Application of Monte Carlo to Heat Transfer Problems

JOHN R. HOWELL*

NASA Lewis Research Center, Cleveland, Ohio

**Present address:* Department of Mechanical Engineering, University of Houston, Houston, Texas.

I. Introduction

A. Definition of Monte Carlo

Monte Carlo has been defined as the technique of solving a problem by putting in random numbers and getting out random answers; this may be as valid a definition as any other.

Kahn (*1*) has given the following more useful definition which seems to incorporate the salient ideas: "The expected score of a player in any reasonable game of chance, however complicated, can in principle be estimated by averaging the results of a large number of plays of the game. Such estimation can be rendered more efficient by various devices which replace the game with another known to have the same expected score. The new game may lead to a more efficient estimate by being less erratic, that is, having a score of lower variance or by being cheaper to play with the equipment on hand. There are obviously many problems about probability that can be viewed as problems of calculating the expected score of a game. Still more, there are problems that do not concern probability but are none the less equivalent for some purposes to the calculation of an expected score. The Monte Carlo method refers simply to the exploitation of these remarks."

This definition also provides a good outline for use of the method. Indeed, what must be done for a specific problem is to set up a game that has the same expected outcome as the physical problem which the game simulates; make the game as simple and fast to play as possible; play the game many times; and find the average outcome. After some remarks on the history of the method and the philosophy of the approach being taken here to summarize and outline it, we shall apply just this formalism to problems in various fields of heat transfer.

B. History

The history of "experimental mathematics" can be traced about as far into the past as the truth seeker cares to pursue it. Hammersley and Handscomb (*2*) give references to over 300 works dealing with Monte Carlo and closely related material published over the last six decades, and mention a determination of the value of π by a mathematical experiment performed some thousands of years ago (*3*). However, the great bulk of the literature has appeared since 1950.

Many early workers actually carried out numerical experiments by such means as throwing dice or playing card games many times over to determine the probability of a given outcome, but useful results from such methods awaited the unique abilities of high-speed digital computers. These machines could play simulations of the game at a high rate and thus compile accurate averages in a reasonable time.

Credit for development of Monte Carlo techniques as we presently use them in engineering and science goes to the extremely competent group of physicists and mathematicians who gathered at Los Alamos during the early work on nuclear weapons, including especially von Neumann and Ulam. The groundwork of the Los Alamos group stimulated a vast outpouring of literature on the subject and encouraged the use of Monte Carlo for a variety of problems. Unfortunately, the method has been applied to many problems to which it is probably ill suited in comparison with other methods. This has caused some criticism of Monte Carlo practitioners as being akin to the proverbial doctor who prescribes penicillin for every ailment.

C. GENERAL REFERENCES

Referring to "the" Monte Carlo method is probably meaningless, although we shall apply such terminology. Any specific problem more likely entails "a" Monte Carlo method, as the label has been placed on a large class of loosely related techniques.

Because Monte Carlo has been applied to such a wide variety of problems, useful shortcuts often appear in references that escape even the most meticulous searcher of the literature.

A number of general books and monographs are available that detail methods and/or review the literature. A valuable early outline is given by Metropolis and Ulam (*4*), which is the first work to use the term Monte Carlo for the approach being considered here. For clarity and usefulness, both the work of Kahn (*1*) and Hammersley and Handscomb (*2*) are valuable, as are the general texts by Cashwell *et al.* (*5*); Schreider (*6*, *6a*) (who gives 282 references, many to the foreign literature); Brown (*7*); and the many excellent papers gathered in the symposium volume edited by Meyer (*8*). Other more specialized monographs and articles are mentioned in following sections.

No attempt is made in this review to exhaust the literature on any of the topics covered; however, it is hoped that a sufficient number of representative works are referenced so that the interested reader can easily find his way to those papers of direct consequence.

Topics to be covered are those areas of heat transfer analysis that are amenable to Monte Carlo analysis. These include radiation, rarefied gas energy transfer, conduction, and some mathematical techniques. Other physical problems where little work has as yet been done will be pointed out.

D. PHILOSOPHY OF PRESENT REVIEW

The references cited in the preceding section give rigorous mathematical justification for the methods employed in Monte Carlo. Those who cannot sleep without such reassurance are urged to read these works carefully. Here,

however, it is hoped to give arguments based on physical foundations, with emphasis on why the mathematical forms evolve. No attempt to provide proofs of statistical laws will be made; the standard texts in statistics carry out these proofs in detail.

The object of this review is rather to present a view of the problems that have been solved by Monte Carlo in certain areas of heat transfer, and the techniques used in their solution.

Furthermore, only the straightforward Monte Carlo approach will be given. The many refinements that can shorten computation time by increasing accuracy will only be mentioned in passing.

In addition, a word needs to be said about the machine running time of Monte Carlo programs. No definitive method of predicting running time exists for most problems. The time used will depend, of course, on the machine used, and perhaps more strongly on the ability of the programmer to pick methods and shortcuts that will reduce the burden on the machine. An example of such a shortcut is the use of special subroutines for computation of such functions as sine and cosine. These routines sacrifice some accuracy to a gain in speed. If problem answers accurate to a few per cent are desired, then the use of eight-place functions from a relatively slow subroutine is a needless luxury, especially if the subroutine is to be used tens of thousands of times.

Finally, only this paragraph will be devoted to the fruitless argument as to whether Monte Carlo or some other method is a "better" way of attacking a given problem. Suppose that a set of integral equations must be solved simultaneously in order to obtain an analytical solution to a given physical problem. A Monte Carlo solution of a physical analog may lead to a lengthy computer run. The question facing the programmer is then: Is it better to program the solution of the integral equations by finite difference techniques, with the possibility that convergence will not be attained, or by Monte Carlo, which, though long running, will give the answer sooner or later? There can be no reply to this question. Only the background and intuition of the individual researcher can give some clue as to the most likely direction of attack. It is hoped that the following material will provide a basis for such decisions.

II. Details of the Method

A. The Random Walk

Any reader looking into the background of the material to be presented here will soon encounter the term *Markov chain*. A Markov chain is simply a chain of events occurring in sequence. The probability of each succeeding

event in the chain is uninfluenced by prior events. The usual example of this is a totally inebriated gentleman who begins a walk through a strange city. At each street corner that he reaches, he becomes confused. In continuing his walk, he chooses completely at random one of the streets leading from the corner. In fact, he may walk up and down the same block several times before he chances to move off down a new street. The history of his walk is then a Markov chain, as his decision at any point is not influenced by where he has been.

Because of the randomness of his choice at each intersection, it might be possible to simulate a sample walk by constructing a "four-holer," that is, a roulette wheel with only four positions, each corresponding to a possible direction. The probability of the gentleman starting at his hotel bar and reaching any point on the city limits could then be found by simulating a large number of histories, using the four-holer to determine the direction of the walk at each decision point in each history.

It might be noted that the probability of the man reaching intersection (l, m) on a square grid representing the city street map is simply

$$P(l, m) = \tfrac{1}{4}[P(l+1, m) + P(l-1, m) + P(l, m+1) + P(l, m-1)] \tag{1}$$

where the factors in the square brackets are the probabilities of his being at each of the adjacent four intersections. This is because the probability of reaching $P(l, m)$ from adjacent intersections is one fourth. The ramifications of this observation will be seen in Section V, C.

As we shall find, this type of random walk is a convenient model for certain heat transfer processes. However, the probability of a certain occurrence for other processes is usually not as immediately obvious as is the case for Eq. (1). More often, the probability of an event must be determined from physical constraints, and then the decision as to what event will occur is made on the basis of this probability. Some of the basic methods of choosing an event from a known probability distribution of events will now be examined. Also, means of constructing these distributions will be discussed.

B. CHOOSING FROM PROBABILITY DISTRIBUTIONS

Consider a very poor archer firing arrows at a target with an outer radius of 10 ft. After firing many arrows, the frequency with which his arrows are found to have struck the target in a small radius increment $\Delta\xi$ at some radius ξ is given by some distribution, perhaps similar to that of Fig. 1. What we desire is a method of simulating further shots by assigning an expected value of ξ to the arrow he is now fitting to his bow, and to each of a group of succeeding arrows. We wish further that the ξ values we assign will correspond to the

frequency distribution of Fig. 1. (We assume that all his arrows hit *somewhere* on the target.)

This situation is analogous to that encountered in many Markov processes. We know what distribution of values occurs in a given physical process, and we desire a method of assigning *individual* values so that the distribution of all our *assigned* values will agree with the required distribution. In radiant heat

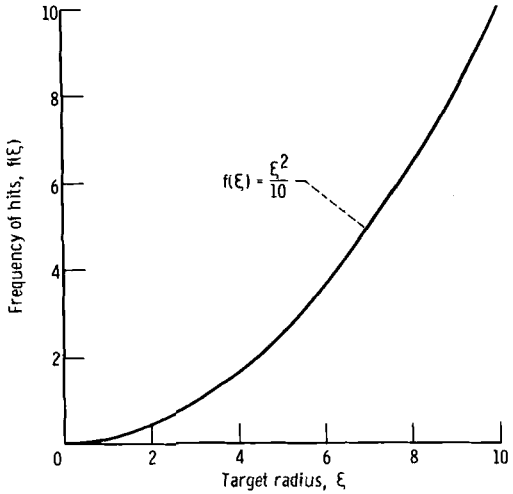

FIG. 1. Smoothed frequency distribution of arrows at various target radii.

transfer, for example, we know that the spectral energy emitted by a blackbody must follow the Planck spectral emission curve. How do we assign individual photons an energy so that, after a large number of photon histories are followed, the distribution of energies that we have assigned is indeed Planckian? Further, in a Markov process, we must make sure that our values at each step are assigned in some random manner so that each decision in the chain is independent.

Following our archer's progress, we shall see how to do this. The frequency curve given in Fig. 1 can (luckily) be approximated by the analytical expression

$$f(\xi) = \xi^2/10 \tag{2}$$

in the interval $0 \leqslant \xi \leqslant 10$, and $f(\xi) = 0$ elsewhere, because all the arrows struck the target. Let us normalize this by the area under the frequency curve, obtaining the relation

$$P(\xi) = \frac{f(\xi)}{\int_0^{10} f(\xi)\, d\xi} = 3\xi^2/1000 \tag{3}$$

Taking the frequency with which arrows have struck the target as the basis for our estimate of the probability with which the next set will strike, then the *probability density function* defined by Eq. (3) is the distribution that our assigned ξ values must satisfy. This relation is plotted in Fig. 2, and it is interpreted physically as the fraction of values (arrows) that lie in the region $\Delta\xi$ around ξ.

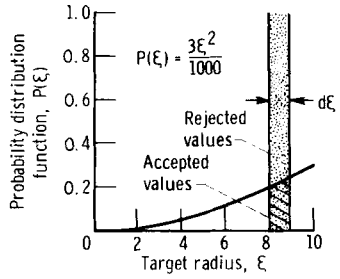

FIG. 2. Probability distribution of arrows on a target.

To assign ξ values, we may proceed as follows: Choose two numbers, R_A and R_B, from a large set of numbers evenly distributed in the range 0 to 1. (How we obtain these numbers in a practical calculation is discussed in Section II, C.) The two *random numbers* are then used to select a point $(P(\xi), \xi)$ on Fig. 2 by setting

$$P(\xi) = R_A; \qquad \xi = (\xi_{\max} - \xi_{\min})R_B = 10R_B$$

This point is then compared to the value of $P(\xi)$ at ξ computed from Eq. (3). If the randomly selected value lies above the computed value of $P(\xi)$, then the randomly selected value of ξ is rejected and two new random numbers are selected. Otherwise, the value of ξ that has been found is used. Referring again to Fig. 2, it is seen that such a procedure assures that the correct fraction of ξ values selected for use will lie in each increment $\Delta\xi$ after a large number of selections is made.

The difficulty with such an event-choosing procedure is that in some cases a large portion of the values of ξ may be rejected because they lie above the curve. A more efficient method for choosing ξ is therefore desirable.

One such method is to integrate the probability density function $P(\xi)$ using the general relation

$$R = \int_{-\infty}^{\xi} P(\xi')\, d\xi' \tag{4}$$

where R can only take on values in the range (0–1) because of the properties of $P(\xi)$. Equation (4) is the general definition of the *cumulative distribution function.* A plot of R against ξ from Eq. (4) shows the probability of an event occurring in the range ($-\infty$ to ξ). For our purposes, the function R is taken to be a random number, and values of ξ are obtained by choosing R at random and solving Eq. (4) for the corresponding value of ξ. To show that the probability density of ξ chosen in this way corresponds to the required $P(\xi)$, we

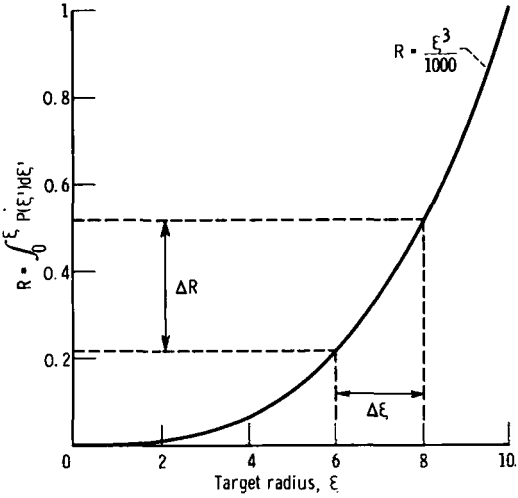

FIG. 3. Cumulative distribution of arrows on target.

can again examine the probability density function of Fig. 2. Inserting the example $P(\xi)$ of Eq. (3) into Eq. (4) gives

$$R = \int_0^{\xi} P(\xi')\,d\xi' = \xi^3/1000 \tag{5}$$

where $P(\xi) = 0$ for $\xi \leqslant 0$. Equation (5) is shown plotted in Fig. 3. Divide the range of ξ into a number of equal increments $\Delta\xi$. Suppose that M values of R are now chosen in the range (0–1) and that these M values are chosen at equal intervals along R. There will be M values of ξ which correspond to these M values of R. The fraction of the M values of ξ which occurs per given increment $\Delta\xi$ is then

$$(M_{\Delta\xi}/M)/\Delta\xi = \Delta R/\Delta\xi \tag{6}$$

But $\Delta R/\Delta\xi$ is, of course, an approximation of $(dR/d\xi)$ if a large enough value is chosen for M and small increments $\Delta\xi$ are examined. But $dR/d\xi$ can be seen from Eq. (5) to be simply $P(\xi)$; and it has been shown, therefore, that

by choosing values of ξ in this manner the required probability distribution is indeed satisfied.

