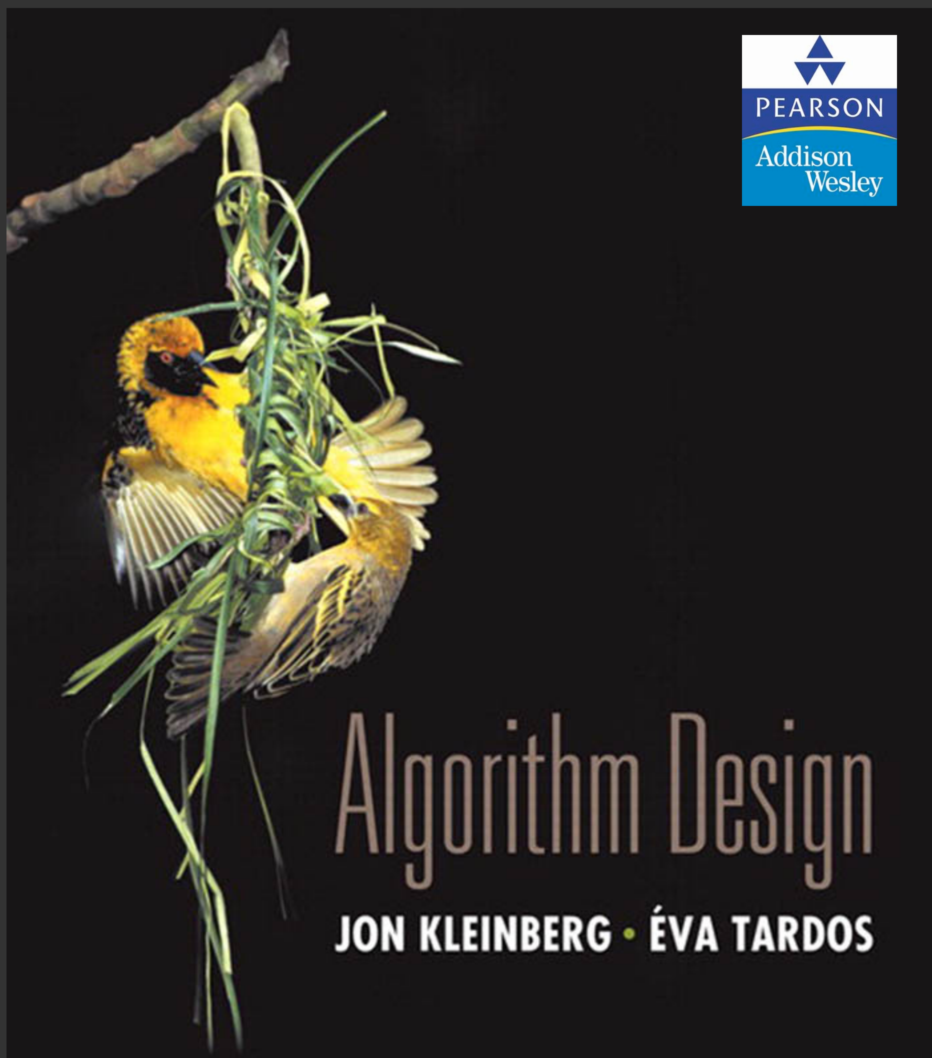

5. Network Flows

- *Ford–Fulkerson pathological example*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

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Ford–Fulkerson pathological example

Intuition. Let r satisfy $r^2 = 1 - r$.

- Initially, some residual capacities are 1 and r.
- After two augmenting paths, some residual capacities are r and r^2.
- After two more augmenting paths, some residual capacities are r^2 and r^3.
- After two more, some residual capacities are r^3 and r^4.
- If augmenting paths choreographed carefully, infinitely many residual capacities arise!

