Complex Analysis

Second Edition

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Juvenes Print Oulu 2017
9 8 7 6 5 4 3 2 1

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Preface to the first edition

These lecture notes are based on the two 4 credit courses given at University of Oulu during Spring 2015 semester. We follow the general outline of the courses given by Jorma Arhippainen in 2008–2012 and the courses given by the author at Moscow State University in the end of 1990s. The major difference in this text is the discussion of extended complex plane and the concept of complex infinity. The text contains problems which range from very easy to somewhat difficult. Exercises are formulated at the end of each course.

After the courses ended this text has been modified as follows. Cauchy theorem and Cauchy integral formula has been moved to Part I. A new chapter on principle of the argument and Rouche's theorem has been added to Part II. Another new chapter on calculation of series by residue theory has also been added to Part II. The Casorati-Sokhotski-Weierstrass theorem has been added to Chapter 5 of Part II.

Oulu, August 2015 Valery Serov

Preface to the second edition

In two years since the first edition of this book appeared some new suggestions for improving the text was proposed. Completely new part, Part III, consisting of two chapters: Conformal mappings and Laplace transform, has been added. After this addition this new edition can be considered as a standard university course in Complex Analysis for mathematics students. In addition to this some corrections and adjustments throughout the book are done, and the following important topics have been added: (1) Cauchy integral formula is formulated now in its most general form using principal value integrals (see Theorem 5.7 in Part I), (2) Taylor expansion at infinity, (3) Jordan's lemma is now formulated for each half plane of the complex plane: upper, lower, left and right and this lemma in its new form is applied in the chapter on Laplace transform (4) Numerous new problems are formulated now in Chapters 1 and 2 of Part III. Together with the list of exercises in Parts I and II they form an integral part of the new edition. The total number of problems and exercises is 167. The readers are asked to investigate and solve most of the problems and exercises.

The last but not the least is: this edition as well as the first one could not have appeared without participation in content and typing of my colleague Adj. Prof. Markus Harju.

Oulu, October 2017 Valery Serov

Contents

Part I

Chapter 1

Complex numbers and their properties

Definition 1.1. The ordered pair (x, y) of real numbers x and y is called a *complex number* $z = (x, y)$ if the following properties are satisfied:

1. $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$. In particular, $z = (x, y) = 0$ if and only if $x = y = 0$.

2. $z_1 \pm z_2 = (x_1 \pm x_2, y_1 \pm y_2)$

3. $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

The notation is: $x = \operatorname{Re} z, y = \operatorname{Im} z$

The complex number $z = (x, 0)$ is identified with real number x, and complex number $z = (0, y)$ is called *purely imaginary*.

Definition 1.2. The complex numbers $(0, 0), (1, 0)$ and $(0, 1)$ are called *zero, unit* and *imaginary unit* and are identified with $0, 1$ and i, respectively.

It is easy to check that

$$\mathrm{i}^2 = (-1, 0), \quad \mathrm{i}(b, 0) = (0, b). \tag{1.1}$$

Indeed,

$$\mathrm{i}^2 = (0, 1) \cdot (0, 1) = (-1, 0)$$

and

$$\mathrm{i}(b, 0) = (0, 1) \cdot (b, 0) = (0, b)$$

by Definition 1.1 .

Since

$$z = (x, y) = (x, 0) + (0, y)$$

using (1.1) we obtain that

$$z = (x, 0) + (0, 1) \cdot (y, 0) = x + \mathrm{i}y$$

such that

$$z_1 + z_2 = x_1 + \mathrm{i}y_1 + x_2 + \mathrm{i}y_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$$

and

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = x_1x_2 + \mathrm{i}y_1x_2 + \mathrm{i}x_1y_2 + \mathrm{i}^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + \mathrm{i}(y_1x_2 + x_1y_2) \end{aligned}$$

that is, these operations (addition and multiplication) are performed as in the usual analysis.

We denote the set of all complex numbers by $\mathbb{C}$.

The division is defined as the operation which is inverse to multiplication. Namely, if $z_2 \neq 0$ (i.e. $x_2 \neq 0$ or $y_2 \neq 0$, so $x_2^2 + y_2^2 > 0$) then

$$\frac{z_1}{z_2} = a + \mathrm{i}b \quad \text{if and only if} \quad z_1 = (a + \mathrm{i}b)z_2.$$

It means that

$$x_1 + \mathrm{i}y_1 = (a + \mathrm{i}b)(x_2 + \mathrm{i}y_2)$$

or

$$\begin{cases} x_1 = ax_2 - by_2 \\ y_1 = bx_2 + ay_2. \end{cases}$$

Solving this for a and b gives

$$a = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \quad b = \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}.$$

Hence

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \mathrm{i}\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}. \tag{1.2}$$

Definition 1.3. For given complex number $z = x + \mathrm{i}y$

1. the number $\overline{z} := x - \mathrm{i}y$ is called the *complex conjugate* to z.
2. the nonnegative (real) number $|z| := \sqrt{x^2 + y^2}$ is called the *modulus* of z.

The following properties can be checked straightforwardly:

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$$

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2\mathrm{i}}$$

$$|z| = 0 \quad \text{if and only if} \quad z = 0$$

$$|z|^2 = z \cdot \overline{z}, \quad |z| = |\overline{z}|, \quad |z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \text{but}$$

$$\Big||z_1| - |z_2|\Big| \le |z_1 \pm z_2| \le |z_1| + |z_2| \tag{1.3}$$

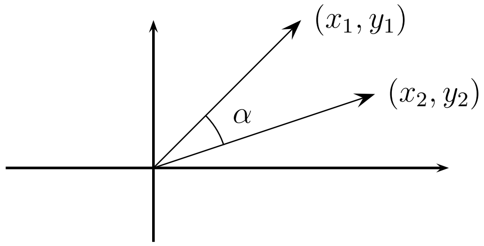
$$|\operatorname{Re} z| \le |z|, \quad |\operatorname{Im} z| \le |z|$$

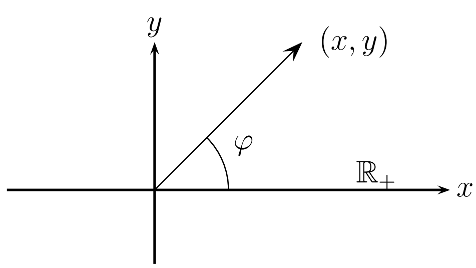
Problem 1.4.

1. Prove that
$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{|z_2|^2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \mathrm{i}\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}.$$

2. Prove that
$$|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2|z_1| \cdot |z_2| \cos\alpha,$$
where α is the angle between the two vectors $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ on the plane $\mathbb{R}^2$.

3. Prove the inequalities (1.3).





Definition 1.5. The angle φ formed by the vector $z = (x, y), z \neq 0$ and the positive real line $\mathbb{R}_+$ is said to be an *argument* of z and denoted by $\varphi = \operatorname{Arg} z, z \neq 0$. The argument of $z = 0$ is not defined.

Remark. It is clear that $\operatorname{Arg} z$ is not defined uniquely. More precisely, it is defined up to $2\pi n, n = 0, \pm 1, \pm 2, \dots$ i.e.

$$\operatorname{Arg} z = \varphi + 2\pi n,$$

where $\varphi \in (0, 2\pi]$ or $\varphi \in (-\pi, \pi]$. This value of φ is called the *main argument* and it is denoted as

$$\arg z = \varphi.$$

Let us assume in the future that

$$\arg z = \varphi \quad \text{with} \quad \varphi \in (-\pi, \pi].$$

In this case the Pythagorian theorem says that

$$\operatorname{Re} z = |z| \cos\varphi \quad \text{and} \quad \operatorname{Im} z = |z| \sin\varphi$$

i.e.

$$z = |z|(\cos\varphi + \mathrm{i}\sin\varphi), \quad z \neq 0. \tag{1.4}$$

Problem 1.6. Prove that

1. $z_1 = z_2$ if and only if $|z_1| = |z_2|$ and $\varphi_1 = \varphi_2$
2. $$\begin{aligned} \varphi \in (0, \pi) & \quad \text{if and only if} \quad \operatorname{Im} z > 0 \\ \varphi \in (-\pi, 0) & \quad \text{if and only if} \quad \operatorname{Im} z < 0 \\ \varphi = 0 & \quad \text{if and only if} \quad \operatorname{Im} z = 0, \operatorname{Re} z > 0 \\ \varphi = \pi & \quad \text{if and only if} \quad \operatorname{Im} z = 0, \operatorname{Re} z < 0. \end{aligned}$$

Problem 1.7. Prove the following statements:

1. $\arg \overline{z} = -\arg z$
2. $$\arg z = \begin{cases} \arctan \frac{\operatorname{Im} z}{\operatorname{Re} z}, & \operatorname{Re} z > 0 \\ \arctan \frac{\operatorname{Im} z}{\operatorname{Re} z} + \pi, & \operatorname{Re} z < 0, \operatorname{Im} z > 0 \\ \arctan \frac{\operatorname{Im} z}{\operatorname{Re} z} - \pi, & \operatorname{Re} z < 0, \operatorname{Im} z < 0 \\ \frac{\pi}{2}, & \operatorname{Re} z = 0, \operatorname{Im} z > 0 \\ -\frac{\pi}{2}, & \operatorname{Re} z = 0, \operatorname{Im} z < 0. \end{cases}$$

Problem 1.8. Prove the following properties:

1. $z_1 \cdot z_2 = |z_1| \cdot |z_2|(\cos(\varphi_1 + \varphi_2) + \mathrm{i}\sin(\varphi_1 + \varphi_2))$

2. $z_1/z_2 = |z_1|/|z_2|(\cos(\varphi_1 - \varphi_2) + \mathrm{i}\sin(\varphi_1 - \varphi_2))$
3. $z^n = |z|^n(\cos(n\varphi) + \mathrm{i}\sin(n\varphi))$ (*De Moivre formula*)

We will use the shorthand notation (which will be proved later)

$$\mathrm{e}^{\mathrm{i}\varphi} := \cos\varphi + \mathrm{i}\sin\varphi.$$

Then (1.4) can be written as

$$z = |z|\mathrm{e}^{\mathrm{i}\varphi}. \tag{1.5}$$

Definition 1.9. The form (1.5) is called the *trigonometric representation* of the complex numbers.

The equality (1.5) is called *Euler's formula*. Using (1.5) we may rewrite the above formulas in shorter way:

$$\begin{aligned} z_1 \cdot z_2 &= |z_1| \cdot |z_2|\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} \\ z_1/z_2 &= |z_1|/|z_2|\mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)} \\ z^n &= |z|^n\mathrm{e}^{\mathrm{i}n\varphi}. \end{aligned}$$

Definition 1.10. The complex number z_0 is said to be the *root of* nth *degree* of the complex number z if

$$z_0^n = z.$$

We denote this by $z_0 = \sqrt[n]{z}$. There are n solutions of the above equation and they are given by

$$(z_0)_k = |z|^{1/n}\mathrm{e}^{\mathrm{i}(\varphi/n+2\pi k/n)}, \quad k = 0, 1, \dots, n-1. \tag{1.6}$$

Problem 1.11. Prove (1.6) using De Moivre formula.

Let us consider in the Euclidean space $\mathbb{R}^3$ the sphere S with center $(0, 0, 1/2)$ and radius $1/2$ in the coordinate system (ξ, η, ζ), i.e.

$$\xi^2 + \eta^2 + (\zeta - 1/2)^2 = 1/4$$

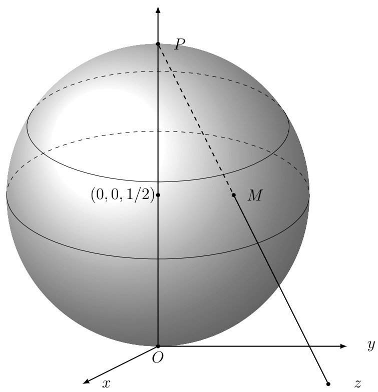
or

$$\xi^2 + \eta^2 + \zeta^2 - \zeta = 0. \tag{1.7}$$

Let us draw a ray from the point $P = (0, 0, 1)$ which intersects the sphere S at the point $M = (\xi, \eta, \zeta)$ and complex plane $\mathbb{C}$ at the point $z = x + \mathrm{i}y$.

The point M is called *stereographic projection* of the complex number z on the sphere S. Since the vectors $\overrightarrow{PM}$ and $\overrightarrow{Pz}$ are colinear we have

$$\frac{\xi}{x} = \frac{\eta}{y} = \frac{1-\zeta}{1}.$$



Thus, using (1.7) we have

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}$$

so that

$$\xi = \frac{x}{1+|z|^2}, \quad \eta = \frac{y}{1+|z|^2}, \quad \zeta = \frac{|z|^2}{1+|z|^2}. \tag{1.8}$$

Definition 1.12. The formulas (1.8) are called the formulas of the stereographic projection.

The formulas (1.8) allow us to introduce "ideal" complex number $z = \infty$ as follows. Since there is one-to-one correspondence between $\mathbb{C}$ and $S \setminus P$ then we may supplement this correspondence by one more, namely

$$P(0,0,1) \longleftrightarrow \infty.$$

In this case

$$S \longleftrightarrow \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$

and, by (1.8),

$$\frac{1}{\infty} = 0, \quad \frac{1}{0} = \infty, \quad z \cdot \infty = \infty, z \neq 0, \quad z + \infty = \infty, \quad \frac{z}{\infty} = 0, z \neq \infty. \tag{1.9}$$

Remark. The set $\overline{\mathbb{C}}$ is called the *extended complex plane*.

Problem 1.13. Prove that the spherical distance between $z_1, z_2 \in \overline{\mathbb{C}}$ can be calculated as

$$\rho_S(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

The *neighborhood* of $z_0 \in \mathbb{C}$ is defined as

$$U_\delta(z_0) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$$

and the neighborhood of $z = \infty$ is defined as

$$U_R(\infty) = \{z \in \mathbb{C} : |z| > R\}.$$

Definition 1.14.

1. The complex number $z_0 \in \overline{\mathbb{C}}$ is called the *limiting point* of some set $M \subset \overline{\mathbb{C}}$ if for any $\delta > 0$ it is true that
$$(U_\delta(z_0) \setminus z_0) \cap M \neq \emptyset$$
(or for any $R > 0$ it holds that $(U_R(\infty) \setminus \infty) \cap M \neq \emptyset$).
2. The set $M \subset \overline{\mathbb{C}}$ is called *closed* if it contains all its limiting points.
3. Denoting all limiting points of $M \subset \mathbb{C}$ by M' we define the *closure* of M as
$$\overline{M} = M \cup M'.$$
4. The *boundary* ∂M of the set $M \subset \overline{\mathbb{C}}$ is defined as
$$\partial M = \overline{M} \cap (\overline{\overline{\mathbb{C}} \setminus M}).$$
5. The point $z_0 \in \overline{\mathbb{C}}$ is called *interior* of some set M if there exists $U_\delta(z_0)$ (or $U_R(\infty)$) such that $U_\delta(z_0) \subset M$ (or $U_R(\infty) \subset M$). If all points of M are interior then M is called an *open set*.

Problem 1.15. Prove that $M \subset \overline{\mathbb{C}}$ is open if and only if $\overline{\mathbb{C}} \setminus M$ is closed.

Definition 1.16. The complex number $z_0 \in \mathbb{C}$ is said to be the *limit of sequence* $\{z_n\}_{n=1}^\infty \subset \mathbb{C}$, denoted by $z_0 = \lim_{n\to\infty} z_n$, if for any $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon, z_0) \in \mathbb{N}$ such that
$$|z_n - z_0| < \varepsilon$$
for all $n \geq n_0$.

We say that $\infty = \lim_{n\to\infty} z_n$ if for any $R > 0$ there is $n_0 = n_0(R) \in \mathbb{N}$ such that $|z_n| > R$ for all $n \geq n_0$.

Proposition 1.17.

1. $z_0 = \lim_{n\to\infty} z_n, z_0 \neq \infty$ *if and only if*
$$\operatorname{Re} z_0 = \lim_{n\to\infty} \operatorname{Re} z_n \quad \textit{and} \quad \operatorname{Im} z_0 = \lim_{n\to\infty} \operatorname{Im} z_n$$

2. $\infty = \lim_{n\to\infty} z_n$ *if and only if* $\lim_{n\to\infty} |z_n| = \infty$.

Proof. 1. If $z_0 = \lim_{n\to\infty} z_n$ then for any $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$|z_n - z_0|^2 < \varepsilon^2, \quad n \geq n_0.$$

It means that

$$(\operatorname{Re} z_n - \operatorname{Re} z_0)^2 + (\operatorname{Im} z_n - \operatorname{Im} z_0)^2 < \varepsilon^2, \quad n \geq n_0.$$

It follows that

$$|\operatorname{Re} z_n - \operatorname{Re} z_0| < \varepsilon, \quad |\operatorname{Im} z_n - \operatorname{Im} z_0| < \varepsilon, \quad n \geq n_0$$

or

$$\operatorname{Re} z_0 = \lim_{n\to\infty} \operatorname{Re} z_n, \quad \operatorname{Im} z_0 = \lim_{n\to\infty} \operatorname{Im} z_n.$$

Conversely, if $a = \lim_{n\to\infty} \operatorname{Re} z_n$ and $b = \lim_{n\to\infty} \operatorname{Im} z_n$ then for any $\varepsilon > 0$ there exist $n_1(\varepsilon), n_2(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} |\operatorname{Re} z_n - a| &< \varepsilon/2, \quad n \geq n_1 \\ |\operatorname{Im} z_n - b| &< \varepsilon/2, \quad n \geq n_2. \end{aligned}$$

Denoting $n_0 = \max(n_1, n_2)$ we obtain for all $n \geq n_0$ that

$$|z_n - (a + \mathrm{i}b)| \leq |\operatorname{Re} z_n - a| + |\operatorname{Im} z_n - b| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

2. Follows immediately from Definition 1.16.

□

Remark. In part 2) of Proposition 1.17 we cannot say anything more. Indeed, let z_n be defined as follows:

$$z_n = \begin{cases} n + \mathrm{i}/n, & n = 2k \\ 1/n + \mathrm{i}n, & n = 2k + 1. \end{cases}$$

Then $|z_n| = \sqrt{n^2 + 1/n^2} \to \infty$ as $n \to \infty$ but $\operatorname{Re} z_n \not\to \infty$ and $\operatorname{Im} z_n \not\to \infty$.

The Bolzano–Weierstrass Principle If the sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is bounded i.e. there exists $M > 0$ such that

$$|z_n| \leq M, \quad n = 1, 2, \ldots$$

then there is a subsequence z_{k_n} which converges to some point $z_0 \in \mathbb{C}$ i.e.

$$\lim_{n\to\infty} z_{k_n} = z_0.$$

Indeed, since $|z_n| \le M$ then $|\operatorname{Re} z_n| \le M$ and $|\operatorname{Im} z_n| \le M$. Using the Bolzano–Weierstrass principle to the real sequence $\operatorname{Re} z_n$ we find $\operatorname{Re} z_{k_n}$ such that there exists $a \in \mathbb{R}$ with

$$a = \lim_{n\to\infty} \operatorname{Re} z_{k_n}.$$

If we consider now $\operatorname{Im} z_{k_n}$ then it is also bounded and hence there exists a subsequence, say $\operatorname{Im} z_{k_n}^{(1)}$ which has a limit

$$b = \lim_{n\to\infty} \operatorname{Im} z_{k_n}^{(1)}.$$

Thus,

$$\lim_{n\to\infty} (\operatorname{Re} z_{k_n}^{(1)} + \mathrm{i} \operatorname{Im} z_{k_n}^{(1)}) = \lim_{n\to\infty} \operatorname{Re} z_{k_n}^{(1)} + \mathrm{i} \lim_{n\to\infty} \operatorname{Im} z_{k_n}^{(1)} = a + \mathrm{i}b.$$

If the sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is not bounded, i.e. for all $M > 0$ there exists $n_M \in \mathbb{N}$ such that $|z_{n_M}| > M$, then there is a subsequence z_{k_n} such that

$$\lim_{n\to\infty} |z_{k_n}| = \infty.$$

The proof of this fact is the same as in real analysis.

There is one more useful property:

$$z_n \to \infty$$

(i.e. $|z_n| \to \infty$) if and only if

$$\lim_{n\to\infty} \frac{1}{z_n} = 0.$$

Cauchy criterion The sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ converges if and only if it is a *Cauchy sequence*, i.e. for any $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that

$$|z_n - z_m| < \varepsilon, \quad n, m \ge n_0.$$

The proof follows from the Cauchy criterion of real analysis.

Arithmetic operations with convergent sequences If

$$\lim_{n\to\infty} z_n = z_0, \quad \lim_{n\to\infty} w_n = w_0$$

then

$$\lim_{n\to\infty} (z_n \pm w_n) = z_0 \pm w_0$$

$$\lim_{n\to\infty} z_n \cdot w_n = z_0 \cdot w_0$$

$$\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z_0}{w_0}, w_0 \ne 0.$$

If

$$\lim_{n\to\infty} z_n = \infty, \qquad \lim_{n\to\infty} w_n = \infty$$

then

$$\lim_{n\to\infty} z_n \cdot w_n = \infty.$$

Problem 1.18.

1. Let $\lim_{n\to\infty} z_n = z_0, z_0 \neq 0, z_0 \neq \infty$ and $\lim_{n\to\infty} w_n = \infty$. Prove that

$$\lim_{n\to\infty} z_n \cdot w_n = \infty, \qquad \lim_{n\to\infty} (z_n \pm w_n) = \infty, \qquad \lim_{n\to\infty} z_n/w_n = 0.$$

2. Let $\lim_{n\to\infty} z_n = \infty$ and $\lim_{n\to\infty} w_n = \infty$. Prove that the limits

$$\lim_{n\to\infty} (z_n \pm w_n) \quad \text{and} \quad \lim_{n\to\infty} z_n/w_n$$

might not exist.

Series The series of the complex numbers

$$\sum_{k=1}^{\infty} z_k$$

is said to be *convergent* if the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} z_k$$

exists. Then this limit is denoted by

$$\sum_{k=1}^{\infty} z_k.$$

It is equivalent to the convergence of the real series

$$\sum_{k=1}^{\infty} \operatorname{Re} z_k \quad \text{and} \quad \sum_{k=1}^{\infty} \operatorname{Im} z_k$$

and in that case

$$\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} \operatorname{Re} z_k + \mathrm{i} \sum_{k=1}^{\infty} \operatorname{Im} z_k.$$

The series $\sum_{k=1}^{\infty} z_k$ is said to be *absolutely convergent* if

$$\sum_{k=1}^{\infty} |z_k| < \infty$$

or

$$\sum_{k=1}^{\infty} |\operatorname{Re} z_k| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} |\operatorname{Im} z_k| < \infty.$$

The latter conditions follow from

$$|z| \le |\operatorname{Re} z| + |\operatorname{Im} z| \quad \text{and} \quad |\operatorname{Re} z|, |\operatorname{Im} z| \le |z|.$$

The absolute convergence implies convergence but not vice versa.

Example 1.19 (*Geometric series*)**.** Since

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}, \quad z \ne 1$$

then the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} z^k$$

exists if and only if $\lim_{n\to\infty} z^{n+1}$ exists and $z \ne 1$. But the latter limit exists if and only if $|z| < 1$ and in that case it equals 0. Thus the series

$$\sum_{k=0}^{\infty} z^k$$

converges if and only if $|z| < 1$ and

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}. \tag{1.10}$$

Example 1.20 (Exponential function)**.** The *exponential function* $\mathrm{e}^z, z \in \mathbb{C}$ can be defined as the following series:

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{1.11}$$

From real analysis we know that

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = \mathrm{e}^{|z|}.$$

Therefore the series (1.11) is well-defined for all $z \in \mathbb{C}$. Even more is true. For $z = x \in \mathbb{R}$ we know that

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Using (1.11) we obtain for purely imaginary $z = \mathrm{i}y$ that

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}y} &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}y)^n}{n!} = \sum_{k=0}^{\infty} \frac{(\mathrm{i}y)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(\mathrm{i}y)^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + \mathrm{i} \sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} = \cos y + \mathrm{i} \sin y.
\end{aligned}$$

This proves formula (1.5).

Now we would like to show that actually the function (1.11) can be represented (or understood) as

$$\mathrm{e}^z = \mathrm{e}^x (\cos y + \mathrm{i} \sin y),$$

where $\mathrm{e}^x, \cos y$ and $\sin y$ are from real analysis. Indeed, by the binomial formula,

$$\begin{aligned}
\mathrm{e}^z &= \mathrm{e}^{x+\mathrm{i}y} = \sum_{n=0}^{\infty} \frac{(x+\mathrm{i}y)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k (\mathrm{i}y)^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k (\mathrm{i}y)^{n-k} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{x^k}{k!} \frac{(\mathrm{i}y)^{n-k}}{(n-k)!} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^k}{k!} \frac{(\mathrm{i}y)^m}{m!} \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{m=0}^{\infty} \frac{(\mathrm{i}y)^m}{m!} = \mathrm{e}^x (\cos y + \mathrm{i} \sin y).
\end{aligned}$$

Problem 1.21. Show that

1. $\mathrm{e}^{z_1} \mathrm{e}^{z_2} = \mathrm{e}^{z_1 + z_2}$
2. $\mathrm{e}^{z + \mathrm{i}2\pi k} = \mathrm{e}^z, k \in \mathbb{Z}$
3. $\mathrm{e}^{-z} = 1/\mathrm{e}^z$ or $\mathrm{e}^z = 1/\mathrm{e}^{-z}$
4. $(\mathrm{e}^z)^n = \mathrm{e}^{nz}, n \in \mathbb{Z}$
5. $|\mathrm{e}^z| \le \mathrm{e}^{|z|}$.

Chapter 2

Functions of complex variable

The complex-valued function of one real variable is the mapping

$$f : (a, b) \to \mathbb{C} \quad \text{or} \quad f : [a, b] \to \mathbb{C}$$

such that

$$z = f(t) = f_1(t) + \mathrm{i} f_2(t),$$

where $t \in (a, b)$ or $t \in [a, b]$. Here, the open interval (a, b) might be infinite but the closed interval $[a, b]$ is considered only for finite a and b.

The notions of limit, continuity, differentiability and integrability are defined coordinate-wise. i.e. for two real-valued functions $f_1(t)$ and $f_2(t)$ of one real variable t.

Definition 2.1.

1. The continuous mapping $f : [a, b] \to \mathbb{C}, z = f(t)$ is called the *Jordan curve* if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$. If in addition $z(a) = z(b)$ then this curve is called *closed*.

2. The Jordan curve is called *piecewise smooth* if there are points

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that $z = f(t)$ is continuously differentiable on the intervals $[t_{j-1}, t_j]$ for $j = 1, 2, \ldots, n$ and $f'(t) \neq 0$.

3. If $n = 1$ above then the Jordan curve is called *smooth*.

We will use the following statement proved by Jordan (we accept it like axiom, without proof):

Any closed Jordan curve divides $\overline{\mathbb{C}}$ into two domains (regions): internal (not containing $z = \infty$) and external (containing $z = \infty$). They are denoted as $\operatorname{int}\gamma$ and $\operatorname{ext}\gamma$, respectively, so that

$$\overline{\mathbb{C}} = \operatorname{int}\gamma \cup \gamma \cup \operatorname{ext}\gamma.$$

Definition 2.2.

1. A set $D \subset \mathbb{C}$ is called *connected* if for any points $z_1, z_2 \in D$ there is a Jordan curve connecting these points and lying in D.

2. A set $D \subset \mathbb{C}$ is called a *domain* if it is connected and open.

We consider a complex-valued function w of one complex variable z as follows. Let us have two copies of the complex plane, one in z and one in w. Let D be a domain in z and G a domain in w. Then a function $w = f(z)$ is the mapping

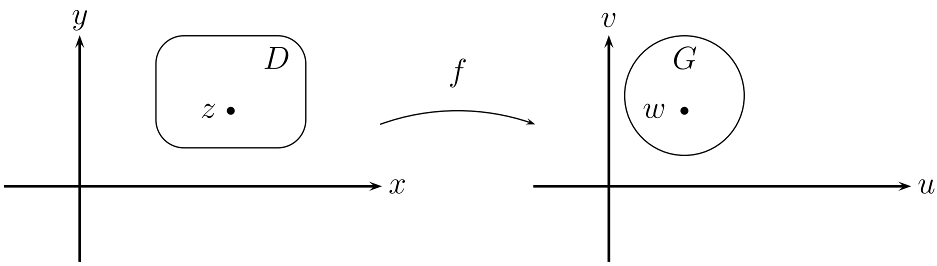
$$f : D \to G$$

such that

$$w = u + \mathrm{i}v = f(z) = f_1(x, y) + \mathrm{i}f_2(x, y).$$

This is equivalent to the definition of two real-valued functions u and v of two real variables x and y such that $w = f(z)$ if and only if

$$u(x, y) = \operatorname{Re} w \quad \text{and} \quad v(x, y) = \operatorname{Im} w.$$



In particular, we have that

$$b = \lim_{z \to z_0} f(z), \quad b \neq \infty \tag{2.1}$$

if and only if

$$\operatorname{Re} b = \lim_{(x,y)\to(x_0,y_0)} \operatorname{Re} f(z) \quad \text{and} \quad \operatorname{Im} b = \lim_{(x,y)\to(x_0,y_0)} \operatorname{Im} f(z).$$

Also,

$$\infty = \lim_{z \to z_0} f(z)$$

if and only if

$$\lim_{z \to z_0} |f(z)| = +\infty$$

i.e. for all $R > 0$ there exists $\delta(R) > 0$ such that $|f(z)| > R$ whenever $|z| > \delta$.

Here (2.1) means that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, z_0) > 0$ such that

$$|f(z) - b|_{\mathbb{C}} < \varepsilon$$

whenever $|z - z_0| < \delta$ i.e. $|(x, y) - (x_0, y_0)|_{\mathbb{R}^2} < \delta$. Therefore, the arithmetic operations for complex-valued functions of one complex variable are satisfied, i.e. if

$$\lim_{z \to z_0} f(z) = a \quad \text{and} \quad \lim_{z \to z_0} g(z) = b$$

then

1. $$\lim_{z \to z_0} (f(z) \pm g(z)) = a \pm b$$

   i.e.

   $$\lim_{(x,y) \to (x_0,y_0)} (\operatorname{Re} f \pm \operatorname{Re} g) = \operatorname{Re} a \pm \operatorname{Re} b$$

   and

   $$\lim_{(x,y) \to (x_0,y_0)} (\operatorname{Im} f \pm \operatorname{Im} g) = \operatorname{Im} a \pm \operatorname{Im} b$$

2. $$\lim_{z \to z_0} f(z) \cdot g(z) = a \cdot b$$

   i.e.

   $$\lim_{(x,y) \to (x_0,y_0)} \operatorname{Re}(f \cdot g) = \operatorname{Re}(a \cdot b)$$

   and

   $$\lim_{(x,y) \to (x_0,y_0)} \operatorname{Im}(f \cdot g) = \operatorname{Im}(a \cdot b)$$

3. $$\lim_{z \to z_0} f(z)/g(z) = a/b, \quad \text{if}\, b \neq 0$$

   i.e.

   $$\lim_{(x,y) \to (x_0,y_0)} \operatorname{Re}(f/g) = \operatorname{Re}(a/b)$$

   and

   $$\lim_{(x,y) \to (x_0,y_0)} \operatorname{Im}(f/g) = \operatorname{Im}(a/b)$$

Definition 2.3. A function $w = f(z)$ is called *univalent* if $f : D \to G$ onto (is surjective) and if for any $z_1, z_2 \in D, z_1 \neq z_2$

$$w_1 = f(z_1) \neq w_2 = f(z_2) \quad \text{(injectivity)}.$$

In this case there is an *inverse function* f^{-1} which maps as

$$f^{-1} : G \to D$$

onto (surjectively) such that $f^{-1}(w) = z$ if $w = f(z)$, i.e.

$$z = f^{-1}(f(z)), \quad w = f(f^{-1}(w)), \quad z \in D, w \in G.$$

This inverse function f^{-1} is also univalent (bijective).

Summarizing, we have

$$z = f^{-1}(f(z)) \quad \text{for all}\, z \in D$$

and

$$w = f(f^{-1}(w)) \quad \text{for all}\, w \in G.$$

Definition 2.4.

1. A function $w = f(z)$ is *continuous* at $z = z_0 \neq \infty$ if $f(z)$ is well-defined in a neighborhood $U_\delta(z_0)$ and if for any $\varepsilon > 0$ there exists $\delta(\varepsilon, z_0) > 0$ such that
$$|f(z) - f(z_0)| < \varepsilon$$
whenever $|z - z_0| < \delta$.
2. A function $w = f(z)$ is continuous at $z = \infty$ if $f(z)$ is well-defined for $|z| > A$ and there exists $b \in \mathbb{C}$ such that for any $\varepsilon > 0$ there is $R(\varepsilon, b) > 0$ such that
$$|f(z) - b| < \varepsilon$$
for any $|z| > R$. In that case $f(\infty) = b$.
3. A function $w = f(z)$ is continuous on the set $A \subset \overline{\mathbb{C}}$ if it is continuous at any point $z_0 \in A$.
4. A function $w = f(z)$ is *uniformly continuous* on the set $A \subset \overline{\mathbb{C}}$ if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
$$|f(z_1) - f(z_2)| < \varepsilon$$
whenever $|z_1 - z_2| < \delta$ and $z_1, z_2 \in A$.

Remark. Since

$$|z - z_0|_{\mathbb{C}} < \delta$$

if and only if

$$|(x, y) - (x_0, y_0)|_{\mathbb{R}^2} < \delta$$

and

$$|f(z) - f(z_0)| < \varepsilon$$

if and only if

$$|u(x, y) - u(x_0, y_0)| < \varepsilon, \quad \text{and} \quad |v(x, y) - v(x_0, y_0)| < \varepsilon$$

then the continuity of $f(z)$ is equivalent to the continuity of $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ as functions of two variables (x, y).

Problem 2.5. Show that $\mathrm{e}^z \neq 0$ for any $z \in \mathbb{C}$ and the limit $\lim_{z\to\infty} \mathrm{e}^z$ does not exist (finite or infinite).

Problem 2.6. Investigate the continuity at 0 of the functions
a) $z^2/|z|^2$, b) $(z\,\mathrm{Re}\,z)/|z|$, c) $(\mathrm{Im}\,z)/z$ d) e^{-1/z^2}.

Example 2.7. A *linear-fractional* (*bilinear*) function is defined for $z \in \mathbb{C}$ as

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0, c \neq 0. \tag{2.2}$$

It is well-defined if $z \neq -d/c$. Since

$$w_1 - w_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d} = \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$$

then this function is univalent in the domain $\mathbb{C} \setminus \{-d/c\}$. The inverse function $z = z(w)$ is also bilinear and defined by

$$z = \frac{dw-b}{a-cw}$$

and it is well-defined (and univalent) in the domain $\mathbb{C} \setminus \{a/c\}$. If we define

$$w(-d/c) = \infty \quad \text{and} \quad w(\infty) = a/c$$

then the bilinear function maps $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ bijectively. The same is true for the inverse function.

Let us show that the bilinear function (2.2) is continuous everywhere in $\overline{\mathbb{C}} \setminus \{-d/c\}$. Indeed, if first $z_0 \neq -d/c, z_0 \neq \infty$ then

$$|w(z) - w(z_0)| = \left|\frac{(ad-bc)(z-z_0)}{(cz+d)(cz_0+d)}\right| = \frac{|ad-bc||z-z_0|}{|cz_0+d+c(z-z_0)||cz_0+d|}.$$

Since $|cz_0+d| > 0$ then we may choose $|z-z_0| < \delta$ and $|z-z_0| \leq \frac{|cz_0+d|}{2|c|}$. In this case

$$|cz+d| = |cz_0+d+c(z-z_0)| \geq |cz_0+d| - |c||z-z_0| \geq \frac{|cz_0+d|}{2}$$

and

$$|w(z) - w(z_0)| < \frac{|ad-bc|\delta}{\frac{|cz_0+d|^2}{2}} \leq \varepsilon.$$

If for arbitrary $\varepsilon > 0$ we will define

$$\delta = \min\left\{\frac{|cz_0+d|}{2|c|}, \frac{\varepsilon|cz_0+d|^2}{2|ad-bc|}\right\}$$

then the condition $|z - z_0| < \delta$ implies $|w(z) - w(z_0)| < \varepsilon$ i.e. the bilinear function is continuous at any such point z_0.

If now we have $z_0 = \infty$ then we may choose $|z| > 2|d/c|$ and obtain

$$|w(z) - w(\infty)| = \left|\frac{az+b}{cz+d} - \frac{a}{c}\right| = \frac{|ad-bc|}{|c||cz+d|} = \frac{|ad-bc|}{|c|^2|z+d/c|}$$
$$\leq \frac{|ad-bc|}{|c|^2(|z| - |d/c|)} \leq \frac{2|ad-bc|}{|c|^2|z|} = \varepsilon.$$

Hence, if for arbitrary $\varepsilon > 0$ we will choose

$$R = \frac{2|ad-bc|}{|c|^2\varepsilon}$$

then the condition $|z| > R$ implies $|w(z) - w(\infty)| < \varepsilon$ i.e. the bilinear function is continuous also at ∞.

Remark. For $c = 0$ the bilinear function reduces to the linear function

$$w(z) = \frac{a}{d}z + \frac{b}{d}, \quad d \neq 0.$$

It is easy to check that this is continuous on $\mathbb{C}$ (but not at ∞) and univalent on $\mathbb{C}$.

Example 2.8. The squared function is defined for $z \in \overline{\mathbb{C}}$ as

$$w = w(z) = z^2, \quad w(\infty) = \infty. \tag{2.3}$$

Since

$$w_1 - w_2 = z_1^2 - z_2^2 = (z_1 - z_2)(z_1 + z_2)$$

then we may conclude that $z_1 \neq z_2$ if and only if $w_1 \neq w_2$ (because $w_1 = w_2$ if and only if $z_1 = z_2$ or $z_1 = -z_2$). Thus the squared function (2.3) is not univalent on $\overline{\mathbb{C}}$.

But if we consider two subdomains

$$D_+ = \{z \in \overline{\mathbb{C}} : \operatorname{Im} z > 0\}$$

and

$$D_- = \{z \in \overline{\mathbb{C}} : \operatorname{Im} z < 0\}$$

then in each of these two subdomains the squared function (2.3) is univalent. It is very easy to check that in both domains $z_1 \neq -z_2$. Indeed, $z_1 = -z_2$ if and only if $\operatorname{Re} z_1 = -\operatorname{Re} z_2$ and $\operatorname{Im} z_1 = -\operatorname{Im} z_2$ i.e. these equalities are impossible in D_+ or in D_-.

In order to define the inverse of $w = z^2$ in $D_\pm$ we proceed as follows:

$$w_1 + \mathrm{i}w_2 = z^2 = x^2 - y^2 + 2\mathrm{i}xy$$

if and only if

$$w_1 = x^2 - y^2, \quad x = \frac{w_2}{2y}.$$

So

$$w_1 = \frac{w_2^2}{4y^2} - y^2$$

or

$$4y^4 + 4y^2 w_1 - w_2^2 = 0.$$

Hence

$$y^2 = \frac{-2w_1 + \sqrt{4w_1^2 + 4w_2^2}}{4}.$$

It yields

$$y = \sqrt{\frac{\sqrt{w_1^2 + w_2^2} - w_1}{2}} \quad \text{in } D_+$$

and

$$y = -\sqrt{\frac{\sqrt{w_1^2 + w_2^2} - w_1}{2}} \quad \text{in } D_-.$$

Consequently,

$$x = \frac{w_2}{2\sqrt{\frac{\sqrt{w_1^2+w_2^2}-w_1}{2}}} \quad \text{in } D_+$$

and

$$x = -\frac{w_2}{2\sqrt{\frac{\sqrt{w_1^2+w_2^2}-w_1}{2}}} \quad \text{in } D_-.$$

Remark. As we can see, in $D_\pm$, $w_2 = 0$ if and only if $x = 0$ i.e. $\operatorname{Im} w = 0$ if and only if $\operatorname{Re} z = 0$ and in this case $\operatorname{Re} w = -(\operatorname{Im} z)^2$ i.e. $w_1 = -y^2 < 0$.

So finally we have

$$z_+ = \frac{w_2}{\sqrt{2}\sqrt{\sqrt{w_1^2 + w_2^2} - w_1}} + \mathrm{i}\frac{\sqrt{\sqrt{w_1^2 + w_2^2} - w_1}}{\sqrt{2}}$$

$$z_- = -\frac{w_2}{\sqrt{2}\sqrt{\sqrt{w_1^2 + w_2^2} - w_1}} - \mathrm{i}\frac{\sqrt{\sqrt{w_1^2 + w_2^2} - w_1}}{\sqrt{2}}.$$

We may simplify these formulas to obtain

$$z_+ = \sqrt{\frac{w_1 + |w|}{2}} + \mathrm{i}\frac{w_2}{\sqrt{\frac{w_1+|w|}{2}}}, \quad z_- = -z_+. \tag{2.4}$$

In these formulas, z_+ is called $(\sqrt{w})_+$ with $\operatorname{Im} z_+ > 0$ and z_- is called $(\sqrt{w})_-$ with $\operatorname{Im} z_+ < 0$ so that we have two *branches* for inverse function.

For the case $x = 0$ we obtain easily from Remark above that

$$z_+ = \mathrm{i}\sqrt{-w_1}, \quad \text{and} \quad z_- = -\mathrm{i}\sqrt{-w_1}. \tag{2.5}$$

For the case $\operatorname{Im} z = 0$ we have real-valued (and nonnegative) function of one real variable x i.e.

$$w_1 = x^2.$$

Its inverse also has two branches

$$x_+ = \sqrt{w_1}, \quad x_- = -\sqrt{w_1}, \quad w_1 \geq 0. \tag{2.6}$$

The formulas (2.4)-(2.6) can be written shortly (compare with (1.6)) as

$$z_\pm = \sqrt{|w|}\mathrm{e}^{\mathrm{i}\arg w/2} \quad \text{and} \quad z_\mp = \sqrt{|w|}\mathrm{e}^{\mathrm{i}(\arg w/2+\pi)} = -\sqrt{|w|}\mathrm{e}^{\mathrm{i}\arg w/2}, \tag{2.7}$$

where $\arg w \in (-\pi, \pi]$. Here $\pm$ depend on $\arg w$. More precisely, if $\arg w \in (0, \pi)$ then $z_+ \in D_+$ and $z_- \in D_-$, but if $\arg w \in (-\pi, 0)$ then $z_+ \in D_-$ and $z_- \in D_+$.

