# Examples and Counterexamples

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## 1 Introduction

Here're some basic examples and counterexamples in elementary algebraic geometry.

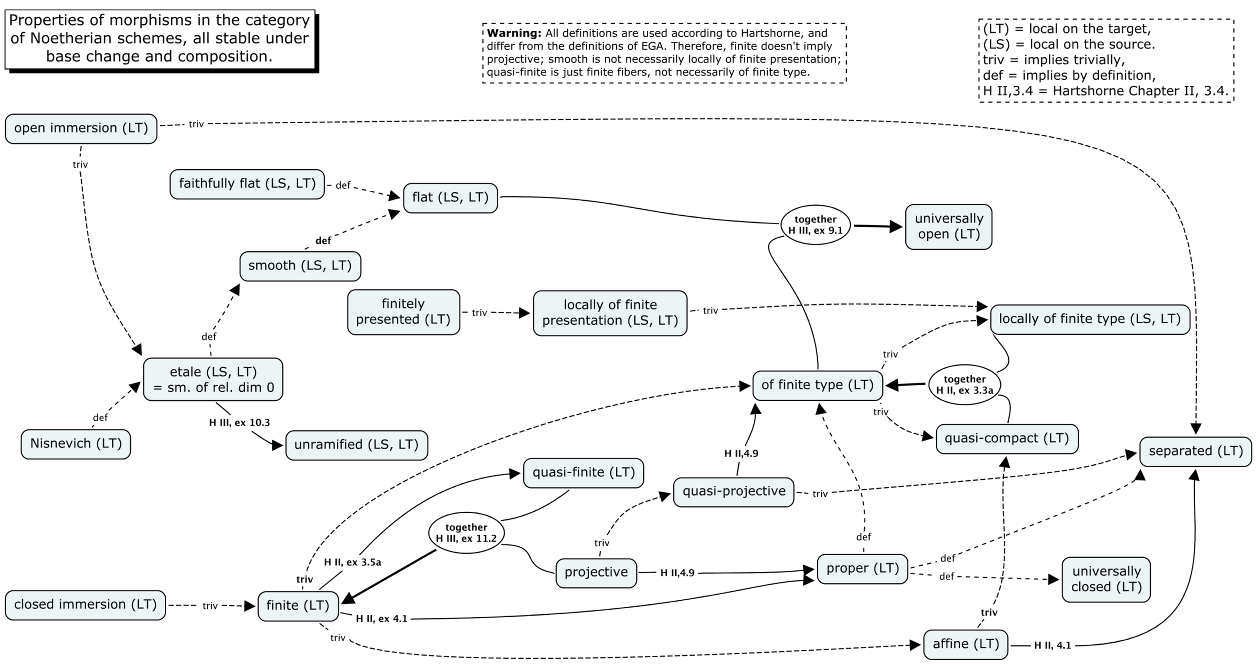



Figure 1: Properties of scheme morphisms; Konrad Voelkel

## 2 Counterexamples in algebraic geometry

### 2.1 Basechange

**Example 2.1** (Reduced scheme but not geometrically reduced)**.** *Let*

$$K = \mathbb{F}_p(T^{\frac{1}{p}})$$

$$k = \mathbb{F}_p(T).$$

*Then* $\mathrm{Spec}(K)$ *is reduced but* $\mathrm{Spec}(K \otimes_k K)$ *is not, because* $1 \otimes T^{\frac{1}{p}} - T^{\frac{1}{p}} \otimes 1$ *is nilpotent.*

**Example 2.2** (a universally closed morphism but not a closed embedding). *We have tons of example. Say*

$$\mathbb{P}^1_k \to \mathbb{P}^1_k; z \mapsto z^n.$$

*Or something like*

$$\mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[t^2, t^3]).$$

**Remark** (integral fibres v.s connected fibres). *Integral is actually a quite strong condition, if a scheme is integral, we can easily prove it's irreducible and hence connected. But the converse is not true.*

**Remark** (integral scheme of finite type over an algebraically closed field $k$ v.s Noetherian scheme). *Just check the definition, you'll find an integral scheme of finite type over an algebraically closed field k is not just noetherian(can be covered by finitely many affine noetherian schemes), and it's also integral(...yeah, sure..)*

**Example 2.3** (Hartshorne III.12.4). *Let $Y$ be an integral scheme of finite type over an algebraically closed field $k$ and $f : X \to Y$ be a flat projective morphism whose fibres are integral schemes(hence connected, this is crucial to see that in this problem the push-forward of a line bundle is still a line bundle, without this assumption, we know the push-forward is a vector bundle, but not necessarily a line bundle). Let $\mathscr{L}, \mathscr{M}$ be line bundles on $X$, and assume that $\mathscr{L}_y \cong \mathscr{M}_y$ on the fibre $X_y$, Then there exists a line bundle $\mathscr{N}$ on $Y$, such that*

$$\mathscr{L} \cong \mathscr{M} \otimes f^*\mathscr{N}.$$

*Just use Grauert theorem, we know $f_*\mathscr{F} := f_*(\mathscr{L} \otimes \mathscr{M}^{-1})$ is a a vector bundle with rank $h^0(X_t, \mathscr{L}_t \otimes \mathscr{M}_t^{-1})$, together with the fact that fibres are all connected, we know it's a line bundle. And*

$$f_*\mathscr{F}(y) \to H^0(X_y, \mathscr{F}_y)$$

*is an isomorphism. But fibrewise $\mathscr{F}_y$ is globally generated(it's just the trivial bundle in this situation), so this isomorphism just means we have a surjection on the fibres of*

$$f^*f_*\mathscr{F} \to \mathscr{F}.$$

*But it's also an injection simply because stalks have finite dimensions. Thus we conclude that*

$$f^*f_*\mathscr{F} \cong \mathscr{F}$$

*now let $\mathscr{N}$ be $f_*\mathscr{F}$, we get the desired property.*

**Remark** ($f_*\mathcal{O}_X \cong \mathcal{O}_Y$). *Use the same method as above, we get a sequence*

$$\mathcal{O}_{y,Y} \to k(y) \otimes f_*\mathcal{O}_X \to H^0(X_t, \mathcal{O}_{X_t}) \cong k.$$

*On the stalk level, the first map is an injection simply because $f_*\mathcal{O}_X$ is locally free. But they both have rank 1, so it's an isomorphism.*

**Example 2.4** (fibres being integral is necessary, actually flatness matters). *We want to compare the following two conditions*

- *$X$ is reduced and fibres are connected*
- *$f : X \to Y$ is flat projective with integral fibres.*

*The example is*

$$X = \mathrm{Spec}(k[x, y]/(x^2, xy)) \to Y = \mathrm{Spec}(k[y])$$
$$k[y] \to k[x, y]/(x^2, xy); y \mapsto y.$$

*This map is not flat($x$ is a zero divisor), but it's projective(since finite), $X, Y$ are noetherian schemes and $Y$ is regular. But $f_*\mathcal{O}_X \neq \mathcal{O}_Y$, and $f_*\mathcal{O}_X$ is not an invertible module over $k[x]$, $x$ is a zero divisor.*

**Example 2.5** (push-forward a vector bundle along a projective morphism might not be a vector bundle, even if cohomology groups of fibres are constant). *If you want to use cohomologe and base change, make sure $Y$ is integral. Consider the example*

$$f : X = \mathbb{P}^1_Y \to \operatorname{Spec}(k[x]/(x^2)) = Y$$

*Then it's a projective morphismbetween noetherian schemes. Just consider the 'Euler sequence' on $\mathbb{P}^1_Y$, then we have*

$$0 \to \mathcal{O}_{\mathbb{P}^1_Y}(-2) \to \mathcal{O}^2_{\mathbb{P}^1_Y}(-1) \to \mathcal{O}_{\mathbb{P}^1_Y} \to 0$$

*This is not what we want, we compute*

$$Ext^1(\mathcal{O}_{\mathbb{P}^1_Y}, \mathcal{O}_{\mathbb{P}^1_Y}(-2)) = H^1(\mathbb{P}^1_Y, \mathcal{O}_{\mathbb{P}^1_Y}(-2)) \cong k[t]/(t^2).$$

*We consider $t \in Ext^1$, then we have*

$$0 \to \mathcal{O}_{\mathbb{P}^1_Y}(-2) \to E \to \mathcal{O}_{\mathbb{P}^1_Y} \to 0$$

*And note that $H^0(X, E)$ is given by the kernel of*

$$H^0(X, \mathcal{O}_{\mathbb{P}^1_Y}) \to H^1(X, \mathcal{O}_{\mathbb{P}^1_Y}(-2))$$

*which is just a multiplication by $t$. Thus $H^0(X, E)$ is the ideal $(x)$, as a $k[x]/(x^2)$-module, it's isomorphic to $k[x]/(x^2)$, then $f_*E = H^0(X, E)$, it's not free. So we have a projective flat morphism between noetherian schemes, and a sheaf $E$ is flat over $Y$(since it's a vector bundle over $X$ and $X$ is flat over $Y$), but $f_*E$ is not a locally free(vector bundle). The only condition missing is that in the cohomology and base-change theorem, we need $Y$ to be an integral scheme.*

**Remark** ($Ext^1$ and maps in the corresponding extension).

**Example 2.6** (constant fibre $\neq$ locally free). *This example is silly, just consider the identity map*

$$id : X = \operatorname{Spec}(k[t]/(t^2)) \to k[t]/(t^2) = X$$

*and the sheaf(actually module) $\mathscr{F} = k[t]/(t)$ over $k[t]/(t^2)$, $y = (t)$ is the unique point in $X$. Then we have*

$$dim_{k(y)} H^0(X, \mathscr{F}) = dim_k k[t]/(t) = 1$$

*which is a constant. But $k[t]/(t)$ is not a free $k[t]/(t^2)$-module. You moght argue it's because $k[t]/(t)$ is not a flat $k[t]/(t^2)$-module(it's interesting to have fun with different criteria of flatness in this over simplified example).*

**Example 2.7** (Hartshorne, Example III.10.0.2). *If $X$ is integral Noetherian then*

$$dim_{k(x)} \Omega_{X/Y} \otimes k(x) = n$$

*is equivalent to $\Omega_{X/Y}$ is locally free of $n$. Hartshorne's book has a standard proof, here a silly remark, if in some case $\Omega_{X/Y}$ is a flat sheaf on $X$, then we can apply the cohomology and base change theorem to the identity morphism $id : X \to X$, then the theorem says(in this special situation) exactly the equivalence of these two condition.*

**Example 2.8** (infinite disjoint union of affine schemes is not affine in general). *We note that*

$$\amalg_{i=1}^{\infty} \mathbb{A}^1_k \neq \operatorname{Spec}(\oplus_{i=1}^{\infty} k[x]) \neq \operatorname{Spec}(\Pi_{i=1}^{\infty} k[x]).$$

*Note that $\sqcup_{i=1}^{\infty} \mathbb{A}^1_k$ is not quasi-compact, thus not affine. The latter two are affine, as a consequence, they're quasi-compact.*

**Example 2.9** (affine push-forward is exact). *Simply because the direct images are associated to*

$$H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)})$$

*it's vanishes for any $i \geq 1$ if this morphism is affine.*

**Remark.** *some naive but useful fact*

- *push-forward along a finite morphism is exact.*

**Example 2.10** (closed subscheme of an affine scheme is affine but an open subscheme might not). *Consider a morphism $f : X \to Y$ where $Y$ is affine and $X$ is Noetherian. Then we know $f$ is affine, then for any quasi-coherent sheaf $\mathscr{F}$ on $X$, we have*

$$H^i(X, \mathscr{F}) = H^i(Y, f_*\mathscr{F}); \forall i \geq 0.$$

*Since $X$ is Noetherian, $f_*\mathscr{F}$ is a quasi-coherent sheaf on $Y$. Then use Serre's criterion for affineness.*

**Example 2.11** ($\dim(X) = 1$ with one or two closed points).

**Example 2.12** ($\dim \Pi_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$). *The short answer is that*

$$\dim \Pi_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z} = +\infty.$$

*https://mathoverflow.net/questions/90980/what-is-the-dimension-of-the-product-ring-prod-mathbb-z-2n-mathbb-z*

*https://math.stackexchange.com/questions/364479/explicitly-represent-a-representable-functor*

**Example 2.13** (0-dimensional affine scheme, infinitely many closed points).

**Remark** (of finite type is important). *Let $S$ be a nonzero finite type algebra of a field $k$, then $\dim(S) = 0$ if and only if $S$ has finitely many primes. http://stacks.math.columbia.edu/tag/02IF*

**Example 2.14** (constant sheaf, not quasi-coherent; skyscraper sheaf, not quasi-coherent). *We just note the following subtlities*

- *the skyscraper sheaf $k$ over $\mathbb{A}_k^1 = \mathrm{Spec}(k[T])$,supported at the origin, is a quasi-coherent sheaf.*
- *the skyscraper sheaf $k(T)$ over the affine line $\mathbb{A}_k^1 = \mathrm{Spec}(k[T])$,supported at the origin, is* **NOT** *a quasi-coherent sheaf.*
- *the skyscraper sheaf $k$ over $\mathbb{P}_k^1 = \mathrm{Proj}(k[T])$ is* **NOT** *a quasi-coherent sheaf. Up to isomorphism, quasi-coherent sheaf on $\mathbb{P}^1$ with global section $k$ has to be $\mathcal{O}_{\mathbb{P}^1}$.*
- *the constant sheaf $\underline{k}$ is almost never quasi-coherent.*
- *the constant sheaf $\underline{k}$ is always flasque. So for Čech cohomology we do have $\check{H}^i(\mathfrak{U}, \underline{k}) = 0, \forall i \geq 1$*

**Remark** (sheaf cohomology, Zariski topology, étale topology). *Note the following facts*

- 
- *Grothendieck vanishing theorem is true for Noetherian topological spaces(in Zariski topology) and sheaves of abelian groups.*
$$H^2_{Zar}(\mathbb{P}^1, \underline{k}) = 0, H^2_{\acute{e}t}(\mathbb{P}^1, \underline{k}) \cong k.$$
- *Čech cohomology also relies on the topology. For example, $\mathbb{P}^1 = U_0 \cap U_1$. If we want to compute the Čech cohomology, we have two diffculties*

- *this cover is not fine, no matter in Zariski topology or étale topology. This is not an issue, because we can use spectral sequence to compute* $\check{H}^2(X, \underline{k})$
-

**Example 2.15** (punctured spectrum of a valuation ring)**.** *'Counterexample' to the Grothendieck vanishing theorem Kollar's counterexample*

**Example 2.16** (Hodge numbers of a blow-up)**.** *We give several computations of the hodge number of* $Bl_C(\mathbb{P}^3)$ *(over* $\mathbb{C}$*), where* $C$ *is the twisted cubic parametrized by* $[u^3, u^2v, uv^2, v^3]$*.*

- $\chi^{p,q} = (-1)^{p+q}h^{p,q}$ *is additive. Then we know*

$$\chi^{p,q}(Bl_C(\mathbb{P}^3)) = \chi^{p,q}(\mathbb{P}^1 \times \mathbb{P}^1) + \chi^{p,q}(\mathbb{P}^3) - \chi^{p,q}(C)$$

  *Then we know the Hodge diamond is given by*

$$\begin{pmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{pmatrix}$$

  *For computations of each terms, see the reference below.*

- *Mixed Hodge structure.*
- *Decomposition theorem.*
- *Computer algebra. Macaulay2. See here.*

**Remark** (refenrence)**.** *computations of some Hodge numbers*

**Example 2.17** (push-forward of a coherent sheaf not coherent)**.**

$$f : X = \operatorname{Spec}(k(x)) \to \operatorname{Spec}(k) = Y.$$

*Then* $f_*\mathcal{O}_X$ *is not coherent over* $\operatorname{Spec}(k)$*.*

**Example 2.18** (push-forward of a quasi-coherent sheaf not quasi-coherent)**.** *Consider*

$$f : X = \amalg_{i=1}^{\infty} \operatorname{Spec}(\mathbb{Z}) \to \operatorname{Spec}(\mathbb{Z}) = S$$

*which is given by the identity map on each component. We claim* $f_*\mathcal{O}_X$ *is not quasi-coherent on* $S$*. Since we know if we have a quasi-coherent sheaf* $\mathscr{F}$ *on* $S$*, we must have a bijection*

$$\mathscr{F}(S) \otimes_{\mathbb{Z}} \mathcal{O}_S(U) \cong \mathscr{F}(U).$$

*This is not true in this example, consider $U = D_+(2)$, then the corresponding map is*

$$(\Pi_{i=1}^{\infty}\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \to \Pi_{i=1}^{\infty}\mathbb{Z}[\frac{1}{2}].$$

*Note that the image of this morphism consists of sequences of rational numbers with a fixed denominator $2^M$, however this is not all of the RHS, since a sequence of rational numbers with odd numerators and denominators $2^{k_i}, k_i \to +\infty$. Thus this map is not surjective, $f_*\mathcal{O}_X$ is not a quasi-coherent sheaf on* $\mathrm{Spec}(\mathbb{Z})$*. At the same time, it's very easy to construct some non-quasi-coherent sheaves, for example on $\mathbb{A}^1_k$ define a sheaf $\mathscr{F}$ by*

$$\mathscr{F} = \mathcal{O}_X(U), 0 \notin U;$$

$$\mathscr{F} = 0, 0 \in U.$$

*It's easy to check it's a sheaf, $\mathscr{F}(\mathbb{A}^1_k) = 0$, but this sheaf is not zero. Thus it's cannot be a quasi-coherent sheaf. The intuition is that 'relations between quasi-coherent sheaves and modules' looks quite similar like 'relations between affine schemes and rings.'*

**Example 2.19** (push-forward might have different cohomology)**.** *Consider*

$$f : \mathbb{P}^1_k \to \mathrm{Spec}(k).$$

*Then $f_*\mathcal{O}_{\mathbb{P}^1_k}(-2)$ is a quasi-coherent sheaf on* $\mathrm{Spec}(k)$*.* $\mathrm{Spec}(k)$ *is noetherian affine, then we have*

$$H^1(\mathrm{Spec}(k), f_*\mathcal{O}_{\mathbb{P}^1_k}(-2)) = 0$$

*on the other hand, we have*

$$H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2)) = kx^{-1}y^{-1}.$$

*Thus we see if $f : X \to Y$ is not affine, even if $X$ has very nice properties(not to say Noetherian, separated),*

$$H^i(X, \mathscr{F}) \neq H^i(Y, f_*\mathscr{F}).$$

**Remark** (push-forward from a noetherian scheme)**.** *In general we have*

- *$f : X \to Y$, where $X$ is noetherian, then the push-forward of a quasi-coherent sheaf is quasi-coherent.*
- *If moreover $X$ is separated, then*
$$\check{H}^p(\mathfrak{U}, \mathscr{F}) = H^p(X, \mathscr{F})$$

**Example 2.20** (proper push-forward v.s finite flat push-forward)**.** *Compare the followings*

- *A theorem from EGA says that if you have a morphism $f : X \to Y$ between two proper noetherian schemes and let $\mathscr{F}$ be a coherent sheaf on $X$, then all direct images $R^if_*\mathscr{F}$ are coherent. Here you really have to be careful, just think about*
$$\mathrm{Spec}(k(x)) \to \mathrm{Spec}(k)$$
*again, the pushforward of the structure sheaf is not coherent since* $\mathrm{Spec}(k(x))$ *is NOT proper over* $\mathrm{Spec}(k)$*. Although topologically, it's just a map between two points.*
- *Let $f : X \to Y$ be a finite morphism, then $f_*\mathcal{O}_X$ is locally free if and only if $f$ is flat.*
- *Any surjective morphism to a regular one dimensional scheme (in practice, just means a curve) is automatically flat.*
- *Miracle flatness theorem.*

**Example 2.21** (push-forward of a trivial bundle not trivial). *You can find more information in my notes 'bundles'. Here I just want to talk about an easy example.*

$$f : \mathbb{P}^1 \to \mathbb{P}^1; [x, y] \mapsto [x^2, y^2].$$

*Then we know $f_*\mathcal{O}_{\mathbb{P}^1}$ is a vector bundle on $\mathbb{P}^1$, use the push-pull formula to compute*

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n) \otimes f_*\mathcal{O}_{\mathbb{P}^1})$$

*we get*

$$f_*\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}.$$

**Remark** (good things happen sometimes). *In some special but important situation, we do have some thing like*

$$f_*\mathcal{O}_D = \oplus_{i=1}^n \mathcal{O}_{f_*D}.$$

*For example, if $k$ is algebraically closed and $f : X \to Y$ is a finite morphism between smooth projective curves, $D = \sum n_i p_i$ and $f_*D$ is defined to be $\sum n_i f(p_i)$. This is because $\mathcal{O}_D$ is just a skyscraper-type sheaf, just use the definition and the fact that this morphism is finite, we get $f_*\mathcal{O}_D = \oplus_{i=1}^n \mathcal{O}_{f_*D}$*

**Example 2.22** ($f^*f_*\mathscr{F}$ and $f_*f^*\mathscr{G}$). *In general, it's always enlightening to think about*

- *the affine case, or*
- *the degenerated case, for example $f : X \to \operatorname{Spec}(k)$.*

*Then we have natural maps*

$$f^*f_*\mathscr{F} \to \mathscr{F}.$$

$$\mathscr{G} \to f_*f^*\mathscr{G}$$

*can be understood and remembered easily.*

- *If we're in the affine case, $f : \operatorname{Spec}(A) \to \operatorname{Spec}(B)$, and $M$ is an $A$-module, and denote $M'$ be the same abelian group as $M$ but with the $B$-module structure induced by the ring map $f^\# : B \to A$. Then*

  $$f_*M = M', f^*f_*M = A \otimes_B M'$$

  *the point is that since the tensor product is over $B$, in general we cannot move $A$ to the right-side, but we do have a natural map*

  $$A \otimes_B M' \to M; a \otimes m \to am.$$

  *If $N$ is a $B$-module, then*

  $$f^*N = A \otimes_B N, f_*f^*N = (A \otimes_B N)'$$

  *The last term means view $A \otimes_B N$ as a $B$-module by the ring map $f^\#$, then we do have*

  $$N \to (A \otimes_B N)', n \mapsto 1 \otimes n.$$

- *In the degenerated case,*

  $$f_*\mathscr{F} = H^0(X, \mathscr{F}), f^*f_*\mathscr{F} = \mathcal{O}_X \otimes_k H^0(X, \mathscr{F})(= \mathcal{O}_X^{h^0(X,\mathscr{F})}).$$

  *We do have a natural map*

  $$\mathcal{O}_X \otimes_k H^0(X, \mathscr{F}) \to \mathscr{F}$$

  *Similarly, a coherent sheaf over $k$ just means a finite dimensional vector space $V$, we have*

  $$f_*f^*V = (V \otimes_k \mathcal{O}_X)'$$

*As usual, the last term means we view it as a $k$-module. Then we have a natural map*

$$V \to (V \otimes_k \mathcal{O}_X)'; v \mapsto v \otimes 1.$$

*If $f$ is of finite type, you may think about it secretly as*

$$V \to V \otimes_k k[x]; v \mapsto v \otimes 1.$$

**Remark** (decomposition of $f^*f_*\mathscr{F}$ and $f_*f^*\mathscr{G}$)**.** *Qirui told me that if you have a nice enough(unramified, finite) morphism $f : X \to Y$, and a finite group $G$ acts on $X$ and compatible with $f$, $\mathscr{F}$(resp. $\mathscr{G}$) is a coherent sheaf on $X$(resp. $Y$), then we have something like*

$$f^*f_*\mathscr{F} = \oplus_{g\in G} g^*\mathscr{F}$$

$$f_*f^*\mathscr{G} = \mathscr{G}^{\oplus deg(f)}.$$

**Example 2.23** (push-forward...)**.** *Typically, we always need to consider push-forward along*

- *a birational proper morphism.*
- *a flat projective morphism.*
- *a finite flat morphism.*
- *a finite morphism.*

**Example 2.24** (what prevent $f_*\mathcal{O}_X$ to be $\mathcal{O}_Y$ ?)**.** *First we have many special situations*

- *If $f_*\mathcal{O}_X$ is a line bundle on $Y$, then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*
- *If $f : X \to Y$ is a proper birational morphism, $Y$ is normal, then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

**Example 2.25** (pullback is a locally free sheaf, but not a locally free sheaf itself)**.** *Consider*

$$f : X = \mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z}) = Y$$

*and the module $\mathscr{F} = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ over $Y$, then $f^*\mathscr{F} \cong \mathbb{Q}$, it's a free sheaf of rank 1 over $X$ but itself is obviously not free over* Spec($\mathbb{Z}$)*, so in many situations, being faithfully flat is essential.*

## 2.2 F

First recall some basic notations with 'f':

- quasi-finite
- finite
- flat
- faithfully flat
- of finite type
- locally of finite type
- finite presentation

**Example 2.26** (quasi-finite,surjective, of finite type, but not finite). *Consider*

$$k[x] \to k[x, \frac{1}{x}] \oplus k.$$

*Then this map is of finite type, surjective, quasi-finite, however* $k[x, \frac{1}{x}] \oplus k$ *is not a finitely generated* $k[x]$*-module.*

**Example 2.27** (locally of finite type but not of finite type).

$$\amalg_{i \in I} \mathbb{A}^1_k \to \mathbb{A}^1_k$$

*where* $I$ *is an infinite index set.*

**Example 2.28** (quasi-finite(and fibre has same cardinality) but not flat). *Consider the normalization of the cuspidal curve*

$$k[t^2, t^3] \to k[t].$$

*Then every fibre contains only one point, however* $k[t]$ *is not a flat module, since*

$$I = (t^2, t^3) \otimes_{k[t^2,t^3]} k[t] \to k[t^2, t^3] \otimes_{k[t^2,t^3]} k[t] \cong k[t]$$

*is not injective,* $t^2 \otimes t \neq t^3 \otimes 1$ *on the left, but they have the same image.*

**Remark.** *Think about the 'Miracle Flatness'.*

**Example 2.29** (of finite type but not of finite presentation). *This is quite common for nonnoetherian schemes,*

$$\mathrm{Spec}(k) \to \mathrm{Spec} k[t_1, t_2 \dots].$$

*This is of finite type because it's actually just a closed embedding of a single point,* $R \to R/I$*, wo do have a surjection* $R[x_1, \dots, x_n] \to R/I$*, you can choose whatever* $n$ *you like, but the kernel is not finitely generated.*

**Remark** (what about noetherian scheme?). *For a noetherian ring* $R$*,* $R[x_1, \dots, x_n]$ *is alos noetherian, thus any ideal is finitely genrated.*

**Remark** (Formal power series ring is Noetherian). *Hilbert basis theorem. Note that* $R[[t]]$ *and* $R[t_1, t_2, \dots]$ *are very different.*

**Example 2.30** (of finite type but not finite).

$$\mathbb{A}^n_k, \mathbb{P}^n_k; n \geq 1$$

*are* $k$*-schemes of finite type, but not finite* $k$*-scheme, so you know, most schemes you know of finite type are not finite.*

**Remark** (finite $\Rightarrow$ of finite type).

**Example 2.31** (proper but not finite).

$$\mathbb{P}^1_k \to \mathrm{Spec}(k).$$

**Remark** (finite v.s proper). *We have the following*

- *finite* $\Rightarrow$ *quasi-finite.*
- *finite* $\Rightarrow$ *projective.*

- *finite $\Rightarrow$ proper. We only need to prove it for* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$*, this follows form the valuation criterion*

$$\begin{tikzcd}
K & B \arrow[l, "v"'] \arrow[dl, "?"] \\
R \arrow[u, "i"] & A \arrow[l, "u"'] \arrow[u, "f"']
\end{tikzcd}$$

  *Since $B$ is finite hense integral over $A$, so $v(B)$ is integral over $u(A)$, however discrete valuation ring $R$ is integrally closed in its fractional field, thus $v(B) \subset R$, then by the valuation criterion, a finite morphism is always proper.*

- *proper + affine $\Rightarrow$ finite.*

- *proper + quasi-finite $\Rightarrow$ finite.*

- *finite $\Rightarrow$ of finite type.*

**Example 2.32** (a affine and projective morphism)**.**

**Example 2.33** (morphism;affine but not of finite type )**.**

$$\mathrm{Spec}(k[x_1, x_2, \dots]) \to \mathrm{Spec}(k).$$

**Example 2.34** (morphism; of finite type but not affine)**.**

$$\mathbb{P}^1_k \to \mathrm{Spec}(k).$$

**Example 2.35** (finite, dominant morphism between normal varieties but not flat)**.** *Consider*

$$X = \mathrm{Spec}(k[x,y]), Y = \mathrm{Spec}(k[x^2, xy, y^2])$$

*and assume char$(k) \neq 2$, then $X, Y$ are both normal, the point is that $X$ is not the normalization of $Y$ in its functional field $K(Y)$, but it's the normalization of $Y$ in $L = k(x,y) \supset K(Y)$. So, it's a normalization map but not 'the normalization'. It's not flat since*

$$X_0 = \mathrm{Spec}(k \otimes_{k[x^2,xy,y^2]} k[x,y]) \cong \mathrm{Spec}(k[x,y]/(x^2, xy, y^2)).$$

*So the fibre over $0$ has length $3$. On the other hand, if $a^2, b^2 \neq 0$*

$$X_{(a^2,ab,b^2)} = \mathrm{Spec}(k \otimes_{k[x^2,xy,y^2]} k[x,y])$$

$$\cong \mathrm{Spec}(k[x,y]/(x^2 - a^2, y^2 - b^2, xy - ab)) \cong \mathrm{Spec}(k[x,y]/(x-a, y-b) \oplus k[x,y]/(x+a, y+b)).$$

*Thus the fibre over a point other than $(0,0,0)$ has length $2$. Thus this is not a flat morphism.*

**Remark** (Miracle flatness theorem)**.** *The miracle flatness theorem says that if you have a morphism $f : X \to Y$ and if $X$ is Cohen-Macaulay, $Y$ is regular, and if the fibres are equidimensional, then $f$ is flat. From the example above we know*

- *$Y$ being normal is not enough if dim$(Y) > 1$, because we have many normal varieties but not regular. But if we just want to consider curves(i.e $\dim(Y) = 1$) then normal=regular, to be more precise, normal implies regular in codimension $1$.*

- *If we consider the morphism above as*

$$\mathbb{P}^1_k \to \mathrm{Proj}(k[x,y,z]/(xz - y^2))$$

$$[u, v] \mapsto [u^2, uv, v^2]$$

  *then it's a flat morphism! Simply because $0$ disappears in this projective picture.*

- *In practice, especially when we want to deal with regular varieties, then finiteness implies flatness just by the miracle flatness theorem.*

**Example 2.36** ($X$ being Cohen-Macaulay is necessary in the miracle flatness theorem)**.** *In Hartshorne* III.9.3*, the following morphism is not flat*

$$f : X = \mathrm{Spec}(k[x, y, z, w]/((z, w) \cap (x + z, y + w))) \to Y = \mathrm{Spec}(k[x, y])$$

$$(a, b, 0, 0) \mapsto (a, b); (a, b, -a, -b) \mapsto (a, b).$$

*To see it's not flat, we only need to compute the length, say at the point $p = (0, 0)$, the scheme theoretical fibre is given by*

$$k[x, y]/(x, y) \otimes_{k[x,y]} k[x, y, z, w]/((z, w) \cap (x + z, y + w))$$

*which is isomorphic to*

$$k[z, w]/(z^2, zw, w^2).$$

*In other words, the fibre over $p = (0, 0)$ has degree 3, but it's easy to compute all other fibres has degree 2! Note that in this example, $Y$ is regular and fibres have the same dimension 0. Also note that we also have examples where $Y$ is normal and $X$ is regular, and fibres have the same dimension but the morphism between them is not flat.*

**Remark** (Why?)**.** *$k[x, y, z, w]/((x.y) \cap (x - z, y - w))$ is not Cohen-Macaulay. To prove this use Hartshorne's connectness theorem. The fact that the fibre over $p = (0, 0)$ has degree 3(if I computed it correctly) is not intuitively trivial for me, in many nice situations, flatness is like a topological concept, but it's actually far more than some topological intuitions. Just consider the 1-dimensional analogue of the morphism above*

$$f : \mathrm{Spec}(k[x, y]/(y(y - x))) \to \mathrm{Spec}(k[x])$$

*it's FLAT. Although topologically, I cannot see any big differences.*

**Example 2.37** (flat but not open; of finite type is necessary)**.** *Consider the natural morphism*

$$\mathrm{Spec}(k(x)) \to \mathrm{Spec}(k[x]).$$

*It's flat since $k[x]$ is a PID and $k(x)$ is torsion-free as a $k[x]$-module. But it's not open(we cannot just invert some polynomials in $k[x]$ to get $k(x)$, or simply because the image is just a point).*

**Example 2.38** (not finite not surjective, sometimes still flat)**.** *The most obvious examples are open immersions like*

$$\mathbb{A}^1_k - \{0\} \to \mathbb{A}^1_k$$

*is flat, but not finite , not surjective.*

**Example 2.39** ($X \to X_{red}$ is not flat)**.** *We consider the natural morphism*

$$f : X = \mathrm{Spec}(k[x, y, z, w]/(z^2, zw, w^2, xz - yw)) \to \mathrm{Spec}(k[x, y]) = Y,$$

*then it's NOT flat, although $Y$ is regular, $X$ has no embedded points, and fibres are equidimensional. So you know, the reason is that $k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$ is not Cohen-Macaulay, not to say regular. To see it's not flat, just check the definition*

$$I = (x, y) \to k[x, y]$$

*is an injection of $k[x, y]$-modules, but*

$$I \otimes_{k[x,y]} k[x, y, z, w]/(z^2, zw, w^2, xz - yw) \to k[x, y] \otimes_{k[x,y]} k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$$

*is not an injection. Because, for example, $x \otimes z - y \otimes w$ is not zero on the left($1 \notin I$, you cannot move $x$ or $y$ to the other side), but its image is $x \otimes z - y \otimes w = xz - yw = 0$(since $1 \in k[x, y]$).*

**Example 2.40** (Hartshorne III.9.8.4III.12.4, flat families, dimension jump). *The first family of twisted cubic curves is given by*

$$\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^3_{\mathbb{A}^1}$$
$$[u, v] \mapsto [u^3, u^2v, auv^2, v^3]$$

*Actually, in computation the corresponding ring homomorphism is given by*

$$k[x, y, z, w, t] \to k[u, v, a]$$
$$[x, y, z, w, t] \mapsto [u^3, u^2v, auv^2, v^3, a].$$

