Chapter 7

REPRESENTATIONS OF THE CLASSICAL GROUPS

In this chapter we will derive the complex representation rings of the classical compact Lie groups. We will also enquire if each group has any irreducible representations which are real or quaternionic. For this purpose we consider the following maps:

$$K(G) \xrightarrow{1+t} K(G) \xrightarrow{1-t} K(G).$$

We define

$$H = \mathrm{Ker}(1-t)/\mathrm{Im}(1+t).$$

7.1 **PROPOSITION.** H is an algebra over Z_2, and the irreducible representations of G which are self-conjugate yield a Z_2-base for H.

The proof is immediate from Chapter 3. We may therefore measure the incidence of self-conjugate irreducible representations by computing H.

We will also use the following lemma.

7.2 LEMMA For any complex representation V, $V^* \otimes V \cong \mathrm{Hom}(V,V)$ is real.

<u>Proof</u>. It carries the bilinear form

$$\mathrm{Tr}(\alpha\beta) = \mathrm{Tr}(\beta\alpha)$$

(see 3.38); this form is symmetric, non-singular and invariant. Now use 3.50.

We now begin to study the groups U(n) and SU(n). Each has an obvious representation with $V = C^n$; we write $\lambda^1, \lambda^2, \ldots, \lambda^n$ for the exterior powers of this operation. Let us write

$$z_j = \mathrm{Exp}(2\pi i x_j),$$

so that the typical element in our maximal torus is

$$D = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{bmatrix};$$

then the character $\chi(\lambda^k)$ of λ^k is the kth elementary symmetric function of $z_1, z_2, \ldots, z_n$. (See the proof of 3.61.) The Weyl group acts by permuting $z_1, z_2, \ldots, z_n$. Thus the character $\chi(\lambda^k)$ is the elementary symmetric sum

$$S(x_1 + x_2 + \ldots + x_k).$$

Since $\chi(\lambda^k)$ consists of a single elementary symmetric sum. λ^k is irreducible. The representation λ^n of $U(n)$ is one-dimensional, and is essentially $\det : U(n) \to S^1$. In particular, it is invertible. The restriction of λ^n to $SU(n)$ is trivial.

7.3 **THEOREM.** The complex representation ring $K(U(n))$ is the tensor product of the polynomial ring generated by $\lambda^1, \lambda^2, \ldots, \lambda^{n-1}$ and the ring of finite Laurent series in λ^n. The algebra H is polynomial on generators $\lambda^i \lambda^{n-i}/\lambda^n$ for $2 \leq 2i \leq n$. These generators are real.

There is of course no suggestion that the modules $\lambda^i \lambda^{n-i}/\lambda^n$ are irreducible; indeed they are not.

<u>Proof of 7.3</u>. By a classical theorem, the ring of symmetric polynomials in $z_1, z_2, \ldots, z_n$ is a polynomial ring generated by the elementary symmetric functions $\chi(\lambda^1), \ldots, \chi(\lambda^n)$. Now take any finite Laurent series which is symmetric; by multiplying it with a suitably high power of $z_1 z_2 \ldots z_n$, we obtain a symmetric polynomial. Hence $K(T)_W$ is as described.

The result for $K(U(n))$ follows by 6.20.

Since we have an obvious pairing

$$\lambda^i \otimes \lambda^{n-i} \to \lambda^n,$$

the dual of λ^i is λ^{n-i}/λ^n. (This also follows from an easy calculation with characters.) Hence the conjugate of

$$(\lambda^1)^{\nu_1}(\lambda^2)^{\nu_2} \ldots (\lambda_n)^{\nu_n}$$

is

$$(\lambda^1)^{\nu_{n-1}}(\lambda^2)^{\nu_{n-2}} \ldots (\lambda^{n-1})^{\nu_1}(\lambda^n)^{-\nu_1-\nu_2\ldots-\nu_n}.$$

So t permutes the monomials in $\lambda^1, \lambda^2, \ldots, \lambda^n$; and we easily see that the only monomials which are fixed under t are the polynomials in

$$\lambda^i\lambda^{n-i}/\lambda^n \qquad (1 \le i \le \tfrac{1}{2}n).$$

These are real by 7.2, since

$$\mathrm{Hom}(\lambda^i, \lambda^i) \cong \lambda^i\lambda^{n-i}/\lambda^n.$$

7.4 THEOREM. The complex representation ring $K(SU(n))$ is a polynomial ring generated by $\lambda^1, \lambda^2, \ldots, \lambda^{n-1}$. The algebra H is polynomial on generators $\lambda^i\lambda^{n-i}$ for $2 \le 2i < n$ and, if $n = 2m$, a generator λ^m. The generators $\lambda^i\lambda^{n-i}$ are real; the generator λ^m is real for m even, quaternionic for m odd.