Problem 2.9. Show that (2.4)-(2.6) and (2.7) are equivalent.

The squared function (2.3) is continuous at any point $z_0 \in \mathbb{C}$ since

$$|w(z) - w(z_0)| = |z^2 - z_0^2| = |z - z_0||z + z_0| < \delta|z + z_0| < \delta(\delta + 2|z_0|) = \varepsilon$$

so, if for arbitrary $\varepsilon > 0$, we choose

$$\delta = -|z_0| + \sqrt{|z_0|^2 + \varepsilon} > 0$$

then the condition $|z - z_0| < \delta$ implies $|w(z) - w(z_0)| < \varepsilon$. So $w = z^2$ is continuous at $z_0 \neq \infty$. At $z_0 = \infty$ this function is not continuous since $w(\infty) = \infty$.

Problem 2.10. Investigate the function $w = z^3$ by the same manner as in Example 2.8 and Problem 2.6.

Example 2.11. The *Zhukovski function* is defined for any $z \neq 0$ and $z \neq \infty$ as

$$w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{2.8}$$

or $z^2 - 2zw + 1 = 0$. We define

$$w(\infty) = w(0) = \infty.$$

Since

$$w(z_1) - w(z_2) = \frac{1}{2}(z_1 - z_2)\left(1 - \frac{1}{z_1 z_2}\right)$$

then $w(z_1) \neq w(z_2)$ if and only if $z_1 \neq z_2$ and $z_1 z_2 \neq 1$. Thus, the Zhukovski function (2.8) is univalent if and only if $z_1 z_2 \neq 1$, for example, if either $|z| < 1$ or $|z| > 1$ i.e. in the domains

$$D_1 = \{z \in \mathbb{C} : |z| < 1\}, \quad D_2 = \{z \in \mathbb{C} : |z| > 1\}.$$

On the unit circle $|z| = 1$ there are always two different points z_1 and z_2 such that $z_1 z_2 = 1$. Indeed, if $z_1 = \mathrm{e}^{\mathrm{i}\varphi_1}, \varphi_1 \in (-\pi, \pi)$ then if we consider $z_2 = \mathrm{e}^{-\mathrm{i}\varphi_1}$ we have $z_1 z_2 = 1$, but $z_1 \neq z_2$. In this consideration the case $z_1 = \mathrm{e}^{\mathrm{i}\pi} = -1$ is excluded.

For any $z = r\mathrm{e}^{\mathrm{i}\varphi}$ we have that

$$w(z) = \frac{1}{2}\left(r\mathrm{e}^{\mathrm{i}\varphi} + \frac{1}{r}\mathrm{e}^{-\mathrm{i}\varphi}\right) = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\varphi + \frac{\mathrm{i}}{2}\left(r - \frac{1}{r}\right)\sin\varphi.$$

It implies

$$|w|^2 = \frac{1}{4}\left(r + \frac{1}{r}\right)^2 + \frac{\cos\varphi - 1}{2}$$

and hence

$$|w|^2 \leq \frac{1}{4}\left(r + \frac{1}{r}\right)^2, \quad |w|^2 \geq \frac{1}{4}\left(r + \frac{1}{r}\right)^2 - 1.$$

Using (2.8) we obtain that in the domain D_1 the inverse function is given by

$$z = w - \sqrt{w^2 - 1}$$

and by

$$z = w + \sqrt{w^2 - 1}$$

in the domain D_2, depending on the choice of $\sqrt{w^2 - 1}$.

Zhukovski function is continuous at any point $z_0 \neq 0, \infty$. Indeed, for such z_0 we have,

$$\begin{aligned} |w(z) - w(z_0)| &= \frac{1}{2}|z - z_0|\left|1 - \frac{1}{zz_0}\right| = \frac{1}{2}|z - z_0|\left|1 - \frac{1}{((z - z_0) + z_0)z_0}\right| \\ &\leq \frac{|z - z_0|}{2}\left(1 + \frac{1}{|(z - z_0) + z_0||z_0|}\right) \\ &\leq \frac{|z - z_0|}{2}\left(1 + \frac{1}{|z_0|(|z_0| - |z - z_0|)}\right) \\ &\leq \frac{|z - z_0|}{2}\left(1 + \frac{1}{|z_0||z_0|/2}\right) \end{aligned}$$

if $|z - z_0| \le |z_0|/2$. Thus, for any $\varepsilon > 0$ and $|z - z_0| < \min(\delta, |z_0|/2)$ we have

$$|w(z) - w(z_0)| < \frac{\delta}{2}\left(1 + \frac{2}{|z_0|^2}\right) = \varepsilon.$$

So choosing

$$\delta = \min\left(\frac{2\varepsilon}{1 + 2/|z_0|^2}, \frac{|z_0|}{2}\right)$$

the condition $|z - z_0| < \delta$ implies $|w(z) - w(z_0)| < \varepsilon$. At $z = 0$ or $z = \infty$ the Zhukovski function is not continuous since $w(0) = w(\infty) = \infty$.

Problem 2.12. Show that the Zhukovski function maps real numbers into real numbers and purely imaginary numbers to purely imaginary numbers.

Problem 2.13. Show that the Zhukovski function maps the unit circle $|z| = 1$ into $\cos(\arg z)$.

As a consequence of the notion of limit we may formulate and prove (as in real analysis) the following general statements:

Proposition 2.14. *Assume that f and g are continuous at some point z_0 (or on a set A). Then*

1. $f \pm g$
2. $f \cdot g$
3. $\frac{f}{g}$, *if $g(z_0) \neq 0$ (or $g(z) \neq 0$ for all $z \in A$)*
4. $|f|$

are continuous at z_0 (or on the set A).

Proposition 2.15. *Let $w = f(z)$ be continuous on a set A and $g(w)$ continuous on the set $f(A)$. Then the* composite function

$$\eta = g(f(z)) = (g \circ f)(z)$$

is continuous on the set A.

Corollary 2.16. *If $w = f(z)$ is univalent and continuous on a domain D, then the inverse function $z = f^{-1}(w)$ is continuous on the domain $G = f(D)$.*

Proof. Since for any $z \in D$ we have

$$z = f^{-1}(f(z))$$

and f is continuous on D then $f^{-1}(w)$ is continuous on $G = f(D)$ because z is continuous. $\square$

Weierstrass theorems

1. If $D \subset \mathbb{C}$ is compact (i.e. closed and bounded) and f is continuous on D then f is bounded and f is uniformly continuous on D.

2. The previous statement holds also for compact $D \subset \overline{\mathbb{C}}$ (see stereographic projection).

3. If $D \subset \overline{\mathbb{C}}$ is compact and f is continuous on D then $|f|$ achieves maximum and minimum on D.

Chapter 3

Analytic functions (differentiability)

Definition 3.1. Let $w = f(z)$ be well-defined on a domain $D \subset \mathbb{C}$ and $z_0 \in D$. If the limit

$$\lim_{D \ni z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists then this limit is called the *derivative* of $f(z)$ at the point z_0 and it is denoted as $f'(z_0)$. In this case f is called *differentiable* at z_0 with

$$\lim_{D \ni z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0). \tag{3.1}$$

We say that $f'(\infty)$ exists if f is continuous at $z = \infty$ and there is $g'(0)$ for $g(z) = f(1/z)$. This is equivalent to

$$g'(0) = \lim_{\zeta \to \infty} \zeta \left[f(\zeta) - f(\infty) \right] =: f'(\infty).$$

This definition is equivalent to the existence of the limit

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{u(x,y) - u(x_0,y_0) + \mathrm{i}(v(x,y) - v(x_0,y_0))}{(x - x_0) + \mathrm{i}(y - y_0)}.$$

In particular, if $x = x_0$ and $y \to y_0, y \neq y_0$ the latter limit equals

$$\begin{aligned} &\lim_{y \to y_0} \frac{u(x_0,y) - u(x_0,y_0) + \mathrm{i}(v(x_0,y) - v(x_0,y_0))}{\mathrm{i}(y - y_0)} \\ &\qquad = \frac{1}{\mathrm{i}} \frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) - \mathrm{i}\frac{\partial u}{\partial y}(x_0,y_0). \end{aligned} \tag{3.2}$$

In the case $y = y_0$ and $x \to x_0, x \neq x_0$ the limit equals

$$\lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0) + \mathrm{i}(v(x, y_0) - v(x_0, y_0))}{x - x_0} = \frac{\partial u}{\partial x}(x_0, y_0) + \mathrm{i}\frac{\partial v}{\partial x}(x_0, y_0). \quad (3.3)$$

Since the limit (3.1) is unique we obtain from (3.2) and (3.3) that we must necessarily have

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \qquad (3.4)$$

The equalities (3.4) are called the *Cauchy-Riemann conditions*. We have proved that they are necessary for existence of $f'(z)$. Actually they are also sufficient. More precisely, let $u(x, y)$ and $v(x, y)$ be differentiable at the point (x_0, y_0). If the conditions (3.4) are satisfied then $f'(z_0)$ exists. Indeed, we have

$$\begin{aligned} &u(x, y) - u(x_0, y_0) \\ &\quad = \frac{\partial u}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial u}{\partial y}(x_0, y_0)(y - y_0) + o(\sqrt{(x - x_0)^2 + (y - y_0)^2}) \end{aligned}$$

and

$$\begin{aligned} &v(x, y) - v(x_0, y_0) \\ &\quad = \frac{\partial v}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial v}{\partial y}(x_0, y_0)(y - y_0) + o(\sqrt{(x - x_0)^2 + (y - y_0)^2}), \end{aligned}$$

where $o(\cdot)$ means that $o(s)/s \to 0$ as $s \to 0$. Therefore we have, using (3.4)

$$\begin{aligned} u(x, y) - u(x_0, y_0) + \mathrm{i}(v(x, y) - v(x_0, y_0)) = &\\ \frac{\partial u}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial u}{\partial y}(x_0, y_0)(y - y_0)&\\ + \mathrm{i}\left(\frac{\partial v}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial v}{\partial y}(x_0, y_0)(y - y_0)\right)&\\ + o(\sqrt{(x - x_0)^2 + (y - y_0)^2})&\\ = \left[\frac{\partial u}{\partial x}(x_0, y_0) + \mathrm{i}\frac{\partial v}{\partial x}(x_0, y_0)\right] [(x - x_0) + \mathrm{i}(y - y_0)]&\\ + o(\sqrt{(x - x_0)^2 + (y - y_0)^2})& \end{aligned}$$

or

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{o(|z - z_0|)}{z - z_0}.$$

This representation implies that the limit

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \frac{\partial f}{\partial x}(x_0, y_0) = f'(z_0) \tag{3.5}$$

exists. In a similar manner we obtain

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = -\mathrm{i}\frac{\partial f}{\partial y}(x_0, y_0) = f'(z_0). \tag{3.6}$$

Thus we have proved the following fundamental result.

Theorem 3.2. *The function $w = f(z)$ is differentiable at the point z_0 if and only if $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ are differentiable at the point (x_0, y_0) as real-valued functions of two real variables x and y and the Cauchy-Riemann conditions* (3.4) *are satisfied.*

Remark. Formulas (3.5) and (3.6) imply that

$$\begin{aligned} f'(z_0) &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right) =: \frac{\partial f}{\partial z} \\ 0 &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right) =: \frac{\partial f}{\partial \overline{z}}. \end{aligned} \tag{3.7}$$

Hence, the Cauchy-Riemann conditions are equivalent to

$$\frac{\partial f}{\partial z}(z_0) = f'(z_0) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}}(z_0) = 0. \tag{3.8}$$

Example 3.3. Consider the function

$$f(z) = \overline{z}.$$

Then $u(x, y) = x$ and $v(x, y) = -y$. The partial derivatives in this case are

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1$$

so that

$$1 = \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} = -1, \quad 0 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

Thus Cauchy-Riemann conditions are not satisfied and therefore $f(z) = \overline{z}$ has no derivative.

Example 3.4. Let us consider

$$f(z) = |z|^2 = z\overline{z} = x^2 + y^2.$$

Then

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0.$$

Hence the Cauchy-Riemann conditions are

$$2x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad 2y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

i.e. $x = y = 0$. Thus (3.4) are satisfied only for $z = 0$ and they are not satisfied for $z \neq 0$ i.e.

$$\lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0}$$

exists (and equals 0) if and only if $z_0 = 0$. So

$$(|z|^2)'(0) = 0.$$

Problem 3.5. Let

$$f(z) = R(x, y)\mathrm{e}^{\mathrm{i}\theta(x,y)},$$

where R and θ are real-valued. Prove that Cauchy-Riemann conditions can be written in this case as

$$\frac{\partial R}{\partial x} = R\frac{\partial \theta}{\partial y} \quad \text{and} \quad \frac{\partial R}{\partial y} = -R\frac{\partial \theta}{\partial x}. \tag{3.9}$$

Problem 3.6. Let

$$w = \frac{az + b}{cz + d}, \quad ad \neq bc, c \neq 0$$

be a bilinear function. Show that

$$w'(z) = -\frac{bc - ad}{(cz + d)^2}$$

for any $z \neq -d/c$.

Problem 3.7. Let

$$w = \frac{az + b}{cz + d}, \quad ad \neq bc, c \neq 0.$$

Show that $w'(\infty) = (bc - ad)/c^2$.

Problem 3.8. Let

$$w = \mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i} \sin y).$$

Show that

$$(\mathrm{e}^z)' = \mathrm{e}^z$$

at any point $z \neq \infty$.

Problem 3.9. Let

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Show that

$$w'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right), \quad z \neq 0, z \neq \infty.$$

Show also that $w'(\infty)$ does not exist but

$$\lim_{z \to 0} w'(z) = \infty, \quad \lim_{z \to \infty} w'(z) = 1/2.$$

Proposition 3.10. *If $w = f(z)$ is differentiable at $z = z_0$ then $f(z)$ is also continuous at z_0 but not vice versa.*

Proof. Since the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists then

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + o(z - z_0).$$

This implies that

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The function $f(z) = \overline{z}$ provides an example of a function which is continuous but not differentiable. □

Proposition 3.11. *Let $\eta(z) = g(f(z))$ be the composition of functions $w = f(z)$ and $\eta = g(w)$. If $f(z)$ is differentiable at $z = z_0$ and $g(w)$ is differentiable at $w_0 = f(z_0)$ then $\eta(z)$ is differentiable at $z = z_0$ and*

$$\eta'(z_0) = g'(w_0)f'(z_0) = g'(f(z_0))f'(z_0). \tag{3.10}$$

Proof. By definition we have

$$\frac{\eta(z) - \eta(z_0)}{z - z_0} = \frac{g(f(z)) - g(f(z_0))}{z - z_0} = \frac{g(w) - g(w_0)}{w - w_0} \cdot \frac{f(z) - f(z_0)}{z - z_0},$$

where $w = f(z)$ and $w_0 = f(z_0)$. If $z \to z_0$ then $w \to w_0$ by Proposition 3.10. Then due to conditions of this Proposition we have

$$\lim_{z \to z_0} \frac{\eta(z) - \eta(z_0)}{z - z_0} = \lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} \cdot \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = g'(w_0)f'(z_0)$$

or $\eta'(z_0) = g'(w_0)f'(z_0)$. □

Corollary 3.12. *Let $w = f(z)$ be univalent on a domain D. Then f is differentiable on D if and only if the inverse function $z = f^{-1}(w)$ is differentiable on $G = f(D)$ and*

$$f'(z) = \frac{1}{(f^{-1})'(w)}, \quad w = f(z). \tag{3.11}$$

In particular, both derivatives are not equal to zero.

Proof. The claim follows from the representations

$$z = f^{-1}(f(z)), z \in D \quad \text{and} \quad w = f(f^{-1}(w)), w \in G$$

and Proposition 3.11. Indeed,

$$1 = (z)' = (f^{-1})'(w)f'(z),$$

where $w = f(z)$ and both derivatives are not equal to zero necessarily. □

Example 3.13. Consider the Zhukovski function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Then (3.11) leads to

$$(f^{-1})'(w) = \frac{2}{1 - 1/z^2} = \frac{2}{1 - 1/(2wz - 1)} = \frac{2wz - 1}{wz - 1} = 1 + \frac{w}{w - 1/z},$$

where $z = w \pm \sqrt{w^2 - 1}$. So

$$(f^{-1})'(w) = 1 \pm \frac{w}{\sqrt{w^2 - 1}}$$

depending on the domains D_1 and D_2, see Example 2.11. In the domains D_1 and D_2 we have $w \neq \pm 1$ and therefore the latter formula is well-defined.

Example 3.14. Let us introduce some new functions:

$$\begin{aligned} \sin z &:= \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}, \quad \cos z := \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2} \\ \sinh z &:= \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}, \quad \cosh z := \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}. \end{aligned} \tag{3.12}$$

These functions are compositions of e^z and $\mathrm{e}^{\mathrm{i}z}$. That's why we have

$$\begin{aligned} (\sin z)' &= \frac{(\mathrm{e}^{\mathrm{i}z})' - (\mathrm{e}^{-\mathrm{i}z})'}{2\mathrm{i}} = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}z} + \mathrm{i}\mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}} = \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2} = \cos z \\ (\cos z)' &= \frac{(\mathrm{e}^{\mathrm{i}z})' + (\mathrm{e}^{-\mathrm{i}z})'}{2} = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}z} - \mathrm{i}\mathrm{e}^{-\mathrm{i}z}}{2} = -\frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}} = -\sin z \\ (\sinh z)' &= \frac{(\mathrm{e}^{z})' - (\mathrm{e}^{-z})'}{2} = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2} = \cosh z \\ (\cosh z)' &= \frac{(\mathrm{e}^{z})' + (\mathrm{e}^{-z})'}{2} = \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2} = \sinh z. \end{aligned}$$

There are also some useful equalities:

$$\cos^2 z + \sin^2 z = \frac{e^{2iz} + 2 + e^{-2iz}}{4} - \frac{e^{2iz} - 2 + e^{-2iz}}{4} = 1$$

and

$$\cosh^2 z - \sinh^2 z = \frac{(e^z + e^{-z})^2}{4} - \frac{(e^z - e^{-z})^2}{4} = 1.$$

Also we obtain the equalities

$$\begin{aligned} \cos(-z) &= \cos z, \quad \sin(-z) = -\sin(z) \\ e^{iz} &= \cos z + i \sin z \\ e^{-iz} &= \cos z - i \sin z. \end{aligned} \tag{3.13}$$

Remark. Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}$$

then

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!}, \quad e^{-iz} = \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}.$$

So we obtain using (3.12) that

$$\begin{aligned} \cos z &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{i^n + (-i)^n}{n!} z^n \\ &= \frac{1}{2} \left(\sum_{k=0}^{\infty} \frac{i^{2k} + (-i)^{2k}}{(2k)!} z^{2k} + \sum_{k=0}^{\infty} \frac{i^{2k+1} + (-i)^{2k+1}}{(2k+1)!} z^{2k+1} \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \end{aligned}$$

because

$$i^{2k} + (-i)^{2k} = (-1)^k + (-1)^k = 2(-1)^k$$

and

$$i^{2k+1} + (-i)^{2k+1} = i(-1)^k - i(-1)^k = 0.$$

So

$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}, \quad z \in \mathbb{C}. \tag{3.14}$$

In a similar fashion we obtain

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}, \quad z \in \mathbb{C}. \tag{3.15}$$

Problem 3.15. Show that

$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \quad \sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}, \quad z \in \mathbb{C}. \tag{3.16}$$

Problem 3.16. Show that

1. $\cos z = \cosh(\mathrm{i}z)$ and $\sin z = -\mathrm{i}\sinh(\mathrm{i}z)$
2. $|\mathrm{e}^z| = \mathrm{e}^x$ and $\overline{(\mathrm{e}^z)} = \mathrm{e}^{\overline{z}}$ for $z = x + \mathrm{i}y$
3. $|\cos z| = \sqrt{\cosh^2 y - \sin^2 x}$
4. $|\sin z| = \sqrt{\sinh^2 y + \sin^2 x}$
5. $|\cos z|^2 + |\sin z|^2 = \cosh^2 y + \sinh^2 y = 1 + 2\sinh^2 y$.

Problem 3.17. Calculate the derivative of the function $f(z) = \mathrm{e}^{z^2}$ using (3.10).

Problem 3.18. Calculate the derivative of the inverse function for $w = z^n$ using (3.11).

Definition 3.19.

1. A function $f(z)$ is said to be *analytic* in a domain D if for each $z \in D$ the derivative $f'(z)$ exists and is continuous in D. The set of all analytic functions in D will be denoted by $H(D)$.
2. A function $f(z)$ is said to be analytic in the point $z_0 \in D$ if $f(z)$ is analytic in some neighborhood $U_\delta(z_0) \subset D$ of z_0.
3. A function $f(z)$ is said to be analytic at $z = \infty$ if $g(z) = f(1/z)$ is analytic at the point $z = 0$.

From this definition and the definition of the derivative it follows that

1. If $f_1, f_2 \in H(D)$ then

$$f_1 \pm f_2, f_1 \cdot f_2, \frac{f_1}{f_2} \in H(D)$$

too. In the last case we assume $f_2 \neq 0$.

2. If $f \in H(D)$ and $g \in H(G)$, where $G = f(D)$ then $g \circ f \in H(D)$.

Example 3.20. The function

$$P_n(z) := a_0 + a_1 z + \cdots + a_n z^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0$ is called the *polynomial* of order n. It is clear that $P_n(z) \in H(\mathbb{C})$ but it is not analytic at $z = \infty$ if $n \geq 1$.

If $P_n(z_0) = 0$ then z_0 is called the root of this polynomial and $P_n(z) = (z - z_0)P_{n-1}(z)$, where P_{n-1} is a polynomial of order $n - 1$.

Example 3.21. The function

$$R(z) = \frac{P_n(z)}{Q_m(z)}, \quad Q_m(z) \neq 0$$

is called the *rational function*. It follows that $R(z)$ is analytic everywhere in

$$\mathbb{C} \setminus \{z_0^{(1)}, \dots, z_0^{(k)}\},$$

where

$$P_n(z_0^{(j)}) \neq 0 \quad \text{and} \quad Q_m(z_0^{(j)}) = 0.$$

Example 3.22. The tangent function is defined by

$$\tan z := \frac{\sin z}{\cos z}, \quad \cos z \neq 0.$$

The zeros of $\cos z$ satisfy $\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z} = 0$. So $\mathrm{e}^{2\mathrm{i}z} = -1$ or

$$\mathrm{e}^{2\mathrm{i}x} = -\mathrm{e}^{2y}.$$

Comparing real and imaginary parts we see that

$$\cos 2x = -\mathrm{e}^{2y}, \quad \sin 2x = 0$$

or

$$2x = \pi k, k \in \mathbb{Z}, \quad \cos(\pi k) = -\mathrm{e}^{2y}.$$

So

$$x = \pi k/2, \quad (-1)^k = -\mathrm{e}^{2y}, \quad k \in \mathbb{Z}$$

or

$$x = \pi k/2, \quad 1 = \mathrm{e}^{2y}, \quad k = \pm 1, \pm 3, \dots$$

Thus

$$y = 0, \quad x = \frac{\pi}{2}(2m+1), \quad m \in \mathbb{Z}.$$

We denote

$$z_m = -\frac{\pi}{2} + m\pi + \mathrm{i}0, \quad m \in \mathbb{Z}.$$

Since $\sin z_m = \pm 1 \neq 0$ then $\tan z$ is analytic everywhere in $\mathbb{C}$ except at z_m. In this domain

$$(\tan z)' = \frac{(\sin z)' \cos z - \sin z (\cos z)'}{\cos^2 z} = \frac{\cos^2 z + \sin^2}{\cos^2 z} = \frac{1}{\cos^2 z}.$$

Problem 3.23. Show that $\sin z = 0$ if and only if $z = \pi k + \mathrm{i}0, k \in \mathbb{Z}$.

Example 3.24. The function

$$\tanh z := \frac{\sinh z}{\cosh z}, \quad \cosh z \neq 0$$

is called hyperbolic tangent function. The zeros of $\cosh z$ satisfy

$$\mathrm{e}^{2x} = -\mathrm{e}^{-\mathrm{i}2y}$$

or $x = 0, y = \pi/2 + \pi m, m \in \mathbb{Z}$. Hence $\tanh z$ is analytic everywhere in $\mathbb{C} \setminus \{z_m\}_{m=-\infty}^{\infty}$, where

$$z_m = 0 + \mathrm{i}(\pi/2 + \pi m), \quad m \in \mathbb{Z}$$

and

$$(\tanh z)' = \frac{1}{\cosh^2 z}.$$

Problem 3.25. Show that $\sinh z = 0$ if and only if $z = 0 + \mathrm{i}\pi k, k \in \mathbb{Z}$.

Example 3.26. Let us consider the exponential function

$$w = \mathrm{e}^z$$

and let us try to find its inverse. Since

$$w = |w|\mathrm{e}^{\mathrm{i}\arg w}, \quad w \neq 0, \arg w \in (-\pi, \pi]$$

and $\mathrm{e}^z = \mathrm{e}^x\mathrm{e}^{\mathrm{i}y}$ then

$$|w| = \mathrm{e}^x \quad \text{and} \quad \arg w = y + 2\pi k, k \in \mathbb{Z}.$$

So

$$x = \log|w| \quad \text{and} \quad y = \arg w + 2\pi k, k \in \mathbb{Z}.$$

Thus

$$z = \log|w| + \mathrm{i}\arg w + \mathrm{i}2\pi k, k \in \mathbb{Z}.$$

We see that the inverse of the function $w = \mathrm{e}^z$ is not single-valued, namely we have infinitely many branches

$$z_k = \log|w| + \mathrm{i}\arg w + \mathrm{i}2\pi k, \quad k \in \mathbb{Z}.$$

The multivalued function is

$$z = \operatorname{Log} w := \log|w| + \mathrm{i}\arg w + \mathrm{i}2\pi k, \quad k \in \mathbb{Z}.$$

Its main branch is

$$z = \log w := \log|w| + \mathrm{i}\arg w, \quad \arg w \in (-\pi, \pi].$$

The *logarithmic function* $w = \operatorname{Log} z, z \neq 0$ is analytic everywhere in $\mathbb{C} \setminus \overline{\mathbb{R}_-}$ since $\arg z$ has a jump over negative real axis. Moreover,

$$(\operatorname{Log} z)' = \frac{1}{(\mathrm{e}^w)'} = \frac{1}{\mathrm{e}^w} = \frac{1}{z}.$$

Therefore it is also continuous in $\mathbb{C} \setminus \overline{\mathbb{R}_-}$ (compare with Corollary 3.12).

Remark. Since

$$\operatorname{Log} z = \log z + \mathrm{i}2\pi k, \quad k \in \mathbb{Z}$$

then the derivative of $\operatorname{Log} z$ is the same ($(\operatorname{Log} z)' = 1/z$) for all branches of the multivalued logarithmic function.

Example 3.27. The function

$$z^{m/n} := \mathrm{e}^{\frac{m}{n}\operatorname{Log} z}, \quad z \neq 0$$

is called the *rational power function.* Since

$$\operatorname{Log} z = \log|z| + \mathrm{i}\arg z + \mathrm{i}2\pi k, \quad k \in \mathbb{Z}$$

then

$$z^{m/n} = \mathrm{e}^{\frac{m}{n}(\log|z| + \mathrm{i}\arg z + \mathrm{i}2\pi k)} = \mathrm{e}^{\frac{m}{n}\log|z|}\mathrm{e}^{\mathrm{i}\frac{m}{n}\arg z + \mathrm{i}\frac{2\pi km}{n}}.$$

The expression

$$\mathrm{e}^{\mathrm{i}\frac{2\pi km}{n}}$$

has different values only for $k = 0, 1, \ldots, n-1$ (we have assumed that m/n is uncancelled fraction). That's why we have n different branches of

$$z^{m/n} = |z|^{\frac{m}{n}}\mathrm{e}^{\mathrm{i}\left(\frac{m}{n}\arg z + \frac{2\pi km}{n}\right)}, \quad k = 0, 1, \ldots, n-1.$$

Its derivative is

$$(z^{m/n})' = (\mathrm{e}^{\frac{m}{n}\operatorname{Log} z})' = \mathrm{e}^{\frac{m}{n}\operatorname{Log} z}\frac{m}{n}(\operatorname{Log} z)' = \frac{m}{n}z^{m/n-1}.$$

Example 3.28. The function

$$z^{\alpha} := \mathrm{e}^{\alpha\operatorname{Log} z}, \quad z \neq 0, \alpha \in \mathbb{R} \setminus \mathbb{Q}$$

is called the *irrational power function.* It is actually equal to

$$z^{\alpha} = \mathrm{e}^{\alpha(\log|z| + \mathrm{i}\arg z + \mathrm{i}2\pi k)} = |z|^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\arg z + \mathrm{i}\alpha 2\pi k}, \quad k \in \mathbb{Z}$$

and we have infinitely many branches since α is not rational number. Its derivative is

$$(z^{\alpha})' = (\mathrm{e}^{\alpha\operatorname{Log} z})' = \alpha\mathrm{e}^{\alpha\operatorname{Log} z}(\operatorname{Log} z)' = \alpha z^{\alpha-1}.$$

The definition of irrational power can be easily generalized for any complex power $\alpha = \alpha_1 + \mathrm{i}\alpha_2$. Namely, the function

$$z^{\alpha} := \mathrm{e}^{\alpha\operatorname{Log} z}, \quad z \neq 0, \alpha \in \mathbb{C}$$

is called the *general power function.* As before, it is equal to

$$\begin{aligned} z^{\alpha} &= \mathrm{e}^{\alpha\operatorname{Log} z} = \mathrm{e}^{\alpha(\log|z| + \mathrm{i}\arg z + \mathrm{i}2\pi k)} = \mathrm{e}^{(\alpha_1 + \mathrm{i}\alpha_2)(\log|z| + \mathrm{i}\arg z + \mathrm{i}2\pi k)} \\ &= \mathrm{e}^{\alpha_1\log|z| - \alpha_2(\arg z + 2\pi k)}\mathrm{e}^{\mathrm{i}(\alpha_2\log|z| + \alpha_1(\arg z + 2\pi k))} \end{aligned}$$

and we have infinitely many branches. The derivative is again $(z^{\alpha})' = \alpha z^{\alpha-1}$.

Example 3.29. Let us find the inverse of $w = \sin z$. From

$$w = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}$$

we obtain

$$2\mathrm{i}w = \mathrm{e}^{\mathrm{i}z} - \frac{1}{\mathrm{e}^{\mathrm{i}z}}$$

or $(\mathrm{e}^{\mathrm{i}z})^2 - 2\mathrm{i}w\mathrm{e}^{\mathrm{i}z} - 1 = 0$. It implies

$$\mathrm{e}^{\mathrm{i}z} = \mathrm{i}w + \sqrt{1 - w^2}.$$

So

$$\mathrm{i}z = \operatorname{Log}(\mathrm{i}w + \sqrt{1 - w^2})$$

or

$$z = -\mathrm{i}\operatorname{Log}(\mathrm{i}w + \sqrt{1 - w^2}),$$

where Log denotes the multivalued function. The inverse of $\sin z$ is hence

$$z = z(w) = -\mathrm{i}\operatorname{Log}(\mathrm{i}w + \sqrt{1 - w^2}) =: \arcsin w$$

and it has infinitely many branches. Its derivative is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}w}\arcsin w &= \frac{\mathrm{d}}{\mathrm{d}w}\frac{1}{\mathrm{i}}\operatorname{Log}(\mathrm{i}w + \sqrt{1 - w^2}) = \frac{1}{\mathrm{i}}\frac{1}{\mathrm{i}w + \sqrt{w^2 - 1}}\left(\mathrm{i} - \frac{w}{\sqrt{1 - w^2}}\right) \\ &= \frac{1}{\mathrm{i}}\frac{1}{\mathrm{i}w + \sqrt{w^2 - 1}}\frac{\mathrm{i}\sqrt{1 - w^2} - w}{\sqrt{1 - w^2}} = \frac{1}{\sqrt{1 - w^2}}, \quad w \neq \pm 1.\end{aligned}$$

Problem 3.30. Show that

1. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
2. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
3. $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$
4. $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.

Problem 3.31. Show that

$$\operatorname{Log}(z_1 \cdot z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$$

for any $z_1 \neq 0$ and $z_2 \neq 0$.

We will finish this chapter by the following very useful rule which is called *L'Hôpital's rule*.

Proposition 3.32. *Suppose f and g are analytic at z_0. If $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$ then*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Proof. Because $g'(z_0) \neq 0$ then g is not identically equal to zero and there is a neighborhood $U_\delta(z_0)$ in which $g'(z) \neq 0$. Therefore the quotient

$$\frac{f(z)}{g(z)} = \frac{f(z) - f(z_0)}{g(z) - g(z_0)}$$

is defined for all $z \in U_\delta(z_0)$ and

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \frac{f'(z_0)}{g'(z_0)}.$$

□

Problem 3.33. Using L'Hôpital's rule calculate the limits

$$\lim_{z \to 0} \frac{\log^2(1+z)}{z^2} \quad \text{and} \quad \lim_{z \to 0} \frac{1 - \cos z}{\sin^2 z}.$$

Chapter 4

Integration of functions of complex variable (curve integration)

Let γ be a smooth Jordan curve i.e.

$$\gamma : z = z(t), \quad t \in [a, b].$$

Assuming that $f(z)$ is a continuous function we may define two types of *curve integrals* along γ as

$$\begin{aligned}
\int_\gamma f(z)\mathrm{d}z &:= \int_a^b f(z(t))z'(t)\mathrm{d}t \\
&= \int_a^b (u(z(t)) + \mathrm{i}v(z(t)))(x'(t) + \mathrm{i}y'(t))\mathrm{d}t \\
&= \int_a^b [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]\mathrm{d}t \\
&+ \mathrm{i}\int_a^b [v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]\mathrm{d}t \\
&= \int_\gamma (u(x, y)\mathrm{d}x - v(x, y)\mathrm{d}y) + \mathrm{i}\int_\gamma (v(x, y)\mathrm{d}x + u(x, y)\mathrm{d}y) \qquad (4.1)
\end{aligned}$$

and

$$\begin{aligned}
\int_\gamma f(z)|\mathrm{d}z| &:= \int_a^b f(z(t))\sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t = \int_a^b f(z(t))|z'(t)|\mathrm{d}t \\
&= \int_a^b u(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t \\
&+ \mathrm{i}\int_a^b v(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t. \qquad (4.2)
\end{aligned}$$

The first integral (4.1) is called *line integral of the second kind*, and the second integral (4.2) is called the *line integral of the first kind*.

Example 4.1. Let $f(z) = z$.

1. Let $\gamma : z(t) = t + \mathrm{i}t^2$ for $t \in [0, 1]$. Then

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= \int_\gamma z\mathrm{d}z = \int_0^1 (t + \mathrm{i}t^2)(1 + 2\mathrm{i}t)\mathrm{d}t = \int_0^1 (t + 3\mathrm{i}t^2 - 2t^3)\mathrm{d}t \\ &= \left(\frac{t^2}{2} + 3\mathrm{i}\frac{t^3}{3} - 2\frac{t^4}{4}\right)\bigg|_0^1 = \frac{1}{2} + \mathrm{i} - \frac{1}{2} = \mathrm{i}.\end{aligned}$$

2. Let $\gamma : z(t) = \alpha + \mathrm{i}t$ for $t \in [0, 1]$. Then

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma z\mathrm{d}z = \int_0^1 (\alpha + \mathrm{i}t)\mathrm{i}\mathrm{d}t = \mathrm{i}\alpha - \frac{1}{2}.$$

3. Let $\gamma : z(t) = t^2 + \mathrm{i}\beta$ for $t \in [0, 1]$. Then

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma z\mathrm{d}z = \int_0^1 (t^2 + \mathrm{i}\beta)2t\mathrm{d}t = \left(2\frac{t^4}{4} + \mathrm{i}\beta t^2\right)\bigg|_0^1 = \frac{1}{2} + \mathrm{i}\beta.$$

Remark. It can be easily checked that in all integrations in Example 4.1 the final result depends only on the value of the function $z^2/2$ at the ends of the curve γ. Namely, the result is

$$\frac{(z(1))^2}{2} - \frac{(z(0))^2}{2}.$$

Example 4.2. Let $f(z) = z$.

1. Let $\gamma : z(t) = \alpha + \mathrm{i}t$ for $t \in [0, 1]$. Then

$$\int_\gamma f(z)|\mathrm{d}z| = \int_\gamma z|\mathrm{d}z| = \int_0^1 (\alpha + \mathrm{i}t)dt = \alpha + \frac{\mathrm{i}}{2}.$$

2. Let $\gamma : z(t) = t^2 + \mathrm{i}\beta$ for $t \in [0, 1]$. Then

$$\int_\gamma f(z)|\mathrm{d}z| = \int_\gamma z|\mathrm{d}z| = \int_0^1 (t^2 + \mathrm{i}\beta)2t\mathrm{d}t = \left(2\frac{t^4}{4} + \mathrm{i}\beta t^2\right)\bigg|_0^1 = \frac{1}{2} + \mathrm{i}\beta.$$

Example 4.3. Let $\gamma : z(t) = a + r\mathrm{e}^{\mathrm{i}t}, t \in (-\pi, \pi]$. Then

1.
$$\begin{aligned}\int_\gamma (z-a)^n \mathrm{d}z &= \int_{-\pi}^{\pi} (r\cos t + r\mathrm{i}\sin t)^n r(-\sin t + \mathrm{i}\cos t)\mathrm{d}t \\ &= r^{n+1}\int_{-\pi}^{\pi} (\cos(nt) + \mathrm{i}\sin(nt))(-\sin t + \mathrm{i}\cos t)\mathrm{d}t \\ &= r^{n+1}\int_{-\pi}^{\pi} \left[-\cos(nt)\sin t - \sin(nt)\cos t\right]\mathrm{d}t \\ &+ \mathrm{i}r^{n+1}\int_{-\pi}^{\pi} \left[\cos(nt)\cos t - \sin(nt)\sin t\right]\mathrm{d}t \\ &= -r^{n+1}\int_{-\pi}^{\pi} \sin(n+1)t\mathrm{d}t + \mathrm{i}r^{n+1}\int_{-\pi}^{\pi}\cos(n+1)t\mathrm{d}t \\ &= r^{n+1}\left(\frac{\cos(n+1)t}{n+1} + \mathrm{i}\frac{\sin(n+1)t}{n+1}\right)\Bigg|_{-\pi}^{\pi} = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1. \end{cases}\end{aligned}$$

2.
$$\begin{aligned}\int_\gamma (z-a)^n |\mathrm{d}z| &= \int_{-\pi}^{\pi} r^n \mathrm{e}^{\mathrm{i}nt} r\mathrm{d}t = r^{n+1}\left(\int_{-\pi}^{\pi}\cos(nt)\mathrm{d}t + \mathrm{i}\int_{-\pi}^{\pi}\sin(nt)\mathrm{d}t\right) \\ &= \begin{cases} 0, & n \neq 0 \\ 2\pi r, & n = 0. \end{cases}\end{aligned}$$

Problem 4.4. Let $f(z) = \overline{z}$. Calculate
$$\int_\gamma \overline{z}\mathrm{d}z,$$
where

1. $\gamma : z(t) = t + \mathrm{i}t^2, t \in [0,1]$
2. $\gamma : z(t) = \alpha + \mathrm{i}t, t \in [0,1]$
3. $\gamma : z(t) = t^2 + \mathrm{i}\beta, t \in [0,1]$
4. $\gamma = \gamma_1 \cup \gamma_2$, where $\gamma_1 : z(t) = t + \mathrm{i}t^2$ and $\gamma_2 : z(t) = (1-t) + \mathrm{i}(1-t)$ for $t \in [0,1]$.

If γ is a piecewise smooth Jordan curve then the integrals along this curve are defined as
$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &:= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} f(z(t))z'(t)\mathrm{d}t \\ \int_\gamma f(z)|\mathrm{d}z| &:= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} f(z(t))|z'(t)|\mathrm{d}t.\end{aligned} \tag{4.3}$$

Using the properties of Riemann integral we obtain that

1.
$$\int_\gamma (c_1 f_1(z) + c_2 f_2(z))\mathrm{d}z = c_1 \int_\gamma f_1(z)\mathrm{d}z + c_2 \int_\gamma f_2(z)\mathrm{d}z$$

2.
$$\int_{\gamma_1\cup\gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z$$

3.
$$\int_\gamma (c_1 f_1(z) + c_2 f_2(z))|\mathrm{d}z| = c_1 \int_\gamma f_1(z)|\mathrm{d}z| + c_2 \int_\gamma f_2(z)|\mathrm{d}z|$$

4.
$$\int_{\gamma_1\cup\gamma_2} f(z)|\mathrm{d}z| = \int_{\gamma_1} f(z)|\mathrm{d}z| + \int_{\gamma_2} f(z)|\mathrm{d}z|$$

If $\gamma : z(t), t \in [a, b]$ is a piecewise smooth Jordan curve we can run the curve *backwards* as follows. Let us consider the curve

$$\gamma_1 : \tilde{z} = \tilde{z}(s) = z(a + b - s), \quad s \in [a, b].$$

The curve γ_1 is denoted by $-\gamma$ i.e. $\gamma_1 = -\gamma$ and

$$\begin{aligned}\int_{\gamma_1} f(\tilde{z})\mathrm{d}\tilde{z} &= \int_a^b f(\tilde{z}(s))\tilde{z}'(s)\mathrm{d}s = -\int_a^b f(z(a+b-s))z'(a+b-s)\mathrm{d}s \\ &= \int_b^a f(z(t))z'(t)\mathrm{d}t = -\int_a^b f(z(t))z'(t)\mathrm{d}t = -\int_\gamma f(z)\mathrm{d}z\end{aligned}$$

i.e.
$$\int_{-\gamma} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z.$$

Definition 4.5. A function $f(z)$ is said to have a *primitive* $F(z)$ on $D \subset \mathbb{C}$ if $F(z)$ is differentiable on D and $F'(z) = f(z)$ everywhere on D.