*With the help of Macaulay2, we can get the ker of this map(that is the ideal of this flat family) is given by*

$$I = (xwt - yz, y^2t - xz, y^3 - x^2w, ywt^2 - z^2)$$

*This is a flat family simply because $t$ is not a zero divisor in $k[x, y, z, w, t]/I$. Then if $t \neq 0$, the corresponding fibre is just an ordinary twisted cubic. But the fibre over $t = 0$ is given by*

$$I_0 = (-yz, -xz, y^3 - x^2w, -z^2)$$

*the primary decomposition is*

$$I_0 = (z, y^3 - x^2w) \cap (x, z^2, yz, y^3).$$

*It's clear that this fibre contains an embedded point. Now we can compute the sheaf cohomology of $\mathscr{I}$ corresponding to $I$. My computation shows that*

$$h^0(\mathbb{P}^3, \mathscr{I}_0) = h^3(\mathbb{P}^3, \mathscr{I}_0) = 0;$$
$$h^1(\mathbb{P}^3, \mathscr{I}_0) = 1, h^2(\mathbb{P}^3, \mathscr{I}_0) = 1.$$

*However, if $t \neq 0$, the cohomology is given by*

$$h^i(\mathbb{P}^3, \mathscr{I}_t) = 0, \forall i.$$

*So we can see $h^1, h^2$ jump at the point $t = 0$, however, the Euler characteristic (of sheaves $\mathscr{I}_t$) is a constant $0$. The second flat family is actually quite similar, just consider*

$$[u^4, u^3v, au^2v^2, uv^3, v^4] \subset \mathbb{P}^4_{\mathbb{A}^1}.$$

*Then you can see similar phenomena.*

**Remark** (How to use semi-continuity theorem?). *In the example above, we use the semi-continuity theorem for the trivial family*

$$\mathbb{P}^3 \times \mathbb{A}^1 \to \mathbb{A}^1$$

*and the ideal sheaf $\mathscr{I}$ on $\mathbb{P}^3 \times \mathbb{A}^1$. The fact that we have a flat family of twisted cubic curves is used to show that $\mathscr{I}$ is flat over $\mathbb{A}^1$. Although you might also use semicontinuity theorem for the structure sheaf on the flat family itself, that's not what I want.*

**Remark** (A flat family of hypersurfaces of the same degree has constant $h^i$'s). *As we have seen above, in general, if $\{X_t\}$ is a flat family of schemes, $h^i(X, \mathcal{O}_X)$ jumps at some points, but if you have a flat family of hypersurfaces of the same degree $X \subset \mathbb{P}^n_T$, then $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function. To see this we first assume $T = Spec(R)$, then $\mathcal{O}_X$ is given by $R[x_0, \ldots, x_n]/(f)$. Consider the fibre over $\mathfrak{p} \in \mathrm{Spec}(T)$. let $k = T_\mathfrak{p}/\mathfrak{p}T_\mathfrak{p}$ be the corresponding quotient field. Then by a change of coordinates, we may assume*

$$\bar{f} = a_n x_n^d + p_{n-1}x_n^{n-1} + \ldots, +p_0$$

*where $p_i$'s are polynomials in variables $x_0, \ldots, x_{n-1}$, $a_n \in k$. Since this procedure only need to invert finitely many elements in $T$, thus this can be done in some basic open subset of $\mathrm{Spec}(T)$, corresponding to a ring $R'$. Then over this open subset $\mathcal{O}_X$ is given by $R'[x_0, x_1, \ldots, x_n]$ which is a free module over $R'[x_0, \ldots, x_{n-1}]$, hence free over $R'$, naturally flat. Then this proves if you have a family of hypersurfaces in $\mathbb{A}^n_T$, then it's naturally flat over $T$, then cover $\mathbb{P}^n_T$ by some affine charts, and think about what is $h^i$...then you'll see $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function.*

**Example 2.41** (an application of the semicontinuity theorem + flat family ???? imcomplete)**.** *It's impossible to have a family of rank 2 vector bundles parametrized by $\mathbb{A}^1_k$ on $\mathbb{P}^1$, such that all fibres are trivial bundles except the fibre over $t = 0$. Then let the fibre over $t = 0$ be $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$. Since the Euler characteristic in a flat family is a constant, thus we have*

- *for $n$ big enough, we have $2n + 2 = a + b + 2n$, thus $a + b = 0$.*
- *for $n = 0$, we have*
$$h^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = 2$$
$$h^1(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = 0$$
*and*
$$h^0(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)) = a + 1$$
$$h^1(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$$
*then we know $a \geq 1$.*

**Remark.** *Can we just use semi-continuity theorem?*

**Example 2.42** (locally free module v.s locally free sheaf; why we need Noetherian condition)**.** *Here we have an example of a locally free module but not locally free as a sheaf over the corresponding scheme. What we really need is the following*

- *$\mathscr{F} = \widetilde{M}$ is locally free as a sheaf if and only if $M$ is projective.*
- *A finitely generated module $M$ over a noetherian ring is projective if and only if it's locally free as a module.*

**Remark** (coherent sheaf $\Leftrightarrow$ finitely generated module, Hartshorne II.5.5)**.** *What matters is the 'finite generation of the kernel'.*

**Example 2.43** (A finitely generated flat module but not finitely presented, not flat)**.** *Let $R = \Pi_{i=1}^{\infty}\mathbb{F}_2$.*

**Example 2.44** (locally free module but not projective)**.** *Consider*
$$R = \mathbb{Z}, M = \mathbb{Z}[\ldots, \frac{1}{p}, \ldots]$$

**Example 2.45** (dimension jumps, semicontinuity theorem)**.** *Let $E$ be an elliptic curve, consider $X = E \times_k E$ and $E \times O - \Delta$, where $\Delta$ is the diagonal, and we have a natural morphism*
$$\pi_1 : E \times O - \Delta \to E \times O$$
*Then we know fibre are degree 0 divisors on $E$ and*
$$l(D) = 1 \Leftrightarrow D \sim 0$$
*this only happens at the fibre over $O$ itself, so the dimension of $H^0$ jumps.*

**Example 2.46** (Hartshorne III.9.7, a flat family but corresponding family of projective cones is not flat)**.** *Consider the family of three points in $\mathbb{P}^2_k$,*
$$X_t = \{[1, 0, 0], [0, 1, 0], [1, 1, t]\}$$
*Then we know the corresponding ideal is*
$$I = (y, z) \cap (x, z) \cap (x - y, tx - z) = (xz - yz, yzt - z^2, xyt - yz, x^2y - xy^2).$$

*It's a flat family since t is not zero divisor, thus this family is flat(or you can use the miracle flatness theorem, or the Euler characteristic criterion, or since this family is parametrized by a regular* $1$*-dimensional scheme, and every irreducible component of* $X$ *dominant* $Y$*..). Thus this also defines the unique flat family in* $\mathbb{P}^3_{k[t]}$ *which is the closure of projective cones for* $t \neq 0$*, now consider* $I_0$*, it's the cone in* $\mathbb{P}^3$ *defined by*

$$I_{Y_0} = (xz, yz, z^2, x^2y - xy^2)$$

*it's not the ideal of the projective cone w.r.t* $X_0$*,*

$$I_{C(X_0)} = (z, x^2y - xy^2)$$

*To see they define different schemes, just consider the intersection with* $V(x, y)$*, surely,* $\mathrm{Proj}(k[z, w]/(z^2)) \neq \mathrm{Proj}(k[z, w]/(z))$*.*

**Remark.** *We need to mention two remarks*

- *If a family is very flat, then it's flat. And the corresponding family of projective cones is also very flat.*
- *If we have an algebraic family of projectively normal varieties in* $\mathbb{P}^n_k$*, parametrized by a nonsingular curve* $T$ *over an algebraically closed field, then it's a very flat family.*

*Actually, it takes me a while to understand the second statement, the logic is that, first apply Theorem* III.9.9 *in Hartshorne's book, we know an algebraic family of normal varieties is a falt family, then we know the Hilbert polynomials are the same, recall*

$$P_t(m) = dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m))$$

*now use the fact that* $X_t$ *is projectively normal, thus*

$$dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m)) = dim_{k(t)}(S_t/I_t)_m$$

*So we know* $dim_{k(t)}(S_t/I_t)_m$ *are the same. So we get a very flat family by definition.*

**Example 2.47** (Fibre product of two schemes doesn't exist(we mean $= \emptyset$))**.** *Just consider*

$$\mathrm{Spec}(\mathbb{Z}/2\mathbb{Z}) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}/3\mathbb{Z}) \cong \mathrm{Spec}(0).$$

*The zero ring corresponding to the empty set. Geometrically, this means* $\mathrm{Spec}(\mathbb{Z}/2\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z})$ *and* $\mathrm{Spec}(\mathbb{Z}/3\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z})$ *have different images, there's no fibre product for sure.*

**Example 2.48** (Flatness over $\mathrm{Spec}(k[\epsilon]/(\epsilon^2))$)**.** *Let* $M$ *be a module over* $\mathrm{Spec}(k[\epsilon]/(\epsilon^2))$*, then to check its flatness, we only need to check that*

$$(\epsilon) \otimes_{k[\epsilon]/(\epsilon^2)} M \to M$$

*is injective. Note that this condition here doesn't mean* $M$ *is a torsion-free* $k[\epsilon]/(\epsilon^2)$*-module, for example, view the ring of dual numbers as a module over itself, then* $\epsilon$ *is definitely a zero-divisor, but it's a flat module over itself for sure. This detail is actually essential in some situations, for example, to prove* $T_Z\mathrm{Hilb}(X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z)$*. In its affine case, let* $I = (f_1, \ldots, f_m)$ *and given a morphism* $\phi \in \mathrm{Hom}_R(I, R/I), \phi(f_i) = g_i$*, we claim* $(f_i + \epsilon g_i)$ *defines a subscheme of* $X \times \mathrm{Spec}(k[\epsilon]/(\epsilon^2))$ *flat(this is the point) over* $\mathrm{Spec}(k[\epsilon]/(\epsilon^2))$*. First note that if* $[g] = [h] \in R/I$*, then* $(f_i + \epsilon g_i) = (f_i + \epsilon h_i)$*, this is trivial for if* $g_i - h_i = \sum u_i f_i$*, then* $\epsilon(g_i - h_i) = \sum \epsilon u_i(f_i + \epsilon g_i)$*. To prove* $M = R[\epsilon]/(\epsilon^2, f_i + \epsilon g_i)$ *is flat over* $k[\epsilon]/(\epsilon^2)$*, we have to check that*

$$(\epsilon) \otimes_{k[\epsilon]/(\epsilon^2)} M \to M$$

*is injective. Note that on the left hand side,* $\epsilon \otimes \epsilon g = 0$*, thus we only need to check that* $\forall f \in R, \epsilon f \in (f_i + \epsilon g_i) \Rightarrow f \in (f_i)$*, this is indeed true because*

$$\epsilon f = \sum (a_i + \epsilon b_i)(f_i + \epsilon g_i) = \sum a_i f_i + \epsilon(a_i g_i + b_i f_i)$$

$$\Rightarrow \sum a_i f_i = 0.$$

*The the existense of $\phi$ tells us*

$$\sum a_i g_i = \sum a_i \phi(f_i) = \phi(\sum a_i f_i) = 0$$

$$\Rightarrow f = \sum b_i f_i \in (f_i).$$

*Conversely, if $(f_i + \epsilon g_i)$ defines a flat family over the dual numbers, we claim $f_i \mapsto [g_i]$ gives us a well-defined element in $\mathrm{Hom}_R(I, R/I)$. We have to show if $f_i + \epsilon u \in (f_i + \epsilon g_i)$, then $[u] = [g_i]$. The argument above also works here, since $\epsilon(g_i - u) \in (f_i + \epsilon g_i)$, then the flatness tells us $g_i - u \in (f_i)$, this is exactly what we want.*

**Remark** (Torsion-free modules over $\mathrm{Spec}(k[\epsilon]/(\epsilon^2))$)**.** *If we define a torsion-free module over a ring to be a module such that $0$ is the only element annihilated by a regular element (non zero-divisor) of the ring. Then any module over $k[\epsilon]/(\epsilon^2)$ is torsion-free. Since regular elements in this ring give us $k$.*

## 2.3 P

**Example 2.49** (proper but not projective)**.** *I think the identity morphism*

$$\mathbb{A}^1_k \to \mathbb{A}^1_k$$

*is proper but not projective. Actually we have a proper morphism between affine barieties must be finite.*

**Example 2.50** (why we need the 'separated' condition? Picard group of an affine line with double origins)**.** *Let $X$ be $\overline{\mathbb{A}_k} - \{0\}$, the affine line with double origins. Here we want to compute*

- $\mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^*)$.
- $\mathrm{CaCl}(X) := H^0(X, \mathcal{M}^*/\mathcal{O}_X^*)/H^0(X, \mathcal{M}^*)$.
- $\mathrm{Cl}(X) := Div(X)/div(k(x))$.

*My computations shows that*

- $\mathrm{Pic}(X) \cong \mathbb{Z}$.
- $\mathrm{CaCl}(X) \cong \mathbb{Z}$.
- $\mathrm{Cl}(X) \cong \mathbb{Z}$.

*So it seems to me even without the separateness condition, we still can talk about everything(Noetherian, normal),*

- *since $X$ is factorial* , $\mathrm{CaCl}(X) \cong \mathrm{Cl}(X)$,
- *$X$ is integral,* $\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$

**Remark** (Relations between $\mathrm{Cl}, \mathrm{CaCl}, \mathrm{Pic}$)**.** *The short answer is that*

- *To define all these concepts, we need noetherian(for finiteness) integral(anyway, we have to deal with fractional field), (separated), regular in codimension $1$(for example, normal, because we need codimension $1$ local rings are all $DVR$).*
- $\mathrm{CaCl}(X) \to \mathrm{Cl}(X)$ *is in general injective, to be an isomorphism, we need to realize every divisor(height $1$ primes as Catier divisors, that is locally principle), this is possible if the scheme is factorial(every local ring is a UFD), since in any $UFD$, height $1$ primes are principle.*

- $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ *is in general injective(even without the Noetherian assumption), to be an isomorphism, we only need $X$ to be Noetherian and reduced(or in practice, integral). The reason is that to realize every line bundle as a subsheaf of $\mathcal{M}$, we need*

$$i : \mathcal{O}_X(U) \to \mathcal{O}_X(U \cap U')$$

*to be injective, what matters is the nilradicals. If we assume $X$ is reduced, then, nilradicals are $0$, $f \in \mathcal{O}_X(U)$ is determined by its values at generic points, then $i$ would be an monomorphism.*

**Example 2.51** (Another way to compute the Picard group of an affine line with double origins)**.**

**Remark.** *We can talk about* $\mathrm{CaCl}(X)$ *and* $\mathrm{Pic}(X)$ *in general even without the Noetherian condition, because they're just some cohomology groups of sheaves on schemes. Only for* $\mathrm{Cl}(X)$*, we need quite a lot of conditions.*

**Example 2.52** (Hartshorne, Example II.6.11.4, $\mathrm{CaCl}^0$)**.**

**Example 2.53** (Hartshorne, II.6.9, Picard groups of singular curves)**.**

**Example 2.54** (Hartshorne II.7.13, a complete nonprojective surface, ??? still some problems)**.** *This construction can be described as 'constructing a twisted family of $\mathbb{P}^1_k$ parametrized by the cuspidal curve $C$'. To be more precise, let $k$ be an algebraically closed field and $char(k) \neq 2$ consider*

$$C : y^2 z = x^3 \subset \mathbb{P}^2_k.$$

*We identify two copies of $C \times_k \mathbb{A}^1_k$ along $C \times \mathbb{A}^1_k - \{0\}$ by the morphism $(x, t) \sim (\phi_t(x), t^{-1})$. Denote the surface by $X$. Then we have a natural morphism*

$$\pi : X \to \mathbb{P}^1_k.$$

*Since properness is local on base, and over $D_+(x)$ or $D_+(y)$ this morphism looks like $C \times \mathbb{A}^1_k \to \mathbb{A}^1_k$, which is just the base change of $C \to \mathrm{Spec}(k)$, thus is proper. We know $X$ is a complete surface(i.e proper over $k$). Next we use the method in [Hartshorne* II.6.9*], and we can get a similar short exact sequence, the normalization of $C \times \mathbb{A}^1_k$ is just $f : \mathbb{P}^1_k \times \mathbb{A}^1_k \to C \times \mathbb{A}^1_k$, we have*

$$0 \to \oplus_{p \in C \times \mathbb{A}^1_k} \widetilde{\mathcal{O}}_p / \mathcal{O}_P \to \mathrm{Pic}(C \times \mathbb{A}^1_k) \to \mathrm{Pic}(\mathbb{P}^1_k \times \mathbb{A}^1_k) \to 0$$

*We only need to consider the codimension $1$ singular locus $p = 0 \times \mathbb{A}^1_k$*

$$\widetilde{\mathcal{O}}_p = k[t, z]_{(t)}, \mathcal{O}_p = k[t^2, t^3, z]_{(t^2, t^3)}$$

*The only difference is degree $1$ monomials $ct, c \in \mathbb{A}^1_k$, then we actually have*

$$0 \to \mathbb{G}_a \to \mathrm{Pic}(C \times \mathbb{A}^1_k) \to \mathbb{Z} \to 0.$$

*We can easily construct a splitting on the RHS. Thus we have*

$$\mathrm{Pic}(C \times \mathbb{A}^1_k) \cong \mathbb{G}_a \times \mathbb{Z}.$$

*To compute* $\mathrm{Pic}(C \times \mathbb{A}^1_k) - \{0\}$*, we can use the sequence as above*

$$0 \to \oplus_{p \in C \times \mathbb{A}^1_k - \{0\}} \widetilde{\mathcal{O}}_p / \mathcal{O}_P \to \mathrm{Pic}(C \times \mathbb{A}^1_k - \{0\}) \to \mathrm{Pic}(\mathbb{P}^1_k \times \mathbb{A}^1_k - \{0\}) \to 0$$

*together with the fact that $A[z, z^{-1}]^* \cong A^* \times \mathbb{Z}$ for any ring $A$, we know similar argument gives us*

$$\mathrm{Pic}(C \times \mathbb{A}^1_k - \{0\}) \cong \mathbb{G}_a \times \mathbb{Z} \times \mathbb{Z}.$$

*Next we consider the restriction map* $\mathrm{Pic}(C\times\mathbb{A}^1_k)\to\mathrm{Pic}(C\times\mathbb{A}^1_k-\{0\})$. *We have plenty of space, so I decide to draw the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{p\in C\times\mathbb{A}^1_k}\widetilde{\mathcal{O}}_p/\mathcal{O}_P \arrow[r] \arrow[d, "i"] & \mathrm{Pic}(C\times\mathbb{A}^1_k) \arrow[r] \arrow[d, "r"] & \mathrm{Pic}(\mathbb{P}^1_k\times\mathbb{A}^1_k) \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & \oplus_{p\in C\times\mathbb{A}^1_k-\{0\}}\widetilde{\mathcal{O}}_p/\mathcal{O}_P \arrow[r] & \mathrm{Pic}(C\times\mathbb{A}^1_k-\{0\}) \arrow[r] & \mathrm{Pic}(\mathbb{P}^1_k\times\mathbb{A}^1_k-\{0\}) \arrow[r] & 0
\end{tikzcd}$$

*where* $i:\mathbb{G}_a\to\mathbb{G}_a\times\mathbb{Z}$ *is just the injection into the first component, and* $j$ *is an isomorphism. So we know the restriction is given by*

$$r:\mathbb{G}_a\times\mathbb{Z}\to\mathbb{G}_a\times\mathbb{Z}\times\mathbb{Z}$$

$$(c,n)\to(c,0,n).$$

*Our final step is to compute* $\mathrm{Pic}(X)$.

**Remark** (why $\mathrm{Pic}(X)\cong\mathbb{G}_a$ implies $X$ is not projective?)**.**

**Remark.** *Since most schemes in this example are not regular in codimension* 1*, so we cannot use the 'stratification trick', which means, we* **DON'T** *have*

$$\mathrm{Pic}(Y)\to\mathrm{Pic}(X)\to\mathrm{Pic}(X-Y)\to 0.$$

*Actually this sequence comes from* $\mathrm{Cl}(X)$*, in our situation, we can't even talk about* $\mathrm{Cl}(X)$.

**Example 2.55** (Hironaka varieties, smooth complete nonprojective 3-folds)**.**

**Example 2.56** ($\mathcal{O}_{\mathbb{P}\mathscr{E}}(1)$ not necessarily ample)**.**

**Remark** (degree of $\mathcal{O}_{\mathbb{P}\mathscr{E}}(1)$)**.** *What the degree of* $\mathcal{O}_{\mathbb{P}\mathscr{E}}(1)$ *and compare it with the curve case, what can you say?*

**Example 2.57** ($\mathrm{Cl}(X)$ of quadric hypersurfaces, Hartshorne II.6.5)**.** *Let* $k$ *be a field, char*$(k)\neq 2$*, and let* $X$ *be the affine quadric hypersurfaces* $\mathrm{Spec}(k[x_0,x_1\dots,x_n]/(x_0^2+\cdots+x_r^2))$*. Then* $X$ *is normal if* $r\geq 2$*(Hartshorne II.*6.4*) and*

- *if* $r=2$*, for example* $\mathrm{Spec}(k[x,y,z]/(xy-z^2))$*,then* $\mathrm{Cl}(X)\cong\mathbb{Z}/2\mathbb{Z}$.
- *if* $r=3$*, for example* $\mathrm{Spec}(k[x,y,z,w]/(xy-z^2-w^2))$*, then* $\mathrm{Cl}(X)\cong\mathbb{Z}$.
- *if* $r\geq 4$*, for example* $\mathrm{Spec}(k[x,y,z,w,u]/(xy-z^2-w^2-u^2))$*, then* $\mathrm{Cl}(X)=0$.

- *For the first statement just consider the example, let* $Y$ *be the prime divisor defined by* $(y,z)$*, we localize at the vertex* $(y,z)$*, then the* DVR *is given by*

  $$(k[x,y,z]/(xy-z^2))_{(y,z)},\mathfrak{m}=(z),(y)=(z^2).$$

  *Then consider the short exact sequece*

  $$\mathbb{Z}\to\mathrm{Cl}(X)\to\mathrm{Cl}(X-Y)\to 0$$

  *since* $X-Y\cong\mathrm{Spec}(k[x,y,z]/(xy-z^2)_y)=\mathrm{Spec}(k[y,y^{-1},z])$*,* $k[y,y^{-1},z]$ *is normal and is a* UFD*, thus* $\mathrm{Cl}(X-Y)=0$*, and we know* $(y)=2Y$ *is principle. Since* $X$ *is normal and if* $\mathrm{Cl}(X)=0$*, we must have* $(y,z)$ *is a principle ideal in* $k[x,y,z]/(xy-z^2)$*, which is not true by considering* $\mathfrak{m}_p/\mathfrak{m}_p^2$*, which is a* $k$*-vector space generated by* $\bar{x},\bar{y},\bar{z}$*. Thus we know* $\mathrm{Cl}(X)=\mathbb{Z}/2\mathbb{Z}$.

- *Fot the second statement, we have to use the fact that $X$ is the affine cone over $V = \mathrm{Proj}(k[x,y,z,w]/(xy - z^2 - w^2))$, and Hartshorne* II.6.3(*b*) *tells us we have*

$$\mathbb{Z} \to \mathrm{Cl}(V) \to \mathrm{Cl}(X) \to 0$$

*where the first map is just sending 1 to a hyperplane section $H \cap V$ as long as $V$ is not contained in $H$. And together with the fact that $V \cong \mathbb{P}^1 \times \mathbb{P}^1$, we know $\mathrm{Cl}(V) \cong \mathbb{Z} \oplus \mathbb{Z}$, then we know we actually have*

$$\mathbb{Z} \to \mathrm{Cl}(V)$$

$$1 \mapsto (1,1).$$

- *If $r \geq 4$, use the Lefschetz hyperplane theorem(Grothendieck's version, if we're allowed to do so), then we know*

$$\mathbb{Z} \to \mathrm{Cl}(V)$$

*is an isomorphism, thus the cokernel $\mathrm{Cl}(X)$ is forced to be 0.*

*Then we consider the projective varieties defined by the same equations as above, we still denote them by $X$. Then*

- *If $r = 2$, then $X \cong \mathbb{P}^1$ is just a conic in $\mathbb{P}^2$, thus $\mathrm{Cl}(X) \cong \mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$.*
- *If $r = 3$, then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by the Segre embedding, thus $\mathrm{Cl}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.*
- *If $r \geq 4$, use Lefschetz hyperplane theorem, we know $\mathrm{Cl}(X) \cong \mathrm{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$. As a consequence, since we know $\mathrm{Spec}(k[x_0,,x_n]/(x_0^2 + \dots x_r^2)), r \geq 4$ is normal with vanishing class group, so $k[x_0,,x_n]/(x_0^2 + \dots x_r^2)$ is a* UFD. *Thus $\mathrm{Cl}(X)$ is given by the cokernel of this map, which is isomorphic to $\mathbb{Z}$.*

**Remark** (Hartshorne II.6.3, relations between $\mathrm{Cl}(V)$ and $\mathrm{Cl}(C(V))$)**.**

**Remark** (Lefschetz hyperplane theorem)**.** *From the discussion above, we can prove Klein's theorem, which says that if $r \geq 4$, and $Y$ is an irreducible subvariety of codimension 1 on $Q$, then $Y$ is a complete intersection of $Q$ and a hyperplane $H \subset \mathbb{P}^n_k$. From*

$$0 \to \mathcal{O}_{\mathbb{P}^r_k}(-2) \to \mathcal{O}_{\mathbb{P}^r_k} \to \mathcal{O}_Q \to 0$$

*we know $H^0(\mathcal{O}_{\mathbb{P}^r_k}(n)) \to H^0(\mathcal{O}_Q(n))$ is an isomorphism, thus pick a random hyperplane $H$, then we have $H \cap Q \sim Y$, thus $f \in K(Q)$, such that $(f)_0 = H \cap Q, f_\infty = Y$, the only possibility is that $f = \frac{l_1}{l_2}$, where $l_1, l_2$ are just linear functions, thus $Y = Q \cap V(l_2)$, that is $Y$ is a complete intersection. Also note that since the affine variety $X = \mathrm{Spec}(k[x_0, \dots, x_n]/(x_0^2 + x_1^2 + \dots + x_n^2))$ is normal, and $\mathrm{Cl}(X) = 0$, by Hartshorne* II.6.2, *we know $k[x_0, \dots, x_n]/(x_0^2 + \dots + x_n^2)$ is a* UFD.

**Example 2.58** (smooth cubic surface $X \subset \mathbb{P}^3$, $\mathrm{Cl}(X) \cong \mathbb{Z}^7$)**.**

The following few examples need a result I learned from *V.L.Popov*'s paper, 'Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles', which say that if $\mathrm{Pic}(G) = 0$(for example, if $G$ is a connected, simply connected linear algebraic group), then we have a short exact sequence

$$0 \to \mathrm{X}_G(H) \to \mathrm{X}(H) \to \mathrm{Pic}(G/H) \to 0$$

where $\mathrm{X}_G(H)$ means characters($\mathrm{Hom}(H, \mathbb{G}_m)$) of $H$ which can be extended to $G$. In practice, it's also useful to know that $\mathrm{X}(H)/\mathrm{X}_G(H)$ is just $\mathrm{X}([G,G] \cap H)$. And we assume in the applications of this method, $k = \mathbb{C}$.

**Example 2.59** ($\mathrm{Pic}(X)$ of flag varieties)**.** *Consider the complete flag variety $F = SL_n/B_+$, then we know $\mathrm{X}(B_+) \cong \mathbb{Z}^{n-1}$ and $\mathrm{X}(SL_n) = 0$, thus we have*

$$\mathrm{Pic}(F_n) \cong \mathbb{Z}^{n-1}.$$

**Remark** (Pic$(G/B)$)**.** *By the Bruhat decomposition and some elementary argument(see my notes 'examples in intersection theory' or Iversen's paper), we know the Picard group of the generalized flag variety $G/B$ is given by*

$$\mathrm{Pic}(G/B) \cong \mathbb{Z}^{|\Phi_+|}$$

*where $|\Phi_+|$ means the cardinality of the set of positive roots. And the corresponding Weil divisors are given by $\overline{Uw_0 s_\alpha B}$.*

**Example 2.60** (Computing $\mathrm{Cl}(X)$ of quadrics from a different point of view)**.** *It turns out that* $\mathrm{Pic}(Q_n)$ *depends on $n$ can be interpreted as we have several sporadic isogenies to orthogonal groups.*

- $\mathrm{Proj}(k[x,y,z]/(x^2+y^2+z^2)) = Q_2 \subset \mathbb{P}^2$, *$SO(3,\mathbb{C})$ acts transitively on $Q_2$, but to apply the theorem, we have to work in $Spin(3,\mathbb{C})$, however it's just $SL(2,\mathbb{C})$($SL(2,\mathbb{C})$ acts on $\mathfrak{sl}(2,\mathbb{C})$ equipped with the Killing form $< x,y >= tr(xy)$ by the adjoint representation, then you can check it's a double cover $SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$). Then the isotropic group of a given point on $Q_2$ is isomorphic to $SO(2,\mathbb{C})$ which is just $\mathbb{C}^\times$ ($SO(2,\mathbb{C}) = \{e^{iz}|z \in \mathbb{C}\}$), thus we know the isotropic group of a given point in $SL(2,\mathbb{C})$ is just the $B_+$, the Borel subgroup, thus we know*

$$\mathrm{Pic}(\mathrm{Q}_2) \cong \mathrm{X}(B_+) = \mathrm{X}(\mathbb{C}^\times) \cong \mathbb{Z}.$$

- $\mathrm{Proj}(k[x,y,z,w]/(x^2+y^2+z^2+w^2)) = Q_2 \subset \mathbb{P}^3$, *this time we have*

$$SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to SO(4,\mathbb{C}).$$

*This is obtained by the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on $V = M_2(\mathbb{C})$ by $(g,h) \bullet A = gAh^{-1}$, $V$ is equipped with a symmetric bilinear form*

$$< x,y >= tr(xwy^t w^{-1}); w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

*The isotropic subgroup of a given point on $Q_2$ is the preimage of the corresponding $SO(3,\mathbb{C})$ in $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, then it's given by $B_+ \times B_-$, thus we get*

$$\mathrm{Pic}(Q_3) = \mathrm{X}(B_+ \times B_-) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

**Remark** (????)**.** *For $r \geq 4$, I know how to carry out this method for $SO(5,\mathbb{C})$ and $SO(5,\mathbb{C})$, but I still don't know how to handle $Spin(n+1,\mathbb{C})$ in general, so maybe latter.*

**Remark** (when $\mathrm{X}(G) = 0$?)**.**

**Example 2.61** ($\mathrm{Cl}(X)$ of Grassmannian)**.** *Consider $Gr(V,k)$,* $\dim(V) = n$. *Then $SL_n$ acts transitively on it and the isotropic subgroup of a point is isomorphic to $GL_m \hookrightarrow SL_n$, know that $\mathrm{X}(GL_m) \cong \mathbb{Z}$, and since $[SL_n, SL_n] = SL_n$ we know $\mathrm{X}(SL_n) = 0$, thus*

$$\mathrm{Pic}(Gr(V,m)) \cong \mathbb{Z}.$$

**Remark** (Chow ring of Grassmannian)**.** *Use some basic knowledge of the Chow ring of Grassmannian, we know there's only one possible Young diagram corresponding to a codimension 1 cycle, then we also know* $\mathrm{Pic}(Gr(V,k)) = \mathbb{Z}$.