$1 - r$

$r - r^2$

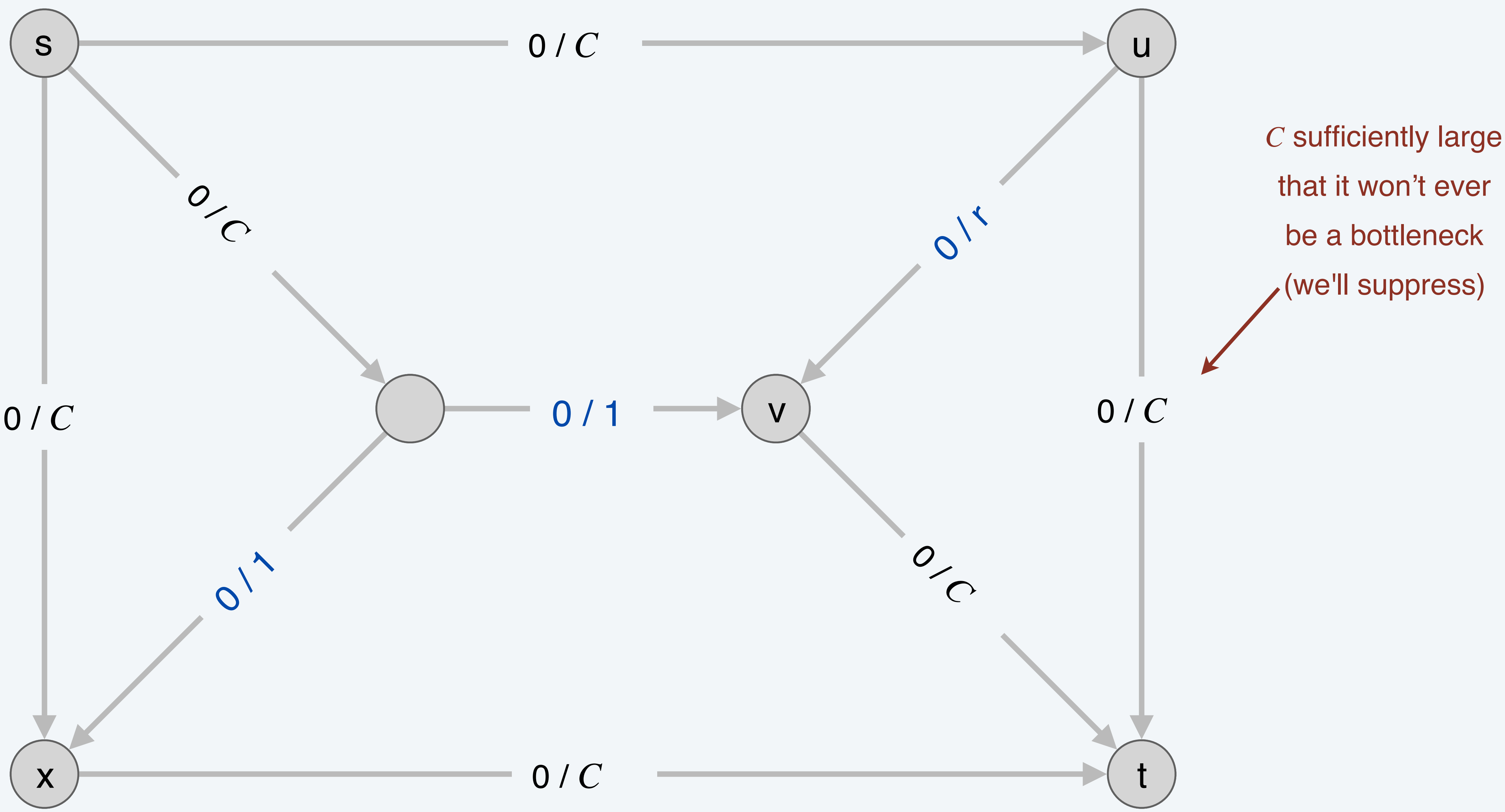
$r^2 - r^3$

$$\boxed{r = \frac{\sqrt{5} - 1}{2} \implies r^2 = 1 - r}$$

$$r \approx 0.618 \implies r^4 < r^3 < r^2 < r < 1$$

Ford–Fulkerson pathological example

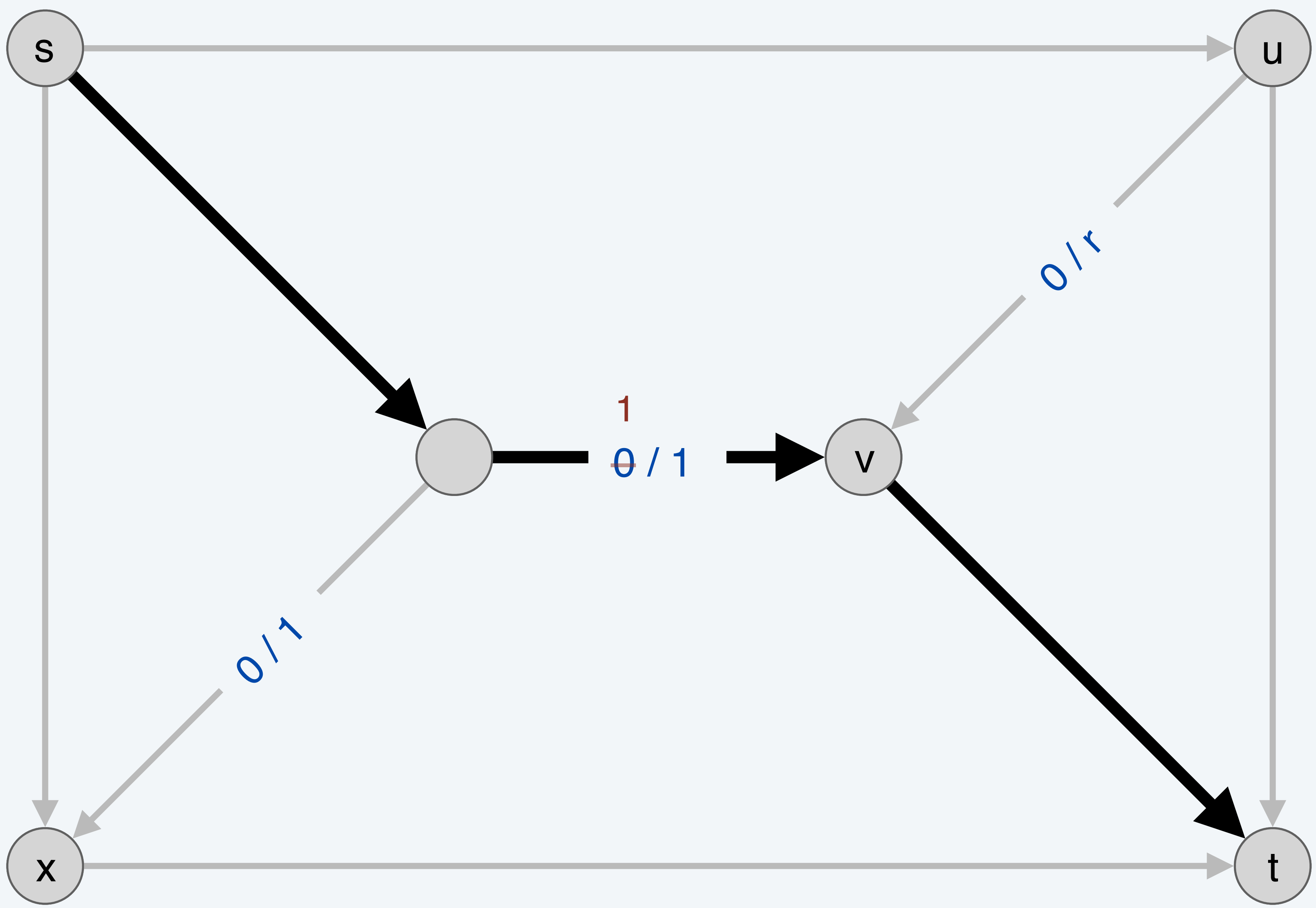
flow network G



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