Theorem 4.6. *If a continuous function $f(z)$ has a primitive $F(z)$ on $D \subset \mathbb{C}$ then for any smooth Jordan curve $\gamma : z(t), t \in [a, b]$ in D it holds that*

$$\int_\gamma f(z)\mathrm{d}z = F(z(b)) - F(z(a)). \tag{4.4}$$

Thus, this integral does not depend on γ, but on the endpoints of γ. In particular, if γ is closed and f has a primitive then

$$\int_\gamma f(z)\mathrm{d}z = 0. \tag{4.5}$$

Proof. Let $\gamma : z(t), t \in [a, b]$ be a smooth Jordan curve. Then for any continuous function $f(z)$ the composition $f(z(t))$ and the product $f(z(t))z'(t)$ are continuous and

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

But $f(z(t))z'(t) = (F(z(t)))'$, where F is a primitive of f. Hence,

$$\int_a^b f(z(t))z'(t)\mathrm{d}t = \int_a^b (F(z(t)))'\mathrm{d}t = F(z(t))|_a^b = F(z(b)) - F(z(a)).$$

This proves the theorem. □

Corollary 4.7. *If $\gamma : z(t), t \in [a, b]$ is a piecewise smooth Jordan curve then*

$$\int_\gamma f(z)\mathrm{d}z = F(z(b)) - F(z(a))$$

too, where F is a primitive of f in the domain D.

Proof. By (4.3) we have

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(z(t))z'(t)\mathrm{d}t = \sum_{j=0}^{n-1} (F(z(t_{j+1})) - F(z(t_j)))$$
$$= F(z(t_n)) - F(z(t_0)) = F(z(b)) - F(z(a))$$

and this proves the claim. □

Theorem 4.8. *Let $\gamma : z(t), t \in [a, b]$ be a piecewise smooth Jordan curve and let f be a continuous function. Then*

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \le \int_\gamma |f(z)||\mathrm{d}z| \le \max_{z\in\gamma} |f(z)|L(\gamma), \tag{4.6}$$

where $L(\gamma) = \int_\gamma |\mathrm{d}z|$ denotes the length *of γ.*

Proof. We have

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

Since this Riemann integral can be understood as limit of integral sums then we obtain

$$\left|\int_a^b f(z(t))z'(t)\mathrm{d}t\right| = \left|\lim_{\Delta t\to 0} \sum_{j=1}^{n} f(z(t_j^*))z'(t_j^*)\Delta t_j\right|$$
$$\le \lim_{\Delta t\to 0} \sum_{j=1}^{n} |f(z(t_j^*))||z'(t_j^*)|\Delta t_j = \int_a^b |f(z(t))||z'(t)|\mathrm{d}t = \int_\gamma |f(z)||\mathrm{d}z|$$
$$\le \max_{z\in\gamma} |f(z)| \int_\gamma 1|\mathrm{d}z| = \max_{z\in\gamma} |f(z)|L(\gamma).$$

□

Theorem 4.9 (Change of variable)**.** *Let $g(z)$ be analytic in the domain $D \subset \mathbb{C}$ and $R(f) \subset D'$. Suppose $\gamma : z(t), t \in [a,b]$ is a piecewise smooth Jordan curve in D and $\gamma' = g(\gamma), \tilde{z}(t) = g(z(t)), t \in [a,b]$ is the transformed curve in D'. Then for all continuous functions f on D we have*

$$\int_\gamma f(g(z))g'(z)\mathrm{d}z = \int_{\gamma'} f(w)\mathrm{d}w. \tag{4.7}$$

Proof. We know that

$$\begin{aligned}\int_\gamma f(g(z))g'(z)\mathrm{d}z &= \int_a^b f(g(z(t)))g'(z(t))z'(t)\mathrm{d}t \\ &= \int_a^b f(g(z(t)))(g(z(t)))'\mathrm{d}t = \int_a^b f(\tilde{z}(t))\tilde{z}'(t)\mathrm{d}t = \int_{\gamma'} f(w)\mathrm{d}w.\end{aligned}$$

□

Example 4.10. Let $\gamma : z(t) = t + \mathrm{i}t^2, t \in [0,1]$. Using $g(z) = z^2$ we get

$$\begin{aligned}\int_\gamma \sin(z^2)z\mathrm{d}z &= \frac{1}{2}\int_{\gamma'} \sin w\mathrm{d}w = -\frac{1}{2}\cos w\Big|_{w(0)}^{w(1)} \\ &= -\frac{1}{2}\cos w\Big|_0^{2\mathrm{i}} = \frac{1}{2}(1-\cos(2\mathrm{i})) = \frac{1}{2}(1-\cosh 2).\end{aligned}$$

Here we have used the notation $w(t) = g(z(t))$.

Problem 4.11. Let $\gamma : z(t) = \alpha + \mathrm{i}t^2, t \in [0,1]$. Calculate

$$\int_\gamma \mathrm{e}^{\sin z}\cos z\mathrm{d}z.$$

Problem 4.12. Let $\gamma : z(t) = 1 + \mathrm{i}t, t \in [0,1]$. Calculate

$$\int_\gamma \log z\frac{1}{z}\mathrm{d}z.$$

Chapter 5

Cauchy theorem and Cauchy integral formulae

Definition 5.1. A bounded domain $D \subset \mathbb{C}$ is called *simply connected* if for any closed Jordan curve $\gamma \subset D$ the internal domain $(\operatorname{int}\gamma)$ belongs to D too. Otherwise D is called *multiply connected*. The number of connected components of the boundary is said to be the *connected order* of D.

Theorem 5.2 (Cauchy theorem)**.** *Let D be a bounded simply connected domain with the boundary ∂D which is a piecewise smooth closed Jordan curve γ. Then for any function $f \in H(D)$ which is continuous in $\overline{D}$ we have*

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Proof. Since $f \in C(\overline{D})$ then $\int_\gamma f(z)\mathrm{d}z$ is well-defined and it is equal to

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = \int_\gamma u\mathrm{d}x - v\mathrm{d}y + \mathrm{i}\int_\gamma v\mathrm{d}x + u\mathrm{d}y.$$

Using now Green's theorem (or Stoke's theorem) we obtain that the integrals in the right hand side are equal to

$$\iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y + \mathrm{i}\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\mathrm{d}y = 0$$

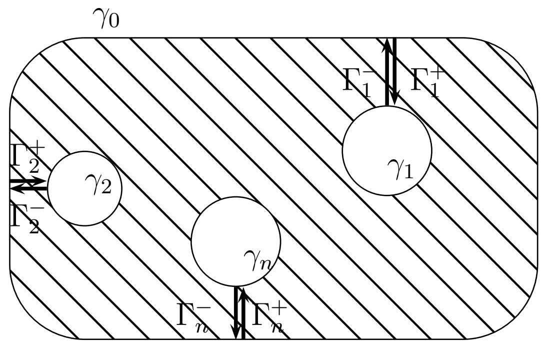
because of Cauchy-Riemann equations. Thus theorem is proved. □

Remark. If the domain D is simply connected then the Cauchy theorem holds not only for the boundary ∂D but also for any closed piecewise smooth Jordan curve γ such that $\gamma \subset D$.

Corollary 5.3. *Let D be a bounded $(n+1)$-connected domain such that $\partial D = \cup_{j=0}^{n}\gamma_j$, where γ_j are closed piecewise smooth Jordan curves, $\operatorname{int}\gamma_j \cap \operatorname{int}\gamma_k = \emptyset, k \neq j$ and $\gamma_1, \ldots, \gamma_n \subset \operatorname{int}\gamma_0$. If $f \in H(D) \cap C(\overline{D})$ then*

$$\int_{\partial D} f(z)\mathrm{d}z = \int_{\gamma_0} f(z)\mathrm{d}z - \sum_{j=1}^{n} \int_{\gamma_j} f(z)\mathrm{d}z = 0.$$

Proof. By the conditions of this Corollary, the domain D has the form depicted below.



Let us join $\gamma_j, j = 1, 2, \ldots, n$ with γ_0 by the smooth Jordan curves Γ_j such that any $\Gamma_j, j = 1, 2, \ldots, n$ is passed twice in opposite directions. In this case we obtain simply connected domain D_1 with the boundary

$$\partial D_1 = (\cup_{j=0}^{n}\gamma_j) \cup (\cup_{j=1}^{n}\Gamma_j^{\pm}).$$

Thus, applying Cauchy theorem to the domain D_1 we obtain

$$\begin{aligned} 0 = \int_{\partial D_1} f(z)\mathrm{d}z &= \int_{\partial D} f(z)\mathrm{d}z + \sum_{j=1}^{n}\int_{\Gamma_j^+} f(z)\mathrm{d}z + \sum_{j=1}^{n}\int_{\Gamma_j^-} f(z)\mathrm{d}z \\ &= \int_{\gamma} f(z)\mathrm{d}z - \sum_{j=1}^{n}\int_{\gamma_j} f(z)\mathrm{d}z. \end{aligned}$$

Here we have used the fact that

$$\int_{\Gamma_j^+} f(z)\mathrm{d}z + \int_{\Gamma_j^-} f(z)\mathrm{d}z = 0$$

and that the positive direction of integration is the direction in which the internal domain is on the left. □

If the domain D is multiply connected then the Cauchy theorem does not hold for arbitrary closed piecewise smooth Jordan curve. In this case it is necessary to integrate over the whole boundary of D. Indeed, let

$$D = \{z : 1 < |z| < 3\}$$

and $\gamma = \{z : |z| = 2\}$. Then $\gamma \subset D$ but

$$\int_\gamma \frac{1}{z} \mathrm{d}z = 2\pi \mathrm{i}.$$

Corollary 5.4. *Let D be a domain which satisfies either the conditions of Theorem 5.2 or Corollary 5.3. If f is analytic in D and continuous in $\overline{D}$ except the points $z_1, \ldots, z_m \in D$ with*

$$\lim_{z \to z_k} (z - z_k) f(z) = 0, \quad k = 1, 2, \ldots, m$$

then

$$\int_{\partial D} f(z) \mathrm{d}z = 0.$$

Proof. For simplicity and without loss of generality we assume that $m = 1$. Then for any $\varepsilon > 0$ there is $\delta(z_1, \varepsilon) > 0$ such that for all z with $0 < |z - z_1| < \delta$ it follows that

$$|z - z_1||f(z)| < \varepsilon.$$

Let $D_1 := D \setminus \{z : |z - z_1| \le \delta\}$ assuming that $\delta > 0$ is so small that $\{z : |z - z_1| \le \delta\} \subset D$. Then for the domain D_1 Cauchy theorem holds and therefore

$$0 = \int_{\partial D_1} f(z) \mathrm{d}z = \int_{\partial D} f(z) \mathrm{d}z - \int_{|z - z_1| = \delta} f(z) \mathrm{d}z.$$

But

$$\begin{aligned} \left| \int_{|z-z_1|=\delta} f(z) \mathrm{d}z \right| &\le \int_{|z-z_1|=\delta} |f(z)||\mathrm{d}z| \\ &= \int_{|z-z_1|=\delta} |z - z_1||f(z)| \frac{|\mathrm{d}z|}{|z - z_1|} < \varepsilon \frac{1}{\delta} \int_{|z-z_1|=\delta} |\mathrm{d}z| = 2\pi\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary then we may let $\varepsilon \to 0$ and obtain

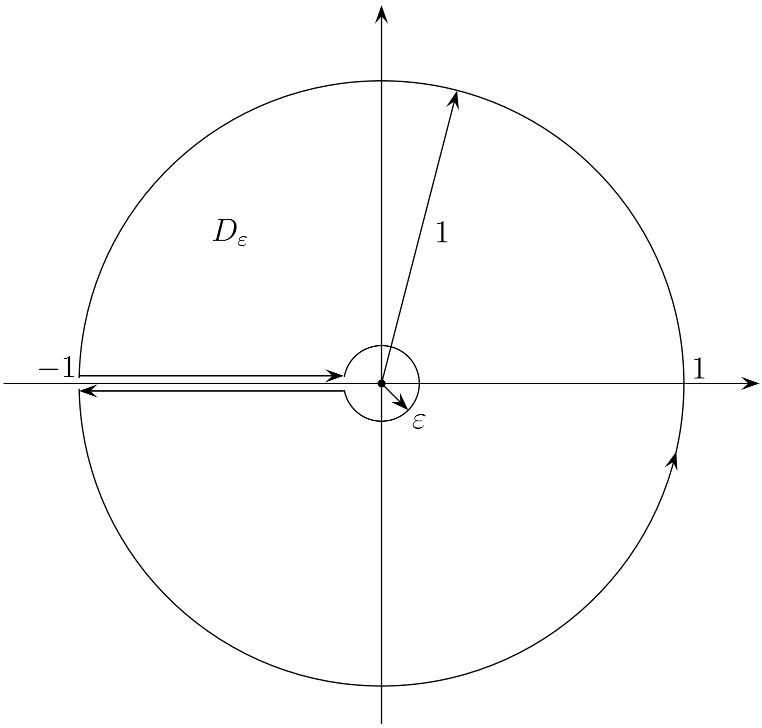
$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \int_{\partial D_1} f(z) \mathrm{d}z = \lim_{\varepsilon \to 0} \left(\int_{\partial D} f(z) \mathrm{d}z - \int_{|z-z_1|=\delta} f(z) \mathrm{d}z \right) \\ &= \int_{\partial D} f(z) \mathrm{d}z - \lim_{\varepsilon \to 0} \int_{|z-z_1|=\delta} f(z) \mathrm{d}z = \int_{\partial D} f(z) \mathrm{d}z. \end{aligned}$$

$\square$

Example 5.5. If we calculate $\int_{|z|=1} \log z \mathrm{d}z$ then using the parametrization $z = \mathrm{e}^{\mathrm{i}\theta}, \theta \in [-\pi, \pi]$ and integration by parts we obtain

$$\begin{aligned}\int_{|z|=1} \log z \mathrm{d}z &= \int_{-\pi}^{\pi} (\log|z| + \mathrm{i}\theta)\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}^2 \int_{-\pi}^{\pi} \theta \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta \\ &= -\int_{-\pi}^{\pi} \theta\cos\theta\mathrm{d}\theta - \mathrm{i}\int_{-\pi}^{\pi}\theta\sin\theta\mathrm{d}\theta = -2\mathrm{i}\int_0^{\pi}\theta\sin\theta\mathrm{d}\theta \\ &= 2\mathrm{i}\left(\theta\cos\theta\big|_0^{\pi} - \int_0^{\pi}\cos\theta\mathrm{d}\theta\right) = -2\pi\mathrm{i}.\end{aligned}$$

It shows that Cauchy theorem does not hold in this case. But we know that $\log z$ is analytic and has a removable singularity at $z = 0$. This phenomenon can be explained as follows: $\log z$ has a jump $2\pi\mathrm{i}$ over the negative real line i.e. it is not continuous in the unit disk and therefore it is not analytic. Even more is true, it is not univalent there. In order to eliminate this problem we proceed as follows. Let us consider the following domain D_ε for $\varepsilon > 0$ small enough.



In this domain D_ε the function $\log z$ is not only analytic but also univalent.

Applying the Cauchy theorem (see Theorem 5.2) we obtain

$$\begin{aligned}0 &= \int_{\partial D_\varepsilon} \log z \mathrm{d}z = \int_{-\pi}^{\pi} (\log 1 + \mathrm{i}\theta)\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta + \int_{-1}^{-\varepsilon} (\log|x| + \mathrm{i}\pi)\mathrm{d}x \\ &- \int_{-\pi}^{\pi} (\log\varepsilon + \mathrm{i}\theta)\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta + \int_{-\varepsilon}^{-1} (\log|x| - \mathrm{i}\pi)\mathrm{d}x \\ &= -2\pi\mathrm{i} + \int_{-1}^{-\varepsilon} \log|x|\mathrm{d}x + \mathrm{i}\pi(1-\varepsilon) + 2\pi\mathrm{i}\varepsilon + \int_{-\varepsilon}^{-1} \log|x|\mathrm{d}x - \mathrm{i}\pi(-1+\varepsilon) = 0\end{aligned}$$

for any $\varepsilon > 0$. Taking $\varepsilon \to +0$ we obtain that

$$\int_{\partial D} \log z \mathrm{d}z := \lim_{\varepsilon \to +0} \int_{\partial D_\varepsilon} \log z \mathrm{d}z = 0,$$

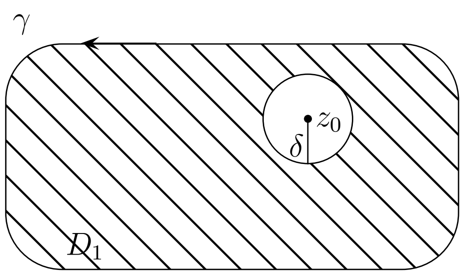
where ∂D is the unit circle with a cut along the negative real line.

Example 5.6. Let γ be a piecewise smooth closed Jordan curve and $z_0 \in \operatorname{int}\gamma$. Then

$$\int_\gamma \frac{1}{z - z_0}\mathrm{d}z = 2\pi\mathrm{i}.$$

Indeed, if we consider the domain

$$D_1 := \operatorname{int}\gamma \setminus \{z : |z - z_0| \le \delta\}$$



then by Corollary 5.3 we have

$$0 = \int_\gamma \frac{\mathrm{d}z}{z - z_0} - \int_{|z - z_0| = \delta} \frac{\mathrm{d}z}{z - z_0}.$$

But

$$\int_{|z - z_0| = \delta} \frac{\mathrm{d}z}{z - z_0} = \int_{-\pi}^{\pi} \frac{\mathrm{i}\delta\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta}{\delta\mathrm{e}^{\mathrm{i}\theta}} = 2\pi\mathrm{i}.$$

This example can be generalized to the multiply connected domain D also, i.e. if $z_0 \in D$ then

$$\int_{\partial D} \frac{1}{z - z_0}\mathrm{d}z = 2\pi\mathrm{i}.$$

Theorem 5.7 (Cauchy integral formula)**.** *Let $D \subset \mathbb{C}$ be a bounded domain with the boundary ∂D which satisfies all conditions of Corollary 5.3. Then for any function $f \in H(D) \cap C(\overline{D})$ and any $z_0 \in \mathbb{C}$ we have*

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f(z)}{z - z_0} \mathrm{d}z = \begin{cases} 0, & z_0 \notin \overline{D} \\ f(z_0), & z_0 \in D \\ \frac{1}{2} f(z_0), & z_0 \in \partial D. \end{cases}$$

Proof. If $z_0 \notin \overline{D}$ then the function

$$h(z) := \frac{f(z)}{z - z_0}$$

is analytic in D and continuous in $\overline{D}$. Then Corollary 5.3 leads to

$$0 = \int_{\partial D} h(z) \mathrm{d}z = \int_{\partial D} \frac{f(z)}{z - z_0} \mathrm{d}z.$$

If $z_0 \in D$ then we consider the function

$$h(z) := \frac{f(z) - f(z_0)}{z - z_0}.$$

It is clear that $h \in H(D \setminus z_0) \cap C(\overline{D} \setminus z_0)$ and $\lim_{z \to z_0} (z - z_0) h(z) = 0$. Thus, using Corollary 5.4 we obtain

$$0 = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} h(z) \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f(z)}{z - z_0} \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f(z_0)}{z - z_0} \mathrm{d}z$$

or

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f(z)}{z - z_0} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} f(z_0) \int_{\partial D} \frac{1}{z - z_0} \mathrm{d}z.$$

But Example 5.6 implies that

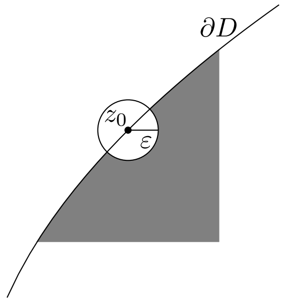
$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f(z)}{z - z_0} \mathrm{d}z = f(z_0).$$

If $z_0 \in \partial D$ then the integral in the left-hand side must be understood as the principal value integral

$$\text{p. v.} \int_{\partial D} \frac{f(z)}{z - z_0} \mathrm{d}z := \lim_{\varepsilon \to +0} \int_{\partial D \setminus \{z : |z - z_0| < \varepsilon\}} \frac{f(z)}{z - z_0} \mathrm{d}z$$

if this limit exists. For $z_0 \in \partial D$ we consider the domain

$$D_\varepsilon = D \setminus (D \cap \{z : |z - z_0| < \varepsilon\})$$



It is clear that the function

$$h(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

is analytic in D_ε and continuous up to the boundary of D_ε. Thus, using again Corollary 5.4 we obtain

$$0 = \frac{1}{2\pi\mathrm{i}} \int_{\partial D_\varepsilon} h(z)\mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \int_{\partial D_\varepsilon} \frac{f(z)}{z - z_0}\mathrm{d}z - \frac{1}{2\pi\mathrm{i}} \int_{\partial D_\varepsilon} \frac{f(z_0)}{z - z_0}\mathrm{d}z.$$

So

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial D_\varepsilon} \frac{f(z)}{z - z_0}\mathrm{d}z = \frac{f(z_0)}{2\pi\mathrm{i}} \int_{\partial D_\varepsilon} \frac{1}{z - z_0}\mathrm{d}z \approx \frac{f(z_0)}{2\pi\mathrm{i}}\pi\mathrm{i} = \frac{f(z_0)}{2}$$

as $\varepsilon \to +0$ since only half of the circle is presented. □

Example 5.8. Let us calculate the integral

$$\int_\gamma \frac{\mathrm{e}^{z^2}}{z(z^2 + 4)}\mathrm{d}z,$$

where $\gamma = \{z : |z| = 3\}$. We parametrize this smooth closed Jordan curve as $\gamma : z(t) = 3\mathrm{e}^{\mathrm{i}t}, t \in [-\pi, \pi]$. Next,

$$\frac{1}{z(z^2 + 4)} = \frac{1}{z(z - 2\mathrm{i})(z + 2\mathrm{i})} = \frac{1}{4} \cdot \frac{1}{z} - \frac{1}{8} \cdot \frac{1}{z - 2\mathrm{i}} - \frac{1}{8} \cdot \frac{1}{z + 2\mathrm{i}}.$$

Hence, applying Cauchy integral formula,

$$\begin{aligned}\int_\gamma \frac{\mathrm{e}^{z^2}}{z(z^2 + 4)}\mathrm{d}z &= \frac{1}{4} \int_\gamma \frac{\mathrm{e}^{z^2}}{z}\mathrm{d}z - \frac{1}{8} \int_\gamma \frac{\mathrm{e}^{z^2}}{z - 2\mathrm{i}}\mathrm{d}z - \frac{1}{8} \int_\gamma \frac{\mathrm{e}^{z^2}}{z + 2\mathrm{i}}\mathrm{d}z \\ &= 2\pi\mathrm{i}\frac{1}{4}\mathrm{e}^0 - 2\pi\mathrm{i}\frac{1}{8}\mathrm{e}^{(-2\mathrm{i})^2} - 2\pi\mathrm{i}\frac{1}{8}\mathrm{e}^{(2\mathrm{i})^2} = 2\pi\mathrm{i}\frac{1 - \mathrm{e}^{-4}}{4} = \pi\mathrm{i}\frac{1 - \mathrm{e}^{-4}}{2}.\end{aligned}$$

Example 5.9. Let us calculate the integral

$$\int_0^\pi \mathrm{e}^{a\cos t}\cos(a\sin t)\mathrm{d}t.$$

Since the integrand is even and sine is odd we have

$$\begin{aligned}\int_0^\pi \mathrm{e}^{a\cos t}\cos(a\sin t)\mathrm{d}t &= \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{e}^{a\cos t}\cos(a\sin t)\mathrm{d}t \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{e}^{a\cos t}(\cos(a\sin t)+\mathrm{i}\sin(a\sin t))\mathrm{d}t \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{e}^{a\cos t}\mathrm{e}^{\mathrm{i}a\sin t}\mathrm{d}t.\end{aligned}$$

For $z(t)=\mathrm{e}^{\mathrm{i}t}, t\in[-\pi,\pi]$ we have $\mathrm{d}t=\frac{\mathrm{d}z}{\mathrm{i}z}$. Then the latter integral can be interpreted as the curve integral over the closed Jordan curve $\gamma : z(t)=\mathrm{e}^{\mathrm{i}t}, t\in[-\pi,\pi]$. That's why it is equal to

$$\begin{aligned}\frac{1}{2}\int_{-\pi}^{\pi} \mathrm{e}^{a(\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t})/2}\mathrm{e}^{\mathrm{i}a(\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{-\mathrm{i}t})/2\mathrm{i}}\mathrm{d}t &= \frac{1}{2}\int_\gamma \mathrm{e}^{\frac{a}{2}(z+1/z)}\mathrm{e}^{\frac{a}{2}(z-1/z)}\frac{\mathrm{d}z}{\mathrm{i}z} \\ &= \frac{1}{2\mathrm{i}}\int_\gamma \frac{\mathrm{e}^{az}}{z}\mathrm{d}z = \frac{1}{2\mathrm{i}}2\pi\mathrm{i}\mathrm{e}^0 = \pi\end{aligned}$$

by Cauchy integral formula.

Example 5.10. Let us calculate the integral

$$\int_\gamma \frac{2z}{z^2+2}\mathrm{d}z,$$

where $\gamma=\{z : |z-\mathrm{i}|=1\}$. First we have

$$\frac{2z}{z^2+2} = \frac{1}{z-\mathrm{i}\sqrt{2}} + \frac{1}{z+\mathrm{i}\sqrt{2}}$$

and therefore

$$\int_\gamma \frac{2z}{z^2+2}\mathrm{d}z = \int_\gamma \frac{1}{z-\mathrm{i}\sqrt{2}}\mathrm{d}z + \int_\gamma \frac{1}{z+\mathrm{i}\sqrt{2}}\mathrm{d}z = 2\pi\mathrm{i}$$

since $\mathrm{i}\sqrt{2}\in\operatorname{int}\gamma$ but $-\mathrm{i}\sqrt{2}\notin\operatorname{int}\gamma$.

Let us consider now a piecewise smooth Jordan curve (not necessarily closed) γ and continuous function $f(z)$ on this curve. If $z\notin\gamma$ then the function

$$F(z) := \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta \tag{5.1}$$

is well-defined on $\mathbb{C}\setminus\gamma$. This function $F(z)$ is called a *Cauchy type integral*.

Theorem 5.11. *The Cauchy type integral* (5.1) *is analytic function in* $\mathbb{C} \setminus \gamma$, *it has derivatives of any order* $n \in \mathbb{N}$ *and the formula*

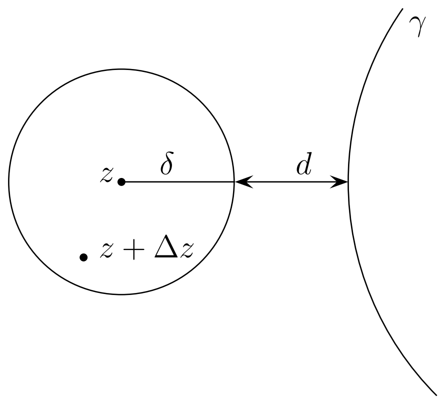
$$F^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta \tag{5.2}$$

holds.

Proof. Let $z \notin \gamma$ and $z + \Delta z \notin \gamma$ too. Then

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - z - \Delta z)} \mathrm{d}\zeta.$$

Since $z \notin \gamma$ and $z + \Delta z \notin \gamma$ then there is $\delta > 0$ and $d > 0$ such that $z + \Delta z \in U_\delta(z), U_\delta(z) \cap \gamma = \emptyset$ and $|\zeta - z| \geq d > 0, |\zeta - z - \Delta z| \geq d > 0$ for any $\zeta \in \gamma$. (Actually $d = \mathrm{dist}(\gamma, |\zeta - z| = \delta)$).



In that case we have

$$\begin{aligned}\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta \right| &= \frac{1}{2\pi} \left| \int_\gamma \frac{\Delta z f(\zeta)}{(\zeta - z)^2 (\zeta - z - \Delta z)} \mathrm{d}\zeta \right| \\ &\leq \frac{1}{2\pi} |\Delta z| \int_\gamma \frac{|f(\zeta)||\mathrm{d}\zeta|}{|\zeta - z|^2 |\zeta - z - \Delta z|} \\ &\leq \frac{1}{2\pi} |\Delta z| M \frac{1}{d^3} \int_\gamma |\mathrm{d}\zeta| = \frac{|\Delta z| M L}{2\pi d^3},\end{aligned}$$

where L is the length of γ and $M = \max_\gamma |f(\zeta)| < \infty$. Letting $\Delta z \to 0$ this estimate shows that

$$\lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta$$

or

$$F'(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta.$$

After this (5.2) can be proved by induction. Indeed,

$$
\begin{aligned}
&\frac{F^{(n-1)}(z+\Delta z)-F^{(n-1)}(z)}{\Delta z}-\frac{n!}{2\pi \mathrm{i}}\int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta-z)^{n+1}}\\
&=\frac{1}{2\pi \mathrm{i}}\int_\gamma f(\zeta)\left[\frac{(n-1)!}{\Delta z}\left(\frac{1}{(\zeta-z-\Delta z)^n}-\frac{1}{(\zeta-z)^n}\right)-\frac{n!}{(\zeta-z)^{n+1}}\right]\mathrm{d}\zeta\\
&=\frac{(n-1)!}{2\pi \mathrm{i}}\int_\gamma f(\zeta)\left[\frac{(\zeta-z)^n-(\zeta-z-\Delta z)^n}{\Delta z(\zeta-z-\Delta z)^n(\zeta-z)^n}-\frac{n}{(\zeta-z)^{n+1}}\right]\mathrm{d}\zeta\\
&=\frac{(n-1)!}{2\pi \mathrm{i}}\int_\gamma f(\zeta)\left[\frac{\sum_{k=1}^n(-1)^{k+1}\binom{n}{k}(\Delta z)^k(\zeta-z)^{n-k}}{\Delta z(\zeta-z-\Delta z)^n(\zeta-z)^n}-\frac{n}{(\zeta-z)^{n+1}}\right]\mathrm{d}\zeta\\
&=\frac{(n-1)!}{2\pi \mathrm{i}}\int_\gamma f(\zeta)\\
&\times\left[\frac{\sum_{j=0}^{n-1}(-1)^j\binom{n}{j+1}(\Delta z)^j(\zeta-z)^{n-j}-n\sum_{j=0}^{n}(-1)^j\binom{n}{j}(\Delta z)^j(\zeta-z)^{n-j}}{(\zeta-z-\Delta z)^n(\zeta-z)^{n+1}}\right]\mathrm{d}\zeta\\
&=\frac{(n-1)!}{2\pi \mathrm{i}}\int_\gamma f(\zeta)\\
&\quad\times\left[\frac{\sum_{j=1}^{n-1}(-1)^j(\Delta z)^j(\zeta-z)^{n-j}\left(\binom{n}{j+1}-n\binom{n}{j}\right)+n(-1)^{n+1}(\Delta z)^n}{(\zeta-z-\Delta z)^n(\zeta-z)^{n+1}}\right]\mathrm{d}\zeta.
\end{aligned}
$$

This representation implies that

$$
\begin{aligned}
&\left|\frac{F^{(n-1)}(z+\Delta z)-F^{(n-1)}(z)}{\Delta z}-\frac{n!}{2\pi \mathrm{i}}\int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta-z)^{n+1}}\right|\\
&\qquad\le\frac{(n-1)!}{2\pi}O(|\Delta z|)\int_\gamma\frac{|f(\zeta)||\mathrm{d}\zeta|}{|(\zeta-z-\Delta z)^n||(\zeta-z)^{n+1}|}\\
&\qquad\qquad\le\frac{(n-1)!}{2\pi}O(|\Delta z|)\frac{ML}{d^{2n+1}}.
\end{aligned}
$$

This estimate completes the proof of (5.2) by induction. □

Corollary 5.12. *Let $D\subset\mathbb{C}$ be a domain (not necessarily simply connected) and $f\in H(D)$. Then f is infinitely many times differentiable in D and*

$$f^{(n)}(z)=\frac{n!}{2\pi \mathrm{i}}\int_\gamma\frac{f(\zeta)}{(\zeta-z)^{n+1}}\mathrm{d}\zeta \tag{5.3}$$

for any $n=1,2,\ldots$, where γ is an arbitrary piecewise smooth closed Jordan curve such that $\operatorname{int}\gamma\subset D$ *and* $z\in\operatorname{int}\gamma$*.*

Proof. Let $z \in D$. Let also γ be an arbitrary piecewise smooth closed Jordan curve such that $\operatorname{int}\gamma \subset D$ and $z \in \operatorname{int}\gamma$. Then by the Cauchy integral formula (see Theorem 5.7) we have

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z}.$$

But the right hand side is Cauchy type integral since f is continuous on γ. Applying Theorem 5.11 we obtain that for any $n = 1, 2, \dots$ we have

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z)^{n+1}}$$

and f is infinitely many times differentiable in D. □

Remark. Formula (5.3) holds also for the boundary ∂D of a domain that satisfies all conditions of Corollary 5.3 if we assume that $f \in H(D) \cap C(D)$. Moreover, as the simplest case, formula (5.3) holds and is very applicable for $\gamma = \{\zeta : |\zeta - z| = \delta\}$ with $\delta > 0$ small enough i.e.

$$\begin{aligned} f^{(n)}(z) &= \frac{n!}{2\pi \mathrm{i}} \int_{|\zeta - z| = \delta} \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z)^{n+1}} = \frac{n!}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{f(z + \delta \mathrm{e}^{\mathrm{i}\theta})\delta \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}{\delta^{n+1}\mathrm{e}^{\mathrm{i}\theta(n+1)}}\mathrm{d}\theta \\ &= \frac{n!}{2\pi}\delta^{-n} \int_{-\pi}^{\pi} f(z + \delta \mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}\theta n}\mathrm{d}\theta. \end{aligned}$$

Problem 5.13. Evaluate the derivative of $F(z)$ from (5.1) at $z = \infty$. Show first that $F(z)$ is continuous at $z = \infty$ and $F(\infty) = 0$. Show also that

$$F'(\infty) = -\frac{1}{2\pi \mathrm{i}} \int_\gamma f(\zeta)\mathrm{d}\zeta.$$

Example 5.14. Let us calculate the integral

$$\int_\gamma \frac{\sin z}{(z - \pi/6)^3}\mathrm{d}z,$$

where $\gamma = \{z : |z| = 1\}$. Since $|\pi/6| < 1$ then applying (5.3) we obtain

$$\int_\gamma \frac{\sin z \mathrm{d}z}{(z - \pi/6)^3} = \frac{2\pi \mathrm{i}}{2!} \left.(\sin z)''\right|_{z=\pi/6} = \pi \mathrm{i} \left.(-\sin z)\right|_{z=\pi/6} = -\pi \mathrm{i} \sin(\pi/6) = -\frac{\pi \mathrm{i}}{2}.$$

Example 5.15. Let us calculate the integral

$$\int_\gamma \frac{\mathrm{d}z}{(z - a)^4(z - b)},$$

where $\gamma = \{z : |z| = r\}$ and $|a| < r < |b|$. Since $z \neq b$ for all $|z| \le r$ then this integral is equal to

$$\int_\gamma \frac{\frac{1}{z-b}\mathrm{d}z}{(z - a)^4} = \frac{2\pi \mathrm{i}}{3!} \left.\left(\frac{1}{z - b}\right)'''\right|_{z=a} = \frac{2\pi \mathrm{i}}{6} \left.\left(-\frac{6}{(z - b)^4}\right)\right|_{z=a} = -\frac{2\pi \mathrm{i}}{(a - b)^4}.$$

Example 5.16. Let f be analytic in a simply connected domain D and $z_1, z_2 \in D, z_1 \neq z_2$. Then for any piecewise smooth closed Jordan curve γ such that $z_1, z_2 \in \operatorname{int}\gamma$ we have

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z_1)(\zeta - z_2)}.$$

Since

$$f(z_j) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z_j}, \quad j = 1, 2$$

then

$$f(z_2) - f(z_1) = \frac{1}{2\pi\mathrm{i}} \int_\gamma f(\zeta) \left(\frac{1}{\zeta - z_2} - \frac{1}{\zeta - z_1} \right) \mathrm{d}\zeta = \frac{z_2 - z_1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z_1)(\zeta - z_2)}.$$

Example 5.17. Let us show that

$$\int_{|z|=1} \mathrm{e}^z \overline{\mathrm{d}z} = -2\pi\mathrm{i}$$

or

$$\int_{|z|=1} \overline{\mathrm{e}^z} \mathrm{d}z = 2\pi\mathrm{i}.$$

We have

$$\begin{aligned} \int_{|z|=1} \mathrm{e}^{\overline{z}} \mathrm{d}z &= \int_{|z|=1} \mathrm{e}^{1/z} \mathrm{d}z \\ &= -\int_{|\zeta|=1} \mathrm{e}^\zeta \mathrm{d}\left(\frac{1}{\zeta}\right) = \int_{|\zeta|=1} \frac{\mathrm{e}^\zeta}{\zeta^2} \mathrm{d}\zeta = 2\pi\mathrm{i}(e^\zeta)'|_{\zeta=0} = 2\pi\mathrm{i}, \end{aligned}$$

where we have also changed the direction of integration when changing variables.

Appendix A

Exercises

1. Calculate
   a) i^k, b) i^{-k}
   for $k = 0, 1, 2, \ldots$.

2. Find $\operatorname{Re} z$ and $\operatorname{Im} z$, when
   a) $z = (2+3\mathrm{i})(-3+2\mathrm{i})$, b) $z = \dfrac{4+2\mathrm{i}}{3-4\mathrm{i}}$, c) $z = \overline{(1+\mathrm{i})} \cdot \dfrac{1}{2-\mathrm{i}}$.

3. Solve z from the equation

   a) $(3+4\mathrm{i})\overline{z} = 1-2\mathrm{i}$,
   b) $\mathrm{i}z + 2\overline{z} = 3 - \mathrm{i}$,
   c) $z^2 = -5 + 12\mathrm{i}$.

4. Prove that
   $$|\operatorname{Re} z| \leq |z|, \quad |\operatorname{Im} z| \leq |z|.$$
   Show also that the equalities hold if and only if z is real or pure imaginary, respectively.

5. Prove that $|z_1 - z_2| = |1 - \overline{z_1} z_2|$, where $z_1, z_2 \in \mathbb{C}$ and $|z_1| = 1$ or $|z_2| = 1$.

6. Express $z \in \mathbb{C}$ in trigonometric form when
   a) $z = -3\mathrm{i}$, b) $z = \sqrt{3} - \mathrm{i}$, c) $z = 2 - \mathrm{i}\sqrt{12}$.

7. Calculate $(1-\mathrm{i}\sqrt{3})^{15}$, $(1+\mathrm{i})^{11}$ and $\dfrac{(1+\mathrm{i})^5}{(1-\mathrm{i}\sqrt{3})^7}$.

8. Let $z \in \mathbb{C}, |z| = 1, z \neq -1$. Prove that z can be written in the form $z = \dfrac{1+\mathrm{i}t}{1-\mathrm{i}t}$ for some $t \in \mathbb{R}$.

9. Solve the equations
   a) $z^4 = -1$, b) $z^6 = 1$, c) $z^3 = -\mathrm{i}$.

10. Prove that the set $\{z \in \mathbb{C} \mid |z - z_0| > r\}$ is open ($z_0 \in \mathbb{C}, r > 0$ are given).

11. Let $A = \{\mathrm{i}, \frac{\mathrm{i}}{2}, \frac{\mathrm{i}}{3}, \ldots\} \subset \mathbb{C}$. Determine if A is bounded, closed or open. Find A' and $\overline{A}$.

12. Find the following limits (if they exist)
a) $\lim\limits_{n\to\infty} \dfrac{\mathrm{i}^n}{n}$, b) $\lim\limits_{n\to\infty} \mathrm{i}^n$, c) $\lim\limits_{n\to\infty} \dfrac{(1+\mathrm{i})^n}{n}$, d) $\lim\limits_{n\to\infty} \dfrac{2n - \mathrm{i}n^2}{(1+\mathrm{i})n - 1}$.

13. Let the sequence $(z_n) \subset \mathbb{C}$ be defined as $z_0 = 3$ and $z_{n+1} = \frac{1}{3}z_n + 2\mathrm{i}$. Show that (z_n) converges and find its limit.

14. Determine which of the following functions are bijective $D \to G$ and find $f^{-1} : G \to D$ whenever it is possible.
a) $f(z) = \overline{z} + \mathrm{i}, z \in \mathbb{C}$, b) $f(z) = \frac{1}{z}, z \in \mathbb{C} \setminus \{0\}$,
c) $f(z) = z^2 + \mathrm{i}, z \in \mathbb{C}$, d) $f(z) = z^2 + \mathrm{i}, 0 \le \arg z < \pi$.

15. Let $f : D \to \mathbb{C}$ be a function such that $f(z) = z^3 + \mathrm{i}, 0 \le \arg z < 2\pi/3$. Determine if f is bijective $D \to \mathbb{C}$. Find $f^{-1}(1)$.

16. Express the function $f(z) = f(x + \mathrm{i}y)$ in the form $f(z) = u(x, y) + \mathrm{i}v(x, y), z \in D$, when
a) $f(z) = z^3, z \in \mathbb{C}$, b) $f(z) = \frac{1}{z}, z \ne 0$, c) $f(z) = \mathrm{e}^{\mathrm{i}z}, z \in \mathbb{C}$.

17. Investigate the existence of the limit of $f(z)$ at the point $z = 0$, when
a) $f(z) = \dfrac{\operatorname{Re} z}{z}$, b) $f(z) = \dfrac{z}{|z|}$, c) $f(z) = \dfrac{z \operatorname{Re} z}{|z|}$.

18. Find the limit $\lim\limits_{z\to z_0} \dfrac{z^3 + z^2 + z + 1}{z - z_0}$, when
a) $z_0 = -1$, b) $z_0 = \mathrm{i}$, c) $z_0 = -\mathrm{i}$. d) Find the limit $\lim\limits_{z\to \mathrm{i}} \dfrac{z^3 + \mathrm{i}}{z - \mathrm{i}}$.

19. Prove using the definition of continuity that the function $f(z) = z^2 + 2z, z \in \mathbb{C}$ is continuous for all $z_0 \in \mathbb{C}$ but it is not continuous at $z_0 = \infty$.

20. Show that the function $f(z) = z^2$ is uniformly continuous on the set $|z - \mathrm{i}| < 2$. Is f uniformly continuous on $\mathbb{C}$?