**Example 2.62** (relations between $\mathrm{K}(X)$ and $\mathrm{Cl}(X)$)**.**

**Example 2.63** (Cl($X$) of connected algebraic groups)**.** *In short, all connected semisimple simply connected linear algebraic have trivial Picard. And Picard groups of connected semisimple adjoint linear algebraic group can be realized as the cokernel of their Cartan matrices. In general, you have to consider the cokernel of the character map*

$$0 \to \mathrm{X}(G) \to \mathrm{X}(T) \to \mathrm{Pic}(G/B) \to \mathrm{Pic}(G) \to 0.$$

**Example 2.64** (Pic($X$) of determinantal varieties $R_{m\times n,s}$)**.** *$R_{m\times n,s}$ denotes all $m \times n$ matrices with rank less or equal to $s$, we assume $s \leq m \leq n$, note that $R_{m\times n,s}$ is not projective anymore, but it's quasi-projective.*

- *If $s = m \leq n$, then $R_{m\times n,m} \cong \mathbb{A}_k^{nm}$, thus we have*

$$\mathrm{Pic}(R_{m\times n,m}) = 0.$$

- *If $s < m \leq n$, then we claim*

$$\mathrm{Pic}(R_{m\times n,s}) \cong \mathbb{Z}.$$

*First note that $R_{m\times n,s-1}$ is a closed subvariety of $R_{m\times n,s}$, the complement $U_s = R_{m\times n,s} - R_{m\times n,s-1}$ is given by all matrices have rank exactly $s$, then we have*

$$\mathrm{Pic}(R_{m\times n,s-1}) \to \mathrm{Pic}(R_{m\times n,s}) \to \mathrm{Pic}(U_s) \to 0.$$

*Now we want to compute* $\dim(R_{m\times n,s})$*, consider the group action of $G = SL_m \times SL_n$ on $U_s$, $(g,h)\bullet R \mapsto gRh^{-1}$, then we know this action is transitive, since $G$ is connected and irreducible, we know $U_s$ is irreducible, $R_{m\times n,s}$ is irreducible. We first compute the isotropic group $H \subset G$ of the matrix*

$$\begin{bmatrix} I_{s\times s} & 0_{s\times(n-s)} \\ 0_{(m-s)\times s} & 0_{(m-s)(n-s)} \end{bmatrix}$$

*then $H$ is given by*

$$\{(g,h) = (\begin{bmatrix} A_{s\times s} & C_{s\times(m-s)} \\ 0_{(m-s)\times s} & D_{(m-s)(m-s)} \end{bmatrix}, \begin{bmatrix} A_{s\times s} & 0_{s\times(n-s)} \\ F_{(n-s)\times s} & H_{(n-s)(n-s)} \end{bmatrix}) \in SL_m \times SL_n\}$$

*Then we get*

$$\begin{gathered} \dim(R_{m\times n,s}) = \dim(U_s) = \dim(SL_m \times SL_n) - \dim(H) \\ = (m^2-1) + (n^2-1) - (s^2 + (m-s)^2 - 1 + s(m-s)) - ((n-s)^2 + s(n-s)) \\ = (m+n-s)s. \end{gathered}$$

*Now*

$$\dim(R_{m\times n,s}) - \dim(R_{m\times n,s-1}) = m + n - 2s + 1 \geq 3.$$

*As a consequence we know* $\mathrm{Pic}(R_{m\times n,s-1}) \to \mathrm{Pic}(R_{m\times n,s})$ *is just the zero map, thus we know*

$$\mathrm{Pic}(R_{m\times n,s}) \cong \mathrm{Pic}(U_s) = \mathrm{X}(H) \cong \mathbb{Z}$$

*where the last isomorphism is given by the character*

$$det : (\begin{bmatrix} A_{s\times s} & C_{s\times(m-s)} \\ 0_{(m-s)\times s} & D_{(m-s)(m-s)} \end{bmatrix}, \begin{bmatrix} A_{s\times s} & 0_{s\times(n-s)} \\ F_{(n-s)\times s} & H_{(n-s)(n-s)} \end{bmatrix}) \in SL_m \times SL_n \mapsto det(A).$$

**Remark.** *The discussion above shows that $R_{m\times n,s}$ has a natural stratification, but we cannot use Kleiman's theorem here since each strata isn't isomorphic to some $\mathbb{A}_k^N$, although you can get the same answer if you 'apply' Kleiman's theorem.*

**Example 2.65** (Pic($X$) of the variety $S_{n,m}$)**.** *First recall that $S_{n,m}$ is defined to be the set of all unordered $m$-frames in an $n$-dimensional vector space. $GL(n)$ acts transitively on it, and we can realize it as a homogeneous space which is equipped with a canonical algebraic structure. Then the isotropic subgroup $H$ of $p = \{e_1, \dots, e_m\}$ is given by matrices*

- *If $m = 1$, $H \cong GL_{n-1}$, $H' \cong SL_{n-1}$, thus*

$$\mathrm{Pic}(S_{n,1}) = 0.$$

  *This is not surprising at all, since $S_{n,1} \cong \mathbb{A}_k^n$.*

- *If $n = m$, then $H \cong S_n$, $H' \cong A_m$. Moreover*
  - *if $n = m = 1$ or $2$, then $A_1, A_2$ are just the trivial group. Hence*

$$\mathrm{Pic}(S_{1,1}) \cong \mathrm{Pic}(S_{2,2}) = 0$$

    *Note that $S_{1,1} \cong \mathbb{A}_k^1$.*
  - *If $n = m = 3$, then $A_3 \cong \mathbb{Z}/3\mathbb{Z}$, thus*

$$\mathrm{Pic}(S_{3,3}) \cong \mathrm{X}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}.$$

  - *If $n = m = 4$, then $H' = A_4$ is not abelian anymore, but $[H', H']$ is given by the Klein group which is isomorphic to $K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, thus we know*

$$\mathrm{Pic}(S_{4,4}) = \mathrm{X}(A_4) \cong \mathrm{X}(A_4/[A_4, A_4]) \cong \mathbb{Z}/3\mathbb{Z}.$$

  - *If $n = m \geq 5$, then $H = A_m$ is simple, $H' = H$, thus $A_m$ doesn't have any nontrivial $1$-dimensional representations, we get*

$$\mathrm{Pic}(S_{n,n}) = 0; n \geq 5.$$

- *If $n > m > 1$,*

$$\begin{bmatrix} S & * \\ 0 & M \end{bmatrix}$$

  *where $S$ is a $m \times m$ unimodular matrix(i.e just the permutations of vectors in a given frame, thus it's isomorphic to $S_m$), and $M$ is an invertible $(n-m) \times (n-m)$ matrix. And we know $\mathrm{Pic}(GL(n)) = 0$, thus we know*

$$\mathrm{Pic}(S_{n,m}) \cong \mathrm{X}([GL_n, GL_n] \cap H) = \mathrm{X}(SL_n \cap H).$$

  *Now let $H' = H \cap SL_n$, to construct $1$-dimensional representations of $H'$, we only need to compute $[H', H']$, then we can get*

$$[H', H'] = \{\begin{bmatrix} A_m & * \\ 0 & SL_{n-m} \end{bmatrix}\}$$

  *Thus we know*

$$\mathrm{Pic}(S_{n,m}) \cong \mathrm{X}(H'/[H', H']) \cong \mathbb{Z}/2\mathbb{Z}.$$

# 3 curves

**Example 3.1** (unirational curves are all rational, birational $\neq$ isomorphism)**.** *Consider the cuspidal curve $C: y^2z = x^3$ again, we have a birational morphism*

$$\mathbb{P}^1_k \to C; [u, v] \mapsto [u^2v, u^3, v^3].$$

*Then it's bijective but not an isomorphism, of course it's birational.*

**Remark** (smooth rational curve$\Rightarrow \mathbb{P}^1_k$ )**.** *This is because we have a dominant(birational) map*

$$\mathbb{P}^1_k \to C$$

*then by Riemann-Hurwitz formula we know $g(C) \leq 0$, thus $C$ is actually $P^1_k$.*

**Remark** (morphisms from $\mathbb{P}^1_k$ to a group scheme are constant)**.** *This is because if $f : \mathbb{P}^1_k \to X$ is not a constant, then the image is a unirational curve, but by Lüroth's theorem it's rational, so it's reduced to the case*

$$\mathbb{P}^1_k = \widetilde{C} \to C \subset X.$$

*Now for some $y \in \mathbb{P}^1_k$, we have*

$$df : T_y\mathbb{P}^1_k \to T_{f(y)}X$$

*is not zero, but $\Omega^1_X$ is trivial, we can thus find an $\omega \in \Gamma(X, \Omega^1_X)$, such that $w(y) \neq 0$, then $f^*\omega \in \Gamma(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k})$ which is not zero at $y$, this is ipossible, since $\Omega^1_{\mathbb{P}^1_k} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, no nonzero global 1-forms. You can use other ways to state this fact(math is the art of giving different names to the same thing),e.g*

- *on abelian surfaces, you cannot find any lines.*
- *on abelian varieties, you cannot find any linear subspace of dimension at least 1.*
- *cubic surfaces are not abelian surfaces, since you can always find lines on a cubic surface.*
- *sometimes, this fact is actually very useful, see the paper 'on a rank 2 bundles on $\mathbb{P}^4$ with 15000 symmetries' by Horrocks and Mumford, in the final argument that every smooth abelian surface is the zero locus of a section of the Horrocks-Mumford bundle, they actually compute the base locus of the complete linear system corresponding to the global sections of that bundle, which turns out to be 25 lines, but no line can live on an abelian surface, so the base locus is supported at finitely many points, in other words codim $\geq 2$, this is crucial for the proof.*

**Example 3.2** (Every point is a inflection point)**.** *It's an exercise in Hartshorne's book, consider the curve over a field with characteristic 3.*

$$C : x^3y + y^3z + z^3x = 0.$$

*Then $C$ is a smooth curve but every point on this curve is an inflection point. The easiest way to see this is to compute the Hessian of $C$. But we can also use a naive way to see this, namely, compute the intersection number of the tangent line $L$ at $(a, b) \in C$ and $C$. Which means, we only need to compute the lenth*

$$k[x, y]/(x + a^3y + b, x^3y + y^3 + x).$$

*And since $char(k) = 3$ and $a^3b + b^3 + a = 0$, we have*

$$(-a^3y + b)^3y + y^3 + (-a^3y - b) = (y - b)^3(-a^9y + 1).$$

*So in the local ring at $p = (a, b)$, the scheme structure of the intersection is*

$$\mathrm{Spec}(k[y]/((y - b)^3)); if\ -a^9b + 1 \neq 0;$$

$$\mathrm{Spec}(k[y]/(y-b)^4); if\ -a^9b+1=0.$$

*In the second situation, we must have*

$$1+a^{28}+a^{21}=0$$

*The derivative is given by $a^{27}$, and since $a=0$ is not a solution, we know there're exactly 28 second type of points if $k$ is algebraically closed.*

**Remark.** *It's well-known that if $char(k)=0$, for a smooth degree $d$ plane curve $C$, we can find $3d(d-1)$ inflection points, this examples show it fails badly if $char(k)\neq 0$.*

**Example 3.3** (Wikipedia, a curve, polar curve and dual curve)**.**

$$C: 4y^2z = x^3xz^2.$$

*The polar curve corresponding to $C$ w.r.t the point $p=(0.9,0,1)$ is*

$$C': 4y^2 = 2.7x^2 - 2xz - 0.9z^2.$$

*The dual curve is given by*

$$C^*: 4X^4Y^2+Y^6+64X^5Z+24XY^4Z+120X^2Y^2Z^2-64X^3Z^3-108Y^2Z^4\subset\mathbb{P}^2.$$

*Computing the equation of $C^*$ is by elimination*

$$eliminate(\{p,q,r,\lambda\},(X=\lambda\frac{\partial f(p,q,r)}{\partial x},Y=\lambda\frac{\partial f(p,q,r)}{\partial y},Z=\lambda\frac{\partial f(p,q,r)}{\partial z},Xp+Yq+Zr)).$$

**Example 3.4** (extension of a morphism)**.** *Compare the following two statements*

- *If $C$ is a regular curve over $k$ and $X$ a complete variety, $p\in C$ is a point on $C$, then any morphism*

  $$f: C-\{p\}\to X$$

  *can be extended to $C$.*

- *Consider the natural projection from the total space of $\mathcal{O}_{\mathbb{P}^1_k}(1)$ to $\mathbb{P}^1_k$, that is*

  $$\mathbb{P}^2_k-[0,0,1]\to\mathbb{P}^1_k$$

  $$[x,y,z]\mapsto[x,y].$$

  *This cannot be extended simply because there's no non-constant morphism $\mathbb{P}^N_k\to\mathbb{P}^n_k$ if $N>n$(this is because such a morphism is given by some homogeneous polynomials, but $n$ homogeneous polynomials with $N$ variables must have have base point, that is common zeros).*

**Example 3.5** (cohomology of $d$ lines intersecting transitively in $\mathbb{P}^2_k$, degree-genus formula)**.** *Note that no matter a degree $d$ curve $C$ is degenerated or not, we always have*

$$0\to\mathscr{I}=\mathcal{O}_{\mathbb{P}^2}(-d)\to\mathcal{O}_{\mathbb{P}^2}\to\mathcal{O}_C\to 0.$$

*Then you can get the arithmetic genus is always given by*

$$\chi(C)=\frac{d^2-3d+4}{2}$$

*Then we get the arithmetic genus of $C$ is always $\frac{(d-1)(d-2)}{2}$. If you don't want do this easy computation, that's fine, what we're saying here is just that arithmetic genus(equivalently, Euler characteristic) is constant* *in a flat family. Then you can normalize the intersecting lines and get a disjoint union of $d$ lines, then its genus is $-d+1$, since we have to normalize $\frac{(d-1)d}{2}$ points, we have*

$$g_C - \frac{(d-1)d}{2} = -d+1.$$

*This is also a proof of the degree-genus formula(another proof is to use the adjunction formula, or use cut-paste procedure).*

**Example 3.6** (regular differentials and degree-genus formula for smooth curves)**.** *The geometric genus $g = \dim H^0(C, \omega_C)$ of a curve $C$ can be computed explicitly by finding a basis of $H^0(C, \Omega_C)$. Let's first do some warm-ups, consider the affine curve $X$ defined by $y^2 = x^3+x$, the sheaf $\omega_X$ of regular differential(regular differential) is a $k[X]$-module. $2ydy = (3x^2+1)dx$, we also say that*

$$\omega = \frac{dy}{3x^2+1} = \frac{dx}{2y}$$

*is a regular differential on $X$. Shouldn't it be a $k[X]$-module, why we have fractions here? Let $\omega = fdx + gdy$, we need*

$$2y\omega = dx, (3x^2+1)\omega = dy.$$

*This requires*

$$(2y)f + (3x^2+1)g = 1$$

*Well, $X$ is smooth, thus $X \cap V(\frac{\partial f}{\partial x} = 3x^2+1, \frac{\partial f}{\partial y} = 2y) = \emptyset$, Hilbert's Nullstellensatz tells us that this is equivalent of saying that we can always find $f, g \in k[X]$ such that $(2y)f + (3x^2+1)g = 1$, in this example we have*

$$(2y)(-\frac{9}{4}xy) + (3x^2+1)(\frac{3}{2}x^2+1) = 1.$$

*What we really mean by $\omega = \frac{dy}{3x^2+1} = \frac{dx}{2y}$ is that*

$$\omega = -\frac{9}{4}xydx + (\frac{3}{2}x^2+1)dy.$$

*Then we can glue regular differentials on every affine open subset to get a regular differential on a general curve. Then for a curve $C \subset \mathbb{P}^2 = \mathrm{Proj}(k[X,Y,Z])$ defined by $F(x,y) = 0$ in $D_+(z)$, let $f = \frac{\partial f}{\partial x}, g = \frac{\partial f}{\partial y}$ we can use the same argument and show that all the regular differentials are of the form*

$$\omega = \frac{-hdx}{g} = \frac{hdy}{f}, \deg(h) \leq d-3.$$

*Why $\deg(h) \leq d-3$ ? This is because on $D_+(y)$ with coordinates $u = \frac{X}{Y}, v = \frac{Z}{Y}$, we have $x = \frac{u}{v}, y = \frac{1}{v}$, hence*

$$\frac{hdy}{f} = \frac{h(\frac{u}{v}, \frac{1}{v})(\frac{-1}{v^2}dv)}{f(\frac{u}{v}, \frac{1}{v})}.$$

*We only need both factors be polynomials in $u, v$. Thus $\deg(h) + 2 \leq d-1$. To get the degree-genus formula, a homogeneous polynomial of degree $d$ in 3 variables has $\binom{d+2}{2} = \frac{(d+2)(d+1)}{2}$ coefficients, thus we know*

$$g = \dim H^0(C, \omega_C) = \frac{((d-3)+2)((d-3)+1)}{2} = \frac{(d-1)(d-2)}{2}.$$

**Example 3.7** (Hartshorne $IV$.3.7, not every plane curve with nodes a projection of a regular curve in $\mathbb{P}^3$)**.** *Consider the curve $C$ over an algebraically closed field $k$, $char(k) \neq 2$*

$$C : xyz^2 + x^4 + y^4$$

*is not a projection of any regular curve in $\mathbb{P}^3$.*

**Example 3.8** (singular 'strange' curves)**.** *Here 'strange' means there exists a point which lies on all the tangent lines at nonsingular points of the curve.*

hyperelliptic

**Example 3.9** (Stacks Project Example 46.11.3, genus changes under a purely inseparable morphism between smooth projective curves)**.**

**Example 3.10** (non-hyperelliptic curves)**.** *We know all genus* 2 *smooth projective curves are hyperelliptic, to construct some non-hyperelliptic curves, just remember a curve $C$ is hyperelliptic if and only if the canonical divisor $K$ is* **not** *very ample. Then we know many curves have ample canonical divisors, for example*

- *degree* 4 *plane curves, like $x^4 + y^4 + z^4 = 0$, $g = 3$, then*

$$\omega_C = \mathcal{O}_{\mathbb{P}^2}(4-3)|_C = \mathcal{O}_{\mathbb{P}^2}(1)|_C$$

- *any curve of $g \geq 2$ which can be realized as a complete intersection in some $\mathbb{P}^n$. Then*

$$\omega_C = \mathcal{O}_{\mathbb{P}^n}((\sum d_i) - n - 1)|_C$$

  *since $deg(K) = 2g - 2 \geq 2$, we know $(\sum d_i) - n - 1 \geq 0$, thus $K$ is very ample. More concretly, consider complete intersection of surfaces in $\mathbb{P}^3$, then we have*

$$g = p_a = \frac{1}{2}ab(a+b-4) + 1$$

  *so, if you want to get a genus* 4*, non-hyperelliptic curve, just let $a = 3, b = 2$.*

**Example 3.11** (a family of elliptic curves over $\mathbb{Q}$ with on rational points)**.** *In poonen's paper, 'an explicit family of genus* 1 *curves violating Hasse principal', we can find several examples*

- *around 1940, Lind*

$$C : 2y^2 = 1 - 17x^4, g = 1$$

  *Based on this example, we can construct a family of elliptic curves violating the Hasse principal, however, with constant $j$-invariant.*

$$X_t : 2y^2 = 1 - [(t^2 + t + 3)^4 + 16(t^2 + t + 1)^4]x^4.$$

- *Selmer, diagonal plane cubic curve*

$$C : 3x^3 + 4y^3 + 5z^3 = 0$$

- *1962, Swinnerton-Dyer wrote a 2-page long paper, in this paper he constructed a cubic surface vialating the Hasse principal*

$$t(t+x)(2t+x) = \Pi_i(x + \theta_i y + \theta_i^2 z)^3, \theta^3 - 7\theta + 14\theta^2 - 7 = 0.$$

- *1966, Cassel and Guy discovered a smooth cubic surface which violating the Hasse principal*

$$X : 5x^3 + 9y^3 + 10z^3 + 12w^3 = 0.$$

- *Based on Cassel and Guy's example, Poonen constructed a non-trivial(non-constant $j$-invariant) family of elliptic curves violating the Hasse Principal.*

$$X_t : 5x^3 + 9y^3 + 10z^3 + 12(\frac{t^2 + 82}{t^2 + 22})^3(x+y+z)^3 = 0.$$

- *Based on Swinnerton-Dyer's example and Poonen's method, I also constructed a family of elliptic curves violating the Hasse principal.*

$$X_t : (\frac{t^2+54}{t^2+1})((\frac{t^2+54}{t^2+1})+1)((\frac{t^2+54}{t^2+1})+2)x^3 = \Pi_i(x+\theta_i y+\theta_i^2 z)^3, \theta^3 - 7\theta + 14\theta^2 - 7 = 0.$$

*For details of these examples and Poonen's method, see his original paper. What I want to do here is to follow Swinnerton-Dyer's method and prove that the cubic surface*

$$t(t+x)(2t+x) = \Pi_i(x+\theta_i y+\theta_i^2 z)^3, \theta^3 - 7\theta + 14\theta^2 - 7 = 0.$$

*violating the Hasse principal.*

- *if* $p \neq 2, 3, 7$, *then the discriminant is not zero, thus by the Hasse bound, we have*

  $$|\#X(\mathbb{F}_p) - q - 1| \leq 2\sqrt{q}.$$

  *That means, we can always find smooth* $\mathbb{F}_p$*-point, then use Hensel's lemma, we can always find smooth* $\mathbb{Q}_p$*-point.*

- *check directly that for* $p = 2, 3$ *or* $7$, *we can still find smooth point.*

  $$p = 2, [0,0,0,1]; p = 3, [0,1,0,-1]; p = 7, [3,0,0,1].$$

  *Then apply Hensel's lemma again, for any prime* $p$, $X$ *contains a smooth* $\mathbb{Q}_p$*-point.*

- *To prove that* $X$ *has no rational points, we need some very basic knowledge of class field theory. First consider the field extension* $\mathbb{Q} \subset \mathbb{Q}(\theta)$ *determined by*

  $$f(\theta) = \theta^3 - 7\theta^2 + 14\theta - 7 = 0.$$

  *Then it's an abelian cubic extension over* $\mathbb{Q}$ *with discriminant* $49$. *The only prime number ramifies in* $\mathbb{Q}(\theta)$ *is* $7$, *denote it by*

  $$\mathfrak{p}_7^3 = (7), \mathfrak{p}_7 | (\theta).$$

  *The second relation comes from* $f(\theta)$.

  - *both sides of the defining equation cannot be zero, since if the RHS is zero, we get* $x = y = z = 0$, *and thus* $t = 0$.
  - *we may assume the LHS is an integer and* $(x,t) = 1$. *If both sides are divisible by* $7$, *then we know* $\mathfrak{p}_7 | x \Rightarrow 7 | x$. *But* $(t,x) = 1$, *the LHS cannot be divided by* $7$, *a contradiction.*
  - *now assume neither side is divisible by* $7$. *Since the RHS is a norm, and three factors on the LHS are coprime. Each one of them must be a norm of some distinct ideals. And class field theory tells us, since* $\mathbb{Q} \subset \mathbb{Q}(\theta)$ *corresponding to* $\{\pm 1\} \subset (\mathbb{Z}/7\mathbb{Z})^\times$, *the only rational primes ramify in* $\mathbb{Q}(\theta)$ *are those congruent to* $\pm 1 (mod 7)$, *on the other hand, we have*

    $$t + (t+x) = 2t + x$$

    *so this is actually impossible.*

*In conlusion, we know* $X$ *contains no rational points.*

**Example 3.12** (Jacobian of a curve??)**.** *We know some basic examples of Jacobian varieties*

- $J(\mathbb{P}^n) = pt$, *since* $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.
- $J(E) \cong E$. *And we have* $\mathrm{Pic}(E) \cong E \times \mathbb{Z}$.

- *$J(C)$, where $C$ is the Fermat quartic $x^4+y^4+z^4=0$. By construction* $\dim(J(C)) = g = \frac{(4-1)(4-2)}{2} = 3$*, I just know $J(C)$ is a three dimensional algebraic torus, but what is it explicitly, I have no idea.*

**Remark** (when the coordinate ring of an affine curve a UFD?)**.**

**Remark** ($\mathrm{Cl}^0(X)$ of the nodal curve)**.**

**Example 3.13** (complex tori but not algebraic tori, intersection theory)**.** *We have two standard counterexamples*

- *Fix some $a \in \mathbb{R}, a > 1$ and consider*

$$\mathbb{C}^2 - \{(0,0)\}/((x,y) \sim (a^k x, a^k y)).$$

*Then topologically, this is just $S^3 \times S^1$, by Künneth's theorem, we know $H^2(X) = 0$, however basic intersection theoretic construction, a complex algebraic variety always have divisors with nonzero homology class. We conclude that $S^3 \times S^1$ has a complex structure but has no algebraic structure.*

- *The quotient $X := \mathbb{C}^2/\Lambda$ of $\mathbb{C}^2$ by a generic lattice of rank 4. Topologically, $X \cong (S^1)^4$. See https://sbseminar.wordpress.com/2008/02/14/complex-manifolds-which-are-not-algebraic/.*

**Example 3.14** (degree genus formula for smooth complete intersection)**.** *This is quite straightforward, let $C \subset \mathbb{P}^n$ be a complete intersection of $n-1$ hypersurfaces $X_i = V(f_i)$, $deg(f_i) = a_i$, then consider the tangent sequence*

$$0 \to \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(-a_i)|_C \to \Omega_{\mathbb{P}^n}|_C \to \Omega_C \to 0$$

$$(\phi_1, \ldots, \phi_{n-1}) \mapsto d(\sum_i f_i \phi_i); dg \mapsto d(g|_C)$$

*Then we know*

$$N_{C/X}|_C = \mathcal{O}_C(\sum_i a_i - n - 1)$$

*This tells us that*

$$2g - 2 = deg(K_C) = deg((\sum_i a_i - n - 1)H|_C) = (\sum_i a_i - n - 1)\Pi a_i$$

*thus we get*

$$g = \frac{1}{2}\Pi a_i(\sum a_i - n - 1)$$

*as a special case. we know a complete intersection in $\mathbb{P}^3$ has genus*

$$g = \frac{1}{2}ab(a + b - 4)$$

**Remark** (non-hyperelliptic curves)**.** *If a curve has genus $g \geq 2$, and $C$ can be realized as a complete intersection then $C$ is non-hyperelliptic, $\omega_C = \mathcal{O}_{\mathbb{P}^n}(\sum a_i - n - 1)|_C$, thus $\omega_C$ is very ample.*

**Example 3.15** (faithful group action on a variety with zero derivative)**.** *Let $k$ be a field of characteristic $p > 0$, and consider the $\mathbb{G}_m$ action on $\mathbb{A}^1$ given by*

$$\mathbb{G}_m(R) \times \mathbb{A}^1(R) \to \mathbb{A}^1(R)$$

$$(\alpha, r) \mapsto \alpha^p r.$$

*Note that $d(\alpha^p) = p\alpha^{p-1}d\alpha = 0$, $\alpha^p - 1 = (\alpha - 1)^p = 0 \Leftrightarrow \alpha = 1$.*

**Example 3.16** (Hurwitz's bound for $\#\mathrm{Aut}(C)$ fails in positive characteristics). *Let $k$ be a field of characteristic $p = 2g+1$, consider the curve*

$$y^2 = x^p - x.$$

**Example 3.17** (non-trivial automorphism fix every $\mathbb{F}_p$ point). *Consider curves*

$$C_1 : y^2 = x^p - x, C_2 : y^2 = x^p - x^{p-1} - x + 1.$$

*The automorphism $\sigma \neq id$ is given by $(x,y) \mapsto (x,-y)$, note that in $\mathbb{F}_p$ we have $x^p - x \equiv 0$. Therefore*

- *$\sigma$ fix every $\mathbb{F}_p$ point of $C_1$,*
- *$\sigma$ fix every rational point on $C_2$ and switches $(0,1)$ and $(0,-1)$.*

**Example 3.18** ( the automorphism group a curve over any field with $g \geq 2$ is finite, how to produce vector fields on a variety?). *If $g > 2$, we know $\omega_C$ is ample, we can linearize $\mathrm{Aut}(C)$ by 'the canonical embedding'*

$$j : X \hookrightarrow \mathbb{P}H^0(C, \omega_C^{\otimes m}) =: \mathbb{P}V.$$

*$j$ is an $\mathrm{Aut}(X)$-equivariant embedding, $\mathrm{Aut}(C)$ acts on the right by pulling back differential forms. Now we claim, if $\#\mathrm{Aut}(C) = \infty$, we can produce a nonzero global vector field on $X$, but this contradicts the fact the $c_1(T_C) = 2 - 2g < 0$. How can we construct a vector field on $C$? Well, any element $v$ in $PGL(V)$ generates a vector field on $\mathbb{P}V$. But this need not to be a vector field on $C$, that's not a big problem, if $v$ comes from some 1-parameter subgroup of $\mathrm{Aut}(C) \hookrightarrow PGL(V)$, it does give us a vector field on $C$. Now, we want to ask when I can find an '1'-parameter subgroup of $\mathrm{Aut}(C)$? We know the Zariski closure of $\mathrm{Aut}(C)$ is also an algebraic group, let's call it $G$, $G$ acts on $C$(everything is defined by polynomials of coordinates in $\mathbb{P}V$). Since everything here is of finite type, if $\#\mathrm{Aut}(C) = +\infty$, we must have $\dim(G) \geq 1$, this means $\dim(T_eG) \geq 1$, we can find some non-zero $v \in T_eG$, by the discussion above, it would give me a nonzero global vector field on $C$, which is impossible. We conclude that, a curve of genus at least 2 over arbitrary field has a finite automorphism group.*

**Remark.** *I learned this proof from Daniel Litt's notes here.*

**Remark** (canonical embedding). *We see from the proof above that the canonical embedding plays an important role, mainly because*

- *the canonical embedding linearizes $\mathrm{Aut}(X)$, namely, $\mathrm{Aut}(X)$ is a subgroup of $PGL_n$.*
- *vector fields on $X$ form a subset of vector fields on $\mathbb{P}V$ generated by the linear action of $PGL_n$, that is a non-zero element $v \in T_e(G)$ must gives us a non-zero vector field on $X$.*

*To get more intuitions, just consider*

$$C : x^4 + y^4 + z^4 \hookrightarrow \mathbb{P}^2_k.$$

*Then $\omega_C \cong \mathcal{O}_C(1)$, $H^0(C, \omega_C)$ is generated by $x, y$ and $z$(viewed as functions on $C$). Note that here we cannot give you any global vector field, just because $\mathrm{Aut}(X) \subset PGL_3$ is finite. An old point of view is that $deg(T_C) = 2 - 2g_C = -4$, thus no global section exsits.*

**Example 3.19** (Varieties singular everywhere). *If $X$ is a variety over an algebraically closed field $k$ and $char(k) = 0$, we know we can find an open subset $U \subset X$, such that $U$ is smooth. This is not true for prime characteristics, consider $\overline{\mathbb{F}}_p$*

$$V : x_0^p + x_1^p + \cdots + x_n^p$$

*is singular everywhere, I know this example is silly, but it's an example...*

**Example 3.20** (morphism singular everywhere)**.** *Let $k = \overline{\mathbb{F}}_p$, consider the Frobenius morphism*

$$f : X = \mathbb{P}^n_k \to \mathbb{P}^n_k = Y$$

$$[x_0, \ldots, x_n] \mapsto [x_0^p, \ldots, x_n^p].$$

*Note that we have the conormal sequence*

$$f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

*Since $d(x_i^p) = 0$, the first map in the sequence is 0, $\Omega_{X/Y} \cong \Omega_{X/k}$ is locally free of rank $1 \neq 0 = \dim(X) - \dim(Y)$. Or it's natural to see the induced map on tangent space is not surjective(it's 0)*

**Remark** (normal, conormal, tangent, cotangent, canonical)**.**

**Example 3.21** (nonperfect field, regular $\neq$ smooth)**.** *Let $k = \mathbb{F}_p(t), p > 2$, consider*

$$C : y^2 = x^p - t \subset \mathbb{A}^2_k.$$

*Then every local ring of $C$ is regular but $C$ is not smooth over $k$. First the Jacobian matrix is given by*

$$(2y, 0)$$

*So it only vanishes at the prime ideal $=(y, x^p - t)$. So we only need to prove*

- *the regularity of*
$$\mathcal{O}_{C,\mathfrak{p}} = (k[x, y]/(y^2 - x^p + t))_{(y,x^p-t)}$$
- *$C$ is not smooth after some base change.*

*The first statement is because $\mathcal{O}_{C,\mathfrak{p}}$ is domain and a local ring with dimension 1, its maximal ideal is generated by $y$, thus $\mathcal{O}_{C,\mathfrak{p}}$ is a DVR, hence regular. Then we base change it to $K = \mathbb{F}_p(t^{\frac{1}{p}})$. Then we get*

$$C_K : y^2 = (x - t^{\frac{1}{p}})^p$$

*Then it's not smooth at $\mathfrak{q} = (y, x - t^{\frac{1}{p}})$, it's not even normal, consider*

$$\alpha = \frac{y}{x - t^{\frac{1}{p}}} \in Frac(\mathcal{O}_{C_K,\mathfrak{q}})$$

*Then*

$$\alpha^2 = \frac{y^2}{(x - t^{\frac{1}{p}})^2} = (x - t^{\frac{1}{p}})^{p-2} \in \mathcal{O}_{C_K,\mathfrak{q}}.$$

*Note that the Auslander–Buchsbaum theorem states that every regular local ring is a unique factorization domain, hence normal. $\mathcal{O}_{C_K,\mathfrak{q}}$ is not normal, not to say regular.*

**Remark** (Jacobian criterion)**.**

$$dim_{k(x)}(\Omega_{X/k} \otimes k) = n - \mathrm{rank} J_x$$

*my question is what is $J_x$ at a nonclosed point $x$? Actually you always compute the ordinary Jacobian, but to determine its rank, you have to ask at what point? See $x$ corresponding to a prime ideal $\mathfrak{p}$, and*

$$C := \mathrm{Spec}(k[x_0, x_1, \ldots, x_n]/(f_1, \ldots, f_m))$$

*Let $_m$ be a $m \times m$ minor of the Jacobian matrix, to determine it's smooth or not at $\mathfrak{p}$, you only need to check if $\Delta_m \in \mathfrak{p}$ or not. In general*

$$\textit{Jacobian} \Rightarrow \textit{smooth} \Rightarrow \textit{regular}$$

**Remark** (different definitions of smoothness)**.**

**Example 3.22** (regular $k$-algebra of dimension 1, but not regular after base change to $\overline{k}$)**.**

$$\mathcal{O}_{C,\mathfrak{p}}, \mathcal{O}_{C_K,q}$$

*above.*

**Example 3.23** (Bertini theorem in characteristic $p$)**.** *Consider the Frobenius map again*

$$f : \mathbb{P}^1_k \to \mathbb{P}^1_k; [x, y] \mapsto [x^p, y^p]$$

*Then we know it's corresponding to the* 1*-dimensional linear system* $\mathfrak{d} = \{pP | P \in \mathbb{P}^1_k\}$*, or in other words* $\{(ax + by)^p | a, b \in k\}$*, they're all singular. However Poonen proves a Bertini theorem for quasi-projective varieties over finite fields.*

**Example 3.24** (a normal variety but not smooth, based on Hartshorne II.6.4 )**.** *Just consider the cone* $X$ *in* $\mathbb{A}^3_k$*, assume that* $char(k) \neq 2$

$$X : x_0^2 = x_1^2 + x_2^2$$

$X$ *is singular at the vertex* $(0, 0, 0)$*. To see it's normal, we prove*

- *Let* $f \in k[x_1, \dots, x_n]$ *be a square-free nonconstant polynomial over a field of characteristic* $\neq 2$*. Then,* $k[x_1, \dots, x_n, z]/(z^2 - f)$ *is integrally closed.*

*Observe that*

$$K = \mathrm{Frac}(k[x_1, \dots, x_n]/(z^2 - f)) = k(x_1, \dots, x_n)[z]/(z^2 - f)$$

*It's a Galois extension of degree* 2 *over* $k(x_1, \dots, x_n)$*. If* $\alpha = g + hz \in K$*, where* $g, h \in k(x_1, \dots, x_n)$*. We know the minimal polynomial of* $\alpha$ *is* $\alpha^2 - 2g\alpha + (g^2 - h^2 f)$*, because*

$$(g + hz)^2 - 2g(g + hz) + (g^2 - h^2 f) = g^2 + 2ghz + h^2 z^2 - 2g^2 - 2ghz + g^2 - h^2 f = 0.$$

*Thus,* $\alpha$ *is integral over* $k[x_1, \dots, x_n]$ *if and only if* $2g$ *and* $g^2 - h^2 f \in k[x_1, \dots, x_n]$*, which is true if and only if* $g, h^2 f \in k[x_1, \dots, x_n]$*. Now if* $\alpha$ *is integral, and* $h$ *had nontrivial denominator, then* $h^2 \notin k[x_1, \dots, x_n]$ *since* $f$ *is square-free; hence* $h \in k[x_1, \dots, x_n]$ *so* $\alpha \in k[x_1, \dots, x_n]/(z^2 - f)$*. Thus,* $k[x_1, \dots, x_n]$ *is integrally closed in* $K$*. As a special case we* $k[x_0, x_1, x_2]/(x_0^2 - x_1^2 - x_2^2)$ *is normal and it's easy to check all its local rings are normal.*

**Example 3.25** (A formally smooth morphism but not smooth)**.** *Consider the ring homomorphims*

$$p : k[t^q, q \in \mathbb{Q}_{>0}] \to k$$

$$t^q \mapsto 0.$$

*It's not smooth simply because it's not flat.*

$$0 \to (t) \to k[t^q, q \in \mathbb{Q}_{>0}] \to k[t^q, q \in \mathbb{Q}_{(0,1)}] \to 0.$$

*After tensoring with* $k$*, the left hand side is not injective anymore. To prove it's formally smooth, let* $A$ *be a ring with a square zero ideal* $I$*, consider the following diagram,*

$$\begin{tikzcd} A/I & k \arrow[l, "g"'] \\ A \arrow[u, "\pi"] & k[t^q, q \in \mathbb{Q}_{>0}]. \arrow[l, "f"'] \arrow[u, "p"'] \end{tikzcd}$$

