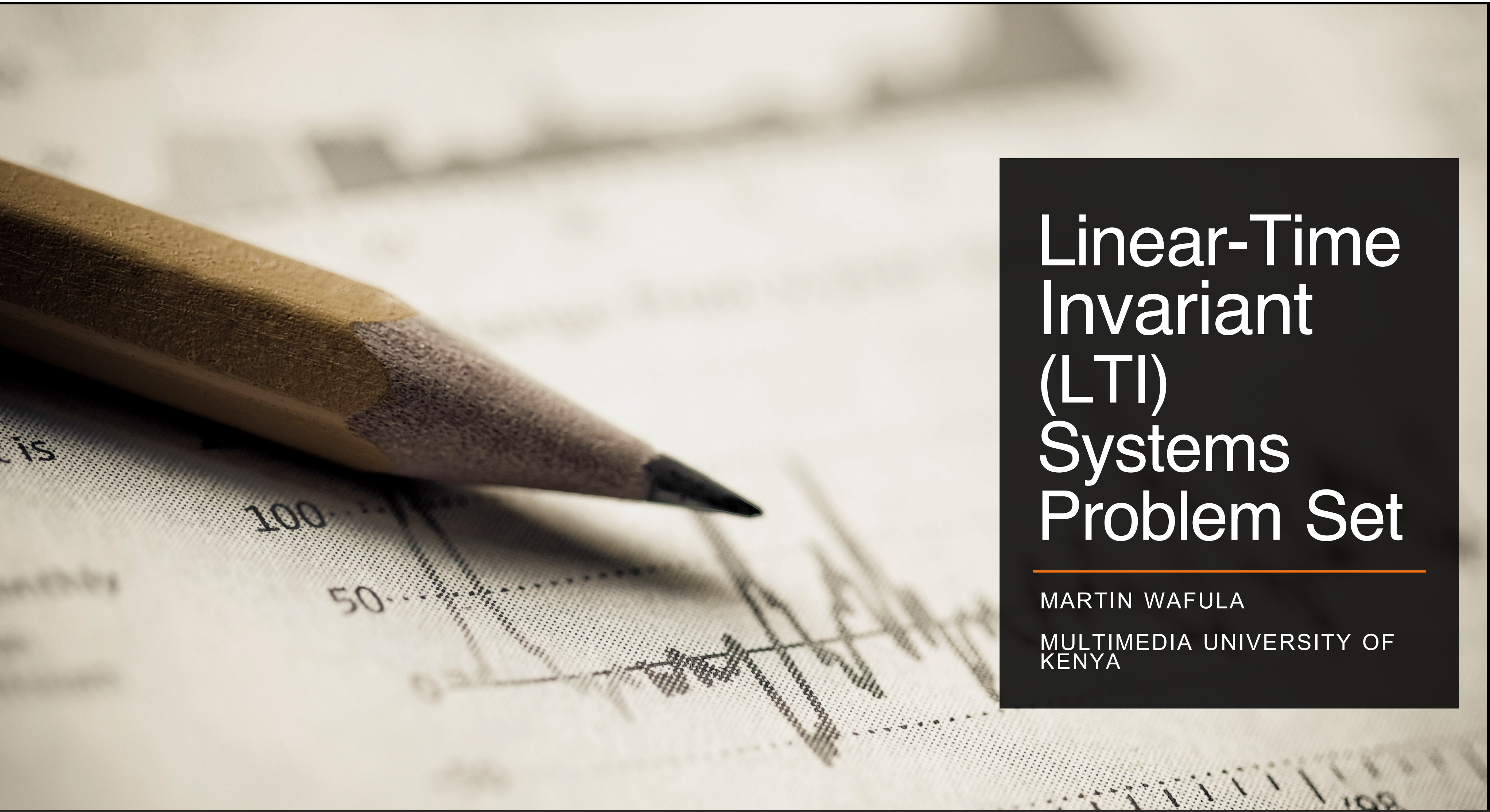

Linear-Time Invariant (LTI) Systems Problem Set

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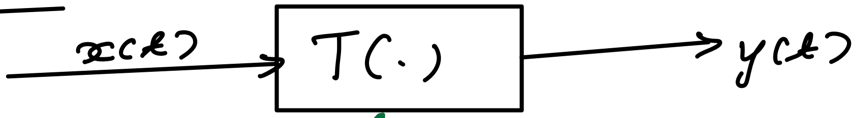
Review of Last Lecture

C.T. Convolution



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$h(t) \to$ impulse response

Describes the output $y(t)$ for any arbitrary input $x(t)$

→ **Remark**: The impulse response $h(t)$ completely determines output $y(t)$ for any arbitrary input $x(t)$ for an LTI System.

Properties of Convolution

i) Commutative: $x(t) * h(t) = h(t) * x(t)$

ii) Associative: $(x(t) * h_1(t)) * h_2(t) = x(t) * (h_1(t) * h_2(t))$

iii) Distributive: $x(t) * (h_1(t) + h_2(t)) = [x(t) * h_1(t)] + [x(t) * h(t)]$

Memoryless Systems, Causality, Stability and Eigenfunctions

① Memoryless system : $y(t) = T(x(t)) = k \cdot x(t) \Rightarrow h(t) = k\,\delta(t)$ (Impulse response of Memoryless system)

$$\boxed{h(t) = 0 \text{ if } t \neq 0} \longrightarrow \text{Memoryless system.}$$

If $h(t) \neq 0$ for $t \neq 0$, then the system is not memoryless.

② Causality : An LTI system is causal if $\boxed{h(t) = 0,\ \text{for } t < 0}$.

$$\therefore\ y(t) = \int_{-\infty}^{\infty} x(t-\tau)\,h(\tau)\,d\tau = \int_{0}^{\infty} x(t-\tau)\,h(\tau)\,d\tau.$$

③ Non-causality : System is non-causal if $\boxed{h(\tau) \neq 0 \text{ for } \tau < 0}$.

④ Anti-causal system : $\boxed{h(\tau) = 0 \text{ for } \tau > 0} \Rightarrow$ $y(t)$ depends only on future values.

⑤ Stability : System is BIBO stable if $\boxed{\int_{-\infty}^{\infty} |h(t)|\,dt < \infty}$.

⑥ Eigenfunction of a C.T. LTI System : $T(e^{\alpha t}) = y(t) = H(\alpha)e^{\alpha t}$ ← eigenfunction

where $H(\alpha) = \int_{-\infty}^{\infty} h(\tau)e^{-\alpha\tau}\,d\tau.$

Differential Equation Description of LTI Systems.