21. Study the uniform continuity of $f(z) = \frac{1}{z}, z \ne 0$ on the set $|z| < 1, z \ne 0$.

22. Investigate if the function $f(z) = z|z|, z \in \mathbb{C}$ has a derivative at any $z_0 \in \mathbb{C}$.

23. Find the derivatives of the following functions (if they exist)
a) $f(z) = \dfrac{z^2 + 1}{(z^2 - 1)^2}, z \ne \pm 1$, b) $f(z) = \mathrm{e}^{\overline{z}}, z \in \mathbb{C}$,
c) $f(z) = \operatorname{Im} z, z \in \mathbb{C}$ d) $f(z) = z \operatorname{Im} z, z \in \mathbb{C}$.

24. Let $f(z) = z^n, 0 \leq \arg z < 2\pi/n, n \geq 2$. Find $f'(z), z \in \mathbb{C}$ and $(f^{-1})(z), z \in D \setminus \{0\}$.

25. Let $f(z) = z^3, 2\pi/3 \leq \arg z < 4\pi/3$. Then $f^{-1} : \mathbb{C} \to D$ exists. Find $(f^{-1})'(\mathrm{i})$ and $(f^{-1})'(-1)$.

26. Let us assume that g is analytic in all of $\mathbb{C}$. Define the function $f : \mathbb{C} \to \mathbb{C}$ by setting
a) $f(z) = g(\overline{z}), z \in \mathbb{C}$, b) $f(z) = \overline{g(\overline{z})}, z \in \mathbb{C}$.
Investigate if f is analytic on $\mathbb{C}$.

27. Let $f(z) = f(x + \mathrm{i}y) = x^3 - 3xy^2 + \mathrm{i}(3x^2y - y^3), z = x + \mathrm{i}y \in \mathbb{C}$. Show that f satisfies the Cauchy–Riemann conditions. Find $f'(z)$.

28. Solve
$$\mathrm{e}^z = 2 + \mathrm{i}.$$

29. Show that the function $f(z) = \frac{1}{z+\mathrm{i}}, z \in \mathbb{C} \setminus \{-i\}$ satisfies the Cauchy–Riemann conditions.

30. Show that the function
$$f(z) = \sin z$$
satisfies the Cauchy–Riemann conditions.

31. Prove that
a) $\mathrm{e}^{\overline{z}} = \overline{\mathrm{e}^z}$, b) $\sin \overline{z} = \overline{\sin z}$, c) $|\mathrm{e}^z| = \mathrm{e}^x$, d) $|\cos z|^2 + |\sin z|^2 = 1 + 2\sinh^2 y$ whenever $z \in \mathbb{C}$.

32. Find
a) $\log(-4)$, b) $\log 3\mathrm{i}$, c) $\log(\sqrt{3} - \mathrm{i})$.

33. Find
a) $\mathrm{i}^{2\mathrm{i}}$, b) $(-\mathrm{i})^{\mathrm{i}}$, c) $\mathrm{i}^{-\mathrm{i}}$.

34. Express the function $f(z) = \operatorname{Log} z, z \neq 0$, in the form $f = u + \mathrm{i}v$. Determine if it satisfies the Cauchy–Riemann conditions.

35. Find the limits
a) $\lim\limits_{z\to 0} \dfrac{\mathrm{e}^{z^2} - 1}{z^2 + 2z}$, b) $\lim\limits_{z\to \frac{\pi}{2}} \dfrac{\cos z}{z - \frac{\pi}{2}}$, c) $\lim\limits_{z\to 0} \dfrac{\cos 2z - 1}{\sin^2 z}$, d) $\lim\limits_{z\to 0} \dfrac{\log^2(1+z)}{z^2}$.

36. Let f be analytic in a domain $A \subset \mathbb{C}$.

    a) Let us assume that $f'(z) = 0$ for all $z \in A$. Show that f is a constant function on A.

    b) Let us assume that $f = u + \mathrm{i}v$ and u is a constant function on A. Show that f is constant on A.

37. Find $\int_\gamma \overline{z}\mathrm{d}z$, where

    a) $\gamma : z(t) = t + \mathrm{i}t^2, t \in [0, 1]$, b) $\gamma : z(t) = t^2 + \mathrm{i}t^4, t \in [0, 1]$.

38. Find $\int_\gamma z^2\mathrm{d}z$, where γ is the line segment from i to $1 + 2\mathrm{i}$.

39. Evaluate the integral

$$\int_\gamma \frac{\mathrm{d}z}{(z - z_0)^n}, \quad n = 2, 3, \ldots,$$

    where γ is closed Jordan curve and a) z_0 is in the interior of γ b) z_0 is in the exterior of γ.

40. Prove that

$$\int_0^{2\pi} \mathrm{e}^{\cos t} \cos(t + \sin t)\mathrm{d}t = \int_0^{2\pi} \mathrm{e}^{\cos t} \sin(t + \sin t)\mathrm{d}t = 0.$$

41. Evaluate the integral $\int_\gamma \sin^2 z\mathrm{d}z$, where γ is the line segment from 0 to i.

42. Evaluate the integrals

    a) $\int_\gamma \frac{\sin z}{z - \mathrm{i}}\mathrm{d}z$, where $\gamma : z(t) = 2\mathrm{e}^{\mathrm{i}t}, t \in [0, 2\pi]$

    b) $\int_\gamma \frac{\sinh z}{z - \mathrm{i}\pi}\mathrm{d}z$, where $\gamma : z(t) = \mathrm{i}\pi + 2\mathrm{e}^{\mathrm{i}t}, t \in [0, 2\pi]$.

43. Evaluate

$$\int_\gamma \frac{\mathrm{e}^z}{z(z - 2\mathrm{i})}\mathrm{d}z,$$

    where a) $\gamma : z(t) = \mathrm{e}^{\mathrm{i}t}, t \in [0, 2\pi]$ b) $\gamma : z(t) = 3\mathrm{e}^{\mathrm{i}t}, t \in [0, 2\pi]$.

44. Evaluate

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{e}^{az}}{z^2 + 1}\mathrm{d}z,$$

    where $\gamma : z(t) = 3\mathrm{e}^{\mathrm{i}t}, t \in \mathbb{R}$ and $a > 0$.

45. Evaluate

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{e}^{az}}{(z^2 + 1)^2}\mathrm{d}z,$$

    where γ and a are as in Exercise 44.

46. Evaluate

a) $\displaystyle\int_\gamma \frac{\mathrm{e}^{\mathrm{i}z}}{z^3}\mathrm{d}z$, where $\gamma : z(t) = 2\mathrm{e}^{\mathrm{i}t}, t \in [0, 2\pi]$

b) $\displaystyle\int_\gamma \frac{\cos z}{(z - \pi/4)^3}\mathrm{d}z$, where $\gamma : z(t) = \mathrm{e}^{\mathrm{i}t}, t \in [0, 2\pi]$

47. Evaluate

$$\int_\gamma \frac{\mathrm{e}^{kz}}{z^{n+1}}\mathrm{d}z \quad \text{and} \quad \int_\gamma \frac{\sin z}{z^{n+1}}\mathrm{d}z,$$

where $\gamma : z(t) = \mathrm{e}^{\mathrm{i}t}, t \in [0, 2\pi]$ and $k \in \mathbb{N}$.

Part II

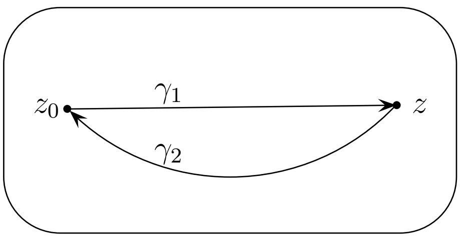
Chapter 1

Fundamental theorem of integration

The Cauchy theorem (as well as the Cauchy integral formula) allows us to prove the *fundamental theorem of integration*. Let f be analytic in the simply connected domain D. Then the integral

$$\int_\gamma f(\zeta)\mathrm{d}\zeta,$$

where γ is a piecewise smooth Jordan curve connecting two points $z_0, z \in D, \gamma \subset D$, is independent on this curve. The reason is: if we consider two different such curves γ_1 and γ_2 (both from z_0 to z) then the curve $\gamma := \gamma_1 \cup \gamma_2$ will be closed



and due to Cauchy theorem (Theorem 5.2) we have

$$0 = \int_\gamma f(\zeta)\mathrm{d}\zeta = \int_{\gamma_1} f(\zeta)\mathrm{d}\zeta - \int_{\gamma_2} f(\zeta)\mathrm{d}\zeta$$

or

$$\int_{\gamma_1} f(\zeta)\mathrm{d}\zeta = \int_{\gamma_2} f(\zeta)\mathrm{d}\zeta.$$

That's why the function

$$F(z) := \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta \tag{1.1}$$

is well-defined since its value is independent on the curve connecting z_0 and z. Even more is true. The function (1.1) is analytic in D and $F'(z) = f(z)$ everywhere in D. Indeed,

$$\begin{aligned}\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) &= \frac{1}{\Delta z}\left(\int_{z_0}^{z+\Delta z} f(\zeta)\mathrm{d}\zeta - \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta\right) - f(z)\\ &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)\mathrm{d}\zeta - f(z)\\ &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} (f(\zeta)-f(z))\mathrm{d}\zeta.\end{aligned}$$

Using the line segment from z to $z+\Delta z$ we obtain

$$\begin{aligned}\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| &\le \frac{1}{|\Delta z|}\left|\int_{z}^{z+\Delta z} |f(\zeta)-f(z)||\mathrm{d}\zeta|\right|\\ &\le \sup_{\zeta\in[z,z+\Delta z]} |f(\zeta)-f(z)| \to 0\end{aligned}$$

as $\Delta z \to 0$. Hence $F(z)$ is analytic in D and $F'(z) = f(z)$ everywhere in D. This fact justifies the following definition.

Definition 1.1. The function $\Phi(z)$ is called the *primitive* for $f(z)$ in D if $\Phi(z) \in H(D)$ and $\Phi'(z) = f(z)$.

So, if f is analytic in D then

$$F(z) = \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta$$

is a primitive for f in D.

Problem 1.2. Show that if Φ_1 and Φ_2 are primitives for f in simply connected D then $\Phi_1(z) - \Phi_2(z) \equiv$ constant in D.

As a consequence we have the fundamental fact: if D is simply connected then

$$\int_{z_1}^{z_2} f(\zeta)\mathrm{d}\zeta = F(z_2) - F(z_1),$$

where F is any primitive for f. This fact is called the *fundamental theorem of complex integration* analogously to the Newton's formula for real integration.

Example 1.3. Let $D \subset \mathbb{C}$ be a simply connected domain such that $0 \notin D$ and $1 \in D$. Then $f(z) = \frac{1}{z}$ is analytic in D and

$$F(z) = \int_{1}^{z} \frac{1}{\zeta}\mathrm{d}\zeta$$

is a primitive for f in D, where D is such that any curve connecting 1 and $z \in D$ does not pass across 0. For example, D can be chosen as

$$D = \mathbb{C} \setminus \{\operatorname{Im} z = 0, \operatorname{Re} z \le 0\}.$$

In this domain we can take the line segment connecting 1 and z. This function $F(z)$ is said to be logarithmic function (or logarithm) i.e.

$$\log z = \int_1^z \frac{1}{\zeta} \mathrm{d}\zeta. \tag{1.2}$$

Problem 1.4. Show that

$$\log z = \log |z| + \mathrm{i} \arg z, \quad z \in D, \tag{1.3}$$

where D is as above i.e. $-\pi < \arg z < \pi$.

Problem 1.5. Let

$$f(z) = \frac{1}{1+z^2}$$

and let D be simply connected such that $\pm\mathrm{i} \notin D$. Show that

$$\arctan z := \int_0^z \frac{\mathrm{d}\zeta}{1+\zeta^2}$$

and D is chosen such that any curve connecting 0 and z does not pass across $\pm\mathrm{i}$ satisfies

$$\arctan z = \frac{1}{2\mathrm{i}} \log \frac{1+\mathrm{i}z}{1-\mathrm{i}z}.$$

The converse statement to Cauchy theorem is also true.

Theorem 1.6 (Morera's theorem)**.** *Let f be a continuous function in a simply connected domain $D \subset \mathbb{C}$. If $\int_\gamma f(\zeta)\mathrm{d}\zeta = 0$ for every piecewise smooth closed Jordan curve in D, then f is analytic in D.*

Proof. We select a point $z_0 \in D$ and define $F(z)$ by

$$F(z) := \int_{z_0}^z f(\zeta)\mathrm{d}\zeta$$

which is well-defined and univalent in D since the result of integration is independent on curve connecting z_0 and z in D. Since f is continuous in D we have (choosing line segment to connect z and $z + \Delta z$)

$$\begin{aligned} \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| &\le \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} |f(\zeta) - f(z)||\mathrm{d}\zeta| \right| \\ &\le \max_{\zeta \in [z, z+\Delta z]} |f(\zeta) - f(z)| \to 0 \end{aligned}$$

as $\Delta z \to 0$. Thus $F'(z) = f(z)$ i.e. F is analytic. But since any analytic function is infinitely many times differentiable then so is f. $\square$

Chapter 2

Harmonic functions and mean value formulae

Let $u(x, y)$ be a real-valued function of two real variables x and y defined on a domain D.

Definition 2.1. If function $u(x, y)$ is twice continuously differentiable in D and satisfies the *Laplace equation*

$$\Delta u = \partial_x^2 u(x, y) + \partial_y^2 u(x, y) = 0$$

in D, then u is said to be *harmonic* in D.

There is a close connection between harmonic and analytic functions. Indeed, if $f = u + \mathrm{i}v$ is analytic in D then (as we proved) f is infinitely many times differentiable in D. So are the functions u and v and the Cauchy-Riemann conditions are satisfied. Then we have $\partial_x u = \partial_y v, \partial_y u = -\partial_x v$. It follows that

$$\partial_x^2 u = \partial_{xy}^2 v, \quad \partial_y^2 u = -\partial_{xy}^2 v$$

and hence

$$\partial_x^2 u + \partial_y^2 u = \partial_{xy}^2 v - \partial_{xy}^2 v = 0$$

i.e. u is harmonic. Similarly

$$\partial_x^2 v + \partial_y^2 v = -\partial_{xy}^2 u + \partial_{xy}^2 u = 0.$$

Thus, if $f \in H(D)$ then $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic in D.

It turns out that the converse is also true. Namely, any harmonic function is the real (or imaginary) part of some analytic function and this connection is unique up to an arbitrary constant. Let u be harmonic in a simply connected domain D. Then we may consider the differential form

$$l := -\partial_y u \mathrm{d}x + \partial_x u \mathrm{d}y.$$

This form is complete differential of some function v since $\partial_y(-\partial_y u) = \partial_x(\partial_x u)$ or $\Delta u = 0$ that is

$$\mathrm{d}v = -\partial_y u \mathrm{d}x + \partial_x u \mathrm{d}y. \tag{2.1}$$

This fact allows us to introduce function $v(x, y)$ as

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\partial_y u \mathrm{d}x + \partial_x u \mathrm{d}y + \text{constant} \tag{2.2}$$

and this definition is correct since the latter integral does not depend on the curve in D connecting the points (x_0, y_0) and (x, y). Even more is true, due to (2.1) and (2.2) we have

$$\partial_x v = -\partial_y u \quad \partial_y v = \partial_x u$$

i.e. the Cauchy-Riemann conditions are satisfied for function $f = u + \mathrm{i}v$. But u and v are twice continuously differentiable in D with Cauchy-Riemann conditions fullfilled. Thus, $f \in H(D)$ and $u = \operatorname{Re} f$. Similarly we may construct uniquely (up to an arbitrary constant) analytic function f such that given harmonic function u is equal to $\operatorname{Im} f$.

Simultaneously we obtained the following important result. By Corollary 5.12 of Part I we know that any analytic function is infinitely many times differentiable. Since any harmonic function is the real (or imaginary) part of some analytic function then any harmonic function is infinitely many times differentiable.

Problem 2.2. Let $f \in H(D)$ and $f \neq 0$ everywhere in D. Prove that $\log |f(z)|$ is harmonic in D.

Let f be analytic in D containing the disk $\{z : |z - z_0| \leq R\}$. Then the Cauchy integral formula yields

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = R} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z_0}.$$

If we parametrize the circle by $\zeta(t) = z_0 + R\mathrm{e}^{\mathrm{i}t}, t \in [-\pi, \pi]$ then $\mathrm{d}\zeta = R\mathrm{i}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t$ and the latter integral transforms to

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \int_{-\pi}^{\pi} \frac{f(z_0 + R\mathrm{e}^{\mathrm{i}t})R\mathrm{i}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t}{R\mathrm{e}^{\mathrm{i}t}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + R\mathrm{e}^{\mathrm{i}t})\mathrm{d}t. \tag{2.3}$$

This formula is called the *mean-value formula* for analytic functions. Since any harmonic function is the real (or imaginary) part of some analytic function then we obtain the mean-value formula also for harmonic function u as

$$\begin{aligned} u(x_0, y_0) &= \operatorname{Re} f(z_0) = \operatorname{Re} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + R\mathrm{e}^{\mathrm{i}t})\mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} f(z_0 + R\mathrm{e}^{\mathrm{i}t})\mathrm{d}t = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x_0 + R\cos t, y_0 + R\sin t)\mathrm{d}t. \end{aligned} \tag{2.4}$$

Remark. Due to periodicity we may replace the integration from $-\pi$ to π by integration from 0 to 2π.

We now prove an important result concerning the modulus of an analytic function.

Theorem 2.3 (Maximum modulus principle)**.** *Let f be analytic and non-constant in a domain D (not necessarily bounded). If $M := \sup_D |f(z)|$ then for any $z \in D$ we have $|f(z)| < M$ i.e. $|f(z)|$ does not attain its supremum at any point $z_0 \in D$.*

Proof. The value M cannot be equal to zero since in this case $f \equiv 0$. It contradicts with the conditions of this theorem. If $M = \infty$ then due to analyticity of f in D we have $|f(z)| < \infty$ for every $z \in D$ i.e. $|f(z)| < M$. That's why we assume now that $0 < M < \infty$.

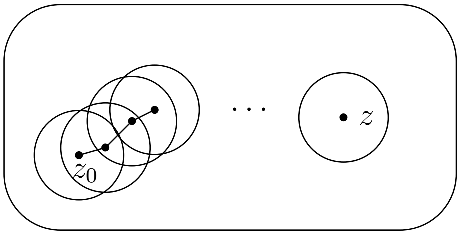
We will assume on the contrary that there is $z_0 \in D$ such that $M = |f(z_0)|$. The mean-value formula (2.3) leads to

$$M = |f(z_0)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(z_0 + R\mathrm{e}^{\mathrm{i}t})\mathrm{d}t\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z_0 + R\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t$$

for any $0 \le R \le R_0$ such that $\{z : |z - z_0| \le R\} \subset D$. Using this we will prove that $|f(z)| = M$ for all $z \in \{z : |z - z_0| \le R\}, 0 \le R \le R_0$. Assume again on the contrary that there is $R > 0$ with $0 \le R \le R_0$ and $t_0 \in [-\pi, \pi]$ such that $|f(z_0 + R\mathrm{e}^{\mathrm{i}t_0})| < M$. Since $|f(z)|$ is continuous there is $\delta > 0$ such that $|f(z_0 + R\mathrm{e}^{\mathrm{i}t})| < M$ for any $t \in (t_0 - \delta, t_0 + \delta)$. If $t_0 = \pm\pi$ then we will consider only either $(t_0, t_0 + \delta)$ or $(t_0 - \delta, t_0)$. These assumptions lead to the following inequalities

$$\begin{aligned}
M &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z_0 + R\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{t_0-\delta} |f(z_0 + R\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t \\
&+ \frac{1}{2\pi}\int_{t_0-\delta}^{t_0+\delta} |f(z_0 + R\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t + \frac{1}{2\pi}\int_{t_0+\delta}^{\pi} |f(z_0 + R\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t \\
&< \frac{1}{2\pi}\left[M(t_0 - \delta + \pi) + 2M\delta + M(\pi - t_0 - \delta)\right] = M.
\end{aligned}$$

This contradiction shows that $|f(z)| \equiv M$ in every disk $\{z : |z - z_0| \le R\}, 0 \le R \le R_0$. Let us show that this equality $|f(z)| = M$ holds in any point $\zeta \in D$. In order to prove it we join z_0 and z by a piecewise smooth Jordan curve $\gamma \subset D$ and denote by $d > 0$ the minimum distance from γ to ∂D. Next, we find consecutive points $z_0, z_1, \dots, z_n = \zeta$ along γ with $|z_{k+1} - z_k| \le d/2$ such that the disks $D_k = \{z : |z - z_k| \le d/2\}, k = 0, 1, \dots, n-1$ are contained in D and cover γ. Each disk D_k contains the center z_{k+1} of the next disk D_{k+1}.



That's why it follows $|f(z)| = M$ for all $z \in D_1$, and inductively, $|f(z)| = M$ for all $z \in D_n$ i.e. $|f(\zeta)| = M$ too. Thus $|f(z)| \equiv M$ everywhere in D.

The last step is to show that $f(z) \equiv$ constant. Indeed, since $u^2 + v^2 \equiv M^2$ then

$$\begin{cases} uu_x + vv_x = 0 \\ uu_y + vv_y = 0. \end{cases}$$

By the Cauchy-Riemann conditions we get

$$\begin{cases} uu_x - vu_y = 0 \\ uu_y + vu_x = 0. \end{cases}$$

Hence $u_x M^2 \equiv 0$ and $u_y M^2 \equiv 0$. Since $0 < M < \infty$ it follows that $u_x \equiv u_y \equiv 0$ in D. These two facts imply immediately that $u \equiv$ constant. Similarly we may obtain that $v \equiv$ constant i.e. $f \equiv$ constant. This contradiction proves the theorem completely. $\square$

Corollary 2.4. *Let D be a bounded domain and let f be analytic in D and continuous in $\overline{D}$. Then either $f \equiv$ constant or $\max_{\overline{D}} |f(z)|$ achieves at the boundary ∂D.*

Proof. Since $f \in C(\overline{D})$ and $\overline{D}$ is compact set in $\mathbb{C}$ then $|f(z)|$ is continuous there too and by Weierstrass theorems there is $\max_{z\in\overline{D}} |f(z)|$ which is achieved at some point $z_0 \in \overline{D}$ i.e.

$$\max_{z\in\overline{D}} |f(z)| = |f(z_0)|.$$

If $f \not\equiv$ constant then Theorem 2.3 implies that for every $z \in D$ we have

$$\max_{z\in\overline{D}} |f(z)| > |f(z)|.$$

Thus $z_0 \in \partial D$ i.e. $|f|$ achieves its maximum at the boundary. $\square$

Corollary 2.5. *Let f_1 and f_2 be analytic in D and continuous in $\overline{D}$, where D is bounded. If $f_1(z) = f_2(z)$ for all $z \in \partial D$ then $f_1(z) = f_2(z)$ everywhere in $\overline{D}$.*

Proof. Let us consider $f(z) := f_1(z) - f_2(z)$. Then Corollary 2.4 implies that $\max_{z\in\overline{D}} |f(z)|$ is achieved at the boundary or $f \equiv$ constant. But $f(z) = 0$ at the boundary. That's why in both cases $f(z) \equiv 0$. □

Corollary 2.6. *Let f be analytic in D. Let us assume in addition that $f(z) \neq 0$ everywhere in D. Then either $f \equiv \text{constant}$ in $\overline{D}$ or $\inf_D |f(z)| < |f(z)|$ for all $z \in D$.*

Proof. Since $f(z) \neq 0$ and analytic in D then $g(z) := 1/f(z)$ is well-defined and analytic in D. Theorem 2.3 implies that either $g \equiv$ constant (so is f) or for every $z \in D$ it follows that

$$|g(z)| < \sup_D |g(z)| = \frac{1}{\inf_D |f(z)|}.$$

This means that $\inf_D |f(z)| < |f(z)|$. □

Since the mean value formula holds also for harmonic functions (see (2.4)) we obtain *maximum principle* for harmonic functions.

Theorem 2.7. *Let $u(x, y)$ be real-valued, harmonic and non-constant in the domain D (not necessarily bounded). If $M = \sup_D u(x, y)$ and $m = \inf_D u(x, y)$ then*

$$m < u(x, y) < M \tag{2.5}$$

for any $(x, y) \in D$.

Proof. The proof literally repeats the proof of Theorem 2.3. □

Remark. In (2.5) it might be that $m = -\infty$ or $M = \infty$.

Corollary 2.8. *Let $u(x, y)$ be real-valued and harmonic in D and continuous in $\overline{D}$, where D is a bounded domain. Then either $u \equiv \text{constant}$ or for any $(x, y) \in D$ we have*

$$\min_{\overline{D}} u(x, y) < u(x, y) < \max_{\overline{D}} u(x, y)$$

i.e. $\min u(x, y)$ *and* $\max u(x, y)$ *are achieved at the boundary ∂D.*

Problem 2.9. Let $f(z) = az + b$ and $D = \{z : |z| < 1\}$. Prove that

$$\max_{|z|\le 1} |f(z)| = |a| + |b|$$

and $\max_{|z|\le 1} |f(z)| = |f(\mathrm{e}^{\mathrm{i}\theta_0})|$ for some real θ_0. Show also that $\theta_0 = \arg b - \arg a$.

Problem 2.10. Let $f(z) = az + b$ with $|b| > |a|$ and $D = \{z : |z| < 1\}$. Prove that $\min_{|z|\le 1} |f(z)| = |b| - |a|$ and $\min_{|z|\le 1} |f(z)| = |f(\mathrm{e}^{\mathrm{i}\theta_0})|$ with $\theta_0 = \arg b - \arg a + \pi$.

Chapter 3

Liouville's theorem and the fundamental theorem of algebra

Theorem 3.1 (Cauchy's inequality)**.** *Let f be analytic in a bounded domain D (not necessarily simply connected) and continuous in $\overline{D}$. Then for any $z_0 \in D$ and for any $n = 0, 1, \ldots$ we have*

$$|f^{(n)}(z_0)| \le \frac{n!M}{R^n}, \tag{3.1}$$

where $M = \max_{\overline{D}} |f(z)|$ and $R = \operatorname{dist}(z_0, \partial D)$.

Proof. Let $z_0 \in D$ be arbitrary and let $R > 0$ be chosen such that we have $\{z : |z - z_0| < R\} \subset D$. Then using Cauchy integral formula we obtain

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta$$

for $n = 0, 1, \ldots$. If we parametrize $|\zeta - z_0| = R$ as $\zeta = z_0 + R\mathrm{e}^{\mathrm{i}t}, t \in [-\pi, \pi]$ then

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{f(z_0 + R\mathrm{e}^{\mathrm{i}t})\mathrm{i}R\mathrm{e}^{\mathrm{i}t}}{R^{n+1}\mathrm{e}^{\mathrm{i}(n+1)t}} \mathrm{d}t = \frac{n!}{2\pi R^n} \int_{-\pi}^{\pi} \frac{f(z_0 + R\mathrm{e}^{\mathrm{i}t})}{\mathrm{e}^{\mathrm{i}nt}} \mathrm{d}t.$$

This representation implies the inequality

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi R^n} \int_{-\pi}^{\pi} \frac{|f(z_0 + R\mathrm{e}^{\mathrm{i}t})|}{|\mathrm{e}^{\mathrm{i}nt}|} \mathrm{d}t \le \frac{n!M}{2\pi R^n} 2\pi = \frac{n!}{MR^n}$$

and concludes the proof. □

Theorem 3.2 (Liouville's theorem)**.** *Let f be analytic in the whole complex plane $\mathbb{C}$ and let $\alpha \geq 0$ be such that*

$$|f(z)| \leq M|z|^{\alpha}, \quad z \in \mathbb{C}$$

with some positive constant M. Then f is a polynomial of order at most $n := [\alpha]$.

Proof. Let $n = [\alpha]$, where $[\alpha]$ denotes the entire part of α. Since f is analytic in $\mathbb{C}$ then for every $R > 0$ and every $z \in \mathbb{C}$ we have by the Cauchy integral formula that

$$f^{(n+1)}(z) = \frac{(n+1)!}{2\pi \mathrm{i}} \int_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z)^{n+2}} \mathrm{d}\zeta.$$

That's why we have the following inequality

$$\begin{aligned} |f^{(n+1)}(z)| &\leq \frac{(n+1)!}{2\pi} \int_{-\pi}^{\pi} \frac{|f(z + R\mathrm{e}^{\mathrm{i}t})| R \mathrm{d}t}{R^{n+2}} \leq \frac{(n+1)!}{2\pi} \frac{M(|z| + R)^{\alpha}}{R^{n+1}} 2\pi \\ &= M(n+1)! R^{\alpha - (n+1)} (1 + |z|/R)^{\alpha} \to 0 \end{aligned}$$

as $R \to \infty$ since $n + 1 > \alpha$. It means that $f^{(n+1)}(z) \equiv 0$ in $\mathbb{C}$. Thus, f is a polynomial of order not bigger than n. □

Problem 3.3. Let f be analytic in $\mathbb{C}$ and $f^{(k)}(z) \equiv 0$ in $\mathbb{C}$ for some $k = 1, 2, \ldots$. Prove that f is a polynomial of order not bigger than $k - 1$.

Corollary 3.4. *Let f be analytic in $\mathbb{C}$ (*entire function*) and bounded in $\mathbb{C}$. Then $f \equiv$* constant.

Proof. Proof follows from the proof of Theorem 3.2 with $\alpha = 0$. □

Problem 3.5. Show that the function $f(z) = \cos z$ is not bounded.

Problem 3.6. Let f be an entire function with the property $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that $f \equiv$ constant.

Theorem 3.7 (The fundamental theorem of algebra)**.** *If P is a polynomial of order $n \geq 1$ then P has at least one zero.*

Proof. Let us assume on the contrary that this polynomial has no roots i.e. $P(z) \neq 0$ for all $z \in \mathbb{C}$. This implies that the function

$$f(z) := \frac{1}{P(z)}$$

is an entire function i.e. it is analytic in the whole space $\mathbb{C}$. Let us write

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0$$

and consider the equation

$$|f(z)| = \frac{1}{|P(z)|} = \frac{1}{|z|^n} \cdot \frac{1}{|a_n + a_{n-1}/z + \cdots + a_0/z^n|}. \tag{3.2}$$

For $k = 1, 2, \ldots, n$ we have $|a_{n-k}|/|z|^k \to 0$ as $|z| \to \infty$. Hence $a_n + a_{n-1}/z + \cdots + a_0/z^n \to a_n$ as $|z| \to \infty$. Thus (3.2) implies $|f(z)| \to 0$ as $|z| \to \infty$. In particular, there is $R > 0$ such that for all $|z| \geq R$ we have

$$|f(z)| \leq 1. \tag{3.3}$$

The next step is: since f is analytic everywhere in $\mathbb{C}$ then $f(z)$ is continuous for all $z \in \mathbb{C}$. In particular, it is continuous in the closed ball $\{z : |z| \leq R\}$ with R as in (3.3). By Weierstrass theorem for continuous functions, $|f(z)|$ is bounded in this closed ball i.e. there is a positive number $M > 0$ such that

$$|f(z)| \leq M, \quad |z| \leq R. \tag{3.4}$$

Combining (3.3) and (3.4) we obtain that

$$|f(z)| \leq \max(1, M)$$

for all $z \in \mathbb{C}$. By Liouville's theorem $f \equiv$ constant and so is P. This contradiction proves the theorem. $\square$

Corollary 3.8. *Let P be a polynomial of order $n \geq 1$. Then P can be represented as*

$$P(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n),$$

where $a_n \neq 0$ and $z_1, z_2, \ldots, z_n$ are the zeros of P counted according to multiplicity.

Problem 3.9. Prove Corollary 3.8.

Chapter 4

Representation of analytic functions via the power series

Recall that the sequence of functions $S_n(z)$, in particular the partial sums

$$\sum_{j=0}^{n} f_j(z)$$

of some series

$$\sum_{j=0}^{\infty} f_j(z),$$

converges to $f(z)$ uniformly on a set $D \subset \mathbb{C}$ if for every $\varepsilon > 0$ there exists an integer $N_0(\varepsilon) > 0$ such that for all $n \geq N_0$ and for all $z \in D$ we have

$$|S_n(z) - f(z)| < \varepsilon$$

(in particular $|\sum_{j=n+1}^{\infty} f_j(z)| < \varepsilon$).

A useful procedure called the Weierstrass M-test can help determine whether an infinite series is uniformly convergent.

Theorem 4.1 (Weierstrass' M-test). *Suppose that the series*

$$\sum_{j=0}^{\infty} f_j(z)$$

has the property that for each $j = 0, 1, \ldots$ it holds that $|f_j(z)| \leq M_j$ for all $z \in D$. If $\sum_{j=0}^{\infty} M_j$ converges then $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly on D.

Proof. Let

$$S_n(z) = \sum_{j=0}^{n} f_j(z)$$

be the nth partial sum of the series. If $n > m$ then for all $z \in D$ we have

$$|S_n(z) - S_m(z)| = \left| \sum_{j=m+1}^{n} f_j(z) \right| \le \sum_{j=m+1}^{n} M_j < \varepsilon$$

for all $n > m \ge N_0(\varepsilon)$. This means that for all $z \in D$ the sequence $\{S_n(z)\}$ is a Cauchy sequence. Therefore there is a function $f(z)$ such that

$$f(z) = \lim_{n\to\infty} S_n(z) = \sum_{j=0}^{\infty} f_j(z).$$

Moreover, this convergence is uniform on D. $\square$

Theorem 4.2. *Suppose that the power series*

$$\sum_{j=0}^{\infty} c_j (z - z_0)^j$$

has radius of convergence $\rho > 0$. Then, for each $r, 0 < r < \rho$ this series converges uniformly on the closed disk $\{z : |z - z_0| \le r\}$ and defines there a continuous function.

Proof. Given $0 < r < \rho$ choose $\zeta \in \{z : |z - z_0| < \rho\}$ such that $|\zeta - z_0| = r$. Due to the properties of the power series we have that

$$\sum_{j=0}^{\infty} c_j (z - z_0)^j$$

converges absolutely for any $z \in \{z : |z - z_0| < \rho\}$. It follows that

$$\sum_{j=0}^{\infty} |c_j(\zeta - z_0)^j| = \sum_{j=0}^{\infty} |c_j| r^j$$

converges. Moreover, for all $z \in \{z : |z - z_0| \le r\}$ we have

$$|c_j(z - z_0)^j| = |c_j||z - z_0|^j \le |c_j| r^j.$$

The conclusion now follows from the Weierstrass' M-test with $M_j = |c_j| r^j$. $\square$

Remark. The *radius of convergence* ρ of the power series can be calculated as

$$\frac{1}{\rho} = \overline{\lim_{j\to\infty}} \sqrt[j]{|c_j|}$$

or

$$\frac{1}{\rho} = \lim_{j\to\infty} \left| \frac{c_{j+1}}{c_j} \right|$$

if the limit exists.

Problem 4.3. Show that the geometric series

$$\sum_{j=0}^{\infty} z^j$$

converges uniformly on the closed disk $\{z : |z| \le r\}$ with any $0 < r < 1$.

Theorem 4.4. *Suppose that the power series*

$$\sum_{j=0}^{\infty} c_j (z - z_0)^j$$

has radius of convergence $\rho > 0$. *Then in the disk* $D_\rho = \{z : |z - z_0| < \rho\}$ *this series defines the function*

$$f(z) := \sum_{j=0}^{\infty} c_j (z - z_0)^j \tag{4.1}$$

which is analytic in D_ρ *and for each* $k = 1, 2, \ldots$ *it holds that*

$$f^{(k)}(z) = \sum_{j=k}^{\infty} c_j j(j-1)\cdots(j-k+1)(z - z_0)^{j-k}. \tag{4.2}$$

Proof. Let $0 < r < \rho$. Then due to Theorem 4.2 in the closed disk $\overline{D_r} = \{z : |z - z_0| \le r\}$ the series (4.1) converges uniformly (and absolutely) and defines a continuous function $f(z)$. That's why we may integrate this series term by term. If $\gamma \subset D_r$ is a piecewise smooth closed Jordan curve then

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=0}^{\infty} c_j \int_\gamma (z - z_0)^j \mathrm{d}z = 0$$

since $(z - z_0)^j$ is analytic for each $j = 0, 1, \ldots$. Applying now Morera's theorem we conclude that f is analytic in D_ρ. Formula (4.2) follows directly by induction and it is based on the properties of power series. □

Problem 4.5. Show that

$$\log(1 - z) = -\sum_{j=1}^{\infty} \frac{z^j}{j}$$

for all $z \in D_1 = \{z : |z| < 1\}$ or

$$\log \zeta = -\sum_{j=1}^{\infty} \frac{(1-\zeta)^j}{j}, \quad 0 < |\zeta| < 2,$$

where log is the main branch.

Problem 4.6. Let f and g have the power series representations

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j, \quad g(z) = \sum_{j=0}^{\infty} d_j (z - z_0)^j \tag{4.3}$$

for $z \in D_\rho = \{z : |z - z_0| < \rho\}$. Show that

$$f(z)g(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j, \quad z \in D_\rho,$$

where $a_j = \sum_{k=0}^{j} c_k d_{j-k}$.

Problem 4.7. Let f and g have the power series representations (4.3) with $d_0 \neq 0$. Show that in some neighborhood of z_0 the function $f(z)/g(z)$ can be represented as the power series

$$\frac{f(z)}{g(z)} = \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

where a_j are uniquely determined from the equations $c_j = \sum_{k=0}^{j} a_k d_{j-k}, j = 0, 1, \ldots$.

Theorem 4.8 (Taylor's expansion)**.** *Suppose that f is analytic in a domain D and that $D_R(z_0) = \{z : |z - z_0| < R\}$ is a disk contained in D. Then f is uniquely represented in $D_R(z_0)$ as a power series*

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j, \quad z \in D_R(z_0), \tag{4.4}$$

where

$$c_j = \frac{f^{(j)}(z_0)}{j!}.$$

Furthermore, for any $r, 0 < r < R$ the convergence is uniform on $\overline{D_r}(z_0) = \{z : |z - z_0| \le r\}$. The power series with such coefficients is called the Taylor series *for f centered at z_0.*

Proof. Let $z_0 \in D$ and let $R = \operatorname{dist}(z_0, \partial D)$ so that $D_R(z_0) = \{z : |z - z_0| < R\} \subset D$. Let $z \in D_r(z_0) = \{z : |z - z_0| < r\}$ with $0 < r < R$. The Cauchy integral formula gives that

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)} \mathrm{d}\zeta.$$

Since $z \in D_r(z_0)$ and $|\zeta - z_0| = r$ we have that

$$\left|\frac{z-z_0}{\zeta - z_0}\right| = \frac{|z-z_0|}{r} < 1.$$

Therefore we have (by geometric series)

$$\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} = \sum_{j=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^j.$$

Moreover, for such fixed z the convergence of this series is uniform on the circle $\{\zeta : |\zeta - z_0| = r\}$. Hence we may integrate this series term by term and obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}}\int_{|\zeta-z_0|=r}\frac{f(\zeta)\mathrm{d}\zeta}{(\zeta-z_0)\left(1-\frac{z-z_0}{\zeta-z_0}\right)} \\ &= \frac{1}{2\pi\mathrm{i}}\int_{|\zeta-z_0|=r}\frac{f(\zeta)}{\zeta-z_0}\sum_{j=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^j \mathrm{d}\zeta \\ &= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi\mathrm{i}}\int_{|\zeta-z_0|=r}\frac{f(\zeta)}{(\zeta-z_0)^{j+1}}\mathrm{d}\zeta\right)(z-z_0)^j = \sum_{j=0}^{\infty}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j. \end{aligned}$$

Since r with $0 < r < R$ and z with $|z - z_0| < r$ are arbitrary we may conclude that the representation (4.4) with the coefficient $c_j = f^{(j)}(z_0)/j!$ holds everywhere in $D_R(z_0)$. Even more is true: the radius of convergence of (4.4) is R and the convergence of (4.4) is uniform in $\overline{D_r}(z_0) = \{z : |z - z_0| \le r\}$ with any $r, 0 < r < R$. The latter fact follows from the properties of the power series. The uniqueness of representation (4.4) follows from the fact that necessarily $c_j = f^{(j)}(z_0)/j!$. □

Corollary 4.9 (Taylor's expansion at ∞). *Let $f(z)$ be analytic for $|z| > R$ (including $z = \infty$). Then $f(z)$ is uniquely represented in $\{z : |z| > R\}$ as the series*

$$f(z) = \sum_{j=0}^{\infty} c_j z^{-j}, \quad |z| > R,$$

where $c_j = \frac{g^{(j)}(0)}{j!}$ for $g(z) := f(1/z)$. Moreover, these coefficients are equal to

$$c_j = \frac{f^{(j)}(\infty)}{j!}, \tag{4.5}$$

where $f^{(j)}(\infty) := (f^{(j-1)})'(\infty)$.

Proof. Let us consider $g(z) := f(1/z)$. Then g is analytic in the domain $\{z : |z| < 1/R\}$. Thus, Taylor's expansion (4.4) at 0 gives

$$g(z) = \sum_{j=0}^{\infty} c_j z^j,$$

where

$$c_j = \frac{1}{2\pi \mathrm{i}} \int_{|z|=\delta} \frac{g(z)\mathrm{d}z}{z^{j+1}} = \frac{g^{(j)}(0)}{j!}.$$

Since $f(z) = g(1/z)$ we obtain for $|z| > R$ that

$$f(z) = \sum_{j=0}^{\infty} c_j z^{-j},$$

where

$$c_j = \frac{g^{(j)}(0)}{j!} = \frac{f^{(j)}(\infty)}{j!}$$

and

$$c_j = \frac{1}{2\pi \mathrm{i}} \int_{|z|=\delta} \frac{f(1/z)\mathrm{d}z}{z^{j+1}} = \frac{1}{2\pi \mathrm{i}} \int_{|z|=1/\delta} f(z) z^{j-1} \mathrm{d}z.$$

It can be mentioned here that the definition of the derivative at ∞ leads to the fact

$$f^{(j)}(\infty) = g^{(j)}(0) = j! c_j.$$

□

Problem 4.10. Using Corollary 4.9 show that

1. $f(\infty) = \lim_{z\to\infty} f(z) = c_0$
2. $f'(\infty) = \lim_{z\to\infty} z[f(z) - f(\infty)]$
3. $f''(\infty) = -\lim_{z\to\infty}[z^3 f'(z) + z f'(\infty)]$
4. $f'''(\infty) = \lim_{z\to\infty}[z^5 f''(z) + 2z^4 f'(z) - z f''(\infty)]$.

