Systems of Linear Equations.

Linear Equations

Def 1.1 (Linear Equations) A linear equation with n unknowns $x_1, \dots, x_n$ is an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, \dots, a_n$ and b are scalars (i.e. numbers). A solution is any choice of values for $x_1, \dots, x_n$ such that (s.t.) the equation is satisfied.

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b}$$

$a_1, \dots, a_n$ & $b \to$ scalars

Example: $n = 2$ linear equation

$$2x_1 - 5x_2 = 3$$

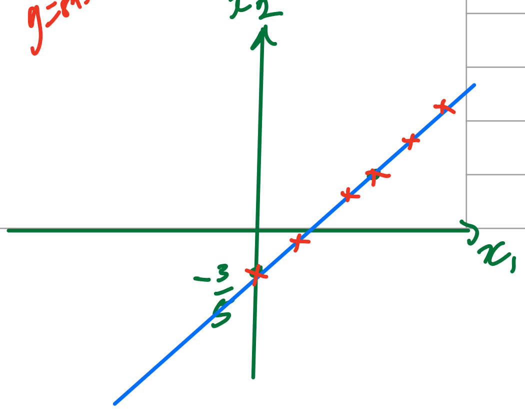
$\Rightarrow$ has infinitely many solutions (x_1, x_2)

$(0, -3/5)$, $(4, 1)$, $(\frac{3}{2}, 10)$, $(-7/2, -2)$, etc.

$$2\left(\frac{-7}{2}\right) - 5(-2) = -7 + 10$$
$$= \underline{3}$$

$5x_2 = 2x_1 - 3$

$y = mx + c$ — $x_2 = \frac{2}{5}x_1 - \frac{3}{5}$



m linear equation $\longrightarrow$ Systems of L.E.

The general System :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m .
\end{aligned}
$$

Can be written in matrix form as

$$A x = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad m \times n \text{ matrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

The system can also be written in augmented matrix form

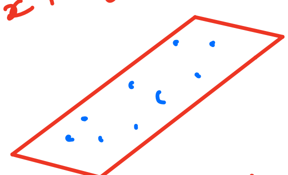
$$Ax = b \rightsquigarrow \underbrace{(A \mid b)}_{\text{augmented matrix form}}$$

Ex:

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

1.1. One equation

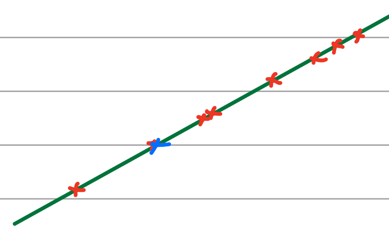
$ax + by + cz = d$



$a_1x_1 + \cdots + a_nx_n = b$

$ax + by = c$ or $a_1x_1 + a_2x_2 = b_1$

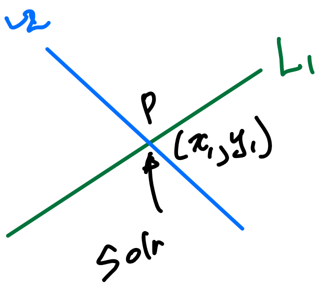
degenerate case: $a = b = 0$ ✗



Any point on the line corresponds to a solution.

($\Rightarrow$ we have infinitely many solutions.)

1.2. Two equations



$$a_1 x + b_1 y = c_1 \quad (L_1)$$
$$a_2 x + b_2 y = c_2. \quad (L_2)$$

$$\Rightarrow \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

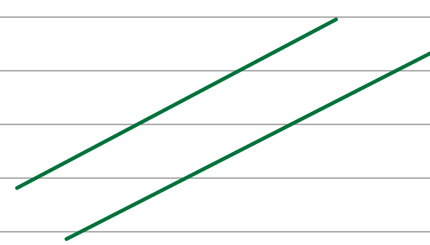
or

$$\left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$$

$\Rightarrow$ Solution corresponds to points that belong to both lines

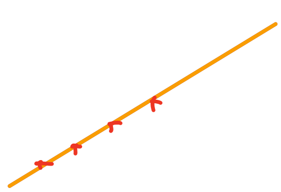
$\Rightarrow$ General case: Here the 2 lines intersects in exactly one point, so the system has a unique solution.