Proof. The result on $K(SU(n))$ is a special case of 6.41;

the identification of the basic weights $\omega_1, \ldots, \omega_k$ mentioned in 6.38 is given in 5.63.

As above, the dual of λ^i is λ^{n-i}. Hence the conjugate of

$$(\lambda^1)^{\nu_1}(\lambda^2)^{\nu_2} \ldots (\lambda^{n-1})^{\nu_{n-1}}$$

is

$$(\lambda^1)^{\nu_{n-1}}(\lambda^2)^{\nu_{n-2}} \ldots (\lambda^{n-1})^{\nu_1}.$$

So t permutes the monomials in $\lambda^1, \lambda^2, \ldots, \lambda^{n-1}$; and we easily see that the only monomials which are fixed under t are polynomials in $\lambda^i\lambda^{n-i}$ $(1 \le i < \frac{1}{2}n)$ and λ^m if $n = 2m$. The representation $\lambda^i\lambda^{n-i}$ is real by 7.2. As for λ^m, the pairing

$$\lambda^m \otimes \lambda^m \to \lambda^{2m} = C$$

has

$$\beta \wedge \alpha = (-1)^m \alpha \wedge \beta ;$$

now use 3.50.

7.5 EXERCISE. Show directly that any representation V of SU(n) extends to U(n). (Hint: It is sufficient to consider an irreducible representation; now consider the action of the centre of SU(n).)

We take next the group Sp(n). It has an obvious representation on $Q^n \cong C^{2n}$; we write $\lambda^1, \lambda^2, \ldots, \lambda^{2n}$ for the exterior powers of this representation. As we have seen in

Chapter 3, λ^k is real for k even, quaternionic for k odd. If we take the element

$$D = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{bmatrix}$$

in T, its action on C^{2n} is given by

$$\begin{bmatrix} z_1 & & & & & & \\ & \bar{z}_1 & & & & & \\ & & z_2 & & & & \\ & & & \bar{z}_2 & & & \\ & & & & \ddots & & \\ & & & & & z_n & \\ & & & & & & \bar{z}_n \end{bmatrix}.$$

Therefore the character $\chi(\lambda^i)$ of λ^i is the ith elementary symmetric function of

$$z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}.$$

7.6 THEOREM. K(Sp(n)) is a polynomial algebra with generators $\lambda^1, \lambda^2, \ldots, \lambda^n$. All the irreducible representations of Sp(n) are self-conjugate.

Proof

(i) It is rather easy to see that $K(T)_W$ is as stated; now use 6.20. Alternatively, use 6.41.

(ii) It follows from the generators given that the whole of $K(Sp(n))$ is self-conjugate. Alternatively, in $Sp(n)$ each element g is conjugate to g^{-1} (see 5.17).

We take next the group $SO(n)$. It has an obvious representation on R^n or C^n; we write $\lambda^1, \lambda^2, \ldots, \lambda^n$ for the exterior powers of this representation. All these representations are real. If we take the element

$$D = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{bmatrix}$$

in $U(n)$ and embed it in $SO(2n)$, its action on C^{2n} is equivalent to that of the diagonal matrix

$$\begin{bmatrix} z_1 & & & & & & \\ & \bar{z}_1 & & & & & \\ & & z_2 & & & & \\ & & & \bar{z}_2 & & & \\ & & & & \ddots & & \\ & & & & & z_n & \\ & & & & & & \bar{z}_n \end{bmatrix}.$$

Therefore the character $\chi(\lambda^i)$ of λ^i is the ith elementary symmetric function of

$$z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1},$$

say σ_i. Similarly, if we embed D in SO(2n + 1), its action on C^{2n+1} is equivalent to that of the diagonal matrix

$$\begin{bmatrix} z_1 & & & & & \\ & \bar{z}_1 & & & & \\ & & \ddots & & & \\ & & & z_n & & \\ & & & & \bar{z}_n & \\ & & & & & 1 \end{bmatrix}.$$

Therefore we have

$$\chi(\lambda^i) = \sigma_i + \sigma_{i-1}.$$

(Here σ_0 is to be interpreted as 1.)

7.7 **THEOREM.** K(SO(2n + 1)) is a polynomial algebra with generators $\lambda^1, \lambda^2, \ldots, \lambda^n$. All the irreducible representations of SO(2n + 1) are real.

Proof

(i) $K(T)_W$ is exactly the same as for Sp(n).

(ii) It follows from the generators given that the whole of K(SO(2n +1)) is real.

