CHAPTER II

Representations of SU(2), SL(2, $\mathbb{R}$) and SL(2, $\mathbb{C}$)

§1. The Unitary Trick

An important class of finite-dimensional representations of SL(2, $\mathbb{C}$) is obtained as follows. Fix an integer $n \geq 0$, and let V_n be the complex vector space of polynomials in two complex variables z_1 and z_2 homogeneous of degree n. Define a representation Φ_n by

$$\Phi_n\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \tag{2.1}$$

Then $\dim V_n = n + 1$, and $\Phi_n\colon \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(V_n)$ is holomorphic. In §2 we shall see that Φ_n is irreducible. It will turn out that there are no other irreducible finite-dimensional holomorphic representations of SL(2, $\mathbb{C}$), up to equivalence.

If we restrict Φ_n to either of the subgroups SL(2, $\mathbb{R}$) or SU(2) of SL(2, $\mathbb{C}$), we obtain a representation of the subgroup. The representation of the subgroup is itself irreducible. In fact, let φ_n be the representation of $\mathfrak{sl}(2, \mathbb{C})$ given by $\varphi_n = d\Phi_n$. Since Φ_n is holomorphic, φ_n is complex linear. Now

$$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{R}) \oplus i\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2), \tag{2.2}$$

and consequently φ_n is determined by its restriction to $\mathfrak{sl}(2, \mathbb{R})$ or to $\mathfrak{su}(2)$. Thus invariant subspaces for $\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{su}(2)$ are invariant for $\mathfrak{sl}(2, \mathbb{C})$, hence for SL(2, $\mathbb{C}$). In other words, the irreducibility of Φ_n for SL(2, $\mathbb{C}$) implies the irreducibility of Φ_n for SL(2, $\mathbb{R}$) and SU(2). This is a special case of Weyl's **unitary trick** given in the following proposition.

Proposition 2.1. Let V be a finite-dimensional complex vector space. Then a representation of any of the following kinds on V leads, via (2.2), to a representation of each of the other kinds. Under this correspondence, invariant subspaces and equivalences are preserved:

(a) a smooth representation of SL(2, $\mathbb{R}$) on V
(b) a smooth representation of SU(2) on V
(c) a holomorphic representation of SL(2, $\mathbb{C}$) on V
(d) a representation of $\mathfrak{sl}(2, \mathbb{R})$ on V

(e) a representation of $\mathfrak{su}(2)$ on V
(f) a complex-linear representation of $\mathfrak{sl}(2, \mathbb{C})$ on V.

Remark. It will be shown in Corollary 3.16 that finite-dimensional representations of Lie groups are automatically smooth. Thus the word "smooth" can be dropped in (a) and (b).

Proof. We can pass from (c) to (a) or (b) by restriction and from (a) or (b) to (d) or (e) by taking differentials. Formula (2.2) allows us to pass from (d) or (e) to (f). Finally Proposition 1.2 shows that SL(2, $\mathbb{C}$) is topologically the product of SU(2) and a Euclidean space. Since

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \alpha \in \mathbb{C}, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

is homeomorphic with the 3-sphere, SU(2) is simply connected. Thus SL(2, $\mathbb{C}$) is simply connected. If φ is given as in (f), then (A.113) asserts the existence of a representation Φ of SL(2, $\mathbb{C}$) with differential φ. Since φ is complex-linear, Φ is holomorphic. Thus we can pass from (f) to (c). If we follow the steps all the way around, starting from (c), we end up with the original representation, since a differential uniquely determines a homomorphism of Lie groups. Thus invariant subspaces and equivalences are preserved.

The name "unitary trick" comes from the fact that item (b) in the list is a representation of a compact group, which can be made unitary by Proposition 1.6. Then Corollary 1.7 yields the following result.

Corollary 2.2. Every finite-dimensional representation of SL(2, $\mathbb{R}$) or of $\mathfrak{sl}(2, \mathbb{R})$ is the direct sum of irreducible representations. The same thing is true of holomorphic finite-dimensional representations of SL(2, $\mathbb{C}$) and of complex-linear representations of $\mathfrak{sl}(2, \mathbb{C})$, with each summand respectively holomorphic or complex-linear.

Corollary 2.3. Every finite-dimensional unitary representation of SL(2, $\mathbb{R}$) is trivial.