Because the distribution of events from which the Monte Carlo choices are made is identical to the distribution governing the process to be modeled, we are assured that our model will give precisely the same results as the actual process.

Often physical problems arise in which the probability density function depends on more than one variable. For example, if the archer in our example suffered from astigmatism, then a dependence on circumferential angle θ might appear in the distribution of arrows on the target in addition to the radial dependence. If the interdependence of the variables is such that the probability density function can be factored into a product form, then we can write

$$P(\xi, \theta) = P(\xi) P(\theta) \tag{7}$$

and values of $P(\xi)$ and $P(\theta)$ can be found by integrating out each variable in turn to give

$$P(\xi) = \int_{\theta_{\min}}^{\theta_{\max}} P(\xi, \theta)\, d\theta \tag{8}$$

$$P(\theta) = \int_{\xi_{\min}}^{\xi_{\max}} P(\xi, \theta)\, d\xi \tag{9}$$

and the methods given above in this section are used to choose values of ξ and θ independently of one another.

If $P(\xi, \theta)$ cannot be placed in the form of Eq. (7) (that is, if $P(\xi)$ and $P(\theta)$ are not independent), then it can be shown (*1*, *6*) that, applying Eq. (5) to the multivariable probability density function $P(\xi, \theta)$, we can solve for ξ and θ by choosing two random numbers, R_1 and R_2. Then we evaluate consecutively the equations

$$\int_{-\infty}^{\xi} \int_{\theta_{\min}}^{\theta_{\max}} P(\xi', \theta')\, d\theta'\, d\xi' = R_1 \tag{10}$$

$$\int_{-\infty}^{\theta} P(\theta', \xi = \text{fixed})\, d\theta' = R_2 \tag{11}$$

where ξ in Eq. (11) is that value obtained from Eq. (10). This procedure may be extended to any number of variables. Equation (10) defines the *marginal distribution function* and Eq. (11) defines the *conditional distribution function*.

C. RANDOM NUMBERS

1. *What They Are*

Formally, we shall take a random number to be a number chosen without sequence from a large set of numbers spaced at equivalued intervals in the range (0–1). If we place the numbers 0, 0.01, 0.02, 0.03, ..., 0.99, 1.00 on slips of paper, and then place the jumbled slips in a hat, we would have fair

assurance that if we pick a few numbers they will be random numbers. If we make many choices, then perhaps smaller intervals and, therefore, more slips should be used; we should replace each slip in that hat after it is used.

For a typical computer problem, as we shall see, random numbers might be needed for 10^5 or more decisions. Of course, it is desirable that we have a rapid way of obtaining them, and that the numbers chosen be truly random.

2. *Where We Get Random Numbers*

On the modern digital computer, it is impractical to fit a mechanical arm and an optical scanner to choose and interpret slips pulled from a hat. To give truly random numbers, one possibility would be to sample a truly random process. Such phenomena as noise in an electronic circuit, radioactive decay particle counts per unit time (and determination of whether the count is odd or even to determine a binary digit), and similar means have been tried; but in the main are found to be too slow for direct computer linkage.

A second means is to obtain or generate tables of random numbers (*9*, *10*) perhaps by one of the processes outlined above, and then enter these tables in the computer memory. This allows rapid access to random numbers, but for complex problems requiring a large quantity of random numbers the use of storage space becomes prohibitive. This method has been widely used, however, when a modest problem is to be solved.

The most widely practiced method used at present for obtaining random numbers for a digital computer is a pseudorandom number generator. This is simply a subroutine that exploits the apparent randomness of groups of digits in large numbers. One simple example of such a routine is to take an eight-digit number, square it, and then choose the middle eight digits of the resulting sixteen-digit number as the required random number. When a new random number is needed, the last random number is squared, and the new random number is taken as the middle eight digits of the result. This process is said by Schreider (*6*) to degenerate after a few thousand cycles by propagating to an all-zero number.

A more satisfactory routine used at the Lewis Research Center of NASA is based on suggestions by Taussky and Todd (*11*). Here a random number is generated by taking the low-order 36 bits of the product $R_{n-1}K$, where $K = 5^{15}$ and R_{n-1} is the previously computed random number. The subroutine is started by taking $R_0 = 1$, or the programmer may give R_0 an arbitrary value. By always starting a given program with the same R_0, it is possible to check solutions through step-by-step tracing of a few histories.

3. *How We Make Sure the Numbers Are Random*

The fact that such subroutines generate pseudorandom numbers immediately raises a danger flag. How do we know such pseudorandomness is

random enough for our purposes ? Do such routines repeat ? After how many numbers ?

Certain standard tests exist that give partial answers to these questions, and a full discussion of them is given in the literature (*2, 11, 12*).

None of these tests is sufficient to establish randomness, although passage of them is necessary. Kendall and Smith (*12*) describe four such tests. The names they ascribe give the flavor of the methods: the frequency test, serial test, poker test, and gap test. These tests are described as "... useful and searching. They are, however, not sufficient...."

Perhaps the safest course to follow is to obtain a standard subroutine whose properties have been established by such tests and use it only within its proven limits.

D. EVALUATION OF ERROR

Because the solutions obtained by Monte Carlo are averages over a number of individual tests, they will in general contain fluctuations about a mean value. As in any process of this type, the mean can be more accurately determined by increasing the number of values used in determining the mean; however, it is not possible to ascribe a 100% confidence in the value we obtain, although we can approach such confidence about as closely as we care to so long as our budget for computer time can stand the strain. More generally we apply some *ad hoc* rules of economy and an estimate of desired accuracy to a given problem, and obtain solutions by trading off within these limits.

To estimate the accuracy of the solutions, we apply any or all of the usual tests applied to statistical data. For example, suppose it is desired to know the probability of the randomly staggering attendee of an engineering convention (who was discussed in Section II, A) reaching a certain bar at the city limits. To exactly determine his success, we would have to follow an infinite number of sample engineers and determine the probability $P(l,m)$ of reaching the boundary point (l,m) as

$$P(l,m) = [S(l,m)/N]_{N\to\infty} \tag{12}$$

where $S(l,m)/N$ is the number of samples $S(l,m)$ reaching the boundary point divided by the total number of samples N. Obviously, following an infinite number of samples would not be economical for researchers unsupported by government contract. What we more generally look for is a probability based on some finite number of samples N, of the order perhaps of 10^2 to 10^6. We need, then, an estimate of the error δ involved in approximating infinity by these relatively small sample sizes.

For a sample size greater than about $N = 20$, it can be shown (*2*, *6*) from application of the "law of large numbers" and the relations governing normal probability distributions that the following relation holds: The probability that the average $S(l,m)/N$ for finite N differs by less than some value δ from $(S(l,m)/N)_{N\to\infty}$ is given by

$$P\left(\left|\frac{S}{N}-\left(\frac{S}{N}\right)_{N\to\infty}\right|\leqslant\delta\right)=\left(\frac{2}{\sqrt{\pi}}\right)\int_0^{\beta/\sqrt{2}}\exp(-\beta'^2)\,d\beta'$$
$$=\operatorname{erf}(\beta/\sqrt{2}) \tag{13}$$

where

$$\beta\approx\delta\{N/[(S/N)(1-S/N)]\}^{1/2} \tag{14}$$

Compilations of the error function (erf) are given in many standard reference tables (*13*, *14*).

Such an analysis of error can be used whenever the samples S in question can be considered to leave a source and either reach a scoring position with probability P or not reach it with probability $(1-P)$.

In many problems, such an error estimation procedure may not be valid. For example, the energy flux at a point might depend on the total energy arriving from many sources. For such situations, the most straightforward way of estimating the error in the value (in the conduction case, the error in the value of flux at the point) is to subdivide the calculation of the mean into a group of submeans. For example, if 100 samples are examined, a mean is calculated on the basis of 100 samples, and 10 submeans of 10 samples each are calculated. The variance or standard deviation of these submeans from the mean of the submeans is then calculated. If P_i is the value of the Ith submean and $\bar{P}$ is the mean of the I submeans, then the *variance* γ^2 of the mean solution $\bar{P}$ is given by

$$\gamma^2=\frac{1}{(I-1)}\left[\sum_{i=1}^{I}(P_i-\bar{P})^2\right]=\frac{1}{(I-1)}\left\{\sum_{i=1}^{I}P_i^2-\left[\left(\sum_{i=1}^{I}P_i\right)^2/I\right]\right\} \tag{15}$$

where the variance is simply the mean square deviation of the submeans about the mean. From the properties of the normal frequency distribution, which a set of results computed by Monte Carlo will in general follow, it is shown in most texts on statistics that the probability of the real mean (that is, the mean calculated for $I\to\infty$) lying within $\pm\gamma$ of the sample mean $\bar{P}$ is about 68%, of lying within $\pm2\gamma$ is about 95%, and of lying within $\pm3\gamma$ is 99.7%. Because γ, the *standard deviation*, is given by the square root of Eq. (15), it is obvious that in order to double our expected accuracy we must quadruple the number of samples which are used in computing the results. This probably means quadrupling the computer time involved unless the term in brackets can somehow be reduced by decreasing the variance (scatter)

of the individual submeans. Much time and ingenuity have been expended in attempts at the latter, under such labels as splitting, Russian Roulette, and selective sampling. These and other variance reducing techniques are discussed by Hammersley and Handscomb (*2*) and Schreider (*6*). The savings in computer time available from application of these techniques is abundant reward for their study, and the reader who intends to use Monte Carlo for any problem of significant complexity is urged to apply them.

III. Application to Thermal Radiative Transfer

A. INTRODUCTION

Many problems arise in the field of heat transfer that are difficult to solve because the equations used in their formulation are placed in a mathematical form descriptive of the entire macroscopic process being considered. Radiative exchange is a prime example.

The microscopic laws governing radiative exchange date back many years. For engineering problems, the most recently derived relation basic to the radiative exchange process is probably Planck's spectral energy distribution, published in 1901. Yet, even with the firm theoretical foundation upon which engineering radiative heat transfer calculations have long been based, it remains difficult to solve these problems. This is at least in part because most researchers have chosen to cast the description of the radiative exchange process into the form of integral equations typified by the so-called "equation of transfer" (*15*, *16*). From this starting point, simplifications are introduced (as, for example, in the diffusion or transparent approximations) in an attempt to reduce the complexity of the rather horrendous equations involved.

Sobolev (*17*) and Ueno (*18*, *19*) have developed a probabilistic treatment of radiative transfer in gases, but elected to cast the equations into a probabilistic form of the equation of transfer. Solutions for various problems are then found by analytical means. This approach again leads to difficulty in solving more complex problems.

By invoking a probabilistic model of the radiative exchange process and also applying Monte Carlo sampling techniques, it is possible to choose a "semimacroscopic"[1] approach, and avoid many of the difficulties inherent in the averaging processes of the usual integral equation formulations. We can then examine the actions of small parts of the total energy on an individual basis, rather than attempting to solve simultaneously for the entire behavior of all the energy involved. Let us examine a model of the radiative exchange process, and then outline the solution of an example problem.

[1] Or, perhaps, "semimicroscopic."

1. *Model of the Radiative Exchange Process*

In engineering radiation calculations, the usual quantities of interest are the local temperatures and energy fluxes. It seems reasonable to model the radiative exchange process by following the progress of discrete amounts of energy, since local energy flux is then easily computed as the number of these energy "bundles" arriving per unit area at some position per unit time. The obvious bundle to visualize is the photon, but the photon has a disadvantage as a basis for our model; its energy depends on its wavelength. Therefore, we choose a model particle more to our liking. This is the "photon bundle," or a bundle carrying a given amount of energy c. We might more easily picture this as a group of photons bound together. If the wavelength of the bundle is specified, then we add enough photons of that wavelength to make the energy of the bundle equal to c.

By assigning equal energies to all bundles, the problem of computing local energy flux becomes trivial; we simply count the number of bundles arriving at the position of interest per unit time and per unit area and multiply by the energy per bundle. Because each bundle of photons must obey the distribution laws for individual photons, we can determine bundle paths and generate bundle histories using the methods of Section II, B, and it is a straightforward procedure to model radiative exchange.

2. *Sample Problem*

For an example, let us look at a rather simple problem outlined by Howell (*20*), and examine the energy transfer between element dA_1 at temperature T_1 and surface A_2, an infinite plane at temperature $T_2 = 0$ (see Fig. 4). Let element dA_1 have emissivity

$$\epsilon_1 = \epsilon_1(\lambda, \eta) \tag{16}$$

and let area 2 have emissivity

$$\epsilon_2 = \epsilon_2(\lambda, \eta) \tag{17}$$

and assume only that the emissivity of both surfaces is independent of circumferential angle θ. This is the case for real surfaces prepared by sandblasting, plating, or etching.

For such a surface element, the total emitted energy per unit time is

$$Q_{e,1} = \epsilon_{T,1}\sigma T_1^4 dA_1 \tag{18}$$

where $\epsilon_{T,1}$ is the total hemispherical emissivity given in this case by

$$\epsilon_{T,1} = \frac{2\pi \int_0^\infty \int_0^{\pi/2} \epsilon_1(\lambda, \eta)\, i_{\lambda,1} \sin \eta \cos \eta\, d\eta\, d\lambda}{\sigma T_1^4} \tag{19}$$

and $i_{\lambda,1}$ is the Planck spectral distribution of blackbody radiant intensity.

If it is assumed that $Q_{e,1}$, the total energy emitted per unit time by dA_1, is composed of N of the energy bundles described above, then the energy assigned to each bundle, c, is simply

$$c = Q_{e,1}/N \tag{20}$$

To determine the energy transferred from element dA_1 to surface A_2, we now follow N bundles of energy through their emission from dA_1, and

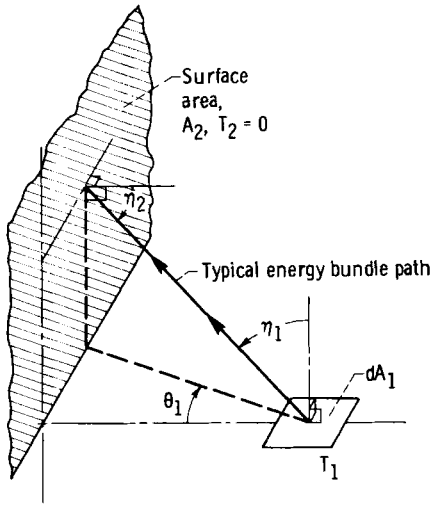

FIG. 4. Radiant interchange between two surfaces.

determine the number S_2 absorbed at A_2. If the energy reflected from A_2 back to dA_1 is neglected, the energy transferred per unit time from dA_1 to A_2 will be

$$Q_{1-2} = cS_2 \tag{21}$$

The next question is how do we determine each individual bundle path and how do we assign a wavelength to each bundle? However this is done, the directions and wavelengths of the N bundles must conform to the constraints given by the emissivity of the surface and the laws governing radiative processes. For example, if we assign wavelengths to N bundles, the spectral distribution of emitted energy generated by the Monte Carlo process (comprised of the energy $cN_\lambda \Delta\lambda$ for discrete intervals $\Delta\lambda$) must closely approximate the spectrum of the actual emitted energy (plotted as $\pi\epsilon_\lambda i_\lambda d\lambda$ versus λ). To assure this, we apply the methods of Section II, B.

