Non-Paramatric Statistics Exercise 3

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Exercise 1.5

Let $(X, \mathcal{A})$ be a measurable space and μ be a probability measure on X. Moreover, let $\mathbf{P}$ be a probability measure on X that has a μ-density $h : X \to [0, \infty)$, and let $L_{\text{dens}} : X \times \mathbb{R} \to [0, \infty)$ be the corresponding loss function. Compute the excess L_{dens}-risks for both $\mathbf{P}$ and μ. Which one is better suited for capturing the intuitive goal of density estimation?

Solution:

h is a μ-density of $\mathbf{P}$ $\Rightarrow$ $\forall A \in \mathcal{A} : \mathbf{P}(A) = \int_A h \,\mathrm{d}\mu$.

Recall from lecture: $\forall x \in X, t \in \mathbb{R} : L_{\text{dens}}(x, t) = |h(x) - t|$.

$\forall x \in X : L_{\text{dens}}(x, h(x)) = 0$ $\Rightarrow$ The Bayes Risk of L_{dens} related to any measure of X is 0. $\quad (*)$

Suppose now we have an estimated density h' of h.

This means that: $h' \geq 0 \;\wedge\; \int_X h' \,\mathrm{d}\mu = 1 \;\wedge\; \forall A \in \mathcal{A} : \mathbf{P}'(A) := \int_A h' \,\mathrm{d}\mu$.

As the excess L_{dens}-risks of μ, we obtain:

$$
\begin{aligned}
\mathcal{R}_{L_{\text{dens}},\mu}(h') - \mathcal{R}^*_{L_{\text{dens}},\mu} &\overset{(*)}{=} \mathcal{R}_{L_{\text{dens}},\mu}(h') \\
&\overset{\text{Def.}}{=} \int_X L_{\text{dens}}(x, h'(x)) \,\mathrm{d}\mu(x) \\
&= \int_X |h - h'| \,\mathrm{d}\mu \\
&= \int_{\{h' \geq h\}} h' - h \,\mathrm{d}\mu \; + \int_{\{h' < h\}} h - h' \,\mathrm{d}\mu \\
&= \int_{\{h' \geq h\}} h' \,\mathrm{d}\mu \; - \int_{\{h' \geq h\}} h \,\mathrm{d}\mu \; + \int_{\{h' < h\}} h \,\mathrm{d}\mu \; - \int_{\{h' < h\}} h' \,\mathrm{d}\mu \\
&= \mathbf{P}'(\{h' \geq h\}) - \mathbf{P}(\{h' \geq h\}) + \mathbf{P}'(\{h' < h\}) - \mathbf{P}(\{h' < h\}) \\
&= \mathbf{P}'(\{h' \geq h\}) - \mathbf{P}(\{h' \geq h\}) + (\,1 - \mathbf{P}'(\{h' \geq h\})\,) - (\,1 - \mathbf{P}(\{h' \geq h\})\,) \\
&= 2\mathbf{P}'(\{h' \geq h\}) - 2\mathbf{P}(\{h' \geq h\}) \\
&= 2(\,\mathbf{P}'(\{h' \geq h\} - \mathbf{P}(\{h' \geq h\})\,).
\end{aligned}
$$

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As the excess L_{dens}-risks of $\mathbf{P}$, we obtain:

$$\begin{aligned}
\mathcal{R}_{L_{\text{dens}},\mathbf{P}}(h') - \mathcal{R}^*_{L_{\text{dens}},\mathbf{P}} &\overset{(*)}{=} \mathcal{R}_{L_{\text{dens}},\mathbf{P}}(h') \\
&\overset{\text{Def.}}{=} \int_X L_{\text{dens}}(x, h'(x))\,\mathrm{d}\mathbf{P}(x) \\
&= \int_X |h - h'|\,\mathrm{d}\mathbf{P} \\
&= \int_{\{h' \geq h\}} h' - h\,\mathrm{d}\mathbf{P} \ + \int_{\{h' < h\}} h - h'\,\mathrm{d}\mathbf{P} \\
&= \int_{\{h' \geq h\}} h'\,\mathrm{d}\mathbf{P} \ - \int_{\{h' \geq h\}} h\,\mathrm{d}\mathbf{P} \ + \int_{\{h' < h\}} h\,\mathrm{d}\mathbf{P} \ - \int_{\{h' < h\}} h'\,\mathrm{d}\mathbf{P} \\
&= \int_{\{h' \geq h\}} h' \cdot h\,\mathrm{d}\mu \ - \int_{\{h' \geq h\}} h^2\,\mathrm{d}\mu \ + \int_{\{h' < h\}} h^2\,\mathrm{d}\mu \ - \int_{\{h' < h\}} h' \cdot h\,\mathrm{d}\mu.
\end{aligned}$$

