Non-Paramatric Statistics Exercise 7

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Exercise 2.16

Show that normal and Cahchy distributions on $\mathbb{R}$ satisfy (2.2.23). Compare the resulting approximation error bounds of Example 2.2.19 with those obtained by Example 2.2.18.

Solution:

Fix arbitrary $\sigma \in \mathbb{R}_{>0}$ and $\mu \in \mathbb{R}$.

The density function of the normal distribution with respect to μ, σ is $h_{\mathcal{N}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$.

The density function of the Cauchy distribution with respect to μ, σ is $h_C(x) = \frac{1}{\pi}\frac{\sigma}{\sigma^2+(x-\mu)^2}$.

We observe that the density functions of both distributions are symmetric with respect to $x = \mu$.
So, we firstly consider the general case, meaning that h is the density of either distributions.
Also fix an arbitrary $s \in (0, 1]$ and consider the partition:

$$\mathbb{R} = (-\infty, \mu - s) \sqcup [\mu - s, \mu) \sqcup [\mu, \mu + s] \sqcup (\mu + s, \infty).$$

For $x \in (\mu + s, \infty) : \omega(h, x, s) = \sup_{x' \in B(x,s)} |h(x) - h(x')| \leq h(x - s) - h(x + s)$.

$$\Rightarrow \int_{(\mu+s,\infty)} \omega(h, x, s)\,\mathrm{d}\lambda(x) \leq \int_{(\mu+s,\infty)} h(x - s) - h(x + s)\,\mathrm{d}\lambda(x) = \mathbf{P}(h \leq \mu + 2s) - \frac{1}{2}.$$

For $x \in [\mu, \mu + s] : \omega(h, x, s) = \sup_{x' \in B(x,s)} |h(x) - h(x')| \leq h(\mu) - h(x + s)$.

$$\Rightarrow \int_{[\mu,\mu+s]} \omega(h, x, s)\,\mathrm{d}\lambda(x) \leq \int_{[\mu,\mu+s]} h(\mu) - h(x + s)\,\mathrm{d}\lambda(x) = s \cdot h(\mu) - (\mathbf{P}(x \leq \mu - 2s) - \mathbf{P}(x \leq \mu + s)).$$

For $x \in [\mu - s, \mu] : \omega(h, x, s) = \sup_{x' \in B(x,s)} |h(x) - h(x')| \leq h(\mu) - h(x - s)$.

$$\Rightarrow \int_{[\mu-s,\mu]} \omega(h, x, s)\,\mathrm{d}\lambda(x) \leq \int_{[\mu-s,\mu]} h(\mu) - h(x + s)\,\mathrm{d}\lambda(x) = s \cdot h(\mu) - (\mathbf{P}(x \leq \mu - s) - \mathbf{P}(x \leq \mu - 2s)).$$

For $x \in (-\infty, \mu - s) : \omega(h, x, s) = \sup_{x' \in B(x,s)} |h(x) - h(x')| \leq h(x + s) - h(x - s)$.

$$\Rightarrow \int_{(-\infty,\mu-s)} \omega(h, x, s)\,\mathrm{d}\lambda(x) \leq \int_{((-\infty,\mu-s)} h(x + s) - h(x - s)\,\mathrm{d}\lambda(x) = \frac{1}{2} - \mathbf{P}(x \leq \mu - 2s).$$

Thus, we obtain in the general case:

$$\begin{aligned}
\int_{\mathbb{R}} \omega(h, x, s)\,\mathrm{d}\lambda(x) &= \int_{(-\infty,\mu-s)\sqcup[\mu-s,\mu)\sqcup[\mu,\mu+s]\sqcup(\mu+s,\infty)} \omega(h, x, s)\,\mathrm{d}\lambda(x) \\
&\leq 2h(\mu) \cdot s + \mathbf{P}(\mu - s \leq x \leq \mu + s) \\
&= 2h(\mu) \cdot s + \int_{[\mu-s,\mu+s]} h(x)\,\mathrm{d}\lambda(x). \\
&\leq 2h(\mu) \cdot s + \int_{[\mu-s,\mu+s]} \sup\nolimits_{x' \in [\mu-s,\mu+s]} h(x')\,\mathrm{d}\lambda(x). \\
&= 4h(\mu) \cdot s.
\end{aligned}$$

Now we consider the specific cases where $h = h_{\mathcal{N}}$ or $h = h_C$.
For $h(x) = h_{\mathcal{N}}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\mu)^2}{2\sigma^2})$: $\displaystyle\int_{\mathbb{R}} \omega(h_{\mathcal{N}}, x, s)\,\mathrm{d}\lambda(x) \leq 4h_{\mathcal{N}}(\mu) \cdot s = \frac{4}{\sqrt{2\pi\sigma^2}} \cdot s$.

Similarly, for $h(x) = h_C(x) = \frac{1}{\pi}\frac{\sigma}{\sigma^2+(x-\mu)^2}$: $\displaystyle\int_{\mathbb{R}} \omega(h_C, x, s)\,\mathrm{d}\lambda(x) \leq 4h_C(\mu) = \frac{4}{\pi\sigma} \cdot s$.