Non-Paramatric Statistics Exercise 5

Osman Ceylan, Jiahui Wang, Zhuoyao Zeng

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Exercise 2.6

Solution:

h is the λ^d-density of $\mathbf{P}$ $\Leftrightarrow$ $\forall B \in \mathcal{B} : \mathbf{P}(B) = \int_B h \,\mathrm{d}\lambda^d$.

h is α-Hölder continuous $\Rightarrow$ $\exists\, |h|_\alpha \in \mathbb{R}_{\geq 0} \forall x_1, x_2 \in \mathbb{R}^d : |h(x_1) - h(x_2)| \leq |h|_\alpha \cdot ||x_1 - x_2||^\alpha$.

Consider the map $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d : x \mapsto t \cdot x$.

$\mathcal{T}$ is a diffeomorphism with $det(J_\mathcal{T}) = det(t \cdot 1^{d\times d}) = t^d$.

Now consider $\mathbf{P}_t$.

$\forall B \in \mathcal{B} : \mathbf{P}_t(B) \stackrel{\text{def}}{=} \mathbf{P}(tB) = \int_{tB} h \,\mathrm{d}\lambda^d = \int_{\mathcal{T}(B)} h \,\mathrm{d}\lambda^d = \int_B h \circ \mathcal{T} \cdot det(J_\mathcal{T}) \,\mathrm{d}\lambda^d = \int_B t^d \cdot h \circ \mathcal{T} \,\mathrm{d}\lambda^d$.

$\Rightarrow$ $h_t = t^d \cdot h \circ \mathcal{T}$.

Now we compare $|h|_\alpha^{\frac{d}{2\alpha+d}} \cdot \big(\lambda^d(\operatorname{supp} h)\big)^{\frac{\alpha+d}{2\alpha+d}}$ and $|h_t|_\alpha^{\frac{d}{2\alpha+d}} \cdot \big(\lambda^d(\operatorname{supp} h_t)\big)^{\frac{\alpha+d}{2\alpha+d}}$ by firstly comparing the terms under exponents.

$\forall x \in \operatorname{supp} h_t : tx \in \operatorname{supp} h$ $\Rightarrow$ $\lambda^d(\operatorname{supp} h_t) = t^{-d} \cdot \lambda^d(\operatorname{supp} h)$.

$$
\begin{aligned}
|h|_\alpha = \sup_{x \neq x' \in \mathbb{R}^d} \tfrac{|h_t(x) - h_t(x')|}{||x-x||^\alpha} &= \sup_{x \neq x' \in \mathbb{R}^d} \tfrac{t^d\,|h(tx) - h(tx')|}{t^{-1}||tx - tx'||^\alpha} \\
&= t^{\alpha+d} \sup_{x \neq x' \in \mathbb{R}^d} \tfrac{|h(tx) - h(tx')|}{||tx - tx'||^\alpha} = t^{\alpha+d} |h|_\alpha.
\end{aligned}
$$

$\Rightarrow$ $|h_t|_\alpha^{\frac{d}{2\alpha+d}} \cdot \big(\lambda^d(\operatorname{supp} h_t)\big)^{\frac{\alpha+d}{2\alpha+d}} = \big(t^{\alpha+d} \cdot |h|_\alpha\big)^{\frac{d}{2\alpha+d}} \cdot \big(t^{-d} \cdot \lambda^d(\operatorname{supp} h)\big)^{\frac{\alpha+d}{2\alpha+d}} = |h|_\alpha^{\frac{d}{2\alpha+d}} \cdot \big(\lambda^d(\operatorname{supp} h)\big)^{\frac{\alpha+d}{2\alpha+d}}$.

$\Rightarrow |h|_\alpha^{\frac{d}{2\alpha+d}} \cdot \big(\lambda^d(\operatorname{supp} h)\big)^{\frac{\alpha+d}{2\alpha+d}}$ is invariant under scaling.