*Then* $f(t^q) \in I$ *since the diagram is commutative. However, every* $t^q$ *is a square, we thus have* $f(t^q) = f((t^{\frac{q}{2}})^2) \in I^2 = 0$*. In other words,* $f$ *factors through* $p$*, this is exactly the formal smoothness of the map* $p$*.*

# 4 Mumford regularity, local cohomology

**Example 4.1** (Serre's example).

**Example 4.2** (why $H^i(X, \mathscr{F}(n)) = 0$?). *$X$ projective scheme over a noetherian ring $A$, $\mathcal{O}_X(1)$ is a very ample invertible sheaf on $X$ over* $\mathrm{Spec}(A)$*, $\mathscr{F}$ is a coherent sheaf on $X$. Then it's common knowledge that*

- *$\forall i \geq 0, H^i(X, \mathscr{F})$ is a finitely generated $A$-module.*
- *$\forall i \geq 1, H^i(X, \mathscr{F}(n)) = 0$ for sufficient large $n$.*

*Why we should expect a theorem like this? Think about $H^n(\mathbb{P}^n, \mathcal{O}(m))$, it means all degree $-n-1-m$ Laurent polynomials $x_0^{i_0} \dots x_n^{i_n}$ with $i_n \leq -1$. Then no matter what $m$ is, if $n$ is big enough, $H^n$ vanishes for the trivial reason, that is, no positive degree Laurent polynomial with $i_k \leq -1$. That's one of the reasons why we should expect such a vanishing theorem. One of the good consequences of the vanishing theorem is that in many general situations, we have to consider derived push-forward:*

$$Rf_*^{\bullet}\mathscr{F} := \sum_{i \geq 0} (-1)^i R^k f_*.$$

*In many cases, if we twist the sheaf $\mathscr{F}$ before pushing forward, we then only need to consider $f_*\mathscr{F}$. For example, [FGA Explained Lemma5.4] says that $\phi : T \to S$ is a morphism of noetherian schemes, $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}_S^n$, and let $\mathscr{F}_T$ be the pull-back of $\mathscr{F}$ w.r.t to the morphism $\mathbb{P}_T^n \to \mathbb{P}_S^n$. Let $\pi_S : \mathbb{P}_S^n \to S$, $\pi_T : \mathbb{P}_T^n \to T$ bethe projections. Then there exists an integer $r_0$ such that the base-change homomorphism*

$$\phi^* \pi_{S,*}\mathscr{F}(r) \to \pi_{T,*}\mathscr{F}_T(r)$$

*is an isomorphism for all $r \geq r_0$. I guess if we drop the twists, we might have a quasi-isomorphism*

$$\phi^* R^{\bullet}\pi_{S,*}\mathscr{F}(r) \to R^{\bullet}\pi_{T,*}\mathscr{F}_T(r).$$

**Example 4.3** (Local duality $\mathrm{H}^i_{\mathfrak{m}}(M) \cong (\mathrm{Ext}^{r+1-i}(M, S(-r-1)))^{\vee}$). *Let $S = k[x,y]$, $\mathfrak{m} = (x,y)$. Consider the $S$-module $R = k[x,y]/(x^2, xy)$, we want to compute the local cohomology $\mathrm{H}^i_{\mathfrak{m}}(R)$, Hilbert polynomial $\chi_R(t)$, and Castelnovo-Mumford regularity of $R$. The Čech complex is given by*

$$0 \to R \xrightarrow{\binom{1}{1}} R[x^{-1}] \oplus R[y^{-1}] \to R[x^{-1}y^{-1}] \to 0.$$

**Remark** (reference). *This example and the original theorem can be found in The Geometry of Syzygies*

**Remark** (Local cohomology, Mumford regularity, Hilbert polynomial). *The local cohomology $H^i_Q(-)$ is defined to be the right derived functor of the $Q$-torsion functor*

$$H_Q^0(M) := \{m \in M | Q^d m = 0 \text{ for some } d\}.$$

*In the following case, the local cohomology can be computed in Čech cohomology. Suppose $R$ is a noetherian ring and $Q = (x_1, \dots, x_t)$, for any $R$-module $M$, the local cohomology $H^i_Q(M)$ is the $i$-th cohomology of the complex*

$$C(x_1, \dots, x_t; M) : 0 \to M \xrightarrow{d} \oplus_{i=1}^t M[x_i^{-1}] \xrightarrow{d} \dots \to \oplus_{\#J=k} M[x_J^{-1}] \to \dots \to M[x_{1,2,\dots,t}^{-1}] \to 0.$$

$$d(m_J \in M[x_J^{-1}]) = \sum_{k \notin J} (-1)^{\#\{i \in J | i < k\}} m_{J \cup \{k\}}$$

*where $m_{J\cup\{k\}}$ denote the localization map $M[x_J^{-1}] \to M[x_{J\cup\{k\}}^{-1}]$.*

**Example 4.4** (regularity theorem is false if the sheaf is not an ideal sheaf)**.** *Let* $\mathscr{F} = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k), k \geq 0$*. Then*

$$\chi(\mathscr{F}(m)) = 2m + 2 = 2\binom{m}{0} + 2\binom{m}{1}$$

*which is independent of* $k$*, however*

$$\mathrm{H}^1(\mathbb{P}^1, \mathscr{F}(m-1)) = 0 \Rightarrow m \geq k - 1.$$

*Thus there's no such polynomial* $F(x_0, x_1)$*, such that* $\mathscr{F}$ *is* $F(a_0, a_1) = F(2,2)$*-regular, since* $F(2,2)$ *is just a constant.*

**Remark.** *This example and the original theorem can be found in Mumford's book 'Lectures on curves on surfaces, 14,15'.*

**Example 4.5** (some relations between (co)homology theories)**.** *We have*

- $H_i^{BM}(X) = H_i(\hat{X}, *)$
- $H_i^{BM}(X) = H_i(\overline{X}, \overline{X} \setminus X)$
- $H_i(X, A) \cong \widetilde{H}_i(X/A)$ *if* $A$ *is a deformation retraction of some neighborhood in* $X$*.*

**Remark** (a resonable space?)**.**

**Example 4.6** (Borel-Moore homology of $\mathbb{R}$ and $\mathbb{R}^2 \setminus \{0\}$)**.** *If we only want to see the results, just apply*

$$H_i^{BM}(\mathbb{R}^1) = H_i(S^1, *) = \widetilde{H}_i(S^1) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

*Similarly, we have*

$$H_i^{BM}(\mathbb{R}^2 \setminus \{0\}) = H_i(S^2, \{0, \infty\}) = \widetilde{H}_i(S^2/\{0, \infty\}) = \begin{cases} \mathbb{Z} & \text{if } n = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

*We can also use locally finite simplicial chains to compute them, take* $\mathbb{R}^2\{0\}$ *as an example, the computation above tells us a generator of* $H_1^{BM}(X)$ *is given by the ray connecting* $0$ *and* $\infty$*, on the other hand a generator of* $H_1(X)$*(the ordinary simplicial homology) is given by a circle surrounding* $0$*. In Borel-Moore homology this circle is a boundary of a locally finite cycle! In other words, the homomorphisms* $H_i(X) \to H_i^{BM}(X)$ *given by the definitions via chain complex are all* $0$*.*

**Remark.** *Note that*

- $H_i^{BM}(X)$ *is not homotopic invariant.*
- $H_i^{BM}$ *doesn't have the naive Poincaré duality.*
- *ordinary push-forward doesn't exist in general for* $H_i^{BM}(X)$*, just think about the inclusion* $\mathbb{R} \setminus \{0\} \to \mathbb{R}$*, then the push-forward of the cycle* $\sum_{i \geq 0} [\frac{1}{2^{i+1}}, \frac{1}{2^i}]$ *is not locally finite at* $0$*.*
- *If* $X$ *is nice enough,* $U \subset X$ *open,* $Y = X \setminus U$*, then we have*

  $$\cdots \to H_k^{BM}(Y) \to H_k^{BM}(X) \to H_k^{BM}(U) \to H_{k-1}^{BM}(Y) \to \dots$$

  *This is deduced form the long exact sequence associated to triple of spaces* $X \subset Y \subset Z$ *and the Poincaré duality for Borel-Moore homology.*

**Example 4.7** (Poincaré duality)**.** *We list several version of Poincaré duality*

- *Let $X$ be a closed subset of a smooth, oriented manifold $M$ with $\dim_{\mathbb{R}}(M) = n$, then we have*

$$H_i^{BM}(X) = H^{n-i}(M, M \setminus X)$$

$$H_{BM}^i(M) \cong H_{n-i}(M).$$

  *Note that we have to distinguish the first isomorphism and the definition*

$$H_i^{BM} = H_i(\hat{X}, *) = H_i(\overline{X}, \overline{X} \setminus X).$$

  *In other words, if $M$ is a compactification of $X$(this implies $X$ itself is smooth), then we must have*

$$H^{n-i}(M, M \setminus X) = H_i(M, M \setminus X).$$

- *Lefschetz duality. Let $M$ be an oriented compact manifold with boundary $X$, then we have*

$$H^k(M, X) = H_{n-k}(M)$$

$$H_k(M) = H_{n-k}(M, X).$$

  *Note that this is quite different from the relative homology used in Borel-Moore homology, since $X$, as the boundary of $M$, is closed in $M$ while in Borel-Moore theory, $M \setminus X$ is open in general.*

**Remark.** *intersection homology*

**Example 4.8** (Perverse cohomology, perverse sheaf)**.**

**Example 4.9** (Intersection cohomology, Grosky-MacPerson)**.**

**Example 4.10** (Intersection cohomology)**.**

$$IH^{n+*}(Y, L) := H^*(Y, IC(L))$$

**Example 4.11** ((co)homology theories related to perverse sheaves)**.** *We want to use some examples to illustrate the following (co)homology theories occur in the study of the decomposition theorem.*

- *ordinary (co)homology $H^i(X, \mathbb{Z})$.*
- *cohomology with support $H_Y(X, \mathscr{F})$.*
- *Borel-Moore homology $H_i^{BM}(X, \mathbb{Z})$.*
- *Sheaf cohomology with compact support $H_c^i(X, \mathscr{L})$.*
- *Stalk cohomology $H_x^i(A^\bullet)$.*
- *Cohomology sheaf $\mathbf{H}^i(A^\bullet)$.*
- $\underline{H}^r(A^\bullet)$.
- *Sheaf complex cohomology $H^i(X, A^\bullet)$ via injective resolution.*
- *Perverse cohomology ${}^{\overline{p}}H^i(A^\bullet)$.*
- *Intersection (co)homology with $\overline{p}$-perversity, $IH_i^{\overline{p}}(X)$.*
- *Relative intersection (cohomology) with perversity $\overline{p}$, $IH_i^p(X, X \setminus \{x\}; \mathscr{L})$.*

**Example 4.12** (Operations related to perverse sheaves)**.** *Some notations need in the theory of perverse sheaves.*

- *perversity.*
- *mapping cone.*
- *link.*
- *cone slice.*
- *injective resolution of a complex of sheaves.*
- *(co)support condition.*
- *truncation functor* ${}^{\overline{p}}\tau_{\leq 0}, {}^{\overline{p}}\tau^{\geq 1}$.
- *truncation functor* ${}^{\overline{p}}\tau_{\leq r}, {}^{\overline{p}}\tau^{\geq r}$.
- $Rj_*\mathscr{L}$ *is a complex of sheaves.*
- *Perverse functors such as* ${}^{\overline{p}}j_*, {}^{\overline{p}}j_!$.
- *support of sheaf.*
- *support of a global section.*
- *Intersection sheaves* $\mathrm{IC}(X, \mathscr{L})$.

**Example 4.13** ($j_*, j_!$)**.** *We encounter $j_*, j_!$ in different situations many times. For example*

- $j_*\mathscr{F}, R^k j_*$ *in ordinary sheaf theory.*
- $j_*\mathscr{F}, j_!\mathscr{F}, R^k j_*, Rj_*, R^\bullet j_*$ *in $K$-theory.*
- $Rj_*(A^\bullet), R^\bullet j_*, R^k j_*$ *in some derived category.*

*Relations between these constructions.*

*Exact sequence associated to these constructions.*

**Example 4.14** (Leray spectral sequence doesn't degenerate)**.** *The following two example are taken from Bhargav Bhatt's notes on perverse sheaves. First in the holomorphic setting Leray spectral sequence doesn't degenerate.*

**Remark.** *Deligne proves that if $f : X \to Y$ is a smooth projective morphisms of varieties over $\mathbb{C}$, then the Leray spectral sequence degenerates:*

$$E_2^{p,q} = H^p(Y, R^q f_* \underline{\mathbb{Q}}) \Rightarrow H^{p+q}(X, \underline{\mathbb{Q}}).$$

**Example 4.15** (Pathologies of non-abelian categories)**.** *Note the following*

- *Consider the category of finitely generated modules over $R = k[x_1, \ldots, x_n, \ldots]$. Then the kernel of the projection $R \to k$ doesn's exist.*
- *Consider the category of free abelian groups, then the cokernel might not exist. For example $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$.*
- *Vanishing of kernel and cokernel do not imply isomorphism.*

**Example 4.16** (Exactness depends on the ambient abelian category)**.** *Let $X = \mathbb{P}^1$, then the Euler sequence*

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathcal{O} \to 0$$

*is not exact in the category of presheaves.*

**Example 4.17** (Derived functors and derived categories). *Some references explicit derived categories Derived category*

**Example 4.18** (Non-isomorphic abelian categories, isomorphic derived categories). *Non-isomorphic abelian categories, isomorphic derived categories*

**Example 4.19** ($\mathbf{R}\Gamma(X, \mathscr{F}), \mathbf{R}\Gamma(A^\bullet), \mathbf{R}f_*(\mathscr{F}), \mathbf{R}f_*(A^\bullet)$).

**Remark** (derived functors(in derived categories, to get a complex), spectral sequence, hypercohomology).

**Example 4.20** (Perverse truncation and perverse cohomology). *To get some feeling about these constructions, first consider smooth algebraic varieties over $\mathbb{C}$, then the perverse truncation is just the ordinary truncation functor. Then if we only have two strata $X = U \amalg Z$, we can glue the two naive t-structures to get a non-trivial t-structure on $X$, and use the method described in Grosky's notes, we can get the perverse truncation functors as some complexes in a distinguish triangle. Let's illustrate what we said in an example.*

**Example 4.21** (Hartshorne II.5.14). *Let $X$ be a connected, normal closed subscheme of $\mathbb{P}^r_A$, where $A$ is a finitely generated $k$-algebra for some field $d$. We consider the relation between the homogeneous coordinate ring $S(X) = A[x_0, \ldots, x_n]/I, I = \Gamma_*(\mathscr{I})$ and $\Gamma_*(X, \mathcal{O}_X)$.*

*Proof.* We'll prove the following statements in steps

- $\Gamma = \Gamma_*(X, \mathcal{O}_X) = \oplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(n))$ is integral over $S = A[x_0, , \ldots, x_n]$.
- $\Gamma$ is integrally closed.
- $\Gamma_d = S_d$ for $d$ large enough.
- $S^{(d)} = \oplus_{n\geq 0} S_{nd}$ is integrally closed for $d$ large enough. Thus the $n$-tuple embedding is projectively normal.

First note that $S$ is an integral domain(otherwise how you can call it normal). And by definition

$$\Gamma = \cap_{i=0}^r S_{x_i}$$

thus for $\forall y \in \Gamma$, we can choose $N$ big enough, such that $x_i^N y \in S$ for all $i$. In other words

$$y \in S \bullet \frac{1}{x_i^N}.$$

Note that $S \bullet \frac{1}{x_i^N}$ is a finite module over $S$, thus we know $y$ is integral over $S$. Secondly to see $\Gamma$ is integrally closed, note that the degree 0 part $S_{((x_i))}$ of $S_{x_i}$ is integrally closed, since it's isomorphic to $A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$(this comes from the assumption that $X$ is normal, thus $S_{((x_i))}$ corresponding to the affine coordinate ring of $S \cap \mathbb{A}^r$, which is normal). And observe that

$$\Gamma^i ;= \{x \in S_{x_i} | deg(x) \geq 0\} \cong S_{((x_i))}[x_i]$$

and

$$\Gamma = \cap_{i=1}^r \Gamma^i$$

Since $S_{((x_i))}$ is integrally closed, so is $\Gamma^i = S_{((x_i))}[x_i]$, and hence $\Gamma$ is integrally closed. So we actually know $\Gamma$ is the integral closure of $S$. Thirdly, simply because $\Gamma$ is a finite module over $S$(general relation between a finitely generated $k$-algebra, domain and its algebraic closure). For example, let $\{y_1, \ldots, y_k\}$ be a basis, by the method above we have $S_{N_i} y_i \subset S$,thus if $d$ is big enough we have $\Gamma_d \subset S_d$, together with the fact that $S \subset \Gamma$, we know $S_d = \Gamma_d$ for large enough $d$.

Finally, to prove $\Gamma^(d)$ is integrally closed for large enough $d$, if $y \in Frac(\Gamma^{(d)})$ and $y$ is integral over $\Gamma^{(d)}$, we know $y \in \Gamma$, use the same trick again, we can prove

$$x_i^{dN} y^n \in S_{\geq dN + deg(y)} \Rightarrow x_i^{dN} y^n \in \Gamma^{(d)}, \forall n$$

$\Rightarrow$ is because $y \in \Gamma \cap Frac(\Gamma^{(d)})$, thus $y$ can be written of the form $\frac{y'}{x_i^{dk_1}}, y' \in \Gamma^{(d)}$, thus $d | dN + ndeg(y)$. So we get $x_i^{dN} y^n \in \Gamma^{(d)}, \forall n$. Then we conclude $y \in \Gamma^{(d)}$, $\Gamma^d$ is integrally closed for *all* $d$, thus if $d$ is large enough, we know $S^{(d)} = \Gamma^{(d)}$ is integrally closed.

As a corollary of the fact that $\Gamma$ is integrally closed, we know if $X \subset \mathbb{P}^r_A$ is normal closed subscheme, then it's projectively normal if and only if

$$\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n)) \to \Gamma(X, \mathcal{O}_X(n))$$

is surjective. For example nonsingular degree $d$ hypersurfaces are all projectively normal. $\square$

**Example 4.22** ($\Gamma_*(X, \mathcal{O}_X) \neq S(X)$)**.** *As a special case of this exercise, we can construct many projective schemes, such that*

$$\Gamma_*(X, \mathcal{O}_X) \neq S(X).$$

*For example consider the smooth quartic curve in* $\mathbb{P}^3$ *(details of this example can be found in my notes 'Hilbert schemes')*

$$C : \mathbb{P}^1 \to \mathbb{P}^3$$

$$[u, v] \mapsto [u^4, u^3 v, uv^3, v^4]$$

*Then we have*

$$H^0(\mathbb{P}^3, \mathscr{I}_C(1)) = 0, H^1(\mathbb{P}^3, \mathscr{I}_C(1)) \cong k$$

*Thus we have*

$$0 \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathbb{P}^3, \mathcal{O}_C(1)) \to k \to 0$$

*Thus $C$ cannot be projectively normal. How come? Actually we know $\{u^4, u^3v, uv^3, v^4\}$ comes from the restriction of $\{x, y, z, w\}$, however $u^2v^2$ doesn't come from this restriction. One way we can see this is by the isomorphism $\mathcal{O}_C(1) \cong \mathcal{O}_{\mathbb{P}^1}(4)$, another way is that we can construct it directly*

$$D_+(x) : u^2v^2 = (\frac{y}{x})^2 \bullet x$$

$$D_+(y) : u^2v^2 = (\frac{x}{y})(\frac{z}{y}) \bullet y$$

$$D_+(z) : u^2v^2 = (\frac{y}{z})(\frac{w}{z}) \bullet z$$

$$D_+(w) : u^2v^2 = (\frac{z}{w})^2 \bullet w.$$

**Remark** (relations between the homogeneous coordinate ring $S(X)$ and $\oplus_{n \geq 0} \mathcal{O}_X(n)$)**.** *Distinguish the following different concepts*

- $X = \mathrm{Proj}(S)$
- *The coordinate ring $S(X)$. This is a relative concept depending on the specific embedding. So in general, $S(X)$ is one of the $S$'s which can give us $X = \mathrm{Proj}(S)$.*
- $\Gamma(X, \mathcal{O}_X) \neq S$.
- $\Gamma_*(X, \mathcal{O}_X) = \oplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$. *Note that in general $\Gamma_*(X, \mathcal{O}_X) \neq S$.*
- *even if $S$ is an integral domain, $\Gamma(X, \mathcal{O}_X) \neq S_{(0)}$, the former is just the degree $0$ part of the latter.*
- *$X \cong Y$, but $\mathcal{O}_X(n) \neq \mathcal{O}_Y(n)$ in general(I mean if $f : X \to Y$, $f^*\mathcal{O}_Y(n) \neq \mathcal{O}_X(n)$ in general.)*

**Example 4.23** ('Cl' of the affine line with double origins)**.** *In Hartshorne's book, it's actually impossible to talk about $\mathrm{Cl}(\bar{\mathbb{A}}^1_k)$, but it seems to me, all definitions work and being nonseparated is bad for other reasons, but we definitely could compute $\mathrm{Cl}(\bar{\mathbb{A}}^1_k)$, $\mathrm{CaCl}(\bar{\mathbb{A}}^1_k)$ and $\mathrm{Pic}(\bar{\mathbb{A}}^1_k)$., using their definitions.*

**Example 4.24** (reduced schemes: functions are determined by their values)**.** *First recall that in a ring $R$, the nilpotent radical is given by*

$$nil(R) = \mathfrak{q} = \cap_{\mathfrak{p},prime}\mathfrak{p}$$

*And the point is that if* $\mathrm{Spec}(R)$ *is irreducible, it's equivalent of saying that the nilradical is a prime ideal, and geometrically just means $\mathfrak{q}$ is the prime ideal of 'the' generic point. In other words,*

- *on an irreducible affine scheme, $f$ vanishes at 'the' generic point means $f$ is nilpotent.*
- *on an reducible affine scheme, $f$ vanishes at 'a' generic point, then $f$ is nilpotent in some open subset of* $\mathrm{Spec}(R)$*.*

*Now if $f(\mathfrak{q}) = 0$ at the generic point, then $f \in \mathfrak{q}$, $f^k = 0$. But now our scheme if reduced thus $f = 0$. And general examples look like*

$$\mathrm{Spec}(k[x,y]/(x^m y^n))$$

- *$nil(R) = 0$. We have two minimal primes $\mathfrak{p}_1 = (x), \mathfrak{p}_2 = (y)$. But be careful $\mathfrak{p}_1$ is the generic point of the $y$-axis, and $\mathfrak{p}_2$ is the generic point of the $x$-axis.*
- *$f = xy$. $f(\mathfrak{p}) = 0, \forall \mathfrak{p} \in \mathrm{Spec}(R)$. However $f \neq 0$, which also means $f \neq nil(R)$.*
- *$f = x$, $f(\mathfrak{p}_2) = x \in k(\mathfrak{p}_2) = k(x)$. And $x \notin nil(R)$. However if you consider $U = D_+(y)$, then*

  $$U = \mathrm{Spec}((k[x,y]/(x^m y^n))_y) \cong \mathrm{Spec}(k[y, y^{-1}, x]/(x^m)).$$

  *Geometrically it can be interpreted as*

  - *$y^{-1}$ occurs since the origin is removed.*
  - *$x^m$ occurs means the nonreduced strcture on the '$y$'-axis is still there.*
  - *$f(\mathfrak{p}_1) = 0$, and we do have $f \in nil(k[y, y^{-1}, x]/(x^m))$.*
  - *If we consider the open subset $U' = D_+(x + y)$, then we can get*

    $$U' \cong \mathrm{Spec}((k[x,y]/(x^m y^n))_{x+y}) \ncong \mathrm{Spec}(k[x, x^{-1}, y]/(y^m) \times k[y, y^{-1}, x]/(x^m)).$$

    *Note that the last formula is $\ncong$. HOw come? Otherwise, you can consider the localization on $D_+((x + y)x)$, then it's clear*

    $$k[x, x^{-1}, y]/(y^n) \ncong (k[x, x^{-1}, y]/(y^n))_{x+y}$$

  - *we have to point out a silly thing,*

    $$k[x, x^{-1}]/(x^m) = 0$$

    *which means if we remove the origin from the nonreduced line* $\mathrm{Spec}(k[x,y]/(y^m))$*, we do get an affine scheme, but it's given by*

    $$\mathrm{Spec}(k[x, x^{-1}, y]/(y^m))$$

  - *More interesting, you can consider*

    $$C_k = \mathrm{Spec}((k[x,y]/(x^m y^n))_{y-x^k})$$

    *it's very surprising(for me, at least), that in general $C_k$'s are not isomorphic as affine schemes. Maybe I misunderstood something? The point is that although the underling topological spaces are the same, but some of them have more invertible functions than others.*

**Remark** (integral = reduced+ irreducible)**.** *So, next time, when you see 'integral', one thing you might think about is that the generic point is corresponding to the nilradical of the ring.*

**Example 4.25** (differences: $\mathbb{A}_k^1 - \{0\} \hookrightarrow \mathbb{A}_k^1$ and $\mathbb{A}_k^2 - \{0\} \hookrightarrow \mathbb{A}_k^2$). *Cover $\mathbb{A}_k^2 - \{0\}$ by $D_+(x)$ and $D_+(y)$, then we know $\Gamma(\mathcal{O}_{\mathbb{A}_k^2 - \{0\}}) = k[x, y]$, it cannot be affine. Base on this, we can also prove $\overline{\mathbb{A}_k^2}$ the affine plane with two origins is not separated. Since the intersection of two affine charts is not affine.*

**Example 4.26** (Separateness: a translation). *A scheme $X$ is separated if and only if $\{U_i\}$, an affine cover of $X$, such that*

- *each intersection is affine and*
- *$\Gamma(\mathcal{O}_X, U_i), \Gamma(\mathcal{O}_X, U_j)$ generate $\Gamma(\mathcal{O}_X, U_i \cap U_j)$.*

*Thus it's obvious now*

- *$\mathbb{P}_{\mathbb{Z}}^n$ is separated.*

*And this also tells us how to prove a scheme is not separated, Since if $X$ is separated over an affine scheme, then $\forall U, V$ open affine subset of $X$, we have $U \cap V = U \times_X V$ is affine. The reason is that by restricting $\Delta : X \to X \times_S X$ to $U \cap V$, we have a closed immersion $\Delta : U \cap V \hookrightarrow U \times_S V$, however $U \times_S V$ is affine, we know $U \cap V = U \times_X V$ is affine, more over we know $\Gamma(\mathcal{O}_X, U_i), \Gamma(\mathcal{O}_X, U_j)$ generate $\Gamma(\mathcal{O}_X, U_i \cap U_j)$. To show some scheme is not affine, this property is easy to use, for example*

- *$\overline{\mathbb{A}_k^1}$, $\Gamma(\mathcal{O}_X, U_1) = k[x], \Gamma(\mathcal{O}_X, U_2) = k[y]$, they can generate $\Gamma(\mathcal{O}_X, U_1 \cap U_2) \cong k[x, x^{-1}]$.*
- *the affine plane(or higher dimensions) with double origins is not separated, since the intersection $\mathbb{A}^2 - \{0\}$ is not affine. Or we can easily show $\Gamma(\mathcal{O}_X, U_1), \Gamma(\mathcal{O}_X, U_2)$ cannot generate $\Gamma(\mathcal{O}_X, U_1 \cap U_2)$.*

**Example 4.27** (generically reduced subscheme of a normal scheme but not reduced).

**Remark.** *This example comes from https://mathoverflow.net/questions/194296/is-being-reduced-a-generic-property-of-schemes/194299*

**Remark** (Some applications). *Use this to prove that*

- *The space of pairs of matrices $(X, Y)$ such that $XY, YX$ are upper-triangular is reduced.*
- *$T^*(\mathfrak{g} \times \mathbb{C}^n)$ is reduced. First you should ask what's your realization of $T^*(\mathfrak{g} \times \mathbb{C}^n)$ ?*

**Example 4.28** ($U \subset X$, open, but $\dim(U) \neq \dim(X)$). *Let $R = \mathbb{Z}_{(2)}[x]$, then $D_+(2) \cong \mathbb{Q}[x]$, however*

$$\dim(U) = 1, \dim(R) = 2$$

**Example 4.29** ($\dim(X) \neq \dim(\mathcal{O}_p)$, $p$ a closed point). *Use the same example as above, and let $p = \mathfrak{m} = (2t - 1)$, then $R/\mathfrak{m} \cong \mathbb{Q}$, we know*

$$\dim(X) = 2, \dim(\mathcal{O}_p) = ht(\mathfrak{m}R_{\mathfrak{m}}) = ht(\mathfrak{m}) = 1.$$

*The last identity is because $R$ is a* UFD, *and any principal prime ideal has height 1. What we want to say is in general*

$$\dim(R) \neq ht(\mathfrak{p}) + dim(A/\mathfrak{p}).$$

*And we define* $\operatorname{codim}(Y, X) = inf\{\mathcal{O}_{p,X} | p \in Y\}$, *then we don't have* $\operatorname{codim}(Y, X) + \dim(Y) = \dim(X)$, *just let $Y = \operatorname{Spec}(R/\mathfrak{m})$. Then* $\operatorname{codim}(Y, X) = 1, \dim(Y) = 0, \dim(X) = 2$.

**Remark.** *Note that the construction above is valid for any local ring $(R, \mathfrak{m})$. Let $\pi \in \mathfrak{m}$ be the uniformizer. Consider $R[T]$ and the maximal ideal $\mathfrak{n} = (\pi T - 1)$. We have another even more elementary method to prove that $\mathfrak{n}$ is maximal. Consider $f(T) \in R[T] \setminus \mathfrak{n}$, apply Euclidean algorithm, we can get $(f, \pi T - 1) = R[T]$. To illustrate just consider $f = T^n$, then $\pi T^n - (\pi T - 1)^{T^{n-1}} = T^{n-1}$, apply this computation inductively together with the fact $f \notin \mathfrak{n}$, we can get a unit in $R[T]$. What's more, we know* $\operatorname{codim}(Y, X) + \dim(Y) = \dim(X)$ *fails trivially if the ring is not integral. But here, $R[T]$ is actually as good as possible: integral domain, Noetherian, universally catenary, regular. You might ask, when this formula is true? We have two situations*

- *A finitely generated integral $k$-algebra.*
- *Cohen-Macaulay local ring (e.g regular local ring).*

*Somebody else also asked this question on math.stackexchange.*

**Example 4.30** (an étale morphism, Hartshorne III.10.6)**.** *We can use two copies of $\mathbb{A}_k^1$ to construct a degree 2 etale cover of the nodal curve*

$$f : X = \operatorname{Spec}(k[s,t]/(t^2 - (s^2-1)^2)) \to \operatorname{Spec}(k[x,y]/(y^2 - x^2(x+1))) = Y$$

$$x \mapsto s^2 - 1, y \mapsto st.$$

*then the fibre is given by*

$$k[x,y]/(x,y) \otimes_{k[x,y]/(y^2-x^2(x+1))} k[s,t]/(t^2 - (s^2-1)^2)$$

*in other words $st = y \otimes 1 = 0, s^2 - 1 = x \otimes 1 = 0$, we get*

$$X_0 = \operatorname{Spec}(k[s,t]/(t^2, s^2 - 1, st))$$

*Note that $x = xy^2 - x(y^2 - 1)$, thus it's actually*

$$\operatorname{Spec}(k[s,t]/(t, s^2-1)) = \operatorname{Spec}(k[s]/(s^2-1))$$

*all other fibres are the same(scheme theoretically), thus we know it's a flat morphism. We can use the fibre criterion for smoothness(this map is naturally of finite presentation), or observe that $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is an isomorphism. we know $f$ is smooth and of relative dimension 0, and $f$ is unramified. So $f$ is a degree 2 etale morphism.*

**Remark.** *Are there any easier ways to check the flatness of $f$ , for example using localization?. Let $\mathfrak{p} = (x,y)$, $\mathfrak{q}_1 = (s-1,t), \mathfrak{q}_2 = (s+1,t)$, then we have*

$$\mathcal{O}_{X,\mathfrak{q}_1} \ldots\ldots$$

**Example 4.31** (étale fundamental group of the nodal curve $y^2 = x^2(x+1)$)**.**

**Example 4.32** (Hartshorne III.10.5, étale locally free implies Zariski locally free)**.** *The proof is not hard, but let me ask*

- *why we need the étale condition?*
- *where we need to use the Nakayama's lemma?*

*For the first question, the answer is we can get a short exact sequence*

$$0 \to f^*kernel \to \mathcal{O}_U^r \to \mathscr{F} \to 0$$

*so you know we actually only need the flatness. For the second question , we have*

$$0 \to f^*kernel \otimes k(x) \to \mathcal{O}_{U,x}^r \otimes k(x) \to f^*\mathscr{F} \otimes k(x) \to 0.$$

*And use Nakayama's theorem, and the freeness of $f^*\mathscr{F}$, we know $f^*kernel \otimes k(x) = 0$. Then we get $f^*kernel = 0$, if $f$ is faithfully flat, we can get kernel $= 0$ directly, but here we don't need it, because firbre, we know $\mathcal{O}_{U,x}^r \to f^*\mathscr{F}$ is an isomorphism, so actually the original $\mathcal{O}_X^r \to \mathscr{F} \to 0$ is an isomorphism fibrewise. Then we know $f^*\mathscr{F}$ is locally free.*

**Example 4.33** (ETALE COHOMOLOGY OF CURVES)**.**

*https://static1.squarespace.com/static/57bf2a6de3df281593b7f57d/t/57bf67e76a49636398ee243f/1472161768284/cohom*

**Example 4.34** ($\mathbb{A}^1$ is not algebraically simply connected)**.**

**Remark** ($\mathbb{P}^1$ is algebraically simply connected)**.**

**Example 4.35** (a group scheme, not smooth, nonperfect field, $char(k) = p > 0$)**.** *Let $k$ be a field of characteristic $p > 0$, $\alpha \in k$ but not a pth power. The closed subgroup scheme*

$$V(x^p + \alpha y^p) \subset \mathbb{G}_{a,k}^2$$

*is reduced and irreducible but not smooth, not even normal.*

**Remark** ($char(k) = p > 0$, non-reduced group schemes)**.** *We can equip* $\mathrm{Spec}(k[x]/(x^p))$ *with two different group scheme structures by the following two exact sequence*

$$0 \to \mu_p \to \mathbb{G}_m \xrightarrow{x \mapsto x^p} \mathbb{G}_m \to 0$$

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{x \mapsto x^p} \mathbb{G}_a \to 0$$