Special Cases 1 (2 lines are parallel)



$\Rightarrow$ The system has no solution.

Special Case 2 (2 lines coincide)

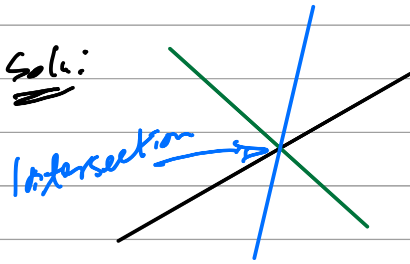


$\Rightarrow$ The system has infinitely many solutions

1.3. Three equations

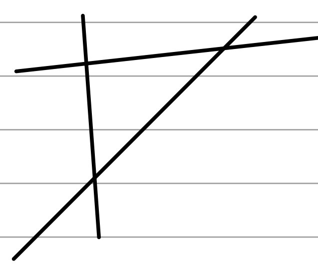
$$\begin{cases} a_1x_1 + b_1x_2 = c_1 \\ a_2x_1 + b_2x_2 = c_2 \\ a_3x_1 + b_3x_2 = c_2 \end{cases} \Rightarrow \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

✓ **Sol:**



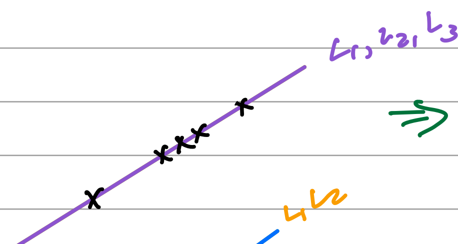
$\Rightarrow$ The solution corresponds to the intersection of the three lines.

① The General case

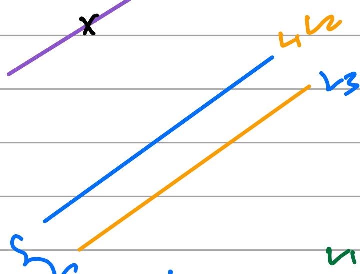


$\Rightarrow$ No solution

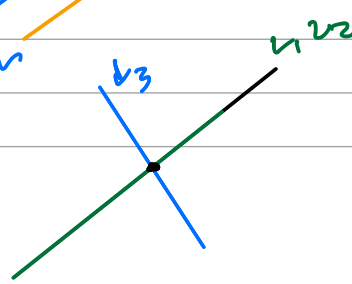
② 2 or more lines Coincide



$\Rightarrow$ Infinitely many solutions

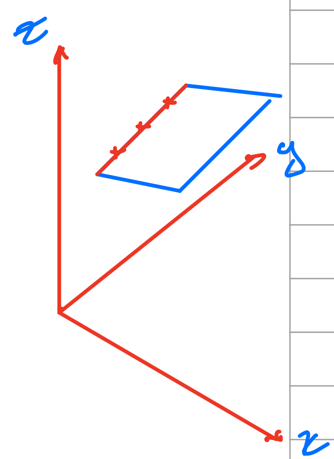


$\Rightarrow$ No solution.



$\Rightarrow$ Unique solution

Rmk: To obtain a unique solution, i.e., unique possible values of n unknowns, we need n equations.



① If m (# equations) $< n$ (# unknowns)

$\Rightarrow$ we don't have enough information & we will find infinite number of solutions.

② If $m > n$

$\Rightarrow$ we have "too much" information, & chances are that some of this information will contract itself.

$\therefore$ We will find no soln.