If our calculations are not wrong, we would say that the excess risk relating to μ should be more suitable for capturing the intuitive goal of density estimation.

Exercise 2.1

Let $X \neq \emptyset$ and $\mathfrak{A} = (A_j)_{j \in J}$ be an at most countable partition of X. Describe the space $\mathcal{L}_0(X)$ of measurable functions $X \to \mathbb{R}$ for the corresponding σ-algebra $\mathcal{A} := \sigma(\mathfrak{A})$.

Solution:

We construct $\Sigma := \{S \subseteq X | \exists I \subseteq J : S = \bigcup_{i \in I} A_i\}$.

$\mathfrak{A} = (A_j)_{j \in J}$ is a countable partition of $X \ \Leftrightarrow\ X = \dot{\bigcup}_{j \in J} A_j$.

$\Rightarrow$ A simple verification of the axioms of σ-algebra shows that Σ is a σ-algebra containing $\mathfrak{A}$.

Meanwhile it is also easy to see that $\Sigma \subseteq \sigma(\mathfrak{A}) \ \Rightarrow\ \Sigma = \sigma(\mathfrak{A})$. (i)

Moreover, recall the following theorem from measure theory:

Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ be mesrable spaces and $f : \Omega_1 \to \Omega_2$.

Also let $(M_n)_{n \in \mathbb{N}}$ a sequence from $\mathcal{F}_1$ and at the same time a partition of Ω_1.

For $\forall n \in \mathbb{N}$ denote $\mathcal{F}_1 \mid M_n := \{A \cap M_n | A \in \mathcal{F}_1\}$.

Then it holds that: f is $\mathcal{F}_1$-$\mathcal{F}_2$-measurable $\Leftrightarrow\ \forall n \in \mathbb{N} : f|_{M_n}$ is $\mathcal{F}_1|M_n$-$\mathcal{F}_2$-measurable.

$(A_j)_{j \in J}$ is paarwise disjoint $\ \Rightarrow\ \forall j \in J : \mathcal{A}|A_j = \{\emptyset, A_j\}$.

$\Rightarrow\ \forall j \in J\, \forall f \in \mathcal{L}_0(X) : f|_{A_j}^{-1}(\mathcal{B}) \in \mathcal{A}|A_j = \{\emptyset, A_j\}$. (ii)

We also know that $\mathcal{B} = \sigma(\mathcal{I}^c_{-\infty})$ with $\mathcal{I}^c_{-\infty} = \{(-\infty, c] \mid c \in \mathbb{R}\}$. (iii)

Combining all considerations , we conclude that:

$$\begin{aligned}
\mathcal{L}_0(X) &\overset{\text{Def}}{=} \{f : X \to \mathbb{R} \mid f^{-1}(\mathcal{B}) \subseteq \mathcal{A}\} \\
&\overset{(i)}{=} \{f : X \to \mathbb{R} \mid \forall B \in \mathcal{B}\ \exists I_B \subseteq J : f^{-1}(B) = \textstyle\bigcup_{i \in I_B} A_i\} \\
&\overset{(ii)}{=} \{f : X \to \mathbb{R} \mid \forall j \in J : f|_{A_j}^{-1}(\mathcal{B}) = \emptyset \ \vee\ f|_{A_j}^{-1}(\mathcal{B}) = A_j\} \\
&\overset{(iii)}{=} \{f : X \to \mathbb{R} \mid \forall j \in J\ \forall c \in \mathbb{R} : f|_{A_j}^{-1}(\,(-\infty, c]\,) = \emptyset \ \vee\ f|_{A_j}^{-1}(\,(-\infty, c]\,) = A_j\}.
\end{aligned}$$