The energy emitted by the surface dA_1 per unit time in the wavelength interval $d\lambda$ about a wavelength λ, and in the angular interval $d\eta$ about η, is

$$d^2Q_{e,1}(\lambda, \eta) = 2\pi \epsilon_1(\lambda, \eta) i_{\lambda,1} \cos \eta \, dA_1 \sin \eta \, d\eta \, d\lambda \tag{22}$$

The total energy emitted by dA_1 per unit time is given by Eq. (18). The probability of emission in a wavelength interval about λ and in an angular interval around η, $P(\lambda, \eta)$, is then the energy in $d\eta \, d\lambda$, Eq. (22), divided by the total energy, Eq. (18):

$$P(\lambda, \eta) \, d\eta \, d\lambda = \frac{d^2Q_{e,1}(\lambda, \eta)}{Q_{e,1}} = \frac{2\pi \epsilon_1(\lambda, \eta) i_{\lambda,1} \cos \eta \sin \eta \, d\eta \, d\lambda}{\epsilon_{T,1} \sigma T_1^4} \tag{23}$$

It is assumed here that the surface properties are product functions of the two variables, angle and wavelength, that is,

$$\epsilon(\lambda, \eta) = \epsilon(\lambda) \epsilon(\eta) \tag{24}$$

(This assumption is probably not valid for many real surfaces, since the angular distribution of emissivity is known to depend on wavelength.[2]) For the assumed form in Eq. (24) it follows that dependence on either variable may be found by integrating out the other variable [see Eq. (8)]. Then the normalized probability of emission occurring in the interval $d\lambda$ is

$$P(\lambda) \, d\lambda = \int_0^{\pi/2} P(\lambda, \eta) \, d\eta \, d\lambda = \frac{2\pi \int_0^{\pi/2} \epsilon_1(\lambda, \eta) \, i_{\lambda,1} \sin \eta \cos \eta \, d\eta \, d\lambda}{\epsilon_{T,1} \sigma T_1^4} \tag{25a}$$

Substituting into Eq. (4) gives

$$R_\lambda = \frac{2\pi \int_0^{\lambda} \int_0^{\pi/2} \epsilon_1(\lambda', \eta) i_{\lambda',1} \sin \eta \cos \eta \, d\eta \, d\lambda'}{\epsilon_{T,1} \sigma T_1^4} \tag{25b}$$

If the number of bundles N is very large, and this equation were solved for λ each time an R_λ were chosen, computing time would become too large for practical calculations. To circumvent this problem equations like Eq. (25b) can be numerically integrated once over the range of λ values and a curve can be fitted to the result. A polynomial curve is often adequate, as we can assume is the case in this problem, giving

$$\lambda = A + BR_\lambda + CR_\lambda^2 + \cdots \tag{26}$$

This equation rather than Eq. (25b) is used in the problem-solving program.

[2] Because of difficulties inherent in surface property measurement, little data exists to prove or disprove this assertion (i.e., simultaneous measurements of both the directional and spectral dependence of emissivity). This in itself might be sufficient justification for the assumption of Eq. (24).

Following a similar procedure for the cone angle of emission η gives the relation

$$R_\eta = \int_0^\eta \int_0^\infty P(\eta', \lambda)\, d\lambda\, d\eta' = \frac{2\pi \int_0^\eta \int_0^\infty \epsilon_1(\lambda, \eta')\, i_\lambda \sin \eta' \cos \eta'\, d\lambda\, d\eta'}{\epsilon_{T,1} \sigma T_1^4} \tag{27}$$

which can again be curve fit to give

$$\eta = A' + B' R_\eta + C' R_\eta^2 + \cdots \tag{28}$$

For a gray diffuse surface, Eq. (25b) reduces to

$$R_{\lambda,\text{gray}} = \frac{\pi \int_0^\lambda i_{\lambda'}\, d\lambda'}{\sigma T_1^4} = F_{0-\lambda} \tag{29}$$

where $F_{0-\lambda}$ is the well-known fraction of blackbody emission in the wavelength interval $(0\text{–}\lambda)$. Equation (27) for this case reduces to

$$R_{\eta,\text{gray}} = 2 \int_0^{\eta_1} \sin \eta' \cos \eta'\, d\eta' = \sin^2 \eta_1 \tag{30a}$$

or

$$\sin \eta_1 = \sqrt{R_\eta} \tag{30b}$$

The point to be made here is that computational difficulty is not greatly different in obtaining λ from either Eq. (26) or Eq. (29), nor is it much different for obtaining η_1 from either Eq. (28) or Eq. (30b). The difference between the nongray nondiffuse case and the gray diffuse case is mainly in the auxiliary numerical integrations of Eqs. (25b) and (27). These integrations are performed once to get the curve fits, and then as far as the main problem-solving program is concerned, the more difficult case might just as well be handled. Thus, increasing problem complexity leads to only gradual increases in the complexity of the Monte Carlo program, and similar gradual increases in computer time.

For emission of an individual energy bundle from surface dA_1, a wavelength λ can be chosen from Eq. (26), and a cone angle of emission η_1 can be chosen from Eq. (28). There remains only specification of the circumferential angle θ_1. Because of the assumption made earlier that emission did not depend on θ_1, it is easily shown by the formalism outlined, and is also fairly obvious from intuition, that θ_1 can be determined by

$$\theta_1 = 2\pi R_\theta \tag{31}$$

where R_θ is again a random number chosen from the range between 0 and 1.

Because the position of plane A_2 with respect to dA_1 is known, it is a simple matter to determine whether a given energy bundle will strike A_2 after leaving dA_1 in direction (η_1, θ_1). (It will hit A_2 whenever $\cos \theta_1 > 0$, as shown

in Fig. 4.) If it misses, another bundle must be emitted from dA_1. If the bundle strikes A_2, it must be determined whether it is absorbed or reflected. To do this, we use geometry to find the angle of incidence η_2 of the bundle onto A_2:

$$cos\ \eta_2 = \sin \eta_1 \cos \theta_1 \tag{32}$$

Knowing the absorptivity of A_2 from Kirchhoff's law

$$\alpha_2(\lambda, \eta) = \epsilon_2(\lambda, \eta) \tag{33}$$

and, having determined the wavelength λ of the incident bundle from Eq. (26) and the incident angle η_2 from Eq. (28), the probability of absorption of the bundle at A_2 can be determined. The probability of absorption is simply the absorptivity of A_2 evaluated at η_2 and λ. This is because the definition of directional spectral absorptivity $\alpha_2(\lambda, \eta)$ is the fraction of energy incident on A_2 (in a given wavelength interval) from a given solid angle that is absorbed by the surface. This is also a precise definition of the expected probability of absorption of an individual bundle. The absorptivity is therefore the probability density function for the absorption of incident energy. It is now easy to determine whether a given incident energy bundle is absorbed by using the first of the two event-choosing methods just outlined, that is, by comparing the surface absorptivity $\alpha_2(\lambda, \eta)$, which corresponds to $P(\xi)$, the probability of absorption, with a random number R_α. If

$$R_\alpha \leqslant \alpha_2(\eta_2, \lambda) \tag{34}$$

the bundle of energy is absorbed, and a counter S_2 in the computer memory is increased by one. Otherwise, the bundle is assumed to be reflected and is henceforth neglected. If it were not neglected, re-reflections from dA_1 would have to be considered. This neglect is reasonable if the absorptivity of A_2 is large, or if the directional reflectivity is such that few bundles are reflected back along the direction of incidence. If such reflections cannot be neglected, angles of reflection must be chosen from known directional reflectivities; and the bundle is followed further along its path until it is absorbed or lost from the system. For the purposes of this example, little is to be gained by following the bundle after reflection from surface A_2, because the derivation of the necessary relations is similar to that already presented, and the bundles are therefore neglected.

A new bundle is now chosen at dA_1, and its history is followed. This procedure is continued until all N bundles have been emitted from dA_1. The energy absorbed at A_2 is then calculated from Eq. (21).

We have now completed derivation of the equations needed for solution of the example problem. In putting together a flow chart to aid in formulating

a computer program (Fig. 5), some methods for shortening machine computing time can be invoked. For example, the angle θ is computed first. If the bundle is not going to strike A_2 on the basis of the calculated θ, there is no point in computing λ and η for that bundle. Alternately, because θ values are

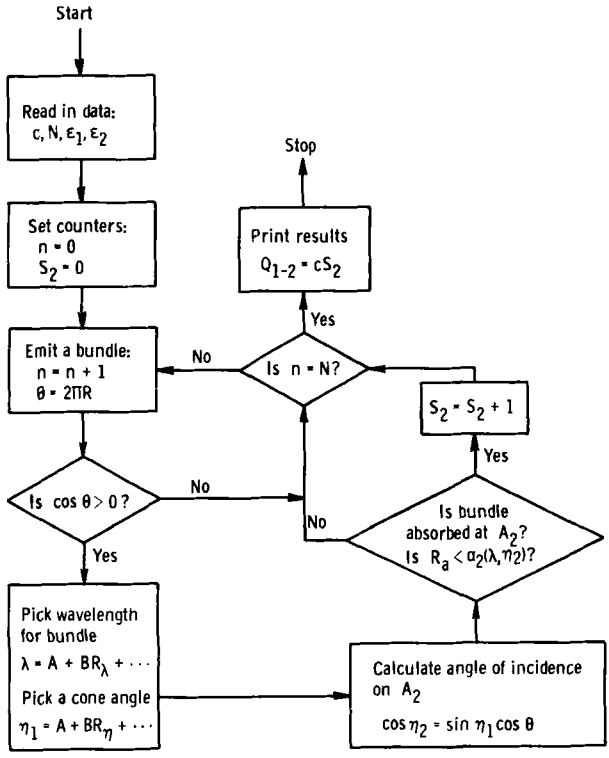

FIG. 5. Computer flow diagram for example radiant interchange problem.

isotropically distributed, it can be noted that exactly half the bundles must strike A_2. Therefore, the calculated θ values can be constrained to the range $-\pi/2 < \theta < \pi/2$.

The solution of this problem by Monte Carlo is now complete. An astute observer will note that this example could be solved without much trouble by standard methods. A more astute observer might note further that extension to only slightly more difficult problems would cause serious consequences for the standard treatments. For example, consider introducing a third surface with directional properties into the problem and accounting for all interactions.

3. *Useful Functions*

A number of useful relations for choosing angles of emission and spectral properties of bundles are given in the previous section. These and other functions from the literature dealing with radiative transfer are summarized in Table I.

B. Application to Surface Interaction Problems

Several means are available in the literature for solving problems of radiative transport between surfaces in the absence of absorbing media. These methods are called standard or conventional methods herein, and include the techniques developed by Poljak (*21*), Hottel (*22*), Oppenheim (*23*), and Gebhard (*24*) as well as formulation in terms of integral equations. Each of these methods has advantages for certain types of problems, and all will outshine the Monte Carlo approach in speed and accuracy over some limited range of radiation calculations. This range is outlined roughly by the complexity of the problem, and the areas of usefulness of the Monte Carlo approach will now be discussed.

The chief usefulness of Monte Carlo to the thermal radiation analyst lies in this fact: Monte Carlo program complexity increases roughly in proportion to problem complexity for radiative interchange problems. This is an important advantage, because the difficulty of carrying out conventional solutions increases roughly with the square of the complexity of the problem due to the matrix form into which such formulations fall. However, because Monte Carlo is somewhat more difficult to apply to the simplest problems, it is most effective in problems where complex geometries and variable properties must be considered. In complex geometries, Monte Carlo has the advantage that simple relations will specify the path of a given energy bundle, whereas most other methods require explicit or implicit integrations over surface areas. Such integrations become difficult when a variety of curved or skewed surfaces are present.

1. *Exchange Factor Computation*

The calculation of radiative exchange factors by standard means involves certain assumptions that limit the usefulness of these factors. Chief among these assumptions are that the surfaces involved are gray diffuse emitters and reflectors, that each surface is isothermal, and that the total flux arriving at and leaving each surface is evenly distributed across the surface. Any of these assumptions may be very poor; most surfaces are neither gray nor diffuse, and in almost no case is the distribution of flux uniform. Where deviations from the assumptions must be considered, calculation of the exchange factors becomes difficult, and if geometries with nonplanar surfaces

are involved, Monte Carlo techniques may become invaluable. It should be noted, however, that unless a parametric study of the interchange of radiant energy within an enclosure with specified characteristics is being carried out, it may well be easier to compute the entire radiative flux distribution by Monte Carlo. This would appear simpler than computing exchange factors by Monte Carlo and then using an auxiliary program to calculate energy exchange by means of these factors.

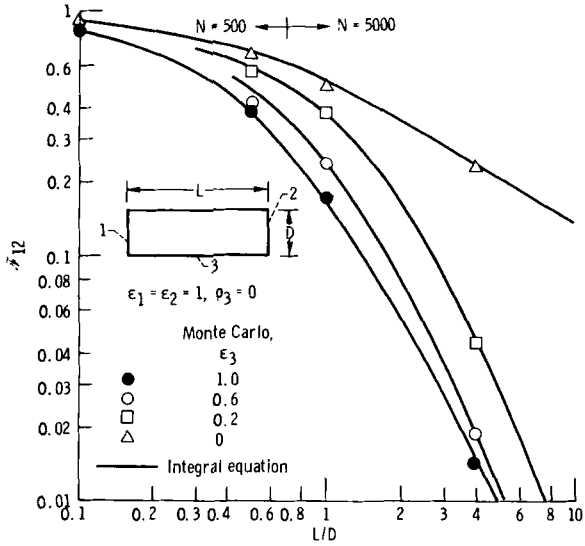

FIG. 6. Radiation exchange factors between the black ends of a diffuse-walled cylinder [from Corlett (25)].

As computed by Monte Carlo, exchange factors are identically equal to the fraction of the total emitted energy bundles from a surface that is incident upon a second surface. No restrictions are made to gray, diffuse surfaces with evenly distributed emitted or reflected flux.