*To be more precise, the functor $\mu_p : S \mapsto \mu_p(S) = \{pth\ roots\ of\ unity\ in\ S\}$ is represented by the scheme* $\mathrm{Spec}(k[x]/(x^p - 1))$*, the functor $\alpha_p : S \mapsto \mu_p(S) = \{pth\ nilpotent\ elements\ in\ S\}$ is represented by the scheme* $\mathrm{Spec}(k[x]/(x^p))$*, however in characteristic $p$, these two schemes are isomorphic, although the two functors(group schemes) are not isomorphic. And this doesn't mean the ring $k[x]/x^p$ has two different group structures, instead, we really have to think about the Hopf algebra structures. For $\mu_p$, we have*

- *composition $c : A \to A \otimes A, x \mapsto U \otimes V$.*
- *unit $e : A \to k, x \mapsto 1$.*
- *inverse $i : A \to A, x \mapsto x^{n-1} = x^{-1}$.*

**Example 4.36** (Hartshorne III.10.7, a linear system with moving singularities outside the base locus)**.** *Let $k$ be an algebraically closed field of characteristic $2$, and $\mathfrak{d}$ be the linear system of cubic curves in $\mathbb{P}_k^2$ passing $\mathbb{P}_{\mathbb{F}_2}^2$ (i.e $7$ points, $[1, 0, 0], \dots$ ).Then we know*

$$\mathfrak{d} = \{a(x^2y + xy^2) + b(x^2z + xz^2) + c(y^2z + yz^2) | a, b, c \in k\}$$

*thus it's a $2$-dimensional linear system. It's easy to check that the base locus contains exactly the $7$ points, we donote it by $T = \{p_1, \dots, p_7\}$, now we get a morphism*

$$\mathbb{P}_k^2 - T \to \mathbb{P}_k^2$$

$$[x, y, z] \mapsto [x^2y + xy^2, x^2z + xz^2, y^2 + yz^2].$$

*Then it determines a field extension, we can compute it by restricting to any open subset, say $D_+(z)$, we get*

$$\phi : k(u, v) \hookrightarrow k(x, y)$$

$$u \mapsto \frac{x(x + y)}{y + 1}, v \mapsto \frac{x(x + 1)}{y(y + 1)}$$

*We don't distinguish $u, v$ and their images under $\phi$. Then this field extension is generated by $x$ and $y$, however*

$$yv + u = x$$

*So $k(x, y) = k(u, v)[y]$, and just plug $x = yv + u$ in $\phi$, we get*

$$u = \frac{(yv + u)(yv + u + y)}{y + 1} \Rightarrow y^2 - \frac{u^2 + u}{v^2 + v} = 0$$

*this is the minimal polynomial of $y$ and it's inseparable since its derivative is $0$, in conclusion we get a purely inseparable morphism $\mathbb{P}_k^2 - T \to \mathbb{P}_k^2$ with degree $2$. Every curve in $\mathfrak{d}$ at a unique point $p = (\sqrt{a}, b, \sqrt{c})$, then you can check if $p$ is $p_i$ for some $i$, then $C$ is the union of three lines passing $p_i$, otherwise it's a cuspidal cubic and $p \neq p_i$, thus we see the singularities is moving around in $\mathbb{P}_k^1$. For the ordinary Bertini theorem to be true, the characteristic of the field is essential.*

**Example 4.37** (Hartshorne III.10.8, a linear system with moving singularities contained in the base locus, any characteristic). *In $\mathbb{P}^3_k = \mathrm{Proj}(k[x, y, z, w])$, consider*

$$\{uw + t((x - z)^2 + y^2 - z^2)|u, v \in k\}$$

*Then it's a 1-dimensional linear system, and the base locus is the conic $\{w = 0, (x - z)^2 + y^2 - z^2\}$ plus the $z$-axis with singularities moving along the $w$-axis.*

## 4.1 rationality

**Example 4.38** (a geometrically rational curve but not rational). *Consider the curve defined over $k = \mathbb{Q}$*

$$C : x^2 + y^2 = pz^2$$

*where $p$ is a prime number congruent to $3(mod 4)$. Then we know $C$ has no rational points, thus we cannot have a birational map from $\mathbb{P}_{\mathbb{Q}^1}$ to $C$. However if we base change to $K = \mathbb{Q}(\sqrt{p})$, we have a birational morphism*

$$\mathbb{P}^1_K \to C_K; t(t^2 - 1, 2t, \frac{1}{\sqrt{p}}(t^2 + 1)).$$

*Thus we know $C_K = C \times_k K$ is a rational curve over $K$.*

**Remark.** *See the following link for more information. https://rigtriv.wordpress.com/2008/09/27/rational-varieties-an-introduction-through-quadrics/*

*And maybe it's good to know*

- *A geometrically rational smooth curve can be embedded into $\mathbb{P}^2_k$ as a smooth conic. Simply because*

$$H^0(C_K, T_{C_K}) \cong H^0(C, T_C) \otimes_k K.$$

- *By the Hasse-Minkowski theorem, a smooth quadric hypersurface defined over the rational numbers is rational over $\mathbb{Q}$ if and only if it has a point defined over $\mathbb{R}$.*
- *Quadric hypersurfaces (which are not the union of two planes) are always rational over finite fields.*
- *Cubic surfaces over an algebraically closed field are always rational.*

# 5 Ampleness

**Example 5.1** (an effective divisor but not ample). *Consider any smooth cubic surface $X \subset \mathbb{P}^3$, for example*

$$X : x^3 + y^3 + z^3 + w^3.$$

*Then $L : \{x + y = 0, z + w = 0\}$ lies on $X$, however by the adjunction formula*

$$-2 = K_L = (K_X + L)L = -HL + L^2 \Rightarrow L^2 = -1.$$

*Then we know $L$ is an effective divisor but not an ample divisor.*

**Remark.** *This example just says that $N_{L/X}|_L \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, so actually any effective divisor has similar properties serves as a counterexample, for example*

- *A line on a special quadric surface*

$$X : xz - yw = 0$$
$$L : x = y = 0$$

*then $X$ is smooth, and by similar argument, we get $L^2 = 0$, note that this is just $X = \mathbb{P}^1 \times \mathbb{P}^1$, so we know two generators of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ are effective, but not ample. However $L_1 + L_2$ is ample, actually, very ample.*

- *the exceptional divisor of $BL_p(\mathbb{P}^2)$.*

**Example 5.2.** *(effective cycles, negative self-intersection numbers) Note that, we can't talk about ampleness if a cycle is not a divisor(codimension 1), but we can still get many interesting phenomenons,*

- *Lines on quintic three-folds. We know a general quintic three-fold contains 2875 lines, but not all smooth quintic three-folds have this property, but for our purpose, this is actually an advantage, for example consider the Fermat three-fold*

$$X : x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$$

*$X$ contains infinitely many lines(divide variables into several groups of size at least 2, and make sure one of them is of size as least 3).*

$$L = [t, 0, -t, s, -s] : (x_0 + x_2, x_1, x_3 + x_4)$$

*is one of those lines, as you can see, the adjunction formula doesn't tell us what is $L^2$ anymore,*

$$-2 = K_L = K_X|_L \otimes \wedge^2 N_{L/X}|_L$$

*we can only get $\wedge^2(N_{L/X})|_L \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, to compute $L^2$, we know it's 0 by the definition of the Chow ring, we can also see this from*

$$L^2 = c_2(N_{L/X}) = 0$$

*We point out that here since $L$ is not a divisor on $X$, we* **can't** *use the ordinary adjunction formula:*

$$-2 = K_L = (K_X|_L + L)L = L^2$$

- *Conics on quintic three-folds.*

- *I also know an example of a rational curve on a quintic three-fold, i.e*

$$i : \mathbb{P}^1 \to \mathbb{P}^4$$

$$[x, y] \to [x^2, -x^2, y^2, -y^2, xy].$$

*Then the image is a rational curve on*

$$X : x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - x_0x_1x_2x_3x_4 = 0$$

*has the property that*

$$N_{C/X}|_C \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

*Let me review the computation briefly, by a linear transformation, $C$ can be parametrized by $[u^2, uv, v^2, 0, 0]$, and its equation is given by $(y^2 - xz)f + sf_1 + tf_2$, we need several steps*

  - *on the level of $\mathbb{P}^4$, we have the Euler sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^4}|_C \to \mathcal{O}_{\mathbb{P}^4}^5(1)|_C \to T_{\mathbb{P}^4}|_C \to 0.$$

  *we know the first map is given by $[x, y, z, s, t]$, so to understand the cokernel $T_{\mathbb{P}^4}|_C$, we only need to ask ourselves, which map kills exactly the image of the first map? It the relations of $[u^2, uv, v^2, 0, 0]$, that is the matrix*

$$\begin{bmatrix} v & -u & & & \\ & v & -u & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

*this is the map* $\mathcal{O}^5_{\mathbb{P}^4}(1) \to T_{\mathbb{P}^4}|_C$. *Then we get*

$$T_{\mathbb{P}^4}|_C = \mathcal{O}_{\mathbb{P}^1}(3)^2 \oplus \mathcal{O}^2_{\mathbb{P}^1}(2).$$

– *on the level of the surface, we have the tangent sequence*

$$0 \to T_X|_C \to T_{\mathbb{P}^4}|_C \to N_{X/\mathbb{P}^4}|_C \to 0$$

*Since we know* $N_{X/\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(5)$, *thus we know* $N_{X/\mathbb{P}^4}|_C = \mathcal{O}_{\mathbb{P}^1}(10)$. *To understand the map*

$$[a, b, c, d] : T_{\mathbb{P}^4}|_C \to N_{X/\mathbb{P}^4}|_C$$

*we need to consider the composition(i.e the image of the genrators of* $T_{\mathbb{P}^4}$, $\frac{\partial}{\partial x_i} \mapsto \frac{\partial F}{\partial x_i}$*)*

$$\mathcal{O}^5_{\mathbb{P}^3}(1)|_C \to T_{\mathbb{P}^4}|_C \to N_{X/\mathbb{P}^4}|_C$$

*so we know this can be written as a multiplication of matrices*

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} v & -u & & & \\ & v & -u & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & \frac{\partial F}{\partial s} & \frac{\partial F}{\partial t} \end{bmatrix}$$

*then we can get*

$$[a, b, c, d] = [-vf, uf, f_1, f_2]$$
$$f = 2u^3v^3, f_1 = u^8 + u^3v^5, f_2 = u^5v^3 + 5v^8.$$

*Now to compute the kernel of the tangent sequence, we have to ask ourselves, this map kills who? This just means what's the relations between* $a, b, c, d$*? Using Macaulay2, we can find* $a, b, c, d$ *satisfy a degree* 8 *relation and* 2 *degree* 11 *relation, which means, the first map in the tangent sequence is given by*

$$\begin{bmatrix} u & * & * \\ v & * & * \\ 0 & ** & ** \\ 0 & ** & ** \end{bmatrix}$$

*where* $*$*'s are some degree* 4 *polynomials in* $u, v$, *and* $**$*'s are some degree* 3 *polynomials.Thus we know the kernel is given by*

$$T_X|_C \cong \mathcal{O}_{\mathbb{P}\not{1}(2)\oplus\mathcal{O}^2_{\mathbb{P}^1}(-1)}.$$

– *on the level of* $C$, *we also have a tangent sequence*

$$0 \to T_C \to T_X|_C \to N_{C/X}|_C \to 0$$

*we know* $T_C = \mathcal{O}_{\mathbb{P}^1}(2)$, *and* $Hom(\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, *we know what we want is given by the cokernel*

$$N_{C/X}|_C = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

**Example 5.3** (normal bundle of the rational cubic curve $i : C \to \mathbb{P}^3$)**.** *This example seems easier at first, it actually very interesting. We give three proofs, one is by direct computation, the second one is by using some basic knowledge of representation theory. First we can use representation theory.*

- $i^*T_{\mathbb{P}^4} = \mathcal{O}^3_{\mathbb{P}^1}(4)$.
- *we have*
$$0 \to T_C \to i^*T_{\mathbb{P}^3} \to N_{C/\mathbb{P}^3}|_C \to 0$$
*computing the degree we know* $N_{C/\mathbb{P}^3}|_C = \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$ *or* $\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6)$.
- *Tensoring with* $\mathcal{O}_{\mathbb{P}^1}(-6)$ *and then compute the cohomology, we get*
$$0 \to H^0(N_{C/\mathbb{P}^3}) \to H^1(\mathcal{O}_{\mathbb{P}^1}(-4))$$
*If we view the twisted cubic curve as*
$$\mathbb{P}(V) \to \mathbb{P}(S^3V)$$
*then as a representation of* $GL(V)$, *we have*
$$H^1(\mathcal{O}_{\mathbb{P}^1}(-4)) \cong H^0(\mathcal{O}_{\mathbb{P}^1}(2))^\vee \cong S^2V.$$
$S^2V$ *is an irreducible representation of* $GL(V)$, *so* $\dim H^0(N_{C/\mathbb{P}^3})$ *is either* $0$ *or* $3 = \dim(S^2V)$, *on the other hand, we have*
$$\dim H^0(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) = 0, \dim H^0(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}) = 1.$$
*Thus we know*
$$N_{C/\mathbb{P}^3}|_C \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5).$$

*I read the second method on Math Stackexchange. If you want to do concrete computation, follow http://math.stackexchange.com/questions/1583326/normal-bundle-of-twisted-cubic/16035251603525*

*This answer is given by Ted Shifrin on Mathstackexchange. "My favorite way to do this is to work with the conormal bundle instead, seeing very concretely what its transition functions are. Here is a bit to get you started. We have*
$$0 \to N^*_{C/\mathbb{P}^3} \to T^*\mathbb{P}^3\big|_C \xrightarrow{\phi} T^*C \to 0.$$
*In the "main" chart* $[1, x, y, z]$, *where* $C$ *is parametrized by* $(t, t^2, t^3)$, *we have*
$$\begin{aligned}\phi(dx) &= dt\\ \phi(dy) &= 2t\,dt\\ \phi(dz) &= 3t^2\,dt,\end{aligned}$$
*and in the chart "at infinity,"* $[X, Y, Z, 1]$, *where* $C$ *is parametrized by* $(s^3, s^2, s)$, *we have*
$$\begin{aligned}\phi(dX) &= 3s^2\,ds\\ \phi(dY) &= 2s\,ds\\ \phi(dZ) &= ds.\end{aligned}$$
*Thus, on the respective charts we have the relations*
$$dy - 2x\,dx = dz - 3x^2\,dx = 0 \qquad dX - 3Z^2\,dZ = dY - 2Z\,dZ = 0.$$
*Now, on the overlap of the two charts, the frame for the conormal bundle in the first chart is given by the two sections*
$$\sigma_1 = -2t\,dx + dy \quad \textit{and} \quad \sigma_2 = -3t^2dx + dz,$$

*and performing the change of coordinates (here is where you have to do some work), the frame in the second chart is given by the two sections*

$$\tau_1 = -3t^{-5}dy + 2t^{-6}dz \quad \textit{and} \quad \tau_2 = t^{-3}dx - 2t^4dy + t^{-5}dz.$$

*Thus, we have*

$$\begin{aligned} \tau_1 &= -3t^{-5}\sigma_1 + 2t^{-6}\sigma_2 \\ \tau_2 &= -2t^{-4}\sigma_1 + t^{-5}\sigma_2. \end{aligned}$$

*Now, the usual row- and column-operation (Smith normal form) game shows that, after change of basis, we obtain frames satisfying*

$$\begin{aligned} \tau_1' &= -3t^{-5}\sigma_1' \\ \tau_2' &= -\frac{1}{3}t^{-5}\sigma_2', \end{aligned}$$

*from which we see that the conormal bundle is $\mathcal{O}_{\mathbb{P}^1}(-5) \oplus \mathcal{O}_{\mathbb{P}^1}(-5)$." Thirdly, I can also give an easier way to do the computation. We only need to understand the sequence*

$$0 \to N_{C/\mathbb{P}^3}|_C \to \mathcal{O}^3_{\mathbb{P}^1}(-4) \to \mathcal{O}_{\mathbb{P}^1}(-2) \to 0$$

*The second map in this sequence is*

$$[3t^2, 2t, t] = [3u^2, 2uv, v^2]$$

*we emphasize that working in homiogeneous coordinates is crucial. Then it's easy to get the relations between $u^2, uv, v^2$, which gives us the matrix form of the first map*

$$\begin{bmatrix} -2v & 0 \\ 3u & -v \\ 0 & 2u \end{bmatrix}$$

*Then since the target is $\mathcal{O}^3_{\mathbb{P}^1}(4)$, we easily get*

$$N^\vee_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(-5) \oplus \mathcal{O}_{\mathbb{P}^1}(-5).$$

**Example 5.4** (some more interesting facts about twisted cubic in $\mathbb{P}^3$)**.** *I don't want to write down the concre computation once more, let me just state the result. Let $X$ be the quadratic surface $X : xw - yz = 0$*

- $i^*T_{\mathbb{P}^3} = \mathcal{O}^3_{\mathbb{P}^1}(4)$.
- $i^*T_X = \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$.
- $i^*N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(4)$.
- *By basic intersection theoretic fact, we have*

  $$[C]^2 = i_*c_1(N_{C/X}|_C) = 4 \in A(X).$$

  *Note that the ambient space is important, if you work in $A(\mathbb{P}^3)$, we know $[C]^2 = 0$.*

- *$X \cong \mathbb{P}^1 \times \mathbb{P}^1$, actually $[C] = (1,2) \in A(\mathbb{P}^1 \times \mathbb{P}^1)$. This can be seen by direct computation(or you can consider the ideal of $X \cap Y$ and remove the residue $L$, you shall get a homegeous polynomial of bidegree $(1,2)$.)*
$$\mathbb{P}^1 \times \mathbb{P}^1 : [a,b] \times [A,B] \mapsto [aA, aB, bA, bB].$$
*Consider two ruling $\mathbb{P}^1$'s,*
$$L_1 = [0,0,A,B]; L_2 = [0,a,0,b]$$
*and the twisted cubic curve*
$$C : [u^3, u^2v, uv^2, v^3]$$
*Thus*
  - *$C \cap L_1 = [0,0,0,1]$, to compute the multiplicity, we can work in $D_+(w)$, thus it's given by*
  $$L_1 : (0,0,z); C : (t^3, t^2, t) \Rightarrow C \cap L_1 = 2$$
  *since their tangent space at the point $(0,0,0)$ are the same!*
  - *$C \cap L_2 = [0,0,0,1]$, however in this situation we have*
  $$L_2 : (0,z,0); C : (t^3, t^2, t) \Rightarrow C \cap L_2 = 1$$
  *since the intersection is transverse.*
- *Let $Y : yw - z^2 = 0, Z : xz - y^2 = 0$, then we know we have a family of quadrics $t_0Y + t_1Z$, the intersection of each quadric in this family with $X$ is given by the union of $C$ and a line $L_t$. And apply Bezout's theorem*
$$([C] + [L])^2 = (2h)^2 = 4h^2 = 4deg(X) = 8.$$
*Thus we have*
$$[C][L] = 2, [L]^2 = 0$$
*The interesting thing here is that $C$ is a cubic curve, and $L$ is a line, but the intersection number on $X$ is 2! To be less surprising, it's simply because each $L$ is tangent to $C$.*
- *I discussed these facts with Shizhang, and he told me we have another beautiful way to understand the geometry here, that is $X = \mathbb{P}^1 \times \mathbb{P}^1$ is actually a $\mathbb{P}^1$ fibration over the twisted cubic curve. And we can get a morphism from a quadric to a twisted cubic curve!*
$$\begin{tikzcd} X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & \mathbb{P}^1 \\ \mathbb{P}^1 & \end{tikzcd}$$
*when restricted to $C$, one of $\pi_1, \pi_2$ is an isomorphism and the other one is a degree 2, and the second map ramifies at 2 points.*

**Example 5.5** (an ample line bundle, but not very ample)**.** *Let $X$ be a smooth cubic curve and consider the line bundle $\mathscr{L}(p)$.*

**Example 5.6** (an ample line bundle with negative degree)**.**

**Example 5.7** (a (smooth) vector bundle but not a (holomorphic, algebraic) vector bundle, Shizhang & Zijun)**.** *Consider*
$$X = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \to \mathbb{P}^1,$$
$$(x,y) \mapsto x,$$
*where $\Delta$ is the diagonal. Then*

- *$X$ is a topological vector bundle and isomorphic to (topologists') $\mathcal{O}_{\mathbb{P}^1}(-2)$.*
- *$X$ is not a holomorphic, not an algebraic line bundle, it's not the total space of (geometers') $\mathcal{O}_{\mathbb{P}^1}(-2)$.*
- *$X$ is diffeomorphic to the holomorphic bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$ and also diffeomorphic to $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2))^{an}$.*

*The first statement comes from we have an Segre embedding*

$$s : \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \to \mathbb{P}^3,$$

$$([x, y], [z, w]) \mapsto [xz, xw, yz, yw].$$

*Denote the coordinates on $\mathbb{P}^3$ be $[X, Y, Z, W]$, then the image a closed subvariety in the complement of the hyperplane $V(Y - Z)$, thus affine. Then we know no section $\sigma : \mathbb{P}^1 \to X$ exists! $X$ is not a vector bundle in the world of algebraic geometry, it's just an affine variety, has a morphism to $\mathbb{P}^1$, every fibre over a closed point is isomorphic to $\mathbb{A}^1$, that's it. However, in a topologist's eyes, the picture is different. Let's work over $\mathbb{C}$. We do have a section(this section is not algebraic, not holomorphic)*

$$\sigma : \mathbb{P}^1_{\mathbb{C}} \cong S^2 \to S^2 \times S^2 \setminus \Delta,$$

$$x \mapsto (x, -x).$$

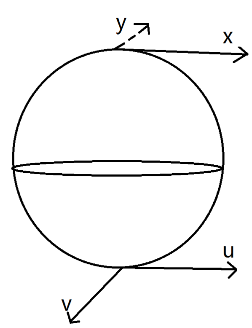



Figure 2: An orientation of $S^2$

Now we have to prove it's topologists' $\mathcal{O}(-2)$. We first have to talk about 'an orientation on $S^2$', for me, this means:

find charts in a compatible way, i.e $\det J > 0$.

For $S^2$, we have spherical projections and it has a natural complex structure, then $z = x + iy, w = u + iv$ gives us a natural orientation

$$(U_0, (x, y)), (U_1, (u, v)); u = \frac{x}{x^2 + y^2}, v = \frac{-y}{x^2 + y^2}.$$

We can define a deformation of the antipodal section

$$\sigma_t : S^2 \to S^2 \times S^2 \setminus \Delta,$$

$$x \mapsto (x, \rho_t(-x)),$$

where $\rho_t$ is the rotation w.r.t to the line connecting the north($N$) and south($S$) poles by an angle $t$ (later, it'll be clear the flow or deformation we choose is not important, it's doesn't matter it's orientation preserving or not, the intersection number of two sections are important). To do computations, we have to work in the $(x, y, u, v)$ coordinate system. Near the north pole of the source two sections are given by

$$\sigma : x = x, y = y, u = -x, v = y,$$

$$\sigma_t : x = x, y = y, u = (-x)cos(t) + ysint(t), v = (-x)(-sin(t)) + ycos(t).$$

$$\sigma \cap \sigma_t = \{p = (N, S), q = (S, N)\} \subset S^2 \times S^2.$$

We only need to compute the index of the intersection. It's given by the sign of

$$A = \begin{pmatrix} \frac{\partial \sigma}{\partial x} \\ \\ \frac{\partial \sigma}{\partial y} \\ \\ \frac{\partial \sigma_t}{\partial x} \\ \\ \frac{\partial \sigma_t}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ \\ 0 & 1 & 0 & 1 \\ \\ 1 & 0 & -cos(t) & sin(t) \\ \\ 0 & 1 & sint(t) & cos(t) \end{pmatrix}.$$

$$\det(A) = -2 + sin(2t).$$

Since $t$ is small, this is negative, the index is $-1$! Same computation for the other intersection point $q$. Thus we get the degree of the bundle is $-2$. All our claims are clear now.

**Remark.** *Think about the definition of the section*

$$\sigma : x = x, y = y, u = -x, v = y,$$

*actually, it's more clear if we write it as*

$$\sigma : x = x', y = y', u = -x', v = y',$$

*where $(x', y')$ is the coordinates in the source around the north pole $N$, $(x, y, u, v)$ is the coordinates in the target around the point $(N, S)$. We can also use de Rham cohomology to compute everything, maybe we want to leave it as an exercise to the readers who have plenty of free time....*

**Remark** (index, degree)**.** *Compare the following concepts*

- *The index of a fixed point of an automorphism $f : X \to X$.*

  $$i(p) := 1; \det(1 - f_*|_{T_pX}).$$

- *Lefschtz index of $f : X \to X$*

  $$\Lambda(f) = \sum_k (-1)^k \mathrm{Tr}(f^* : H^k(X) \to H^k(X)).$$

  *Actually, we have $\Lambda(f) = \sum_{p \in \mathrm{Fix}} i(p)$.*

- *index of a vector field. Poincaré–Hopf theorem for compact manifold.*

- *index of the intersection of two maps $f : X \to Y$, $g : Z \to Y$.*

- *degree of a map $f : X \to Y$ (in differential topology) v.s degree of a morphism $f : X \to Y$ (in algebraic geometry).*
- *degree of 'the' top Chern class.*
- *intersection index.*
- *intersection number(in differential topology) v.s. intersection number (in algebraic geometry). As far as I can see, for those concepts in topology, we only need to remember the definition of the intersection index of two smooth maps. For example, the degree of a smooth map $f : X \to Y$ is actually the intersection index of cycles $[X \times y]$ and $[\Gamma] = [(x, f(x))]$ in $X \times Y$. More specially, $[\Delta][\Gamma]$ gives us $\sum_{p \in \mathrm{Fix}} i(p)$.*

**Example 5.8** (Pull-back a bundle along an isomorphism might not get an isomorphic bundle)**.** *Let $E$ be an elliptic curve, then for $p, a \in E$, $\mathcal{O}(p) \cong \mathcal{O}(p + a)$ if and only if $a = 0 \in E$. Let $\tau_a$ be the translation by $a$. It's an isomorphism of $E$. However*

$$\mathcal{O}(p) \neq \mathcal{O}(p + a) = \tau_a^*(\mathcal{O}(p)).$$

*On the other hand, if $X$ is a smooth variety and $G$ the finite automorphism group of $X$. We have*

$$\mathrm{Pic}^G(X) \cong \{\textit{Equivariant Weil divisor}\}/\mathrm{div}\{G\textit{-invariant functions}\}.$$

*First by Hilbert 90, we know $H^1(G, \mathcal{O}(X)^*) = 0$, this tells us $\mathrm{Pic}^G(X)$ is embedded into $\mathrm{Pic}(X)$. Then we define a homomorphism*

$$\phi : \mathrm{Pic}(X) \to \{\textit{Equivariant Weil divisor}\}/\mathrm{div}\{G\textit{-invariant functions}\}$$

$$D + \mathrm{div}(f) \mapsto \frac{1}{|G|} \sum g^* D + \mathrm{div}(\frac{1}{|G|} \sum g^* f)$$

**Example 5.9** (Horrocks–Mumford bundle)**.** *We compute the Euler characteristic of the bundle $\mathscr{F}$. Since we know*

$$c(\mathscr{F}) = 1 + 5h + 10h^2.$$

*It's standard to apply the Hirzebruch–Riemann–Roch theorem whch says*

$$\chi(\mathscr{F}(n-5)) = \{Ch(\mathscr{F}(n-5)Td(T_X))\}_0.$$

*The computation is done with the help of Macaulay2,*

$$Ch(\mathscr{F}(n-5)) = \frac{1}{12}(n-5)^4h^4 + \frac{5}{3}(n-5)^3h^3 + \frac{125}{12}(n-5)^2h^2 + \frac{125}{6}(n-5)h + 2$$

$$Td(X) := Td(T_X) = h^4 + \frac{25}{12}h^3 + \frac{35}{12}h^2 + \frac{5}{2}h + 1.$$

*Hence*

$$\chi(\mathscr{F}(n-5)) = \frac{(n^2-1)(n^2-24)}{12}.$$

*However, we can also compute $\chi(\mathscr{F}(n-5))$ by the construction of the bundle directly, namely we have*

- $\mathscr{F} := ker(q)/im(p)$

$$0 \to \mathcal{O}(2) \otimes V_1 \xrightarrow{p} \wedge^2 T_X \otimes W \xrightarrow{q} \mathcal{O}(3) \otimes V_3 \to 0$$

- *Koszul complex of the tangent bundle*

$$0 \to \mathcal{O} \to \mathcal{O}(1) \otimes V \to \mathcal{O}(2) \otimes \wedge^2 V \to \wedge^2 T_X \to 0.$$

*Then we have*

$$\begin{aligned}\chi(\mathscr{F}(n-5)) &= \chi(\wedge^2 T_X(n-5) \otimes W) - \chi(\mathcal{O}(n-3) \otimes V_1) - \chi((\mathcal{O}(n-2) \otimes V_3)) \\ &= 2\chi(\wedge^2 T_X) - 5\chi(\mathcal{O}(n-3)) - 5\chi(\mathcal{O}(n-2)) \\ &= 2(10\chi(\mathcal{O}(n-3)) + \chi(\mathcal{O}(n-5)) - 5\chi(\mathcal{O}(n-4))) \\ &\quad -5\chi(\mathcal{O}(n-3)) - 5\chi(\mathcal{O}(n-2)).\end{aligned}$$

*Recall that*

$$\chi(\mathcal{O}_{\mathbb{P}^4}(n)) = \binom{n+4}{4}$$

*we get*

$$\begin{aligned}\chi(\mathscr{F}(n-5)) &= 15\binom{n+1}{4} - 10\binom{n}{4} + 2\binom{n-1}{4} - 5\binom{n+2}{4} \\ &= \frac{(n^2-1)(n^2-24)}{12}.\end{aligned}$$

**Example 5.10** ($H^0(N_{Z/X}) \neq H^0(N_{Z/X}|_Z)$)**.** *This is silly, consider $I = (x), R = \mathbb{C}[x]$, then we know*

$$\mathrm{Hom}_R(I/I^2, R) = 0.$$

*This is because $\forall f \in \mathrm{Hom}_R(I/I^2, R), f(x^2) = xf(x) = 0 \Rightarrow f(x) = 0$. This also tells us as a $R$-module, $(I/I^2)^\vee = 0$(note that $\dim_{\mathbb{C}} H^0(I/I^2) = 1$). However,*

$$\mathrm{Hom}_{R/I}(I/I^2, R/I) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}.$$

*For example, we have*

$$T_Z\mathrm{Hilb}(X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) = H^0(N_{Z/X}|_Z).$$

*If $X$ is smooth and 1-dimensional, $Z$ is a 0-dimensional subscheme, together with Serre duality, we have*

$$T_Z\mathrm{Hilb}(X) = H^0(T_X^\vee \otimes \mathcal{O}_Z)^\vee.$$

*Specially for $I = (x^n)$, we know $H^0(T_X^\vee \otimes \mathcal{O}_Z) = \{x, x^2, \ldots, x^n\}$, as a torus representation, we get*

$$T_{(x^n)}\mathrm{Hilb}(\mathbb{C}) = t + \cdots + t^n.$$

*This also agree with the global coordinates of $\mathrm{Hilb}(\mathbb{C}) \cong \mathbb{C}^n$ given by $(a_1, \ldots, a_n)$.*

$$I = (f), f = x^n + a_1 x^{n-1} + \cdots + a_0$$

**Example 5.11** ($K$-theory is 'finer' than Chow ring)**.** *Just consider three intersecting lines $X = V(xyz = 0)$ in $\mathbb{P}^2$. Then we have*

$$[X] = 2[line],$$

*but in the $K$-theory, we have*

$$[\mathcal{O}_X] = 3[\mathcal{O}_{line}] - 3[\mathcal{O}_{pt}].$$

*In other words, the Chow ring only captures the lowest degree(codimension) part of the $K$-theory.*

**Example 5.12** (Segre class of the second Veronese embedding). *First recall that the Segre classes $s(V, M)$ are characterized by*

- *if $V \subset M$ is a regular embedding then*

$$s(V, M) := c(N_V M)^{-1} \cap [V].$$

- *if $\pi : M' \to M$ is a proper birational map, and $\pi^{-1}(V) \to V$ is the restriction of $\pi$, then*

$$s(V, W) := p_*(s(\pi^{-1}V, M')).$$

*Now we consider the second Veronese embedding $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$. The tangent sequence gives us*

$$s(\nu_2(\mathbb{P}^2), \mathbb{P}^5) = \frac{(1+h)^3}{(1+H)^6} \cap [\mathbb{P}^2] = \frac{(1+h)^3}{(1+2h)^6} = [\mathbb{P}^2] - 9[\mathbb{P}^1] + 51[\mathbb{P}^0].$$

**Example 5.13** (Chern-Fulton v.s. Chern-Fulton-Johnson).

**Example 5.14** (Morse theory on Grassmannian). *We use two approaches to study the Morse theory on complex Grassmannians,*

- *Morse-Smale dynamics and the gradient flow.*
- *Localization.*

*Let's start with the localization method. We have a natural torus action on $Gr(k, n)$ defined as the right multiplication of $t \in T$*

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix} \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \dots & \\ & & & t_n \end{pmatrix} = \begin{pmatrix} t_1 a_{11} & t_2 a_{12} & \dots & t_n a_{1n} \\ t_1 a_{21} & t_2 a_{22} & \dots & t_n a_{2n} \\ \dots & & & \\ t_1 a_{k1} & t_2 a_{k2} & \dots & t_n a_{kn} \end{pmatrix}$$

*Consider the standard open covering of $Gr(k, n)$, we know fixed points are exactly unimodular matrices, for example*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & \dots & 0 \end{pmatrix}$$

*Hence we have $\binom{n}{k}$ fixed point of the torus action corresponding to the coordinates $\{x_J\}$ of the Plücker embedding. Let $[W]$ be a fixed point, $T_{[W]}(Gr(k, n)) \cong Hom(W, V/W)$ which can be naturally identified with* *the rest $(n-k)$ columns in the echelon form. And as a representation of the torus, the weight decomposition is given by*

$$T_{[W]}(Gr(k,n)) \cong \oplus_{j\notin J,i} V(i,j), \mathrm{weight}(V(i,j)) = t_j t_i^{-1}.$$

*where $V(i,j)$ is the the one-dimensional space at the position $(i,j)$. We only need to compute the index at each fixed point $x_J = x_{j_1,\ldots,j_k}$, this is the same as*

$$2\#\{j \notin J, j > i\} = 2[(n-k-j_1+1) + (n-k-j_2+2) + \cdots + (n-k-j_k+k)]$$

$$= 2(n-k)k + 2(\frac{(k+1)k}{2} - \sum_\alpha j_\alpha) =: d_J.$$

*Thus we get*

$$P_t(Gr(k,n)) = \sum_J t^{2d_J}.$$

*Specially, for $\mathbb{P}^n_{\mathbb{C}}$,*

$$P_t(\mathbb{P}^n_{\mathbb{C}}) = 1 + t^2 + t^4 + \cdots + t^{2n}.$$

*Note that $d_J$ has a natural combinatorial interpretation: $0 \le (j_1 - 1) \le (j_2 - 2) \ldots (j_k - k) \le n-k$ gives us a partition inside a $k \times (n-k)$ rectangle, $d_J$ is the size of the complement, which is still the size of a partition inside this rectangle. So we can write it as*

$$P_t(Gr(k,n)) = \sum_{\lambda \subset \square} t^{2|\lambda|}.$$

*If we consider those partitions according to $j_k - k = n-k$ or $j - k \neq n-k$(which means the last row in the partition is the whole row or not), we get a recursive formula*

$$P_t(k,n) = P_t(k,n-1) + t^{2(n-k)} P_t(k-1,n-1)$$

*Hence we get a expression which is convenient for down-to-earth computations,*

$$P_t(k,n) = P_t(Gr(k,n)) = \frac{\Pi_{i=1}^n (1-t^{2i})}{(\Pi_1^k(1-t^{2i}))(\Pi_{i=1}^{n-k}(1-t^{2i}))}.$$

*Let me create a notation*

$$P_t(Gr(k,n)) = \binom{n, 1-t^{2i}}{k}.$$

*Now we consider a more familiar version of the Morse theory on Grassmannians, which means, we construct a Morse function explicitly(in the localization method, the Morse function is given by the moment map of the torus action, but we don't need to write it down, since we know critical points are just those fixed points of the torus action, for more details, see Nakajima's book on Hilbert schemes,Chapter* 5*).*

**Example 5.15** (Moment maps on $\mathrm{Hilb}^n(\mathbb{C}^2)$)**.**

**Example 5.16** (Moment maps on smooth nested Hilbert schemes)**.**

**Remark** (adjoint orbit, symplectic form, moment maps)**.**

**Example 5.17** (Zariski closure of an algebraic group, not even a group)**.** *Consider the nodal curve*

$$C : y^2 = x^2(x+1).$$

*Then*

$$C \setminus \{0\} \cong \mathrm{Spec}(k[t,t^{-1}]) \cong \mathbb{G}_m.$$

*Its closure is just $C$ which is not an algebraic group(since it's not smooth). This silly counterexample is made for the following property of algebraic groups*

> *If $G$ is an algebraic group and $H$ an abstract subgroup, the Zariski closure of $H$ is a group.*

*For example, use this we can easily prove that $SL(2,\mathbb{Z})$ is Zariski-dense in $SL(2,\mathbb{C})$. For the following reasons*

- $U_+ = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \subset \overline{SL(2,\mathbb{Z})}$ *because all matrices of the form* $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ *do. Same for $U_-$.*
- *The maximal torus* $T = \{\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}\} \cong \mathrm{Spec}(k[t,t^{-1}])$, $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z})$, *now*

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{x} & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & -\frac{1}{x} \end{pmatrix},$$

  *use the property above we now know $T \subset \overline{SL(2,\mathbb{Z})}$,*
- *Thus $B = TU_+ \subset \overline{SL(2,\mathbb{C})}$ by the property above, Bruhat decomposition tells us*

$$SL(2,\mathbb{C}) = U_-B \bigsqcup U_-sB.$$

  *Use the property again, we know*

$$\overline{SL(2,\mathbb{Z})} = SL(2,\mathbb{C}).$$

**Example 5.18** ($SL(2,\mathbb{Z}$ is Zariski-dense in $SL(2,\mathbb{C})$, another proof, Sam)**.** *Sam told me another proof different from the one uses Bruhat decomposition. Fix an embedding $\mathbb{Q}_5 \hookrightarrow \mathbb{C}$. The argument goes as follows*

- *$SL(2,\mathbb{Z})$ is dense in $SL(2,\mathbb{Z}_5)$ in the 5-adic topology, hence in the Zariski topology.*
- *$SL(2,\mathbb{Z}_5)$ is an open dense subset of $SL(2,\mathbb{Q}_5)$ in the 5-adic topology, hence in the Zariski topology.*
- *Since $\mathbb{Q}_5$ contains a 4th root of unit, the closure of $SL(2,\mathbb{Z})$ contains $SL(2,\mathbb{Q}(i))$.*
- *$SL(2)$ over any field contains an open $(\mathbb{A}^1 \setminus \{0\})^3$ embedded as*

$$(x,y,z) \mapsto \begin{pmatrix} x & y \\ \frac{xz-1}{y} & z \end{pmatrix}.$$

- *$\mathbb{Q}(i)$ points of $(\mathbb{A}^1 \setminus \{0\})^3$ are dense in its $\mathbb{C}$-points in euclidean topology and hence in the Zariski topology, we're done.*

# 6 Schemes as functors, sheaves on categories

It's highly recommended to read David Mumford's book Lectures on curves on an algebraic surface.