Problem 4.11. Show that

1.
$$\mathrm{e}^z = 1 + z + \cdots + \frac{z^n}{n!} + \cdots$$

2.
$$\sin z = z - \frac{z^3}{3!} + \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \cdots$$

3.
$$\cos z = 1 - \frac{z^2}{2!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$$

4.
$$\sinh z = z + \frac{z^3}{3!} + \cdots + \frac{z^{2n-1}}{(2n-1)!} + \cdots$$

5.
$$\cosh z = 1 + \frac{z^2}{2!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots$$

and all these Taylor series converge for any $|z| < \infty$.

Problem 4.12. Let f and g be analytic in a domain D and $f(z) = g(z)$ on the set $E \subset D$ which has a limiting point in D. Show that $f(z) = g(z)$ for all $z \in D$.

Definition 4.13. Let f be analytic in a domain D. If
$$f(z_0) = 0, f'(z_0) = 0, \ldots, f^{(k_0-1)}(z_0) = 0$$
but $f^{(k_0)}(z_0) \neq 0$ for some $z_0 \in D$ and $k_0 \geq 1$, then z_0 is called the *zero* of f of order k_0.

Problem 4.14. Let f be analytic in a domain D and let $f \not\equiv 0$. Show that all zeros of f in D are isolated i.e. for any bounded domain D_1 with $\overline{D_1} \subset D$ there are only at most finitely many zeros of f in D_1.

Problem 4.15. Let
$$f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0. \end{cases}$$
Show that f is analytic everywhere in $\mathbb{C}$ and find its Taylor expansion centered at 0.

Problem 4.16. Suppose that
$$f(z) = \sum_{j=0}^{\infty} c_j z^j$$
is an entire function. Show that $\overline{f(\overline{z})}$ is entire too. When $\overline{f(\overline{z})} = f(z)$?

Problem 4.17. Let
$$f(z) = \begin{cases} \mathrm{e}^{-1/z^2}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$
Show that f is not continuous at 0 and that it has no Taylor expansion at 0.

Problem 4.18. Let f be as in Problem 4.17. Define the Taylor expansion for f at any point $z_0 \neq 0$.

Chapter 5

Laurent expansions

If f is analytic in the disk $\{z : |z - z_0| < R\}$ then we have only the Taylor's representation for this function. But if f is analytic in the deleted neighborhood i.e. the punctured disk $\{z : 0 < |z - z_0| < R\}$ then what kind of representation we may have for this function?

Let us consider the series (formally for the moment)

$$\sum_{j=-\infty}^{\infty} c_j (z - z_0)^j = \sum_{j=-\infty}^{-1} c_j (z - z_0)^j + \sum_{j=0}^{\infty} c_j (z - z_0)^j =: s_2(z) + s_1(z). \quad (5.1)$$

The first term $s_2(z)$ is called the power series with negative degrees. The second series in (5.1) defines the analytic function $s_1(z)$ in the disk $\{z : |z - z_0| < R\}$, where

$$R^{-1} = \overline{\lim_{j\to\infty}} \sqrt[j]{|c_j|} = \sup_j \sqrt[j]{|c_j|}. \quad (5.2)$$

It makes sense to consider the first series for $|z - z_0| > 0$. Thus, if we change the variables as

$$\zeta = \frac{1}{z - z_0}, \quad z = z_0 + \frac{1}{\zeta}$$

we obtain for $s_2(z)$ the representation

$$s_2(z) = s_2(z_0 + 1/\zeta) = \sum_{j=1}^{\infty} c_{-j} \zeta^j =: s_2^*(\zeta), \quad (5.3)$$

where $\zeta = 0$ corresponds to $z = \infty$. So, we have the power series with respect to positive degrees of ζ with radius of convergence $1/r$ which satisfies (see (5.2))

$$r = \overline{\lim_{j\to\infty}} \sqrt[j]{|c_{-j}|}$$

such that $s_2^*(\zeta)$ is analytic function (this series converges) for any $|\zeta| < 1/r$. Equivalently, $s_2(z)$ is analytic in $\{z : |z - z_0| > r\}$.

If it turns out that $r < R$ then $s(z) = s_2(z) + s_1(z)$ is analytic in the annulus $\{z : r < |z - z_0| < R\}$ centered at z_0 with radii r and R. In this case the series (5.1) is said to be a *Laurent expansion* for $s(z)$ in the annulus. The opposite statement also holds.

Example 5.1. Let us find three different Laurent expansions involving powers of z for the function

$$f(z) = \frac{3}{2 + z - z^2}.$$

This function has singularities at $z = -1$ and $z = 2$ and is analytic in the disk $\{z : |z| < 1\}$, in the annulus $\{z : 1 < |z| < 2\}$ and in the region $\{z : |z| > 2\}$. We start by writing

$$f(z) = \frac{3}{(1+z)(2-z)} = \frac{1}{1+z} + \frac{1}{2-z} = \frac{1}{1+z} + \frac{1}{2} \cdot \frac{1}{1 - z/2}.$$

We have three cases:

1. for $|z| < 1$ we have

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{j=0}^{\infty} (-z)^j = \sum_{j=0}^{\infty} (-1)^j z^j$$

and

$$\frac{1}{2} \frac{1}{1 - z/2} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} z^j.$$

Hence we have the Taylor expansion

$$f(z) = \sum_{j=0}^{\infty} \left((-1)^j + \frac{1}{2^{j+1}} \right) z^j$$

2. for $1 < |z| < 2$ we have

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1 + 1/z} = \frac{1}{z} \sum_{j=0}^{\infty} (-1)^j \frac{1}{z^j} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{z^j}$$

and

$$\frac{1}{2} \frac{1}{1 - z/2} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j = \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} z^j.$$

So

$$f(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{z^j} + \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} z^j$$

This is a Laurent expansion in the annulus $1 < |z| < 2$.

3. for $|z| > 2$ we have
$$\frac{1}{1+z} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{z^j}$$
and
$$\frac{1}{2} \cdot \frac{1}{1-z/2} = -\frac{1}{z} \cdot \frac{1}{1-2/z} = -\frac{1}{z} \sum_{j=0}^{\infty} \left(\frac{2}{z}\right)^j = -\sum_{j=1}^{\infty} \frac{2^{j-1}}{z^j}.$$
Therefore
$$f(z) = \sum_{j=1}^{\infty} ((-1)^{j+1} - 2^{j-1}) z^{-j}.$$
This is a Laurent expansion at ∞ (or Taylor expansion).

Problem 5.2. Find the Laurent expansion for e^{-1/z^2} centered at $z_0 = 0$.

Theorem 5.3. *Suppose that f is analytic in the annulus $\{z : r < |z - z_0| < R\}$ with $0 \le r < R$. Then for every $z \in \{z : r < |z - z_0| < R\}$ we have*
$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j, \tag{5.4}$$
where the coefficients c_j are uniquely determined by
$$c_j = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z_0)^{j+1}}, \quad j = 0, \pm 1, \pm 2, \ldots \tag{5.5}$$
with a piecewise smooth closed Jordan curve $\gamma \subset \{z : r < |z - z_0| < R\}$ and $z_0 \in \operatorname{int} \gamma$. Moreover, the convergence in (5.4) *is uniform on any closed subannulus $\{z : r < r_1 \le |z - z_0| \le R_1 < R\}$.*

Proof. Let $z \in \{z : r < |z - z_0| < R\}$. Then we can find $r_1 > r$ and $R_1 < R$ such that $z \in \{z : r_1 < |z - z_0| < R_1\}$. Using the Cauchy integral formula for multiply connected domain we obtain
$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = R_1} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} - \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = r_1} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} \\ &= \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = R_1} \frac{f(\zeta)}{\zeta - z_0} \frac{\mathrm{d}\zeta}{1 - \frac{z - z_0}{\zeta - z_0}} + \frac{1}{2\pi\mathrm{i}} \int_{|\zeta - z_0| = r_1} \frac{f(\zeta)}{z - z_0} \frac{\mathrm{d}\zeta}{1 - \frac{\zeta - z_0}{z - z_0}}. \end{aligned}$$
Since
$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(\zeta - z_0)^j}, \quad |z - z_0| < R_1$$

and

$$\frac{1}{1-\frac{\zeta-z_0}{z-z_0}} = \sum_{j=0}^{\infty} \frac{(\zeta-z_0)^j}{(z-z_0)^j}, \quad |z-z_0| > r_1$$

we may integrate term by term in these series (since these series converge uniformly on the circles $|\zeta - z_0| = R_1$ and $|\zeta - z_0| = r_1$, respectively) and obtain

$$\begin{aligned} f(z) &= \sum_{j=0}^{\infty} (z-z_0)^j \frac{1}{2\pi \mathrm{i}} \int_{|\zeta-z_0|=R_1} \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta-z_0)^{j+1}} \\ &+ \sum_{j=0}^{\infty} \frac{1}{(z-z_0)^{j+1}} \frac{1}{2\pi \mathrm{i}} \int_{|\zeta-z_0|=r_1} \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta-z_0)^{-j}} \\ &= \sum_{j=-\infty}^{-1} (z-z_0)^j \frac{1}{2\pi \mathrm{i}} \int_{|\zeta-z_0|=r_1} \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta-z_0)^{j+1}} \\ &+ \sum_{j=0}^{\infty} (z-z_0)^j \frac{1}{2\pi \mathrm{i}} \int_{|\zeta-z_0|=R_1} \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta-z_0)^{j+1}} \\ &= \sum_{j=-\infty}^{\infty} (z-z_0)^j \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta-z_0)^{j+1}}, \end{aligned}$$

where the integrals are considered for an arbitrary piecewise smooth closed Jordan curve $\gamma \subset \{z : r < |z - z_0| < R\}$ and $z_0 \in \operatorname{int}\gamma$. We have used the fact that these integrals are independent on such curves due to Cauchy theorem for multiply connected domains.

Thus, we proved the Laurent expansions (5.4)-(5.5). It is evident that this representation is unique since we may obtain necessarily (5.5). Uniform convergence of (5.4) for $z \in \{z : r < r_1 \le |z - z_0| \le R_1 < R\}$ follows from the arbitrariness of r_1 and R_1 in the preceding considerations. □

Definition 5.4. The series (5.4) with the coefficients (5.5) is called the *Laurent expansion* (representation) of the analytic function f in the annulus $\{z : r < |z - z_0| < R\}$ and

$$f_1(z) = \sum_{j=0}^{\infty} c_j (z-z_0)^j, \quad f_2(z) = \sum_{j=1}^{\infty} c_{-j} (z-z_0)^{-j}$$

are called the *regular* and *main parts* of this expansion, respectively.

If f is analytic in the annulus $\{z : 0 < |z-z_0| < r\}$ with some $r > 0$ then z_0 is said to be an *isolated singular point* of f. Then Theorem 5.3 says that $f(z)$ in this annulus can be represented via the Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z-z_0)^j,$$

where c_j are calculated by (5.5).

Example 5.5. Let us find the Laurent expansion for

$$f(z) = \frac{\cos z - 1}{z^4}$$

that involves powers of z. Since

$$\cos z - 1 = -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

and this representation is valid for all $|z| < \infty$ then

$$f(z) = \frac{\cos z - 1}{z^4} = -\frac{1}{z^2 2!} + \frac{1}{4!} - \frac{z^2}{6!} + \cdots = -\frac{1}{2} \cdot \frac{1}{z^2} + \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+4)!}.$$

This is the Laurent expansion in the neighborhood of $z = 0$ and $z = \infty$ both.

Problem 5.6. Find the Laurent expansion for

$$f(z) = \frac{\sin 2z}{z^4}$$

that involves powers of z.

Problem 5.7. Find three Laurent expansion for

$$f(z) = \frac{1}{z^2 - 5z + 6}$$

centered at $z_0 = 0$.

Problem 5.8. Find two Laurent expansions for

$$\frac{1}{z(4-z)^2}$$

that involves powers of z.

Definition 5.9. If the number of nonzero coefficients (5.5) for $j < 0$ is

1. empty
2. finite
3. infinite

then z_0 is called a *removable point*, a *pole* and an *essentially singular point* for f, respectively.

Let z_0 be removable for f. Then its Laurent expansion has the form

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j,$$

where $z \in \{z : 0 < |z - z_0| < r\}$. But this series, as a power series, converges in the whole disk $\{z : |z - z_0| < r\}$ and it is equal to $f(z)$ for all $z \neq z_0$. If we define f at the point z_0 as

$$f(z_0) := c_0 = \lim_{z \to z_0} \sum_{j=0}^{\infty} c_j (z - z_0)^j$$

then we obtain a new function in the whole disk $\{z : |z - z_0| < r\}$ which is analytic there. In particular, f is bounded in the closed disk $\{z : |z - z_0| \leq r_1\}$ with $r_1 < r$. The opposite property is also true. The following theorem holds.

Theorem 5.10. *Let f be analytic in the annulus $\{z : 0 < |z - z_0| < r\}$ for some $r > 0$. Then z_0 is a removable singular point of f if and only if f is bounded in the deleted neighborhood of z_0.*

Proof. It remains to prove this theorem only in the opposite direction. Let us assume that f is bounded in some deleted neighborhood of z_0 i.e. there is $M > 0$ such that

$$|f(z)| \leq M, \quad 0 < |z - z_0| < \delta.$$

Due to Theorem 5.3 we have that for all $z \in \{z : 0 < |z - z_0| < r\}$ it holds that

$$f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j,$$

where

$$c_j = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = \delta} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \mathrm{d}\zeta, \quad 0 < \delta < r.$$

Thus

$$|c_j| \leq \frac{1}{2\pi} \max_{|\zeta - z_0| = \delta} |f(\zeta)| \delta^{-j} 2\pi \leq M \delta^{-j}, \quad j = 0, \pm 1, \pm 2, \ldots \tag{5.6}$$

But for $j < 0$ it follows that $c_j = 0$ because we may let $\delta \to 0$ in these estimates. Hence, z_0 is a removable singular point. $\square$

Remark. The estimate (5.6) has an independent interest.

If f is analytic in some domain $D \subset \mathbb{C}$ then $z_0 \in D$ is a root of order m of f if in some neighborhood $U_\delta(z_0) \subset D$ f admits the representation

$$f(z) = (z - z_0)^m \varphi(z), \tag{5.7}$$

where $\varphi(z)$ is analytic in $U_\delta(z_0)$ and $\varphi(z_0) \neq 0$. It is equivalent to

$$f(z_0) = 0, f'(z_0) = 0, \dots, f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \neq 0.$$

Another thing is: if $f \not\equiv 0$ is analytic in $D \subset \mathbb{C}$ and $f(z_0) = 0, z_0 \in D$ then the order of the root is always finite i.e. there is $m \in \mathbb{N}$ such that (5.7) holds. If we assume on the contrary that $f^{(k)}(z_0) = 0, k = 0, 1, \dots$ then by the Taylor expansion we have

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j \equiv 0$$

for all $z \in U_\delta(z_0)$. Further, using the procedure of continuation (see proof of Theorem 2.3) for any $z_1 \in D$ we may obtain $f(z_1) = 0$ i.e. $f(z) \equiv 0$ in D. This contradiction proves the fact.

Another consequence is: if $f(z_n) = 0$ and $z_n \to z_0, z_n \neq z_0$ with $z_0, z_n \in D$ then $f \equiv 0$ in D.

Let us assume that z_0 is a pole of f. Then f has the Laurent expansion of the form

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j + \sum_{j=1}^{m} c_{-j}(z - z_0)^{-j}, \tag{5.8}$$

where $c_{-m} \neq 0$. Then we say that z_0 is a *pole* of order m.

Theorem 5.11. *Let f be analytic in the annulus $\{z : 0 < |z - z_0| < r\}$. Then f has a pole of some order m at z_0 if and only if $\lim_{z\to z_0} |f(z)| = \infty$.*

Proof. Let us assume first that z_0 is a pole of order $m \in \mathbb{N}$. Then the following representation holds

$$f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j + \sum_{j=1}^{m} c_{-j}(z - z_0)^{-j},$$

where $c_{-m} \neq 0$. Then for the function

$$F(z) := (z - z_0)^m f(z)$$

we have

$$F(z) = c_{-m} + c_{-m+1}(z - z_0) + \cdots + c_{-1}(z - z_0)^{m-1} + c_0(z - z_0)^m + \cdots$$

i.e. $F(z)$ has a removable singularity at z_0. Moreover, there exists

$$\lim_{z\to z_0} F(z) = c_{-m} \neq 0.$$

This fact implies that there is $0 < \delta < r$ such that

$$|F(z)| > \frac{|c_{-m}|}{2} > 0$$

for all $0 < |z - z_0| < \delta$ and therefore

$$|f(z)| > \frac{|c_{-m}|}{2}|z - z_0|^{-m}$$

i.e.

$$\lim_{z\to z_0} |f(z)| = \infty. \tag{5.9}$$

Conversely, if (5.9) holds then

$$\lim_{z\to z_0} \frac{1}{|f(z)|} = \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

This fact can be interpreted as follows: a new function

$$g(z) := \frac{1}{f(z)}$$

is analytic in $\{z : 0 < |z - z_0| < \delta\}$ and the function g can be extended as an analytic function everywhere in $\{z : |z - z_0| < \delta\}$ and z_0 is a root of analytic function g. But the root of a analytic function (if it is not identically zero) is of finite order, say m. That's why

$$g(z) = (z - z_0)^m \varphi(z),$$

where $\varphi(z)$ is analytic and $\varphi(z_0) \neq 0$. Hence

$$f(z) = \frac{1}{(z - z_0)^m} \frac{1}{\varphi(z)},$$

where $1/\varphi(z)$ is analytic in the neighborhood of z_0 and $1/\varphi(z_0) \neq 0$. This condition allows us to represent $1/\varphi(z)$ via its Taylor expansion for $|z - z_0| < \delta$ as

$$\frac{1}{\varphi(z)} = \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

where $a_0 = 1/\varphi(z_0) \neq 0$. This implies that Laurent expansion for f is

$$f(z) = \frac{a_0}{(z - z_0)^m} + \cdots + \frac{a_{m-1}}{(z - z_0)} + \sum_{j=0}^{\infty} a_{m+j}(z - z_0)^j.$$

This means that z_0 is a pole of order m for f. □

Corollary 5.12. *z_0 is a pole of order m for function f which is analytic in the annulus $\{z : 0 < |z - z_0| < \delta\}$ if and only if z_0 is a root of $1/f$ of order m and this function is analytic in $\{z : |z - z_0| < \delta\}$.*

Theorem 5.13. *Let f be analytic in the annulus $\{z : 0 < |z - z_0| < r\}$. Then f has an essentially singular point at z_0 if and only if there is no $\lim_{z \to z_0} f(z)$ (finite or infinite).*

Proof. If we assume on the contrary that there is $\lim_{z \to z_0} f(z)$ finite or infinite then in the first case z_0 is a removable singularity and in the second case it is a pole of some order. This contradiction proves the theorem. □

As the consequence of this fact we can obtain that there exist two different sequences z_n' and z_n'' converging to z_0 such that $f(z_n')$ is bounded and $\lim_{n \to \infty} f(z_n'') = \infty$. Even more is true.

Theorem 5.14 (Casorati-Sokhotski-Weierstrass)**.** *Let $z_0 \neq \infty$ be an essential singularity of $f(z)$. Let E be the set of all values of $f(z)$ in the deleted neighbourhood of z_0. Then E is* dense *in $\mathbb{C}$ i.e. for any $\varepsilon > 0, \delta > 0$ and complex number w there exists a complex number z with $0 <\mid z - z_0 \mid< \delta$ such that $\mid f(z) - w \mid< \varepsilon$.*

Proof. Let α be an arbitrary point of $\mathbb{C}$. Let us assume that for any

$$z \in \{z \in \mathbb{C} : 0 <\mid z - z_0 \mid< \delta\}$$

we have $f(z) \neq \alpha$. Otherwise there is

$$z' \in \{z \in \mathbb{C} : 0 <\mid z - z_0 \mid< \delta\}$$

such that $f(z') = \alpha$ and everything is proved.

Then in the deleted neighbourhood of z_0 the function

$$g(z) := \frac{1}{f(z) - \alpha}$$

is well-defined. For this function $g(z)$ the point z_0 will be also essential singularity. Indeed, if z_0 is a pole or removable singularity for $g(z)$ then z_0 is a zero or removable singularity for $f(z) - \alpha$, respectively. In both cases we have a contradiction with essential singularity at z_0 for $f(z)$. Thus $g(z)$ cannot be bounded in this neighbourhood and therefore there is a sequence z_n converging to z_0 such that $\lim_{n \to \infty} g(z_n) = \infty$ or $\lim_{n \to \infty} f(z_n) = \alpha$. It means that $\alpha \in \overline{E}$ and the theorem is proved. □

Remark. If $z_0 = \infty$ is an essential singularity for $f(z)$ then (by definition) zero is the essential singularity for $\varphi(z) = f(1/z)$. Then, Theorem 5.14 holds for $\varphi(z)$ in the deleted neighbourhood of zero which is equivalent to the fact that Theorem 5.14 holds for $f(z)$ in the neighbourhood of ∞.

There is a substantial strengthening of Theorem 5.14 which only guarantees that the range of $f(z)$ is dense in $\mathbb{C}$. Namely, the following Great Picard's Theorem holds. We give it without proof.

Theorem 5.15 (Picard)**.** *If analytic function $f(z)$ has an essential singularity at z_0 then on any deleted neighbourhood of z_0 the function $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.*

Example 5.16.

1. The function $f(z) = \mathrm{e}^{1/z}$ has an essential singularity at $z = 0$ but still never attains the value 0.

2. The function
$$f(z) = \frac{1}{1 - \mathrm{e}^{1/z}}$$
has an essential singularity at $z = 0$ and attains the value ∞ infinitely often in any neighbourhood of 0 ($z_n = \mathrm{i}/2\pi n, n = \pm 1, \pm 2, \ldots$). However, it does not attain the values 0 or 1, since $\mathrm{e}^{1/z} \neq 0$.

Example 5.17.

1. Consider the function
$$f(z) = \frac{\sin z}{z}.$$
Since
$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$
for $|z| > 0$ then we can remove the singularity at $z = 0$ if we define $f(0) = 1$ since then f will be analytic at $z = 0$.

2. Consider the function
$$g(z) = \frac{\cos z - 1}{z^2}.$$
Since again for all $|z| > 0$ we have
$$\frac{\cos z - 1}{z^2} = \frac{1}{z^2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) = -\frac{1}{2} + \frac{z^2}{4!} - \frac{z^4}{6!} + \cdots$$
then defining $g(0) = -1/2$ we obtain function that is analytic for all z.

Example 5.18. Consider the function
$$f(z) = \frac{\sin z}{z^3}.$$
Since for all $|z| > 0$ we have
$$\frac{\sin z}{z^3} = \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots$$
then $c_{-2} = 1 \neq 0$. Therefore $f(z)$ has a pole of order 2 at 0.

Example 5.19. Consider the function

$$f(z) = z^2 \sin \frac{1}{z}.$$

Since for all $|z| > 0$ we have

$$z^2 \sin \frac{1}{z} = z^2 \left(\frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - \cdots \right) = z - \frac{1}{z 3!} + \frac{1}{z^3 5!} - \cdots$$

then the Laurent expansion has infinitely many negative powers of z. Hence $z = 0$ is essentially singular point for f.

Problem 5.20. Suppose that f has a removable singularity at z_0. Show that the function $1/f$ has either a removable singularity or a pole at z_0.

Problem 5.21.

1. Let f be analytic and have a zero of order k at z_0. Show that f' has a zero of order $k-1$ at z_0.
2. Let f be analytic and have a zero of order k at z_0. Show that f'/f has a simple pole (pole of order 1) at z_0.
3. Let f have a pole of order k at z_0. Show that f' has a pole of order $k+1$ at z_0.

Problem 5.22. Find the singularities of

$$f(z) = \frac{1}{\sin \frac{1}{z}}.$$

Let f be analytic in the region $\{z : |z| > R\}$. Then the function

$$\varphi(z) := f(1/z)$$

is analytic in the annulus $\{z : 0 < |z| < 1/R\}$. Hence $z = 0$ might be an isolated singular point for φ. The Laurent expansion for φ gives

$$\varphi(z) = \sum_{j=-\infty}^{\infty} c_j z^j, \quad 0 < |z| < \frac{1}{R}.$$

Thus we have the following expansion for f

$$f(z) = \varphi(1/z) = \sum_{j=-\infty}^{\infty} c_j z^{-j} = \sum_{j=-\infty}^{\infty} c_{-j} z^j, \quad |z| > R. \tag{5.10}$$

Definition 5.23. If $z = 0$ is a removable singularity, a pole or an essential singularity for $\varphi(z)$ then $z = \infty$ is called a removable singularity, a pole or an essential singularity for $f(z)$, respectively.

Remark. This definition implies that if the number of coefficients (5.10) for $j > 0$ is empty, finite or infinite then $z = \infty$ is a removable singularity, a pole or an essential singularity, respectively.

Example 5.24.

1. Let f be a polynomial of order n i.e.

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad a_0 \neq 0.$$

   Then $z = \infty$ is a pole of order n.

2. Let f be analytic in the whole space $\mathbb{C}$. If $z = \infty$ is a removable singularity then $f \equiv$ constant and if $z = \infty$ is a pole of order n then f is a polynomial of order n.

Problem 5.25. Consider the function

$$f(z) = z^2 \sin \frac{1}{z}.$$

Show that f has a pole of order 1 at $z = \infty$. Compare this result with second part of Example 5.24.

Chapter 6

Residues and their calculus

Recall that if a piecewise smooth closed Jordan curve $\gamma : z(t), a \le t \le b$ is parametrized so that $\operatorname{int}\gamma$ is kept on the left as $z(t)$ moves around γ then we say that γ is *oriented positively.* Otherwise, γ is said to be *oriented negatively*.

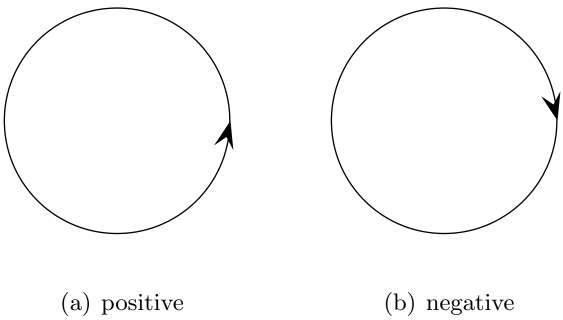


Figure 6.1: Orientation illustrated with circles

Let z_0 be an isolated singular point for a function f i.e. $f(z)$ is analytic in the annulus $\{z : 0 < |z - z_0| < r\}$ if $z_0 \neq \infty$ and in the region $\{z : |z| > R\}$ if $z_0 = \infty$.

Definition 6.1. The *residue* of f at the point z_0 is defined by

$$\operatorname*{Res}_{z=z_0} f := \frac{1}{2\pi \mathrm{i}} \int_\gamma f(\zeta) \mathrm{d}\zeta, \tag{6.1}$$

where $z_0 \in \operatorname{int}\gamma$, $\gamma \subset \{z : 0 < |z - z_0| < \delta\}$ and γ is positively oriented if $z_0 \neq \infty$ and $0 \in \operatorname{int}\gamma$, $\gamma \subset \{z : |z| > R\}$ and γ is negatively oriented if $z_0 = \infty$.

Remark. Due to Cauchy theorem for multiply connected domains the integral in (6.1) is independent on the corresponding curve and thus, the residue can be rewritten as

$$\begin{aligned}\operatorname*{Res}_{z=z_0\neq\infty} f &= \frac{1}{2\pi\mathrm{i}}\int_{|\zeta-z_0|=\delta} f(\zeta)\mathrm{d}\zeta \\ \operatorname*{Res}_{z=\infty} f &= -\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=R} f(\zeta)\mathrm{d}\zeta\end{aligned} \tag{6.2}$$

If $z_0 \neq \infty$ then the formulas (5.5) show us that

$$\operatorname*{Res}_{z=z_0} f = \frac{1}{2\pi\mathrm{i}}\int_{|\zeta-z_0|=\delta} f(\zeta)\mathrm{d}\zeta = c_{-1}, \tag{6.3}$$

where c_{-1} is the coefficient in front of $(z-z_0)^{-1}$ of the Laurent expansion for f. If $z_0 = \infty$ then

$$\operatorname*{Res}_{z=\infty} f = -\frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=R} f(\zeta)\mathrm{d}\zeta = -c_1, \tag{6.4}$$

where c_1 is the coefficient in front of z^{-1} of the Laurent expansion for f.

Example 6.2. Let us find $\operatorname{Res}_{z=0} g$ of

$$g(z) = \frac{3}{2z + z^2 - z^3}.$$

Since

$$\frac{3}{2z+z^2-z^3} = \frac{1}{z}\cdot\frac{3/2}{1+z/2-z^2/2}$$

and

$$g_1(z) = \frac{3/2}{1+z/2-z^2/2}$$

is analytic in the neighborhood of $z = 0$ and such that $g_1(0) = 3/2$ then the Laurent expansion for g has the form

$$g(z) = \frac{3/2}{z} + \sum_{j=0}^{\infty} c_j z^j.$$

Thus

$$\operatorname*{Res}_{z=0} g = 3/2.$$

Example 6.3. If $f(z) = \mathrm{e}^{2/z}$ then the Laurent expansion of f about the point 0 has the form

$$\mathrm{e}^{2/z} = 1 + \frac{2}{z} + \frac{2^2}{z^2 2!} + \cdots$$

and $\operatorname{Res}_{z=0} f = 2$. At the same time (by definition) $\operatorname{Res}_{z=\infty} f = -2$.

Theorem 6.4 (Residues at poles)**.** *If f has a pole of order k at $z_0 \neq \infty$ then*

$$\operatorname*{Res}_{z=z_0} f = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \Big((z-z_0)^k f(z) \Big). \tag{6.5}$$

Proof. Suppose that f has a pole of order k at $z_0 \neq \infty$. Then f can be written as

$$f(z) = \frac{c_{-k}}{(z-z_0)^k} + \frac{c_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{c_{-1}}{(z-z_0)} + \sum_{j=0}^{\infty} c_j (z-z_0)^j, \quad c_{-k} \neq 0.$$

Multiplying both sides by $(z-z_0)^k$ gives

$$(z-z_0)^k f(z) = c_{-k} + \cdots + c_{-1}(z-z_0)^{k-1} + \sum_{j=0}^{\infty} c_j (z-z_0)^{j+k}.$$

If we differentiate both sides $k-1$ times we get

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} ((z-z_0)^k f(z)) = (k-1)! c_{-1} + \sum_{j=0}^{\infty} c_j (j+k) \cdots (j+2)(z-z_0)^{j+1}.$$

Letting $z \to z_0$ the result is

$$\lim_{z \to z_0} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} ((z-z_0)^k f(z)) = (k-1)! c_{-1}.$$

By (6.3) this leads to (6.5). □

Corollary 6.5. *Let $f = \varphi/\psi$ be such that*

$$\varphi(z_0) \neq 0, \quad \psi(z_0) = 0, \quad \psi'(z_0) \neq 0.$$

Then f has a pole of order 1 at z_0 and

$$\operatorname*{Res}_{z=z_0} f = \frac{\varphi(z_0)}{\psi'(z_0)}. \tag{6.6}$$

Proof. The conditions for φ and ψ show that z_0 is a pole of order 1 for $f = \varphi/\psi$. Hence

$$\operatorname*{Res}_{z=z_0} f = \lim_{z \to z_0} \left((z-z_0) \frac{\varphi(z)}{\psi(z)} \right) = \lim_{z \to z_0} \frac{\varphi(z)}{\frac{\psi(z)-\psi(z_0)}{z-z_0}} = \frac{\varphi(z_0)}{\psi'(z_0)}$$

by Theorem 6.4. □

Corollary 6.6. *Let $f = \varphi/\psi$ be such that*

$$\varphi(z_0) \neq 0, \quad \psi(z_0) = \psi'(z_0) = 0, \quad \psi''(z_0) \neq 0.$$

Then

$$\operatorname*{Res}_{z=z_0} f = \frac{2\varphi'(z_0)}{\psi''(z_0)} - \frac{2\varphi(z_0)\psi'''(z_0)}{3(\psi''(z_0))^2}.$$

Corollary 6.7. *If f has a pole of order k at $z_0 = \infty$ then*

$$\operatorname*{Res}_{z=\infty} f = -\frac{1}{(k+1)!} \lim_{z\to 0} \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}} (z^k f(1/z)). \tag{6.7}$$

Problem 6.8. Prove Corollaries 6.6 and 6.7.

Problem 6.9. Find the residue of

$$f(z) = \frac{\pi \cot(\pi z)}{z^2}$$

at $z_0 = 0$.

Theorem 6.10 (Cauchy's residue theorem)**.** *Let $D \subset \mathbb{C}$ be a simply connected domain and let γ be a piecewise smooth closed Jordan curve which is positively oriented and lies in D. If f is analytic in D except the points $z_1, z_2, \ldots, z_n \in \operatorname{int}\gamma$ then*

$$\int_\gamma f(\zeta)\mathrm{d}\zeta = 2\pi\mathrm{i} \sum_{j=1}^n \operatorname*{Res}_{z=z_j} f. \tag{6.8}$$

Proof. Since there are finitely many singular points in $\operatorname{int}\gamma$ there exists $r > 0$ such that the positively oriented circles $\gamma_j := \{z : |z - z_j| = r\}, j = 1, 2, \ldots, n$ are mutually disjoint and all lie in $\operatorname{int}\gamma$. Applying the Cauchy theorem for multiply connected domain we obtain

$$\int_\gamma f(\zeta)\mathrm{d}\zeta + \sum_{j=1}^n \int_{-\gamma_j} f(\zeta)\mathrm{d}\zeta = 0$$

or

$$\int_\gamma f(\zeta)\mathrm{d}\zeta = \sum_{j=1}^n \int_{\gamma_j} f(\zeta)\mathrm{d}\zeta = 2\pi\mathrm{i} \sum_{j=1}^n \operatorname*{Res}_{z=z_j} f.$$

□

Corollary 6.11. *Let $D \subset \mathbb{C}$ be a multiply connected bounded domain with the boundary ∂D which is a combination of finitely many disjoint piecewise smooth closed Jordan curves. If f is analytic in D and continuous in $\overline{D}$ except the points $z_1, z_2, \ldots, z_n \in D$ then*

$$\int_{\partial D} f(\zeta)\mathrm{d}\zeta = 2\pi\mathrm{i} \sum_{j=1}^n \operatorname*{Res}_{z=z_j} f,$$

where the integration holds over positively oriented curves.

Corollary 6.12. *Let f be analytic in $\mathbb{C}$ except $z_1, z_2, \ldots, z_n, z_0 = \infty$. Then*

$$\sum_{j=0}^n \operatorname*{Res}_{z=z_j} f = 0. \tag{6.9}$$

Proof. Let $R > 0$ be chosen so that $z_1, z_2, \dots, z_n \in \{z : |z| < R\}$. Theorem 6.10 gives that

$$\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=R} f(\zeta) \mathrm{d}\zeta = \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} f,$$

where the circle $\{\zeta : |\zeta| = R\}$ is positively oriented. But

$$\frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=R} f(\zeta) \mathrm{d}\zeta = - \operatorname*{Res}_{z=\infty} f$$

by (6.2). □

Example 6.13. Let us find the isolated singular points and the residues at these points for

$$f(z) = \frac{\mathrm{e}^z}{1 - \cos z}.$$

Since $\mathrm{e}^z \neq 0$ for all $z \in \mathbb{C}$ then the singular points of f may appear only when $1 - \cos z = 0$ or $\mathrm{e}^{\mathrm{i}z} = 1$. So the singular points are

$$z_n = 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots.$$

At the same time we have

$$(1 - \cos z)'|_{z=z_n} = \sin z|_{z=z_n} = 0, \quad (1 - \cos z)''|_{z=z_n} = \cos z|_{z=z_n} = 1.$$

It means that all these points z_n are roots of order 2 of the denominator. Therefore all these points z_n are poles of order 2 for $f(z)$. From these considerations it follows also that $z = \infty$ is not an isolated singular point (it is not classified). By Theorem 6.4 we have that

$$\begin{aligned}
\operatorname*{Res}_{z=z_n} f &= \lim_{z \to z_n} \frac{\mathrm{d}}{\mathrm{d}z} \left((z - z_n)^2 \frac{e^z}{1 - \cos z} \right) = \lim_{\zeta \to 0} \frac{\mathrm{d}}{\mathrm{d}\zeta} \left(\zeta^2 \frac{\mathrm{e}^{\zeta + 2\pi n}}{1 - \cos \zeta} \right) \\
&= \lim_{\zeta \to 0} \frac{\mathrm{d}}{\mathrm{d}\zeta} \left(\zeta^2 \frac{\mathrm{e}^{\zeta + 2\pi n}}{\zeta^2/2! - \zeta^4/4! + \cdots} \right) = \mathrm{e}^{2\pi n} \lim_{\zeta \to 0} \frac{\mathrm{d}}{\mathrm{d}\zeta} \left(\frac{\mathrm{e}^{\zeta}}{1/2 - \zeta^2/4! - \cdots} \right) \\
&= \mathrm{e}^{2\pi n} \lim_{\zeta \to 0} \left(\frac{\mathrm{e}^{\zeta}}{1/2 - \zeta^2/4! + \cdots} - \frac{\mathrm{e}^{\zeta}(-2\zeta/4! + 4\zeta^3/6! - \cdots)}{1/2 - \zeta^2/4! + \zeta^4/6! - \cdots} \right) = 2\mathrm{e}^{2\pi n}
\end{aligned}$$

for $n = 0, \pm 1, \pm 2, \dots$. This can be proved also using Corollary 6.6.

Example 6.14. Let P be a polynomial of degree at most 2. Let us show that if a, b and c are distinct complex numbers then

$$f(z) = \frac{P(z)}{(z - a)(z - b)(z - c)} = \frac{A}{z - a} + \frac{B}{z - b} + \frac{C}{z - c},$$

where

$$A = \frac{P(a)}{(a-b)(a-c)} = \operatorname*{Res}_{z=a} f,$$

$$B = \frac{P(b)}{(b-a)(b-c)} = \operatorname*{Res}_{z=b} f$$

and

$$C = \frac{P(c)}{(c-a)(c-b)} = \operatorname*{Res}_{z=c} f.$$

Indeed, since

$$\frac{P(z)}{(z-a)(z-b)(z-c)} = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c}$$

then $z = a, z = b$ and $z = c$ are singular points of f (if, of course, a, b and c are not roots of P). That's why the terms

$$\frac{A}{z-a}, \quad \frac{B}{z-b}, \quad \frac{C}{z-c}$$

are the main parts of the Laurent expansion for f around a, b and c, respectively. Thus,

$$A = \operatorname*{Res}_{z=a} f, \quad B = \operatorname*{Res}_{z=b} f, \quad C = \operatorname*{Res}_{z=c} f$$

and

$$A = \lim_{z\to a} \frac{P(z)(z-a)}{(z-a)(z-b)(z-c)} = \frac{P(a)}{(a-b)(a-c)}$$

$$B = \lim_{z\to b} \frac{P(z)(z-b)}{(z-a)(z-b)(z-c)} = \frac{P(b)}{(b-a)(b-c)}$$

$$C = \lim_{z\to c} \frac{P(z)(z-c)}{(z-a)(z-b)(z-c)} = \frac{P(c)}{(c-a)(c-b)}.$$

Problem 6.15. Show that if P has degree of at most 3 then

$$f(z) = \frac{P(z)}{(z-a)^2(z-b)} = \frac{A}{(z-a)^2} + \frac{B}{z-a} + \frac{C}{z-b},$$

where

$$A = \operatorname*{Res}_{z=a}((z-a)f), \quad B = \operatorname*{Res}_{z=a} f, \quad C = \operatorname*{Res}_{z=b} f.$$

Problem 6.16. Let γ be a piecewise smooth closed Jordan curve and let f be analytic in $\operatorname{int}\gamma$. Let $z_0 \in \operatorname{int}\gamma$ be the only zero of f and of order k. Show that

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(\zeta)}{f(\zeta)} \mathrm{d}\zeta = k = \operatorname*{Res}_{z=z_0} \frac{f'}{f}.$$

Chapter 7

The principle of the argument and Rouche's theorem

Let G be a domain on the complex plane and D be a bounded subdomain of G such that $\overline{D} \subset G$. The domain D needs not be simply connected but the boundary ∂D of this domain is a combination of finitely many disjoint piecewise smooth closed Jordan curves. Let f be an analytic function on G. Consequently, f is analytic on the closed domain $\overline{D}$.

Proposition 7.1. *Let the domains D and G be as above and let f be analytic on G, except finite number of poles $z_k \in D$ of order μ_k for $k = 1, 2, \ldots, n$. Let us assume in addition that $f(z) \neq 0$ on $\overline{D}$ except finite number of zeros $w_k \in D$ of order $\lambda_k, k = 1, 2, \ldots, m$. Then the function*

$$\frac{f'(z)}{f(z)}$$

is analytic on $\overline{D}$ except the points $\{z_k\}_{k=1}^{n}$ and $\{w_k\}_{k=1}^{m}$ (which are poles of order 1 for f'/f) and

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f'(\zeta)\mathrm{d}\zeta}{f(\zeta)} = N - P, \tag{7.1}$$

where $N = \sum_{k=1}^{m} \lambda_k$ and $P = \sum_{k=1}^{n} \mu_k$.