Proof. Let $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{sl}(2, \mathbb{R})$. Then $\mathfrak{su}(2) = \mathfrak{k} \oplus i\mathfrak{p}$. For a given finite-dimensional representation of SL(2, $\mathbb{R}$), form the associated family of representations as in Proposition 2.1. In a suitable inner product, SU(2) acts by unitary operators, by Proposition 1.6, and thus $\mathfrak{su}(2)$ acts by skew-Hermitian operators. Hence $i\mathfrak{p}$ acts by diagonable operators with imaginary eigenvalues, and $\mathfrak{p}$ acts by diagonable operators with real eigenvalues. If the representation of SL(2, $\mathbb{R}$) is unitary in some inner product, then $\mathfrak{p}$ acts with imaginary eigenvalues. Hence $\mathfrak{p}$ acts as 0. Since $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, all of $\mathfrak{sl}(2, \mathbb{R})$ acts as 0.

§2. Irreducible Finite-Dimensional Complex-Linear Representations of $\mathfrak{sl}(2, \mathbb{C})$

Corollary 2.2 reduces the study of several classes of finite-dimensional representations to the study of irreducible ones, and Proposition 2.1 says that it is enough to study the irreducible complex-linear representations of $\mathfrak{sl}(2, \mathbb{C})$.

We shall make repeated use of the basis $\{h, e, f\}$ of $\mathfrak{sl}(2, \mathbb{C})$ over $\mathbb{C}$ given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These elements satisfy the bracket relations

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{2.3}$$

Theorem 2.4. For each integer $m \geq 1$ there exists up to equivalence a unique irreducible complex-linear representation π of $\mathfrak{sl}(2, \mathbb{C})$ on a space V of dimension m. In V there is a basis $\{v_0, \ldots, v_{m-1}\}$ such that (with $n = m - 1$)

(1) $\pi(h)v_i = (n - 2i)v_i$
(2) $\pi(e)v_0 = 0$
(3) $\pi(f)v_i = v_{i+1}$ with $v_{n+1} = 0$
(4) $\pi(e)v_i = i(n - i + 1)v_{i-1}$ with $v_{-1} = 0$.

Moreover, the representation π can be realized as the differential of the representation Φ_n in (2.1).

Remark. Property (1) gives the eigenvalues of $\pi(h)$. Notice for $\mathfrak{sl}(2, \mathbb{C})$ that the smallest eigenvalue is the negative of the largest. The eigenvalues of $\pi(h)$ will generalize in Chapter IV to "weights," and the weight corresponding to n will be the "highest weight" of π.

Proof of uniqueness. Let π be a complex-linear irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on V with $\dim V = m$. Let $v \neq 0$ be an eigenvector for $\pi(h)$, say with $\pi(h)v = \lambda v$. Then $\pi(e)v, \pi(e)^2 v, \ldots$ are also eigenvectors because

$$\begin{aligned} \pi(h)\pi(e)v &= \pi(e)\pi(h)v + \pi([h, e])v && \text{by (1.6)} \\ &= \pi(e)\lambda v + 2\pi(e)v && \text{by (2.3)} \\ &= (\lambda + 2)\pi(v). \end{aligned}$$

Since $\lambda, \lambda + 2, \lambda + 4, \ldots$ are distinct, these eigenvectors are independent (or 0). By finite-dimensionality we can find v_0 in V with (λ redefined and)

(a) $v_0 \neq 0$
(b) $\pi(h)v_0 = \lambda v_0$
(c) $\pi(e)v_0 = 0$.

Define $v_i = \pi(f)^i v_0$. Then $\pi(h)v_i = (\lambda - 2i)v_i$, by the same argument as above, and so there is a minimum integer n with $\pi(f)^{n+1}v_0 = 0$. Then $v_0, \ldots, v_n$ are independent and

(1) $\pi(h)v_i = (\lambda - 2i)v_i$
(2) $\pi(e)v_0 = 0$
(3) $\pi(f)v_i = v_{i+1}$ with $v_{n+1} = 0$.

We claim $V = \operatorname{span}\{v_0, \ldots, v_n\}$. It is enough to show $\operatorname{span}\{v_0, \ldots, v_n\}$ is stable under $\pi(e)$. In fact, we show

(4) $\pi(e)v_i = i(\lambda - i + 1)v_{i-1}$ with $v_{-1} = 0$.

We proceed by induction for (4), the case $i = 0$ being (2). Assume (4) for case i. To prove case $i + 1$, we write

$$\begin{aligned}\pi(e)v_{i+1} &= \pi(e)\pi(f)v_i = \pi([e,f])v_i + \pi(f)\pi(e)v_i \\ &= \pi(h)v_i + \pi(f)\pi(e)v_i \\ &= (\lambda - 2i)v_i + \pi(f)(i(\lambda - i + 1))v_{i-1} \\ &= (i + 1)(\lambda - i)v_i,\end{aligned}$$

and the induction is complete.