Corlett (*25*) has computed modified exchange factors corresponding to Hottel's $\overline{F}$ (*22*) for including the effects of refractory surfaces. This was done for a variety of geometries, including louvres, and circular and square ducts with various combinations of diffusely and specularly reflecting interior surfaces and ends. One set of results, the interchange factors between the black ends of a cylinder with diffusely reflecting internal surface, is shown in Fig. 6.

TABLE I

CONVENIENT FUNCTIONS FOR THERMAL RADIATION PROBLEMS

Phenomenon	Variable	Condition	Relation
Emission from a surface (assumed to have no dependence on circumferential angle, θ)	Cone angle, η	Diffuse emitter	$\sin \eta = R^{1/2}$
		Directional gray emitter	$R = \dfrac{2\int_0^{\eta} \epsilon(\eta') \sin \eta' \cos \eta' \, d\eta'}{\epsilon_T}$
		Directional nongray emitter	$R = \dfrac{2\pi \int_0^{\eta}\int_0^{\infty} \epsilon(\lambda', \eta') i_\lambda \sin \eta' \cos \eta' \, d\lambda \, d\eta'}{\epsilon_T \sigma T^4}$
	Circumferential angle, θ	Diffuse emitter	$\theta = 2\pi R$
	Wavelength, λ	Black or gray	$F_{0-\lambda} = R$
		Nongray diffuse	$R = \dfrac{\pi \int_0^{\lambda} \epsilon(\lambda) i_\lambda \, d\lambda}{\epsilon_T \sigma T^4}$

		Nongray nondiffuse	$R = \dfrac{2\pi \int_0^{\lambda} \int_0^{\pi/2} \epsilon(\lambda', \eta') i_\lambda \sin \eta' \cos \eta' \, d\eta' \, d\lambda}{\epsilon_T \sigma T^4}$
Emission from a volume element of gas with absorption coefficient κ_λ	Cone angle, η		$\cos \eta = 1 - 2R$
	Circumferential angle, θ		$\theta = 2\pi R$
	Wavelength, λ	Gray	$F_{0-\lambda} = R$
		Nongray	$R = \dfrac{\int_0^{\lambda} \kappa_\lambda i_\lambda \, d\lambda}{\int_0^{\infty} \kappa_\lambda i_\lambda \, d\lambda}$
Absorption in a gas of absorption coefficient κ_λ	Path length, r	Properties constant along r	$r = -(1/\kappa_\lambda) \ln R$
		Properties vary along r	$-\int_0^{r} \kappa_\lambda(r') \, dr' = \ln R$

Weiner *et al.* (*26*) carried out the Monte Carlo evaluation of some simple exchange factors for comparison with analytical solutions. They then considered energy exchange within a five-sided enclosure, each side being assumed to have a directional emissivity dependent upon cone angle of emission and each having a specular reflectivity. They also worked out the case of interchange within a simulated optical system. This system is constructed of an enclosure made up of a combination of spherical and conical surfaces that encloses a cylindrical specular reflector with two surfaces. This is obviously an interchange problem to cause many unhappy hours of analyzing integral limits in the usual formulations.

2. *Cavity Properties*

At least one Monte Carlo solution exists in the literature for a surface interaction with a distant source. This is the case of a conical cavity with diffusely reflecting inner surface. Polgar and Howell (*27*) analyzed the bidirectional reflectivity of the cavity when exposed to a beam of parallel

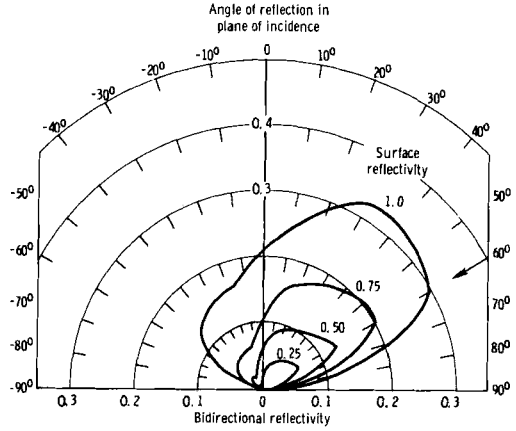

FIG. 7. Bidirectional reflectivity of a diffusely reflecting conical cavity. Cone angle, 30° angle of incidence of radiation, 60° [from Polgar and Howell (27)].

incident radiation, and also determined the directional emissivity of the cavity. Parameters varied were the angle of incidence, cone angle, and emissivity of the inner surface of the cone. One set of representative results is shown in Fig. 7. No results were found in the literature for direct comparison of the computed directional properties; however, the hemispherical emissivity results were obtained by integrating the directional values, and were

compared by Polgar and Howell (*28*) to analytical results of Sparrow and Jonsson (*29*). The comparison is shown in Fig. 8.

Because of the small number of energy bundles reflected from the cone interior through any given area element on a unit hemisphere imagined over

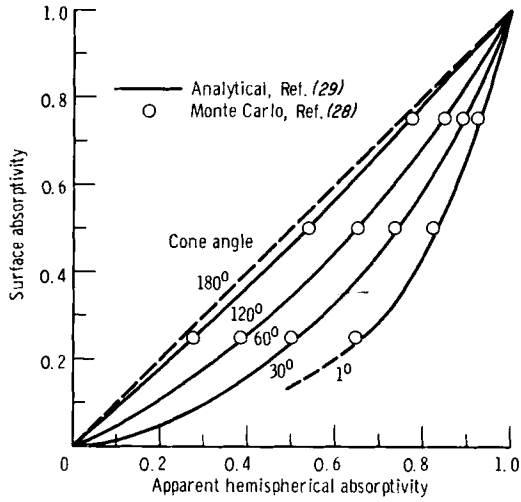

FIG. 8. Comparison of Monte Carlo results for the apparent absorptivity of conical cavities [Polgar and Howell (28)] with analytical results [Sparrow and Jonsson (29)].

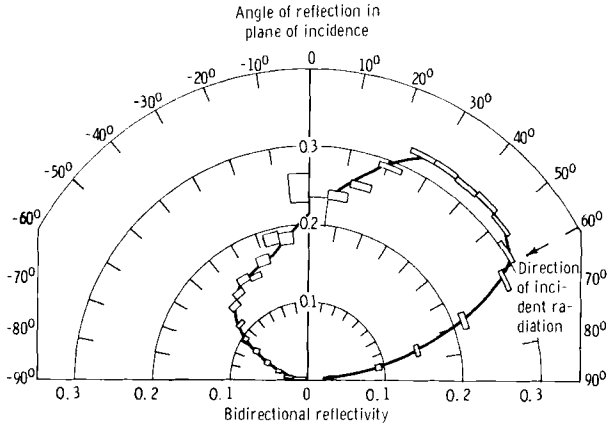

FIG. 9. Expected standard deviation of results for the bidirectional reflectivity of diffuse conical cavities; Cone angle, 30° surface reflectivity, 1.0 [from Polgar and Howell (27)].

the conical cavity, the bidirectional results of Polgar and Howell (*27*) had pronounced scatter. An example of this is shown in Fig. 9, where the standard deviation of the computed reflectivity at various angles of reflection is shown. The solid angle subtended by area elements on the hemisphere varies with the sine of the angle of reflection so that the number of sample energy bundles per unit solid angle near the cone axis becomes very small. This leads to larger errors at angles near the cone axis.

3. *Extension to Directional and Spectral Surfaces*

Because of the difficulty of mathematically treating the exchange of energy between surfaces with properties that depend on wavelength and direction, these variables are often assumed unimportant without justification. Attempts to predict the effect of these variables for a given material and surface have not been notably successful because of the difficulty of prescribing the random characteristics that the surfaces may possess.

Nelson (*29a*) has employed a Monte Carlo analysis of ray reflections from dielectrics, assuming the reflections to occur from a surface prescribed by various periodic and randomly oriented surface elements. The distribution of reflected energy is built up by considering the reflection of many rays. Kaye *et al.* (*29b*) have considered the reflection from highly reflecting paints by examining the scattering process occurring from particles within the layer of pigmented coating.

Toor (*29c*) and Viskanta *et al.* (*29d*) have used the directional and spectral properties of surfaces to compare the energy exchange using these properties to the exchange under various simplifying assumptions. Considerable variation between the results for directional properties and assumed distributions was found.

Application of Monte Carlo to surfaces having both directional and spectral variations have not yet appeared in the literature. The reasons for this omission seem two-fold.

First, accurate and complete directional and spectral properties, especially the former, are not often found in the literature. A researcher desiring to include such effects in his analysis might thus be unable to find the requisite data for any given system.

Second, when solutions are obtained for such problems, they are often so specialized that little interest exists to warrant their wide dissemination in the open literature.

As pointed out by Dunn *et al.* (*30*), when the radiative properties become available, the methods for handling such surface radiative energy exchange problems now exist, and Monte Carlo appears to be one of the better-suited techniques.

C. APPLICATION TO RADIATION IN ABSORBING-EMITTING MEDIA

1. *Introduction*

Monte Carlo has more obvious utility in solving problems of radiative transfer through absorbing-emitting media than for surface radiation interchange problems. This is because complete definition of the local radiation balance in a gas or other absorbing-emitting media requires integration of the incoming radiation, not only from the surrounding surfaces, but from all volume elements of the surrounding medium. Such problems are difficult to solve at best, and much effort has been expended in attempts at developing standard solution methods and simplifications. This is often done by making as many assumptions, reasonable if possible, as are necessary to obtain an answer and philosophically accepting the resulting loss of accuracy, if not validity. Surfaces that are black, gray, diffuse, or specular, and gases that are opaque, transparent, gray, or isothermal are assumptions that fall into this category. Few problems in radiative transfer are solved without explicitly or implicitly making one or more of these assumptions.

By applying the model of radiative energy exchange outlined previously, it is possible to account for a large variety of effects in gas radiation problems. This can be done without resort to the simplifying assumptions that are often necessary in the analytical approaches as typified by the work of Usiskin and Sparrow (*31*), Hottel and Cohen (*32*), and Viskanta and Grosh (*33*).

The additional step to be introduced to the model previously discussed is a determination of the path length that an individual bundle travels before it is absorbed, scattered, or lost from the system. The required relations are given in Table I. It is possible to allow for variations in gas properties along the bundle path; indeed, it is in principle possible to account for variations in the refractive index of the medium by causing the bundles to travel curved paths, although this has not been done to date.

If a problem is solved in which radiative equilibrium can be assumed, then whenever a bundle is absorbed in the medium, a new bundle must be emitted from the same point in the medium to ensure no accumulation of energy. The functions required for determination of the angles and wavelengths of emission are shown in Table I. The new emitted bundle in the medium may be considered as merely the continuation of the history of the absorbed bundle and the history is continued until the energy bundle reaches a bounding surface.

Under these conditions, the total energy Q_e emitted by a volume element dV is given by

$$Q_e = 4 \int_0^\infty \kappa_\lambda e_\lambda \, dV \, d\lambda \tag{35}$$

But the energy contained in the bundles emitted by the volume must be equal to the energy contained in the bundles absorbed, or,

$$Q_e = cS_{dV} \tag{36}$$

where S_{dV} is the number of bundles absorbed in dV. Then, if we note that

$$\bar{\kappa}_p \equiv \frac{\int_0^\infty \kappa_\lambda e_\lambda \, d\lambda}{\sigma T_{dV}^4} \tag{37}$$

where $\bar{\kappa}_p$ is the Planck mean absorption coefficient for radiant energy, we can substitute Eq. (37) into Eq. (35) to eliminate the integral, and then equate (35) and (36) to give

$$T_{dV} = \left[\frac{cS_{dV}}{4\bar{\kappa}_p \sigma \, dV}\right]^{1/4} \tag{38}$$

This allows determination of the local temperature in the gas in terms of gas properties and known Monte Carlo quantities. If $\bar{\kappa}_p$ is independent of T_{dV}, then the determination of T_{dV} is complete. Otherwise, a new iteration is run on the basis of gas properties evaluated using the temperature distribution just calculated. The process is repeated until the temperatures converge.

There are so many variations possible on this model, many of which might lead to increased efficiency, that they cannot all be mentioned. One of the most frequently suggested is the fractional absorption of energy when a bundle reaches a surface of known absorptivity. Using such a scheme, the bundle energy is reduced after each reflection. The bundle history is then followed until a sufficient number of reflections have occurred to reduce the bundle energy below some predetermined level. This level is chosen so that the effect of the bundle in succeeding reflections would be negligible. The history is then terminated. Such a procedure leads to better accuracy for many problems, because a bundle history extends on the average through many more events, and a given number of bundles provides a larger number of events for compiling averages.

The obvious rule of thumb is the following: Use whatever shortcuts can be applied to the case in question, and do not be bound by cookbook rules.

Some problems that have been solved by Monte Carlo techniques will now be examined.

2. *Radiation through Gray Gases*

a. Gray Gases between Infinite Parallel Planes. This might be called the "Fanny Hill problem" of radiative transfer, as almost every researcher in the field has tried it at least once (see, for example, the literature (*31–42*)). Because of the wealth of solutions available in the literature, almost every

new method of solution is tried in this geometry and then compared to the results of one or more of the exact analytical approaches typified by the work of Usiskin and Sparrow (*31*) and Heaslet and Warming (*35*).

The Monte Carlo method is no exception, and in the work of Howell and Perlmutter (*42*) the local gas emissive power and the net energy transfer between the plates is calculated. Parameters are the plate emissivity ϵ (taken equal for both plates), and various values of the gas optical thickness τ. The optical thickness is the product of the gray gas absorption coefficient, κ, and the plate spacing D. Two cases are examined, the first being a gas with no

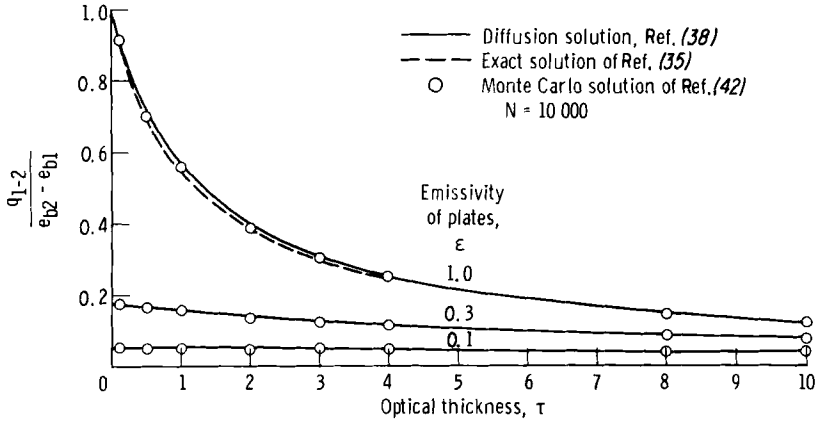

FIG. 10. Heat transfer between infinite parallel plates separated by a gray gas.

internal energy generation contained between plates at different temperatures and the second a gas with uniformly distributed energy sources between plates at equal temperatures. Figure 10 indicates the accuracy that can be obtained by Monte Carlo solutions in such idealized situations. The calculated energy transfer values have a 99.99% probability of lying within $\pm 5\%$ of the midpoints shown.