**Example 6.1** ($\mathbb{A}^1_{\mathbb{Z}}$)**.**

**Example 6.2** ($\mathbb{P}^n_{\mathbb{Z}}$)**.**

**Example 6.3** (Fano scheme and its tangent space)**.**

**Example 6.4** (Grassmannian)**.**

**Remark.** *The graph of a morphism to a separated scheme is closed.*

**Example 6.5** (Quotient stack $[\mathbb{A}^n/GL_n]$)**.**

*https://mathoverflow.net/questions/184253/the-quotient-stack-mathbban-mathrmgl-n?rq=1*

*Geometric picture*

$$-\bullet$$

**Example 6.6** (Examples of algebraic stacks without coarse moduli space)**.**

*https://mathoverflow.net/questions/3742/examples-of-algebraic-stacks-without-coarse-moduli-space?rq=1*

**Example 6.7** ($x^2 + y^3 = z^7$)**.**

*http://math.mit.edu/ poonen/papers/pss.pdf*

**Example 6.8** ($(p-1)/24$ supersingular elliptic curves in characteristic $p$)**.**

*https://mathoverflow.net/questions/24573/is-there-a-nice-proof-of-the-fact-that-there-are-p-1-24-supers*

**Example 6.9** (Mumford, Picard groups of moduli problems)**.**

*http://www.mathcs.emory.edu/ brussel/Scans/mumfordpicard.pdf*

*https://www.daniellitt.com//expository-notes/*

**Example 6.10** (tangent space of the Picard scheme)**.**

**Example 6.11** (isotrivial but non-trivial family of elliptic curves)**.** *Consider the family of elliptic curves over $k^*$*

$$X := \mathrm{Spec}(k[x,y,t,t^{-1}]/(y^2 - x^3 + t)) \to \mathrm{Spec}k[t,t^{-1}] = k^*.$$

*This is a family of elliptic curves with constant $j \equiv 0$. This is not a trivial bundle because the total space $E$ is affine, that is*

$$\mathrm{Spec}(k[x,y,t,t^{-1}]/(y^2 - x^3 + t)) = D_+(y^2 - x^3) \subset \mathbb{A}^2_k.$$

*If this family is trivial $X \cong E \times k^*$, then we get a dominant morphism*

$$\mathbb{P}^2 \dashrightarrow E \times k^* \to E,$$

*this implies that $E$ is unirational, by Luroth's theorem, this means $E$ is rational, contradicts the fact that $g(E) = 1$.*

**Remark** (trivialization in étale topology)**.** *Consider the étale open subset*

$$\phi : \mathrm{Spec}(k[t^{\frac{1}{6}}, t^{\frac{-1}{6}}]) \to Spec(k[t,t^{-1}]),$$

*which means if we let $x = t^{\frac{1}{3}}x', y = t^{\frac{1}{2}}y'$, we get an isomorphism between $y^2 = x^3 - t$ and $(y')^2 = (x')^3 - 1$.*

**Example 6.12** ($pt \to B\mathbb{G}_m$ is an fppf(even smooth and surjective) cover)**.** *I read this from Daniel Litt's notes 'Picard groups of moduli problems'. We try to parametrize all line bundles, that means we 'remember'*

- *'isomorphism', namely, let $\{U_i\}$ be a cover of $X$. $\mathcal{L}, \mathcal{L}'$ are two line bundles on $X$, and $f_i : \mathcal{L}|_{U_i} \to \mathcal{L}'|_{U_i}$ are isomorphisms, so that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then $f$ descents to(i.e gives us ) an isomorphism $f : \mathcal{L} \to \mathcal{L}'$. In other words* $\mathrm{Isom}(\mathcal{L}, \mathcal{L}')$ *is an* fpqc *sheaf: if $i : U \to X$ is a* fpqc *morphism, we have an exact sequence*
$$\mathrm{Isom}(\mathcal{L}, \mathcal{L}')(X) \to \mathrm{Isom}(\mathcal{L}, \mathcal{L}')(U) \rightrightarrows \mathrm{Isom}(\mathcal{L}, \mathcal{L}')(U \times_X U).$$

- *'automorphism', that's just the cocycle condition. Let $B\mathbb{G}_m(X)$ be the category of line bundles on $X$, with isomorphisms as morphisms, for any* fpqc *morphism $U \to X$, we have an exact sequence in the sense of groupoids.*

$$B\mathbb{G}_m(X) \to B\mathbb{G}_m(U) \rightrightarrows B\mathbb{G}_m(U \times_X U) \overset{\rightarrow}{\rightrightarrows} B\mathbb{G}_m(U \times_X U \times_X U).$$

*A morphism $f : T \to B\mathbb{G}_m$ is the same as a line bundle on $T$(but different morphisms may have isomorphisms between them). Let $\mathcal{L}, \mathcal{L}'$ be a bundle on $T$ and $T'$ respectively, corresponding to morphisms $f, f'$, consider*

$$\begin{tikzcd} T \times_{B\mathbb{G}_m} T' \arrow[r] \arrow[d] & T' \arrow[d, "f'"] \\ T \arrow[r, "f"] & B\mathbb{G}_m \end{tikzcd}$$

*Then a morphism $X \to T \times_{B\mathbb{G}_m} T'$ is the same as*

- *morphisms $g : X \to T, g' : X \to T'$ and*

- *$f \circ g = f' \circ g'$, let the bundle corresponding to $f \circ g$ be $\mathcal{M}$, then we know the condition means '$\mathcal{M}(U \to X) \cong \mathcal{L}(U \to X \to T)$'(to be more precise, to really get a line bundle on $X$, we mean the inverse sheaf tensored with the structure sheaf of $X$), that is $\mathcal{M} \cong g^*\mathcal{L}$. In other words $f \circ g = f' \circ g'$ is the same as*
$$g^*\mathcal{L} \cong g'^*\mathcal{L}'.$$

*Then we can justify the fact $pt \to B\mathbb{G}_m$ is a* fppf. *cover.*

$$\begin{tikzcd} T \times_{B\mathbb{G}_m} pt \arrow[r] \arrow[d] & pt \arrow[d, "f'"] \\ T \arrow[r, "f"] & B\mathbb{G}_m \end{tikzcd}$$

*where $f'$ is given by $\mathcal{O}_{pt}$ and $f$ is given by a line bundle on $T$. By our discussion, a morphism $X \to T \times_{B\mathbb{G}_m} pt$(i.e a 'point' of this functor) is the same as*

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "g"] & pt \arrow[d, "f'"] \\ T \arrow[r, "f"] & B\mathbb{G}_m \end{tikzcd}$$

$$\textit{an isomorphism} : g^*\mathcal{L} \cong g'^*\mathcal{O}_{pt} = \mathcal{O}_X.$$

*Now what does $(T \times_{B\mathbb{G}_m} pt)(X)$ mean? It means all possible isomorphisms $g^*\mathcal{L} \cong g'^*\mathcal{O}_{pt} = \mathcal{O}_X$, so we get*

- *if $g^*\mathcal{L} \cong \mathcal{O}_X$, then*
$$(T \times_{B\mathbb{G}_m} pt)(X) = \mathbb{G}_m(X) = \mathcal{O}_X^*,$$

- *if $g^*\mathcal{L} \not\cong \mathcal{O}_X$, then*
  $$(T \times_{B\mathbb{G}_m} pt)(X) = \emptyset.$$
  *$T \times_{B\mathbb{G}_m} pt$ is actually a scheme. To see this, let $f_i : U_i \to X$ be morphisms such that $f_i^*\mathcal{L} \cong \mathcal{O}_X$, we know this functor is 'glued' together by its restrictions to $U_i \times pt$, that is 'gluing'(in the sence above, two descents) $\mathbb{G}_m(U_i)$ by automorphisms of $\mathbb{G}_m(U_i \cap U_j)$ which satisfies the cocycle conditions, that's the same as the ordinary gluing procedure of $\mathcal{L}$, namely $\mathcal{L}|_{U_i} \cong \mathcal{L}|_{U_j}$, in short*

  *we're now gluing $\mathbb{G}_m(U_i)$ by the same construction of $\mathcal{L}$*

  *this tells us*
  $$T \times_{B\mathbb{G}_m} pt \cong \mathrm{Tot}(\mathcal{L}) \setminus \{0\}.$$
  *That's just the $\mathbb{G}_m$-bundle associated to $\mathcal{L}$ and $\{0\}$ means the zero section.* $\mathrm{L} \setminus \{0\}$ *is indeed faithfully flat over $T$.*

**Remark** ($T' \to B\mathbb{G}_m$ representable?(i.e any pull-back along this morphism is a scheme))**.** *The answer is YES. By the diagonal criterion in Daniel's Notes, we only need to check the following pull-back is representable*

$$\begin{tikzcd} B\mathbb{G}_m \times_{B\mathbb{G}_m \times B\mathbb{G}_m} T \arrow[r] \arrow[d] & T \arrow[d, "(\mathcal{L}_1, \mathcal{L}_2)"] \\ B\mathbb{G}_m \arrow[r, "\Delta"] & B\mathbb{G}_m \times B\mathbb{G}_m \end{tikzcd} .$$

*This is true, just repeat the discussion above, we should get*

$$B\mathbb{G}_m \times_{B\mathbb{G}_m \times B\mathbb{G}_m} T \cong \mathrm{Isom}(\mathcal{L}, \mathcal{L}') \cong \mathrm{Tot}(\mathcal{L}' \otimes \mathcal{L}^\vee) \setminus \{0\}.$$

**Remark** ('glue')**.** *When considering problems related to stacks, 'glue' actually means something slightly different: we don't mean we want to identify things to get some equivalent classes, but to remember their relations, i.e*

- *Descent for isomorphisms (between two 'things')*
- *Descent for auotomorphisms (defining a 'thing')*

*For quotient stack, specially, we need to connect 'points' in an orbit by the 'isomorphism' between them ($x = gy$) and we also want to remember the stablizer(automorphism, $gx = x$) at every point $x$.*

**Example 6.13** (quotient morphism $X \to X/G$ with no Zariski-local sections)**.** *We have the following examples*

- *finite Galois covers of a curve of genus $\geq 1$.*

**Example 6.14** ($\mathrm{Pic}(BG)$ over an algebraically closed field $k$)**.** *$BG$ is just the quotient stack $[pt/G]$. We can define line bundles on $BG$ directly, we can also define them by descending from $pt$.*

- *a line bundle on $BG$ is defined by associating to each map $X \to BG$(i.e a $G$-torsor on $X$) a line bundle $\mathcal{L}_{\mathcal{G}}$, 'glue' here means for every morphism $h : T' \to T$ over $BG$, we have to* **specify** *an isomorphism $h^*\mathcal{L}_{\mathcal{G}} \cong \mathcal{L}_{\mathcal{G}'}$. Specially, for an automorphism $f : T \to T$ over $BG$(this is much more than an automorphism of $T$ itself), it's the identity on $T$, so it's an automorphism of $\mathcal{G}$, it's $G$. In other words, for any $g \in G$ we have to specify an element in $\mathrm{Aut}(\mathcal{L}_{\mathcal{G}}) \cong k^*$. All constructions are compatible with compositions, so we actually have a character map for any $G$-torsor $\mathcal{G}$*
  $$\chi_{\mathcal{G}} : G \to k^*.$$

- *any other way to view line bundles on $BG$ is that every line bundle on $BG$ comes from a line bundle and descent data on pt(this is because $k = \bar{k}$, we don't have any non-trivial étale cover of pt). $\mathcal{O}_{pt}$ is the only line bundle on pt, the descent data here means for any $g \in G$, we have to specify an automorphism of $\mathcal{O}_{pt}$ which satisfies the cocycle condition, i.e a map(of set) $\alpha : G \to \mathbb{G}_m$ satisfies the cocycle condition, note that we only have one bundle, but we have $|G|$ automorphisms, think about it, it just means, the composition of $\alpha(g_2) \circ \alpha(g_1) : \mathcal{O}_{pt} \xrightarrow{\alpha(g_1)} \mathcal{O}_{pt} \xrightarrow{\alpha(g_2)} \mathcal{O}_{pt}$ has to agree with $\alpha(g_1 g_2) : \mathcal{O}_{pt} \to \mathcal{O}_{pt}$, this exactly means $\alpha$ is a group homomorphism! This tells us*

$$\mathrm{Pic}(BG) \cong \mathrm{Hom}(G, k^*) = H^1(G, k^*).$$

*Combine these two, the homomorphism in the second point of view is the same as $\chi_{\mathcal{G}} : G \to \mathcal{G}_m$ associated to the trivial $G$-torsor $\mathcal{G}$ as in the first point of view. That is*

$$\mathrm{Tot}(\mathcal{L}_{G\times pt}) = \mathcal{G} \times_G \mathbb{A}^1_k,$$

*where $G$ acts on $\mathbb{A}^1_k$ trivially. In general if we have $\chi : G \to k^*$, the corresponding $G$-torsor is given by*

$$\mathrm{Tot}(\mathcal{L}_{\mathcal{G}}) \cong \mathcal{G} \times_G \mathbb{A}^1_k,$$

*but here $G$ acts on $\mathbb{A}^1_k$ by the character $\chi$. If $G = \mathcal{G}_m$, we get the 'classical' result*

$$\mathrm{Pic}(B\mathbb{G}_m) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z} = H^2(\mathbb{CP}^\infty, \mathbb{Z}).$$

**Example 6.15** (Pic($\mathscr{M}_{1,1}$) over a field $k$, $char(k) \neq 2, 3$, algebraic method)**.** *We follow Mumford's original paper 'Picard groups of moduli problems', it's very readable and enlightening, thus we highly recommend interested readers of reading it. We want to find some numerical invariants of a line bundle on $\mathscr{M}_{1,1}$. To be more clear, our goal is to define a homomorphism*

$$\mathrm{Pic}(\mathscr{M}_{1,1}) \to \mathbb{Z}/12\mathbb{Z}.$$

*Consider a line bundle $\mathscr{L}$ on the moduli problem $\mathscr{M}_{1,1}$ and a family of elliptic curves $\pi : \mathscr{X} \to S$, then the family has a natural involution(order 2 automorphism), namely the inversion $\rho$:*

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\rho"] \arrow[d, "\pi"] & \mathscr{X} \arrow[d, "\pi"] \\ S \arrow[r, "id"] & S \end{tikzcd}.$$

*By the definition of a line bundle on a moduli problem, we get an automorphism $\mathscr{L}(\rho)$(that is the restriction map corresponding to the 'inclusion of open sets' $\rho$) of $\mathscr{L}_\pi$(the line bundle on $S$ corresponding to $\mathscr{L}$ and the family $\pi$). For example if $S = \mathrm{Spec}(k), \mathscr{X} = C$, then $\mathscr{L}(\rho)$ is determined by an element $\alpha(C) \in H^0(C, \mathcal{O}_C^*) \cong k^*$. $\alpha^2(C) = 1$, thus $\alpha(C) = 1$ or $\alpha(C) = -1$. Similarly, for a general family $\pi : \mathscr{X} \to S$, then $\mathscr{L}(\rho)$ is given by an element $\alpha \in H^0(S, \mathcal{O}_S^*)$, the restriction to every fibre $\pi^{-1}(s)$ over a closed point $s \in S$ is just the ordinary inversion, this tells us $\alpha|_s = \alpha(\pi^{-1}(s))$, thus locally, this is a constant either $= 1$ or $= -1$. Note that we can put all elliptic curves in a single family over a connected base. For example*

$$E_t : y^2 = x(x-1)(x-t), t \neq 0, 1, \infty.$$

*We define $\alpha(\mathscr{L}) := \alpha(E_t)$, We thus get a homomorphism(this doesn't depends on $E_t$, by the compatibility of $\mathscr{L}$ with pull-backs in this Grothendieck topology)*

$$\alpha : \mathrm{Pic}(\mathscr{M}_{1,1}) \to \mathbb{Z}/2\mathbb{Z}.$$

*Note that the line bundle $\mathscr{L}$ gives us a line bundle for every family, and some family has more automorphisms, this actually helps us to construct more homomorphisms from* Pic($\mathscr{M}_{1,1}$) *to more concrete groups. To be more precise, let*

$$C_1 = V(y^2 - x(x+1)(x-1)), j = 0,$$

$$C_2 = V(y^2 - x(x-\omega)(x-\omega^2)), \omega^3 = 1, j = 12^3 = 1728.$$

*Consider two families*

$$\pi_1 : C_1 \to \mathrm{Spec}(k), \pi_2 : C_2 \to \mathrm{Spec}(k).$$

*We know* $\mathrm{Aut}(C_1) \cong \mathbb{Z}/4\mathbb{Z}$ *with a generator* $\sigma$

$$x \mapsto -x, y \mapsto iy.$$

$\mathrm{Aut}(C_2) \cong \mathbb{Z}/6\mathbb{Z}$ *with a generator* $\tau$

$$x \mapsto \omega x, y \mapsto -y.$$

*But our discussion above,* $\mathscr{L}(\sigma)$ *acts on* $\mathscr{L}_{\pi_1}$ *by multiplication of a fourth root of unity, similarly* $\mathscr{L}_{\pi_2}$ *acts on* $\mathscr{L}_{\pi_1}$ *by multiplication of a sixth root of unity. Then we can define a homomorphism*

$$\beta : \mathrm{Pic}(\mathscr{M}_{1,1}) \to \mathbb{Z}/4\mathbb{Z} \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z}.$$

*This is because we have relations* $\mathscr{L}(\sigma)^2 = \mathscr{L}(\tau)^3 = \alpha(\mathscr{L})$*, we can fix a* $12^{th}$ *primitive root* $\zeta$*, then there's a unique integer* $\beta(\mathrm{mod} 12)$ *such that*

$$\zeta^{6\beta} = \alpha(\mathscr{L}), \zeta^{3\beta} = \mathscr{L}(\sigma), \zeta^{2\beta} = \mathscr{L}(\tau).$$

*This* $\beta$ *gives us the homomorphism we want. We first prove that* $\beta$ *is surjective, to do this , recall that the Hodge bundle of a family* $\pi : \mathscr{X} \to S$ *is defined to be the direct image sheaf*

$$R^1\pi_*\mathcal{O}_{\mathscr{X}} \to S.$$

*For any elliptic curve* $C$

$$\mathrm{dim} H^1(C, \mathcal{O}_C) = \mathrm{dim} H^0(C, \omega_C) = g_C = 1,$$

*then by the cohomology and base change theorem, we know* $R^1\pi_*\mathcal{O}_{\mathscr{X}}$ *is a line bundle over* $S$*. Note that the cohomology and base change theorem not just tells us the rank of* $R^1\pi_*\mathcal{O}_{\mathscr{X}}$ *is* $1$ *but also the compatibility w.r.t Cartesian diagrams we need. Therefore we do get a line bundle on* $\mathscr{M}_{1,1}$ *by defining*

$$\Lambda(\mathscr{X} \to S) = R^1\pi_*\mathcal{O}_{\mathscr{X}}.$$

*Now we only need to compute the action of* $\sigma$ *on* $H^0(C_1, \omega_{C_1})$ *and of* $\tau$ *on* $H^0(C_2, \omega_{C_2})$*. By basic knowledge of regular differentials, we know* $H^0(C_1, \omega_{C_1})$ *is a one dimensional vector space generated by* $w = \frac{dx}{2y} = \frac{dy}{3x^2-1}$*.*

$$\sigma : \frac{dx}{2y} \mapsto \frac{-dx}{idy} = i\frac{dx}{2y}, \mathrm{ord}(\sigma) = 4.$$

*Similarly,* $H^0(C_2, \omega_{C_2})$ *is a one dimensional vector space generated by* $w = \frac{dx}{2y} = \frac{dy}{3x^2}$*.*

$$\tau : \frac{dx}{2y} \mapsto \frac{\omega dx}{-dy} = -\omega\frac{dx}{2y}, \mathrm{ord}(\tau) = 6.$$

$\beta$ *is thus surjective. We can make a one-and-for-all choice such that* $\beta(\Lambda) = 1$*. To prove* $\beta$ *is injective, we let* $\beta(\mathscr{L}) = 0$*, we have to prove that* $\mathscr{L} \cong \mathcal{O}_{\mathscr{M}_{1,1}}$*. What does this mean? This means we can find an étale cover* $S$ *of* $\mathscr{M}_{1,1}$ *which trivilizes* $\mathscr{L}$*, such that if we fix a trivilization* $\phi : \mathscr{L}|_S \cong \mathcal{O}_S$*, the descent data of* $\mathscr{L}$ *commutes with that of* $\mathscr{O}_{\mathscr{M}_{1,1}}$ *under this trivilization. We already know* $S = \mathscr{E}_t \to \mathbb{A}^1_t \setminus \{0,1\}$ *above is an étale cover of* $\mathscr{M}_{1,1}$ *of degree* $12$*, any line bundle on* $S$ *is trivial since it's an open subset of* $\mathbb{A}^1_t$*. Now we have to prove the compatibility on the fibre product*

$$\begin{tikzcd} S \times_{\mathscr{M}_{1,1}} S \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & S \\ S. & \end{tikzcd}$$

*That is, the following diagram commutes*

$$\begin{tikzcd} \pi_1^*\mathscr{L} \arrow[r] \arrow[d, "\pi_1^*\phi"] & \pi_2^*\mathscr{L} \arrow[d, "\pi_2^*\phi"] \\ \pi_1^*\mathcal{O}_S \arrow[r] & \pi_2^*\mathcal{O}_S. \end{tikzcd},$$

*where $\phi$ is an isomorphism $\phi : \mathcal{O}_S \to \mathscr{L}|_S$ given by a global section $\theta \in H^0(S, \mathscr{L})$. This can be checked on stalks over closed points in $S \times_{\mathscr{M}_{1,1}} S$, a closed point*

$$s = \{\mathrm{Spec}(k) \to S \times_{\mathscr{M}_{1,1}} S\}$$

*corresponding to two closed points $s_1, s_2$ in $S$ and an isomorphism $\psi_s : \pi^{-1}(s_1) \cong \pi^{-1}(s_2)$. By the definition of a line bundle, we get $\mathscr{L}(\psi_s) : \mathscr{L}_{s_1} \to \mathscr{L}_{s_2}$. Now consider a group action on $S$ generated by $g_1, g_2$:*

$$g_1 : \lambda \mapsto \frac{1}{\lambda}, g_2 : \lambda \mapsto 1 - \lambda.$$

*The point is that $\mathscr{L}_{\psi_s}$ comes from an antomorphism(provided by the group action) of the family $\pi : \mathscr{X} \to S - \mathbb{A}^1_j \setminus \{0,1\}$. $H^0(S, \mathcal{O}_S^*) = k^*$, so $\beta$ is defined on this family and restricted $\alpha(C)$ on every fibre $C$. Since $\beta(\mathscr{L}) = 0$, we know the induced action $H^0(S,\mathscr{L}) \to H^0(S,\mathscr{L})$ is just the identity map. For the diagram to commute, we only need to check $\mathscr{L}(\psi_s) : \pi^*\theta|_{s_1} \to \pi_2^*\theta|_{s_2}$,since this isomorphism comes from an antomorphism of $S$, we need to check $\theta(s_1) = \theta(s_2)$, this is true if $\theta \in H^0(S,\mathscr{L})^G$, $H^0(S,\mathscr{L})^G \neq \emptyset$ because the existence of the $j$-invariant. Thus it's always possible to choose such a global section $\theta$. Therefor $\beta$ is injective. In conclusion, we get an isomorphism*

$$\beta : \mathrm{Pic}(\mathscr{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}.$$

**Remark** (Hodge bundle, $S \times_{\mathscr{M}_{1,1}} S$ and degree of the étale cover)**.**

**Example 6.16** ($\mathrm{Pic}(\mathscr{M}_{1,1})$ over a general scheme)**.** *Fulton.*

**Example 6.17** (Non-reductive group action on affine varieties)**.** *Kirwan.*

# 7 Hodge theory

**Remark** (singular (co)homology, sheaf (co)homology, de Rham cohomology an Hodge decomposition)**.** *We want to clarify the following concepts*

- *$H^*(X,\mathbb{Z})$ and $H^*(X,\underline{\mathbb{Z}})$.(Same question for $\mathbb{C},\mathbb{R}$)*
- *what do we mean by $d\bar{z}$ when we're talking about an algebraic(or holomorphic) curve?*
- *$H^n(X,\mathbb{C}) \cong \oplus_{p+q=n} H^{p,q}(X)$ and $H^n(X,\underline{\mathbb{C}}) \cong \oplus_{p+q=n} H^p(X, \wedge^q\Omega_X)$?*
- *Do we have $H^p(X,\wedge^q\Omega_X) = \overline{H^q(X,\wedge^p\Omega_X)}$, in what sense?*
- *Any relations between $H^*(X,\wedge^k\Omega_X)$, $H^*_{dR}(X,\mathbb{C})$ and $H^*(X,\mathbb{C})$?*
- *algebraic description of the Betti lattice $H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{C})$?*

**Example 7.1** (William Lang, Hoge spectral sequence in characteristic 3)**.** *Let $k = \mathbb{F}_3$. Consider*

$$X : y^2z = x^3 - tz^2 \subset \mathbb{P}^3_k.$$

*Then*
$$b^1_{\mathrm{dR}} = \dim_k(H^1_{\mathrm{dR}}(X/k)) = 3.$$
*However*
$$h^{0,1} = h^{1,0} = 1,$$
*where* $h^{p,q} = \dim_k(H^q(X, \wedge^p \Omega_{X/k}))$*. That means in positive characteristics, the ordinary relation between the Hodge cohomology groups* $H^q(X,^p\Omega_X)$ *and the algebraic de Rham cohomology groups* $H^n_{\mathrm{dR}}(X/k)$ *fails.*

**Remark.** *This example is provided by William Lang in his thesis 'Quasi-elliptic surfaces in characteristic three', I read this from Alex Youcis's blog post here.*

**Remark** ($X$ irreucible, then $H^i_{\mathrm{dR}}(X/k) = 0, \forall i > 0$?(This cannot be true))**.** *Recall our definition of the algebraic de Rham cohomology*
$$0 \to \underline{k} \to \mathcal{O}_X \to \Omega^1_{X/k} \to \cdots \to \wedge^n \Omega_{X/k} \to 0$$
$$H^i_{\mathrm{dR}}(X/k) := \mathbb{H}^i(\Omega^\bullet_{X/k})$$
*The de Rham complex above is a resolution of* $\underline{k}$*, then by abstract nonsense, we have*
$$\mathbb{H}^i(\Omega^\bullet_{X/k}) \cong H^i(X, \underline{k}).$$
*If* $X$ *is irreducible, then the constant sheaf* $\underline{k}$ *is flasque, therefore* $H^i(X, \underline{k}) = 0 \Rightarrow H^i_{\mathrm{dR}}(X/k) = 0, \forall i > 0$*, this is ridiculous! What's wrong? Actually, we don't have any type of Poicare lemma in algebraic settings, to put it more directly*

*the de Rham complex is not a resolution of the constant sheaf* $\underline{k}$ *in Zariski topology.*

*In general*
$$H^i(X, k) \neq \mathbb{H}^i(\Omega^\bullet_{X/k}).$$

**Example 7.2** (Kähler manifold but not algebraic)**.**

**Example 7.3** ($\operatorname{rank}(H^{2k}(X_t, \mathbb{Z}) \cap H^{k,k}(X_t))$ not constant)**.** *Dani, Litt*

**Example 7.4** (Family of elliptic curves, Gauss-Manin connection and Picard-Fucus function)**.**

**Example 7.5** (Polarized variation of Hodge structure over the punctured disk)**.**

**Example 7.6** (variation of Hodge structures, Siegel upper half plane)**.** *Milne.*

**Example 7.7** (Mixed Hodge structure, nodal curve)**.**

**Example 7.8** (Mixed Hodge structure, split over $\mathbb{R}$)**.**

**Example 7.9** (Unimodular lattices)**.** *tony Feng.*

**Example 7.10** (Mixed Hodge structure on $\mathfrak{gl}(V_{\mathbb{C}})$)**.**

**Example 7.11** (a stable sheaf but not geometrically stable)**.**

**Example 7.12** ($\Omega_{\mathbb{P}^n}$ is stable)**.**

**Example 7.13** (Harder-Narasimhan filtration, Gieseker nonstable but Mumford-Takemoto semistable)**.** *On the Gieseker Harder-Narasimhan filtration for principal bundles*

**Example 7.14** ($\mathbb{P}^1 \times \mathbb{P}^1$ and change of polarization)**.** *Notions of stability of sheaves*

**Example 7.15** (a rank 2 moduli space, K3 surface of degree 8 in $\mathbb{P}^5$ )**.** *page 7*

**Example 7.16** (Hom, $\mathscr{H}om, Ext^i_A(-,-), \mathscr{E}xt^i_-(-,-), Tor^i_-(-,-)$ with Macaulay2)**.** *We learned these examples from*

- *Computing with sheaves and sheaf cohomology in algebraic geometry*
- *Computing with sheaves and sheaf cohomology in algebraic geometry*
- $\mathscr{E}xt^i(\mathcal{O}_{L_1}, \mathcal{O}_{L_2})$ *for two lines* $L_1, L_2$ *in* $\mathbb{P}^3$

**Example 7.17** ($Ext^1$ and extension of vector bundles)**.** *Consider the following example*

- *vector bundles in* $Ext^1(\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{O}_{\mathbb{P}^1}(2))$
- *Extensions of vector bundles on* $\mathbb{P}^1$

**Example 7.18** ($S$-equivalence)**.**

**Example 7.19** (Jordan-Holder filtration v.s. Harder-Narasimhan filtration)**.** *Jordan-Holder filtration v.s. Harder-Narasimhan filtration*

**Example 7.20** (Strong semistability is not an open property)**.** *Strong semistability is not an open property*

**Example 7.21** (Chow ring of $\overline{\mathrm{M}}_{0,n}$)**.** *Let's start with* $\overline{\mathrm{M}}_{0,4} \cong \mathbb{P}^1$*, then we have* $\delta_{1,2} = \delta_{1,3} = \delta_{1,4}$*, and* $\delta_{1,2}\delta_{1,3} = 0$*. We get the Chow ring*

$$\mathrm{CH}^*(\mathbb{P}^1) \cong \mathbb{Z}[\delta]/(\delta^2).$$

*How about* $\overline{\mathrm{M}}_{0,5}$*? We have* $\binom{5}{2} = 10$ *generators* $\delta_{i,j}$ *with the relations*

- *For distinct i,j,k,l*
$$\delta_{i,j} + \delta_{k,l} = \delta_{i,k} + \delta_{j,l} = \delta_{i,l} + \delta_{j,k}$$
- *multiply the equations above by* $\delta_{i,j}$*, and apply the third relation in Keel's description, we get double products*
$$\delta_{i,j}^2 = \delta_{k,l}^2 = -\delta_{a,b}\delta_{c,d}$$
- *all triple products and above vanish by dimensional argument. where* $a, b, c, d$ *are any* 4 *distinct elements in* $\{1, 2, 3, 4, 5\}$.