1.4. How to solve system in General

Some systems will be easy to solve: triangular systems

$$\begin{aligned} ax + by &= d \\ 0x + cy &= e \end{aligned} \quad \text{or} \quad \begin{aligned} ax + by + cz &= g \\ 0x + dy + ez &= h \\ 0x + 0y + fz &= k \end{aligned}$$

$y = e/c$

$ax + be/c = d$

$x = \dfrac{d - be/c}{a}$

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix} \qquad \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} g \\ h \\ k \end{pmatrix}$$

$\Rightarrow$ Entries below the diagonal are zero.

Example

$$3x + 2y = 4 \quad \cdots (1)$$
$$0x - y = 1 \quad \cdots (2)$$

From (2) $-y = 1 \Leftrightarrow y = -1$

Substitute back into eqn (1)

$$3x + 2(-1) = 4$$
$$3x = 4 + 2$$
$$\underline{\underline{x = 2}}$$

Back substitution

Example: Determine the solutions (if any) to the following equations:

$$3x + y - 2z = -2 \quad \cdots (1)$$
$$x + y + z = 2 \quad \cdots (2)$$
$$2x + 4y + z = 0 \quad \cdots (3)$$

Soln

From (2) $z = \boxed{2 - x - y} \rightarrow (*)$

Subst $*$ into (3)

$$2x + 4y + (2 - x - y) = 0$$
$$x + 3y + 2 = 0$$
$$x + 3y = -2 \quad \cdots (4)$$

Subst $*$ into (1) $\quad 3x + y - 2(2 - x - y) = -2$

$$5x + 3y - 4 = -2$$
$$\Rightarrow 5x + 3y = 2 \quad \cdots (5)$$

$x + 3y = -2 \quad \text{--- (4)}$

$5x + 3y = 2 \quad \text{--- (5)}$

Substract eqn (4) from (5)

$4x = 4 \Rightarrow \underline{\underline{x = 1}}$

eqn (4) $x + 3y = -2$

$\Rightarrow 1 + 3y = -2$

$3y = -3$

$\underline{\underline{y = -1}}$

$z = 2 - x - y$

$= 2 - 1 - (-1)$

$= \underline{\underline{2}}$

Soln $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Example:

$$\left.\begin{aligned} 2x + 3y - z &= 2 \quad \text{--- (1)} \\ -2y + 3z &= 4 \quad \text{--- (2)} \\ 7z &= -5 \quad \text{.. (3)} \end{aligned}\right\} \text{triangular system}$$

(3) $z = -5/7$ (From last eqn)

Back substitution to second eqn, we get y.

$-2y + 3\left(\frac{-5}{7}\right) = 4$

$-2y - 15/7 = 4 \Rightarrow$

$$y = \frac{4 + \frac{15}{7}}{-2}$$

Back subst into (1) ~~to~~ to get x.

Elementary Row Operations (EROs)

There are 4 types of EROs that do not change the solution set:

① Any equation of the form $0 = 0$ can be removed.

Ex:

$$\begin{cases} R_1 = 0 \\ 0 = 0 \end{cases} \iff R_1 = 0$$

② Equations/rows can be interchanged.

Ex

$$\begin{cases} R_1 = 0 \\ R_2 = 0 \end{cases} \text{swap} \iff \begin{cases} R_2 = 0 \\ R_1 = 0 \end{cases}$$

③ Any equation can be multiplied by a non-zero scalar (constant).

Ex:

$$\begin{cases} R_1 = 0 \\ R_2 = 0 \end{cases} \iff \begin{cases} \lambda R_1 = 0 \\ R_2 = 0 \end{cases}, \quad \lambda \neq 0$$

④ Any multiple of any equation/row can be added to any other equation/row.