In Fig. 11, the emissive power distribution within the gas is shown. Comparison with the exact solution of Usiskin and Sparrow (*31*) is quite good; however, some trends common to all straightforward Monte Carlo solutions in gas radiation problems are evident.

First, the calculated points in Fig. 11 reveal increasing scatter with decreasing optical thickness. This reflects the smaller fraction of energy bundles being absorbed in a given volume element as the optical thickness of the gas decreases. As the number of absorbed bundles decreases, the expected accuracy of the local emissive power becomes less, and more scatter naturally appears in the results. Conversely, as the optical thickness increases, scatter

becomes less, and in Fig. 11 the curve of results for an optical thickness of ten is quite smooth.

A second effect mentioned by Howell and Perlmutter (*42*) is not evident from Fig. 11, and that is that the machine time required for solution of problems involving large optical thickness, say larger than ten, becomes quite large. This is simply because the free path of an energy bundle, as defined in Table I, becomes very short for large optical thickness; therefore, many absorptions occur during a typical bundle history.

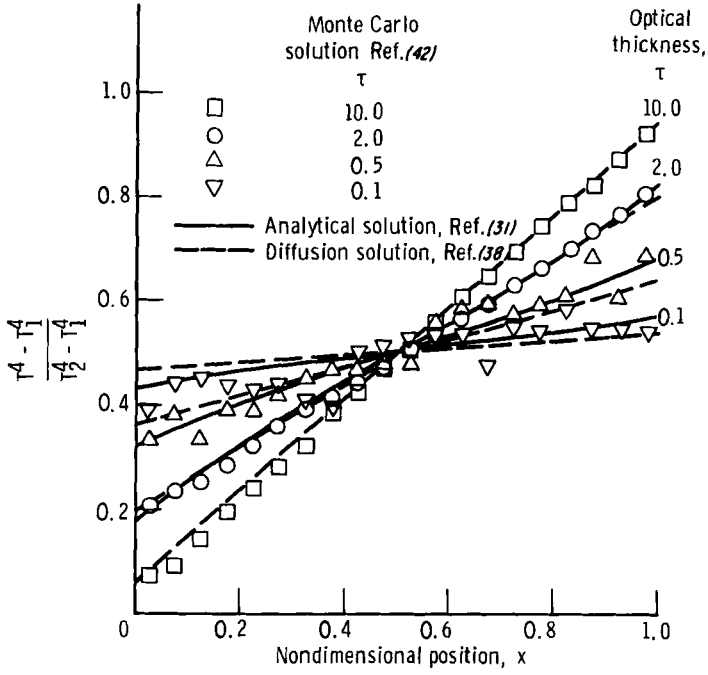

FIG. 11. Emissive power distribution in a gray gas between black infinite parallel plates.

Two limits are now obvious. For small optical thickness accuracy becomes poor, and for large optical thickness running time becomes excessive. From a practical viewpoint, these are not serious limitations, as the so-called transparent and diffusion approximations to the exact analytical formulation become valid in just those regions where the straightforward Monte Carlo approach begins to fail. In addition, the range of optical thickness over which a Monte Carlo solution can be effectively utilized is easily extended by a variety of techniques, including those with the graphic names of "splitting,"

"Russian roulette," and a large number of specialized schemes for specific solutions (*6–8*). Many of these involve biasing the free path length to increase the number of bundles absorbed in otherwise weakly absorbing regions. Such techniques can be looked upon as analogous to applying mathematical sleight-of-hand to increase the rate of convergence of a set of integral or finite difference equations, in which case the tricks may appear somewhat less obscene to the analytically minded.

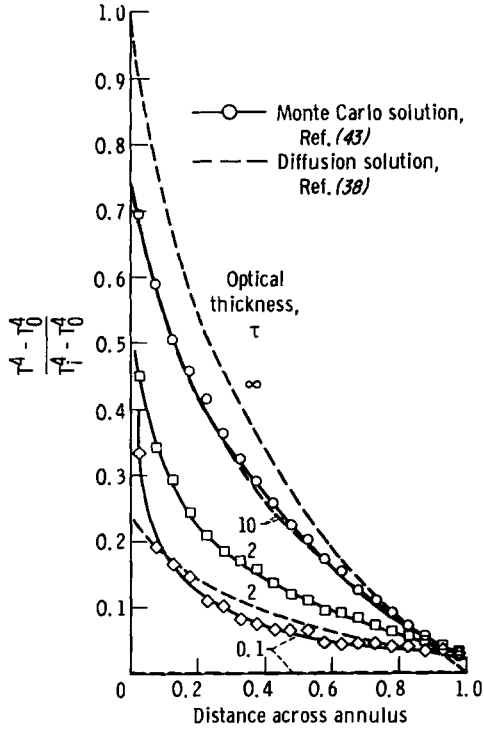

FIG. 12. Emissive power distribution in a gray gas in the annulus between black concentric cylinders of radius ratio 0.1.

b. Gray Gas between Infinitely Long Concentric Cylinders. A more difficult problem than infinite parallel plates to treat analytically is the determination of the emissive power distribution and local energy flux in a gray gas within the annulus between concentric cylinders. The transfer equation for determination of the emissive power at any radius involves an integral with the local radius appearing as one of the limits. The set of integral equations describing the local emissive power for each finite volume of gas thus have

integrals with limits that are different for every equation in the set. This set must be solved simultaneously.

The Monte Carlo approach differs only slightly from that for parallel planes. The only additional complication is the determination of the bundle position in terms of cylindrical coordinates.

Some Monte Carlo results for this case are shown in Fig. 12 as taken from Perlmutter and Howell (*43*). Because of the analytical difficulties of this case, no exact formulation using integral techniques is available in the literature. Comparison of the results is therefore made with a modified diffusion solution (*38*). Trends in accuracy similar to those noted for the infinite plate case are evident.

c. Radiation between Adjacent Gray Regions. A Monte Carlo solution has also appeared dealing with the interaction of radiative energy between two

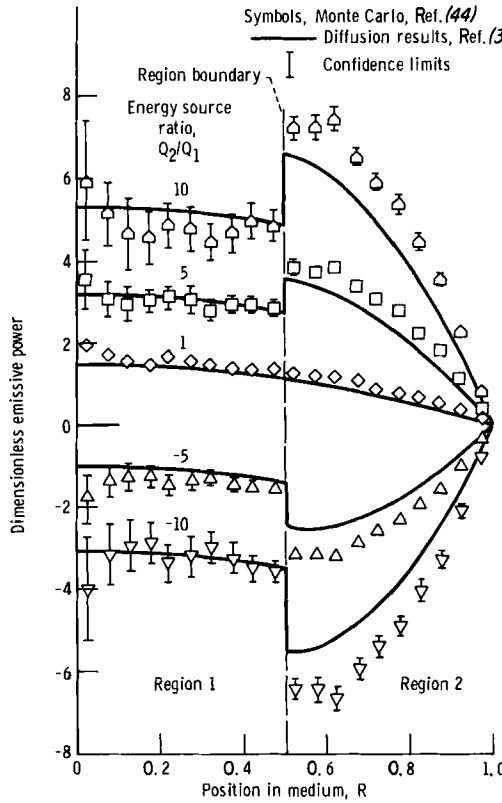

FIG. 13. Emissive power distribution in coaxial gas regions with differing energy sources. Diameter ratio 0.5; optical thickness 2 in both regions.

regions, each of which has individual radiative properties and internal energy generation rates (*44*). This is of interest because it gives another insight into sources of error due to geometrical effects. Figure 13 shows the emissive power distribution in two concentric cylindrical regions of similar optical thickness but different rates of energy generation. The vertical bars are the 95% confidence limits on the results.

The emissive power in the gas bears a direct proportionality to the number of energy bundles absorbed in a given volume element. The volume elements used in the calculation of the results shown in Fig. 13 are of equal radius and, therefore, of different volume. The elements near the center ($R \to 0$) have the smallest volume and, consequently, the smallest number of bundle absorptions. This is reflected in the increasing variance (decreasing accuracy) at these points.

3. *Consideration of Radiative Property Variations*

The greatest criticism leveled against many treatments of radiative transfer in gases is the inability to accurately account for the strong spectral, temperature, and pressure dependence of the radiative absorption coefficient. Such coefficients can be computed with reasonable accuracy by quantum-mechanical methods, but few analyses have been able to completely include the effect of all variables. Most treatments are limited to gray gases or the consideration of various types of mean absorption coefficients.

Monte Carlo is well suited to consideration of property variations. It involves very little extra effort to assign wavelengths to individual energy bundles and to allow the paths of the bundles to depend on the local spectral absorption coefficient. The relations necessary to achieve this are given in Table I.

If property variations with temperature are considered, an iterative solution is usually necessary because the temperature distribution within the medium is not generally know a priori. Determination of the path length becomes more difficult also, because the absorption coefficient varies with position. By applying the formalism outlined in Section II, B, the path length r is found to be given by

$$\ln R = -\int_0^r \kappa_\lambda(r')\, dr' \tag{39}$$

To evaluate this integral for r along a fixed line after choosing a random number R is time consuming but at least feasible. Howell and Perlmutter (*45*) used this approach, reducing the complexity somewhat by considering temperature- and wavelength-dependent hydrogen absorption coefficients in the simple geometry of the gas contained between infinite parallel plates. They considered energy transfer through the gas between plates at different

temperatures, and also the case of internal energy generation with a parabolic distribution of source strengths in the gas. To evaluate the path length, they approximated Eq. (39) by dividing the gas into plane increments of thickness Δx. The path length through a given increment was then

$$\Delta r = \Delta x / \cos \eta$$

where η is the angle between the bundle path and the perpendicular to the plates. Equation (35) could then be written

$$\ln R + \Delta r \sum_{p=1}^{i} \kappa_{\lambda,p} > 0 \tag{40}$$

and the summation was carried out until a value of i was reached that satisfied the inequality. This i value would be related to the increment number in which absorption occurs. The values of $\kappa_{\lambda,p}$ were assumed, and then recalculated on the basis of the newly computed local temperatures. This procedure was continued until convergence.

Figure 14 shows the property variations involved, and Fig. 15 shows a set of emissive power distributions calculated as outlined. Again, accuracy becomes poorer, as evidenced by increased scatter, in those regions of low temperature. This is due to the decrease with temperature of the absorption coefficient and, therefore, number of absorptions.

Other Monte Carlo solutions have included property variations; but because they also included other factors, these solutions will be discussed in succeeding sections.

4. *Coupling with Other Heat Transfer Modes*

When radiative transfer in a gas is coupled with convective or conductive energy transfer, solution becomes even more difficult. Combined energy transfer is occurring by radiation, where fourth-power temperatures are governing, and by conduction and/or convection, where first-power temperature differences are governing. Further, the radiative terms in the energy equation for such problems take the form of multiple integrals, while conduction terms contain second derivatives. Further still, the radiative surface properties which appear may be functions of wavelength, direction, and temperature. If gases are involved, these variables plus pressure can strongly affect the local gas-radiation physical properties. The complete energy balance on each element of the system then takes the form of a nonlinear integrodifferential equation.

Although in principle it is possible to treat the entire coupled problem by a Monte Carlo approach, such a treatment has not appeared in the literature. In those references treating coupled modes of heat transfer (*46–49*), the convective and conductive terms are treated as lumped energy sources or

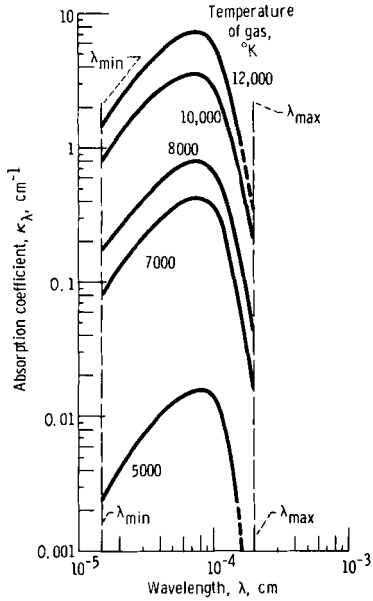

FIG. 14. Spectral absorption coefficient of hydrogen at 1000 atm [from Howell and Perlmutter (45)].

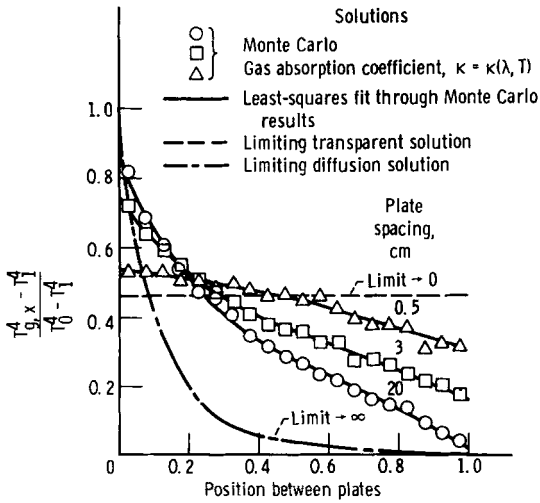

FIG. 15. Emissive power distribution in hydrogen between infinite parallel plates at temperatures $T_0 = 9500°K$, $T_1 = 4500°K$ [from Howell and Perlmutter (45)].

sinks; and Monte Carlo was used only to evaluate the remaining radiative terms on the basis of an assumed temperature distribution. With the radiative terms evaluated and substituted into the original equations, conventional numerical techniques were applied to the resulting differential equations and a local temperature distribution was generated. This was used as a basis for re-evaluating the radiative integrals, and the procedure was continued until convergence.

An example of the power of the Monte Carlo approach in such problems is given by Howell *et al.* (*49*). Here the local temperature distribution as a function of length and radius and the axial heat flux distribution in a conical rocket nozzle were determined under conditions expected in a gas-core nuclear propulsion system. Variations in physical properties with local temperature, pressure, and wavelength were examined, albeit not simultaneously, and coupled radiation and convection were considered. In addition, the ability of a layer of optically thick gas injected along the nozzle wall to attenuate the extreme predicted radiative fluxes was demonstrated.