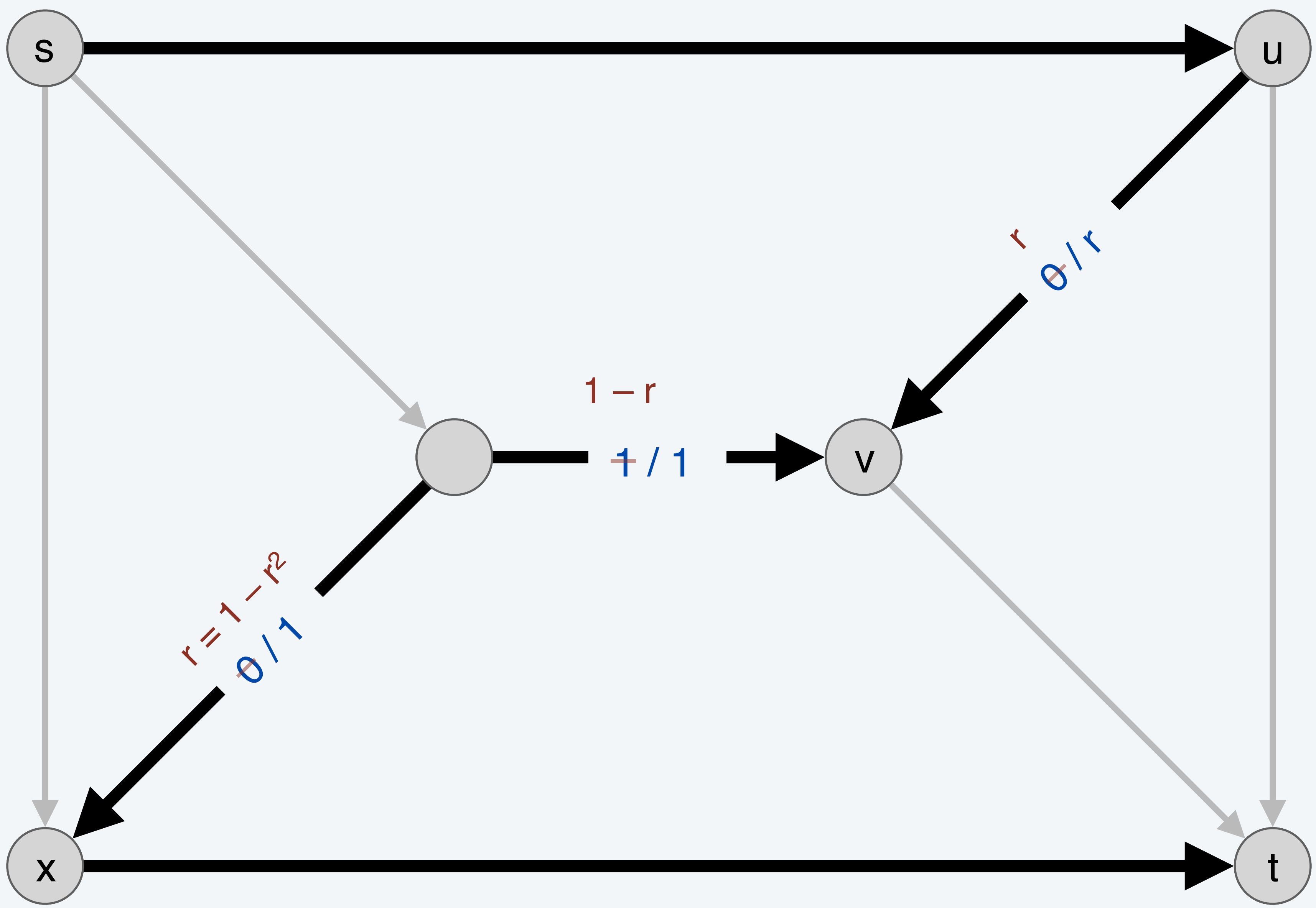
augmenting path 1: s→w→v→t (bottleneck capacity = 1)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

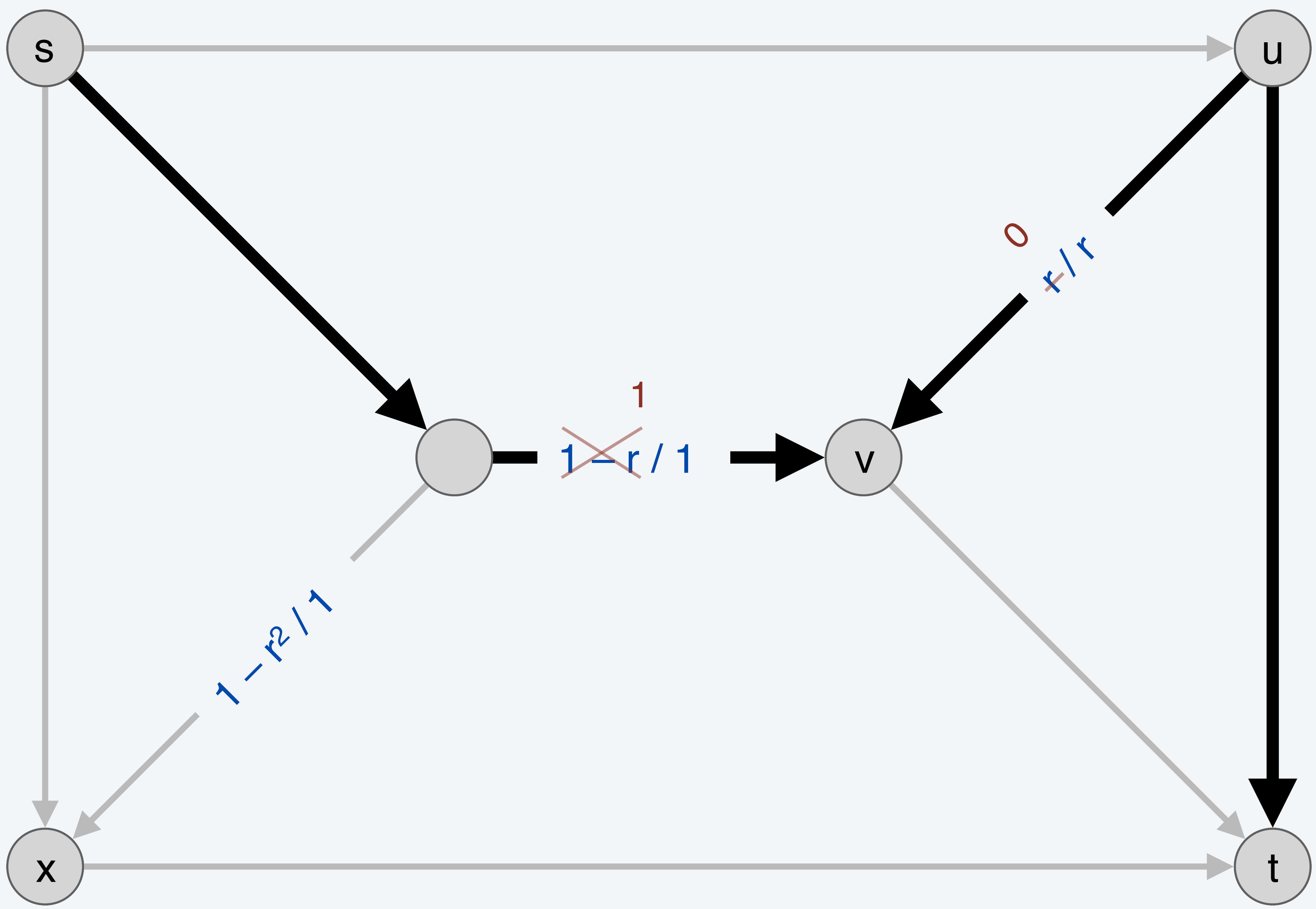
augmenting path 2: s→u→v→w→x→t (bottleneck capacity = r)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