To finish the proof of uniqueness, we show $\lambda = n$. We have

$$\operatorname{Tr} \pi(h) = \operatorname{Tr}(\pi(e)\pi(f) - \pi(f)\pi(e)) = 0.$$

Thus $\sum_{i=0}^n (\lambda - 2i) = 0$, and we find $\lambda = n$.

Proof of existence. From the proof of uniqueness and from Corollary 2.2, it follows that $\pi(h)$ cannot have eigenvalue $n = m - 1$ in a reducible complex-linear representation of dimension m. Form the differential $d\Phi_n$ of the representation Φ_n of (2.1). Here $d\Phi_n$ is complex-linear and has dimension m. Also

$$d\Phi_n(h)(z_2^n) = \frac{d}{dt}\Phi_n\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}(z_2^n)\bigg|_{t=0} = \frac{d}{dt}e^{nt}z_2^n\bigg|_{t=0} = nz_2^n,$$

so that $d\Phi_n(h)$ does have n as an eigenvalue. Consequently $d\Phi_n$ is irreducible, and existence follows.

§3. Finite-Dimensional Representations of $\mathfrak{sl}(2, \mathbb{C})$

The previous section dealt only with complex-linear finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$. But $\mathfrak{sl}(2, \mathbb{C})$ has other finite-dimensional representations, e.g., $\Phi\begin{pmatrix} a & b \\ c & -a\end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a}\end{pmatrix}$. The relevant fact for handling them is the following proposition.

Proposition 2.5. For the complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ is $\mathbb{C}$-isomorphic as a Lie algebra to $\mathfrak{g} \oplus \mathfrak{g}$.

Proof. Let J denote multiplication by $\sqrt{-1}$ in $\mathfrak{g}$, and define an $\mathbb{R}$-linear map $L: \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g} \oplus \mathfrak{g}$ by

$$L(X + iY) = (X + JY, X - JY).$$

Then it is easy to see that L is one-one and preserves brackets. Hence L is an $\mathbb{R}$-isomorphism. Moreover, L satisfies

$$L(i(X + iY)) = (J(X + JY), \; -J(X - JY)).$$

This equation exhibits L as a $\mathbb{C}$-isomorphism of $\mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{g} \oplus \bar{\mathfrak{g}}$, where $\bar{\mathfrak{g}}$ is the same real Lie algebra as $\mathfrak{g}$ but where the multiplication by $\sqrt{-1}$ is defined as multiplication by $-i$. Then complex conjugation of matrices exhibits $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ as $\mathbb{C}$-isomorphic, and the result follows.

We can handle finite-dimensional representations of $SL(2, \mathbb{C})$ by means of Proposition 2.5 and a suitable unitary trick. The finite-dimensional representations of $SL(2, \mathbb{C})$ and of $\mathfrak{sl}(2, \mathbb{C})$ correspond to one another since $SL(2, \mathbb{C})$ is simply connected. (We are implicitly using the automatic smoothness of the representations of the group, as proved in Chapter III.) A finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ yields by Proposition 2.5 a complex-linear representation of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, which is fully reducible because $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ is the complexification of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. An irreducible unitary representation of $SU(2) \times SU(2)$ has to be a tensor product of irreducible representations of the factors, by an easy application of the Peter-Weyl Theorem. Thus an irreducible complex-linear finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ has to be a tensor product and must be given, according to Theorem 2.4, by a pair of nonnegative integers.

Consequently the finite-dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$ are direct sums of irreducible representations, and the irreducible representations are parametrized by pairs of nonnegative integers. The representation is complex-linear exactly when the second integer is 0 and is complex conjugate linear exactly when the first integer is 0. A global realization of the irreducible representation $\Phi_{m,n}$ of $SL(2, \mathbb{C})$ with parameters (m, n) is in the vector space of polynomials in $z_1, z_2, \bar{z}_1, \bar{z}_2$ that are homogeneous of degree m in (z_1, z_2) and homogeneous of degree n in $(\bar{z}_1, \bar{z}_2)$, with action

$$\Phi_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right). \tag{2.4}$$

§4. Irreducible Unitary Representations of SL(2, ℂ)

The **principal series,** or **unitary principal series,** of SL(2, ℂ) is a family of representations in $L^2(\mathbb{C})$ that is indexed by pairs (k, iv) with $k \in \mathbb{Z}$ and $v \in \mathbb{R}$. The representation $\mathscr{P}^{k,iv}$ is given by

$$\mathscr{P}^{k,iv}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = |-bz + d|^{-2-iv}\left(\frac{-bz+d}{|-bz+d|}\right)^{-k} f\left(\frac{az-c}{-bz+d}\right)$$

for $z \in \mathbb{C}$ and $f \in L^2(\mathbb{C})$.