Such a system, involving combined heat transfer modes in a flowing system with variable physical properties and an arbitrary axisymmetric bounding surface would at best be a laborious test of the more conventional approaches to radiative transfer. Yet the Monte Carlo program was able to predict temperature profiles in the propellant, converged to within 0·1% at all points, along with axial heat-flux distributions at the nozzle wall. All this was done in less than 10 min of machine time on an IBM 7094 computer.

5. *Transient Radiation Problems*

Chief contributors to the development of Monte Carlo techniques for radiative transfer under transient conditions have been Fleck (*50*, *51*) and Campbell (*52*) and Campbell and Nelson (*53*). The model they apply is essentially that outlined in previous sections, with the additional proviso that the flight times between events in the history of each bundle are computed, which adds considerable complexity to the problem. Bundles are followed along their paths, and their position at some time t is used to determine the distribution of energy at that time. The bundles of course travel at a velocity equal to the velocity of light in the medium.

The review article by Fleck (*50*) gives a comprehensive discussion of applications, including the effects of scattering under transient conditions.

6. *Incorporation of Scattering Phenomena*

Scattering of radiation is easily treated by Monte Carlo for any given distribution of scattering angles. It is analyzed exactly as absorption and non-isotropic re-emission in a gas volume, except that the scattering process does not affect the energy balance on a volume element.

Collins and Wells (*54*) and Plass and Kattawar (*54a*) have studied the transmission of light when scattering is an important phenomenon by using Monte Carlo. They examined the effects of Rayleigh scattering, and of Mie scattering from particles of a given size distribution. Multiple scattering within an atmosphere of arbitrarily described density distribution and effects due to ground and cloud reflections were included.

D. COMMENTS ON MONTE CARLO FOR THERMAL RADIATION

1. *Disadvantages of Monte Carlo Technique*

Monte Carlo calculations give results that fluctuate around the "real" answer, because the method is a repetitive experiment on a mathematical model used in place of the actual physical situation. The uncertainty can be found by applying the standard statistical tests; and the uncertainty can be reduced in the same manner as experimental error, that is, by averaging over more tests (bundle histories), or by reducing the variance of individual tests.

No rigorous test exists to guarantee the convergence of Monte Carlo results to valid solutions. This has not as yet proven to be a difficulty in thermal radiation problems. It would often be immediately obvious that convergence to invalid solutions was occurring because of the limiting solutions and physical constraints that are known for most radiation problems.

Most of the other problems that do arise in sampling techniques have to do with obtaining an optimum sample size. For example, the free path of an energy bundle between absorptions becomes quite short in optically thick gases. Any energy bundle undergoes a large number of absorptions and emissions in the gas before traversing the distance between the plates. This means that the computer time needed to obtain a sample of bundles transferred between the plates becomes prohibitively large in such a case. Conversely, for nearly transparent gases, relatively few absorptions occur in any given element of gas, and an insignificant energy sample is available to accurately calculate the temperature of the gas element.

Both these problems are common enough in transport processes that are mathematically related to radiative transport that special methods of "weighting" the free paths of bundles have been developed to obtain adequate samples, thereby saving computer time and gaining accuracy at the expense of added complexity.

In the gas-radiation problems available so far in the literature, it has been found that problems which cannot be accurately handled by either the transparent or diffusion approximations fall into the range of mean free paths that allow efficient Monte Carlo programming without reverting to weighting techniques.

2. *Closing Remarks*

Monte Carlo is discussed in the preceding section as a method suitable for use in the solution of complex problems in radiative transfer. A sample problem is outlined to demonstrate its application, and some of the advantages and disadvantages of the technique are discussed along with pertinent literature references.

From all this, certain conclusions emerge. First, Monte Carlo appears to have a definite advantage over other radiative-transfer calculation techniques when the difficulty of the problem being treated lies above some undefined level. Just where this level is cannot be established, as it is probably a function of the experience, competence, and prejudice of the individual working the problem. However, problems above this nebulous benchmark in complexity can be treated by Monte Carlo with greater flexibility, simplicity, and speed. Monte Carlo does lack a kind of generality common to other approaches in that each problem may require an individual technique and a dash of ingenuity often helps. This places a greater burden on the programmer's backlog of experience and intuition where standard methods may allow programming through "cookbook" application of their formalism if they can be applied at all.

Second, for thermal-radiation problems, the parameters and mathematical relations involved lie in ranges which allow straightforward Monte Carlo programming without the need of the more exotic schemes occasionally necessary in other Monte Carlo transport studies.

Third, with all its advantages, the method suffers from certain problems. The worst of these are the statistical nature of the results and the lack of guaranteed convergence. It should be noted that the latter fault is common to all methods when complex problems are treated, because rigorous mathematical criteria to guarantee convergence are available only in very specialized cases.

Finally, it must be commented that the person using Monte Carlo techniques often develops a physical grasp of the problems encountered, because the model being analyzed is simple, and the mathematics describing it are therefore on a less sophisticated basis. This is in contrast to the rather poor physical interpretations and predictions which we can make when working with, say, a matrix of nonlinear second-order integrodifferential equations.

IV. Rarefied Gas Dynamics and Heat Transfer

A. PHYSICAL PROBLEM AND THE NEED FOR MONTE CARLO

The analogy between rarefied gas and radiant heat transfer has often been exploited (*55*) to attain solutions for one type of problem from available work on the other. This analogy holds also for Monte Carlo solutions, which have

been applied to both phenomena for essentially the same reason: It is much easier to deal with the actions of individual molecules or photons than with the integral equations describing the motions of the entire set.

In rarefied gas problems, the velocity distribution of the molecules in thermodynamic equilibrium is given by the Maxwellian velocity distribution. However, when significant energy transfer occurs, departures from equilibrium may become pronounced, and the already difficult Boltzmann equation of transfer becomes hopeless from the standpoint of obtaining closed-form analytical solutions.

B. MOLECULAR FLOW PROBLEMS WITH COLLISIONS

Haviland (*56*), who seems to have been the first to successfully apply Monte Carlo methods to molecular flow problems, gives an excellent review and discussion of Monte Carlo techniques. He presents solutions to the problems of heat transfer between parallel walls and the determination of

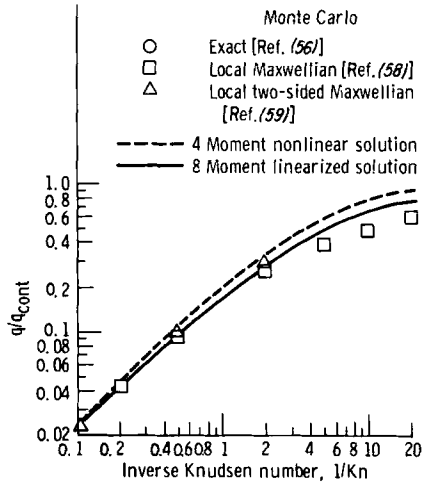

FIG. 16. Rarefied gas heat transfer between infinite parallel plates compared to heat transfer by a continuum gas.

shock layer profiles. Fifty references are given to related work up to the date of publication early in 1965. For that reason, no detailed outline will be presented here.

Haviland's work is an attempt to generate the actual local velocity distributions in gases with intermolecular collisions. A direct analog to the collision process is used, and sample molecules are followed throughout their histories.

Hicks and Aggarwal (*57*) approach the solution of the Boltzmann equation by evaluating the collision integral via a Monte Carlo method, and then solving the remaining differential form of the equation by standard numerical means. The integral is then re-evaluated, and the procedure is continued until convergence. Such a procedure can no longer be considered a physical analog of the molecular interactions, and it is an example of the auxiliary use of Monte Carlo techniques in problem solving.

In a recent series of papers (*58–60*), Perlmutter has assumed various approximate forms for the local velocity distributions. These distributions are converged iteratively through adjusting certain parameters that appear in the distributions. This procedure should in principle allow faster convergence and therefore shorter running times than does Haviland's approach. A good comparison to more exact solutions has been demonstrated, as is shown in Fig. 16. At an inverse Knudsen number of two, the results of Haviland (*56*) and Perlmutter (*58, 59*) are indistinguishable.

C. Free Molecule Flow

When no intermolecular collisions are present, the molecular flow problem is considerably simplified. Such a situation is called free molecule flow. Monte Carlo has been used in the practical case of vacuum pumping through various geometries. The work of Davis (*61*) and Ballance (*62*) are representative. Perlmutter has extended the method to account for the effects of various magnetic field orientations on ion paths (*63–65*).

V. Solution of Heat Conduction Problems

The relation between the differential equations describing the conduction of heat and a simple stochastic model describing this phenomenon has been known for some time. Chandrasekhar (*66*) described this relation in detail over 25 years ago, and notes in his paper that Lord Rayleigh (*67*) had treated an analogous problem in 1880.

Perhaps because of the ease with which the physical model can be grasped, a variety of workers has applied Monte Carlo techniques to conduction problems (*68–74*). However, because of the powerful standard analytical tools available to any engineer with a copy of the book by Carslaw and Jaeger (*75*), it has generally been found that Monte Carlo solutions require excessive time and effort in comparison. A few classes of problems remain in which Monte Carlo is competitive, and, as digital computers gain in speed and new methods are found to increase program efficiency and further reduce running time, conduction problems may again be fair game. Keeping in mind the

limitations noted, let us examine the method of attack. We can then see clearly those classes of conduction problems to which Monte Carlo can be profitably applied at present.

A. DERIVATION OF MODEL FROM THE HEAT EQUATION

Let us write the heat equation for a two-dimensional problem as

$$a\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]=\frac{\partial T}{\partial t} \tag{41}$$

where a is the thermal diffusivity of the material. By the usual finite difference approximations this can be placed in the form

$$\frac{a}{4(\Delta x)^2}[T(l+1,m,s)-2T(l,m,s)+T(l-1,m,s)]$$

$$+\frac{a}{4(\Delta y)^2}[T(l,m+1,s)-2T(l,m,s)+T(l,m-1,s)]$$

$$=\frac{T(l,m,s+1)-T(l,m,s)}{\Delta t} \tag{42}$$

where l, m, and s are lattice points on a three-dimensional grid representing space coordinates in x and y and a time coordinate t, respectively. If we choose Δt, Δy, and Δx such that

$$\frac{a\,\Delta t}{(\Delta y)^2}=\frac{a\,\Delta t}{(\Delta x)^2}=1 \tag{43}$$

then eq. (42) can be written as

$$T(l,m,s+1)=\tfrac{1}{4}[T(l+1,m,s)+T(l-1,m,s)+T(l,m+1,s)+T(l,m-1,s)] \tag{44}$$

Now suppose for a moment that the random walk pedestrian of Section II, A is walking on a similar grid. If his choice of direction at each grid point is equally probable, then if he arrives at point $(l,m,s+1)$ his probability of being there must be one-fourth of the probability of being at the surrounding points at the previous step. Therefore

$$P(l,m,s+1)=\tfrac{1}{4}[P(l+1,m,s)+P(l-1,m,s)+P(l,m+1,s)+P(l,m-1,s)] \tag{45}$$

which is obviously of the same form as Eq. (44). Then, by replacing P by T, we see that the temperature at point (l,m) and time (s) can be found by solving the differential equation describing the conduction process by means of the appropriate random walk scheme.

B. EXAMPLE PROBLEM

Suppose, for example, that we prescribe a thin plate of finite size with known properties and an initial temperature distribution $T(l,m,s=0)$. We require the distribution of temperature at time $t = s_f \Delta t$. We choose a grid of sufficient size to ensure that our random walkers will reach an edge of the plate in reasonable time, yet provide enough lattice points to give a good indication of the distribution of temperatures.[3] The time and distance increments are chosen to satisfy Eq. (43).

If we further specify that the edges of the plate are insulated, then the net number of walkers crossing an edge at any position must be zero. This type of boundary condition becomes clear if we consider the random walkers to be heat "particles" travelling through the lattice. This interpretation comes from consideration of the derivation of the heat equation, which is simply a balance of energy on a volume element. The random walkers must therefore obey boundary conditions applicable to energy flux, and a particle crossing an insulated edge must be balanced by another particle crossing the edge in the opposite direction. Thus a net flux of zero is maintained at the boundary. If net flux other than zero is prescribed, then enough walkers are allowed to cross the boundary to satisfy the condition.

A walker reaching an insulated boundary (zero net flux) can be considered to be reflected from the boundary. In the problem at hand, the history of the walker is continued until s_f steps have been taken.

To determine the final temperature distribution in the plate, we start some total number N of bundles through their histories. The fraction of these histories originating at lattice point $(l,m,0)$ is determined by the relation, assuming equal lattice spacing over the region,

$$\frac{n(l,m,0)}{N} = P(l,m,0) = \frac{T(l,m,0)}{\sum_m \sum_l T(l,m,0)} \tag{46}$$

Since the temperature is a measure of the internal energy of a body, the right-hand term is the fraction of the total internal energy that is initially ascribed to each lattice point. Modification of Eq. (46) becomes necessary if certain regions (i.e., near boundaries) have lattice spacings that are not equivalent to those in the interior.

To solve our problem, each particle is started from its origin point and allowed to continue its random walk until s_f steps have been taken. The lattice point at which the particle is located after step s_f is then recorded. After all N

[3] This sentence blithely hides a very large number of possible pitfalls, as anyone who has solved finite difference equations may testify.

particle histories have terminated, the new temperature distribution is given by

$$T(l,m,s_f) = \frac{n(l,m,s_f)}{N} \sum_{m=0}^{m_{\max}} \sum_{l=0}^{l_{\max}} T(l,m,s_f) \tag{47}$$

In comparison with other numerical approaches, this procedure has the advantage that the running time to solution is a very weak function of the shape of the boundary. It is almost as simple to solve for the transient behavior of the temperature distribution in a plate with oddly scalloped boundaries as in such a simple case as a disk. Further, retention of an energy source term adds little difficulty, involving the production (source) or disappearance (sink) of individual energy particles at specified positions and times. Further still, boundary conditions of almost any type can be included by modest extensions of the model.

C. LAPLACE'S EQUATION

When the heat equation (41) is written without source or transient terms (Laplace's equation) and is then placed in finite difference form, it becomes

$$T(l,m) = \tfrac{1}{4}[T(l+1,m) + T(l-1,m) + T(l,m+1) + T(l,m-1)] \tag{48}$$

if we set

$$(\Delta x/\Delta y) = 1 \tag{49}$$

This is identical in form to the original random walk equation (1), so that we can model not only conduction but all processes that are described by Laplace's equation with a simple time-independent random walk.