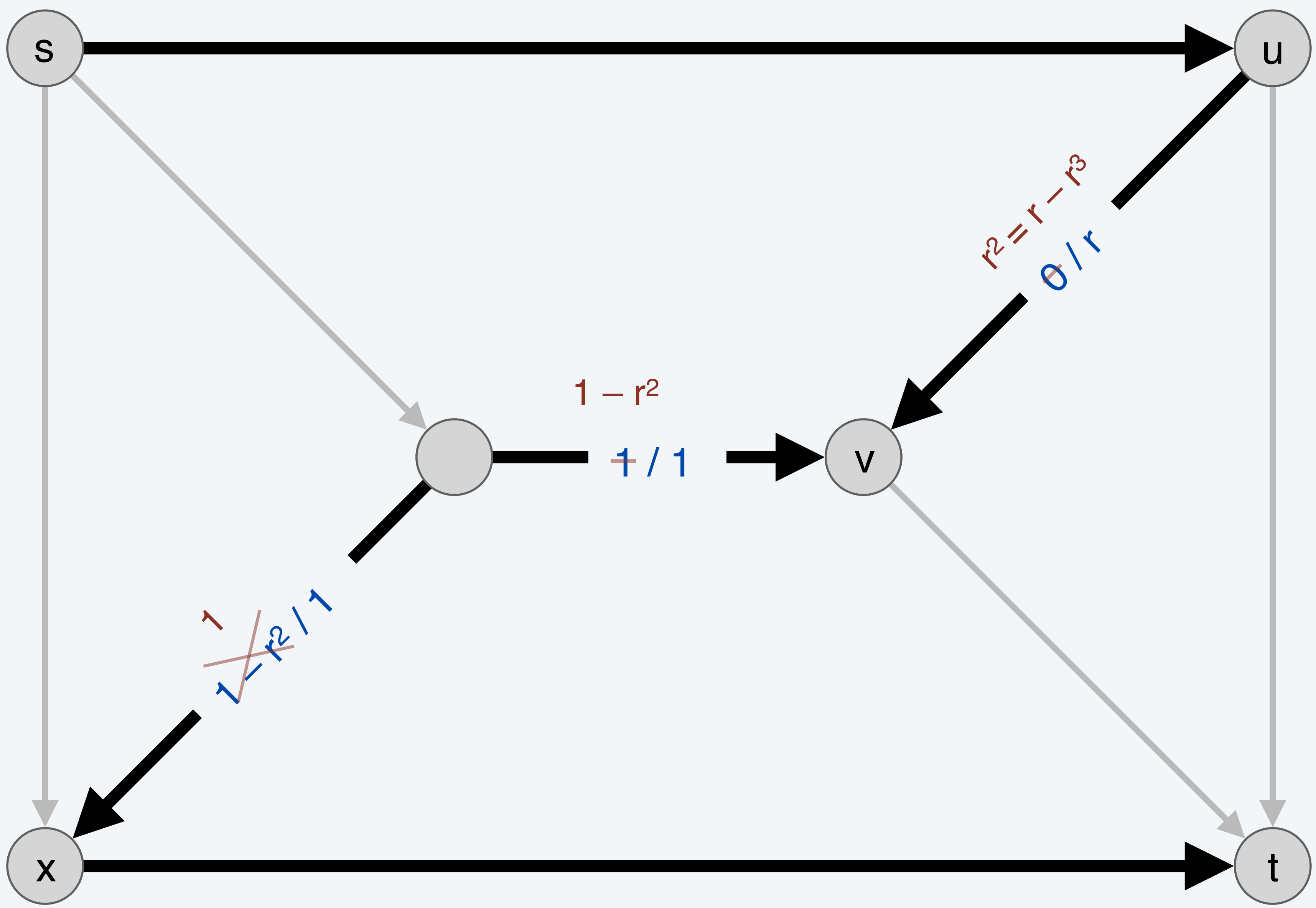
augmenting path 3: s→w→v→u→t (bottleneck capacity = r)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