- Input-output relationship of several systems can be represented by using differential equations (D.E.).

- The general form of such a representation is:

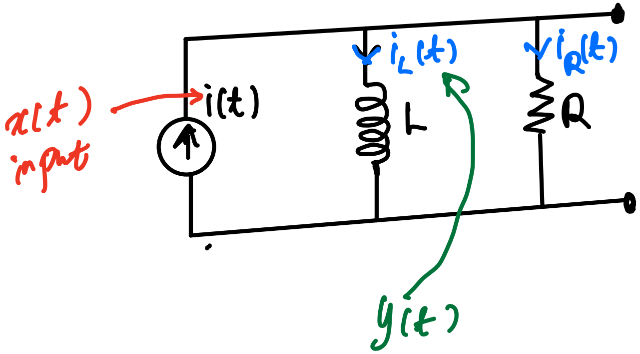
$$\boxed{\sum_{k=0}^{M} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}}$$

where $x(t)$ – input of system

$y(t)$ – output of system.

→ M^{th} order Linear Constant Coefficient DE (LCCDE)

Example (LR-circuit)



$$i(t) = i_L(t) + i_R(t)$$

This can be described with a DE

$$\underbrace{\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i(t)}_{\text{D.E representation of I/O relation.}}$$

We need the solution of this system

What does solution mean?

output $y(t)$ for a given input signal $x(t)$.

Soln

$$y(t) = \underbrace{y_p(t)}_{\text{Particular Soln}} + \underbrace{y_h(t)}_{\text{homogeneous solution}}$$

Any general solution can be represented as sum of particular solution and homogeneous solution.

Homogeneous solution is given as

$$\sum_{k=0}^{M} a_k \frac{d^k y_h(t)}{dt} = 0 \cdots (i)$$

- To solve (i), we need M auxilliary conditions / boundary conditions to determine the output signal.
- These boundary/auxilliary conditions are given as:

$$\underbrace{y(t), \frac{dy(t)}{dt}, \frac{d^2y(t)}{dt^2}, \ldots, \frac{d^{M-1}y(t)}{dt^{M-1}}}_{\text{at some point } t_0.}$$

Linearity

Qn: For a system described by the DE, when is the system linear?

$\Rightarrow$ If all auxilliary conditions are zero.

Time-invariance:

Qn: When is the system time-invariant?

$\Rightarrow$ For time-invariance, the system has to be at initial rest condition (satisfy)

$\Rightarrow$ If $x(t) = 0$ for $t \le t_0$

Assume $y(t) = 0$ for $t \le t_0$.

Initial conditions become:

$$y(t_0) = \left.\frac{dy}{dt}\right|_{t=t_0}, \ldots, \frac{d^{M-1}y(t)}{dt^{M-1}} = 0$$

Example: (Page 45, lecture notes)

Consider the DE: $\frac{dy}{dt} + 2y(t) = x(t)$

where $x(t) = Ke^{3t}u(t)$ and K is some constant.

Solution:

$y(t) = y_p(t) + y_h(t)$

Homogeneous sol'n, $y_h(t)$

$y_h(t)$ satisfies the following D.E.

$$\underbrace{\frac{dy_h}{dt} + 2y_h(t) = 0}_{\text{Homogeneous DE}} \quad --- \quad (ii)$$

$\mapsto$ has no driving force/fn $x(t)$

$\rightarrow$ For the eqn of this form (where sums of derivatives of $y_h(t)$ have to sum to 0), a reasonable guess would be that $y_h(t)$ takes the form

$$y_h(t) = Ae^{mt}, \text{ for some } m \in \mathbb{C} \cdots (iii)$$

Substituting this into homogeneous eqn (i)

$$\frac{d}{dt}\left(Ae^{mt}\right) + 2Ae^{mt} = 0$$

$$\Rightarrow mAe^{mt} + 2Ae^{mt} = 0$$

$$\Rightarrow (m+2)Ae^{mt} = 0.$$

$$\Rightarrow m + 2 = 0$$

$$\Rightarrow \underline{\underline{m = -2}}$$

$y_h(t) = Ae^{-2t}$, for any constant A.

Particular Solution $y_p(t)$

$$\frac{dy_p(t)}{dt} + 2y_p(t) = Ke^{3t}u(t) \quad \cdots (iv)$$

- It seems reasonable to try $y_p(t) = Be^{3t}$, for some constant B.

Substituting into (iv) and evaluating; we have

$$3Be^{3t} + 2Be^{3t} = Ke^{3t} \quad \text{for } t \geq 0$$

$$\Rightarrow (3+2)Be^{3t} = Ke^{3t}$$

$$\Rightarrow 5Be^{3t} = Ke^{3t}$$

$$\Rightarrow 5B = K \Rightarrow B = K/5$$

$$\therefore y_p(t) = \frac{K}{5}e^{3t} \quad \text{for } t \geq 0.$$

Together:

$$y(t) = y_h(t) + y_p(t) = Ae^{-2t} + \frac{K}{5}e^{3t} \quad \text{for } t > 0$$

To determine the coefficients, we need initial conditions e.g.

e.g. Let the system be at rest until input is applied (i.e. $y(t) = 0$ until $x(t)$ becomes non-zero), we have $y(t) = 0$ for $t < 0$. Suppose we are given the initial condition $y(0) = 0$.

Then,

$$y(0) = A + K/5 \Rightarrow A = -K/5$$

Thus, with given initial conditions we have

$$y(t) = \frac{K}{5}\left(e^{3t} - e^{-2t}\right)u(t).$$

Therefore to solve a DE of form

$$\sum_{k=0}^{M} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x}{dt^k}.$$

1) Find the homogeneous solution to equation:

$$\sum_{k=0}^{M} a_k \frac{d^k y_h}{dt^k} = 0.$$

by hypothesizing that $y_h(t) = Ae^{mt}$ for some $m \in \mathbb{C}$.

- If M different values of m are given denoted by $m_1, m_2, \ldots, m_M$ for which the proposed form holds, we take

$$y(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t} + \cdots + A_M e^{m_M t},$$

where $A_1, A_2, \ldots, A_M$ are to be determined from initial conditions.

2) Next find the particular solution to equation

$$\sum_{k=0}^{M} a_k \frac{d^k y_p}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x}{dt^k},$$

where $x(t)$ is some given function.