If a region is encompassed by an irregular boundary with some variation of temperature along the boundary, then we can find the temperature at any interior point $T(l,m)$ by following a particle from that interior point until a boundary is reached. The individual history is then terminated, and the temperature at the boundary point, $T_b(l,m)$, is recorded. This procedure is repeated for N bundles, so that N boundary temperatures are obtained. Let us now see how such a procedure allows us to determine the temperature of our origin point, $T(l, m)$.

Consider for a moment the inverse problem. If N bundles are started from lattice points on the boundary with the number originating at each lattice point determined by an equation such as Eq. (46), and then the number that reach the interior point of interest is found, we could determine the temperature of the interior point. This is effectively a stochastic application of the Cauchy integral formula. By reversing the bundle paths, then, the temperature at the interior point is found by starting all bundles at the interior and

allowing them to reach the boundary. A rigorous proof of this intuitive argument is given by Brown (*7*) and by Chandrasekhar (*66*).

The temperature at the interior point is then the average of the N boundary temperatures reached by the random walkers, or

$$T_{\text{int}} = T_b(l, m)/N \tag{50}$$

D. Extensions of the Method

We have seen how a simple random walk process leads easily to the determination of the temperature at any interior point. This capability in itself is interesting; we did not need to solve for the temperatures at all interior points, as we would with common relaxation techniques. However, because each point (l', m') is simply the location of an event in a Markov chain, each particle that passes through an intermediate lattice point (l', m') could be considered to be a history originating at that point. It is then possible to quickly generate the entire temperature distribution within the region by use of the information contained in such "subhistories" without solving for each interior temperature separately.

In addition, we have seen that the temperature at an interior point only depends on the final boundary points of the N particles. We can then use Eq. (50) to solve for T_{int} for *any* distribution of temperature on the boundary *after* solving for the final (l, m) values once and for all. Also, by truncation or extension of individual histories, the effect of slightly different boundary locations on interior temperatures can be examined. However, solution for the temperature at any interior point can sometimes be obtained by direct numerical solution of the Cauchy integral, which depends solely on the boundary conditions. This is almost invariably more efficient than the Monte Carlo solution of the same problem when it can be carried out.

For a lattice that has very small grid spacing in relation to the region of interest, the straightforward random walk approach can lead to long computer runs. The properties of the Markov chain allow some shortcuts to be taken with a view to reducing computer time with no loss of accuracy.

Brown (*7*) has noted that there is equal probability of a particle reaching any point on the perimeter of a circle drawn completely within the region boundary and around the origin of the bundle. If the circle is made as large as possible within the boundary, then all of the random wanderings within the circle can be eliminated by placing the particle on the circle perimeter at some randomly chosen point. The process can be repeated until the particle is quite close to the boundary. Haji-Sheikh and Sparrow (*68*, *71*) and Haji-Sheikh (*70*) have used essentially this approach, terming it the "floating

random walk," and have applied it to transient problems and examples with moving boundaries and nonlinear boundary conditions. They have demonstrated considerable reductions over the computer time used for the conventional random walk approach.

Emery and Carson (*69*) have modified the standard random walk approach to give what they call the "Exodus method." A large number of walkers are followed simultaneously in directions that depend on the probabilities of proceeding from one node to its neighbors. By so doing, no random number generator is required. For example, if forty walkers are dispatched from a given lattice point, ten are sent to each neighbor in the square grid. The original nodal point may in turn receive up to forty walkers from its neighbors, the number being less as some of the original walkers reach boundaries. This procedure is continued until most of the walkers have reached boundaries, and local temperatures are computed as for the standard random walk.

E. DISCUSSION

From the outline of the method as applied to heat conduction problems, we can make the following observations: If a closed-form analytical solution to a problem exists, it of course should be used. If the problem must be solved numerically, then it is worthwhile to examine Monte Carlo and the modified methods as possible approaches. The most likely applications of Monte Carlo are to problems with unusual boundary conditions, meaning unusual with respect to geometry or to the specification of conditions. Applications to problems where only the conditions at a specific interior point are of interest may be solved by Monte Carlo if a variety of boundary conditions are to be studied for a given geometry. In such a case, it was shown that one Monte Carlo solution is sufficient to establish the conditions at the interior point for *any set* of conditions at the boundary. Other numerical methods require an individual solution for each set of boundary conditions.

VI. Other Uses in Heat Transfer

Aside from the direct probabilistic modeling of radiative, rarefied gas and conduction heat transfer that was discussed in Sections III, IV, and V, respectively, it is possible to carry out a number of other computations via Monte Carlo. Most of these are at best of peripheral interest to the heat transfer analyst, but a few have direct applications in the field. Some of the techniques will be briefly outlined.

A. EVALUATION OF INTEGRALS

The essence of the Monte Carlo method of evaluating integrals is best demonstrated by a simple example. Consider the integral

$$\text{int} = \int_0^1 f(x)\,dx = \int_0^1 x\,dx \tag{51}$$

shown in Fig. 17. This integration is trivial to carry out directly, but suppose that we desired an answer using Monte Carlo. One approach is to pick N pairs of random numbers. Since the ordinate and abscissa in the example run

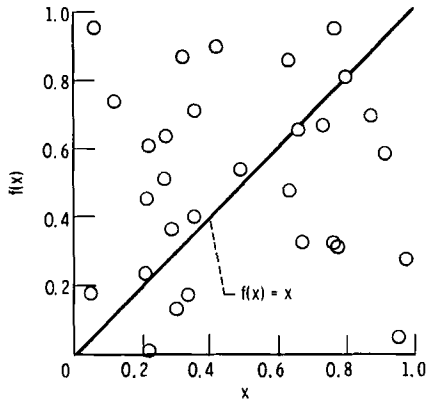

FIG. 17. Sample points in the Monte Carlo integration of a simple function.

between 0 and 1, these N pairs define N points distributed at random on the $x-y$ plane in the region of interest. We can easily normalize the random numbers to any other span.

By using a table of random numbers, we can obtain the set of points shown in Table II. These are plotted on Fig. 17. The fraction of the points lying below the curve $f(x) = x$ must be proportional to the fraction or area under the curve, or

$$S/N = \int_0^1 x\,dx/1 \tag{52a}$$

and, using the 30 points in the table, we find

$$\int_0^1 x\,dx \approx 13/30 = 0.43 \tag{52b}$$

To increase accuracy, we can take more points. For 100 points from the same table, the answer becomes 0.44. This is still not a very good approximation to the analytical value of 0.5. A better method is obviously desirable.

TABLE II

RANDOM NUMBERS FOR EXAMPLE INTEGRATION

x	y	x	y	x	y
.30188	.13165	.42327	.89186	.27240	.51475
.29260	.36702	.97498	.27870	.21999	.45430
.12698	.73654	.63795	.85377	.78064	.31331
.87590	.68966	.05510	.18719	.76529	.32568
.33776	.17733	.77028	.94252	.63704	.47794
.66923	.65455	.21726	.24004	.35958	.70388
.22851	.61064	.49863	.53279	.32119	.86375
.73937	.66302	.22601	.01597	.35595	.40183
.06605	.94854	.80177	.80420	.91128	.58026
.67392	.32869	.22712	.63769	.95480	.05856

A faster and more accurate technique for performing the integration in simple cases is to randomly choose N values of x, and determine values of $f(x)$ that correspond to them. The integral is then approximated by

$$\text{int} = \int_0^1 f(x)\,dx \approx \sum_{i=1}^{N} [f(x)]_i \Big/ N \tag{53}$$

For the example, since $f(x) = x$, we can carry out the indicated operations, applying the values in either column of Table II. Using the $f(x)$ column directly gives

$$\text{int} \approx \sum_{i=1}^{30} [f(x)]_i \Big/ 30 = 0.515 \tag{54}$$

This is a much better estimate than that of the first method for an equal number of values. Using all 60 values in the table gives int = 0.503.

Hammersley and Handscomb (*2*), Schreider (*5*), and Brown (*7*), discuss means of increasing accuracy and of applying the outlined methods to complex cases. These methods can be extended to a j-dimensional integral, and we can find for example the fraction of randomly chosen points in $(j+1)$-dimensional space that are enclosed by the volume described by the integral. In fact, it is only for such multidimensional integrals that a Monte Carlo evaluation becomes valuable, because the number of random points required to evaluate the integral with a given accuracy is not strongly related to the dimension of the integral. Most other numerical techniques for evaluating integrals require the total number of nodal values M to depend on the dimensionality j as

$$M = m^j$$

where m is the number of nodes required to adequately describe the function for the least well-behaved variable.

Brown (7) discusses a number of variance reduction techniques that can be used to increase the expected accuracy of the result. It should be remembered that, although we do not obtain "analytical" answers by Monte Carlo, almost any numerical technique for evaluating multiple integrals leads to truncation or round-off errors that can accumulate to *unknown* magnitude. The Monte Carlo result has the virtue of providing a measure of just how good (or bad) the answer is through application of the appropriate statistical tests.

B. Other Mathematical Techniques

The literature abounds with applications of the Monte Carlo methodology to matrix inversion routines, the related problem of the solution of simultaneous equations and the solution of partial differential equations of various types. The chief usefulness of Monte Carlo arises with those problems where conventional numerical techniques become intractable due to propagation of round-off or truncation errors or where the conventional techniques lead to unwieldy numbers of operations in order to obtain acceptable accuracy. When such situations arise, the analyst should be aware that methods have been developed that can be applied with high hopes of success. These techniques will not be discussed further, as they are of marginal interest in heat transfer, and are well covered in the general literature.

C. Other Physical Problems

Stochastic methods to determine equations of state for gases were an early use of Monte Carlo (*76*, *77*). Also, Kirmse (*78*) has used the method in a study of the complex phenomena associated with turbulent diffusion. This should be a fruitful field for further application.

Montroll and Shuler (*79*) have used a one-dimensional random walk to trace the passage of the energy states of a molecule through successive values, using known transition probabilities between the states. The transitions are assumed to be caused by molecular interactions with a surrounding "bath" of constant-temperature molecules. A final level of activation of the energy states is assumed to correspond to the occurrence of a chemical reaction, and the rate at which molecules reach this activation level gives a measure of the chemical reaction rate.

Other applications can be envisioned in nonequilibrium radiation problems, where the coupling between the radiative and collisional excitation of atoms or molecules must be considered.

A final category of heat transfer problems that seem amenable to efficient solution by Monte Carlo means is the analysis of systems. If the distribution of the characteristics of system components is known, and the behavior of the system as a whole can be described mathematically, then the average and extreme behavior of the system can be analyzed. This is done by assigning randomly chosen component characteristics to each component, thus constructing a sample system. The characteristics of a number of such systems are then analyzed.

Woodward (*80*) has applied this methodology to the problem of the interaction of gas turbine performance characteristics when a number of the turbines are exhausted through a common duct.

Such problems as thermal analysis of a process plant, accounting for performance degradation of components due to scaling and downtime for repairs, would seem a useful application of Monte Carlo; optimization of cleaning and overhaul schedules to maintain maximum performance might be carried out.

VII. Concluding Remarks

The material presented here tends to bear out the frequent assertion that Monte Carlo is a method of last resort. If almost any analytical or standard numerical technique can reasonably be applied to a problem, then it will in all likelihood produce more accurate results more quickly than will Monte Carlo. However, this leaves a very large number of problems in the field of heat transfer as valid targets for Monte Carlo attacks.

The power of the method in dealing with a large number of simultaneous effects and with multidimensional problems without great increase in cost or complexity over simplified problems is a great asset, and indeed such problems provide the usual area of application. As improved high-speed digital computers continue to bring reductions in both cost and running time, the range of problems that can be treated efficiently by methods such as those discussed here will continue to expand.

SYMBOLS

A	surface area
a	thermal diffusivity
c	energy carried by a sample Monte Carlo bundle
D	spacing between parallel plates
e	emissive power
$F_{0-\lambda}$	fraction of total energy emitted by a blackbody in the wavelength range 0–λ
$f(\xi)$	frequency distribution of events occurring at ξ
I	total number of subsets used to compute mean
int	example integral of Eq. (51)
i	radiant intensity
j	dimensionality of an integral
k	thermal conductivity

l, m	lattice indices in square mesh corresponding to x, y positions, respectively
N	total number of sample bundles
n	individual sample index
$\bar{P}$	mean of calculated values of P
P	probability density function
Q	energy per unit time
R	number chosen at random from an evenly distributed set of numbers in the range 0–1; a random number
S	number of events occurring at some position
s	lattice index corresponding to time displacement
T	temperature
t	time
V	volume
x, y	positions in Cartesian coordinate system
α	radiative surface absorptivity
β	function defined by Eq. (14)
γ	standard deviation, defined by Eq. (15)
δ	probable error
ϵ	radiative surface emissivity
η	cone angle
θ	circumferential angle
κ	radiative gas absorption coefficient
$\bar{\kappa}_p$	Planck mean radiative gas absorption coefficient, Eq. (47)
λ	wavelength
ξ	variable
σ	Stefan–Boltzmann constant
τ	optical thickness; κD

Subscripts

b	evaluated at boundary
dV	evaluated in volume element dV
e	emitted
i	increment index
T	integrated over all λ
λ	spectrally dependent
1, 2	at surface 1 or 2

REFERENCES

1. H. Kahn, Applications of Monte Carlo. Rept. AECU 3259. Rand Corp., Santa Monica, California, 1956.
2. J. M. Hammersley and D. C. Handscomb, "Monte Carlo Methods." Methuen, London, 1964.
3. The Holy Bible, 1 Kings 7:23 and 2 Chronicles 4:2.
4. N. Metropolis and S. Ulam, The Monte Carlo method. *J. Am. Statist. Assoc.* **44**, No. 247, 335 (1949).
5. E. D. Cashwell, C. J. Everett, and O. W. Rechard, A practical manual on the Monte Carlo method for random walk problems. Rept. LA-2120. Los Alamos Sci. Lab., Los Alamos, New Mexico; "A Practical Manual on the Monte Carlo Method for Random Walk Problems." Pergamon Press, Oxford, 1959.
6. Yu. A. Schreider, ed., "Method of Statistical Testing—Monte Carlo Method." American Elsevier, New York, 1964.

6a. Yu. A. Schreider, ed., "The Monte Carlo Method: The Method of Statistical Trials." Macmillan (Pergamon), New York, 1966.