Proposition 2.6. For each pair (k, iv), $\mathscr{P}^{k,iv}$ is an irreducible unitary representation of SL(2, ℂ). Moreover, $\mathscr{P}^{k,iv}$ is unitarily equivalent with $\mathscr{P}^{-k,-iv}$.

Remarks. The proof of irreducibility may be approached in several ways. This time we use the Euclidean Fourier transform. In Chapter VII we shall indicate a more elementary argument.

Proof. A straightforward change of variables shows that $\mathscr{P}^{k,iv}(g)$ is isometric for each g, and it is easy to check that $g \to \mathscr{P}^{k,iv}(g)$ is a homomorphism. For f in $C^\infty_{\text{com}}(\mathbb{C})$, the continuity at 1 of $g \to \mathscr{P}^{k,iv}(g)f$ as a map of G into $L^2(\mathbb{C})$ follows by dominated convergence. Since $C^\infty_{\text{com}}(\mathbb{C})$ is dense in $L^2(\mathbb{C})$, the strong continuity of $g \to \mathscr{P}^{k,iv}(g)$ follows. Thus $\mathscr{P}^{k,iv}$ is a unitary representation.

For the irreducibility we shall show that actually $\mathscr{P}^{k,iv}$ is irreducible when restricted to the lower triangular group. On that subgroup, the representation is given by

$$\mathscr{P}^{k,iv}\begin{pmatrix} 1 & 0 \\ z_0 & 1 \end{pmatrix} f(z) = f(z - z_0) \tag{2.5}$$

$$\mathscr{P}^{k,iv}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f(z) = |a|^{2+iv}\left(\frac{a}{|a|}\right)^{k} f(a^2 z). \tag{2.6}$$

Motivated by Proposition 1.5, let B be a bounded linear operator on $L^2(\mathbb{C})$ commuting with $\mathscr{P}^{k,iv}(g)$ for all g. Since B commutes with (2.5), B is given on the Fourier transform side by multiplication by a bounded measurable function m:

$$\widehat{Bf}(\zeta) = m(\zeta)\hat{f}(\zeta) \qquad \text{for } f \in L^2(\mathbb{C}). \tag{2.7}$$

Here the Fourier transform is given by

$$\hat{f}(\zeta) = \int_{\mathbb{C}} e^{-2\pi i z \cdot \zeta} f(z)\, dz,$$

where $z \cdot \zeta$ denotes $x\xi + y\eta$ if $z = x + iy$ and $\zeta = \xi + i\eta$. Since

$$\mathscr{P}^{k,iv}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} B = B\mathscr{P}^{k,iv}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

(2.6) gives $(Bf)(a^2 z) = B(f(a^2 \cdot))(z)$ for all $a \neq 0$. Multiplying by $e^{-2\pi i z \cdot \zeta}$ and integrating and using (2.7), we obtain from the left side

$$\begin{aligned} \int_{\mathbb{C}} e^{-2\pi i z \cdot \zeta}\, Bf(a^2 z)\, dz &= |a|^{-4} \int_{\mathbb{C}} e^{-2\pi i z \cdot a^{-2}\zeta}\, Bf(z)\, dz \\ &= |a|^{-4}\widehat{Bf}(a^{-2}\zeta) \\ &= |a|^{-4} m(a^{-2}\zeta)\hat{f}(a^{-2}\zeta). \end{aligned} \tag{2.8}$$

From the right side we get

$$\begin{aligned} m(\zeta) f(a^2 \cdot)\hat{\ }(\zeta) &= m(\zeta) \int_{\mathbb{C}} e^{-2\pi i z \cdot \zeta} f(a^2 z)\, dz \\ &= |a|^{-4} m(\zeta) \int_{\mathbb{C}} e^{-2\pi i z \cdot a^{-2}\zeta} f(z)\, dz \\ &= |a|^{-4} m(\zeta)\hat{f}(a^{-2}\zeta). \end{aligned} \tag{2.9}$$

The equality of (2.8) and (2.9) for all f means that

$$m(a^{-2}\zeta) = m(\zeta) \tag{2.10}$$

for almost all ζ for each a. By Fubini's Theorem, (2.10) holds for some $\zeta = \zeta_0 \neq 0$ for almost every a. Since $a^{-2}\zeta_0$ sweeps out $\mathbb{C} - \{0\}$, m is constant almost everywhere. Thus B is scalar, and $\mathscr{P}^{k,iv}$ is irreducible.