Example (Pg 46 Lecture Notes) Consider the DE $y''(t) + y'(t) - 6y(t) = x'(t) + x(t)$ where $x(t) = e^{4t}u(t)$.

Soln

Homogeneous Solution

$y_h(t) = Ae^{mt}$ satisfying:

$$y_h''(t) + y_h'(t) - 6y_h(t) = 0.$$

$$m^2Ae^{mt} + mAe^{mt} - 6Ae^{mt} = 0$$

$$(m^2 + m - 6)Ae^{mt} = 0$$

$$\Rightarrow m^2 + m - 6 = 0 \quad \text{(Auxilliary eqn)}$$

$$\Rightarrow m = -3 \text{ or } m = 2$$

$$y_h(t) = A_1e^{-3t} + A_2e^{2t}.$$

where A_1 and A_2 are constants that can be determined by initial conditions

Particular Solution

For $t > 0$, $x'(t) + x(t) = 4e^{4t} + e^{4t}$

$$= 5e^{4t}$$

$\therefore y_p(t)$ is of form $y_p(t) = Be^{4t}, t > 0$

Substituting in

$$y_p''(t) + y_p'(t) - 6y_p(t) = x'(t) + x(t)$$

$$16Be^{4t} + 4Be^{4t} - 6Be^{4t} = 5e^{4t}$$

$$\Rightarrow (16 + 4 - 6)Be^{4t} = 5e^{4t}$$

$$\Rightarrow 14Be^{4t} = 5e^{4t}$$

$$\Rightarrow B = 5/14$$

$$\Rightarrow y_p(t) = \frac{5}{14}e^{4t} \text{ for } t > 0.$$

Overall Soln

$$y(t) = y_h(t) + y_p(t) = A_1 e^{-3t} + A_1 e^{2t} + \tfrac{5}{14} e^{4t}, \text{ for } t>0.$$

→ If system is at rest until start or input is applied and that $y(0) = y'(0) = 0$, we have:

$$y(0) = A_1 + A_2 + 5/14 = 0$$

$$y'(0) = -3A_1 + 2A_2 + 20/14 = 0$$

$$\Rightarrow A_1 = 1/7, \quad A_2 = -\tfrac{1}{2}$$

Thus solution

$$y(t) = \frac{1}{7} e^{-3t} - \frac{1}{2} e^{+2t} + \frac{5}{14} e^{4t} \text{ for } t>0.$$

$$= \left(\frac{1}{7} e^{-3t} - \frac{1}{2} e^{2t} + \frac{5}{14} e^{4t}\right) u(t).$$

Thm (Homogeneous DE) Let $ak^2+bk+c=0$ (1) be auxilliary equation of the ODE:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \ldots (2)$$

Case 1: If the auxilliary equation (A.E) has 2 real roots k_1 and k_2, then

$$y_n = Ae^{k_1 x} + Be^{k_2 x}.$$

Case 2: If AE has (unique) double real solution k, then

$$y_n = Axe^{kx} + Be^{kx} = (Ax+B)e^{kx}.$$

Case 3: If A.E. has 2 (non-real) complex conjugate roots, $\alpha \pm j\beta$, then

$$y_n = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Rmk: To find the particular integral soln y_p for $ay''+by'+cy=f(x)$, we need:

When $f(x)$ is	try $y_p(x) =$
① A polynomial of degree r	A polynomial of the (same) degree r.
② Ae^{kx} (A constant)	αe^{kx} (α constant)
③ $A\cos kx + B\sin kx$ (A, B constants)	$\alpha\cos kx + \beta\sin kx$ α, β constant.

Properties of Discrete Time LTI Systems

Impulse Response

Recall from sifting property : $x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]$

$\delta[n] \longrightarrow \boxed{T(\cdot)} \longrightarrow h[n]$ ← impulse response

discrete unit impulse

$$\boxed{h[n] = T(\delta[n])}$$

is Response to the unit impulse.

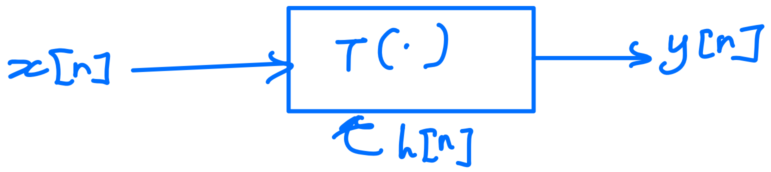
→ Impulse response is key as it helps characterise ~~the~~ output to any arbitrary input signal.

Response to an arbitrary input signal $x[n]$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\,h[n-m] = x[n] * h[n]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\,x[n-m] = h[n] * x[n]$$

Commutativity of convolution.



$$y[n] = T(x[n]) = T\left(\sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]\right)$$

$$= \sum_{m=-\infty}^{\infty} x[m]\,T(\delta[n-m]) \quad \text{(follows from linearity)}$$

$$= \sum_{m=-\infty}^{\infty} x[m]\,h[n-m] \quad \text{(follows from time-invariance)}$$

$$= x[n] * h[n]$$

Properties of Convolution

Similar to C.T.

1. Commutative
2. Associative
3. Distributive

Example: Determine the function of an LTI system if its impulse response is $h[n] = 0.5\,\delta[n] + 0.5\,\delta[n-1]$.

Soln

$$y[n] = x[n] * \delta[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

$$= \sum_{m=-\infty}^{\infty} x[m]\left(0.5\,\delta[n-m] + 0.5\,\delta[n-1-m]\right)$$

$$= \sum_{m=-\infty}^{\infty} x[m]\, 0.5\,\delta[n-m] + \sum_{m=-\infty}^{\infty} x[m]\, 0.5\,\delta[n-1-m] \quad \text{(Distributivity of conv)}$$

$$= 0.5\, x[n] + 0.5\, x[n-1]$$

$$= 0.5\left(x[n] + x[n-1]\right)$$

Rmk: This system computes the mean value of two input samples, current value and past value.

① Memoryless D.T. System.

$h[n] = k\,\delta[n]$ → Impulse response of a memoryless D.T. LTI system.

② Causal D.T. LTI System.

⇒ Output depends only on present and past inputs.

⇒ D.T LTI System is causal iff

$h[n] = 0$ for $n < 0$.

③ BIBO Stability

DT LTI System is BIBO stable if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$ i.e. absolutely summable/finite

④ Eigenfunction of D.T. LTI System.