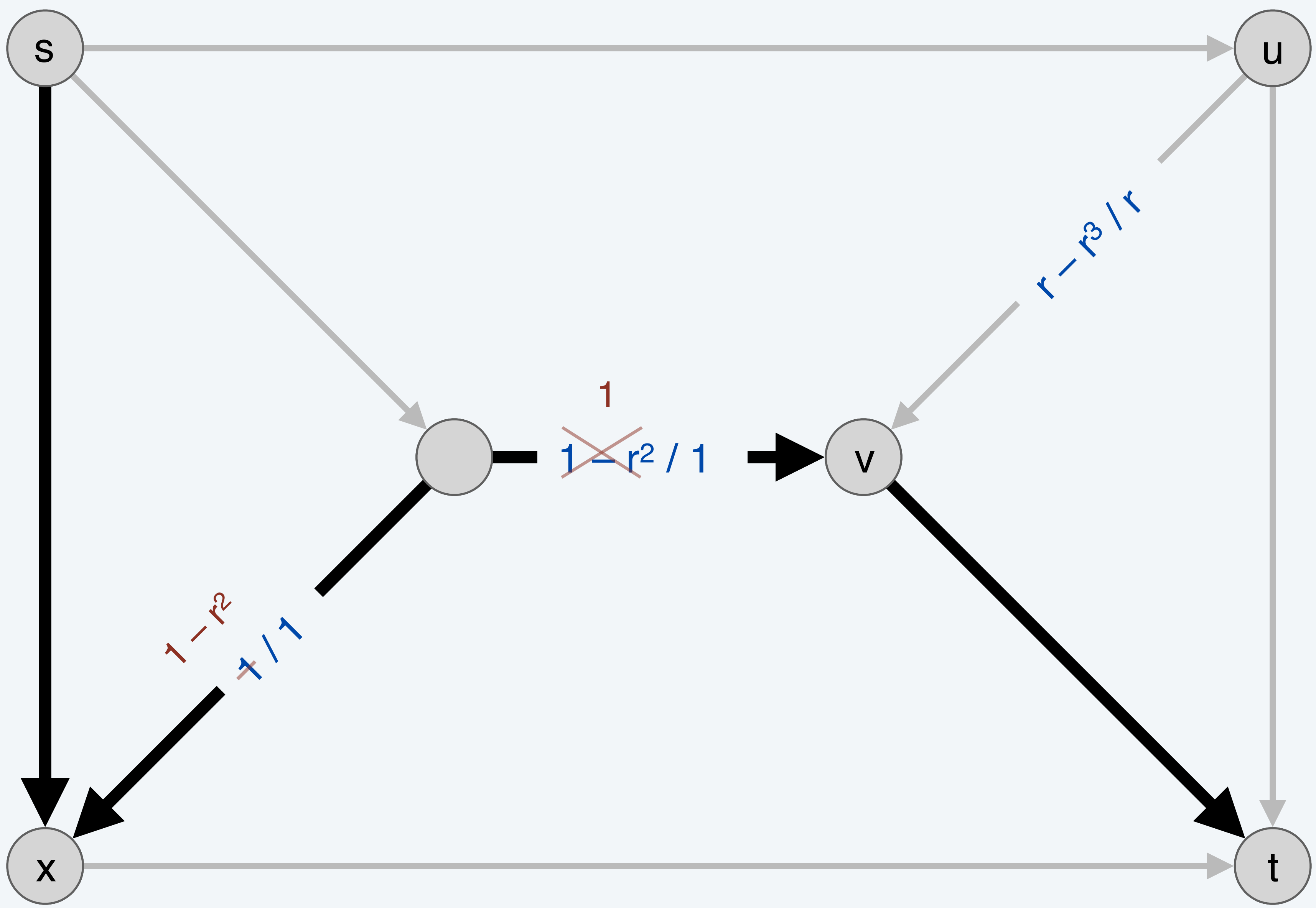
augmenting path 4: s→u→v→w→x→t (bottleneck capacity = r^2)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

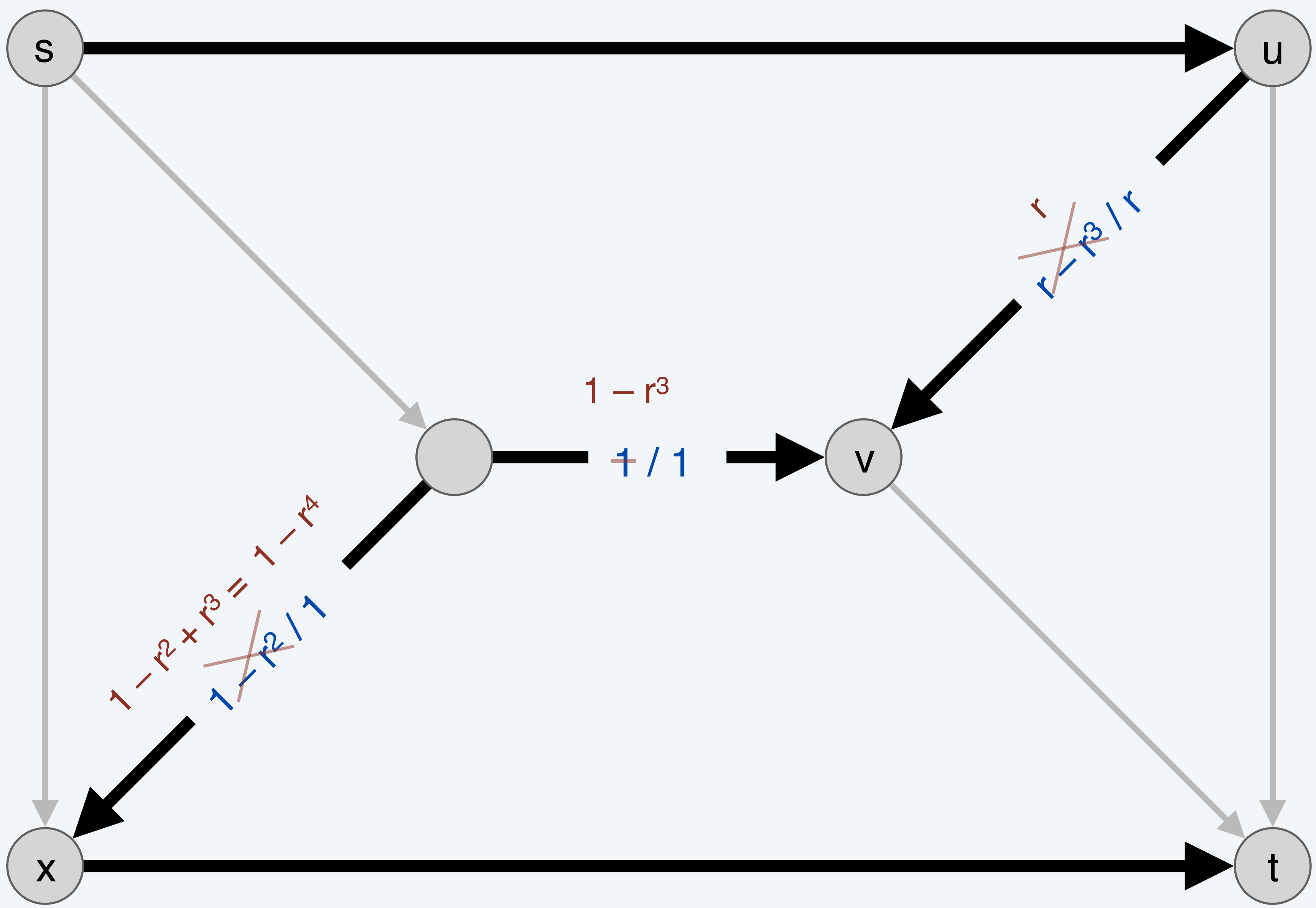
augmenting path 5: s→x→w→v→t (bottleneck capacity = r^2)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

