Linear Algebra

→ Matrix algebra / Matrix calculus

→ Matrix inverses (Inversion)

→ Systems of equations

→ Back substitution

→ More systematics methods

Elementary Row Operations (EROs)

i) Gaussian elimination

ii) Gauss-Jordan elimination

→ Row echelon form

upper triangular matrix

→ Reduced row echelon form (RRE)

diagonal matrix.

$$A = \begin{array}{c} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ \\ R_i \\ R_{i+1} \\ \\ R_m \end{array}$$

Def 1 (Row echelon form) A matrix $A = (a_{ij})$ is in row echelon form or is a row echelon matrix if the following conditions hold:

(i) There are no zero's above non-zero rows.

(ii) If R_i and R_{i+1} are non-zero consecutive rows of A, then the first non-zero entry of row R_{i+1} is situated (in a column) to the right of (the column of) the first non-zero entry of R_i.

Rmk:

The first non-zero entry in each row of a row echelon matrix is called a pivot.

Example (Row echelon matrices)

Consider the following matrices:

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & 9 \\ 0 & 0 & 4 & 0 & -2 \\ 0 & 0 & 0 & 6 & -7 \end{bmatrix} \begin{array}{l} R_1 \quad i=1 \\ R_2 \\ R_3 \end{array}$$

→ Matrix A is a row echelon matrix and its pivots are $1, 4$ & 6

Apply elementary row operations to generate a row echelon matrix.

Not Row Echelon matrix

$$B = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 7 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

R_1, R_2, R_3 Non-zero rows

$R_4 \to$ Zero row.

$R_4 = (0,0,0,0)$ zero row vector

$R_4 = (0) \in \mathbb{R}^{1\times 4}$

i) satisfied $\to$ No zeros above non-zero rows (R_1, R_2 & R_3)

ii) R_1 & R_2 ✓

R_2 & R_3 $\to$ 7 which is ~~its~~ the first non-zero entry of R_3 is not to the right of the first non-zero entry of R_2.

$\Rightarrow$ Condition (ii) is not satisfied.

$\Rightarrow \therefore B$ is not a row echelon matrix.

$$C = \begin{bmatrix} 1 & 0 & 0 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{array}{l} R_1 \to \text{Non-zero row} \\ R_2 \to \text{Zero-row} \\ R_3 \to \text{Non-zero row} \\ R_4 \to \text{Non Zero row} \end{array}$$

$\Rightarrow$ Not a row echelon matrix

$\Rightarrow$ Cond (i) is not satisfied. R_2 is a zero row above R_3 & R_4 which are non-zero rows.

ERO's

Interchange

$R_2 \leftrightarrow R_4$

The new matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

will be row echelon matrix.

Example

A system (Solve).

$$\begin{array}{ll} x_1 + x_2 + x_3 = 3 & R_1 \\ x_1 - x_2 + 2x_3 = 2 & R_2 \\ 2x_1 + x_2 - x_3 = 2 & R_3 \end{array}$$

Soln

a) Gaussian elimination (GE) method)

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 2 \\ 2 & 1 & -1 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & -1 & -3 & -4 \end{array}\right]}_{R_2 \leftrightarrow R_3} \xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2}$$

$$-1 - \tfrac{1}{2}(-2) \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3.5 & -3.5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -4 \\ 0 & -2 & 1 & -1 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2 \ \checkmark} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -4 \\ 0 & 0 & 7 & 7 \end{array}\right]$$

Row echelon matrix

Pivots

1, -1, 7

rank = 3

Back substitution:

$R_3: \ -3.5x_3 = -3.5$

$$x_3 = 1$$

$R_2: \ -2x_2 + x_3 = -1$

$\Rightarrow -2x_2 = -1 - x_3$

$= -1 - 1$

$= -2$

$\Rightarrow x_2 = 1$

$R_3: \ x_1 + x_2 + x_3 = 3$

$x_1 = 3 - x_2 - x_3 = 3 - 1 - 1 = 1$

$$x_1 = 1, \ x_2 = 1, \ x_3 = 1$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$7x_3 = 7$

$\Rightarrow x_3 = 1$

$-x_2 - 3x_3 = -4$

$-x_2 - 3(1) = -4$

$-x_2 = -4 + 3$

$-x_2 = -1$

$x_2 = 1$

x_1 ✓

$\rightarrow$ diagonal matrix

(b) Gauss-Jordan (G-J) elimination.