Let us consider $x[n] = z^n$ and

$$y[n] = T(x[n]) = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, z^{(n-m)}$$

$$= z^n \underbrace{\sum_{m=-\infty}^{\infty} h[n]\, z^{-m}}_{H(z)}$$

$H(z) \rightarrow$ a function that depends only on impulse response.

$$\boxed{y[n] = T(z^n) = H(z)\, z^n}$$

$\rightarrow$ This is an eigenfunction since the input is simply the scaled version of input.

Systems Described by Difference Equations

Similar to C.T. LTI systems given by differential equations, D.T. LTI systems can be described by difference equations.

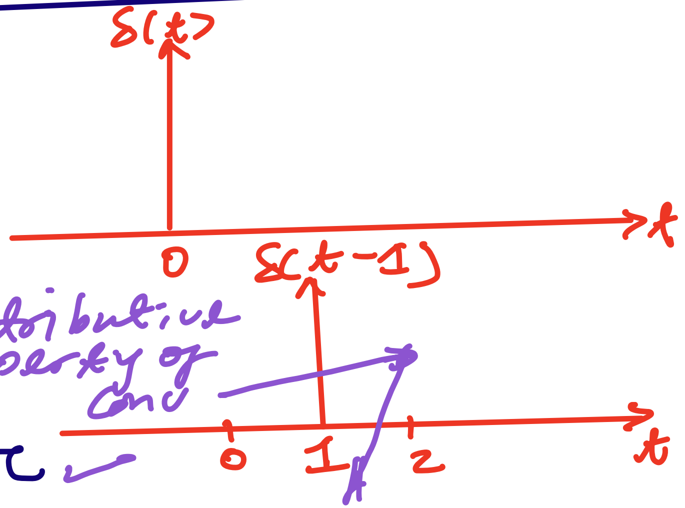
If $x[n]$ is the input signal and $y[n]$ is output signal:

$$\underbrace{\sum_{k=0}^{M} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]}_{\text{standard canonical form of the difference equation.}}$$

Problem Set 1 : C.T. LTI SYSTEMS

① Determine the function of a LTI C.T. system if its impulse response is $h(t) = \delta(t) + \delta(t-1)$.

$$x(t) \longrightarrow \boxed{h(t) = \delta(t) + \delta(t-1)} \longrightarrow y(t)$$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\delta(t-\tau) + \delta(t-\tau-1)\right] d\tau \Rightarrow \text{Distributive property of conv}$$

$$= \underbrace{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, d\tau}_{\text{From sifting property}} + \int_{-\infty}^{\infty} x(\tau)\,\delta(t-1-\tau)\, d\tau$$

From sifting property $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, d\tau$

$$x(t) * (h_1(t) + h_2(t))$$
$$= x(t) * h_1(t) + x(t) * h_2(t)$$

$$= x(t) + x(t-1)$$

$\Rightarrow$ This system computes the sum of inputs at 2 time instants, one at current and the other at current time minus 1

② Determine the function of an LTI C.T. System if its impulse response $h(t) = 0.1\,[u(t) - u(t-10)]$.

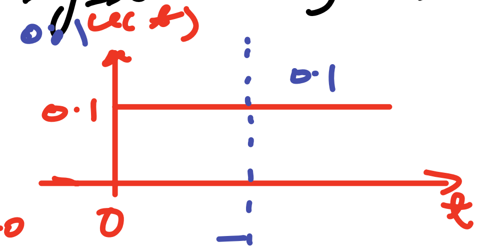
Output function

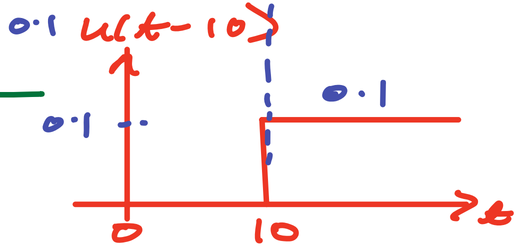
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

$$0.1\,u(t) = \begin{cases} 0.1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$0.1\,u(t-10) = \begin{cases} 0.1 & \underline{t \geq 10} \\ 0 & t < 10. \end{cases} \quad t-10 \geq 0$$

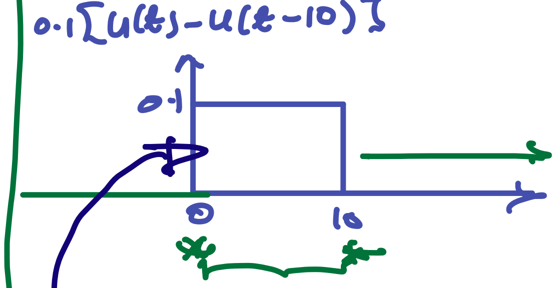




$$y(t) = \int_{-\infty}^{\infty} 0.1[u(\tau) - u(\tau-10)]\,x(t-\tau)\,d\tau \qquad u(t) - u(t-10) = \begin{cases} 1, & 0 \leq t \leq 10 \\ 0, & 0.\omega. \end{cases}$$

$$= 0.1 \int_{-\infty}^{\infty} \underline{[u(\tau) - u(\tau-10)]}\,x(t-\tau)\,d\tau$$

$$= 0.1 \int_{0}^{10} x(t-\tau)\,d\tau \quad \cdots$$



rectangular pulse

→ System averages input values from current time minus 10 to current time.

③ Compute the output $y(t)$ if the input is $x(t) = e^{-at}u(t)$ with $a > 0$ and the LTI system's impulse response is $h(t) = u(t)$. Discuss the stability and causality of the system.

$x(\tau) = e^{-a\tau}u(\tau) \Rightarrow u(\tau) = \begin{cases} 1, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$ ✓

$h(t-\tau) = u(t-\tau)$

hence integrate from 0 instead $-\infty$.