*Well, what is this ring exactly? Let's do more trivial computations, if we label* $\{\delta_{1,2}, \delta_{1,3}, \ldots, \delta_{2,3}, \ldots, \delta_{4,5}\}$ *lexicographically as* $\{e_1, \ldots, e_{10}\}$*, the only complicated part of* $\mathrm{CH}^*(\overline{\mathrm{M}}_{0,5})$ *is the* $\mathrm{Pic}(\overline{\mathrm{M}}_{0,5})$*, use the first relation above, it's given by the cokernel of the matrix in the first figure below. Carry out the standard elimination algorithm, we get the matrix in the second figure below, this tells us that*

$$\mathrm{rank(Pic)}(\overline{M}_{0,5}) = 5$$

*To be more precise, we have*

$$\delta_{1,2} = \delta_{1,3} + \delta_{2,4} - \delta_{3,4}$$
$$\delta_{1,3} = \delta_{1,4} + \delta_{2,3} - \delta_{2,4}$$
$$\delta_{1,4} = \delta_{1,5} + \delta_{3,4} - \delta_{3,5}$$
$$\delta_{2,3} = \delta_{2,5} + \delta_{3,4} - \delta_{4,5}$$
$$\delta_{2,4} = \delta_{2,5} + \delta_{3,4} - \delta_{3,5}$$
$$\delta_{1,5}, \delta_{2,5}, \delta_{3,4}, \delta_{3,5}, \delta_{4,5} \text{ are free generators}$$

*These computations tell us, Keel's description of* $\mathrm{CH}^*(\overline{M}_{0,n})$ *is not necessarily an easy and clean one. On the other hand, we can give a more geometric description of the computation of* $\mathrm{CH}^*(\overline{\mathrm{M}}_{0,5})$ *here by realizing*

| № | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | -1 | | | | -1 | | 1 | | |
| 2 | 1 | | -1 | | -1 | | | 1 | | |
| 3 | 1 | | | -1 | -1 | | | | 1 | |
| 4 | 1 | -1 | | | | | -1 | | 1 | |
| 5 | 1 | | -1 | | | | -1 | | | 1 |
| 6 | 1 | | | -1 | | -1 | | | | 1 |
| 7 | | 1 | -1 | | | | | | -1 | 1 |
| 8 | | 1 | | -1 | | | | -1 | | 1 |
| 9 | | | | | 1 | -1 | | | -1 | 1 |
| 10 | | | | | 1 | | -1 | -1 | | 1 |

Figure 3: Linear relations among $\delta_{i,j}$

$\mathrm{CH}^*(\overline{\mathrm{M}}_{0,5})$ *as a blow-up of* $\mathbb{P}^1 \times \mathbb{P}^1$ *along* 3 *distinct points on the diagonal. That is the universal curve diagram*

$$\begin{tikzcd}
\mathscr{U}_{g,n+1} \arrow[r, "contraction"] \arrow[d, "\pi_{n+1}"] & \mathscr{U}_{g,n} \times_{\overline{\mathrm{M}}_{g,n}} \mathscr{U}_{g,n} \arrow[r] \arrow[d] & \mathscr{U}_{g,n} \arrow[d, "\pi_n"] \\
\overline{\mathrm{M}}_{g,n+1} \arrow[r, "\cong"] & \mathscr{U}_{g,n} \arrow[r, "\pi_n"] & \overline{\mathrm{M}}_{g,n}
\end{tikzcd}$$

*specializes to*

$$\begin{tikzcd}
\overline{\mathrm{M}}_{0,5} \cong Bl_{(0,0),(1,1),(\infty,\infty)}(\mathbb{P}^1 \times \mathbb{P}^1) \arrow[r, "contraction"] \arrow[d, "\pi_4"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d, "\pi_3"] \\
\overline{\mathrm{M}}_{0,4} \arrow[r, "\cong"] & \mathbb{P}^1 \arrow[r, "\pi_3"] & \overline{\mathrm{M}}_{0,3} \cong pt
\end{tikzcd}$$

*The blow-up of* $\mathbb{P}^2$ *at two points is isomorphic to the blow-up of* $\mathbb{P}^1 \times \mathbb{P}^1$ *at one point. To see this, think about the geometry here* $Bl_{2pts}(\mathbb{P}^2)$ *contains three* $(-1)$*-divisors, namely,* $E_1, E_2$ *and* $H - E_1 - E_2$*(the strict transformation of the line passing the two points). If we blow down the divisor* $H - E_1 - E_2$*, Consider the ample divisor* $2H - E_1 - E_2$*, this corresponding to conics passing the two points, we get a degree 2 smooth surface in* $\mathbb{P}^3$*, over an algebraically closed field, it's isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$ *which just contracts the strict transformation of the line passing through the two points. Then we know* $\overline{\mathrm{M}}_{0,5} \cong Bl_{4pts}(\mathbb{P}^2)$*, the Chow ring structure is given by*

$$\mathrm{CH}^*(\overline{\mathrm{M}}_{0,5}) \cong \mathbb{Z}[H, E_1, E_2, E_3, E_4]/(H^2 - 1, HE_i, E_i^2 + 1, E_iE_j).$$

*Finally, we want to ask, what are those* $\delta_{i,j}$*'s in this ring?*

| № | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | -1 | 0 | 0 | 0 | -1 | 0 | 1 | 0 | 0 |
| 2 | 0 | 1 | -1 | 0 | -1 | 1 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | -1 | 1 | 0 |
| 4 | 0 | 0 | 0 | 0 | 1 | 0 | -1 | -1 | 0 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 1 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Figure 4: Linear relations among $\delta_{i,j}$

**Remark** (Keel, Chow ring of $\overline{\mathrm{M}}_{0,n}$)**.** *Sean Keel computed* $\mathrm{CH}^*(\overline{\mathrm{M}}_{0,n})$ *in the paper Intersection theory of moduli space of stable n-pointed curves of genus* 0*. Which says that* $\mathrm{CH}^*(\overline{\mathrm{M}}_{0,n})$ *is generated by boundary divisors* $\{\delta_S | S \subset \{1,2,\dots,n\}, \#S \geq 2, \#S^c \geq 2\}$ *subject to the following relations*

- $\delta_S = \delta_{S^c}$.
- *For any distinct* $i,j,k,l \in \{1,2,\dots,n\}$

$$\sum_{i,j\in S,k,l\in S^c} \delta_s = \sum_{i,k\in S,k,l\in S^c} \delta_s = \sum_{i,l\in S,j,k\in S^c} \delta_s$$

- $\delta_S\delta_T = 0$ *unless* $S \subset T, S \subset T^c, S^c \subset T, S^c \subset T^c$.

**Remark** (Blow-ups of $\mathbb{P}^2$). *Here, we have to be a little bit careful. We know $Bl_{2pts}(\mathbb{P}^2)$ is isomorphic to $Bl_{pt}(\mathbb{P}^1\times\mathbb{P}^1)$, but note that $\mathbb{F}_1 \cong Bl_{pt}(\mathbb{P}^2) \ncong \mathbb{P}^1\times\mathbb{P}^1$ because on $\mathbb{P}^1\times\mathbb{P}^1$ we don't have any $(-1)$-curve! This fact also tells us that $\mathbb{F}_1 := \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))$ is not minimal, all other $\mathbb{F}_n$'s are minimal! To conclude,*

- $Bl_{3 \text{ distinct points on } \Delta}(\mathbb{P}^1\times\mathbb{P}^1) \cong Bl_{4 \text{ general points}}(\mathbb{P}^2)$
- $Bl_{3 \text{ general points}}(\mathbb{P}^1\times\mathbb{P}^1) \cong Bl_{3 \text{ distinct points on } \Delta}(\mathbb{P}^1\times\mathbb{P}^1)$
- $Bl_{4 \text{ general points}}(\mathbb{P}^2) \ncong Bl_{4pts,\ 3 \text{ on a line}}(\mathbb{P}^2)$
- *If we blow-up three points on a line $L$ in $\mathbb{P}^2$, the total transformation $L'$ of the $L$ is in the class $H-E_1-E_2-E_3$, thus $L'^2=-2$, thus we get a $(-2)$-curve, the ordinary blow-up doesn't have any $(-2)$-curve.*

**Example 7.22** (Betti numbers and topological Euler characteristics of $\overline{\mathrm{M}}_{0,n}$).

**Example 7.23** (Canonical divisor of $\overline{\mathrm{M}}_{0,n}$).

**Example 7.24** ($\mathrm{rank}(H^2(\overline{\mathrm{M}}_{0,n},\mathbb{Z}))$).

**Remark** ($\mathrm{rank}(\mathrm{Pic}(\mathscr{K}_g))$).

**Example 7.25** (another computation of $\mathrm{Pic}(\mathscr{M}_{1,1})$).

**Example 7.26** (a reflexive sheaf but not locally free).

*reflexive sheaf*

**Example 7.27** (a coherent sheaf but not a quotient of locally free sheaf). *Consider $X=\overline{\mathbb{A}^n}$, the affine plane with double origins, $n\geq 2$. Then any locally free sheaf on $X$ is trivial. Because we know any locally free sheaf on $\mathbb{A}^n$ is free(Quillen-Suslin Theorem), to get a locally free sheaf on $X$, we need to glue two copies of locally free sheaf on $\mathbb{A}^n$ along $\mathbb{A}^n\setminus\{0\}$. $\Gamma(A^n\setminus\{0\})=\Gamma(\mathbb{A}^n)$, thus this isomorphism extends to $\mathbb{A}^n$, which means we can only glue two isomorphic free sheaves. Then 'any coherent sheaf on $X$ is a quotient of locally free sheaves' is equivalent of saying that 'every coherent sheaf on $X$ is globally generated', by a proposition in Hartshorne, this is equivalent of saying $\mathcal{O}_X$ is ample $\Rightarrow H^i(X,\mathscr{F})=0, \forall i\geq 1$, apply Serre's criterion of affiness, this tells us $X$ is affine, which is absurd. Let's denote the two copies of $\mathbb{A}^n$ by $U_0, U_1$, two origins $p,q$, consider the ideal sheaf $\mathscr{I}$ of $p$,*

$$U_0 : (x_1,\dots,x_n)$$

$$U_1 : k[y_1,\dots,y_n]$$

$$U_0 \cong U_1 : x_i \to y_i.$$

*Then $1\in\mathscr{L}_q$. But any global section of $\mathscr{L}$ must be of the form $\{(f,f)|f(0)=0\}$, thus $\mathscr{I}$ is not globally generated.*

**Remark.** *More discussions about this fact can be found here. We do have*

- *on a quasi-projective variety, any coherent sheaf is a quotient of a locally free sheaf.*

# 8 Hironaka's example

**Example 8.1** (Hironaka's example). *First we give the construction of Hironaka's example. Let $X$ be a projective 3-fold, $C,D$ be two curves in $X$ intersecting at two points $P,Q$ transversely. To be more precise, we can just consider*

$$X=\mathbb{P}^3_k=\mathrm{Proj}(k[x,y,z,w]), C=\mathrm{Proj}(k[x,y,z,w]/(xy-z^2,w)), D=\mathrm{Proj}(k[x,y,z,w]/(xy-w^2,z)),$$

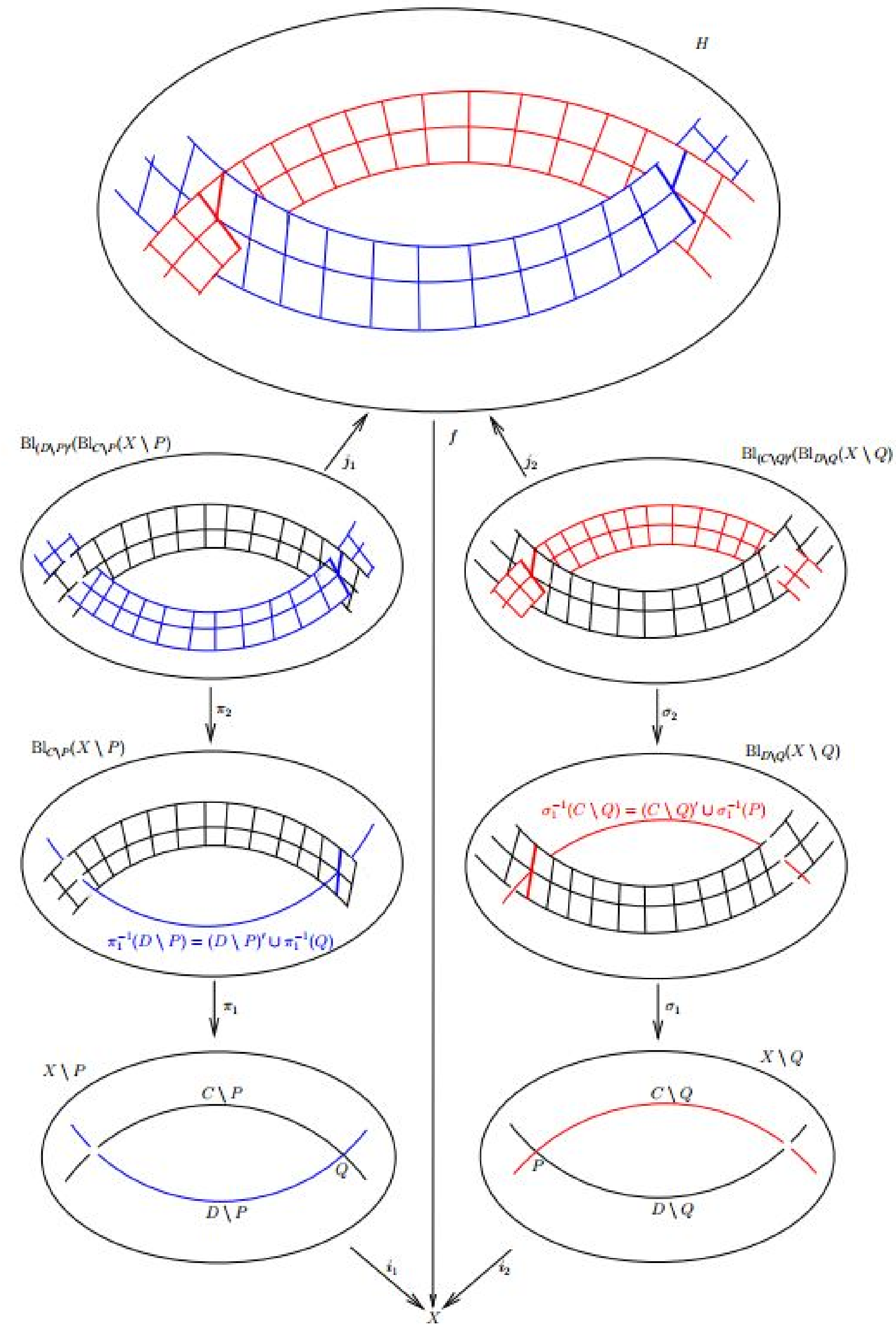



Figure 5: Hironaka's example of a complete but non-projective variety; Ulrich Thiel

$$P = [1, 0, 0, 0], Q = [0, 1, 0, 0].$$

*We have a $\mathbb{Z}/2\mathbb{Z} = \{1, g\}$-action on $X$ which switches $C, D$ and $P, Q$.*

$$g([x, y, z, w]) = [y, x, z, w].$$

*The construction is clear in the figure above(blow-up two open subscheme in different orders). Let $\pi = \pi_2 \circ \pi_1, \sigma = \sigma_2 \circ \sigma_1$. On $U = X \setminus \{P, Q\}$, the order of the blow-ups doesn't matter, so we can glue the two blow-ups along $\pi^{-1}(U) \cong \sigma^{-1}(U)$. Then we get a variety $H$. Let's do some local computations to find out the exceptional divisors. Locally,*

- *The first blow-up looks like $Bl_{(x,y)}\mathbb{A}^3_k$, which is just the closed variety in $\mathbb{A}^3_k \times \mathbb{P}^1_k$ defined by $xv - yv = 0$. The local picture of the exceptional divisor is a hypersurface given by $\mathrm{Proj}(k[x, y, z][u, v]/(xv - yu)) \cong \mathbb{A}^1_k \times \mathbb{P}^1_k$.*
- *The total transformation of $D$. Consider two affine charts of $Bl_C X$, $U_1 = D_+(u), U_2 = D_+(V)$, then we can write down the local equations*
  - $$Bl_C X \cap U_1 \cap \pi_1^{-1}(D) = \{y = vx, y = 0, z = 0\} = \{x = y = z\} \cup \{y = z = v = 0, x \neq 0\} \cong D',$$

    *where $D'$ is the strict transformation of $D$.*
  - $$Bl_C X \cap U_2 \cap \pi_1^{-1}(D) = \{yu = x, y = 0, z = 0\} = \{x = y = z\} = \pi^{-1}(Q) \cong \mathbb{P}^1_k := M_1.$$
- *Note that all the blow-ups are smooth, thus the local picture of $\pi_2$(also $\sigma_2$), but give us a different line as a new exceptional divisor $M_2$ over $Q$.*
- *Denote the exceptional hypersurfaces of $\pi_1, \pi_2, \sigma_1, \sigma_2$ by $E_1, E_2, F_1, F_2$, then we can glue $E_1$ and $F_2$ which gives us the red surface $S_1$ , glue $E_2$ and $F_1$ gives us the blue surface $S_2$, in the figure above.*
- *Let $f : H \to X$, then $f^{-1}(P) = \{L_1, L_2\} \subset S_1$, $f^{-1}(Q) = \{M_1, M_2\} \subset S_2$. Denote $f_1 = f|_{S_1}, f_2 = f|_{S_2}$.*

*Now we can prove $H$ is not projective nut complete.*

- *For projectivity, since $C, D$ are just plane quadrics, they're isomorphic to $\mathbb{P}^1_k$(not to say rational). Consider $A \in C \setminus \{P, Q\}, B \in D \setminus \{P, Q\}$, by pulling back to $S := S_1 \cup S_2$(the union of the red and the blue surface), we get*

  $$A \sim_C Q \Rightarrow f_1^{-1}(A) \sim_{S_1} M_1 + M_2,$$
  $$B \sim_D P \Rightarrow f_2^{-1}(B) \sim_{S_2} L_1 + L_2.$$

  *Similarly,*

  $$A \sim_C P \Rightarrow f_1^{-1}(A) \sim_{S_1} L_2,$$
  $$B \sim_D Q \Rightarrow f_2^{-1}(B) \sim_{S_2} M_2.$$

  *Now consider these relations in $S$, we get*

  $$L_1 + M_1 \sim_S 0.$$

  *We get an effective algebraic cycle equivalent to $0$! This is impossible for a projective variety. We conclude that $H$ is not projective.*
- *For completeness, we only need to prove that $\alpha : H \times Y \to Y$ is closed for any variety $Y$. Let $\beta X \times Y \to Y$ be the projection, then we have $\alpha = \beta \circ (f \times id)$. $\beta$ is a closed morphism since $X$ is projective. When restricted to $U_1 (f \times id)^{-1}(X \setminus P), U_2 = (f \times id)^{-1}(X \setminus Q)$, $f \times id$ is just a composition of blow-ups, locally, this gives us an isomorphism or the projection $\mathbb{A}^3_k \times \mathbb{P}^1_k \to \mathbb{A}^3_k$, which is closed. In short, $f \times id$ is closed and $H$ is universally closed, $H$ is compelte.*

**Remark.** *Note that*

$$\mathrm{Proj}(k[x,y,z,u,v]/(xv-yu)) \not\cong \mathrm{Proj}(k[x,y,z] \oplus (x,y) \oplus (x,y)^2 \oplus \dots).$$

*But we do have*

$$k[x,y,z] \oplus (x,y) \oplus (x,y)^2 \oplus \dots \cong \mathrm{Proj}(k[x,y,z][u,v]/(xv-yu)).$$

*Consider the natural map*

$$\mathrm{Proj}(k[x,y,z][u,v]/(xv-yu)) \to \mathrm{Proj}(k[u,v]),$$

*the fibre over* $(v)$ *is given by* $\mathrm{Proj}(k[x,y,z][u]/(yu))$*, but this is the same as* $\mathrm{Spec}(k[x,y,z])$ *since if a relevant homogeneous prime ideal contains* $yu$*, it must contains* $y$*. However, this remark is kind of redundant since the geometry is quite clear: the normal bundle* $N_C X|_C \cong \mathcal{O}(1)^2$*, or it just means planes passing* $L=(x,y)$ *are parametrized by* $\mathbb{P}^1$*.*

**Remark.** *This construction could be simplified by considering two curves intersecting transversely at only one point. Then we should get something like* $L_1 + L_2 \sim M_1, M_1 \sim L_2 \Rightarrow L_1 \sim 0$*.*

**Example 8.2** (Hironaka's example, Moishezen manifold of dimension 3 but not algebraic)**.**

**Example 8.3** (Hironaka's example, a deformation of Kähler manifolds that is not a Kähler manifold)**.** *First note that the obstruction comes from the fact that we can find an effective cycle on a special fibre* $V_0$ *is algebraically equivalent to* $0$*, hence holomorphically equivalent to* $0$*. Hence the integral of any closed* $(1,1)$*-form cannot give a positive definite hermitian metric. This implies* $V_0$ *isn't Kähler.*

*The idea of Hironaka's construction is the same as above: gluing blow-ups on different open subvarieties. Let*

$$X = \mathrm{Proj}(k[x,y,z,w]) = \mathbb{P}^3_k, W_0 = D_+(x), W_1 = D_+(y).$$

$$I_{C_1} = (z,w), I_{C_2} = (xy+yz+zx, w), I_{C_{3,t}} = ((y+w)(x+ty)+yw, z).$$

- $t=0$*, these three curves are smooth with common point* $p=[1,0,0,0], q=[0,1,0,0]$*.*
- $t \neq 0$*,* $C_1 \cap C_2 \cap C_{3,t} = p = [1,0,0,0], C_1 \cap C_{3,t} = \{p, q_t = [-t,1,0,0]\}$*.*

*Now we start constructing the two blow-ups,*

$$T := \mathbb{A}^1_k, H = X \times T$$

$$F_1 = C_1 \times T, F_2 = C_2 \times T, F_3 = C_{3,t},$$

$$P = p \times T, Q = q \times T,$$

$$U_0 = W_0 \times T = \mathrm{Spec}(k[z_1,z_2,z_3,t]), U_1 = W_1 \times T = \mathrm{Spec}(k[y_0,y_2,y_3,t]),$$

$$U' = W_0 \cap D_+(x+ty) = \mathrm{Spec}(k[z_1,z_2,z_3,t,(1+tz_1)^{-1}]). P \subset U'.$$

*We're going to work over* $H = (H \setminus P) \cup U'$ *(they play the roles of* $X-P$ *and* $X-Q$ *in the example above respectively.) Let*

$$x_1(t) = (1+tz_1)(z_1+z_2+z_3+z_1z_2) + z_1z_3, x_2 = z_2, x_3 = z_3.$$

*Then the prime ideals of* $F_1, F_2, F_3$ *in* $U'$ *are given by* $(x_2,x_3),(x_3,x_1(t)),(x_1(t),x_2)$ *respectively. We glue* $Bl_{\cup_{i=1}^3 C_i}(H \setminus P)$ *and* $Bl_{I(t)}(U')$ *along* $U' \cap H \setminus P$*, where*

$$I(t) = (x_1(t)x_2, x_2x_3, x_3x_1(t))(x_1(t), x_2x_3)(x_2, x_3x_1(t))^2(x_3, x_1(t)x_2)^2.$$

*Note that in* $U'$ *we have a primary decomposition*

$$I(t) = (x_2,x_3)^5 \cap (x_3,x_1(t))^4 \cap (x_1(t),x_2)^4 \cap (x_1(t),x_2,x_3)^7.$$

*From this we can prove the following*

- *two blow-ups are isomorphic in the overlap.*
- *the fibre $V_0$ is not projective, not a Kähler manifold.*
- *$V_0$ is smooth.*
- *all other fibres $V_t, t \neq 0$ are smooth projective variety, hence are all Kähler manifold.*

*To demonstrate, we prove $V_0$ is smooth over p, which means we have to prove the blow-up of $\mathbb{A}^3_k$ of the ideal $(x_1x_2, x_2x_3, x_3x_1)(x_1, x_2x_3)(x_2, x_3x_1)^2(x_3, x_1x_2)^2$ is smooth. We first blow-up the ideal $I_1 = (x_1, x_2, x_3, x_3x_1)$ and then the rest three. This is just a local computation*

$$X' = Bl_{I_1}(\mathbb{A}^3_k) = \mathrm{Proj}(k[x_1, x_2, x_3][a, b, c]/(x_1x_2b = x_2x_3a, x_1x_2c = x_3x_1a, x_2x_3c = x_3x_1b)) \subset \mathbb{A}^3_k \times \mathbb{P}^2_k.$$

*This is cyclic symmetric, we only need to consider $D_+(a)$, and let $[u, v] = [\frac{b}{a}, \frac{c}{a}]$, then this first-step blow-up is given by*

$$x_3 = ux_1, x_3 = vx_2, x_2v = x_1u \Rightarrow Bl_{I_1}(X) \cong \{x_2v = x_1u\} \subset \mathbb{A}^4_k.$$

*The origin is the only singularity. Now $(x_1, x_2x_3), (x_2, x_1x_3)$ becomes principal ideals and so in the second step, we only need to blow-up the ideal $(x_3, x_1x_2) = (x_1) \cap (u, x_2)$, that is, just blow-up the ideal $J = (u, x_2)$. Then $X'' = Bl_J(X')$ is a closed subvariety in $X' \times \mathbb{P}^1_k$ given by the equations $u\beta = x_2\alpha$. Then in $D_+(\alpha)$, let $z = \frac{\beta}{\alpha}$, $X''$ is given by*

$$x_1 = vz, x_2 = vz, x_3 = uvz.$$

*This is smooth, isn't it? Hence, we're done. we also have a question*

*Is $H$ smooth, and why?*

**Remark** (Blow-ups)**.** *Note the following*

- *blow-up is not sensible to the non-reduced structures.*
- *the blow-up of the ideal $I = (x_1x_2, x_3)^4 \cap (x_2, x_3)$ is smooth, why we have to argue like 'after the blow-up of the ideal $(x_2, x_3)$, the pull-back becomes a principal ideal or the ideal of a smooth curve, hence the blow-up of the $I$ is smooth?'*

**Remark.** *Why some closed $(1, 1)$-form on a Kähler manifold should give me positive definite hermitian metric by integration against topological 2-cycles?*

**Remark** (why this construction doesn't work with two curves?)**.** *This is a typical phenomenon in algebraic geometry: we can define a family of curves(e.g. quadrics) $X_t$ over some base, although topologically, every fibre is quite clear, however, you might get some non-reduced fibres, this makes it hard to talk about concepts like complex structure, Kähler metric, because the correspondence provided by GAGA doesn't exist anymore. For example, consider a family of skew lines in $\mathbb{A}^3_k$ parametrized by $\mathbb{A}^1_k$ :*

$$X_t = \mathrm{Spec}(k[x, y, z][t]/(x, y) \cap (x - t, z)) = \mathrm{Spec}(k[x, y, z][t]/(yz, xz, xy - yt, x^2 - xt)).$$

*Then it's obvious $\mathcal{O}_{X_0} = k[x, y, z]/(yz, xz, xy, x^2)$ is not reduced!*

**Remark** (symplectic manifold, non-Kähler)**.** *First recall that one of the obstructions comes from the existence of the Hodge decomposition on a Kähler manifold. Kodaira-Thurston's example below is a symplectic 4-manifold with odd first Betti number, thus non-Kähler. i.e.*

$$X = (T^3 \times \mathbb{R})/\mathbb{Z}, n \bullet (x, y, z, t) = (x, y + nx, z, t + n).$$

$$\omega_X = dx \wedge dy + dz \wedge dt.$$

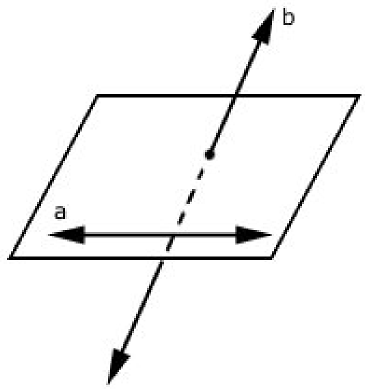



Figure 6: Family of skew lines, non-reduced limit

*We can view $X$ as a symplectic torus fibration over the $(z,t)$-torus or as a Lagrangian torus fibration over the $(x,z)$-torus. Let's use the first fibration. The natural map $\pi : \mathbb{R}^4 \to X$ is the universal covering of $X$, the Deck transformation can be read from*

$$(x, y, z, t) \mapsto (x + a, y + b + d(x + a), z + c, t + d).$$

*i.e we can view it as a (non-commutative) group structure $\Gamma$ defined on the set $\mathbb{Z}^4$,*

$$(a, b, c, d) \star (a', b', c', d') = (a + a', b' - d'a, c + c', d + d').$$

*Thus* $\mathrm{rank}([\Gamma, \Gamma]) = 1$*, the commutator subgroup is generated by* $(0, 1, 0, 0)$*, it gives us the abelianization*

$$H_1(X, \mathbb{Z}) \cong \Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}^3.$$

*This cannot be a Kähler manifold. McDuff also has an example based on Kodaira-Thurston's. For details, see*

$$http : //www.homepages.ucl.ac.uk/\ ucahjde/ST - lectures/lecture11.pdf.$$

**Remark** (a finite subset of a variety need not be contained in an open affine subvariety)**.**

**Example 8.4** (Hironaka's example, a variety with no Hilbert scheme)**.**

**Example 8.5** (Stacks project Tag O8KE, Descent data for schemes need not be effective, even for a projective morphism)**.** *This is still based on Hironaka's example. Let* $k = \mathbb{C}$*. We first introduce the players:*

- $$\mathbb{P}^3 = \mathrm{Proj}(k[x, y, z, w])$$
- *two curves in* $\mathbb{P}^3$
$$C = \mathrm{Proj}(k[x, y, z, w]/(xy - z^2, w))$$
$$D = \mathrm{Proj}(k[x, y, z, w]/(xy - w^2, z)).$$
- *they intersect at*
$$P = [1, 0, 0, 0], Q = [0, 1, 0, 0].$$
- *two lines in* $\mathbb{P}^3$
$$l_1 : [x, x, z, z], l_2 : [x, -x, z, -z].$$

- *a group action of* $G = \mathbb{Z}/2\mathbb{Z} = \{1, g\}$ *on* $\mathbb{P}^3$

$$g \bullet [x, y, z, w] = [y, x, w, z].$$

- $l_1 \cup l_2$ *is the fixed locus of the* $G$*-action, thus* $G$ *acts freely on*

$$Y = \mathbb{P}^3 \setminus \{l_1 \cup l_2\}.$$

- *quotient* $S = Y/G$ *exists as a quasi-projective scheme, explicitly* $S$ *is the image of the open subscheme* $Y$ *under the morphism*

$$\mathbb{P}^3 \to \mathrm{Proj}(k[x, y, z, w]^G) = \mathrm{Proj}(k[u_0, u_1, v_0, v_1, v_2]/(v_0v_1 - v_2^2)),$$

  *where*

$$u_0 = x + y, u_1 = z + w, v_0 = (x - y)^2, v_1 = (z - w)^2, v_2 = (x - y)(z - w).$$

  *Note that* $\mathrm{Proj}(k[u_0, u_1, v_0, v_1, v_2])$ *is the weighted projective pace* $\mathbb{P}(1, 1, 2, 2, 2)$*, not the ordinary one.*

- *Hironaka's construction on* $Y$ *(previously, Hironaka's construction starts with* $\mathbb{P}^3$*, here we delete* $l_1 \cup l_2$ *to make it easier to talk about quotient, nothing essential changes), denote the complete but not projective variety over* $Y$ *be to*

$$\pi : V_Y \to Y.$$

- *an open cover(in any sense you like) of* $Y$*.*

$$X = (Y - P) \sqcup (Y - Q).$$

*Now the group action on* $Y$ *lifts to a group action on* $V_Y$ *by properly switching the preimage of* $C, D$*, this gives us a descent datum* $(V_Y/Y, \phi_Y)$ *relative to the* $G$*-torsor* $Y \to S$ *(what does mean?). Consider the diagram of natural pull-backs*

$$\begin{array}{ccc}
V & \xrightarrow{p} & X \\
\Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\
V_Y & \xrightarrow{\pi} & Y \\
\Big\downarrow{\scriptstyle h'} & & \Big\downarrow{\scriptstyle h} \\
U & \xrightarrow{\theta} & S.
\end{array}$$

*Note that the composition(and the two arrows themselves)* $X \to Y \to S$ *are étale covers.* $p$ *is projective since when restricted to* $Y - P$ *or* $Y - Q$*,* $\pi$ *is given by blow-ups. And the descent datum* $(V_Y/Y, \phi_Y)$ *pulls back to a descent datum* $(V/X, \phi)$*. If the descent datum* $(V/X, \phi)$ *is effective in the category of schemes, then* $U$ *must exist as a scheme and its corresponding descent datum pulls back to* $(V/X, \phi)$*. Then* $U$ *is the quotient of* $V_Y$ *of the* $G$*-action. Let* $E = \pi^{-1}(C \cup D)$*, we use the notation and result in the first example, then we have*

$$\pi^{-1}(P) = L_1 \cup L_2, \pi^{-1}(Q) = \pi^{-1}(gP) = M_1 \cup M_2,$$

$$g(L_1) = M_1, g(L_2) = M_2,$$

$$L_1 + M_1 = L_1 + g(L_1) \sim 0.$$

*By descent of closed subschemes(actually by descent of affine morphims), we can find a copy of* $\mathbb{P}^1 \cong L \subset U$ *such that* $h^{-1}(L) = L_1 \cup g(L_1) = L_1 \cup M_1$*. Chose a complex point* $R$ *(thus not the generic point), thus we can find a function* $f \in \mathcal{O}_{U,R}$ *in the local ring at* $R$ *such that* $f(R) = 0$*, but* $L \nsubseteq V(f)$*. Fix an irreducible component of* $V(f)$ *containing* $R$*, denote it by* $W$*, it has codimension 1 in* $\mathrm{Spec}(\mathcal{O}_{U,R})$*. Then we have*

$$\emptyset \neq h'^{-1}(W) \cap (L_1 \cup g(L_1))$$

$$L_1 \nsubseteq h'^{-1}(W), g(L_1) \nsubseteq h'^{-1}(W).$$

*We give a name to effective divisor $h'^{-1}(W)$, call it $D$. We know $V_Y$ is smooth, thus $\mathcal{O}(D)$ is a line bundle on $V_Y$. We naturally get a line bundle on $E = \pi^{-1}(C \cup D)$((the 'red and blue surface'). This line bundle on $E$ has a positive intersection number on the 1-cycle $L_1 \cup g(L_1)$, however we have $L_1 + g(L_1) \sim 0$! This contradicts the fact that on any proper smooth schemes over a field, the degree of a line bundle is well defined. The only possibility is that $U$ doesn't live in the category of schemes.The descent datum $(V/X, \phi)$ is not effective.*