Proof. Consider the function f'/f in the neighbourhood of the pole z_k. Then $f(z)$ can be represented there as

$$f(z) = (z - z_k)^{-\mu_k} f_1(z),$$

where $f_1(z)$ is analytic in this neighbourhood and $f_1(z_k) \neq 0$.This implies that

$$\frac{f'(z)}{f(z)} = \frac{-\mu_k (z - z_k)^{-\mu_k - 1} f_1(z) + (z - z_k)^{-\mu_k} f_1'(z)}{(z - z_k)^{-\mu_k} f_1(z)} = -\frac{\mu_k}{z - z_k} + \frac{f_1'(z)}{f_1(z)}, \tag{7.2}$$

where the second term $f_1'(z)/f_1(z)$ in the latter sum is analytic in this neighbourhood of z_k since $f_1(z_k) \neq 0$. The representation (7.2) shows that z_k is a pole of order 1 for f'/f and

$$\operatorname*{Res}_{z=z_k} \frac{f'(z)}{f(z)} = -\mu_k. \tag{7.3}$$

Consider now the function f'/f in the neighbourhood of a zero w_k. Then we have that

$$f(z) = (z - w_k)^{\lambda_k} f_2(z),$$

where $f_2(z)$ is analytic in this neighbourhood and $f_2(w_k) \neq 0$. Thus, we have

$$\frac{f'(z)}{f(z)} = \frac{\lambda_k (z - w_k)^{\lambda_k - 1} f_2(z) + (z - w_k)^{\lambda_k} f_2'(z)}{(z - w_k)^{\lambda_k} f_2(z)} = \frac{\lambda_k}{z - w_k} + \frac{f_2'(z)}{f_2(z)}, \tag{7.4}$$

where the second term in the latter sum is analytic in this neighbourhood since $f_2(w_k) \neq 0$. The representation (7.4) shows also that w_k is a pole of order 1 for f'/f and

$$\operatorname*{Res}_{z=w_k} \frac{f'(z)}{f(z)} = \lambda_k. \tag{7.5}$$

Since the function f'/f is analytic on $\overline{D}$ except the points $\{z_k\}_{k=1}^n, \{w_k\}_{k=1}^m$ (where it has the simple poles) then applying the Cauchy's residue theorem (see Theorem 6.10) we obtain (see (7.3) and (7.5))

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f'(\zeta) \mathrm{d}\zeta}{f(\zeta)} = \sum_{k=1}^n \operatorname*{Res}_{z=z_k} \frac{f'(z)}{f(z)} + \sum_{k=1}^m \operatorname*{Res}_{z=w_k} \frac{f'(z)}{f(z)} = -\sum_{k=1}^n \mu_k + \sum_{k=1}^m \lambda_k = N - P.$$

This finishes the proof. □

Corollary 7.2. *Suppose that $f(z)$ is analytic on $\overline{D}$ and $f(z) \neq 0$ on $\overline{D}$ except the zeros $w_k \in D$ of order $\lambda_k, k = 1, 2, \ldots, m$. Then*

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial D} \frac{f'(\zeta) \mathrm{d}\zeta}{f(\zeta)} = N. \tag{7.6}$$

Let γ be a piecewise smooth closed Jordan curve and let $f(z)$ be analytic on $\overline{\operatorname{int} \gamma}$

Definition 7.3. Let ζ_0 be a point of γ and $\varphi_0 = \operatorname{Arg} f(z)$ at ζ_0. Let also $\varphi_1 = \operatorname{Arg} f(z)$ at ζ_0 after going around once along this curve from ζ_0 to ζ_0 in positive direction. Then the value $\varphi_1 - \varphi_0$ is called the *variation* of $\operatorname{Arg} f(z)$ along curve γ and it is denoted by

$$\varphi_1 - \varphi_0 = \operatorname*{Var}_{\gamma} \operatorname{Arg} f.$$

Theorem 7.4 (The principle of argument). *Let f be analytic on $\overline{\operatorname{int}\gamma}$, where γ is a piecewise smooth closed Jordan curve, except the poles $\{z_k\}_{k=1}^n \subset \operatorname{int}\gamma$ of order μ_k. Assume that $f(z) \neq 0$ on $\overline{\operatorname{int}\gamma}$ except the zeros $\{w_k\}_{k=1}^m \subset \operatorname{int}\gamma$ of order λ_k. Then*

$$\frac{1}{2\pi} \operatorname*{Var}_{\gamma} \operatorname{Arg} f(z) = N - P, \tag{7.7}$$

where $N = \sum_{k=1}^m \lambda_k$ and $P = \sum_{k=1}^n \mu_k$.

Proof. Since $f(z) \neq 0$ on γ we may consider the multivalued function

$$\operatorname{Log} f(z) = \log|f(z)| + \mathrm{i} \operatorname{Arg} f(z).$$

Moreover, this function is analytic in the neighbourhood of γ and

$$(\operatorname{Log} f(z))' = \frac{f'(z)}{f(z)}.$$

Proposition 7.1 says that

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma (\operatorname{Log} f(\zeta))' \mathrm{d}\zeta = N - P.$$

It is equivalent to the changes of $\log f(\zeta)$ after going around once along γ from ζ_0 to ζ_0 i.e.

$$\begin{aligned} N - P &= \frac{1}{2\pi\mathrm{i}} [\operatorname{Log} f(\zeta)]_{\zeta=\zeta_0}^{\zeta=\zeta_0} = \frac{1}{2\pi\mathrm{i}} [\log|f(\zeta)| + \mathrm{i} \operatorname{Arg} f(\zeta)]_{\zeta=\zeta_0}^{\zeta=\zeta_0} \\ &= \left.\frac{\operatorname{Arg} f(\zeta)}{2\pi}\right|_{\zeta=\zeta_0}^{\zeta=\zeta_0} = \frac{\operatorname{Var}_\gamma \operatorname{Arg} f(\zeta)}{2\pi}. \end{aligned}$$

□

Theorem 7.5 (Rouche). *Let G be a simply connected domain, γ be a piecewise smooth closed Jordan curve in G and f and g be analytic functions on G except finitely many poles which are located in $\operatorname{int}\gamma$. If $|f(\zeta)| > |g(\zeta)|$ on γ then*

$$N_{f+g} - P_{f+g} = N_f - P_f, \tag{7.8}$$

where N_f, N_{f+g}, P_f and P_{f+g} denote the number of zeros or poles (taking into account their multiplicity) for functions f and $f+g$, respectively.

Proof. The conditions for f and g on γ show that $|f(\zeta)| > 0$ and $|f+g| \geq |f| - |g| > 0$ on γ i.e. f and $f+g$ are not equal to zero on γ. That's why we may apply Theorem 7.4 and obtain

$$\frac{1}{2\pi} \operatorname*{Var}_{\gamma} \operatorname{Arg}(f+g) - \frac{1}{2\pi} \operatorname*{Var}_{\gamma} \operatorname{Arg}(f) = (N_{f+g} - P_{f+g}) - (N_f - P_f).$$

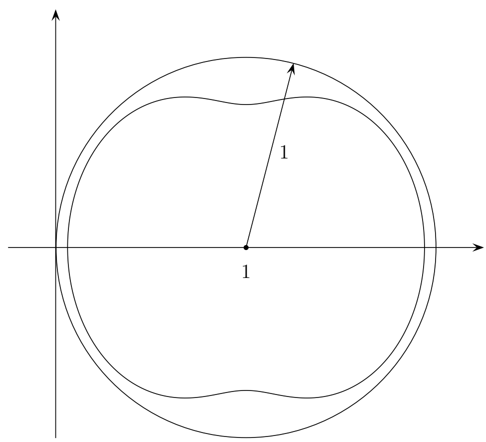
But the left hand side of the latter equality is equal to (see the proof of Theorem 7.4)

$$\frac{1}{2\pi}\operatorname*{Var}_{\gamma}\operatorname{Arg}\frac{f+g}{f}=\frac{1}{2\pi}\operatorname*{Var}_{\gamma}\operatorname{Arg}(1+g/f).$$

We will show now that this value is equal to zero. Indeed, since on γ we have

$$|g/f+1-1|=|g/f|<1$$

then the value $g/f+1$ on γ changes inside the circle $\{w:|w-1|<1\}$ such that $w=0$ does not belong to this set.



Since it does not go around zero along γ then $\operatorname{Var}_\gamma \operatorname{Arg}(1+g/f)=0$. Hence the equality (7.8) holds and Theorem is proved. □

Corollary 7.6. *Suppose that f and g are analytic. Then under the conditions of Theorem 7.5 we have that*

$$N_{f+g}=N_f. \tag{7.9}$$

Example 7.7. Let $P(z)=z^{10}-5z^7+2$. The fundamental theorem of algebra says that this polynomial has exactly 10 roots (taking into account their multiplicities). The question now is: how many of these roots are located in the unit disk $\{z:|z|<1\}$. Indeed, if we denote $g(z)=z^{10}+2$ and $f(z)=-5z^7$ then $P(z)=f(z)+g(z)$. The function f has 7 roots in this disc and for $|z|=1$ we have that

$$|g(z)|=|z^{10}+2|\le|z|^{10}+2=3<5=|f(z)|=5|z|^7.$$

By Rouche's theorem we obtain $N_{f+g}=N_f=7$.

Problem 7.8. Prove fundamental theorem of algebra using Corollary 7.6.

Problem 7.9. Show that the equation

$$a_0 + a_1 \cos\varphi + a_2 \cos 2\varphi + \cdots + a_n \cos n\varphi = 0,$$

where $0 \leq a_0 < a_1 < \cdots < a_n$ has $2n$ simple roots on the interval $(0, 2\pi)$.

Problem 7.10. Show that if $f(z)$ is analytic and univalent in the domain D then $f'(z) \neq 0$ for all $z \in D$.

Chapter 8

Calculation of integrals by residue theory

8.1 Trigonometric integrals

Suppose that we want to calculate an integral of the form

$$\int_0^{2\pi} R(\cos t, \sin t)\mathrm{d}t, \tag{8.1}$$

where $R(u, v)$ is a rational function of two variables u and v i.e.

$$R(u, v) = \frac{\sum_{k,l} a_{kl} u^k v^l}{\sum_{m,n} b_{mn} u^n v^m}$$

and the summation in both sums is finite. Due to periodicity (8.1) is equal to

$$\int_{-\pi}^{\pi} R(\cos t, \sin t)\mathrm{d}t. \tag{8.2}$$

Consider the unit circle $\{z : |z| = 1\}$ which is parametrized as (positive orientation) $\gamma : z(t) = \mathrm{e}^{\mathrm{i}t}, t \in [-\pi, \pi]$. Then

$$\cos t = \frac{\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}}{2} = \frac{z + 1/z}{2} = \frac{z^2 + 1}{2z},$$

$$\sin t = \frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}}{2i} = \frac{z - 1/z}{2i} = \frac{z^2 - 1}{2iz}$$

and

$$\mathrm{d}z = \mathrm{d}(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t}\mathrm{i}\mathrm{d}t$$

or

$$\mathrm{d}t = \frac{\mathrm{d}z}{\mathrm{i}\mathrm{e}^{\mathrm{i}t}} = \frac{\mathrm{d}z}{\mathrm{i}z}.$$

The integral (8.1) transforms to the curve integral

$$\int_{-\pi}^{\pi} R(\cos t, \sin t)\mathrm{d}t = \int_\gamma R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2\mathrm{i}z}\right)\frac{\mathrm{d}z}{\mathrm{i}z} = \int_\gamma \widetilde{R}(z)\mathrm{d}z, \tag{8.3}$$

where

$$\widetilde{R}(z) = \frac{1}{\mathrm{i}z}R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)$$

is a rational function of only one variable z. This rational function $\widetilde{R}$ may have only singular points which are poles (roots of the denominator of $\widetilde{R}$).

Let us consider the poles of $\widetilde{R}$ which are located inside the unit disk $\{z : |z| < 1\}$ and denote them as $z_1, z_2, \ldots, z_m$. The residue theorem gives

$$\int_{-\pi}^{\pi} R(\cos t, \sin t)\mathrm{d}t = \int_\gamma \widetilde{R}(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} \widetilde{R}. \tag{8.4}$$

Example 8.1. Let us evaluate the integral

$$\int_0^{2\pi} \frac{1}{3+2\sin t}\mathrm{d}t.$$

Due to (8.3) we have

$$\int_0^{2\pi} \frac{1}{3+2\sin t}\mathrm{d}t = \int_\gamma \frac{1}{\mathrm{i}z}\frac{1}{3+2\frac{z^2-1}{2\mathrm{i}z}}\mathrm{d}z = \int_\gamma \frac{\mathrm{d}z}{z^2+3\mathrm{i}z-1},$$

where γ is the unit circle. The roots of the denominator are

$$z_{1,2} = \frac{-3\mathrm{i} \mp \mathrm{i}\sqrt{5}}{2}.$$

It is easy to see that $|z_1| > 1$ and $|z_2| < 1$. By (8.4) we get

$$\int_0^{2\pi} \frac{1}{3+2\sin t}\mathrm{d}t = 2\pi\mathrm{i}\operatorname*{Res}_{z=z_2}\frac{1}{z^2+3\mathrm{i}z-1} = 2\pi\mathrm{i}\frac{1}{2z_2+3\mathrm{i}} = \frac{2\pi}{\sqrt{5}}$$

after using also (6.6).

Example 8.2. Let us evaluate the integral

$$I := \int_0^{2\pi} \frac{1}{1+3\cos^2 t}\mathrm{d}t.$$

Repeating the same procedure as above we obtain

$$I = \int_\gamma \frac{1}{\mathrm{i}z}\frac{1}{1+3\left(\frac{z^2+1}{2z}\right)^2}\mathrm{d}z = \frac{1}{\mathrm{i}}\int_\gamma \frac{4z\mathrm{d}z}{3z^4+10z^2+3}.$$

The roots of the denominator are

$$z_1 = \mathrm{i}\sqrt{3}, \quad z_2 = -\mathrm{i}\sqrt{3}, \quad z_3 = \mathrm{i}/\sqrt{3}, \quad z_4 = -\mathrm{i}/\sqrt{3}.$$

It is clear that $|z_1|, |z_2| > 1$ and $|z_3|, |z_4| < 1$. That's why

$$\begin{aligned} I &= 2\pi \left(\operatorname*{Res}_{z=z_3} \frac{4z}{3z^4 + 10z^2 + 3} + \operatorname*{Res}_{z=z_4} \frac{4z}{3z^4 + 10z^2 + 3} \right) \\ &= 2\pi \left(\frac{4z_3}{12z_3^3 + 20z_3} + \frac{4z_4}{12z_4^3 + 20z_4} \right) \\ &= 2\pi \left(\frac{\mathrm{i}/\sqrt{3}}{3(\mathrm{i}/\sqrt{3})^3 + 5\mathrm{i}/\sqrt{3}} - \frac{\mathrm{i}/\sqrt{3}}{3(-\mathrm{i}/\sqrt{3})^3 - 5\mathrm{i}/\sqrt{3}} \right) = \pi. \end{aligned}$$

Problem 8.3. Evaluate

$$\int_0^{2\pi} \frac{\cos(2t)}{5 - 4\cos t} \mathrm{d}t.$$

Problem 8.4. Evaluate

$$\int_0^{2\pi} \frac{\sin^2 t}{5 + 4\cos t} \mathrm{d}t.$$

8.2 Improper integrals of the form $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$

Let $f(x)$ be a continuous real-valued function of $x \in \mathbb{R}$. The *Cauchy principal value of the integral*

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x$$

is defined by

$$\mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} f(x)\mathrm{d}x$$

provided the limit exists. By this definition we obtain

$$\mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0$$

if f is odd and

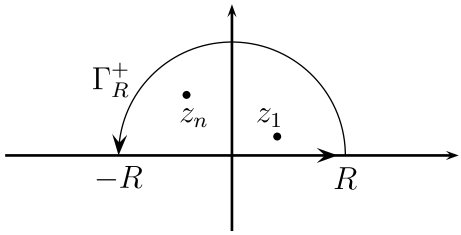
$$\mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\int_0^{\infty} f(x)\mathrm{d}x$$

if f is even.

Theorem 8.5. *Let f be analytic for $\operatorname{Im} z > 0$ and continuous for $\operatorname{Im} z \geq 0$ except for the singular points $z_1, z_2, \ldots, z_n$ with $\operatorname{Im} z_j > 0$ for all $j = 1, 2, \ldots, n$. If $f(z) = o(1/|z|)$ for $z \to \infty, \operatorname{Im} z > 0$ then*

$$\mathrm{p.\,v.} \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi\mathrm{i} \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} f. \tag{8.5}$$

Proof. Let $R > 0$ be chosen such that all points $z_1, z_2, \ldots, z_n$ belong to the region $\{z : |z| < R, \operatorname{Im} z > 0\}$. Let γ_R be the union of the line segment $[-R, R]$ and the upper semicircle Γ_R^+.



The residue theorem gives that

$$\int_{-R}^{R} f(x)\mathrm{d}x + \int_{\Gamma_R^+} f(z)\mathrm{d}z = 2\pi\mathrm{i} \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} f.$$

But

$$\begin{aligned} \left| \int_{\Gamma_R^+} f(z)\mathrm{d}z \right| &= \left| \int_0^{\pi} f(R\mathrm{e}^{\mathrm{i}t}) R\mathrm{e}^{\mathrm{i}t}\mathrm{i}\mathrm{d}t \right| \le \int_0^{\pi} |f(R\mathrm{e}^{\mathrm{i}t})| R\mathrm{d}t \\ &= \int_0^{\pi} o(1/R) R\mathrm{d}t = o_R(1)\pi \to 0 \end{aligned}$$

as $R \to \infty$. That's why

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\mathrm{d}x = 2\pi\mathrm{i} \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} f.$$

$\square$

Example 8.6. Let us evaluate the integral

$$\int_0^{\infty} \frac{1}{x^4+1}\mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4+1}\mathrm{d}x.$$

The singular points of

$$\frac{1}{z^4+1}$$

are

$$z_0 = \mathrm{e}^{\mathrm{i}\pi/4}, \quad z_1 = \mathrm{e}^{\mathrm{i}3\pi/4}, \quad z_2 = \mathrm{e}^{\mathrm{i}5\pi/4}, \quad z_3 = \mathrm{e}^{\mathrm{i}7\pi/4}.$$

It is clear also that $\operatorname{Im} z_0, \operatorname{Im} z_1 > 0$ and $\operatorname{Im} z_2, \operatorname{Im} z_3 < 0$. Hence

$$\begin{aligned}\int_0^\infty \frac{1}{x^4+1}\mathrm{d}x &= \pi\mathrm{i}\left(\operatorname*{Res}_{z=z_0}\frac{1}{z^4+1} + \operatorname*{Res}_{z=z_1}\frac{1}{z^4+1}\right) = \pi\mathrm{i}\left(\frac{1}{4z_0^3}+\frac{1}{4z_1^3}\right)\\ &= \frac{\pi\mathrm{i}}{4}\left(\mathrm{e}^{-3\mathrm{i}\pi/4}+\mathrm{e}^{-9\mathrm{i}\pi/4}\right)\\ &= \frac{\pi\mathrm{i}}{4}\left(\cos\frac{3\pi}{4}-\mathrm{i}\sin\frac{3\pi}{4}+\cos\frac{9\pi}{4}-\mathrm{i}\sin\frac{9\pi}{4}\right)\\ &= \frac{\pi\mathrm{i}}{4}\left(-2\mathrm{i}\sin\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{4}.\end{aligned}$$

Example 8.7. Let us evaluate the integral

$$\int_0^\infty \frac{x^4}{x^6+1}\mathrm{d}x.$$

The singular points of

$$\frac{z^4}{z^6+1}$$

are

$$z_k = \mathrm{e}^{\mathrm{i}(\pi/6+2\pi k/6)}, \quad k=0,1,\ldots,5.$$

It is clear that only z_0, z_1 and z_2 belong to the upper half plane. Thus

$$\begin{aligned}\int_0^\infty \frac{x^4}{x^6+1}\mathrm{d}x &= \pi\mathrm{i}\sum_{j=0}^{2}\operatorname*{Res}_{z=z_j}\frac{z^4}{z^6+1} = \pi\, i\left(\frac{z_0^4}{6z_0^5}+\frac{z_1^4}{6z_1^5}+\frac{z_2^4}{6z_2^5}\right)\\ &= \frac{\pi\mathrm{i}}{6}\left(\frac{1}{z_0}+\frac{1}{z_1}+\frac{1}{z_2}\right) = \frac{\pi\mathrm{i}}{6}\left(\mathrm{e}^{-\mathrm{i}\pi/6}+\mathrm{e}^{-\mathrm{i}\pi/2}+\mathrm{e}^{-\mathrm{i}5\pi/6}\right)\\ &= \frac{\pi\mathrm{i}}{6}\left(\cos\frac{\pi}{6}-\mathrm{i}\sin\frac{\pi}{6}-\mathrm{i}+\cos\frac{5\pi}{6}-\mathrm{i}\sin\frac{5\pi}{6}\right)\\ &= \frac{\pi\mathrm{i}}{6}\left(-2\mathrm{i}\sin\frac{\pi}{6}-\mathrm{i}\right) = \frac{\pi}{3}.\end{aligned}$$

Problem 8.8. Evaluate the integral

$$\int_{-\infty}^\infty \frac{x^2}{(x^2+4)^2}\mathrm{d}x.$$

Problem 8.9. Evaluate the integral

$$\int_{-\infty}^\infty \frac{1}{(x^4+1)^2}\mathrm{d}x.$$

Problem 8.10. Evaluate the integral

$$\text{p. v.}\int_{-\infty}^\infty \frac{1}{x(x^2+1)}\mathrm{d}x.$$

8.3 Improper integrals of the form $\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ax} f(x)\mathrm{d}x$

Theorem 8.11 (Jordan's lemma)**.** *Let us assume that f is continuous in the region $\{z : |z| > R, \operatorname{Im} z > 0\}$ for some $R > 0$. If*

$$\lim_{z\to\infty} f(z) = 0, \quad \operatorname{Im} z > 0$$

then

$$\lim_{R\to\infty} \int_{|\zeta|=R, \operatorname{Im}\zeta>0} \mathrm{e}^{\mathrm{i}a\zeta} f(\zeta)\mathrm{d}\zeta = 0 \tag{8.6}$$

for any $a > 0$.

Proof. Under the conditions for f we have that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$|f(z)| < \varepsilon, \quad |z| > R, \operatorname{Im} z > 0.$$

We parametrize the semicircle as $\gamma : \zeta(t) = R\mathrm{e}^{\mathrm{i}t}, t \in (0, \pi)$. In that case we obtain

$$\begin{aligned}\left|\int_\gamma \mathrm{e}^{\mathrm{i}a\zeta} f(\zeta)\mathrm{d}\zeta\right| &\le \int_\gamma |\mathrm{e}^{\mathrm{i}a\zeta}||f(\zeta)||\mathrm{d}\zeta| < \varepsilon \int_0^\pi \left|\mathrm{e}^{\mathrm{i}aR(\cos t + \mathrm{i}\sin t)}\right| R\mathrm{d}t \\ &= \varepsilon R \int_0^\pi \mathrm{e}^{-aR\sin t}\mathrm{d}t = 2\varepsilon R \int_0^{\pi/2} \mathrm{e}^{-aR\sin t}\mathrm{d}t \\ &< 2\varepsilon R \int_0^{\pi/2} \mathrm{e}^{-aR2t/\pi}\mathrm{d}t\end{aligned}$$

since $\sin t > 2t/\pi$ for $0 < t < \pi/2$ and $a > 0$. The latter integral can be calculated precisely and therefore

$$\left|\int_\gamma \mathrm{e}^{\mathrm{i}a\zeta} f(\zeta)\mathrm{d}\zeta\right| < \frac{\pi\varepsilon}{a}\left(1 - \mathrm{e}^{-aR}\right) < \frac{\pi\varepsilon}{a}.$$

Since $\varepsilon > 0$ was arbitrary we obtain (8.6). □

Corollary 8.12. *Let us assume that f is continuous in the region $\{z : |z| > R, \operatorname{Im} z < 0\}$ for some $R > 0$. If*

$$\lim_{z\to\infty} f(z) = 0, \quad \operatorname{Im} z < 0$$

then

$$\lim_{R\to\infty} \int_{|\zeta|=R, \operatorname{Im}\zeta<0} \mathrm{e}^{\mathrm{i}a\zeta} f(\zeta)\mathrm{d}\zeta = 0 \tag{8.7}$$

for any $a < 0$.

Corollary 8.13. *Let us assume that f is continuous in the regions $\{z : |z| > R, \operatorname{Re} z < 0\}$ or $\{z : |z| > R, \operatorname{Re} z > 0\}$ for some $R > 0$. If*

$$\lim_{z\to\infty\, \operatorname{Re} z<0} f(z) = 0 \quad \textit{or} \quad \lim_{z\to\infty\, \operatorname{Re} z>0} f(z) = 0,$$

then

$$\lim_{R\to\infty} \int_{|\zeta|=R, \operatorname{Re}\zeta<0} \mathrm{e}^{a\zeta} f(\zeta)\mathrm{d}\zeta = 0 \tag{8.8}$$

or

$$\lim_{R\to\infty} \int_{|\zeta|=R, \operatorname{Re}\zeta>0} \mathrm{e}^{a\zeta} f(\zeta)\mathrm{d}\zeta = 0 \tag{8.9}$$

for any $a > 0$ or $a < 0$, respectively.

Theorem 8.14. *Let f be analytic for $\operatorname{Im} z > 0$ and continuous for $\operatorname{Im} z \geq 0$ except at the singular points $z_1, z_2, \ldots, z_n$ with $\operatorname{Im} z_j > 0$ for all $j = 1, 2, \ldots, n$. If $f(z) = o(1)$ for $z \to \infty, \operatorname{Im} z > 0$ then*

$$\text{p. v.} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ax} f(x)\mathrm{d}x = 2\pi\mathrm{i} \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} (\mathrm{e}^{\mathrm{i}az} f) \tag{8.10}$$

for $a > 0$.

Proof. Let $R > 0$ be chosen such that all singular points $z_1, z_2, \ldots, z_n$ belong to the region $\{z : |z| < R, \operatorname{Im} z > 0\}$. Let γ_R be the union of the line segment $[-R, R]$ with the upper semicircle Γ_R^+. The residue theorem gives that

$$\int_{-R}^{R} \mathrm{e}^{\mathrm{i}ax} f(x)\mathrm{d}x + \int_{\Gamma_R^+} \mathrm{e}^{\mathrm{i}az} f(z)\mathrm{d}z = 2\pi\mathrm{i} \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} (\mathrm{e}^{\mathrm{i}az} f).$$

Jordan's lemma (see (8.6)) implies that for $a > 0$ the integral over Γ_R^+ tends to zero as $R \to \infty$. Hence, letting $R \to \infty$ we obtain (8.10). □

Example 8.15. Let us evaluate the integral

$$\int_0^{\infty} \frac{x \sin x}{x^2 + 4} \mathrm{d}x.$$

Indeed, we have

$$\begin{aligned}
\int_0^{\infty} \frac{x \sin x}{x^2 + 4} \mathrm{d}x &= \frac{1}{2} \text{p. v.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4} \mathrm{d}x = \frac{1}{2} \operatorname{Im} \left(\text{p. v.} \int_{-\infty}^{\infty} \frac{x \mathrm{e}^{\mathrm{i}x}}{x^2 + 4} \mathrm{d}x \right) \\
&= \frac{1}{2} \operatorname{Im} \left(2\pi\mathrm{i} \operatorname*{Res}_{z=2\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}z} z}{z^2 + 4} \right) = \frac{1}{2} \operatorname{Im} \left(2\pi\mathrm{i} \frac{\mathrm{e}^{\mathrm{i}2\mathrm{i}} 2\mathrm{i}}{2 \cdot 2\mathrm{i}} \right) \\
&= \frac{1}{2} \operatorname{Im} \left(\frac{\mathrm{e}^{-2}}{2} 2\pi\mathrm{i} \right) = \frac{\pi}{2} e^{-2}.
\end{aligned}$$

Example 8.16. Let us evaluate the integral

$$\int_0^\infty \frac{\cos(ax)}{x^2+1}\mathrm{d}x.$$

Indeed, we have

$$\begin{aligned}\int_0^\infty \frac{\cos(ax)}{x^2+1}\mathrm{d}x &= \frac{1}{2}\,\mathrm{p.\,v.}\int_{-\infty}^\infty \frac{\cos(ax)}{x^2+1}\mathrm{d}x = \frac{1}{2}\operatorname{Re}\left(\mathrm{p.\,v.}\int_{-\infty}^\infty \frac{\mathrm{e}^{\mathrm{i}ax}}{x^2+1}\mathrm{d}x\right)\\ &= \frac{1}{2}\operatorname{Re}\left(2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}az}}{z^2+1}\right) = \frac{1}{2}\operatorname{Re}\left(2\pi\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}a\mathrm{i}}}{2\mathrm{i}}\right) = \frac{\pi}{2}e^{-a}.\end{aligned}$$

Definition 8.17. Let f be a continuous real-valued function of $x \in [a,b]$ except possibly the point $c \in (a,b)$. The principal value of the integral

$$\int_a^b f(x)\mathrm{d}x$$

is defined as

$$\mathrm{p.\,v.}\int_a^b f(x)\mathrm{d}x := \lim_{\varepsilon\to+0}\left[\int_a^{c-\varepsilon} f(x)\mathrm{d}x + \int_{c+\varepsilon}^b f(x)\mathrm{d}x\right]$$

if the limit exits.

Example 8.18. Let us evaluate the principal value integral

$$\mathrm{p.\,v.}\int_a^b \frac{1}{x-c}\mathrm{d}x, \quad a<c<b.$$

By the definition we have

$$\begin{aligned}\mathrm{p.\,v.}\int_a^b \frac{\mathrm{d}x}{x-c} &= \lim_{\varepsilon\to+0}\left[\int_a^{c-\varepsilon}\frac{\mathrm{d}x}{x-c} + \int_{c+\varepsilon}^b \frac{\mathrm{d}x}{x-c}\right]\\ &= \lim_{\varepsilon\to+0}\left[\log|-\varepsilon| - \log|a-c| + \log|b-c| - \log|\varepsilon|\right] = \log\frac{b-c}{c-a}.\end{aligned}$$

Example 8.19. Let us evaluate the integral

$$\int_0^\infty \frac{\sin x}{x}\mathrm{d}x.$$

We have

$$\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{1}{2}\,\mathrm{p.\,v.}\int_{-\infty}^\infty \frac{\sin x}{x}\mathrm{d}x,$$

where principal value integral is considered with respect to ∞ and 0. We have

$$\mathrm{p.\,v.}\int_{-\infty}^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{1}{\mathrm{i}}\,\mathrm{p.\,v.}\int_{-\infty}^\infty \frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x = \frac{1}{\mathrm{i}}\lim_{R\to\infty,\varepsilon\to 0}\left(\int_{-R}^{-\varepsilon}\frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x + \int_\varepsilon^R \frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x\right).$$

Here we have used the fact that

$$\text{p. v.} \int_{-\infty}^{\infty} \frac{\cos x}{x} \mathrm{d}x = 0.$$

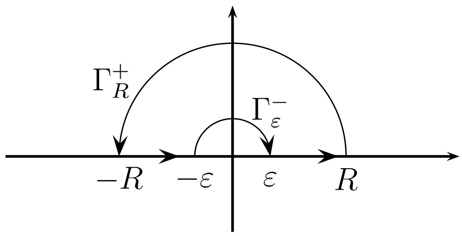
Consider the function

$$f(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{z}.$$

It has only one singular point $z = 0$. That's why we consider the closed curve

$$\gamma = [-R, -\varepsilon] \cup \Gamma_\varepsilon^- \cup [\varepsilon, R] \cup \Gamma_R^+,$$

see Figure below.



Inside of γ the function f is analytic and continuous up to the curve γ. Using the Cauchy theorem we have

$$0 = \int_\gamma \frac{\mathrm{e}^{\mathrm{i}z}}{z} \mathrm{d}z = \int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{\mathrm{i}x}}{x} \mathrm{d}x + \int_{\Gamma_\varepsilon^-} \frac{\mathrm{e}^{\mathrm{i}z}}{z} \mathrm{d}z + \int_\varepsilon^R \frac{\mathrm{e}^{\mathrm{i}x}}{x} \mathrm{d}x + \int_{\Gamma_R^+} \frac{\mathrm{e}^{\mathrm{i}z}}{z} \mathrm{d}z. \tag{8.11}$$

The integral over Γ_R^+ tends to 0 as $R \to \infty$ due to Jordan's lemma. The integral over Γ_ε^- can be calculated as

$$\int_{\Gamma_\varepsilon^-} \frac{\mathrm{e}^{\mathrm{i}z}}{z} \mathrm{d}z = -\int_{\Gamma_\varepsilon^+} \frac{\mathrm{e}^{\mathrm{i}z}}{z} \mathrm{d}z = -\int_0^\pi \frac{\mathrm{e}^{\mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}t}} \mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}t}}{\varepsilon \mathrm{e}^{\mathrm{i}t}} \mathrm{d}t = -\mathrm{i} \int_0^\pi \mathrm{e}^{\mathrm{i}\varepsilon \cos t} \mathrm{e}^{-\varepsilon \sin t} \mathrm{d}t.$$

But the last integral tends to $-\mathrm{i}\pi$ as $\varepsilon \to 0$ due to continuity of the functions $\mathrm{e}^{\mathrm{i}\varepsilon \cos t}$ and $\mathrm{e}^{-\varepsilon \sin t}$ with respect to ε and $t \in [0, \pi]$.

Letting now $\varepsilon \to 0$ and $R \to \infty$ in (8.11) we obtain

$$0 = \lim_{R \to \infty, \varepsilon \to 0} \left(\int_{-R}^{-\varepsilon} \frac{\mathrm{e}^{\mathrm{i}x}}{x} \mathrm{d}x + \int_\varepsilon^R \frac{\mathrm{e}^{\mathrm{i}x}}{x} \mathrm{d}x \right) - \mathrm{i}\pi$$

or

$$\text{p. v.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x} \mathrm{d}x = \mathrm{i}\pi.$$

Therefore

$$\int_0^\infty \frac{\sin x}{x} \mathrm{d}x = \frac{1}{2\mathrm{i}} \text{p. v.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x} \mathrm{d}x = \frac{\pi}{2}.$$

This integral is called the *Dirichlet integral*.

Problem 8.20. Evaluate the *Fresnel integrals*

$$\int_0^\infty \cos(x^2)\mathrm{d}x \quad \text{and} \quad \int_0^\infty \sin(x^2)\mathrm{d}x.$$

Problem 8.21. Prove that

$$\int_0^\infty \frac{\log x}{x^2+a^2}\mathrm{d}x = \frac{\pi \log a}{2a}$$

for any $a > 0$. In particular,

$$\int_0^\infty \frac{\log x}{1+x^2}\mathrm{d}x = 0.$$

Problem 8.22. Show that

$$\int_0^\infty \frac{x^{\alpha-1}}{x+\lambda}\mathrm{d}x = \lambda^{\alpha-1}\frac{\pi}{\sin(\alpha\pi)}$$

for $0 < \alpha < 1$ and $\lambda > 0$.

Chapter 9

Calculation of series by residue theory

There are two results which may work in applications to the calculation of number series by residue theory.

Theorem 9.1. *Let $f(z)$ be analytic in $\mathbb{C}$ except the finite number of points $\{z_j\}_{j=1}^m$ with $\operatorname{Im} z_j \neq 0$. Let us assume in addition that $f(z) \to 0$ as $|z| \to \infty$. Then*

$$\sum_{k=-\infty}^{\infty} (-1)^k f(k) = -\sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} \frac{\pi f(z)}{\sin \pi z}. \tag{9.1}$$

Proof. For any $n \in \mathbb{Z}$ large enough and for $R > 0$ let us consider the curve (rectangle)

$$\begin{aligned} \Gamma_{n,R} = \{z \in \mathbb{C} : & x + \mathrm{i}R, x \in [-n-1/2, n+1/2], \\ & x - \mathrm{i}R, x \in [-n-1/2, n+1/2], -n-1/2+\mathrm{i}y, y \in [-R, R], \\ & n+1/2+\mathrm{i}y, y \in [-R, R]\} \end{aligned}$$

such that all singular points of $f(z)$ belong to $\operatorname{int} \Gamma_{n,R}$. Then the function

$$\frac{\pi f(z)}{\sin \pi z}$$

has the singular points

$$\{z_j\}_{j=1}^m, z_k = k, k = 0, \pm 1, \pm 2, \ldots, \pm n$$

inside $\operatorname{int} \Gamma_{n,R}$. Using now the Cauchy residue theorem for this special domain

$\operatorname{int}\Gamma_{n,R}$ we obtain

$$\begin{aligned}
\int_{\Gamma_{n,R}} \frac{\pi f(z)}{\sin \pi z} \mathrm{d}z &= 2\pi \mathrm{i} \sum_{k=-n}^{n} \operatorname*{Res}_{z=k} \frac{\pi f(z)}{\sin \pi z} + 2\pi \mathrm{i} \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} \frac{\pi f(z)}{\sin \pi z} \\
&= 2\pi \mathrm{i} \left(\sum_{k=-n}^{n} \frac{\pi f(k)}{\pi \cos \pi k} + \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} \frac{\pi f(z)}{\sin \pi z} \right) \\
&= 2\pi \mathrm{i} \left(\sum_{k=-n}^{n} (-1)^k f(k) + \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} \frac{\pi f(z)}{\sin \pi z} \right). \qquad (9.2)
\end{aligned}$$

Now, in order to get (9.1) we need to investigate the curve integral in the left hand side of (9.2). This integral can be represented as the sum of the following four integrals:

$$\begin{aligned}
I_1 &= \int_{-n-1/2}^{n+1/2} \frac{\pi f(x - \mathrm{i}R) \mathrm{d}x}{\sin \pi (x - \mathrm{i}R)} \\
I_2 &= \int_{n+1/2}^{-n-1/2} \frac{\pi f(x + \mathrm{i}R) \mathrm{d}x}{\sin \pi (x + \mathrm{i}R)} \\
I_3 &= \mathrm{i} \int_{-R}^{R} \frac{\pi f(n + 1/2 + \mathrm{i}y) \mathrm{d}y}{\sin \pi (n + 1/2 + \mathrm{i}y)} \\
I_4 &= \mathrm{i} \int_{R}^{-R} \frac{\pi f(-n - 1/2 + \mathrm{i}y) \mathrm{d}y}{\sin \pi (-n - 1/2 + \mathrm{i}y)}.
\end{aligned}$$

Since

$$|\sin \pi (x \pm \mathrm{i}R)| = \left| \frac{\mathrm{e}^{\mathrm{i}\pi x} \mathrm{e}^{\mp \pi R} - \mathrm{e}^{-\mathrm{i}\pi x} \mathrm{e}^{\pm \pi R}}{2\mathrm{i}} \right| \geq \frac{\mathrm{e}^{\pi R} - \mathrm{e}^{-\pi R}}{2} \geq \frac{1}{4} \mathrm{e}^{\pi R}, R > 0$$

then for I_1 and I_2 we have the following estimate

$$|I_1|, |I_2| \leq \frac{4\pi}{\mathrm{e}^{\pi R}} \int_{-n-1/2}^{n+1/2} |f(x \mp \mathrm{i}R)| \mathrm{d}x \leq \frac{4\pi}{\mathrm{e}^{\pi R}} \max_{x \in [-n-1/2, n+1/2]} |f(x \mp \mathrm{i}R)|(2n+1).$$

If we choose $R \geq n$ and take into account that $f(z) \to 0$ as $|z| \to +\infty$ (actually we need here only boundedness of f) then when $R \geq n \to \infty$ the right hand side of the latter inequality tends to zero. Next, since

$$\begin{aligned}
\sin(\pm \pi (n + 1/2 + \mathrm{i}y)) &= \pm \sin(\pi (n + 1/2) \mp \mathrm{i}\pi y) \\
&= \pm (-1)^n \cos(\mathrm{i}\pi y) = \pm (-1)^n \cosh(\pi y)
\end{aligned}$$

then we have the following estimates for I_3 and I_4

$$
\begin{aligned}
|I_3|, |I_4| &\le \pi \int_{-R}^{R} \frac{|f(n+1/2 \pm \mathrm{i}y)|\mathrm{d}y}{\cosh(\pi y)} \\
&\le \pi \max_{y\in[-R,R]} |f(n+1/2 \pm \mathrm{i}y)| \int_{-R}^{R} \frac{\mathrm{d}y}{\cosh(\pi y)} \\
&\le \pi \max_{y\in[-R,R]} |f(n+1/2 + \mathrm{i}y)| \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\cosh(\pi y)} \to 0, \quad n \to \infty
\end{aligned}
$$

due to the fact that $f(z) \to 0$ as $|z| \to +\infty$ and

$$
\int_{-\infty}^{\infty} \frac{\mathrm{d}y}{\cosh(\pi y)} = 1.
$$

If we let now $R \ge n \to \infty$ in (9.2) we obtain that

$$
0 = 2\pi\mathrm{i}\left(\sum_{k=-\infty}^{\infty} (-1)^k f(k) + \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} \frac{\pi f(z)}{\sin \pi z}\right).
$$

It implies (9.1) and therefore Theorem is completely proved. □

Remark. Actually some of the singular points $\{z_j\}_{j=1}^m$ of $f(z)$ may locate on the real line but such that they are not equal to $n \in \mathbb{Z}$.

Theorem 9.2. *Let $f(z)$ be analytic in $\mathbb{C}$ except the finite number of points $\{z_j\}_{j=1}^m$ with $\operatorname{Im} z_j \neq 0$. Let us assume in addition that $zf(z) \to 0$ as $|z| \to \infty$. Then*

$$
\sum_{k=-\infty}^{\infty} f(k) = -\sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} \left(\pi \cot(\pi z) f(z)\right). \tag{9.3}
$$

Proof. Literally the same as for Theorem 9.1. The only difference is

$$
\operatorname*{Res}_{z=k} \pi \cot(\pi z) f(z) = \frac{\pi \cos(\pi k) f(k)}{(\sin \pi z)'\,|_{z=k}} = f(k).
$$

□

Remark. Again (as in Theorem 9.1) actually some singular points $\{z_j\}_{j=1}^m$ of $f(z)$ may locate on the real line such that they are not equal to $n \in \mathbb{Z}$.