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)\,u(t-\tau)\,d\tau$$

$$= \int_{0}^{\infty} e^{-a\tau}\,u(t-\tau)\,d\tau$$

Apply change of variables

Let $t-\tau = \lambda \Rightarrow \underline{\tau = t-\lambda}$

$\underline{d\lambda = -d\tau}$

$$= \int_{t}^{-\infty} e^{-a(t-\lambda)}\,u(\lambda)\,(-d\lambda) \quad \text{--- ①}$$

$$= \int_{-\infty}^{t} e^{-a(t-\lambda)}\,\underline{u(\lambda)}\,d\lambda$$

$$= \int_{-\infty}^{t} e^{-at}e^{a\lambda}\,u(\lambda)\,d\lambda$$

$$= e^{-at}\int_{-\infty}^{t} e^{a\lambda}\,u(\lambda)\,d\lambda$$

$$= e^{-at} \int_{-\infty}^{t} e^{a\lambda} u(\lambda)\, d\lambda$$

When $t < 0$, $u(\lambda) = 0 \xrightarrow{\text{implies that}} y(t) = 0,\ t < 0$

$t > 0$ Integral will be non-zero part

$$u(\lambda) = 1 \quad \text{for} \quad 0 < \lambda \le t$$

$$y(t) = e^{-at} \int_{-\infty}^{t} e^{a\lambda} u(\lambda)\, d\lambda = e^{-at} \int_{0}^{t} e^{a\lambda}\, d\lambda$$

$$= e^{-at} \left[\frac{1}{a} e^{a\lambda} \Big|_{0}^{t} \right]$$

$$e^{-at} \cdot e^{at} = e^{0}$$

$$= e^{-at} \left[\frac{1}{a} \left(e^{at} - 1 \right) \right]$$

$$= \frac{1}{a} \left(1 - e^{-at} \right)$$

when $t > 0$.

Combining the results for $t < 0$ and $t > 0$ we yield

$$y(t) = \frac{1}{a} \left(1 - e^{-at} \right) u(t)$$

$$y(t) = \begin{cases} \frac{1}{a}\left(1 - e^{-at}\right), & t > 0 \\ 0, & t < 0 \end{cases}$$

$\Rightarrow$ BIBO Stable $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \Rightarrow \int_{-\infty}^{\infty} |u(t)|\, dt = \int_{0}^{\infty} 1\, dt = t \Big|_{0}^{\infty} = \infty$, System is NOT BIBO stable.

(ff) Solve $x(t) * u(t-t_0)$.

$$x(t) * u(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, \underbrace{u(t-t_0-\tau)}\, d\tau$$

$$u(t-t_0-\tau) = \begin{cases} 1 & t-t_0-\tau \geq 0 \\ & \text{or } \tau \leq t-t_0 \\ 0 & \tau \geq t-t_0. \end{cases}$$

non-zero only for

$t-t_0-\tau \geq 0 \Rightarrow \underline{\tau \leq t-t_0}$

$$= \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$$

✓✓

(5) Let the input signal $x(t) = u(t)$ and the system impulse response $h(t) = e^{-t}u(t)$. Calculate the corresponding output signal $y(t)$.

$$x(t) = u(t) \longrightarrow \boxed{h(t) = e^{-t}u(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) = h(t) * x(t) \quad \text{(Commutativity of convolution)}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau} \underbrace{u(\tau)}_{\text{nonzero only for } \tau \geq 0} \underbrace{u(t-\tau)}_{\text{nonzero only for } t-\tau \geq 0 \Rightarrow \tau \leq t} d\tau.$$

$\checkmark$ $\therefore$ Integral only survive for $\tau \geq 0$ & $\tau \leq t$

$$= \begin{cases} \int_0^t e^{-\tau}\, d\tau, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$$

$$\int_0^t e^{-\tau}\, d\tau = -e^{-\tau}\Big|_0^t = \underbrace{1 - e^{-t}}_{\text{when } t \geq 0}$$

$$y(t) = \begin{cases} 1 - e^{-t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$= (1 - e^{-t})\, u(t).$$

⑥ Let $x(t) = u(t) - u(t-1)$. What is

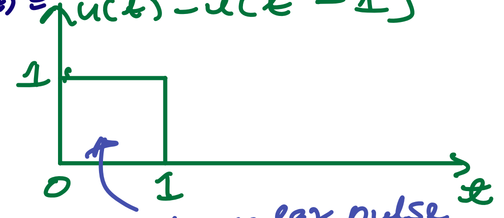
$y(t) = x(t) * x(t)$?

$= \int_{-\infty}^{\infty} x(\tau)x(t-\tau)\,d\tau$

$u(t) = \begin{cases} 1, & t \geq 0 \\ 0 & t < 0 \end{cases}$

$u(t-1) = \begin{cases} 1, & t \geq 1 \\ 0 & t < 1 \end{cases}$

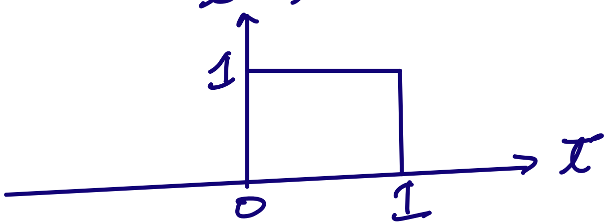
$x(t) = [u(t) - u(t-1)]$

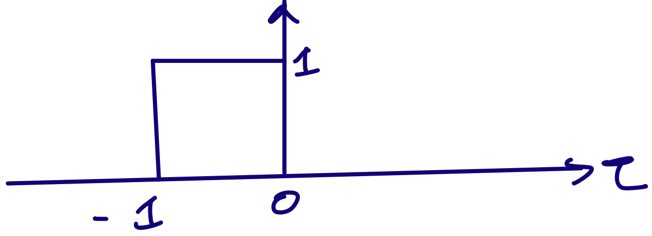


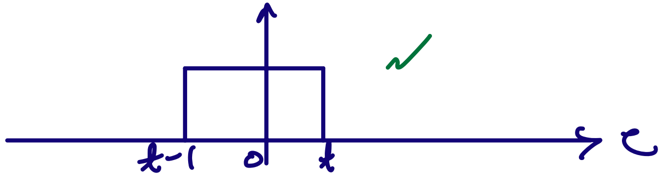
rectangular pulse

$u(t) - u(t-1) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{o.w} \end{cases}$

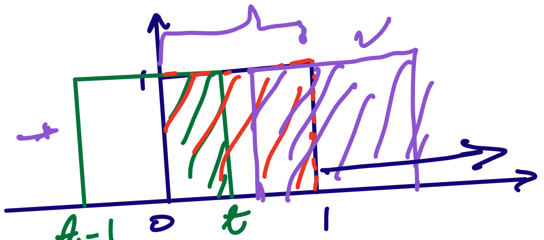
Use graphical method:







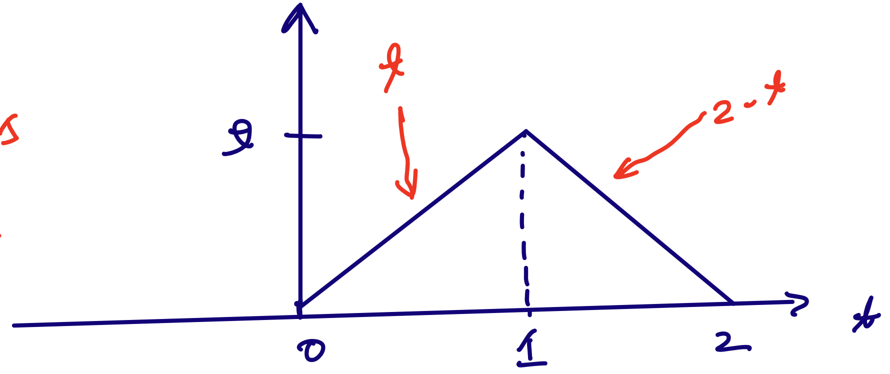
$\int x(\tau)\, x(t-\tau)\, d\tau$



$\int_0^t 1\, d\tau = \tau \Big|_0^t = t$

Increasing linearly with t for $0 \leq t \leq 1$. For $t \geq 1$, Decreases linearly until it reaches 0.

$\frac{1}{2} \cdot 2 \cdot 1 = 1$



$$y(t) = \begin{cases} 1 - |t-1| , & 0 \le t \le 2 \\ 0 , & o.w. \end{cases}$$

$$x(t) = \begin{cases} 1 , & 0 < t \le 1 \\ 0 , & o.w. \end{cases}$$

$$\int_{-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau$$

$$x(\tau) = \begin{cases} 1 , & 0 \le \tau \le 1 \\ 0 , & o.w. \end{cases}$$

$$x(t-\tau) = \begin{cases} 1 , & 0 \le t-\tau \le 1 \\ 0 , & o.w. \end{cases}$$

$0 \le t - \tau$

$\tau \le t$

$t - \tau \le 1$

$t - 1 \le \tau$

$$\int_{t-1}^{t} 1\, d\tau = \tau \Big|_{t-1}^{t}$$

$$= t - (t-1)$$

$$= \underline{1}$$ ✓

NB: The convolution of 2 square pulses of equal width and height (a square pulse and itself) gives a triangular pulse..

(7) [Periodic Convolution] Let $x_1(t), x_2(t)$ be periodic with a common period, T_0. Periodic Convolution is defined as

$$f(t) := \int_0^{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau = x_1(t) * x_2(t)$$

Show that

$$f(t) = \int_{t_0}^{t_0+T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau .$$

Shift the integral by any arbitrary value / small value t_0.

For periodic fn $x(t) = x(t+T_0)$
$T_0 \Rightarrow$ period

Soln Let $\phi_t(\tau) = x_1(\tau)\, x_2(t-\tau)$ for a given t.

$$\therefore \phi_t(\tau+T_0) = x_1(\tau+T_0)\, x_2(t-(\tau+T_0)).$$

$$= x_1(\tau+T_0)\, x_2(t-\tau-T_0)$$

$$= x_1(\tau)\, x_2(t-\tau)$$

$$= \phi_t(\tau) \Rightarrow$$

We have shown that $\phi_t(\tau+T_0) = \phi_t(\tau)$. This implies that $\phi_t(\tau)$ is periodic. ✓

$$\therefore f(t) = \int_0^{T_0} \phi_t(\tau)\, d\tau = \int_{t_0}^{t_0+T_0} \phi_t(\tau)\, d\tau$$

Thus, the integral over any period is the same.

$$= \int_{t_0}^{t_0+T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

This follows from the fact that, for a periodic signal, integral over any period ~~f~~ is the same

$$= \tilde{f}(t_0)$$

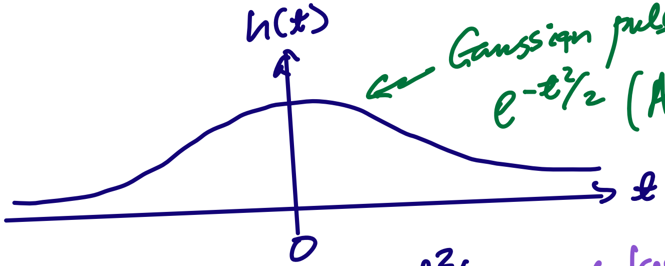
⑧ [BIBO Stability] Consider a system with impulse response

$$h(t) = e^{-t^2/2} \quad -\infty < t < \infty.$$

Is this system BIBO stable?

Soln For BIBO stability, $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$ ⟹ Impulse response must always be absolutely integral.

finite.



Gaussian pulse $e^{-t^2/2}$ (Always positive)

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} e^{-t^2/2}\,dt$$

(Ignore absolute since pulse is always +ve)

$$= \sqrt{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt}_{1}$$

⟹ Gaussian probability distribution function (pdf)

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}$$

⟹ Var $\sigma^2 = 1$, mean $\mu = 0$

$N(0,1)$

$= \sqrt{2\pi} \cdot 1$

$= \sqrt{2\pi} < \infty$

⟹ System with impulse response given by a Gaussian pulse is BIBO stable.

⑨ Consider LTI system

output signal

Input

$$y(t) = \int_{-\infty}^{t} \underline{e^{-j\omega_0(t-\tau)}} \, \underline{x(\tau)} \, d\tau ,$$

$u(t-\tau)$ at $t - \tau \geq 0$

$t \geq \tau$

Find (a) the impulse response $h(t)$.

(b) the eigenfunction for the function $e^{st} u(t)$.

Soln

a) Impulse response $\int_{-\infty}^{t} e^{-j\omega_0(t-\tau)} x(\tau) \, d\tau$ ✓

$$= \int_{-\infty}^{\infty} \underbrace{e^{-j\omega_0(t-\tau)} u(t-\tau)}_{h(t-\tau)} \underbrace{x(\tau)}_{x(\tau)} \, d\tau$$

∴ Impulse response $h(t) = e^{-j\omega_0 t} u(t)$ ✓

b) To find the eigenfunction corresponding to $\underline{e^{st}u(t)}$, we note that this can also be written as

$$h(t) * x(t) = \int_{-\infty}^{\infty} e^{-j\omega_0(t-\tau)} u(t-\tau)x(\tau)\,d\tau$$

$$= x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} e^{-j\omega_0\tau} \underline{u(\tau)}\, x(t-\tau)\,d\tau$$

