The Laplace Transform

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Outline

Definitions and Basics

Laplace Transform Definition

Bilateral Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \quad s = \sigma + j\omega$$

Unilateral Version

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

Eigenfunction Property

For LTI system $h(t) : e^{st} \to H(s)e^{st}$ where $H(s) = \mathcal{L}\{h(t)\}$

Relation to Fourier

When $\sigma = 0$:

$$X(j\omega) = \mathcal{F}\{x(t)\}$$

Laplace transform of $x(t)$ equals Fourier transform of $x(t)e^{-\sigma t}$

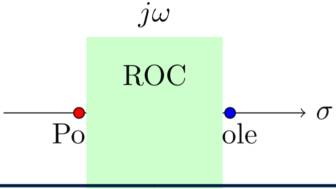
Region of Convergence Properties

Convergence Condition

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt < \infty$$

ROC Properties

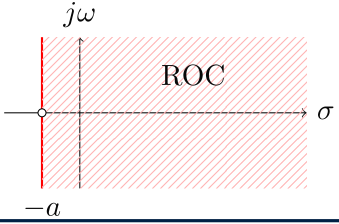
- Strip parallel to $j\omega$-axis
- No poles in ROC
- ROC must be specified with $X(s)$
- ROC determines causality and stability



Example 8.1: Right-Sided Exponential

Given $x(t) = e^{-at}u(t)$:

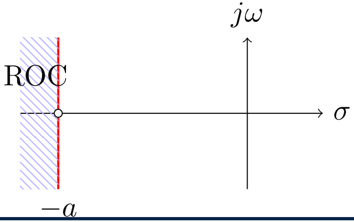
$$X(s) = \int_0^\infty e^{-(s+a)t}dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty = \frac{1}{s+a}, \quad \underbrace{\mathrm{Re}\{s\} > -a}_{ROC}$$



Example 8.2: Left-Sided Exponential

Given $x(t) = -e^{-at}u(-t)$:

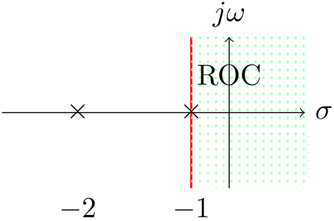
$$X(s) = -\int_{-\infty}^{0} e^{-(s+a)t}dt = \frac{1}{s+a}e^{-(s+a)t}\Big|_{-\infty}^{0} = \frac{1}{s+a}, \quad \underbrace{\mathrm{Re}\{s\} < -a}_{ROC}$$



Example 8.3: Sum of Exponentials

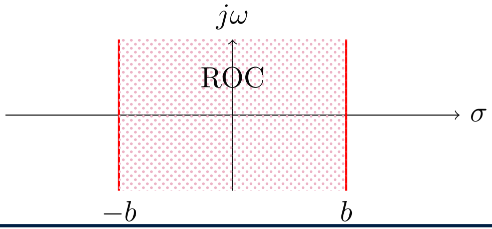
Given $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$:

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{(s+2)(s+1)}, \quad \underbrace{\mathrm{Re}\{s\} > -1}_{ROC}$$

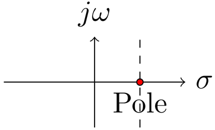


Given $x(t) = e^{-b|t|}$:

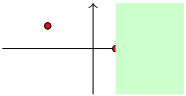
$$x(t) = e^{-bt}u(t) + e^{bt}u(-t), \quad X(s) = \frac{1}{s+b} - \frac{1}{s-b}, \quad \underbrace{-b < \Re\{s\} < b}_{ROC}$$



1. **Strip Property**: ROC is parallel to $j\omega$-axis
2. **Pole Exclusion**: ROC cannot contain poles



3. **Finite Duration**: Entire plane if absolutely integrable
4. **Right-Sided**: Right of rightmost pole



5. **Left-Sided**: Left of leftmost pole
6. **Two-Sided**: Strip between poles

Properties

Key Laplace Transform Properties

Linearity

$$ax(t) + by(t) \leftrightarrow aX(s) + bY(s)$$

Time Shifting

$$x(t - t_0) \leftrightarrow e^{-st_0}X(s)$$

Frequency Shifting

$$e^{at}x(t) \leftrightarrow X(s - a)$$

Key Laplace Transform Properties

Differentiation in Time

$$\frac{d^n x(t)}{dt^n} \leftrightarrow s^n X(s)$$

Integration

$$\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow \frac{1}{s} X(s)$$

Inverse Laplace Transform

- Recover $x(t)$ from $X(s)$ and ROC
- Three common methods:
    1. Inspection
    2. Partial Fraction Expansion
    3. Complex Contour Integration