So far the exterior powers λ^i have given us all the generators we need. It is easy to produce arguments to show that for $SO(2n)$ we need something else.

(i) In $SO(4n+2)$ not every element g is conjugate to g^{-1}. Therefore it is possible to construct a class function f such that $f(g) \neq f(g^{-1})$. Therefore (3.47) $SO(4n+2)$ has at least one representation which is not self-conjugate. But all the λ^i are real.

(ii) Consider the representation λ^n of $SU(2n)$. We have already seen that it is self-conjugate. So its restriction to $SO(2n)$ is self-conjugate for two essentially different reasons: first because λ^n is self-conjugate on $SU(2n)$, and secondly because each exterior power λ^i is real on $SO(2n)$. But we have already seen that an irreducible representation V can have essentially only one isomorphism with V^*. Therefore the representation λ^n of $SO(2n)$ is reducible.

If n is odd this argument is complete in itself; the representation λ^n of $SO(2n)$ is both quaternionic and real, so it cannot be irreducible. If n is even it is desirable to amplify the word "essentially" a little, and this will be done below.

(iii) An alternative argument proceeds by considering the representation λ^n of $O(2n)$. Consider an element g in $O(2n)$

such that $\det(g) = -1$; it is easy to see that its action on C^{2n} is equivalent to that of a diagonal matrix

$$\begin{bmatrix} z_1 & & & & & & \\ & \bar{z}_1 & & & & & \\ & & \ddots & & & & \\ & & & z_{n-1} & & & \\ & & & & \bar{z}_{n-1} & & \\ & & & & & 1 & \\ & & & & & & -1 \end{bmatrix}$$

It is now easy to check that the restriction of $\chi = \chi(\lambda^n)$ to the component of determinant -1 in $O(2n)$ is zero. Let the average value of $\bar{\chi}\chi$ over $SO(2n)$ be ν; then the average value of $\bar{\chi}\chi$ over $O(2n)$ is $\frac{1}{2}\nu$. So $\frac{1}{2}\nu \geq 1$ (3.34) and $\nu \geq 2$. That is, λ^n must split over $SO(2n)$ into at least two summands.

We now amplify argument (ii). Let us define a non-singular bilinear pairing

$$F : \lambda^n(R^{2n}) \otimes \lambda^n(R^{2n}) \to \lambda^{2n}(R^{2n}) = R$$

by $F(v,w) = v \wedge w$. Then F is invariant under $SO(2n)$; indeed for $g \in O(2n)$ we have

$$F(gv, gw) = (\det g)F(v,w).$$

Let us define another non-singular bilinear pairing

$$S : \lambda^n(R^{2n}) \otimes \lambda^n(R^{2n}) \to R$$

by

$$S\Big((v_1 \wedge v_2 \wedge \ldots \wedge v_n) \otimes (w_1 \wedge w_2 \wedge \ldots \wedge w_n)\Big)$$
$$= \sum_\rho \epsilon(\rho)(v'_{\rho(1)}w_1) \ldots (v'_{\rho(n)}w_n).$$

Here ρ runs over all permutations, and $v'w$ is the usual inner product in R^{2n}. Then S is invariant under $O(2n)$. Let us define an automorphism β of $\lambda^n(R^{2n})$ by setting

$$S(\beta v, w) = F(v, w).$$

We easily check that for $g \in O(2n)$ we have

$$\beta g v = (\det g)\beta v.$$

We may describe β as follows. Let $v_1, v_2, \ldots, v_{2n}$ be any orthonormal basis with determinant +1 in R^{2n}; then

$$\beta(v_1 \wedge v_2 \wedge \ldots \wedge v_n) = v_{n+1} \wedge v_{n+2} \wedge \ldots \wedge v_{2n}.$$

Thus $\beta^2 = (-1)^n$.

It follows that $\lambda^n(R^{2n})$ splits into the ± 1 eigenspaces of β if n is even, and into the $\pm i$ eigenspaces of β if n is odd. Of course the latter splitting takes place over C. Elements of $SO(2n)$ preserve the two eigenspaces; elements of determinant -1 in $O(2n)$ interchange the two eigenspaces. In particular, neither eigenspace can be zero.