Convolution is Commutative

↳ Non-zero only if $\tau \geq 0$

$$= \int_{0}^{\infty} e^{-j\omega_0\tau} x(t-\tau)\,d\tau$$

Substitute $x(t) = e^{st}$

$$\Rightarrow x(t-\tau) = e^{s(t-\tau)}$$

$$= \int_0^\infty e^{-j\omega_0\tau}\, \underline{e^{s(t-\tau)}}\,d\tau$$

$$= e^{st}\int_0^\infty e^{-j\omega_0\tau} e^{-s\tau}\,d\tau$$

$$= e^{st}\int_0^\infty e^{-(s+j\omega_0)\tau}\,d\tau$$

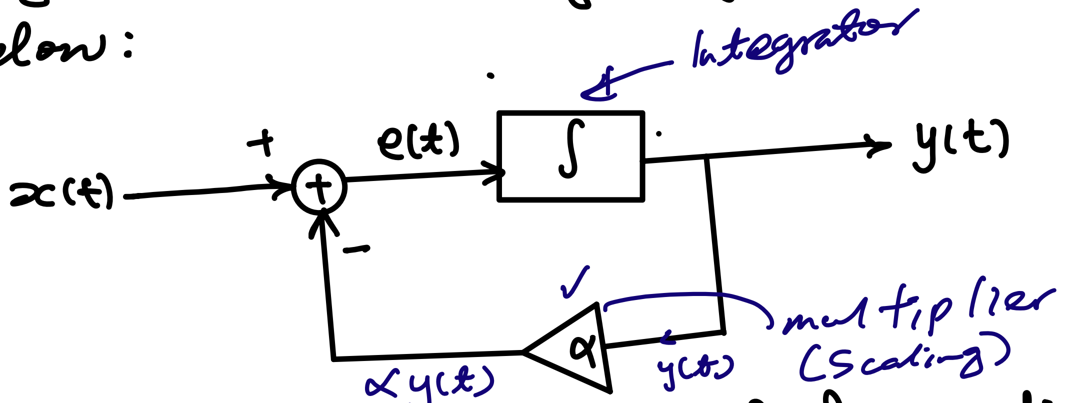
$$= e^{st} \left. \frac{e^{-(s+j\omega_0)\tau}}{-(s+j\omega_0)} \right|_0^\infty$$

$$= e^{st}\left(0 - \frac{1}{-(s+j\omega_0)}\right)$$

to be 0, $s>0 \Rightarrow e^{-st}$ decaying expo-

$$= e^{st} \underbrace{\left(\frac{1}{s+j\omega_0}\right)}_{\text{eigenfunction of } e^{st}.} \text{ if } Re(s) > 0$$

(10) [Systems Described by Differential Equation] Consider the system below:



$x(t)$ – input signal
$y(t)$ – output signal
$e(t)$ – error function

Determine the differential equation (DE) representation of the system above.

$$x(t) - \alpha y(t) = e(t) \quad \cdots (i)$$

$$\int_0^t e(\tau)\, d\tau = y(t) \quad \cdots (ii)$$

Differentiate both sides of (ii)

$$e(t) = \frac{dy(t)}{dt} \quad \cdots (iii)$$

Substitute (iii) into (i)

$$x(t) - \alpha y(t) = \frac{dy(t)}{dt}$$

$$\sum_{k=0}^{M} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_n \frac{d^k x(t)}{dt^k}$$

$$\Rightarrow \underbrace{\frac{dy(t)}{dt} + \alpha y(t) = x(t)}$$

DE describing above system.

(11) Consider the system above, which we have described by

$$\frac{dy(t)}{dt} + \alpha y(t) = x(t).$$

Initial conditions/Boundary condition

Let the auxilliary condition $y(0) = y_0$ and input signal $x(t) = C e^{-\beta t} u(t)$. Obtain the output response $y(t)$.

Soln Recall soln to a DE is

$$y(t) = \underbrace{y_h(t)}_{\text{homogeneous soln}} + \underbrace{y_p(t)}_{\text{particular integral/soln}}$$

For $y_h(t)$ (Homogeneous soln)

Let $y_h(t) = Ae^{st}$

Homogeneous soln is obtained from

$$\frac{dy_h(t)}{dt} + \alpha y_h(t) = 0$$

$$\frac{d}{dt}\left(Ae^{st}\right) + \alpha Ae^{st} = 0$$

$$sAe^{st} + \alpha Ae^{st} = 0$$

$$(s+\alpha) Ae^{st} = 0.$$

$$\Rightarrow s+\alpha = 0 \text{ or } s = -\alpha$$

$$\Rightarrow y_h(t) = Ae^{-\alpha t}.$$

For $y_p(t)$ (Particular soln)

Let $y_p(t) = K e^{-\beta t}$

Particular soln can be obtained from

$$\frac{dy_p(t)}{dt} + \alpha y_p(t) = x(t) = Ce^{-\beta t}$$

$$\frac{d}{dt}(Ke^{-\beta t}) + \alpha Ke^{-\beta t} = Ce^{-\beta t}.$$

$$-\beta K\cancel{e^{-\beta t}} + \alpha K \cancel{e^{-\beta t}} = C\cancel{e^{-\beta t}}$$

$$(-\beta + \alpha) K = C \text{ or } K = \frac{C}{\alpha - \beta}.$$

$$\Rightarrow y_p(t) = \frac{C}{\alpha-\beta} e^{-\beta t}$$

$$\therefore y(t) = y_h + y_p = Ae^{-\alpha t} + \frac{C}{\alpha-\beta} e^{-\beta t}$$

We use the auxilliary conditions to obtain A

$$y(0) = y_0$$

$$\Rightarrow y(0) = Ae^0 + \frac{C}{\alpha-\beta}e^0 = y_0$$

$$\Rightarrow A + \frac{C}{\alpha-\beta} = y_0$$

$$\Rightarrow A = y_0 - \frac{C}{\alpha-\beta}$$

$$\therefore \underbrace{y(t) = \left(y_0 - \frac{C}{\alpha-\beta}\right)e^{-\alpha t} + \frac{C}{\alpha-\beta} e^{-\beta t}}_{\text{output of signal given system and auxilliary conditions}} \checkmark$$

For $t < 0$, $x(t) = 0$

$$\frac{dy(t)}{dt} + \alpha y(t) = 0$$ (only homogeneous soln satisfy this)

$$y(t) = S e^{-\alpha t}, \quad S = \text{constant}.$$

$$y(0) = y_0 \Rightarrow S = y_0$$

For $t < 0$, $y(t) = y_0 e^{-\alpha t}$

output signal for $t < 0$.
(think of this a zero-input signal ie $x(t) = 0$).