7. G. W. Brown, *in* "Modern Mathematics for the Engineer" (E. F. Bechenbach, ed.), Chapter 12. McGraw-Hill, New York, 1956.
8. H. A. Meyer, ed., "Symposium on Monte Carlo Methods." Wiley, New York, 1956.
9. "A Million Random Digits with 100,000 Normal Deviates" (Rand Corp.), Free Press, Glencoe, Illinois, 1955.
10. M. G. Kendall and B. B. Smith, Tables of random sampling numbers. Tracts for Computers No. XXIV. Cambridge Univ. Press, London and New York, 1954.
11. O. Taussky and J. Todd, *in* "Generating and Testing of Pseudo-Random Numbers" (H. A. Meyer, ed.), pp. 15–28. Wiley, New York, 1956.

12. M. G. Kendall and B. B. Smith, Randomness and random sampling numbers. *Roy. Statist. Soc. J.* Pt. 1, 147 (1938).
13. H. B. Dwight, "Mathematical Tables of Elementary and Some Higher Mathematical Functions." 2nd ed. Dover, New York, 1958.
14. E. Jahnke and F. Emde, "Tables of Functions with Formulae and Curves," 4th ed. Dover, New York, 1945.
15. S. Chandrasekhar, "Radiative Transfer." Dover, New York, 1960.
16. V. Kourganoff, "Basic Methods in Transfer Problems; Radiative Equilibrium and Neutron Diffusion." Dover, New York, 1963.
17. V. V. Sobolev, New method in the theory of light scattering. *Astron. Zh.* **28**, 355 (1951).
18. S. Ueno, The formation of absorption lines by coherent and non-coherent scattering—IV. The solution of the equation of transfer by the probabilistic method. *Contrib. Inst. Astrophys. Kyoto* **64**, 1 (1956).
19. S. Ueno, The probabilistic method for problems of radiative transfer—III. Line formation by coherent scattering. *J. Math. Mech.* **7**, 629 (1958).
20. J. R. Howell, Calculation of radiant heat Exchange by the Monte Carlo method. *Am. Soc. Mech. Engrs. Paper* 65-WA/HT-54 (1965).
21. G. Poljak, Analysis of the heat exchange by radiation between gray surfaces by the Saldo-method. *Tech. Phys. USSR* **1**, No. 5/6, 555 (1935).
22. H. C. Hottel, Radiant heat-transmission. *In* "Heat Transmission" (W. H. McAdams, ed.). McGraw-Hill, New York, 1954.
23. A. K. Oppenheim, Radiation analysis by the network method. *Trans. ASME* **78**, 725 (1956).
24. B. Gebhart, "Heat Transfer." McGraw-Hill, New York, 1961.
25. R. C. Corlett, Direct Monte Carlo calculation of radiative heat transfer in vacuum. *J. Heat Transfer* **88**, 376 (1966).
26. M. M. Weiner, J. W. Tindall, and L. M. Candell, Radiative interchange factors by Monte Carlo. *Am. Soc. Mech. Engrs. Paper* 65-WA/HT-51 (1965).
27. L. G. Polgar and J. R. Howell, Directional thermal-radiative properties of conical cavities. *NASA Tech. Note* TN D-2904 (1965).
28. L. G. Polgar and J. R. Howell, Directional radiative characteristics of conical cavities and their relation to lunar phenomena. *In* "Thermophysics and Temperature Control of Spacecraft and Entry Vehicles" (G. B. Heller, ed.), pp. 311–323. Academic Press, New York, 1966.
29. E. M. Sparrow and V. K. Jonsson, Radiant emission characteristics of diffuse conical cavities. *J. Opt. Soc. Am.* **53**, 816 (1963).
29a. H. F. Nelson, Ray reflection from rough dielectric surfaces, Purdue Univ. Rept. AA & ES 66-5, Lafayette, Indiana, 1966.
29b. B. H. Kaye, M. Jackson, and G. A. Zerlaut, Investigation of light scattering in highly reflecting pigmented coatings. Vol. 3. Monte Carlo and other Statistical Investigations. NASA Contractor Rept. CR-61167, 1967.
29c. J. S. Toor, Radiant heat transfer analysis among surfaces having direction dependent properties by the Monte Carlo method. M.S. Thesis, Purdue Univ., Lafayette, Indiana, 1967.
29d. R. Viskanta, J. R. Schornhorst, and J. S. Toor, Analysis and experiment of radiant heat exchange between simply arranged surfaces. Purdue Univ. Rept. AFFDL-TR-67-94, Lafayette, Indiana, 1967.
30. S. T. Dunn, J. C. Richmond, and J. F. Parmer, Survey of infrared measurement techniques and computational methods in radiant heat transfer. *J. Spacecraft Rockets* **3**, 961 (1966).

31. C. M. Usiskin and E. M. Sparrow, Thermal radiation between parallel plates separated by an absorbing-emitting non-isothermal gas. *Intern. J. Heat Mass Transfer* **1**, 28 (1960).
32. H. C. Hottel and E. S. Cohen, Radiant heat exchange in a gas-filled enclosure: Allowance for non-uniformity of gas temperature. *A.I.Ch.E. J.* **4**, 3 (1958).
33. R. Viskanta and R. J. Grosh, Recent advances in radiant heat transfer. *Appl. Mech. Rev.* **17**, 91 (1964).
34. J. E. Hitchcock, Approximate methods for the calculation of radiant heat transfer in the presence of a gray-nonisothermal gas. Ph.D. Thesis, Purdue Univ., Lafayette, Indiana, 1963.
35. M. A. Heaslet and R. F. Warming, Radiative transport and wall temperature slip in an absorbing planar medium. *Intern. J. Heat Mass Transfer* **8**, 979 (1965).
36. R. V. Meghreblian, An approximate analytical solution for the radiation exchange between two flat surfaces separated by an absorbing gas. *Intern. J. Heat Mass Transfer* **5**, 1051 (1962).
37. R. Goulard and M. Goulard, One dimensional energy transfer in radiant media. *Intern. J. Heat Mass Transfer* **1**, 81 (1960).
38. R. G. Deissler, Diffusion approximation for thermal radiation in gases with jump boundary condition. *J. Heat Transfer* **86**, 240 (1964).
39. R. F. Probstein, Radiation slip. *AIAA J.* **1**, 1202 (1963).
40. T. J. Love, Jr., An investigation of radiant heat transfer in absorbing, emitting and scattering media. Rep. ARL63-3. Univ. of Oklahoma Res. Inst., Norman, Oklahoma, 1963.
41. J. H. Ferziger and G. M. Simmons, Application of Case's method to plane parallel radiative transfer. *Intern. J. Heat Mass Transfer* **9**, 987 (1966).
42. J. R. Howell and M. Perlmutter, Monte Carlo solution of thermal transfer through radiant media between gray walls. *J. Heat Transfer* **86**, 116 (1964).
43. M. Perlmutter and J. R. Howell, Radiant transfer through a gray gas between concentric cylinders using Monte Carlo. *J. Heat Transfer* **86**, 169 (1964).
44. J. R. Howell, Radiative interactions between absorbing-emitting and flowing media with internal energy generation. AIAA Paper 66-434. Am. Inst. Aeron. Astronaut., Easton, Pennsylvania, 1966; see also Radiative interactions between absorbing-emitting and flowing media with internal energy generation. *NASA Tech. Note* TN D-3614 (1966).
45. J. R. Howell and M. Perlmutter, Monte Carlo solution of radiant heat transfer in a nongrey nonisothermal gas with temperature dependent properties. *A.I.Ch.E. J.* **10**, 562 (1964).
46. J. R. Howell, M. K. Strite, and H. Renkel, Heat transfer analysis of rocket nozzles using very high temperature propellants. *AIAA J.* **3**, 669 (1965).
47. J. R. Howell and M. K. Strite, Heat transfer in rocket nozzles using high-temperature hydrogen propellant with real property variations. *J. Spacecraft Rockets* **3**, 1063 (1966).
48. J. R. Howell and H. Renkel, Analysis of the effect of a seeded propellant layer on thermal radiation in the nozzle of a gaseous-core nuclear propulsion system. *NASA Tech. Note* TN D-3119 (1965).
49. J. R. Howell, M. K. Strite, and H. Renkel, Analysis of heat transfer effects in rocket nozzles operating with very high-temperature hydrogen. *NASA Tech. Rept.* TR R-220 (1965).
50. J. A. Fleck, Jr., *In* "Methods in Computational Physics" (B. Alder, S. Fernbach, and M. Rotenberg, eds.), Vol. 1, pp. 43–65. Academic Press, New York, 1963.
51. J. A. Fleck, Jr., The calculation of nonlinear radiation transport by a Monte Carlo method. Lawrence Radiation Lab. Rept. UCRL-6698-DEL. Univ. of California, Berkeley, California, 1961.

52. P. M. Campbell, Monte Carlo method for radiative transfer. *Intern. J. Heat Mass Transfer* **10**, 519 (1967).
53. P. M. Campbell and R. G. Nelson, Numerical methods for nonlinear radiation transport calculations. Lawrence Radiation Lab. Rept. UCRL-7838. Univ. of California, Berkeley, California, 1964.
54. D. G. Collins and M. B. Wells, Monte Carlo codes for study of light transport in the armosphere-Vol. I: Description of codes. U.S. Army Electron. Command Tech. Rept. ECOM-00240-F, AD 625115, Fort Monmouth, New Jersey, Radiation Res. Assoc., Inc., 1965.
54a. G. N. Plass and G. W. Kattawar, Monte Carlo calculations of light scattering from clouds. Southwest Center for Advanced Studies Rept. 2, Dallas, Texas (1967).
55. E. R. G. Eckert, Similarities between energy transport in rarefied gases and by thermal radiation. *In* "Modern Developments in Heat Transfer" (W. Ibele, ed.), pp. 159–180. Academic Press, New York, 1963.
56. J. K. Haviland, The solution of two molecular flow problems by the Monte Carlo method. *In* "Methods in Computational Physics" (B. Alder, S. Fernbach, and M. Rotenberg, eds.), Vol. 4, pp. 109–209. Academic Press, New York, 1965.
57. B. L. Hicks and J. K. Aggarwal, Numerical studies of strong shock waves; Pt. I: Illiac solution of a Boltzmann difference equation by Nordsieck's method. Coordinated Sci. Lab. Rept. I-111. Univ. of Illinois, Urbana, Illinois, 1962.
58. M. Perlmutter, Solution of rarefied-gas transport problems. *Am. Soc. Mech. Engrs. Paper* 65-WA/HT-38 (1965).
59. M. Perlmutter, Rarefied-gas heat transfer between parallel plates by a Monte Carlo method. *Proc. Intern. Heat Transfer Conf. 3rd* **I**, p. 225. Science Press, Ephrata, Pennsylvania for A.I.Ch.E., 1966.
60. M. Perlmutter, Analysis of Couette flow and heat transfer between parallel plates enclosing rarefied gas by Monte Carlo. *In* "Rarefied Gas Dynamics" (C. L. Brundin, ed.) Vol. 1, pp. 455–480 Academic Press, New York, 1967.
61. D. H. Davis, Monte Carlo calculation of molecular flow rates through a cylindrical elbow and pipes of other shapes, *J. Appl. Phys.* **31**, 1169 (1960).
62. J. O. Ballance, Transmission probability determination with directed mass motion and with mean free path considerations. *In* "Advances in Vacuum Science and Technology", Vol. 2, Pt. 1, p. 85. Pergamon Press, Oxford, 1967.
63. M. Perlmutter, Monte Carlo solution for the characteristics of a highly rarefied ionized gas flowing through a channel with a transverse magnetic field. *NASA Tech. Note* TN D-2211 (1964).
64. M. Perlmutter, Collisionless flow of an ionized gas through a channel with an imposed magnetic field. *AIAA J.* **3**, 1855 (1965).
65. M. Perlmutter, Monte Carlo solution for the characteristics of a highly rarefied ionized gas flowing through a channel with a transverse magnetic field. *In* "Rarefied Gas Dynamics" (J. H. deLeeuw, ed.), Vol. II, pp. 1-21. Academic Press, New York, 1966.
66. S. Chandrasekhar, Stochastic problems in physics and astronomy. *Rev. Mod. Phys.* **15**, 1 (1943).
67. Lord Rayleigh, On the resultant of a large number of vibrations of the same pitch and of arbitrary phase. *Phil. Mag.* **10**, 73 (1880).
68. A. Haji-Sheikh and E. M. Sparrow, The floating random walk and its application to Monte Carlo solutions of heat equations. *J. SIAM* **14**, 370 (1966).
69. A. F. Emery and W. W. Carson, A modification to the Monte Carlo method—The Exodus method. *Am. Soc. Mech. Engrs. Paper* 66-WA/HT-61. (1966).
70. A. Haji-Sheikh, Application of Monte Carlo methods to thermal conduction problems. Ph.D. Thesis, Univ. of Minnesota, Minneapolis, Minnesota, 1965.

71. A. Haji-Sheikh and E. M. Sparrow, The solution of heat conduction problems by probability methods. *J. Heat Transfer* **89**, 121 (1967).
72. L. W. Ehrlich, Monte Carlo solutions of boundary value problems involving the difference analogue of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{K}{y}\frac{\partial u}{\partial y} = 0$. *J. Assoc. Comput. Mach.* **6**, 204 (1959).
73. W. H. McCrea and F. J. W. Whipple, Random paths in two and three dimensions. *Proc. Roy. Soc. Edinburgh* **60**, 281 (1940).
74. J. Todd, Experiments in the solution of differential equations by Monte Carlo methods. *J. Wash. Acad. Sci.* **44**, 377 (1954).
75. H. S. Carslaw and J. C. Jaeger, "Conduction of Heat in Solids," 2nd ed. Oxford Univ. Press, London and New York, 1959
76. B. J. Alder and T. E. Wainwright, Phase transition for a hard sphere system. *J. Chem. Phys.* **27**, 1208 (1957).
77. W. W. Wood, F. R. Parker, and J. D. Jacobson, Recent Monte Carlo calculations of the equation of state of Lennard-Jones and hard sphere molecules. *Nuovo Cimento Suppl.* **9**, 133 (1958).
78. D. W. Kirmse, A Monte Carlo study of turbulent diffusion. Ph.D. Thesis, Iowa State Univ., Ames, Iowa, 1964.
79. E. W. Montroll and K. E. Shuler, The application of the theory of stochastic processes to chemical kinetics. *Advan. Chem. Phys.* **1**, 361–399 (1958).
80. R. M. Woodward, Investigation of performance of multiple gas generators with a common exhaust. Rept. TM65SE-20. General Electric Corp., West Lynn, Massachusetts, 1966. [Issued as U.S. Army Aviation Materials Lab. Tech. Rept. 66-28 AD 635950, 1966].