**Remark** (descent datum)**.** *Some remarks on basic descent theory*

- *descent for affine morphisms*
- *relative descent datum and group actions.*
- *descent datum and quotients.*

**Remark** (where do we use the condition that $U$ is a scheme?)**.** *when do we know a point is not the generic point of a subvariety?*

**Remark** (Hironaka's example, the quotient of a scheme by a free action of a finite group need not be a scheme)**.** *It's clear from the discussion above.*

**Example 8.6** (Hironaka's example, a scheme of finite type over a field such that not every line bundle comes from a divisor)**.**

**Remark.** *In practice, we only need to assume that $X$ is reduced or projective to avoid this pathological phenomenon.*

$$http : //stacks.math.columbia.edu/tag/08KE$$

$$http : //stacks.math.columbia.edu/tag/08KF$$

$$https : //perso.univ{-}rennes1.fr/matthieu.romagny/GT_Hilb/Nitsure_Construction_of_Hilbert_and_Quot_schemes.pdf$$

$$https : //ncatlab.org/ericforgy/published/Notes+on+Grothendieck+Topologies, +Fibered+Categories+and+Descent+$$

$$https : //arxiv.org/pdf/math/0412512.pdf$$

$$http : //download.springer.com/static/pdf/724/bok(p14)$$

Mumford, Geometric invariant theory

https://books.google.com/books?id=jAWVmIz80A4Cpg=PA11lpg=PA11dq=Kleiman+example+nonreduced++nonproject
$ensa = Xved = 0ahUKEwjIy8vi1OvTAhVh9IMKHTGJAYYQ6AEIIjAAv = onepageq = Kleimanexamplenonreduce$
$falseMUmford, invariantfinitelygenerated, appendixinGIT$

**Example 8.7** (reduced scheme, after base-change nowhere reduced, Maclane)**.** *Let $k = \mathbb{F}_p(u, v)$,*

$$C := \operatorname{Spec}(A) = \operatorname{Spec}(k[x, y]/(ux^p + vy^p - 1)) \subset \mathbb{A}^2_k.$$

*Then*

- *$A$ is Dedekind*
- *$k \subset \operatorname{Frac}(A)$ is relatively algebraically closed*
- *$k' = \mathbb{F}_p(u^{\frac{1}{p}}, v^{\frac{1}{p}})$, then $C \times_{k'} k$ is nowhere reduced.*

**Example 8.8** (strict normal crossing divisor).

**Example 8.9** (A non catenary Noetherian local ring, Stacks project, Tag 02JE).

$$http://stacks.math.columbia.edu/tag/02JE$$

**Remark** (Cohen-Macaulay rings, Dedekind domains, Complete local noetherian rings , Regular rings).

**Example 8.10** (a discrete valuation ring but not a $G$-ring). *BLR, Neron model,page82, example 11. And in Qing Liu's book, Example* 8.2.31

**Example 8.11** (Akizuki's example, a noetherian local integral domain whose normalization is not finite). *Let*

- $A$, *a* DVR
- $t$, *the local parameter*
- $k = A/(t)$, *the residue field*
- $\hat{A}$, *the completion of* $A$ *w.r.t* $(t)$
- *assume* $\hat{A}$ *has a transcendental element over* $A$

*For example, we can take*

- $A = \mathbb{Z}_{(p)}$, *the localization of* $\mathbb{Z}$ *at the prime ideal* $(p)$
- $t = p$, *the local parameter*
- $k = \mathbb{F}_p$
- $\hat{A} = \mathbb{Z}_p$, *the ring of* $p$*-adic integers*

*or*

- $A = k[t]_{(t)}$, *the localization of* $k[t]$ *at the prime ideal* $(t)$
- $t$, *the local parameter*
- $k$, *the residue field*
- $\hat{A} = k[\![t]\!]$, *the ring of formal power series*

*We first give the construction of Akizuki's example,*

- $z = z_0 = a_0 + a_1 t^{n_1} + \cdots + a_k t^{n_k} + \cdots \in \hat{A}$ *such that*
  - $a_i \in A^{\times}$
  - $n_{i+1} > 2n_i + 2$
  - $z$ *is transcendental over* $A$, *thus* $A[z] \subset \hat{A}$ *is just the polynomial ring of one variable over* $A$
  - $A = \mathbb{Z}_{(p)}$, *we may take* $z = \sum \frac{1}{(p+1)^n} p^{n!}$. $A = k[t]$, *we may take* $z = \sum t^{n!}$
- $z_r = \frac{z_0 - \textit{first } r \textit{ terms}}{t^{n_r}} = a_r + a_{r+1} t^{n_{r+1} - n_r} + \ldots$. *Let* $m_r = n_r - n_{r-1}$, *we have* $m_r > n_{r-1} + 2$.
- $z_r - a_r = t^{m_{r+1}} z_{r+1}$
- $t^{n_r} z_r = z_0 - \sum_{i=0}^{r-1} a_i t^{n_i}$

- $$B = A[z_0, z_1, \dots] = A[(z_0 - a_0), (z_1 - a_1) \dots]$$
$$B_m = A[z_0, a_1, \dots]_{(t)}$$
$$C = A[t(z_0 - a_0), \{(z_i - a_i)^2\}_{i=1}^{\infty}]$$
$$C_M = A[t(z_0 - a_0), \{(z_i - a_i)^2\}_{i=1}^{\infty}]_{(t)}$$

- *The final claim is that*

$$C_M \text{ is a 1-dimensional integral noetherian local ring}$$

$$B_m \text{ is the normalization of } C_M \text{ in } \mathrm{Frac}(C_M) = \mathrm{Frac}(B_m) = A(z)$$

$$B_m \text{ is not finite over } C_M$$

*Now let's give the proof. The main observation is that*

$$0 \neq f \in M, \text{ the principal ideal } fC_M \subset C_M \text{ contains some power of } t$$

*First, we should ask, what an element $f \in C$ looks like? Well, it's a polynomial in $t(z_0 - a_0), (z_i - a_i)^2$, topologically, we want to describe $f$ as a point in some small neighborhood of a point in $A$. How to do this? Just by raising the power of $t$. It's not hard because we can*

- *we can always replace $t(z_0 - a_0)$ by $t^{n_i+1}(z_i - a_i)$, the difference lies in $A$. Replacing $t^{n_i+1}(z_i - a_i)$ by $t^{n_j+1}(z_j - a_j)$ if we want to raising the power of $t$ further.*
- *we can replace $(z_{i-1} - a_{i-1})^2$ by $t^{2m_i}(z_i - a_i)^2$ + multiple of $t^{n_i+2}(z_i - a_i)$ + somthing in $A$. This is because (remember $m_i > n_i + 2$)*

$$(z_{i-1} - a_{i-1})^2 = (t^{m_i} z_i)^2 = t^{2m_i}((z_i - a_i)^2 - a_i^2) + 2a_i t^{2m_i} z_i$$

- *That is, any $f \in C, r, N$ positive integers, we can write it as*

$$f = \alpha + \beta t^{n_r+1}(z_r - a_r) + t^N \theta, \alpha, \beta \in A, \theta \in C.$$

- *$0 \neq f \in M$, thus exists $N > 0$ such that $f \notin t^N \hat{A}$, we may choose $r$ such that $n_r + 1 > N$. Then we know $\alpha = t^k u$, and $k < N, u \in A^\times$. Dividing both sides by $u$, we may assume*

$$f = t^k(1 + t^{N-k}\theta) + \beta t^{n_r+1}(z_r - a_r)$$

*Let*
$$g = t^k(1 + t^{N-k}\theta) - \beta t^{n_r+1}(z_r - a_r)$$

*We have*
$$fg = t^{2k}(1 + t^{N-k}\theta)^2 - \beta t^{2n_r+2}(z_r - a_r)^2 = t^{2k}v, v \in C \setminus M$$

*Then we know $t^{2k} \in fC_M$.*

- *It follows that any proper prime ideal in $C_M$ contains $t$, thus $\mathrm{Spec}(C_M) = \{0, C_M\}$, in other words, it's a 1-dimensional local ring.*
- *To see $C_M$ is noetherian, first notice that $C_M/(t^N C_M)$ is a finite $A/(t^N)$-module generated by $\{1, t^{n_r+1} z_{r+1}\}$. Now any nonzero ideal $I \subset C_M$, we know $(t^N) \subset I$, thus $C_M/I$ is a quotient of $C_M/(t^N C_M)$ as an $A$-module, thus Noetherian.*
- *Now it's clear that $C_M$ is an integral, noetherian local ring of dimension 1.*

- *We only need to prove that $B_m$ is not finite over $C_M$. Just consider the ascending chain of submodules generated by $\{z_i - a_i\}_{i=1}^r$, if it's stable, then we must have*

$$z_r - a_r = \sum_{i=1}^{r-1} \theta_i (z_i - a_i), \theta_i \in C_M.$$

*We may assume $\theta_i = \frac{\alpha_i}{\gamma}$, wher $\alpha_i \in C, \gamma \notin M$, we get*

$$\gamma(z_r - a_r) = \sum_{i=1}^{r-1} \alpha_i (z_i - a_i).$$

*Multiplying both sides by $t^{n_r}$, we get an algebraic equation w.r.t $z$*

$$\gamma(z - \sum_{i=0}^{r+1} a_i t^{n_i}) = \sum_{i=1}^{r-1} \alpha_i t^{n_r - n_i} (z - \sum_{j=0}^{j+1} a_j t^{n_j})$$

*Note that $\gamma$ is a unit in $C_M$, coefficients on the right hand side are all divided by some positive power of $t$, every $\alpha_i$ is a polynomial in $z$ over $A$. Thus this equation is an non-trivial polynomial, whcih contradicts the fact that $z$ is transcendental over $A$. Now everything is clear.*

*More details and discussion of Akizuki's example can be found in Akizuki's counterexample*

**Remark.** *For integral schemes of finite type, the normalization is always finite. My personal feeling is that, to consider geometric questions, we have to assume a scheme is of finite type over $k$ or $\mathbb{Z}$ at some point.*

**Remark.** *p-adic completion of the integers equals the completion*

**Example 8.12** (a flat and finite type morphism between noetherian schemes, fibres have different dimensions)**.** *Let $R$ be your favorite DVR (for example, $\mathbb{Z}_{(p)}, k[t]_{(t)}$). Our palyers are*

- *$Y = \mathrm{Spec}(R)$, $(0)$ is the generic point, which is open in $Y$ with residue field $K = \mathrm{Frac}(R)$, $\mathfrak{m}$ is the unique maximal ideal, which is closed in $Y$ with residue field $k = R/\mathfrak{m}$.*
- *Glue $\mathbb{A}_R^1 = \mathrm{Spec}(R[x])$ along $(\mathbb{G}_m)_K \subset \mathbb{A}_R^1$ with a union of $\mathbb{A}_K^1$ and $\mathbb{P}_K^2$ meeting at the origin by inversion. That is, at the generic fibre, we get a union of $\mathbb{P}_K^1$ and $\mathbb{P}_K^2$. Denote the glued scheme by $X$.*
- *$\pi : X \to Y$ gives us a flat morphism since all modifications happen at the generic fibre.*
- *$\pi$ is not proper because $\mathbb{P}_K^2$ is a closed subset of $X$, but the image of it is the genric point which is not closed.*
- $\dim(Y_{(0)}) = 2$*(not pure)*, $\dim(Y_{\mathfrak{m}}) = 1$. *.....*

**Remark.** *In fact, we have*

- *Let $f : X \to Y$ be a proper flat and finitely presented morphism(no assumption on irreducibility, noetherian hypothesis), $y \mapsto \dim X_y$ is a locally constant function.*
- *Let $f : X \to Y$ be a flat and finite type morphism between noetherian schemes, $d := \dim X_{\eta_Y}$, then all geometric fibres $X_{\bar{y}}$ has pure dimension $d$.*

**Example 8.13** (non-Gorenstein variety)**.** *Hyman Bass and Ubiquity: Gorenstein Rings*

**Example 8.14** (valuation rings and discrete valuation rings)**.**

**Example 8.15** (Brian Conrad's notes, page69, a semi-stable curve over a noetherian scheme with smooth generic fibre is normal)**.**

**Example 8.16** (smooth morphism, isomorphic generic fibre, nonisomorphic special fibres, Shizhang)**.** *Let* $V = \mathcal{O} \oplus \mathcal{O}(2), W = \mathcal{O} \oplus \mathcal{O}$ *over* $\mathbb{P}^1_{k[\![t]\!]}$

$$\begin{tikzcd} \mathbb{P}(V) \arrow[dr, "f"] & & \mathbb{P}(W) \arrow[dl, "g"'] \\ & \mathrm{Spec}(k[\![t]\!]) & \end{tikzcd}$$

**Remark** (no fine Moduli space for semi-stable vector bundles)**.** *First note that this example can be extended to all* $F_n$ *surfaces.*

**Example 8.17** (smooth varieties over a DVR, isomorphic generic fibres, nonisomorphic special fibres)**.** *Here*

**Example 8.18** (resolution of singularities of semistable curves over a discrete valuation ring)**.** *Let* $R$ *be a discrete valuation ring(e.g* $k[\![t]\!]$*),* $\pi$ *a uniformizer and* $n \geq 1$*, consider*

$$C_n : \mathrm{Spec} R[x,y]/(xy - \pi^n).$$

*Then the generic fibre is smooth(*$\pi$ *is invertible in* $\mathrm{Frac}(R)$*), its special fibre is singular at a unique point* $p = (x, y)$*, the completion is given by*

$$\widehat{\mathscr{O}_{C_n,p}} = \widehat{R}[x,y]/(xy - \pi^n)$$

*Brian Conrad's notes on alteration page78.*

**Example 8.19** (Henselization)**.** *Stacks project Tag 03QD Wikipedia Henselian rings and example of algebraic formal power series*

**Example 8.20** (page 11, imperfect field geometrically integral curve , $k(x)/k$ not separable)**.**

**Example 8.21** (Complete local rings)**.** *Complete local rings are not hard to get. Just take any local ring and compute the completion. For example*

- $R = k[\![x,y]\!]/(xy)$ *is the completion of the local ring at the node of the nodal curve* $y^2 = x^2(x+1)$*. It's not regular. Since the minimal number of generators of* $\mathfrak{m} = (x,y)$ *is 2. But the Krull dimension is just* 1*. To see this any prime* $\mathfrak{p}$ *is generated by some* $x^n u$ *and* $y^m v$ *where* $u, v$ *are invertible elements in this complete ring. For the ideal to be a prime, the only possibility is that* $n = m = 0$*. Notice that* (0) *is not a prime ideal of this ring. We get* $\dim R = 1$*.*
- $R = \mathbb{Z}_p[\![x,y]\!]/(xy - v)$ *is the completion of* $\mathbb{Z}_p[x,y]/((x^2 - 2 + y^2)(x^2 - y^2) + p^r y)$*.*

**Example 8.22** (A regular local ring does not contain a field)**.** *The localization of* $\mathbb{Z}[X]$ *at the maximal ideal* $(2, x)$*, that is* $\mathbb{Z}[x]_{(2,x)}$ *is a dimension* 2 *regular local ring which does not contain any field.*

# 9 Weights

**Example 9.1** ($X = \mathbb{C}^\times \hookrightarrow \mathbb{P}^1_{\mathbb{C}} = \overline{X}$, weight filtration)**.** $D = \overline{X} \setminus X = \{*, *\}$*. The weight spectral sequence is given by*

$$E_1^{p,q} = \begin{matrix} 2 \xrightarrow{d_1} 1 \\ \\ 0 \qquad 0 \\ \\ 0 \qquad 1 \end{matrix} \Rightarrow E_2^{p,q} = Gr_q H^{p+q}(X) = \begin{matrix} 1 \qquad 0 \\ \\ 0 \qquad 0 \\ \\ 0 \qquad 1 \end{matrix}.$$

*$d_1$ is surjective is because we know $E_2^{p,q}$ computes $H^{p+q}(X)$, which is just the cohomology of $S^1$. The filtration of $H^1(X)$ is given by*

$$0 = W_0H^1(X) \subset 0 = W_1H^1(X) \subset W_2H^1(X) = H^1(X;\mathbb{Q}) = \mathbb{Q}$$

$$Gr_2^W H^1(X;\mathbb{Q}) = W_2H^1(X;\mathbb{Q})/W_1H^1(X;\mathbb{Q}) \cong \mathbb{Q}.$$

**Remark** (Weight filtration, reference, Joana Cirici )**.** *Let $X$ be a smooth complex variety of dimension $n$. Let $j : X \hookrightarrow \overline{X}$ be a smooth compatification of $X$ such that $D = \overline{X}\backslash X$ is a normal crossing divisor. $D = \bigcup_{i=1}^N D_i$ is the decomposition of $D$ into irreducible components. Let $D^{(0)} = \overline{X}$, $D^{(p)} = \{\amalg_{\{i_1,\ldots,i_p\}\subset\{1,2,\ldots,N\}} D_{i_1} \cap \cdots \cap D_{i_p}\}$, note that since $D$ is a normal crossing divisor, $D^{(p)}$ is a smooth projective variety of dimension $n-p$. The weight spectral sequence is given by*

$$E_1^{-p,q} = H^{q-2p}(D^{(p)};\mathbb{Q}) \Rightarrow H^{q-p}(X;\mathbb{Q}).$$

*The differential on the $E_1$-page is given by the sum of Gysin maps*

$$d_1 : E_1^{-p,q} \to E_1^{-p+1,q}$$

$$i_*(j) : H^{q-2p}(D_{i_1} \cap \cdots \cap D_{i_p}) \to H^{q-2p+2}(D_{i_i} \cap \ldots \hat{D}_{ik} \cdots \cap D_{i_p})$$

*, where $i$ is just the natural embedding. The weight spectral sequence degenerates on the $E_2$-page which gives us a filtration of the form*

$$0 = W_{n-1}H^n(X;\mathbb{Q}) \subset W_nH^n(X;\mathbb{Q}) \subset \cdots \subset W_{2n}H^n(X;\mathbb{Q}) = H^n(X;\mathbb{Q}),$$

*where $Gr_q^W H^{p+q} = E_2^{p,q}$ and $W_{k+1}H^n(X;\mathbb{Q})/W_kH^n(X) = Gr_{k+1}^W H^n(X)$. To make this discussion more obvious, see the following diagrams.*

$$E_1^{-p,q}(X) = H^{q-2p}(D^{(p)};\mathbb{Q}) = \begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
H^{(1)}(D^{(2)}) & \xrightarrow{d_1} & H^3(D^{(1)}) & \xrightarrow{d_1} & H^5(\overline{X}) & & 5 \\
H^0(D^2) & \xrightarrow{d_1} & H^2(D^{(1)}) & \xrightarrow{d_1} & H^4(\overline{X}) & & 4 \\
0 & & H^1(D^{(1)}) & \xrightarrow{d_1} & H^3(\overline{X}) & & 3 \\
0 & & H^0(D^{(1)}) & \xrightarrow{d_1} & H^2(\overline{X}) & & 2 \\
0 & & 0 & & H^1(\overline{X}) & & 1 \\
0 & & 0 & & H^0(\overline{X}) & & q=0 \\
D^{(2)} & & D^1 & & D^{(p=0)} = \overline{X} & &
\end{array}$$

$$\Rightarrow E_2^{p,q}(X) = Gr_q H^{p+q}(X) \qquad \begin{array}{cccc}
\cdots & \cdots & \cdots & \\
Gr_5^W H^3(X) & Gr_5^W H^4(X) & Gr_5^W H^5(X) & H^5 \\
Gr_4^W H^2(X) & Gr_4^W H^3(X) & Gr_4^W H^4(X) & H^4 \\
0 & Gr_3^W H^2(X) & Gr_3^W H^3(X) & H^3 \\
0 & Gr_2^W H^1(X) & Gr_2^W H^2(X) & H^2 \\
0 & 0 & Gr_1^W H^1(X) & H^1 \\
0 & 0 & Gr_0^W H^0(X) & H^{q=0} \\
2 & 1 & p=0 &
\end{array}$$

**Remark** (Gysin map)**.**

**Example 9.2** (punctured Riemann surface)**.** *Let $\overline{X}$ be a Riemann surface of genus $g$. $D$ is given by $p$ distinct points, $X = \overline{X} \setminus D$. Then we have*

$$E_1^{p,q} = \begin{array}{cc} p \xrightarrow{d_1} & 1 \\ 0 & 2g \\ 0 & 1 \end{array} \Rightarrow E_2^{p,q} = Gr_q H^{p+q}(X) = \begin{array}{cc} p-1 & 0 \\ 0 & 2g \\ 0 & 1 \end{array}.$$

*Note that we don't need to do any computation, since we know $X$ is homotopic to some wedge product of several $S^1$'s, thus $H^2(X;\mathbb{Q})$ and above all vanish. This gives us the surjectivity of $d_1$. The only interesting part is the filtration of $H^1(X;\mathbb{Q})$ which is given by*

$$0 = W_0 H^1(X;\mathbb{Q}) \subset W_1 H^1(X;\mathbb{Q}) \cong \mathbb{Q}^{2g} \subset W_2 H^1(X;\mathbb{Q}) \cong H^1(X;\mathbb{Q}) \cong \mathbb{Q}^{2g+p-1},$$

$$Gr_1 H^1(X;\mathbb{Q}) \cong \mathbb{Q}^{2g}, Gr_2 H^1(X;\mathbb{Q}) \cong \mathbb{Q}^{p-1}.$$

**Example 9.3** (complement of three intersecting lines in $\mathbb{P}^2$)**.** *Let $\overline{X} = \mathbb{P}^2$, $D$ be the union of three lines intersecting at 3 different points, which is of course a normal crossing divisor. $X = \overline{X} \setminus D$. Now we have*

*$D^{(1)} = \{\mathbb{P}^1, \mathbb{P}^1, \mathbb{P}^1\}, D^{(2)} = \{*, *, *\}$, run the weight spectral sequence we get*

$$E_1^{p,q} = \begin{array}{ccccc} 3 & \xrightarrow{d_1} & 3 & \xrightarrow{d_1} & 1 \\ \\ 0 & & 0 & & 0 \\ \\ 0 & & 3 & \xrightarrow{d_1} & 1 \\ \\ 0 & & 0 & & 0 \\ \\ 0 & & 0 & & 1 \end{array} \Rightarrow E_2^{p,q} = Gr_q H^{p+q}(X) = \begin{array}{ccc} 1 & 0 & 0 \\ \\ 0 & 0 & 0 \\ \\ 0 & 2 & 0 \\ \\ 0 & 0 & 0 \\ \\ 0 & 0 & 1 \end{array}.$$

*Note again, here we don't need to do any computation, topologically, $\mathbb{P}^2 \setminus \mathbb{P}^1 \cong \mathbb{R}^4$, then we need to remove two more intersecting lines in $\mathbb{R}^2 \times \mathbb{R}^2$, for example, let's just remove $0 \times \mathbb{R}^2$ and $\mathbb{R}^2 \times 0$, this tells us $X$ is homotopic to $S^1 \times S^1$. Then the $E_2$-page above is the only possibility. And it's easy to describe the filtration.*

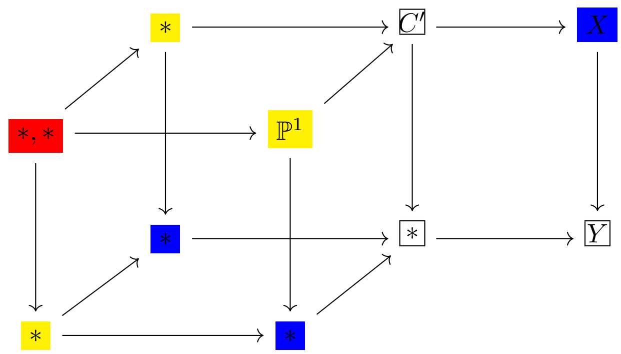
**Example 9.4** (complement of three concurrent lines in $\mathbb{P}^2$)**.** *The point of this example is that three concurrent lines in $\mathbb{P}^2$ is not a normal crossing divisor. However, it's not a problem at all, we can blow them up w.r.t to the intersecting point , let's denote the blow up by $\overline{X} = Bl_p(\mathbb{P}^2)$. Set $D$ to be $\{L_1, L_2, L_3, E\}$, where $E$ is the exceptional divisor. Then $\overline{X} \setminus D$ is isomorphic to the complement of three concurrent lines. Now $D^{(1)} = \{\mathbb{P}^1, \mathbb{P}^1, \mathbb{P}^1, \mathbb{P}^1\}$, $D^{(1)} = \{*, *, *\}$. Apply the weight spectral sequence, we get*

$$E_1^{p,q} = \begin{array}{ccccc} 3 & \xrightarrow{d_1} & 4 & \xrightarrow{d_1} & 1 \\ \\ 0 & & 0 & & 0 \\ \\ 0 & & 4 & \xrightarrow{d_1} & 2 \\ \\ 0 & & 0 & & 0 \\ \\ 0 & & 0 & & 1 \end{array} \Rightarrow E_2^{p,q} = Gr_q H^{p+q}(X) = \begin{array}{ccc} 0 & 0 & 0 \\ \\ 0 & 0 & 0 \\ \\ 0 & 2 & 0 \\ \\ 0 & 0 & 0 \\ \\ 0 & 0 & 1 \end{array}.$$

*First, notice that the 2 in the $E_1$-page is because we have an extra $\mathbb{P}^1$ given by the exceptional divisor. Then we know $Gr_2 H^1(X; \mathbb{Q}) \cong \mathbb{Q}^2$. Note again, by a similar argument as above $X$ is homotopic to $S^1 \vee S^1$.*

**Remark** ($X = \mathbb{C}^2 \setminus \{0\} \subset \mathbb{P}^2$)**.** *Use the same method again(we mean, blow-up), it's not hard to see the weight filtration of $H^n(X; \mathbb{Q})$.*

**Example 9.5** (cubical hyperresolution, It's our guess, we didn't check the precise definition carefully!!!!!!).




**Example 9.6** (Nodal curve). *Consider $X = \{(x, y, z) \subset \mathbb{P}^2 | y^2 z = x^2(x + z)\}$, then the standard resolution is given by*

$$\begin{tikzcd} \tilde{Y} = \{*, *\} \arrow[r] \arrow[d] & \widetilde{X} \cong \mathbb{P}^1 \arrow[d, "f"] \\ Y = \{* = (0,0)\} \arrow[r] & X \end{tikzcd}$$

**Remark** (weight spectral sequence for singular varieties).

**Remark** (Mayer-Vietoris exact sequence).

**Remark** (cubical hyperresolution). *cubical hyperresolution Some Remarks on Hyperresolutions, J.H.M. Steenbrink*

**Example 9.7** (cuspidal singularity).

# 10 Weights in arithmetic geometry

**Example 10.1** (de Rham-Witt complex). *See the following*

- *de Rham-Witt complex*

**Example 10.2** (Tate twist). *p-adic Tate twist*

**Example 10.3** (inertia group). *Ramification group*

**Example 10.4** (étale cohomology of curves). *Daniel Litt's notes.*

**Example 10.5** (Frobenius morphisms).

**Example 10.6** (monodromy operators). *page 14*

**Example 10.7** (Milnor fibration, nearby cycles of a family $X \to D$, reference). *The definition of Milnor fibre exactly means close enough to a point $x \in X_0$, the family over some small punctured disk $S^*$ is trivial, then just choose one typical fibre in this family. And the classical definition or the definition by a base change to the the universal covering of $S^*$ (see here, page 3) is just a way to make out choice in some sense 'canonical' or 'unambiguous'. Consider the family*

$$X = \{(x_1, x_2, t) | x_1 x_2 = t, |t| < 1\} \to D$$

$$(x_1, x_2, t) \mapsto t.$$

*Then by the intuition of 'nearby cycle', we know the Milnor fibre over* $pt = (0,0,0)$ *is given by* $x_1x_2 = \epsilon \neq 0$, *which is just* $(a, \frac{1}{a})$, *which is just* $\mathbb{C}^{\times}$. *This intuition can be verified by the definition, which say that the Milnor fibre over* $pt$ *is given by*

$$F_{pt} = \{(x_1, x_2, u) | |x_1|^2 + |x_2|^2 + |x_1x_2|^2 < \epsilon, \exp(u) = x_1x_2\}.$$

*Note that the domain of* $u$ *is contractible, so* $F_{pt}$ *is a trivial* $S^1$ *fibration over a contractible base. We conclude that*

$$F_p \cong S^1.$$

*And the sheaves of nearby cycles are just skyscraper sheaves supported at the point* $pt$.

$$R\Psi\mathbb{Z} \cong \mathbb{Z}, R^1\Psi\mathbb{Z} \cong \mathbb{Z}_{pt}.$$

**Example 10.8** (Dwork family of elliptic curves, need 'monodromy formula')**.** *Consider the family*

$$f : X = \{(\epsilon, [x, y, z] \in D \times \mathbb{P}^2_{\mathbb{C}} | \epsilon(x^3 + y^3 + z^3) = 3xyz\} \to D.$$

$$X_0$$

*consists of three lines intersecting at* 3 *different points.*

$$P_1 = [1, 0, 0], P_2 = [0, 1, 0], P_3 = [0, 0, 1].$$

*Note that* $R^0\Psi\mathbb{Z} \cong \mathbb{Z}, R^1\Psi\mathbb{Z} \cong \oplus_{i=1}^{3}\mathbb{Z}_{P_i}$ *and higher nearby cycles sheaves vanish because* $R^1\Psi\mathbb{Z}$ *is locally of rank 1. then the* $E_2$*-page of the nearby cycles spectral sequence looks like*

$$\begin{tikzcd}
H^0(X_0, R^1\Psi\mathbb{Z}) \arrow[drr] & H^1(X_0, R^1\Psi\mathbb{Z}) & H^2(X_0, R^1\Psi\mathbb{Z}) \\
H^0(X_0, \mathbb{Z}) & H^1(X_0, \mathbb{Z}) & H^2(X_0, \mathbb{Z})
\end{tikzcd}$$

$$= \begin{tikzcd}
\mathbb{Z}^3 \arrow[drr, "d_2"] & 0 & 0 \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}^3
\end{tikzcd}$$

*Since it degenerates to* $H^i(X_t, \mathbb{Z})$, *we know* $\ker(d_2) \cong \mathbb{Z}$. *Or we can state this fact as the following exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & H^1(X_0, \mathbb{Z}) \arrow[r, "i"] \arrow[d, "="] & H^1(X_t, \mathbb{Z}) \arrow[r, "p"] \arrow[d, "="] & H^0(X_0, R^1\Psi\mathbb{Z}) \arrow[r, "d_2"] \arrow[d, "="] & H^2(X_0, \mathbb{Z}) \arrow[r] \arrow[d, "="] & H^2(X_t, \mathbb{Z}) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] & \mathbb{Z}^2 \arrow[r, "p"] & \mathbb{Z}^3 \arrow[r, "d_2"] & \mathbb{Z}^3 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

*We want to compute the monodromy action on* $H^1(X_t, \mathbb{Z})$. *By the construction in the reference, we know the monodromy action of* $\pi_1(S^*)$ *on* $R^k\Psi\mathbb{Z}$ *is trivial if* $i \geq 1$. *Then as a* $\pi_1(S^*)$*-module, Since the monodromy action doesn't affect the cohomology of the special fibre, we can choose a basis of* $H^1(X_t, \mathbb{Z}) = \{e_1, e_2\}$, *where* $e_1$ *is the image of a generator of* $H^1(X_0, \mathbb{Z})$, *and* $e_2$ *is the pre-image of a generator of* $\ker(d_2)$, *then the monodromy action of a generator* $T$ *of* $\pi_1(S^*)$ *can be written in a form*

$$T = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

*We want to figure out the integer number $m$ above. Apply the 'monodromy formula', we know the dollowing diagram commutes up to sign.*

$$\begin{tikzcd} H^1(X_t,\mathbb{Z}) \arrow[r, "p"] \arrow[d, "1-T"] & H^0(X_0, R^1\Psi\mathbb{Z}) \arrow[d, "\overline{\alpha}"] \\ H^1(X_t,\mathbb{Z}) & H^1(X_0,\mathbb{Z}) \arrow[l] \end{tikzcd}$$

*where $\overline{\alpha}$ comes from the LES associated to the short exact sequence*

$$0 \to \mathbb{Z} \to \oplus_{i=1}^{3}\mathbb{Z}_{L_i} \to R^1\Psi\mathbb{Z} \cong \oplus_{i=1}^{3}\mathbb{Z}_{P_i} \to 0.$$

*Use Čech cohomology, we can compute*

$$\overline{\alpha} : H^0(X_0, R^1\Psi\mathbb{Z}) \to H^1(X,\mathbb{Z})$$

$$[P_i] \mapsto 1.$$

*The point is that $\mathrm{im}(p) = \ker(d_2)$ is $\mathbb{Z}/3\mathbb{Z}$-invariant. Because let $\mathbb{Z}/3\mathbb{Z}$ act on $X$ by permuting the coordinates, then $p$ is naturally invariant almost by definition of the nearby cycle spectral sequence. And the we know $\mathrm{coker}(p) \cong \mathbb{Z}^2$ is torsion free, thus $\mathrm{im}(p)$ is generated by $[P_1]+[P_2]+[P_3]$, there for $\overline{\alpha}([P_1]+[P_2]+[P_3]) = 3$. We conclude that the monodromy action of $\pi_1(S^*)$ on $H^1(X_t,\mathbb{Z})$ is given by*

$$T \mapsto \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

**Remark.** *Every irreducible complex representation of an abelian group is 1-dimensional, but if the group is not compact(finite group, or compact), then we don't necessarily have semi-simplicity.*

# References