$$y(t) = \left(y_0 - \frac{c}{\alpha - \beta}\right) e^{-\alpha t} + \frac{c}{\alpha - \beta} e^{-\beta t}, \text{ for } t \geq 0.$$

$$y(t) = \underbrace{y_0 e^{-\alpha t}}_{y_{zi}(t)} + \underbrace{\frac{c}{\alpha - \beta}\left(e^{-\beta t} - e^{-\alpha t}\right)}_{y_{zs}(t)}$$

zero input

$y_{zi}(t)$: output to zero-input/response to auxilliary condition

zero-state sign

$y_{zs}(t)$: Response with zero auxilliary condition i.e. if $y_0 = 0$,
$\Rightarrow y_{zi} = 0$
$y(t) = y_{zs}(t)$

Example Problems for D.T. LTI Systems

① Let $x[n] = \alpha^n u[n]$ be a D.T. input signal to a system and its impulse response is $h[n] = \beta^n u[n]$. What is the output signal $y[n]$ for the system?

Soln

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m]$$

$$= \sum_{m=-\infty}^{\infty} \alpha^m u[m]\, \beta^{n-m} u[n-m]$$

$u[n-m] = 0$ if $n-m<0$ or $n<m$

$\quad\quad\quad\;\; 1$ if $n-m \geq 0$ or $n \geq m$

$$u[m] = \begin{cases} 0 & \text{if } m<0 \\ 1 & \text{if } m \geq 0 \end{cases}$$

$$= \begin{cases} \sum_{m=0}^{n} \alpha^m \beta^{n-m}, & \text{if } n \geq 0 \\ 0, & \text{if } n<0 \end{cases}$$

$0 \leq m \leq n$.

Integral/Sum $= 0$ if $m<0$ & $m>n$

⇒ Input signal is causal ⇒ it is 0 for $n<0$

⇒ Impulse response is causal ⇒ it is 0 for $n<0$

∴ Output is also going to be naturally causal.

$$= \begin{cases} \sum_{m=0}^{n} \alpha^m \beta^n \cdot \beta^{-m} & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

$$= \begin{cases} \beta^n \sum_{m=0}^{n} \alpha^m \beta^{-m}, & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

$$= \begin{cases} \beta^n \sum_{m=0}^{n} \left(\frac{\alpha}{\beta}\right)^m & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

Geometric series.

$$= \begin{cases} \dfrac{1-(\alpha/\beta)^{n+1}}{1-\alpha/\beta}\beta^n & \text{if } \alpha \neq \beta, n \geq 0 \\ \beta^n (n+1) & \text{if } \alpha = \beta, n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

Please revise geometric series. Good D.T. systems.

② [Causal System] A system is described by the function

$$y[n] = \sum_{k=-\infty}^{n} 2^{k-n} \, x[k+1].$$ Is this system causal?

output signal

input signal

Soln Reminder: D.T. L.T.I system is causal if $\boxed{h[n] = 0 \text{ for } n<0.}$

output depends only on present & past inputs.

$$y[n] = \sum_{k=-\infty}^{n} 2^{k-n} x[k+1] \quad \cdots \quad (1)$$

Let $u = k+1 \Rightarrow k = u-1$ (Change of variable)

Rewrite (1) as

$$y[n] = \sum_{u=-\infty}^{n+1} 2^{u-1-n} x[u]$$

$$= \underline{x[n+1]} + \sum_{k=-\infty}^{n} 2^{u-n-1} x[u]$$

$\Rightarrow$ System is NOT causal since the output $y[n]$ depends on the future input $x[n+1]$

Alternative Method / soln.

One could find impulse response of system.

$\Rightarrow$ Set the input $x[n] = \delta[n]$

$$\therefore \quad h[n] = \sum_{k=-\infty}^{n} 2^{k-n} \underbrace{\delta[k+1]}$$

$$\delta[k+1] = \begin{cases} 1 & \text{if } k=-1 \\ 0 & \text{if } k \neq -1 \end{cases}$$

$$h[n] = \begin{cases} 2^{-1-n} & \text{for } n \geq -1 \\ 0 & \text{o.w} \end{cases}$$

$\Rightarrow$ $y[n]$ is nonzero ($y[n] \neq 0$) iff $n \geq -1$

k takes value -1

$n=-1$

$$h[-1] = 2^{-1-(-1)} = 2^0 = 1$$

$\Rightarrow h[n] \neq 0$ for $n < 0$.

$\therefore$ System is not Causal.

③ [BIBO Stability] Let $h[n] = \alpha^n u[n]$. Is this system BIBO stable?

Recall: D.T. LTI system is BIBO stable if $\underbrace{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}_{\text{finite}}$ → absolutely summable

Soln: $h[n] = \alpha^n u[n]$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\alpha|^n \underbrace{u[n]}$$

→ non zero only if $n \geq 0$

$$= \sum_{n=0}^{\infty} |\alpha^n|$$ → Geometric Series

→ SMP tables.

$$= \begin{cases} \dfrac{1}{1-|\alpha|}, & \text{if } |\alpha| < 1 \\ \infty, & \text{otherwise} \end{cases}$$

⇒ System is BIBO stable iff $|\alpha| < 1$, otherwise it is an unstable system.

4) [Difference Equation] Let a D-T LTI system be described by the difference equation

$$y[n] = \alpha y[n-1] + x[n].$$

Assuming initial rest, what is the impulse response of the system?

Soln Set $x[n] = \delta[n]$ $\Rightarrow$ From defn of impulse response.

From initial rest

$$x[0] = \delta[0] = 1.$$

$$\therefore y[0] = \alpha y[-1] + \delta[0] = 1 \quad (\text{since } y[-1] = 0)$$

$n=1$ $\quad y[1] = \alpha y[0] + \underbrace{\delta[1]}_{=0} = \alpha \cdot 1 + 0 = \alpha.$

$n=2$ $\quad y[2] = \alpha y[1] + \delta[2] = \alpha \cdot \alpha + 0 = \alpha^2$

$n=3$ $\quad y[3] = \alpha y[2] + \delta[3] = \alpha \cdot \alpha^2 + 0 = \alpha^3$

$\vdots$

n $\quad y[n] = \alpha y[n-1] + \delta[n] = \alpha \cdot \alpha^{n-1} + 0 = \alpha^n$

Since our input is ~~the~~ the impulse function

$\delta[n]$

$$y[n] = \boxed{h[n] = \alpha^n}$$

$\Rightarrow$ impulse response of system.