF) for accurate 3D geometry reconstruction. Analysis of the training process reveals deep connections to projective geometry, spatial reasoning in $\mathbb{R}^3$, and the network's perception of three-dimensional space.

Contents

1 Background

1.1 What is 3D reconstruction from images?

The goal of 3D reconstruction is to recover the structure of a 3D scene using only 2D images. Consider a 3D scene represented by a set of points $P \in \mathbb{R}^3$, having coordinates $P(P_x, P_y, P_z)$. For each image taken for the 3D scene contains a set of pixel points $p \in \mathbb{R}^2$, having coordinates $p(p_x, p_y)$. The process of capturing a 3D point in a 2D image I_n can be modeled as a projection function π_n

$$\pi_n : \mathbb{R}^3 \to \mathbb{R}^2, \quad (P_x, P_y, P_z) \mapsto (p_x, p_y)$$

This function represents how a camera maps a 3D point to 2D pixel in the n-th image. To reconstruct the 3D scene, we need to solve the inverse problem π_n^{-1}.

$$\pi_n^{-1}(p) \to P$$

However, this inverse problem is typically **ill-posed**, as multiple 3D points may project to the same 2D pixel, leading to ambiguity. We will detail in Section 1.4.

1.2 Information Encoded in 2D Images

A 2D image I_n can provide multiple information, such as color and texture. The information available from an image includes:

- **Pixel coordinates**: $p(p_x, p_y) \in \mathbb{R}^2$, represent the location of each pixel in the image
- **Color values**: $C_n(p) = [r, g, b] \in [0, 1]^3$, represent the RGB color of each pixel in the image
- **Image gradient**:

$$\nabla I_n(p) = (\nabla r(p), \nabla g(p), \nabla b(p)) = \left(\underbrace{\left[\frac{\partial r}{\partial p_x}, \frac{\partial r}{\partial p_y}\right]^\top}_{\text{red channel}}, \underbrace{\left[\frac{\partial g}{\partial p_x}, \frac{\partial g}{\partial p_y}\right]^\top}_{\text{green channel}}, \underbrace{\left[\frac{\partial b}{\partial p_x}, \frac{\partial b}{\partial p_y}\right]^\top}_{\text{blue channel}} \right) \in \mathbb{R}^6$$

  It captures local changes in intensity, indicating edges or texture information in the image

- **Learned features**: $\phi(I_n)(p) \in \mathbb{R}^d$, represent a high-dimensional features extracted from the image using methods like convolutional neural networks (CNNs) or other feature extracters

These data result from projecting 3D structures through a camera. For example, the color $I_n(p)$ may correspond to the visible surface of a 3D object, while $\nabla I_n(p)$ may hint the edge of the shape of that 3D object, etc.

1.3 The Forward Projection: From 3D world to 2D Image

We formalize the perspective projection process that projects a 3D point $P = [P_x, P_y, P_z]^T \in \mathbb{R}^3$ to a pixel point $p = [p_x, p_y]^T \in \mathbb{R}^2$ using the camera parameters.

Camera parameters:

- **Extrinsic** (world-to-camera transformation)
  - **Camera center**: $C(C_x, C_y, C_z) \in \mathbb{R}^3$, represent the position of the camera in the world coordinate.
  - **Rotation matrix**: $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in SO(3)$, represent a 3x3 matrix that rotate the world to align with the orientation of the camera.
  - **Translation vector**: $t = -RC \in \mathbb{R}^3$, represent the translate to set the camera center be the world origin.
- **Intrinsic** (projection to image plane)
  - **Intrinsic matrix**:
$$K = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$
where f_x, f_y are the focal lengths in (pixels) and c_x, c_y is the principal point (the pixel coordinates where the camera's lens is optically centered)

Forward Projection Pipeline

We uses homogeneous coordinates P_{hom} that an extra coordinates is added to handle scaling. The process involves:

1. **World to camera coordinate**: Transform $P(P_x, P_y, P_z)$ to camera coordinate
$$P \to \underbrace{(P-C)}_{\text{set cemera to origin}} \to \underbrace{R(P-C)}_{\text{rotate the world for alignment}} = \underbrace{RP - RC = RP + t}_{\text{simplify}} = \underbrace{[R|t]\begin{bmatrix} P \\ 1 \end{bmatrix}}_{\text{matrix operation}}$$
2. **Perspective projection**: $P_{hom} = K[R|t]\begin{bmatrix} P \\ 1 \end{bmatrix} = \begin{bmatrix} p'_x \\ p'_y \\ z' \end{bmatrix}$
3. **Normalization**: Convert to 2D pixel coordinates by the scaling fator z'
$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \frac{1}{z'}\begin{bmatrix} p'_x \\ p'_y \end{bmatrix}, \qquad z' \neq 0$$

Result

$$P_{hom} = \begin{bmatrix} p'_x \\ p'_y \\ z' \end{bmatrix} = K[R|t]\begin{bmatrix} P \\ 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \frac{1}{z'}\begin{bmatrix} p'_x \\ p'_y \end{bmatrix}, \qquad z' \neq 0$$

1.4 The Inverse Problem: From 2D Image to 3D world

References

[LWX23] Jingwang Ling, Zhibo Wang, Feng Xu. *ShadowNeuS: Neural SDF Reconstruction by Shadow Ray Supervision*. arXiv: 2211.14086, 2023.