Heaviside Cover-up Method

For $X(s) = \frac{N(s)}{(s+p_1)\cdots(s+p_n)}$ with distinct poles:

$$X(s) = \sum_{i=1}^{n} \frac{k_i}{s + p_i}$$

where $k_i = (s + p_i)X(s)\big|_{s=-p_i}$

Example (Example 8.8 Detailed)

Given $X(s) = \frac{s+5}{(s+1)(s-2)(s+4)}$:

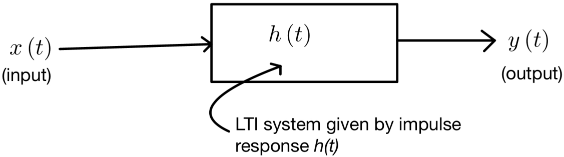
$$k_1 = \left.\frac{s+5}{(s-2)(s+4)}\right|_{s=-1} = \frac{4}{(-3)(3)} = -\frac{4}{9}$$

$$k_2 = \left.\frac{s+5}{(s+1)(s+4)}\right|_{s=2} = \frac{7}{(3)(6)} = \frac{7}{18}$$

$$k_3 = \left.\frac{s+5}{(s+1)(s-2)}\right|_{s=-4} = \frac{1}{(-3)(-6)} = \frac{1}{18}$$

LTI System Analysis and Applications

Transfer Function of LTI System



Recall: Convolution Property

- $y(t) = h(t) * x(t) \Leftrightarrow Y(s) = H(s)X(s)$
- ROC contains $R_H \cap R_X$

Transfer function of the system:

$$H(s) = \frac{Y(s)}{X(s)}$$

$$= \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Causality and Stability

- **Causal:** ROC is right of rightmost pole
- **Stable:** ROC includes $j\omega$-axis
- **Causal and stable:** All poles in left-half plane

Transfer function of LTI system described by the DE

Differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Transfer Function of System

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Example (Example 8.10 Complete)

Given $h(t) = e^{-2t}u(t)$ and $x(t) = e^{-3t}u(t)$:

$$H(s) = \frac{1}{s+2}, \mathrm{Re}\{s\} > -2$$
$$X(s) = \frac{1}{s+3}, \mathrm{Re}\{s\} > -3$$
$$Y(s) = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$
$$y(t) = (e^{-2t} - e^{-3t})u(t)$$

Differentiation Property

$$\mathcal{L}\left\{\frac{d^n x}{dt^n}\right\} = s^n X(s)$$

Example (Example 8.12 Full Solution)

$$\text{Given:} \quad \frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = x(t)$$

Laplace transform:

$$(s^3 + 2s^2 - s - 2)Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s-1)(s+1)(s+2)}$$

Partial fraction expansion:

$$H(s) = \frac{1/6}{s-1} - \frac{1/2}{s+1} + \frac{1/3}{s+2}$$

For causal system:

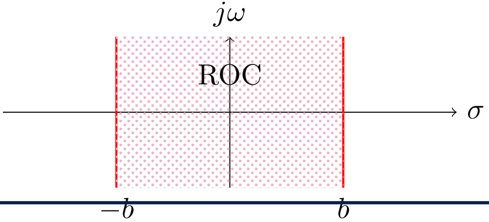
$$h(t) = \left(\frac{1}{6}e^{t} - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}\right)u(t)$$

Example 8.5: Two-Sided Exponential

Given $x(t) = e^{-b|t|}$:

$$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$$

$$X(s) = \frac{1}{s+b} - \frac{1}{s-b} \quad \text{for } -b < \text{Re}\{s\} < b$$

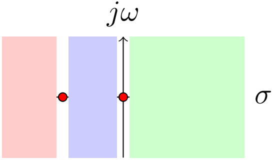


Example 8.6: Multiple Possible ROCs

Given $X(s) = \frac{1}{s(s+1)}$:

Possible ROCs:

1. $\text{Re}\{s\} > 0$
   $\Rightarrow x(t) = (1 - e^{-t})u(t)$
2. $-1 < \text{Re}\{s\} < 0$
   $\Rightarrow x(t) = -u(-t) - e^{-t}u(t)$
3. $\text{Re}\{s\} < -1$
   $\Rightarrow x(t) = (-1 + e^{-t})u(-t)$



Key Takeaways

- Laplace transform extends Fourier analysis to broader signal classes
- ROC is crucial for proper inverse transformation
- System analysis becomes algebraic in s-domain
- Differential equations convert to polynomial equations
- Pole-zero plots provide visual system characterization

Applications

- Circuit analysis
- Control systems
- Signal processing
- Communication systems