EROs

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -4 \\ 0 & 0 & 7 & 7 \end{array}\right] \xrightarrow[R_2 \to R_2 + \frac{3}{7}R_3]{R_1 \to R_1 - \frac{1}{7}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 7 & 7 \end{array}\right]$$

$-3+\frac{3}{7}(7)$

$-4+\frac{3}{7}(7)$

$$\xrightarrow{R_1 \to R_1 + R_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 7 & 7 \end{array}\right]}_{\text{diagonal matrix}}$$

R_3: $7x_3 = 7$

$x_3 = 1$

R_2: $\frac{-x_2}{-1} = \frac{-1}{-1}$

$x_2 = 1$

R_3: $x_1 = 1$

We can go further to create an identity matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 7 & 7 \end{array}\right] \xrightarrow[R_3 \to R_3/7]{R_1 \to R_1,\ R_2 \to -1 R_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]}_{\text{Reduced row echelon form (RRE)}}$$

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{array}\right\} \checkmark$$

Exercise

1. Apply the Gauss-Jordan elimination to the systems of equation below.

a)
$$\begin{aligned} x - 2y &= 2 \\ -\tfrac{1}{2}x + z &= 2 \\ -x - y + 2z &= 0 \\ -3y + 2z &= 2 \end{aligned}$$

Ans $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$ → Reduced row echelon form

b) Use Gaussian elimination:

$$A = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & -3 & 6 & -3 \end{array}\right]$$

Soln $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ · Row echelon matrix

Def 2 (Rank of a $k \times n$ matrix) The rank of a $k \times n$ matrix A is the number of pivots of any row echelon matrix obtained from A using elementary operations. The rank of A is denoted by rank(A).

Example (Rank)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row echelon matrix

pivots: 1, 3

Since this matrix is a row echelon matrix and has 2 pivots. The rank of the matrix A

$$\text{rank}(A) = 2$$

Rmk (How to calculate the rank of a matrix A)

(1) Reduce the matrix A to a row echelon matrix R using Gaussian elimination

(2) The rank of A is the number of pivots in R, i.e.,

$$\boxed{\text{rank}(A) = \text{rank}(R) = \text{number of pivots of } R}$$

Proposition 1 (Consistent & Inconsistent matrices)
Let $[A|b] \in \mathbb{R}^{k \times n+1}$ be an augmented matrix of a system of linear equations. The following assertions hold:

(i) If $\boxed{\text{rank}(A) < \text{rank}([A|b])}$, then the system is inconsistent.

(ii) If $\boxed{\text{rank}(A) = \text{rank}([A|b])}$, then the system is consistent. In this case, the number of independent variables coincides with $n - \text{rank}(A)$, i.e.,

$$\boxed{\text{number of independent variables} = n - \text{rank}(A)}$$

Example: Consider the system below

$$\begin{aligned} x &= 0 \\ (1-i)y - (2i)z &= 1-i \\ y + (1-i)z &= 1 \end{aligned}$$

whose coefficient matrix A and vector b of independent terms are, respectively.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-i & -2i \\ 0 & 1 & 1-i \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1-i \\ 1 \end{bmatrix}$$

a) Find the rank of the matrix A and state whether it is consistent or inconsistent.

b) Find the Hermitian transpose of matrix A.

c) Calculate the determinant of the complex matrix A.

Rmk: Any square matrix A can be decomposed into the sum of symmetric matrix with an anti-symmetric matrix:

$$\boxed{A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)}$$

Exercise [Source: Lina Oliveira Linear Algebra]

1) Show that, given a square matrix A, the matrix $A + A^T$ is symmetric and the matrix $A - A^T$ is anti-symmetric.

Def (Trace of a matrix) Let $A = (a_{ij})$ be a square matrix of order n. The trace of A is the sum of all entries in the diagonal of A, i.e.,

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$$

Example: Consider the following matrices

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 5 \\ -3 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$$

a) Classify A and B as symmetric or anti-symmetric

→ Recall

- diagonal of anti-symmetric matrix is null (all entries are zero)
- diagonal of a symmetric matrix is "like a mirror" reflecting entries on both sides of the diagonal.

b) Calculate the trace of matrix A and B.

$$\operatorname{tr} A = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$
$$= 1 + 4 + (-6) = \underline{-1}$$

$$\operatorname{tr} B = \sum_{i=1}^{3} a_{ii} = 0 + 0 + 0 = 0$$ (trace of an anti-symmetric matrix is always zero.)

Prop 2 (Properties of Trace) Let A, B be $n \times n$ square matrices and let α be a scalar. Then,

(i) $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$

(ii) $\operatorname{tr}(\alpha A) = \alpha \operatorname{tr} A$

(iii) $\operatorname{tr} A = \operatorname{tr} A^T$ → Follows from the fact that diagonal of the transpose of a square matrix remains unchanged.

(iv) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$