Ex

$$\begin{cases} R_1 = 0 \\ R_2 = 0 \end{cases} \iff \begin{cases} R_1 = 0 \\ R_2 + \lambda R_1 = 0 \end{cases}, \quad \text{for any } \lambda$$

Note: If we choose $\lambda = \pm 1$ we get addition and subtraction.

$$\begin{cases} R_1 = 0 \\ R_2 = 0 \end{cases} \Longleftrightarrow \begin{cases} R_1 = 0 \\ R_2 + R_1 = 0 \end{cases} \Longleftrightarrow \begin{cases} R_1 = 0 \\ R_2 = 0 \end{cases} \Longleftrightarrow \begin{cases} R_1 = 0 \\ R_2 - R_1 = 0 \end{cases}$$

(EROs)

Thm 1.2 Using only operations 1 to 4 above any system of m linear equations and n unknowns can be transformed into equivalent triangular systems.

Example: Solve the following system.

$$2x_1 + 4x_2 + 8x_3 = 4$$
$$x_1 + 3x_2 - x_3 = 1$$
$$3x_1 + 3x_2 + 2x_3 = 2.$$

Soln

Augment matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & 8 & 4 \\ 1 & 3 & -1 & 1 \\ 3 & 3 & 2 & 2 \end{array}\right) \xRightarrow{R_1 \to \frac{1}{2}R} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 1 & 3 & -1 & 1 \\ 3 & 3 & 2 & 2 \end{array}\right)$$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & -5 & -1 \\ 0 & -3 & -10 & -4 \end{array}\right) \xRightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & -25 & -7 \end{array}\right)$$

Row Echelon form.

$$-25x_3 = -7 \Longrightarrow x_3 = 7/25$$

$$x_2 - 5x_3 = -1 \Rightarrow x_2 = -1 + 5(7/25) = 2/5$$

$$x_1 + 2x_2 + 4x_3 = 2 \Rightarrow x_1 = 2 - 2\left(\tfrac{2}{5}\right) - 4\left(\tfrac{7}{25}\right)$$
$$= \underline{\underline{2/25}}$$

$$-1 + \tfrac{7}{5}$$

$$-5 + \tfrac{7}{5} = 2/5$$

$$.15$$

$$x_1 = 2/25,\ x_2 = 2/5,\ x_3 = 7/25$$

Ex 2 Solve the system

$$2x + y - z = -3$$
$$3x + 2y + z = 6$$
$$x + y + 2z = 9$$

Soln The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 3 & 2 & 1 & 6 \\ 1 & 1 & 2 & 9 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & 2 & 1 & 6 \\ 2 & 1 & -1 & -3 \end{array}\right)$$

$$\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -1 & -5 & -21 \\ 0 & -1 & -5 & -21 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -1 & -5 & -21 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Last row corresponds to eqn $0 = 0$, any choice of z will solve the eqn.

Set $z = \lambda$.

Back substitution (R_2) $-1y - 5z = -21$

$$-y - 5\lambda = -21$$
$$y = 21 - 5\lambda$$

(R_1) $x + y + 2z = 9$

$$\Rightarrow x = 9 - y - 2z$$
$$= 9 - (21 - 5\lambda) - 2\lambda$$
$$= -12 + 3\lambda$$

$$\boxed{x = -12 + 3\lambda, \; y = 21 - 5\lambda, \; z = \lambda}.$$

This method is called Gaussian Elimination (GE).

Exercise Solve the system using Gaussian elimination method.

①
$$\begin{aligned} x + 3y + 4z &= 3 \\ 2x - y - 2z &= 3 \\ -x + 4y + z &= -5 \end{aligned}$$

Ans $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

②
$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 1 \\ 4x_1 + 6x_2 + 9x_3 &= 1 \\ 2x_1 + 4x_2 + 6x_3 &= 1 \end{aligned}$$

Ans → Not yet done.

③
$$\begin{aligned} x + y + z &= 6 \\ 2x + 4y - 3z &= 1 \\ 3x + 2y - 2z &= 1 \end{aligned}$$

Ans $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Gauss-Jordan (GJ) Elimination

Partition of matrices