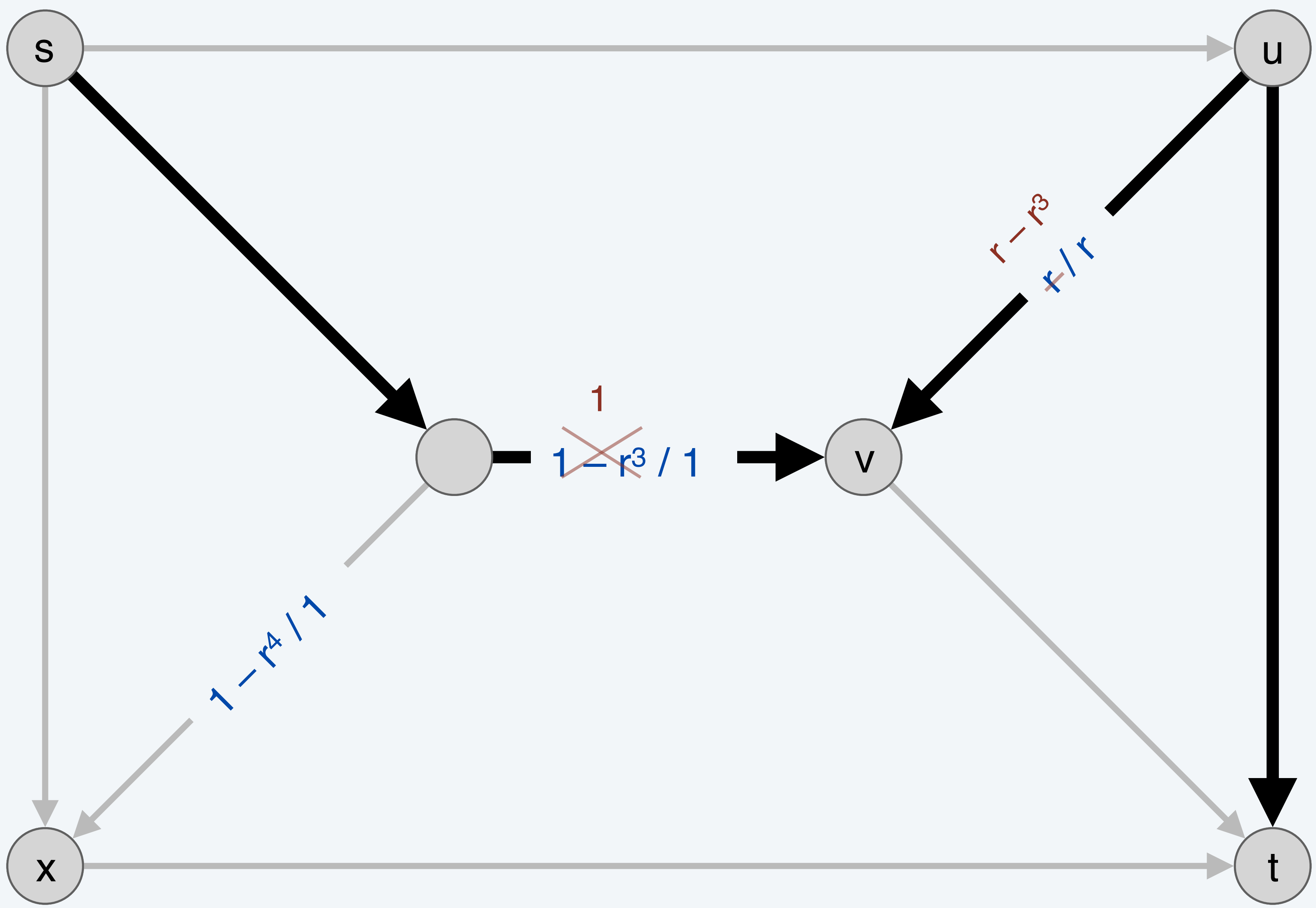
augmenting path 6: s→u→v→w→x→t (bottleneck capacity = r³)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