We now enquire after the characters of the summands (say V and W). The character $\chi(\lambda^n)$ of λ^n is the nth elementary symmetric function of

$$z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}.$$

Let us write

$$a_+ = \Sigma z_1^{\epsilon_1} z_2^{\epsilon_2} \ldots z_n^{\epsilon_n} \mid \epsilon_r = \pm 1 \text{ and } \epsilon_1 \epsilon_2 \ldots \epsilon_n = +1.$$

$$a_- = \Sigma z_1^{\epsilon_1} z_2^{\epsilon_2} \ldots z_n^{\epsilon_n} \mid \epsilon_r = \pm 1 \text{ and } \epsilon_1 \epsilon_2 \ldots \epsilon_n = -1.$$

These are elementary symmetric sums (see 5.17). We have

$$\chi(\lambda^n) = a_+ + a_- + \text{lower terms}.$$

Since the characters of representations are linear combinations with non-negative coefficients of elementary symmetric sums, we have

$$\chi(V) = aa_+ + ba_- + \sigma$$

where a and b are 0 or 1, and σ is a sum of lower terms. Now consider the automorphism θ of $SO(2n)$ obtained by conjugating with an element g of determinant -1 in $O(2n)$, say

$$g = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{bmatrix}.$$

Its effect on T is to invert z_n; thus $\theta a_+ = a_-$, $\theta a_- = a_+$ and $\theta\sigma = \sigma$. Hence

$$\chi(W) = ba_+ + aa_- + \sigma,$$

$$\chi(\lambda^n) = (a + b)(a_+ + a_-) + 2\sigma$$

and $a + b = 1$. It follows that we can name the summands of λ^n so that

$$\chi(\lambda^n_+) = \alpha_+ + \sigma$$
$$\chi(\lambda^n_-) = \alpha_- + \sigma .$$

7.8 COROLLARY. The automorphism θ of $SO(2n)$ is not inner.

<u>Proof</u>. An inner automorphism takes a representation into an equivalent representation.

7.9 THEOREM. $K(SO(2n))$ is a free module over the polynomial ring $Z[\lambda^1, \lambda^2, \ldots, \lambda^n]$ on two generators 1 and λ^n_+ (or equivalently 1 and λ^n_-). If n is even all the irreducible representations of $SO(2n)$ are real. If n is odd,

$$H = Z_2[\lambda^1, \lambda^2, \ldots, \lambda^{n-1}].$$

<u>Proof</u>. We have to study $K(T)_W$, that is, the set of finite Laurent series in $z_1, z_2, \ldots, z_n$ which are symmetric under permutations and under inverting an even number of the z_r. The set S of such symmetric elements admits an automorphism θ: invert an odd number of the z_r. (Of course, θ arises as explained above.) We have $\theta^2 = 1$. So over the rationals, S

splits as the sum of the $+1$ and -1 eigenspaces of θ:

$$s = \frac{1}{2}(1+\theta)s + \frac{1}{2}(1-\theta)s.$$

The $+1$ eigenspace is the ring of polynomials in

$$\chi(\lambda^1), \chi(\lambda^2), \ldots, \chi(\lambda^n)$$

(as in 7.6, 7.7). Suppose given an element $\underline{a}$ in the -1 eigenspace. Then

$$a = \sum_r c_r(z_1, z_2, \ldots, z_{n-1}) z_n^r,$$

where

$$c_{-r} = -c_r,$$

so that

$$a = \sum_1^n c_r(z_1, z_2, \ldots, z_{n-1})(z_n^r - z_n^{-r}).$$

Thus

$$a = a'(z_n - z_n^{-1}).$$

By symmetry $\underline{a}$ is divisible by the remaining $(z_r - z_r^{-1})$; so

$$a = a''(z_1 - z_1^{-1})(z_2 - z_2^{-1}) \ldots (z_n - z_n^{-1}).$$

Here a'' must be an element of the $+1$ eigenspace; so we have

$$a = p(\alpha_+ - \alpha_-)$$

where p is a polynomial in $\chi(\lambda^1), \ldots, \chi(\lambda^n)$. For a general element s in S we have

$$s = \frac{1}{2}(1+\theta)s + \frac{1}{2}p(\alpha_+ - \alpha_-).$$

Since $\alpha_+ + \alpha_-$ lies in the $+1$ eigenspace we may write this

$$s = q + p\alpha_+$$

where q lies in the $+1$ eigenspace and is integral (since s and $p\alpha_+$ are so).

So $K(T)_W$ is as claimed, and the result on $K(SO(2n))$ follows by 6.20.

If n is even all the generators for $K(SO(2n))$ are real. If n is odd $t(\lambda_+^n) = \lambda_-^n$, and the calculation of H is easy. This completes the proof.

For lack of time I have not included anything on the representation-theory of Spin(n). Of course, this is included as a special case of 6.41; but some may prefer to see the basic representations arise more directly. I advise such readers to study Clifford algebras out of [1] and the representations of the Clifford algebras out of [7]. In [7] Eckmann actually studies the representations of a certain finite group G, but the Clifford algebra is an obvious quotient of the group ring R(G), and so the representations of the Clifford algebra are easily read off from the representations of G.