Example 9.3. Show that for real $a \neq 0$ we have

$$
\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \frac{\pi}{a} \coth(\pi a).
$$

Indeed, let

$$f(z) = \frac{1}{z^2 + a^2}, \quad z \in \mathbb{C}.$$

This function has two singular points $z_1 = \mathrm{i}a$ and $z_2 = -\mathrm{i}a$. Then Theorem 9.2 gives that

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = -\left(\operatorname*{Res}_{z=\mathrm{i}a} \frac{\pi \cot(\pi z)}{z^2 + a^2} + \operatorname*{Res}_{z=-\mathrm{i}a} \frac{\pi \cot(\pi z)}{z^2 + a^2}\right)$$
$$= -\left(\frac{\pi \cot(\pi \mathrm{i}a)}{2\mathrm{i}a} + \frac{\pi \cot(-\pi \mathrm{i}a)}{-2\mathrm{i}a}\right) = -\frac{\pi \cot(\pi \mathrm{i}a)}{ia} = \frac{\pi}{a} \coth(\pi a).$$

Example 9.4. Show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Indeed, let $a = \varepsilon > 0$ and small. Then Example 9.3 implies that

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + \varepsilon^2} = \frac{1}{\varepsilon^2} + 2\sum_{k=1}^{\infty} \frac{1}{k^2 + \varepsilon^2} = \frac{\pi}{\varepsilon} \coth(\pi\varepsilon).$$

So

$$2\sum_{k=1}^{\infty} \frac{1}{k^2 + \varepsilon^2} = \frac{\pi}{\varepsilon} \coth(\pi\varepsilon) - \frac{1}{\varepsilon^2} = \frac{\varepsilon\pi(\mathrm{e}^{2\varepsilon\pi} + 1) - \mathrm{e}^{2\varepsilon\pi} + 1}{\varepsilon^2(\mathrm{e}^{2\varepsilon\pi} - 1)}.$$

Using Taylor expansion for e^{ξ} near zero we can easily obtain that the limit of the right hand side of the latter equality is equal to $\pi^2/3$. Thus

$$2\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{3}.$$

Problem 9.5. Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

Problem 9.6. Show that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}.$$

Problem 9.7. Show that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

Appendix A

Exercises

1. Find the primitives of the following functions
   a) $f(z) = \sin z \cos z$ b) $f(z) = \cos^2 z$ c) $f(z) = z\mathrm{e}^{2z}$
   d) $f(z) = z^2 \sin z$ e) $f(z) = z \sin z^2$ f) $f(z) = \mathrm{e}^z \sin z$

2. Let f be analytic in the whole $\mathbb{C}$ such that
$$|f(z)| \leq \left|\frac{z+1}{z-1}\right|$$
for all $z \in \mathbb{C}$. Prove that f is constant function.

3. Let f be analytic in the disk $\{z : |z| < R\}$. Assume that f is non-constant. Let us define the function
$$g(r) := \max_{|z| \leq r} |f(z)|, \quad 0 < r < R.$$
Prove that $g(r_1) < g(r_2)$ whenever $0 < r_1 < r_2 < R$.

4. Let $f(z) = \cos z, z \in \mathbb{C}$. Find $\max_{|z|\leq 1} |f(z)|$.

5. Investigate the convergence of the function sequence $f_n, n = 1, 2, \ldots$ in the set $E \subset \mathbb{C}$ when
   a) $f_n(z) = \dfrac{nz}{z+n}, E = \{z : |z| < 1\}$ b) $f_n(z) = \dfrac{nz}{nz+1}, E = \{z : |z| > 1\}$.
   Is the convergence uniform in E?

6. Find the radius of convergence and disk of convergence for the following series
   a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k+1} z^k$ b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}(z-1)^k$ c) $\displaystyle\sum_{k=0}^{\infty} k^2 z^k$ d) $\displaystyle\sum_{k=0}^{\infty} \frac{k^3}{3^k} z^k$.

7. Find the radius of convergence for the series
$$\sum_{k=0}^{\infty} \left(\frac{1}{1-\mathrm{i}/2}\right)^{k+1} (z-\mathrm{i}/2)^k.$$

Find also the sum of the series.

8. Find the function $f(z) = \sum_{k=0}^{\infty} k z^k$ for $|z| < 1$.

9. Find the Taylor series for $f(z) = \sin z$ around the point $z = \pi/4$.

10. Find the Taylor series for $f(z) = (z-1)^{-2}$ around the point $z = 2$.

11. Find the order of the root $z = 0$ of $f(z) = \mathrm{e}^z - 1 - \sin z$.

12. Problem 4.12. Apply this problem to prove that if f is an analytic function in the unit disk such that

$$f\left(\frac{n}{2n+1}\right) = f\left(\frac{n}{2n+1}\mathrm{i}\right), \quad n = 2, 3, \ldots$$

then $f^{(10)}(0) = 0$.

13. Find the Laurent series for f at $z_0 = 0$ and investigate the type of singular point 0 and evaluate the residue, when
a) $f(z) = \dfrac{1 - \cos z}{z}$ b) $f(z) = \dfrac{\mathrm{e}^{z^2}}{z^3}$

14. Find the Laurent series for $f(z) = \dfrac{1}{z(z+1)(z+2)}$ at $z_0 = 0$.

15. Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{1}{a + \cos t} \mathrm{d}t, a > 1$.

16. Evaluate the integral $\displaystyle\int_0^{\infty} \frac{1}{x^6 + 1} \mathrm{d}x$.

17. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin bx}{x^2 + a^2} \mathrm{d}x, \quad a, b > 0$$

18. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \pi^2} \mathrm{d}x.$$

19. Evaluate the integrals

$$\int_{-\infty}^{\infty} \frac{\sin x}{x - \omega} \mathrm{d}x, \quad \int_{-\infty}^{\infty} \frac{\cos x}{x - \omega} \mathrm{d}x,$$

where $\mathrm{Im}\, \omega \neq 0$.

20. Evaluate the series
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}.$$

21. Evaluate the series
$$\sum_{k=-\infty}^{\infty} \frac{1}{k + \mathrm{i}a}, \quad a \in \mathbb{R} \setminus \{0\}.$$

Part III

Chapter 1

Conformal mappings

We return now to the geometrical properties of non-zero derivative. Let f be analytic in the domain D and let $z_0 \in D$ be an arbitrary point. If $f'(z_0) \neq 0$ then this is equivalent to (see Cauchy-Riemann conditions)

$$|f'(z_0)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} > 0.$$

It means that the Jacobian of the transformation from (x, y) to (u, v) is non-zero at (x_0, y_0) and thus in the neighborhood of $(u_0, v_0) = (u(x_0, y_0), v(x_0, y_0))$ there exists an inverse function $z = x + \mathrm{i}y = f^{-1}(w), w = u + \mathrm{i}v$ such that $z = f^{-1}(w)$ is analytic at $w_0 = u_0 + \mathrm{i}v_0$ and

$$(f^{-1}(w))'(w_0) = \frac{1}{f'(z_0)}.$$

This fact can be interpreted as follows: in the neighborhood of z_0 the function $w = f(z)$ is univalent and analytic. But this property is local (as we can see).

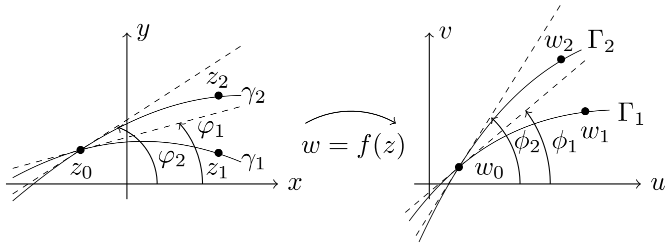
Another geometric property of analytic function with non-zero derivative is the following. Let f be analytic in the domain D and $f'(z_0) \neq 0$ for $z_0 \in D$. Consider two arbitrary curves γ_1 and γ_2 on the z-plane which intersect at the point z_0. Assume that the angle between γ_1 and γ_2 at z_0 is $\varphi_2 - \varphi_1$, and the angle between Γ_1 and Γ_2 at $w_0 = f(z_0)$ in the w-plane is equal to $\phi_2 - \phi_1$, where Γ_j is the image of γ_j under the mapping f for $j = 1, 2$, see Figure below. If $z_1 = z_0 + \Delta z_1 \in \gamma_1$ and $z_2 = z_0 + \Delta z_2 \in \gamma_2$ then $f(z_1) = f(z_0 + \Delta z_1) = f(z_0) + \Delta f_1 = w_0 + \Delta w_1 \in \Gamma_1$ and $f(z_2) = f(z_0 + \Delta z_2) = f(z_0) + \Delta f_2 = w_0 + \Delta w_2 \in \Gamma_2$.

Moreover,

$$\lim_{\Delta z_1 \to 0} \arg \frac{\Delta w_1}{\Delta z_1} = \lim_{\Delta z_1 \to 0} [\arg \Delta w_1 - \arg \Delta z_1] = \phi_1 - \varphi_1$$

and

$$\lim_{\Delta z_2 \to 0} \arg \frac{\Delta w_2}{\Delta z_2} = \lim_{\Delta z_2 \to 0} [\arg \Delta w_2 - \arg \Delta z_2] = \phi_2 - \varphi_2.$$



By the existence of $f'(z_0) \neq 0$ and due to the independence of this derivative with respect to direction we obtain

$$\phi_1 - \varphi_1 = \phi_2 - \varphi_2 = \arg f'(z_0). \tag{1.1}$$

So, we may conclude that the transformation $w = f(z)$ preserves the angles with respect to orientation and magnitude. In addition, since $f'(z_0) \neq 0$ then

$$|\Delta w| = k|\Delta z| + o(|\Delta z|), \quad k = |f'(z_0)|, \tag{1.2}$$

i.e. there is the *factor of stretching* in all directions.

It is also proved earlier (see Problem 7.10 of Part II) that if f is analytic in the domain D and univalent there then $f'(z) \neq 0$ for all $z \in D$.

These properties justify the following definition.

Definition 1.1. The mapping $f : D \to \mathbb{C}$ is called *conformal* at $z_0 \in D$ if it preserves the angles and the factor of stretching at this point. If f is conformal at each point in D then f is called conformal in D.

There is a very deep connection between analytic functions and conformal mappings.

Theorem 1.2. *The mapping $f : D \to \mathbb{C}$ is conformal in D if and only if f is analytic and univalent in D.*

Proof. Let f be analytic and univalent in the domain D. Then applying Problem 7.10 of Part II we conclude that $f'(z) \neq 0$ everywhere in D. Hence, see (1.1) and (1.2), f is conformal at each point $z \in D$ and therefore it is conformal in D.

Conversely, let z_0 be an arbitrary point in D and let $w_0 = f(z_0)$. By the conditions of this theorem we have

$$\arg(w_2 - w_0) - \arg(w_1 - w_0) = \alpha + o(\max(|w_1 - w_0|, |w_2 - w_0|))$$

and

$$\arg(z_2 - z_0) - \arg(z_1 - z_0) = \alpha + o(\max(|z_1 - z_0|, |z_2 - z_0|)),$$

where $\alpha = \varphi_2 - \varphi_1 = \phi_2 - \phi_2$, see Figure above. Moreover,

$$\frac{|w_2 - w_0|}{|z_2 - z_0|} = k + o(1), \quad \frac{|w_1 - w_0|}{|z_1 - z_0|} = k + o(1)$$

as $|z_2 - z_0|, |z_1 - z_0| \to 0$. These equalities imply that

$$\frac{w_2 - w_0}{z_2 - z_0} = k\mathrm{e}^{\mathrm{i}\varphi} + o(1), \quad \frac{w_1 - w_0}{z_1 - z_0} = k\mathrm{e}^{\mathrm{i}\varphi} + o(1),$$

where (since α is the same in both equalities)

$$\arg \frac{w_2 - w_0}{z_2 - z_0} = \varphi + o(1), \quad \arg \frac{w_1 - w_0}{z_1 - z_0} = \varphi + o(1)$$

as $|z_2 - z_0|, |z_1 - z_0| \to 0$. Since γ_1 and γ_2 are arbitrary then z_2 and z_1 are arbitrary too. Hence we may conclude that there exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = k\mathrm{e}^{\mathrm{i}\varphi} = f'(z_0),$$

and $f'(z_0) \neq 0$ (or $k \neq 0$) i.e. f is analytic and univalent in D. $\square$

Remark. Theorem 1.2 says that univalent functions and only they realize conformal mappings.

The next important property of conformal mappings is contained in the following theorem.

Theorem 1.3 (Boundary correspondence principle). *Let D be simply connected domain with the boundary ∂D which is a closed curve γ. Let also $f \in H(D) \cap C(\overline{D})$. Assume that f maps γ to the closed curve $\Gamma := f(\gamma)$ bijectively with the same direction of the circuit as for γ. Then $f : D \to \operatorname{int}\Gamma$ is surjective and conformal.*

Proof. Due to Theorem 1.2 it suffices to show that f is univalent in D and f maps D onto $\operatorname{int}\gamma$. Let us consider two different points $w_1 \in \operatorname{int}\Gamma$ and $w_2 \in \mathbb{C} \setminus \overline{\operatorname{int}\Gamma}$ and two different functions

$$F_1(z) = f(z) - w_1, \quad F_2(z) = f(z) - w_2, \quad z \in D.$$

If z goes over γ then $w = f(z)$ goes over Γ and the direction of the circuit over these curves are the same. Thus, using the principle of argument (see Theorem 7.4 of Part II) we obtain that

$$\frac{1}{2\pi} \operatorname*{Var}_{\gamma} \operatorname{Arg} F_1(z) = N(F_1) = 1, \quad \frac{1}{2\pi} \operatorname*{Var}_{\gamma} \operatorname{Arg} F_2(z) = N(F_2) = 0,$$

where $N(F_1)$ and $N(F_2)$ denote the number of zeros of F_1 and F_2, respectively. It means that for any $w_1 \in \operatorname{int}\Gamma$ there is only one point $z_1 \in D$ such that $w_1 = f(z_1)$ and for any $w_2 \in \mathbb{C} \setminus \overline{\operatorname{int}\Gamma}$ there are no points $z \in D$ such that $w_2 = f(z)$ i.e. f maps D onto $\operatorname{int}\Gamma$ and it is univalent in D. $\square$

There is one more important property of conformal mappings: Schwarz reflection principle (or Schwarz symmetry principle).

Definition 1.4. Let $D \subset \mathbb{C}$ be a domain. The set

$$J(D) = \{\zeta \in \mathbb{C} : \zeta = \overline{z}, z \in D\} \tag{1.3}$$

is called the *conjugate domain*.

This definition implies that $J(D)$ is a domain and that if $f(z)$ is analytic in D then $g(z) := \overline{f(\overline{z})}$ is analytic in $J(D)$. Indeed, since $f(z)$ is analytic in D then for each $z_0 \in D$ the Taylor expansion holds i.e.

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j, \quad a_j = \frac{f^{(j)}(z_0)}{j!}$$

for $|z - z_0| < R$ with $R = \operatorname{dist}(z_0, \partial D)$. Thus, if $\zeta, \zeta_0 \in J(D)$ then $\overline{\zeta}, \overline{\zeta_0} \in D$ and

$$f(\overline{\zeta}) = \sum_{j=0}^{\infty} a_j (\overline{\zeta} - \overline{\zeta_0})^j$$

or

$$g(\zeta) = \overline{f(\overline{\zeta})} = \sum_{j=0}^{\infty} \overline{a_j} (\zeta - \zeta_0)^j$$

i.e. $g(\zeta)$ is analytic too.

Theorem 1.5 (Schwarz reflection principle)**.** *Let D be a domain in the upper half of the complex plane whose boundary includes an interval $I := (a, b)$ of the real axis. Let $f \in H(D) \cap C(\overline{D})$. Suppose that $f(x + \mathrm{i}0)$ is real for all $x \in I$ and define the function*

$$F(z) = \begin{cases} f(z), & z \in D \cup I \\ \overline{f(\overline{z})}, & z \in J(D). \end{cases} \tag{1.4}$$

Then F is analytic on $D \cup I \cup J(D)$.

Proof. Since $f(z) \in H(D)$ and $\overline{f(\overline{z})} \in H(J(D))$ then it remains to show that $F(z)$ is analytic at each point $x_0 \in I$. First we check that $F(z)$ is continuous everywhere in $D \cup I \cup J(D)$. Continuity of F in $D \cup I$ follows from the conditions of the theorem. The definition (1.4) of F and the real-valuedness of $f(x + \mathrm{i}0)$ imply that

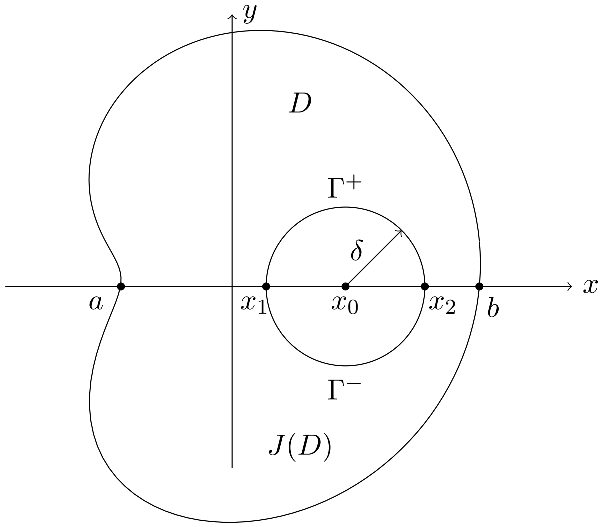
$$F(x - \mathrm{i}0) = \overline{f(\overline{x - \mathrm{i}0})} = f(x + \mathrm{i}0) = F(x + \mathrm{i}0).$$

This proves that F is continuous. Next, we introduce the closed curves

$$\Gamma^+ := \{\zeta : |\zeta - z_0| = \delta, \operatorname{Im} \zeta > 0\} \cup [x_1, x_2]$$

and

$$\Gamma^- := \{\zeta : |\zeta - z_0| = \delta, \operatorname{Im} \zeta < 0\} \cup [x_1, x_2].$$



By Theorem 5.7 in Part I we obtain

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{|\zeta-z_0|=\delta}\frac{F(\zeta)\mathrm{d}\zeta}{\zeta-z_0} &= \frac{1}{2\pi\mathrm{i}}\,\mathrm{p.\,v.}\int_{\Gamma^+}\frac{F(\zeta)\mathrm{d}\zeta}{\zeta-z_0}+\frac{1}{2\pi\mathrm{i}}\,\mathrm{p.\,v.}\int_{\Gamma^-}\frac{F(\zeta)\mathrm{d}\zeta}{\zeta-x_0}\\ &= \frac{1}{2}F(x+\mathrm{i}0)+\frac{1}{2}F(x-\mathrm{i}0)=f(x).\end{aligned}$$

Hence, the Cauchy integral formula yields that F is analytic also at x_0. □

Problem 1.6. Suppose that $f(z)=\sum_{j=0}^{\infty}a_jz^j$ and this series converges for $|z|<r$ and f is real for $x\in(-r,r)$. Show that all a_j are real and $f(z)=\overline{f(\overline{z})}$ for all $|z|<r$.

There is a key question at this point: Is there, in fact, a conformal mapping from a given domain D to some other domain, for example, unit disc? The theoretical answer is the celebrated *Riemann Mapping Theorem* which we give without a proof.

Theorem 1.7 (Riemann)**.** *If D is any simply-connected domain, not equal to the whole complex plane $\mathbb{C}$, then there exists a conformal mapping of D onto $\{w:|w|<1\}$. This mapping is uniquely determined by the value $f(z_0)$ and $\arg f'(z_0)$ at one arbitrary point $z_0\in D$, for example, by the values $f(z_0)=0$ and $f'(z_0)>0$.*

Remark. The assumption that the domain D is not equal to the entire complex plane $\mathbb{C}$ is essential. Indeed, if we assume that there exists a conformal mapping $f(z)$ of the complex plane $\mathbb{C}$ onto the unit disc $\{w:|w|<1\}$ then $f(z)$ is bounded entire function. Hence, due to Liouville theorem $f\equiv$ constant and $f'(z)\equiv 0$. The same is true if $D=\mathbb{C}\setminus\{z_0\}$ with some fixed point $z_0\in\mathbb{C}$

since z_0 is a removable singularity for $f(z)$, therefore again $f(z) \equiv$ constant and $f'(z) \equiv 0$. That's why the equivalent formulation of the Riemann Mapping Theorem includes the assumption that the boundary of $D \subset \mathbb{C}$ has more than two points.

Example 1.8. Let $f(z) = e^{i\frac{\pi}{a}z}, a > 0$. Then f maps $\{z : 0 < \operatorname{Re} z < a\}$ onto $\{w : |w| < 1\}$ conformally. Indeed, if $z = x + iy, 0 < x < a$ then

$$e^{i\frac{\pi}{a}(x+iy)} = e^{-\frac{\pi}{a}y}e^{i\frac{\pi}{a}x} = e^{-\frac{\pi}{a}y}\left(\cos\frac{\pi}{a}x + i\sin\frac{\pi}{a}x\right)$$

so that $|f(z)| = e^{-\frac{\pi}{a}y} \in (0, +\infty)$ if $y \in \mathbb{R}$ and $\arg f(z) = \frac{\pi}{a}x \in (0, \pi)$ if $0 < x < \pi$. Since in addition $f'(z) = i\frac{\pi}{a}e^{i\frac{\pi}{a}z} \neq 0$ for all z and f is one-to-one transformation, then f is conformal.

Example 1.9. Consider a linear-fractional transformation

$$w = f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad c \neq 0.$$

We call it a *non-degenerate* (or *regular*) linear-fractional transformation. This transformation is well-defined and analytic everywhere on $\overline{\mathbb{C}} \setminus \{-d/c\}$. Its derivative is equal to

$$f'(z) = \frac{ad - bc}{(cz + d)^2}, \quad z \neq -d/c, \quad f'(\infty) = \frac{bc - ad}{c^2}$$

and it is not equal to zero everywhere on $\overline{\mathbb{C}} \setminus \{-d/c\}$. We know that f maps $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ bijectively (see Example 2.7 of Part I). So f is conformal. Let us represent it in the form

$$f(z) = \begin{cases} \lambda\frac{\alpha+z}{\beta+z}, & a \neq 0 \\ \lambda\frac{1}{\beta+z}, & a = 0, \end{cases} \tag{1.5}$$

where $\lambda = a/c, \alpha = b/a$ and $\beta = d/c$ if $a \neq 0$ and $\lambda = b, \beta = d/c$ if $a = 0$. The following theorem holds.

Theorem 1.10. *If $z_1 \neq z_2, z_2 \neq z_3, z_1 \neq z_3$ and $w_1 \neq w_2, w_2 \neq w_3, w_1 \neq w_3$ then the correspondence*

$$z_j \to w_j, \quad j = 1, 2, 3$$

defines uniquely a non-degenerate linear-fractional transformation ($a \neq 0$). Moreover,

$$\lambda = \frac{Aw_2 - Bw_1}{A - B}, \quad \alpha = \frac{Bw_1z_2 - Aw_2z_1}{Aw_2 - Bw_1}, \quad \beta = \frac{Bz_2 - Az_1}{A - B}, \tag{1.6}$$

where

$$A = \frac{w_1 - w_3}{w_2 - w_3}, \quad B = \frac{z_1 - z_3}{z_2 - z_3}.$$

Proof. Using (1.5) for $a \neq 0$ we have

$$w_1 - w_3 = \lambda \frac{(z_1 - z_3)(\alpha - \beta)}{(\beta + z_1)(\beta + z_3)}, \quad w_2 - w_3 = \lambda \frac{(z_2 - z_3)(\beta - \alpha)}{(\beta + z_2)(\beta + z_3)}.$$

Here $\beta \neq \alpha$ since $ad \neq bc$. These equalities imply that

$$\frac{w_1 - w_3}{w_2 - w_3} = \frac{z_1 - z_3}{z_2 - z_3} \frac{\beta + z_2}{\beta + z_1}$$

or

$$\beta = \frac{z_2 - z_1 \frac{w_1 - w_3}{w_2 - w_3} \frac{z_2 - z_3}{z_1 - z_3}}{\frac{w_1 - w_3}{w_2 - w_3} \frac{z_2 - z_3}{z_1 - z_3} - 1} = \frac{z_2 - \frac{A}{B} z_1}{\frac{A}{B} - 1} = \frac{Bz_2 - Az_1}{A - B}.$$

It proves (1.6) for β. Next,

$$w_1 = \lambda \frac{\alpha + z_1}{\beta + z_1}, \quad w_2 = \lambda \frac{\alpha + z_2}{\beta + z_2}$$

imply that

$$\frac{w_1}{w_2} = \frac{\alpha + z_1}{\beta + z_1} \frac{\beta + z_2}{\alpha + z_2}$$

or

$$\alpha = \frac{w_2 z_1 (\beta + z_2) - w_1 z_2 (\beta + z_1)}{w_1 (\beta + z_1) - w_2 (\beta + z_2)} = \frac{Bw_1 z_2 - Aw_2 z_1}{Aw_2 - Bw_1} = \frac{Aw_2 z_1 - Bw_1 z_2}{Bw_1 - Aw_2}.$$

This proves (1.6) for α. Finally,

$$\lambda = \frac{w_1(\beta + z_1)}{\alpha + z_1} = \frac{w_1 \left(\frac{Bz_2 - Az_1}{A - B} + z_1 \right)}{\frac{Aw_2 z_1 - Bw_1 z_2}{Bw_1 - Aw_2} + z_1} = \frac{w_1 \frac{B(z_2 - z_1)}{A - B}}{\frac{Bw_1 z_1 - Bw_1 z_2}{Bw_1 - Aw_2}} = \frac{Aw_2 - Bw_1}{A - B}$$

proving the claim for λ. The formulae (1.6) show that α, β and λ are uniquely determined by the correspondence $z_j \to w_j, j = 1, 2, 3$ if z_j and w_j are mutually distinct points. □

Corollary 1.11. *If we denote*

$$w := f(z) = \lambda \frac{\alpha + z}{\beta + z}, \quad a \neq 0$$

then Theorem 1.10 says that

$$\frac{w_1 - w_3}{w_2 - w_3} : \frac{w_1 - w}{w_2 - w} = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z}{z_2 - z}. \tag{1.7}$$

Proof. As it is proved in Theorem 1.10

$$\frac{w_1 - w_3}{w_2 - w_3} = \frac{z_1 - z_3}{z_2 - z_3}\frac{\beta + z_2}{\beta + z_1}.$$

Similarly, if $w \neq w_1, w \neq w_2$ and $z \neq z_1, z \neq z_2$, we obtain

$$\frac{w_1 - w}{w_2 - w} = \frac{z_1 - z}{z_2 - z}\frac{\beta + z_2}{\beta + z_1}.$$

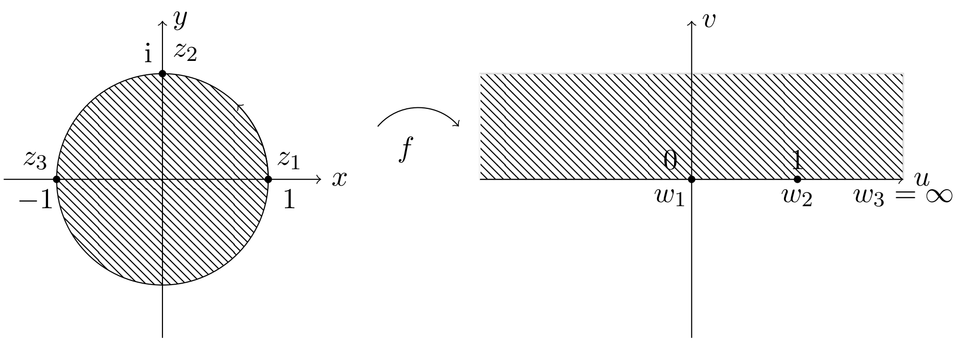
Hence, (1.7) follows straightforwardly from the latter equalities. □

Corollary 1.12. *For the case $a = 0$ ($b \neq 0$ necessarily) instead of three correspondences it is enough to have only two different points $z_1 \neq z_2$ and $w_1 \neq w_2$, respectively with $w_1 \neq 0$ and $w_2 \neq 0$. In that case, see* (1.5)*,*

$$\beta = \frac{w_1 z_1 - w_2 z_2}{w_2 - w_1}, \quad \lambda = \frac{w_1 w_2 (z_1 - z_2)}{w_2 - w_1}. \tag{1.8}$$

Problem 1.13. Show (1.8) for the case $a = 0, b \neq 0, c \neq 0$ in the non-degenerate linear-fractional transformation.

Example 1.14. Let us find $w = f(z)$ which is a conformal mapping of the unit disk $\{z : |z| < 1\}$ onto the domain $\{w : \operatorname{Im} w > 0\}$. Let z_j and w_j be as in the Figure below.



By Theorem 1.10 and Theorem 1.3 we have

$$\frac{0-\infty}{1-\infty} : \frac{0-w}{1-w} = \frac{1+1}{\mathrm{i}+1} : \frac{1-z}{\mathrm{i}-z}.$$

So

$$\frac{1-w}{w} = \frac{2}{\mathrm{i}+1}\frac{z-\mathrm{i}}{z-1}$$

or

$$w = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\frac{z-1}{z+1}.$$

Problem 1.15. Using Example 1.14 show that

$$w = \frac{z + e^{-i\pi/2}}{e^{-i\pi/2} - z}$$

maps conformally the domain $\{z : \operatorname{Im} z > 0\}$ onto the unit disc $\{w : |w| < 1\}$.

Problem 1.16. Show that

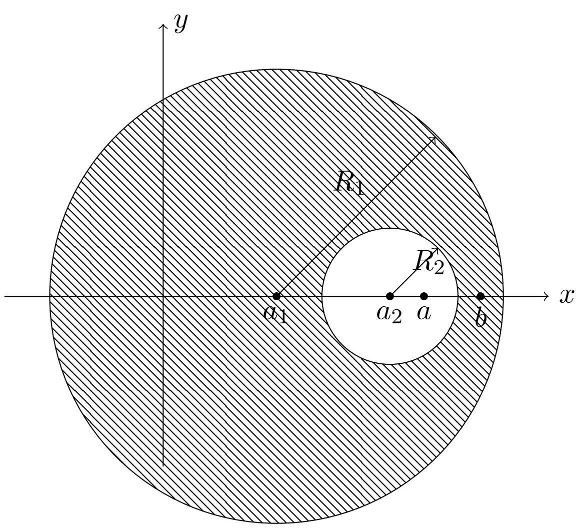
$$w = f(z) = e^{i\alpha} \frac{z - z_0}{z\overline{z_0} - 1}$$

maps conformally the unit disc $\{z : |z| < 1\}$ onto the unit disc $\{w : |w| < 1\}$ such that an arbitrary point $z_0, |z_0| < 1$ is transferred to $w_0 = 0$ and α is an arbitrary real parameter. Show that if $\arg f'(z_0)$ is prescribed then α is uniquely determined.

Problem 1.17. Show that a non-degenerate linear-fractional transformation maps lines and circles on the extended complex plane onto lines or circles.

Problem 1.18. Find the conditions on $0 < r_1 < r_2$ and $0 < R_1 < R_2$ which guarantee the existence of the conformal mapping of the annulus $\{z : r_1 < |z| < r_2\}$ onto the annulus $\{w : R_1 < |w| < R_2\}$.

Example 1.19. Consider a non-concentric ring (annulus), i.e. the set which is formed by two circles $\{z : |z - a_1| = R_1\}$ and $\{z : |z - a_2| = R_2\}$ such that $0 < R_2 < R_1$ and the first circle is located inside of the second one. We assume without loss of generality that a_1 and a_2 are real, see Figure below.



The task is to map conformally this annulus onto the domain $\{w : \operatorname{Im} w > 0\}$. Let now a and b be two real numbers such that they are symmetric with respect to the first and second circle at the same time, i.e. they satisfy the equations

$$(a-a_1)(b-a_1)=R_1^2, \quad (a-a_2)(b-a_2)=R_2^2. \tag{1.9}$$

Solving these equations we can easily obtain a and b uniquely ($a<b$). Then the map

$$w_1=\frac{z-a}{z-b}$$

transfers conformally given non-concentric ring to the concentric one centered at 0. Indeed, if $z-a_1=R_1\mathrm{e}^{\mathrm{i}\varphi}$ then

$$\begin{aligned} w_1&=\frac{z-a}{z-b}=\frac{(z-a_1)-(a-a_1)}{(z-a_1)-(b-a_1)}=\frac{R_1\mathrm{e}^{\mathrm{i}\varphi}-(a-a_1)}{R_1\mathrm{e}^{\mathrm{i}\varphi}-\frac{R_1^2}{a-a_1}}\\ &=\frac{a-a_1}{R_1}\frac{R_1\mathrm{e}^{\mathrm{i}\varphi}-(a-a_1)}{(a-a_1)-R_1\mathrm{e}^{-\mathrm{i}\varphi}}\mathrm{e}^{-\mathrm{i}\varphi}. \end{aligned}$$

This equality implies that

$$|w_1|\Big|_{|z-a_1|=R_1}=\left|\frac{a-a_1}{R_1}\right|=\frac{a-a_1}{R_1}=:r_1$$

Similarly we obtain that

$$|w_1|\Big|_{|z-a_2|=R_2}=\left|\frac{a-a_2}{R_2}\right|=\frac{a-a_2}{R_2}=:r_2.$$

Let us note that for $0<R_2<R_1$ it follows that $r_2<r_1$ since $b>a$. The next step is: we consider

$$w_2=\log w_1$$

with the main branch of logarithm. Under this transformation this symmetric (or concentric) annulus is transferred conformally to the set

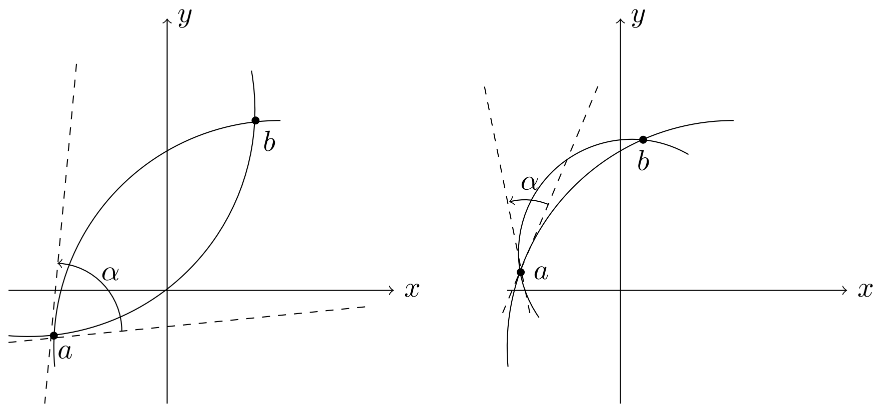
$$\{w_2:\log r_2<\operatorname{Re} w_2<\log r_1\}.$$

Using now Example 1.8 we may conclude that the required conformal mapping is given by

$$w=\mathrm{e}^{\mathrm{i}\frac{\pi}{\log(r_1/r_2)}}\left(\log\frac{z-a}{z-b}-\log r_2\right),$$

where a and b are from (1.9).

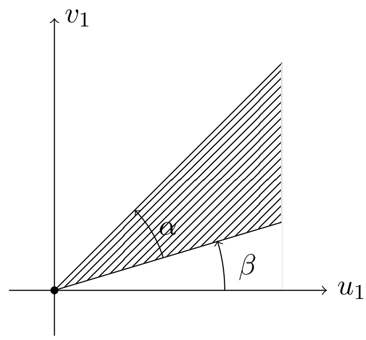
Example 1.20. Let us find the conformal mapping of the crescent shape (lune) formed by two arcs of two different circles.



We consider first

$$w_1 = \frac{z-a}{z-b},$$

where a and b are the two intersecting points of these circles. Then this conformal mapping transfers this lune to the angle of span α (this angle is the same as for lune due to conformality), with the vertex in the origin.



Indeed, if $z = \rho_0 e^{i\varphi}, \varphi_0 \leq \varphi \leq \varphi_0'$ for the part of the first circle in the boundary of the lune and $z = \rho_1 e^{i\varphi}, \varphi_1 \leq \varphi \leq \varphi_1'$ for the second circle then

$$\begin{aligned} \left.\frac{z-a}{z-b}\right|_{z=\rho_0 e^{i\varphi}} &= \frac{\rho_0 e^{i\varphi} - \rho_0 e^{i\varphi_0}}{\rho_0 e^{i\varphi} - \rho_0 e^{i\varphi_0'}} = \frac{e^{i\varphi} - e^{i\varphi_0}}{e^{i\varphi} - e^{i\varphi_0'}} \\ &= e^{i(\varphi_0 - \varphi_0')} \frac{e^{i(\varphi-\varphi_0)/2} - e^{-i(\varphi-\varphi_0)/2}}{e^{i(\varphi-\varphi_0')/2} - e^{-i(\varphi-\varphi_0')/2}} = e^{i(\varphi_0 - \varphi_0')} \frac{\sin(\varphi-\varphi_0)/2}{\sin(\varphi-\varphi_0')/2}. \end{aligned}$$

Similarly

$$\left.\frac{z-a}{z-b}\right|_{z=\rho_1 e^{i\varphi}} = e^{i(\varphi_1 - \varphi_1')} \frac{\sin(\varphi-\varphi_1)/2}{\sin(\varphi-\varphi_1')/2}.$$

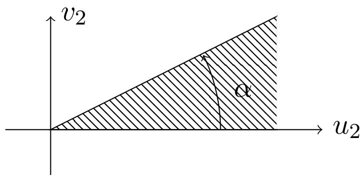
These formulae show that the arcs are mapped to the rays starting from the origin because

$$\frac{v_1}{u_1} = \tan(\varphi_0 - \varphi_0'), \quad \frac{v_1}{u_1} = \tan(\varphi_1 - \varphi_1'),$$

respectively for these two arcs. Next,

$$w_2 = \mathrm{e}^{-\mathrm{i}\beta} w_1$$

maps conformally the sector $\{w_1 : \beta < \arg w_1 < \alpha + \beta\}$ to the sector $\{w_2 : 0 < \arg w_2 < \alpha\}$.



Finally,

$$w = w_2^{\pi/\alpha} = \left(\mathrm{e}^{-\mathrm{i}\beta} \frac{z-a}{z-b}\right)^{\pi/\alpha}$$

maps conformally the latter sector onto the domain $\{w : \operatorname{Im} w > 0\}$. Indeed,

$$w = w_2^{\pi/\alpha} = \mathrm{e}^{\frac{\pi}{\alpha}(\log|w_2| + \mathrm{i}\arg w_2)} = \mathrm{e}^{\mathrm{i}\frac{\pi}{\alpha}\arg w_2} \mathrm{e}^{\frac{\pi}{\alpha}\log|w_2|}.$$

This is equivalent that $\arg w = \frac{\pi}{\alpha}\arg w_2 \in (0, \pi)$, and $\operatorname{Re} w \in (-\infty, \infty), \operatorname{Im} w > 0$. Here we have used the boundary correspondence principle.

Problem 1.21. Show that the Zhukovski function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

maps conformally

1. $\{z : |z| < 1\}$ onto $\mathbb{C} \setminus [-1, 1]$,
2. $\{z : |z| < 1, \operatorname{Im} z < 0\}$ onto $\{w : \operatorname{Im} w > 0\}$ and
3. $\{z : |z| > 1\}$ onto $\mathbb{C} \setminus [-1, 1]$.

There is an application of conformal mappings also in the theory of partial differential equations.

Let D be a simply-connected and bounded domain on the complex plane $\mathbb{C}$.

Definition 1.22. A function $G(z, \zeta)$ is said to be *Green's function* for the Laplace operator Δ in the domain D if the following conditions are satisfied:

1. $G(z,\zeta) = \frac{1}{2\pi}\log|z-\zeta| + g(z,\zeta)$ for $z,\zeta \in D$
2. $\Delta_z g(z,\zeta) = 0$ for $z,\zeta \in D$
3. $g(z,\zeta) = -\frac{1}{2\pi}\log|z-\zeta|$ for $z \in \partial D, \zeta \in D$.

Remark. This definition implies (in particular) that $G(z,\zeta) = 0$ for $z \in \partial D$ and $\zeta \in D$.

With the Green's function in hand, the solution of the *inhomogeneous boundary value problem*

$$\begin{cases} \Delta u(z) = F(z), & z \in D \\ u(z) = u_0(z), & z \in \partial D \end{cases}$$

is given by the superposition principle as

$$u(z) = \int_D G(z,\zeta)F(\zeta)\mathrm{d}\xi\mathrm{d}\eta + \int_{\partial D} \partial_{\nu_\zeta} G(z,\zeta)\mathrm{d}\sigma(\xi,\eta),$$

where $z = x + \mathrm{i}y, \zeta = \xi + \mathrm{i}\eta$ and ∂_ν is the outward normal derivative with respect to ζ on the boundary ∂D.

Using the principles of conformal mappings we may construct the Green's function for arbitrary simply-connected bounded domain D. Indeed, let ζ be an arbitrary fixed point from D. Let $h(z,\zeta)$ be a function which maps conformally D onto the unit disc $\{w : |w| < 1\}$ such that $h(\zeta,\zeta) = 0$. This function exists due to Riemann Mapping Theorem (see Theorem 1.7). Moreover, $h'_z(z,\zeta) \neq 0$ for all $z \in D$ (see Theorem 1.2 of this Part and Problem 7.10 of Part II). Hence, $h(z,\zeta)$ has a zero of order 1 at $z = \zeta$. This fact allows us to represent $h(z,\zeta)$ in the form

$$h(z,\zeta) = (z-\zeta)\psi(z,\zeta), \quad \psi(\zeta,\zeta) \neq 0.$$

It implies that

$$\frac{1}{2\pi}\log|h(z,\zeta)| = \frac{1}{2\pi}\log|z-\zeta| + g(z,\zeta),$$

where $g(z,\zeta) = \frac{1}{2\pi}\log|\psi(z,\zeta)|$. We prove that

$$G(z,\zeta) := \frac{1}{2\pi}\log|h(z,\zeta)|$$

is the Green's function for Δ in D. Indeed, since $h(z,\zeta) \in H(D)$ (ζ is a parameter) and $h'_z(z,\zeta) \neq 0$ for all $z \in D$ then $\psi(z,\zeta) \neq 0$ for all $z \in D$ and analytic there. Thus $g(z,\zeta) = \frac{1}{2\pi}\log|\psi(z,\zeta)|$ is harmonic in D (see Problem 2.2 of Part II). Next, since $|h(z,\zeta)| = 1$ for all $z \in \partial D$ and for all $\zeta \in D$ (see Theorem 1.7) then

$$g(z,\zeta) = -\frac{1}{2\pi}\log|z-\zeta|, \quad z \in \partial D, \zeta \in D.$$

This proves that $G(z,\zeta)$ is the needed Green's function.

Problem 1.23. Show that the Green's function for the unit disc is given by

$$G(z,\zeta) = \frac{1}{2\pi} \log \left| \frac{z-\zeta}{z\bar{\zeta}-1} \right|$$

Hint: Use the fact that a non-degenerate linear-fractional transformation

$$w = \frac{z-\zeta}{z\bar{\zeta}-1}, \quad |\zeta|, |z| < 1$$

maps conformally unit disc onto itself such that $w = 0$ for $z = \zeta$.