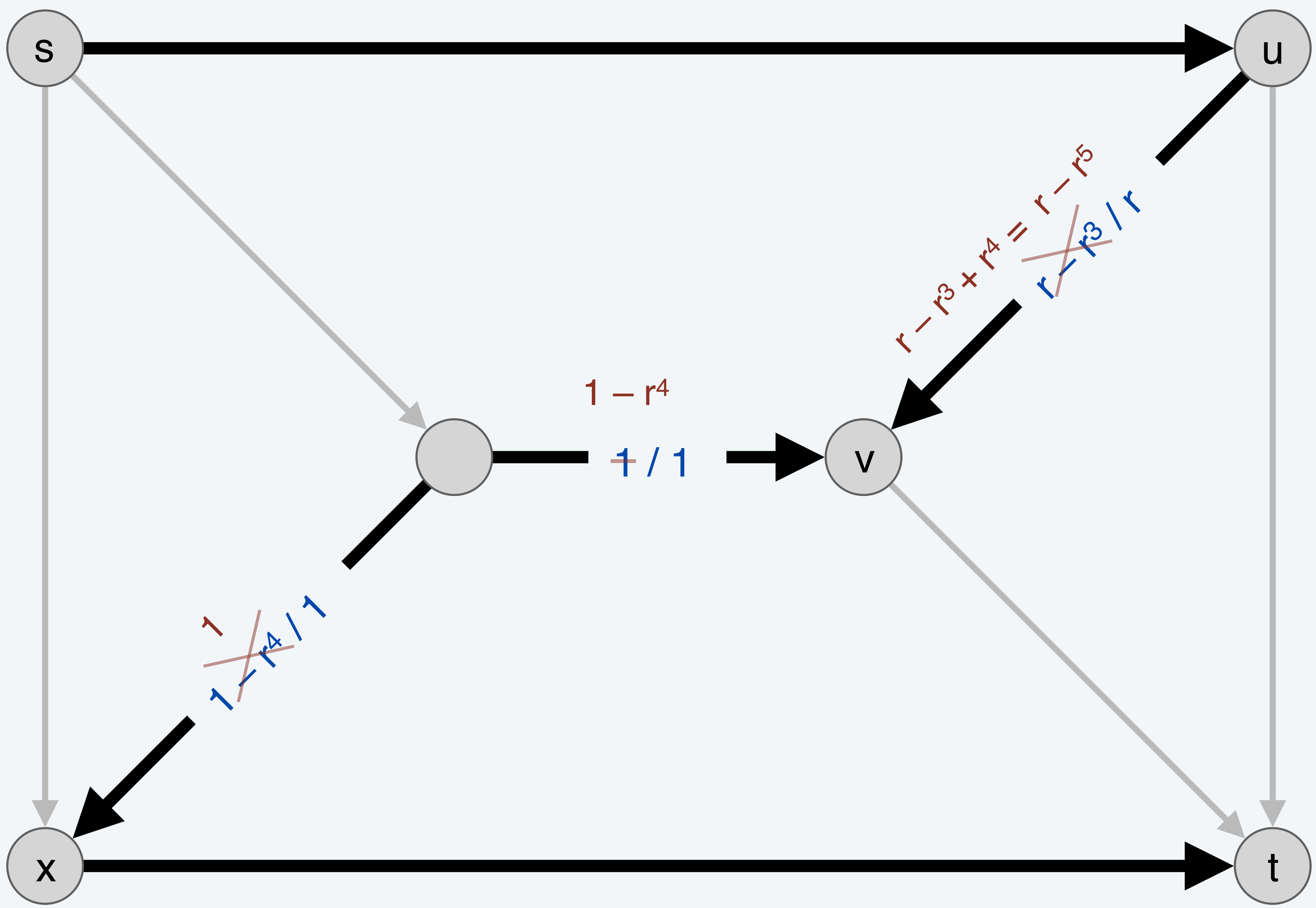
augmenting path 7: s→w→v→u→t (bottleneck capacity = r^3)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

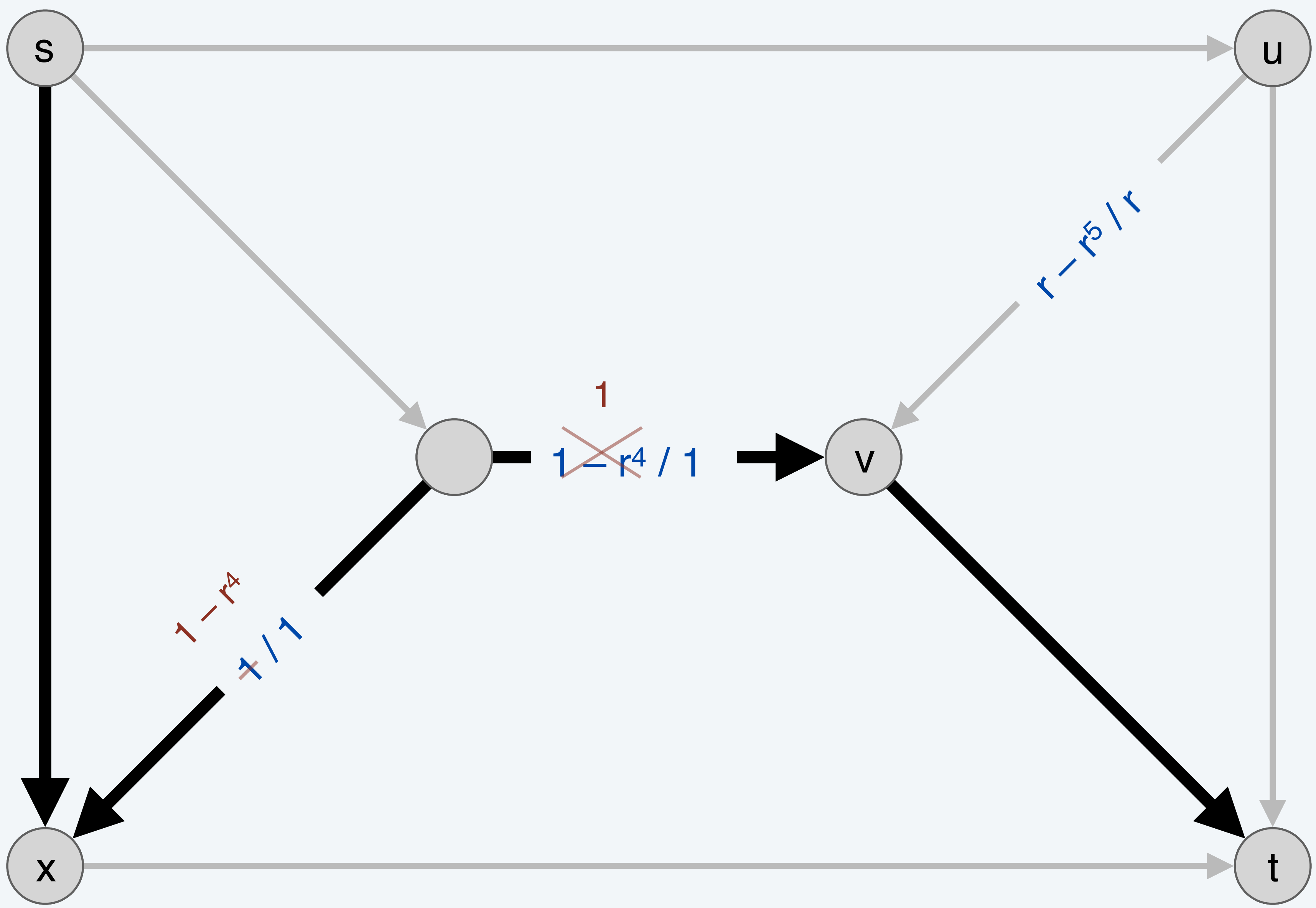
augmenting path 8: s→u→v→w→x→t (bottleneck capacity = r^4)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

