1. SELBERG TRACE FORMULA FOR $GL(2,\mathbf{F}_q)$

- Fourier Analysis on finite groups and applications, Chapter 23.
- Fourier Analysis on finite groups and applications, Chapter 20. Spherical functions.

Theorem 1.1 (Selberg trace formula for finite symmetric spaces G/K)**.** *Suppose that G/K is a finite symmetric space and Γ is a subgroup of G. If f is a K-bi-invariant function on G and $\rho = \mathrm{Ind}_\Gamma^G \mathbb{1}$, then*

$$\sum_{\pi\in\widehat{G}^K} m(\pi,\rho)(\mathscr{F}f)(\pi) = \sum_{\{\gamma\}\in\Xi_\Gamma} \frac{|G_\gamma|}{|\Gamma_\gamma|} I_G(f,\gamma).$$

Definition 1.2 ($\{\gamma\}$, G_γ, Γ_γ, Ξ_γ)**.**

$$\begin{aligned}
\{\gamma\} &= \{x\gamma x^{-1} | x\in G\} = \text{the conjugacy class of } \gamma \text{ in } G\\
\Gamma_\gamma &= \{x\in\Gamma | x^{-1}\gamma x = \gamma\} = \text{the stablizer of } \gamma \text{ in } \Gamma \text{ under the conjugation action}\\
G_\gamma &= \{x\in G | x^{-1}\gamma x = \gamma\} = \text{the stablizer of } \gamma \text{ in } G \text{ under the conjugation action}\\
\Xi_\Gamma &= \text{the set of conjugacy classes in } \Gamma
\end{aligned}$$

Definition 1.3 (Spherical transform $\mathscr{F}f$ and $\widehat{G}^K$)**.** The G-representations that appear in $\mathrm{Ind}_K^G \mathbb{1}$ (or $L^2(G/K)$) is denoted by $\widehat{G}^K$. The **spherical transform** of f is given by

$$(\mathscr{F}f)(\pi) = \sum_{x\in G} f(x) h_\pi(x),$$

where $h_\pi(x)$ is the spherical function associated to $\pi\in\widehat{G}^K$

$$h_\pi(x) = \frac{1}{|K|}\sum_{k\in K}\chi_\pi(kx).$$

Definition 1.4 (Orbital sum $I_G(f,\gamma)$)**.** The orbital sum of f at γ is

$$I_G(f,\gamma) = \sum_{x\in G_\gamma\backslash G} f(x^{-1}\gamma x).$$

Definition 1.5 (Horocycle transform)**.** For $f\in L^2(K\backslash G/K)$, the **horocycle** transform is defined to be

$$(\mathscr{H}f)(y) = \sum_{u\in\mathbf{F}_q^*} f(u + y\sqrt{\delta}).$$

Remark 1.6. $f\in L^2(K\backslash G/K)$ can also be written as a function of hyperbolic distance. For example, for $z = x + y\sqrt{\delta}$, for some $\phi\in\mathbf{F}_q\to\mathbf{C}$, we have

$$f\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}^{-1}\gamma\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right) = \phi(d(\gamma z, z)).$$

This is the function appear in the elliptic terms in the Selberg trace formula for the case below.

2. SELBERG TRACE FORMULA EXAMPLES

- Fourier analysis on finite groups with applications, Chapter 23.
- Harmonic analysis database, $GL(2,\mathbf{F}_q)$ conjugacy classes.

2.1. $G = GL(2,\mathbf{F}_q)$, $\Gamma = GL(2,\mathbf{F}_p)$, $K = \begin{pmatrix} x & y\delta \\ y & x \end{pmatrix}$, $q = p^k$, $p \neq 2$.

Theorem 2.1. *Let $f \in L^2(K\backslash G/K)$, the Selberg trace formula is*

$$\begin{aligned}
\sum_{\pi \in \widehat{G}^K} m(\pi, \mathrm{Ind}_\Gamma^G \mathbb{1})(\mathscr{F} f)(\pi) = {} & \frac{q(q-1)(q^2-1)}{p(p^2-1)} f(\sqrt{\delta}), \textit{\textbf{ central terms}} \\
& + \frac{(q+1)(q-1)^2}{2(p-1)} \sum_{c \in \mathbf{F}_p^\times, c \neq 1} (\mathscr{H} f)(c), \textit{\textbf{ hyperbolic terms}} \\
& + \frac{q(q^2-1)}{p}((\mathscr{H} f)(1) - f(\sqrt{\delta})), \textit{\textbf{ parabolic terms}} \\
& + \begin{cases} \frac{(q^2-1)}{2(p+1)} \sum_{u \in \mathbf{F}_p} \sum_{z \in H_q} \phi\left(\frac{\mathrm{N}(\delta - z^2)}{(u^2-\delta)y^2}\right), & k \text{ odd} \\ \frac{(q+1)(q-1)^2}{2(p^2-1)} \sum_{a,b \in \mathbf{F}_p, b \neq 0} (\mathscr{H} f)\left(\frac{a+\eta b}{a-\eta b}\right), & k \text{ even} \end{cases}, \textit{\textbf{ elliptic terms}}.
\end{aligned}$$

Example 2.2 (Central).

Example 2.3 (Hyperbolic).

Example 2.4 (Parabolic).

Example 2.5 (Elliptic, Type I).

Example 2.6 (Elliptic, Type II).