Problem 1.24. Using Problem 1.23 show that the Green's function for simply-connected bounded domain D can be written as

$$G(z,\zeta) = \frac{1}{2\pi} \log \left| \frac{g(\zeta)-g(z)}{g(z)\overline{g(\zeta)}-1} \right|,$$

where g maps conformally D onto the unit disc.

Chapter 2

Laplace transform

Let f be a function (possibly complex-valued) of one real variable t. We denote by $\mathcal{F}^+$ the class of functions (and write $f \in \mathcal{F}^+$) which satisfy the conditions

1. $f(t) \equiv 0, t < 0$
2. $f(t)$ is continuous for $t \geq 0$
3. there exists $M > 0$ and $a > 0$ such that $|f(t)| \leq M\mathrm{e}^{at}$ for any $t \geq 0$.

The value $s := \inf a$ is called the *growth index* of f.

Problem 2.1. Show that if the growth index of $f \in \mathcal{F}^+$ is equal to $s \geq 0$ then the growth index of $t^\mu f(t)$ for any $\mu \geq 0$ is also equal to s. In particular, the growth index of t^μ for any $\mu > 0$ is equal to zero.

Definition 2.2. Let f be a function from the class $\mathcal{F}^+$. The *Laplace transform* of f, denoted by $\mathcal{L}(f)(p)$ is defined by

$$\mathcal{L}(f)(p) := \int_0^\infty \mathrm{e}^{-pt} f(t)\mathrm{d}t, \quad p \in \mathbb{C}. \tag{2.1}$$

Theorem 2.3 (Existence)**.** *Suppose $f \in \mathcal{F}^+$ with growth index $s \geq 0$. Then the Laplace tranform $\mathcal{L}(f)(p)$ is well-defined analytic function in the domain $\{p \in \mathbb{C} : \operatorname{Re} p > s\}$. Moreover,*

$$\lim_{\operatorname{Re} p \to +\infty} \mathcal{L}(f)(p) = 0 \tag{2.2}$$

uniformly with respect to $\operatorname{Im} p \in \mathbb{R}$.

Proof. Let $p = x + \mathrm{i}y$ and $f \in \mathcal{F}^+$ with growth index $s \geq 0$. Then for any $\varepsilon > 0$ there is $M_\varepsilon > 0$ such that $|f(t)| \leq M_\varepsilon \mathrm{e}^{(s+\varepsilon)t}, t \geq 0$. It implies for any fixed

$x = \mathrm{Re}\, p > s$ that

$$|\mathcal{L}(f)(p)| \le \left| \int_0^\infty \mathrm{e}^{-(x+\mathrm{i}y)t} f(t)\mathrm{d}t \right| \le \int_0^\infty \mathrm{e}^{-xt}|f(t)|\mathrm{d}t$$
$$\le M_\varepsilon \int_0^\infty \mathrm{e}^{-(x-s-\varepsilon)t}|f(t)|\mathrm{d}t = \frac{M_\varepsilon}{x-s-\varepsilon} \qquad (2.3)$$

if ε is chosen such that $0 < \varepsilon < x - s$. This proves well-posedness of (2.1) for $\mathrm{Re}\, p > s$. In addition, (2.3) shows that the integral in (2.1) converges uniformly for all $x = \mathrm{Re}\, p \ge s_0 > s$. Let us prove now that $\mathcal{L}(f)(p)$ is analytic in the domain $\{p \in \mathbb{C} : \mathrm{Re}\, p > s\}$. If p_0 and Δp are chosen so that $\mathrm{Re}\, p_0, \mathrm{Re}(p_0 + \Delta p) > s$ then

$$\frac{\mathcal{L}(f)(p_0 + \Delta p) - \mathcal{L}(f)(p_0)}{\Delta p} = \int_0^\infty \mathrm{e}^{-p_0 t} f(t) \frac{\mathrm{e}^{-\Delta p t} - 1}{\Delta p} \mathrm{d}t.$$

But it is known that

$$\lim_{\Delta p \to 0} \frac{\mathrm{e}^{-\Delta p t} - 1}{\Delta p} = -t.$$

Due to this fact, Problem 2.1 and the fact that the integral in (2.1) converges uniformly for $\mathrm{Re}\, p \ge s_0 > s$ we may consider the limit $\Delta p \to 0$ under the integral sign. Hence we obtain the existence of the limit

$$\lim_{\Delta p \to 0} \frac{\mathcal{L}(f)(p_0 + \Delta p) - \mathcal{L}(f)(p_0)}{\Delta p} = \int_0^\infty \mathrm{e}^{-p_0 t} f(t) \lim_{\Delta p \to 0} \frac{\mathrm{e}^{-\Delta p t} - 1}{\Delta p} \mathrm{d}t$$
$$= -\int_0^\infty \mathrm{e}^{-p_0 t} t f(t) \mathrm{d}t = -\mathcal{L}(tf)(p_0).$$

The latter formula proves the analyticity of $\mathcal{L}(f)(p)$ for all $\mathrm{Re}\, p > s$ and also the equality

$$\mathcal{L}(tf)(p) = -(\mathcal{L}(f))'(p). \qquad (2.4)$$

Finally, (2.2) follows from (2.3) straightforwardly. □

Corollary 2.4. *Formula* (2.4) *can be generalized as*

$$\mathcal{L}(t^n f)(p) = (-1)^n (\mathcal{L}(f))^{(n)}(p), \quad n = 1, 2, \ldots. \qquad (2.5)$$

Proof. Follows from (2.4) by induction using the fact that any analytic function is infinitely many times differentiable. □

Example 2.5. Let us show that

$$\mathcal{L}(t^n)(p) = \frac{n!}{p^{n+1}}, \quad \mathrm{Re}\, p > 0 \qquad (2.6)$$

for any $n = 0, 1, 2, \ldots$. Indeed, Problem 2.1 gives that for each $n = 0, 1, 2, \ldots$ the growth index of $t^n \in \mathcal{F}^+$ is equal to zero. Formula (2.5) yields

$$\begin{aligned}\mathcal{L}(t^n)(p) &= (-1)^n(\mathcal{L}(1))^{(n)}(p) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}p^n}\int_0^\infty \mathrm{e}^{-pt}\mathrm{d}t = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}p^n}\frac{1}{p}\\ &= (-1)^n \frac{(-1)^n n!}{p^{n+1}} = \frac{n!}{p^{n+1}}\end{aligned}$$

for $\operatorname{Re} p > 0$ and $n = 0, 1, 2, \ldots$.

Problem 2.6. Generalize (2.6) and show that, for $\nu \geq 0$,

$$\mathcal{L}(t^\nu)(p) = \frac{\Gamma(\nu+1)}{p^{\nu+1}}, \quad \operatorname{Re} p > 0,$$

where Γ is Euler's gamma function and $p^{\nu+1}$ is the multi-valued analytic function given by

$$p^{\nu+1} = pp^\nu = p\mathrm{e}^{\nu \log p} = p\mathrm{e}^{\nu[\log|p| + \mathrm{i}\operatorname{Arg} p]} = p|p|^\nu \mathrm{e}^{\mathrm{i}\nu \operatorname{Arg} p}.$$

Example 2.7. Let $f \in \mathcal{F}^+$ and $f(t) = \mathrm{e}^{\alpha t}, t \geq 0$ with $\operatorname{Re}\alpha \geq 0$. Then, by definition,

$$\mathcal{L}(\mathrm{e}^{\alpha t})(p) = \int_0^\infty \mathrm{e}^{-(p-\alpha)t}\mathrm{d}t = \frac{1}{p-\alpha}, \quad \operatorname{Re} p > \operatorname{Re}\alpha \tag{2.7}$$

is well-defined in the domain $\{p : \operatorname{Re} p > \operatorname{Re}\alpha\}$. In particular, for real ω we have

$$\mathcal{L}(\mathrm{e}^{\mathrm{i}\omega t}) = \frac{1}{p - \mathrm{i}\omega}, \quad \mathcal{L}(\sin\omega t) = \frac{\omega}{p^2+\omega^2}, \quad \mathcal{L}(\cos\omega t) = \frac{p}{p^2+\omega^2} \tag{2.8}$$

for $\operatorname{Re} p > 0$.

Remark. For $\operatorname{Re}\alpha < 0$ we have $|\mathrm{e}^{\alpha t}| \leq 1$ for $t \geq 0$ and therefore the growth index is $s = 0$. In that case (2.7) holds for $\operatorname{Re} p > 0$ (even for $\operatorname{Re} p \geq 0$).

Problem 2.8. 1. Show that if $f \in \mathcal{F}^+$ is periodic with period $T > 0$ then

$$\mathcal{L}(f)(p) = \frac{1}{1-\mathrm{e}^{-pT}}\int_0^T \mathrm{e}^{-pt}f(t)\mathrm{d}t, \quad \operatorname{Re} p > 0.$$

2. Show that if $a > 0$ then

$$\mathcal{L}(\sinh(at)) = \frac{a}{p^2-a^2}, \quad \mathcal{L}(\cosh(at)) = \frac{p}{p^2-a^2}$$

for $\operatorname{Re} p > a$.

3. Show that if $a > 0$ then
$$\mathcal{L}\left(\frac{\sinh(at)}{t}\right) = \frac{1}{2}\log\frac{p+a}{p-a}$$
for $\operatorname{Re} p > a$.

4. Show that if $f, g \in \mathcal{F}^+$ and $tf(t) = g'(t)$ then
$$\mathcal{L}(f)(p) = \int_p^\infty z\mathcal{L}(g)(z)\mathrm{d}z,$$
where the integral on the right hand side is a primitive (with minus sign) for the analytic function $z\mathcal{L}(g)(z)$. In particular,
$$\mathcal{L}(tf)(p) = \mathcal{L}(g')(p) = -(\mathcal{L}(f))'(p) = p\mathcal{L}(g)(p).$$

5. Show that if $f, g \in \mathcal{F}^+$ and $f(t) = \int_t^\infty g(\tau)\mathrm{d}\tau$ then
$$\mathcal{L}(f)(p) = -\frac{1}{p}\mathcal{L}(g)(p), \quad \operatorname{Re} p > 0.$$

Definition 2.9. Let $f_1, f_2 \in \mathcal{F}^+$. The *convolution* $g := f_1 * f_2 = f_2 * f_1$ of f_1 and f_2 is defined by
$$g(t) = \int_0^t f_1(\tau)f_2(t-\tau)\mathrm{d}\tau = \int_0^t f_2(\tau)f_1(t-\tau)\mathrm{d}\tau. \tag{2.9}$$

Remark. The growth index of $g = f_1 * f_2$ is $\max(s_1, s_2)$, where s_1 and s_2 are the growth indices of f_1 and f_2, respectively.

We collect some properties of the Laplace transform in class $\mathcal{F}^+$ in the following theorem.

Theorem 2.10. *1. Let $f_k \in \mathcal{F}^+$ with growth indices $s_k \geq 0$ for $k = 1, 2, \ldots, m$. Then $f(t) := \sum_{k=1}^m c_k f_k(t), c_k \in \mathbb{C}$ belongs to the class $\mathcal{F}^+$ with the growth index $s = \max(s_1, \ldots, s_m)$ and*
$$\mathcal{L}(f)(p) = \sum_{k=1}^m c_k\mathcal{L}(f_k)(p), \quad \operatorname{Re} p > s.$$

2. Let f_1 and f_2 have growth indices s_1 and s_2, respectively. Then $g = f_1 * f_2 \in \mathcal{F}^+$ with the growth index $s = \max(s_1, s_2)$ and
$$\mathcal{L}(g)(p) = \mathcal{L}(f_1 * f_2)(p) = \mathcal{L}(f_1)(p)\mathcal{L}(f_2)(p), \quad \operatorname{Re} p > s. \tag{2.10}$$

3. Let $f \in \mathcal{F}^+$ with the growth index s and let $f \in C^{(n)}[0,\infty)$. Then $\mathcal{L}(f^{(n)})(p)$ exists for $\operatorname{Re} p > s$ *and*

$$\mathcal{L}(f^{(n)})(p) = p^n \left[\mathcal{L}(f)(p) - \frac{f(0)}{p} - \cdots - \frac{f^{(n-1)}(0)}{p^n}\right]. \tag{2.11}$$

4. If $f \in \mathcal{F}^+$ with the growth index $s \geq 0$ and $\lambda \in \mathbb{C}$ then

$$\mathcal{L}(\mathrm{e}^{-\lambda t} f)(p) = \mathcal{L}(f)(p+\lambda), \quad \operatorname{Re} p > \max(0, s - \operatorname{Re}\lambda). \tag{2.12}$$

Proof. 1. Follows from the linearity of integral and from the fact that for two functions f_1 and f_2 with growth indices s_1 and s_2 the growth index of the sum $f_1 + f_2$ is $\max(s_1, s_2)$.

2. By the definition of convolution we have for $\varepsilon > 0$ small enough that

$$\begin{aligned}
|g(t)| &\leq M_\varepsilon^{(1)} M_\varepsilon^{(2)} \int_0^t \mathrm{e}^{(s_1+\varepsilon)\tau} \mathrm{e}^{(s_1+\varepsilon)(t-\tau)} \mathrm{d}\tau \\
&= M_\varepsilon^{(1)} M_\varepsilon^{(2)} \mathrm{e}^{(s_2+\varepsilon)t} \int_0^t \mathrm{e}^{(s_1-s_2)\tau} \mathrm{d}\tau \\
&= M_\varepsilon^{(1)} M_\varepsilon^{(2)} \mathrm{e}^{(s_2+\varepsilon)t} \frac{\mathrm{e}^{(s_1-s_2)t} - 1}{s_1 - s_2} \\
&= M_\varepsilon^{(1)} M_\varepsilon^{(2)} (\mathrm{e}^{(s_1+\varepsilon)t} - \mathrm{e}^{(s_2+\varepsilon)t}) \frac{1}{s_1 - s_2} \leq M_\varepsilon^{(1)} M_\varepsilon^{(2)} \frac{\mathrm{e}^{(s_1+\varepsilon)t} + \mathrm{e}^{(s_2+\varepsilon)t}}{|s_1 - s_2|}
\end{aligned}$$

for $s_1 \neq s_2$. This shows that the growth index s for g is equal to $\max(s_1, s_2)$. Next, for $\operatorname{Re} p > s$ we have that

$$\begin{aligned}
\mathcal{L}(g)(p) &= \int_0^\infty \mathrm{e}^{-pt} \left(\int_0^t f_1(\tau) f_2(t-\tau) \mathrm{d}\tau \right) \mathrm{d}t \\
&= \int_0^\infty f_1(\tau) \int_\tau^\infty \mathrm{e}^{-pt} f_2(t-\tau) \mathrm{d}t \mathrm{d}\tau \\
&= \int_0^\infty f_1(\tau) \int_0^\infty \mathrm{e}^{-p(\xi+\tau)} f_2(\xi) \mathrm{d}\xi \mathrm{d}\tau \\
&= \int_0^\infty \mathrm{e}^{-p\tau} f_1(\tau) \int_0^\infty \mathrm{e}^{-p\xi} f_2(\xi) \mathrm{d}\xi \mathrm{d}\tau = \mathcal{L}(f_1)(p) \mathcal{L}(f_2)(p).
\end{aligned}$$

We have used here Fubini's theorem and the fact that $\operatorname{Re} p > s = \max(s_1, s_2)$. For the case $s_1 = s_2$ the proof is similar.

3. We proceed by induction with respect to n. For $n = 1$ we assume that $f \in \mathcal{F}^+$ with growth index s and $f' \in C[0,\infty)$. Then for $\operatorname{Re} p > s$ we

obtain formally by integration by parts that

$$\mathcal{L}(f')(p) = \int_0^\infty \mathrm{e}^{-pt} f'(t)\mathrm{d}t$$
$$= \mathrm{e}^{-pt} f(t)\Big|_0^\infty + p\int_0^\infty \mathrm{e}^{-pt} f(t)\mathrm{d}t = -f(0) + p\mathcal{L}(f)(p).$$

The right hand side exists and is finite due to the fact that $f \in \mathcal{F}^+$ with growth index $s \geq 0$ and $\operatorname{Re} p > s$. This proves (2.11) for $n = 1$. Let us assume that (2.11) holds for any $n \geq 1$. Then by induction hypothesis we may write

$$\mathcal{L}(f^{(n+1)})(p) = \mathcal{L}((f^{(n)})')(p) = -f^{(n)}(0) + p\mathcal{L}(f^{(n)})(p)$$
$$= -f^{(n)}(0) + p\left(p^n\left[\mathcal{L}(f)(p) - \frac{f(0)}{p} - \cdots - \frac{f^{(n-1)}(0)}{p^n}\right]\right)$$
$$= p^{n+1}\left[\mathcal{L}(f)(p) - \frac{f(0)}{p} - \cdots - \frac{f^{(n)}(0)}{p^{n+1}}\right].$$

This proves (2.11) by induction.

4. If $f \in \mathcal{F}^+$ with growth index s then for any $\varepsilon > 0$ there is $M_\varepsilon > 0$ such that

$$|\mathrm{e}^{-\lambda t} f(t)| = \mathrm{e}^{-t\operatorname{Re}\lambda}|f(t)|$$
$$\leq M_\varepsilon \mathrm{e}^{(s+\varepsilon)t - t\operatorname{Re}\lambda} \leq M_\varepsilon \begin{cases} \mathrm{e}^{(s+\varepsilon-\operatorname{Re}\lambda)t}, & s > \operatorname{Re}\lambda \\ \mathrm{e}^{\varepsilon t}, & s \leq \operatorname{Re}\lambda. \end{cases}$$

This means that the growth index for $\mathrm{e}^{-\lambda t} f(t)$ is equal to $s_\lambda := \max(0, s - \operatorname{Re}\lambda)$. Next,

$$\mathcal{L}(\mathrm{e}^{-\lambda t} f(t))(p) = \int_0^\infty \mathrm{e}^{-(p+\lambda)t} f(t)\mathrm{d}t = \mathcal{L}(f(t))(p+\lambda)$$

for $\operatorname{Re} p > s_\lambda$.

□

The next result shows how we can recover the original function $f \in \mathcal{F}^+$ if its Laplace transform is known.

Theorem 2.11 (Mellin's formula)**.** *Let $\mathcal{L}(f)(p)$ be the Laplace transform of $f \in \mathcal{F}^+$ with growth index $s \geq 0$. Then*

$$f(t) = \lim_{A\to+\infty} \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} p - \mathrm{i}A}^{\operatorname{Re} p + \mathrm{i}A} \mathrm{e}^{pt} \mathcal{L}(f)(p)\mathrm{d}p$$
$$= \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} p - \mathrm{i}\infty}^{\operatorname{Re} p + \mathrm{i}\infty} \mathrm{e}^{pt} \mathcal{L}(f)(p)\mathrm{d}p =: \mathcal{L}^{-1}(\mathcal{L}(f))(t), \quad (2.13)$$

where the integration is carried out over the line for fixed $\operatorname{Re} p$ *such that* $\operatorname{Re} p > s$ *and where* $\mathcal{L}^{-1}$ *denotes the* inverse Laplace transform.

Proof. Let us define

$$\varphi(t) = \mathrm{e}^{-xt} f(t), \quad x > s.$$

Since $x > s$ then for any $0 < \varepsilon < x - s$ we have

$$|\varphi(t)| \le M_\varepsilon \mathrm{e}^{-(x-s-\varepsilon)t}.$$

It means that φ tends to zero as $t \to +\infty$ exponentially and $\varphi(t) \equiv 0$ for $t < 0$. Using now the Fourier inversion formula

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(\eta) \mathrm{e}^{\mathrm{i}\xi(t-\eta)} \mathrm{d}\eta \mathrm{d}\xi$$

we obtain

$$\begin{aligned} \mathrm{e}^{-xt} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-x\eta} f(\eta) \mathrm{e}^{\mathrm{i}\xi(t-\eta)} \mathrm{d}\eta \mathrm{d}\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-(x+\mathrm{i}\xi)\eta} f(\eta) \mathrm{e}^{\mathrm{i}\xi t} \mathrm{d}\eta \mathrm{d}\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}(f)(x+\mathrm{i}\xi) \mathrm{e}^{\mathrm{i}\xi t} \mathrm{d}\xi. \end{aligned}$$

So

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}(f)(x+\mathrm{i}\xi) \mathrm{e}^{(x+\mathrm{i}\xi)t} \mathrm{d}\xi = \frac{1}{2\pi \mathrm{i}} \int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty} \mathcal{L}(f)(x+\mathrm{i}\xi) \mathrm{e}^{(x+\mathrm{i}\xi)t} \mathrm{d}(\mathrm{i}\xi),$$

where the integral is understood in the sense of principal value at infinity (as in Fourier inversion formula). This proves (2.13). □

Remark. Formula (2.13) shows that the result of inversion is actually independent on $\operatorname{Re} p$ if $\operatorname{Re} p > s$.

Example 2.12. Let us evaluate the inverse Laplace transform of the function

$$\frac{1}{p^3(p^2+1)}, \quad \operatorname{Re} p > 0.$$

Using (2.10) and Examples 2.5 and 2.7 we have

$$\frac{1}{p^3(p^2+1)} = \frac{1}{p^3} \frac{1}{p^2+1} = \mathcal{L}\left(\frac{t^2}{2}\right) \mathcal{L}(\sin t) = \mathcal{L}\left(\frac{t^2}{2} * \sin t\right).$$

Therefore

$$\mathcal{L}^{-1}\left(\frac{1}{p^3(p^2+1)}\right) = \int_0^t \frac{\tau^2}{2} \sin(t-\tau) \mathrm{d}\tau = \frac{t^2}{2} + \cos t - 1.$$

Example 2.13. Let us evaluate the inverse Laplace transform of the function

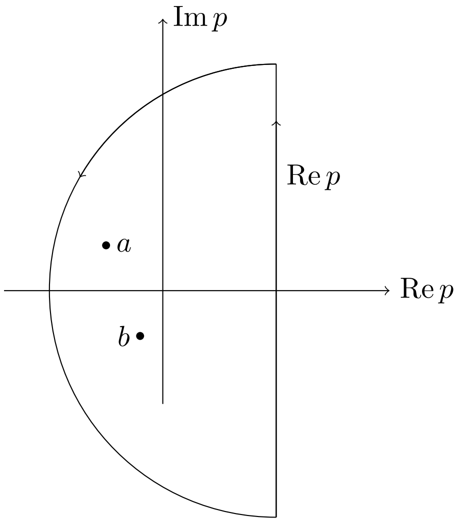
$$\frac{p}{(p+a)(p+b)}, \quad a, b \in \mathbb{C}.$$

Let us first assume that $a \neq b$. Then the Mellin's formula reads as

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} p - \mathrm{i}\infty}^{\operatorname{Re} p + \mathrm{i}\infty} \mathrm{e}^{pt} \frac{p}{(p+a)(p+b)} \mathrm{d}p.$$

Using now Jordan's lemma in the left half plane (see Corollary 8.13 in Part II), where $\operatorname{Re} p > -\operatorname{Re} a, \operatorname{Re} p > -\operatorname{Re} b$ and $\operatorname{Re} p > 0$ we obtain

$$f(t) = \operatorname*{Res}_{p=-a} \frac{p\mathrm{e}^{pt}}{(p+a)(p+b)} + \operatorname*{Res}_{p=-b} \frac{p\mathrm{e}^{pt}}{(p+a)(p+b)} = \frac{b\mathrm{e}^{-bt} - a\mathrm{e}^{-at}}{b-a}.$$



For the second case $a = b$ we may proceed by the same manner or use limiting process $b \to a$ in the latter formula to obtain that

$$f(t) = \mathrm{e}^{-at} - at\mathrm{e}^{-at}.$$

Problem 2.14. Using Mellin's formula find the inverse Laplace transforms of the following functions:

1. $F(p) = \frac{1}{p^4-1}, \operatorname{Re} p > 1$
2. $F(p) = \frac{p}{(p-1)^2}, \operatorname{Re} p > 1$
3. $F(p) = \frac{\mathrm{e}^{-ap} - \mathrm{e}^{-bp}}{p}, 0 \leq a < b, \operatorname{Re} p > 0$
4. $F(p) = \frac{\mathrm{e}^{-ap} - \mathrm{e}^{-bp}}{1+p^2}, 0 \leq a < b, \operatorname{Re} p > 0$

5. $F(p) = \log \frac{p+b}{p+a}, a \neq b, \operatorname{Re} p > \max(0, -\operatorname{Re} a, -\operatorname{Re} b)$

6. $F(p) = p \log \frac{p^2-a^2}{p^2}, a > 0, \operatorname{Re} p > 0.$

Problem 2.15. Show that

$$\mathcal{L}^{-1}(FG) = \mathcal{L}^{-1}(F) * \mathcal{L}^{-1}(G),$$

where F and G satisfy all conditions of Theorem 2.11.

The next theorem (given here without proof) characterizes the set of analytic functions that are Laplace transforms of some function from the class $\mathcal{F}^+$.

Theorem 2.16. *Let $F(p)$ be a function of complex variable p which satisfies the conditions:*

1. *$F(p)$ is analytic for $\operatorname{Re} p > s \geq 0$*

2. *$\lim_{|p| \to +\infty} F(p) = 0$ uniformly in $\arg p$ with $\operatorname{Re} p > s$*

3. *for any $x > s$ we have*

$$\int_{-\infty}^{\infty} |F(x + \mathrm{i}y)| \mathrm{d}y < \infty.$$

Then for any fixed $\operatorname{Re} p > s$ there exists the limit

$$\lim_{A \to +\infty} \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} p - \mathrm{i}A}^{\operatorname{Re} p + \mathrm{i}A} \mathrm{e}^{pt} F(p) \mathrm{d}p =: f(t)$$

such that $F(p) = \mathcal{L}(f)(p)$.

We consider now applications of Laplace transform to differential equations with constant coefficients and to some class of integral equations. Let us consider the initial value problem (or Cauchy problem) of the form

$$\begin{aligned} a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = f(t), \quad t > 0 \\ y(0) = y_0, y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}, \end{aligned} \tag{2.14}$$

where a_j, y_j are given complex constants ($a_0 \neq 0$) and f is a given function. The task is to determine $y(t)$. Due to linearity of (2.14) this problem can be represented as the sum of two separate problems: (a) with homogeneous equation ($f = 0$) and (b) with homogeneous initial conditions ($y_j = 0$). Next, in order to solve problem (a) it suffices to find the fundamental system of solutions i.e. the system $\{\varphi_j(t)\}_{j=0}^{n-1}$ such that

$$a_0 \varphi_j^{(n)}(t) + a_1 \varphi_j^{(n-1)}(t) + \cdots + a_n \varphi_j(t) = 0, \quad j = 0, 1, \ldots, n-1$$

with

$$\varphi_j^{(k)}(0) = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \tag{2.15}$$

for $k = 1, 2, \ldots, n-1$. In that case the solution of (a) is given by

$$u(t) = \sum_{j=0}^{n-1} y_j \varphi_j(t), \tag{2.16}$$

where the constants y_j are from (2.14). Since we know that, see (2.11),

$$\mathcal{L}(\varphi_j^{(k)})(p) = p^k \left[F_j(p) - \frac{\varphi_j(0)}{p} - \cdots - \frac{\varphi_j^{(k-1)}(0)}{p^k} \right], \quad F_j = \mathcal{L}(\varphi_j)$$

then (2.16) implies

$$\mathcal{L}(\varphi_j^{(k)})(p) = \begin{cases} p^k F_j(p), & k \leq j \\ p^k [F_j(p) - \frac{1}{p^{j+1}}], & k > j. \end{cases} \tag{2.17}$$

Using (2.17) and applying the Laplace transform to the homogeneous equation from (2.14) we obtain

$$\begin{aligned} a_0 p^n \left[F_j(p) - \frac{1}{p^{j+1}} \right] + a_1 p^{n-1} \left[F_j(p) - \frac{1}{p^{j+1}} \right] + \cdots \\ + a_{n-j-1} p^{j+1} \left[F_j(p) - \frac{1}{p^{j+1}} \right] + a_{n-j} p^j F_j(p) + \cdots + a_n F_j(p) = 0. \end{aligned}$$

This equation can be rewritten as

$$F_j(p) = \frac{Q_j(p)}{P_n(p)}, \quad j = 0, 1, 2, \ldots, n-1, \tag{2.18}$$

where $P_n(p) = a_0 p^n + a_1 p^{n-1} + \cdots + a_n$ is the characteristic polynomial of the differential operator from (2.14) and

$$Q_j(p) = a_0 p^{n-j-1} + a_1 p^{n-j-2} + \cdots + a_{n-j-1}, \quad j = 0, 1, \ldots, n-1. \tag{2.19}$$

To solve (2.18) with respect to $\mathcal{L}^{-1}(F_j(p))(t)$ we apply Mellin's formula for fixed $\operatorname{Re} p > s$, where $s \geq 0$ is to the right of all singular points of $Q_j(p)/P_n(p)$. We obtain

$$\varphi_j(t) = \mathcal{L}^{-1}(F_j)(t) = \frac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} p - \mathrm{i}\infty}^{\operatorname{Re} p + \mathrm{i}\infty} \mathrm{e}^{pt} \frac{Q_j(p)}{P_n(p)} \mathrm{d}p.$$

Jordan's lemma in the left half plane gives

$$\varphi_j(t) = \sum_{l=1}^{m} \operatorname*{Res}_{p=p_l} \left(\mathrm{e}^{pt} \frac{Q_j(p)}{P_n(p)} \right), \tag{2.20}$$

where $p_l, l = 1, 2, \ldots, m$ are the singular points of $Q_j(p)/P_n(p)$. Now the problem (a) is solved by (2.16) and (2.20).

For solving the problem (b), i.e. the problem (2.14) with non-homogeneous equation ($f \neq 0$) and with homogeneous initial conditions ($y_j = 0$) we use (2.11) and easily obtain

$$P_n(p)\mathcal{L}(v)(p) = \mathcal{L}(f)(p),$$

where P_n is a characteristic polynomial and v is the solution of the problem. Applying Mellin's formula gives

$$v(t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\, p - \mathrm{i}\infty}^{\mathrm{Re}\, p + \mathrm{i}\infty} \mathrm{e}^{pt} \frac{\mathcal{L}(f)(p)}{P_n(p)} \mathrm{d}p, \tag{2.21}$$

where fixed $\mathrm{Re}\, p > s \geq 0$ is to the right of all singular points of $\mathcal{L}(f)(p)/P_n(p)$. Formula (2.21) can be simplified as follows. Since $a_0 \neq 0$ then by (2.19) we have

$$\begin{aligned} \mathcal{L}(v)(p) &= \frac{1}{a_0} \frac{a_0}{P_n(p)} \mathcal{L}(f)(p) = \frac{1}{a_0} \frac{Q_{n-1}(p)}{P_n(p)} \mathcal{L}(f)(p) \\ &= \frac{1}{a_0} \mathcal{L}(\varphi_{n-1}) \mathcal{L}(f)(p) = \frac{1}{a_0} \mathcal{L}(\varphi_{n-1} * f)(p), \end{aligned}$$

where φ_{n-1} is defined in (2.20). The inverse Laplace transform yields

$$v(t) = \frac{1}{a_0} \int_0^t \varphi_{n-1}(\tau) f(t-\tau) \mathrm{d}\tau. \tag{2.22}$$

Combining (2.16) and (2.22) we see that the of solution (2.14) is given by

$$y(t) = u(t) + v(t) = \sum_{j=0}^{n-1} y_j \varphi_j(t) + \frac{1}{a_0} \int_0^t \varphi_{n-1}(\tau) f(t-\tau) \mathrm{d}\tau. \tag{2.23}$$

Example 2.17. Let us solve the initial value problem

$$y^{(4)}(t) + 2y''(t) + y(t) = 0, \quad y(0) = y'(0) = y''(0) = 0, y'''(0) = 1.$$

Formula (2.23) leads in this case to the solution $y(t) = \varphi_3(t)$. But $\varphi_3(t)$ equals

$$\begin{aligned}
\varphi_3(t) &= \operatorname*{Res}_{p=\mathrm{i}} \frac{\mathrm{e}^{pt}}{p^4+2p^2+1} + \operatorname*{Res}_{p=-\mathrm{i}} \frac{\mathrm{e}^{pt}}{p^4+2p^2+1} \\
&= \left(\mathrm{e}^{pt}\frac{1}{(p+\mathrm{i})^2}\right)'\Bigg|_{p=\mathrm{i}} + \left(\mathrm{e}^{pt}\frac{1}{(p-\mathrm{i})^2}\right)'\Bigg|_{p=-\mathrm{i}} \\
&= t\mathrm{e}^{pt}\frac{1}{(p+\mathrm{i})^2}\bigg|_{p=\mathrm{i}} - \mathrm{e}^{pt}\frac{2}{(p+\mathrm{i})^3}\bigg|_{p=\mathrm{i}} + t\mathrm{e}^{pt}\frac{1}{(p-\mathrm{i})^2}\bigg|_{p=-\mathrm{i}} - \mathrm{e}^{pt}\frac{2}{(p-\mathrm{i})^3}\bigg|_{p=-\mathrm{i}} \\
&= \frac{t\mathrm{e}^{\mathrm{i}t}}{(2\mathrm{i})^2} - \frac{2\mathrm{e}^{\mathrm{i}t}}{(2\mathrm{i})^3} + \frac{t\mathrm{e}^{-\mathrm{i}t}}{(-2\mathrm{i})^2} - \frac{2\mathrm{e}^{-\mathrm{i}t}}{(-2\mathrm{i})^3} \\
&= -\frac{t\mathrm{e}^{\mathrm{i}t}}{4} + \frac{\mathrm{e}^{\mathrm{i}t}}{4\mathrm{i}} - \frac{t\mathrm{e}^{-\mathrm{i}t}}{4} - \frac{\mathrm{e}^{-\mathrm{i}t}}{4\mathrm{i}} = -\frac{t}{2}\cos t + \frac{1}{2}\sin t.
\end{aligned}$$

Example 2.18. Let us solve the initial value problem

$$y''(t) + y(t) = \sin t, \quad y(0) = y'(0) = 0.$$

Formula (2.23) leads to the solution

$$y(t) = \int_0^t \varphi_1(\tau)\sin(t-\tau)\mathrm{d}\tau,$$

where

$$\varphi_1(t) = \operatorname*{Res}_{p=\mathrm{i}} \frac{\mathrm{e}^{pt}}{p^2+1} + \operatorname*{Res}_{p=-\mathrm{i}} \frac{\mathrm{e}^{pt}}{p^2+1} = \frac{\mathrm{e}^{\mathrm{i}t}}{2\mathrm{i}} - \frac{\mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}} = \sin t.$$

Thus,

$$\begin{aligned}
y(t) &= \int_0^t \sin\tau\sin(t-\tau)\mathrm{d}\tau = -\frac{1}{2}\int_0^t (\cos t - \cos(2\tau - t))\mathrm{d}\tau \\
&= -\frac{t}{2}\cos t + \frac{1}{2}\frac{\sin(2\tau-t)}{2}\bigg|_0^t = \frac{1}{2}\sin t - \frac{t}{2}\cos t.
\end{aligned}$$

Example 2.19. Let us solve the initial value problem

$$y''(t) + \omega^2 y(t) = \cos(\nu t), \quad y(0) = 0, y'(0) = 1, \nu, \omega \in \mathbb{C}.$$

Let first $\nu \neq \pm\omega$. Then (2.23) gives the solution as

$$y(t) = \varphi_1(t) + \int_0^t \varphi_1(\tau)\cos(\nu(t-\tau))\mathrm{d}\tau,$$

where $\varphi_1(t)$ is defined as

$$\varphi_1(t) = \operatorname*{Res}_{p=\mathrm{i}\omega} \frac{\mathrm{e}^{pt}}{p^2+\omega^2} + \operatorname*{Res}_{p=-\mathrm{i}\omega} \frac{\mathrm{e}^{pt}}{p^2+\omega^2} = \frac{\mathrm{e}^{\mathrm{i}\omega t}}{2\mathrm{i}\omega} + \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{-2\mathrm{i}\omega} = \frac{\sin(\omega t)}{\omega}.$$

For $\omega = 0$ we have $\varphi_1(t) = t$. So for $\omega \neq 0$ we get

$$y(t) = \frac{\sin(\omega t)}{\omega} + \frac{1}{\omega}\int_0^t \sin(\omega\tau)\cos(\nu(t-\tau))\mathrm{d}\tau.$$

Since $\nu \neq \pm\omega$ then the latter integral equals

$$\frac{1}{2\omega}\int_0^t \left[\left(\frac{\cos((\omega-\nu)\tau+\nu t)}{\nu-\omega}\right)' - \left(\frac{\cos((\omega+\nu)\tau-\nu t)}{\nu+\omega}\right)'\right]\mathrm{d}\tau$$
$$= \frac{1}{2}\left[\frac{\cos(\omega t)}{\nu-\omega} - \frac{\cos(\nu t)}{\nu-\omega} - \frac{\cos(\omega t)}{\nu+\omega} + \frac{\cos(\nu t)}{\nu+\omega}\right] = \frac{\cos(\omega t)-\cos(\nu t)}{\nu^2-\omega^2}.$$

Therefore the solution is ($\nu \neq \pm\omega \neq 0$)

$$y(t) = \frac{\sin(\omega t)}{\omega} + \frac{\cos(\omega t)-\cos(\nu t)}{\nu^2-\omega^2}.$$

If $\omega = 0$ and $\nu \neq \pm\omega$ then

$$y(t) = t + \frac{1-\cos(\nu t)}{\nu^2}.$$

In the case $\nu = \pm\omega$ we may use the limiting process to obtain

$$y(t) = \frac{\sin(\omega t)}{\omega} + \frac{t}{2}\frac{\sin(\omega t)}{\omega}.$$

Problem 2.20. Solve the problems

1. $y'(t) + by(t) = \mathrm{e}^t, y(0) = y_0$
2. $y'''(t) + y(t) = 1, y(0) = y'(0) = y''(0) = 0$
3. $y''(t) + y(t) = \sin(\omega t), y(0) = 0, y'(0) = 1$
4. $y^{(4)}(t) + 4y(t) = \sin t, y(0) = y'(0) = y''(0) = y'''(0) = 0$
5. $y''(t) + 4y'(t) + 8y = 1, y(0) = y'(0) = 0$
6. $y''(t) - y(t) = -2t(\mathrm{e}^{-t}+1), y(0) = 0, y'(0) = y_0.$

Example 2.21. Let us solve the integral equation

$$g(t) = f(t) + \lambda\int_0^t K(t-\tau)g(\tau)\mathrm{d}\tau,$$

where $g, f, K \in \mathcal{F}^+$ with the corresponding growth indices. Applying the Laplace transform we obtain

$$\mathcal{L}(g)(p) = \mathcal{L}(f)(p) + \lambda\mathcal{L}(K)(p)\mathcal{L}(g)(p).$$

So we have (formally)

$$g(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(f)}{1-\lambda\mathcal{L}(K)}\right)(t).$$

This formula can be simplified as follows (see Problem 2.15). We have

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left(\mathcal{L}(f) + \lambda\frac{\mathcal{L}(K)}{1-\lambda\mathcal{L}(K)}\mathcal{L}(f)\right)(t) \\ &= f(t) + \lambda\mathcal{L}^{-1}\left(\frac{\mathcal{L}(K)}{1-\lambda\mathcal{L}(K)}\mathcal{L}(f)\right)(t) \\ &= f(t) + \lambda\int_0^t f(t-\tau)\mathcal{L}^{-1}\left(\frac{\mathcal{L}(K)}{1-\lambda\mathcal{L}(K)}\right)(\tau)\mathrm{d}\tau. \end{aligned}$$

This formula gives the solution with any kernel $K(t)$ of the integral equation. For example, if $K(t) = \mathrm{e}^t$ then for $\operatorname{Re} p > 1$ we have

$$\mathcal{L}(K) = \frac{1}{p-1}$$

and so we may conclude that

$$\mathcal{L}^{-1}\left(\frac{\mathcal{L}(K)}{1-\lambda\mathcal{L}(K)}\right)(t) = \mathcal{L}^{-1}\left(\frac{1}{p-\lambda-1}\right)(t) = \mathrm{e}^{(\lambda+1)t}.$$

Therefore, for this particular case the solution of the integral equation

$$g(t) = f(t) + \lambda\int_0^t \mathrm{e}^{t-\tau}g(\tau)\mathrm{d}\tau$$

is equal to

$$g(t) = f(t) + \lambda\int_0^t f(t-\tau)\mathrm{e}^{(\lambda+1)\tau}\mathrm{d}\tau = f(t) + \lambda\int_0^t f(\tau)\mathrm{e}^{(\lambda+1)(t-\tau)}\mathrm{d}\tau.$$

Problem 2.22. Solve the equations

1. $f(t) = \int_0^t \mathrm{e}^{-(t-\tau)}g(\tau)\mathrm{d}\tau$
2. $g(t) = 1 - \int_0^t (t-\tau)g(\tau)\mathrm{d}\tau$
3. $f(t) = \int_0^t \sin^2(t-\tau)g(\tau)\mathrm{d}\tau$

Problem 2.23. 1. Generalize Problem 2.6 for the case $\nu > -1$. Namely, show that

$$\mathcal{L}(t^\nu)(p) = \frac{\Gamma(\nu+1)}{p^{\nu+1}}, \quad \nu > -1,$$

where $\mathcal{L}(t^\nu)(p)$ is understood as the limit

$$\mathcal{L}(t^\nu)(p) := \lim_{\delta\to+0}\int_\delta^\infty t^\nu\mathrm{e}^{-pt}\mathrm{d}t$$

which exists.

2. Using part 1. solve the integral equation

$$g(t) = f(t) + \lambda \int_0^t \frac{g(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau, \quad 0 < \alpha < 1.$$

Problem 2.24 (*Abel's equation*)**.** Let $0 < \alpha < 1$ and

$$f(t) = \int_0^t \frac{g(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau.$$

Show that

$$g(t) = \frac{\sin(\alpha\pi)}{\pi} \left(\frac{f(0)}{t^{1-\alpha}} + \int_0^t \frac{f'(\tau)\mathrm{d}\tau}{(t-\tau)^{1-\alpha}} \right)$$

is a solution of this equation. Hint: Use the first part of Problem 2.23 and the formula

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}, \quad 0 < \alpha < 1.$$

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