augmenting path 9: s→x→w→v→t (bottleneck capacity = r⁴)



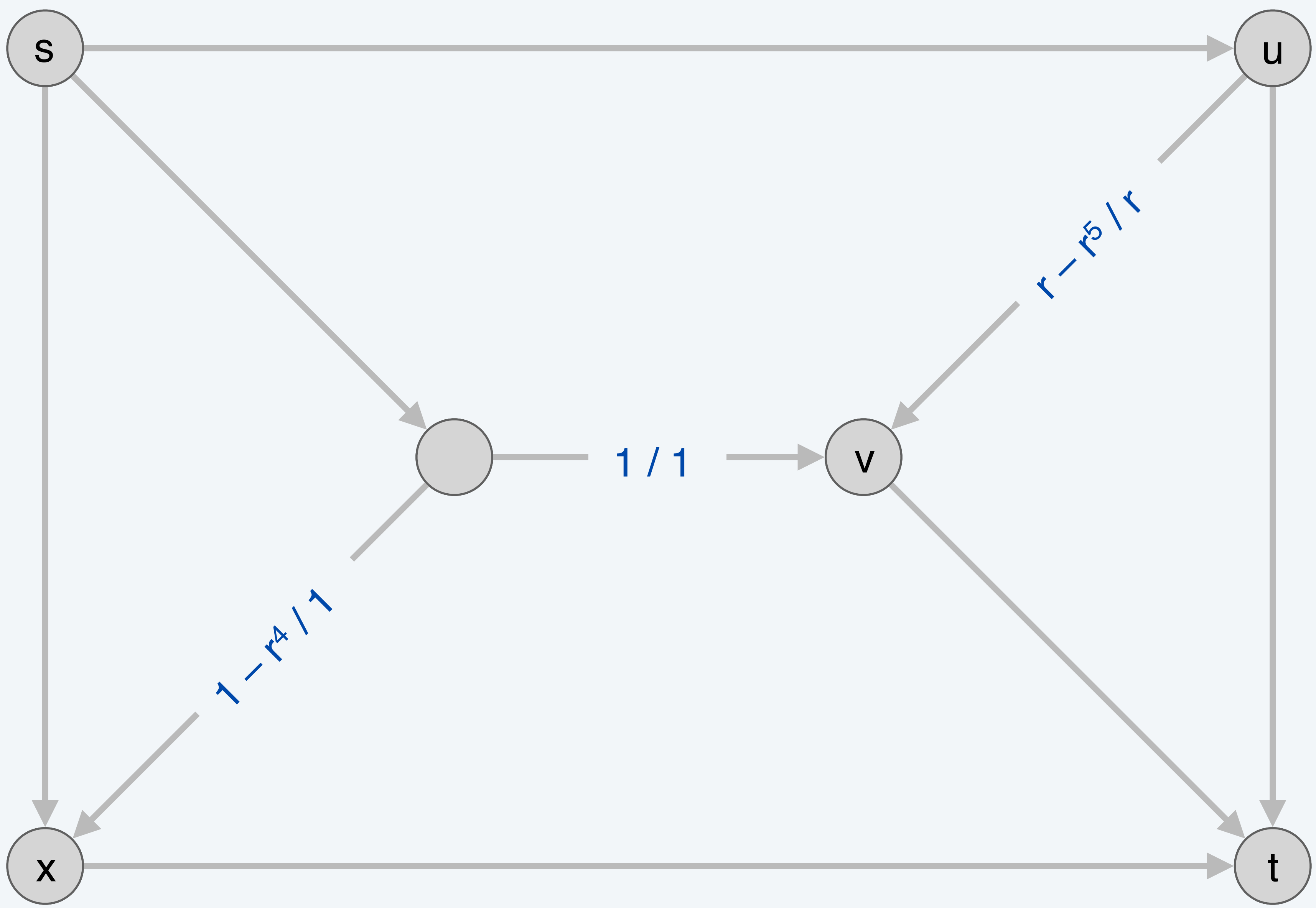
$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

flow after augmenting path 1: { $r - r^1$, 1, $1 - r^0$ } (value of flow = 1)

flow after augmenting path 5: { $r - r^3$, 1, $1 - r^2$ } (value of flow = $1 + 2r + 2r^2$)

flow after augmenting path 9: { $r - r^5$, 1, $1 - r^4$ } (value of flow = $1 + 2r + 2r^2 + 2r^3 + 2r^4$)



$$r^2 = 1 - r$$

Ford–Fulkerson pathological example

Theorem. The Ford–Fulkerson algorithm may not terminate; moreover, it may converge to a value not equal to the value of the maximum flow.

Pf.

- After $(1 + 4k)$ augmenting paths of the form just described, the value of the flow

$$\begin{aligned} &= 1 + 2\sum_{i=1}^{2k} r^i \\ &\le 1 + 2\sum_{i=1}^{\infty} r^i \\ &= 3 + 2r \\ &< 5 \end{aligned}$$

$$r = \frac{\sqrt{5}-1}{2}$$

- Value of maximum flow = $2C + 1$. ▪