Further Fourier analysis, which we omit, will establish the equivalences. An elementary proof will be given in Chapter VII.

The full **nonunitary principal series** of SL(2, ℂ) is a family of representations indexed by pairs (k, w) with $k \in \mathbb{Z}$ and $w = u + iv \in \mathbb{C}$. The representation $\mathscr{P}^{k,w}$ is given by

$$\mathscr{P}^{k,w}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = |-bz + d|^{-2-w}\left(\frac{-bz + d}{|-bz + d|}\right)^{-k} f\left(\frac{az - c}{-bz + d}\right), \tag{2.11}$$

and the Hilbert space is L^2 of $\mathbb{C}$ with respect to the measure $(1 + |z|^2)^{\mathrm{Re}\, w}\, dx\, dy$. Although $\mathscr{P}^{k,w}$ is a representation for every value of w, it is not unitary for this inner product unless w is imaginary.

However, for $k = 0$ and w real, it becomes unitary for $0 < w < 2$ with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{f(z)\overline{g(\zeta)}\, dz\, d\zeta}{|z - \zeta|^{2-w}}, \tag{2.12}$$

and the resulting representations are called **complementary series.** This topic will be taken up further in Chapter XVI.

Up to equivalence the trivial representation, the unitary principal series, and the complementary series are the only irreducible unitary representations of SL(2, ℂ). The only equivalences among members of this list are those in Proposition 2.6. This completeness will be proved in Chapter XVI.

The nonunitary principal series contains all the irreducible finite-dimensional representations of SL(2, ℂ) as subrepresentations. To see this, we first rewrite $\Phi_{m,n}$ in (2.4) by replacing z_1 by 1 and z_2 by z. The polynomials are no longer homogeneous but now are of degree $\leq m$ in z and $\leq n$ in $\bar{z}$. The action becomes

$$\Phi_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(z) = (-bz + d)^m(-\bar{b}\bar{z} + \bar{d})^n P\left(\frac{az - c}{-bz + d}\right). \tag{2.13}$$

Comparing (2.11) and (2.13), we see that

$$\Phi_{m,n} \subseteq \mathscr{P}^{n-m,\ -2-m-n}. \tag{2.14}$$

§5. Irreducible Unitary Representations of SL(2, ℝ)

We begin with a list of some unitary representations of SL(2, ℝ):

(1) Discrete series $\mathscr{D}_n^+$ and $\mathscr{D}_n^-$, $n \geq 2$ an integer. The Hilbert space for $\mathscr{D}_n^+$ is

$$\left\{ f \text{ analytic for } \operatorname{Im} z > 0 \,\middle|\, \|f\|^2 = \iint_{\operatorname{Im} z > 0} |f(z)|^2 y^{n-2}\, dx\, dy < \infty \right\},$$

and the action is

$$\mathscr{D}_n^+\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (-bz + d)^{-n} f\left(\frac{az - c}{-bz + d}\right).$$

The space for $\mathscr{D}_n^+$ is not 0 because $(z + i)^{-n}$ is in it. The representation $\mathscr{D}_n^-$ is obtained by using complex conjugates. All these representations are unitary, and we verify below that they are irreducible. In addition, these representations are **square-integrable** in the sense that some nonzero matrix coefficient is in $L^2(G)$. We shall verify that

$$\int_G |(\mathscr{D}_n^+(g)(z + i)^{-n}, (z + i)^{-n})|^2\, dg < \infty$$

after Proposition 5.28, using another realization of $\mathscr{D}_n^+$ given in §6. Then it will follow (from Proposition 9.6) that every matrix coefficient is in $L^2(G)$.

(2) Principal series $\mathscr{P}^{+,iv}$ and $\mathscr{P}^{-,iv}$, $v \in \mathbb{R}$. The Hilbert space for each of these is $L^2(\mathbb{R})$, and the action is

$$\mathscr{P}^{\pm,iv}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \begin{cases} |-bx+d|^{-1-iv} f((ax-c)/(-bx+d)) & \text{if } +. \\ \operatorname{sgn}(-bx+d)|-bx+d|^{-1-iv} f((ax-c)/(-bx+d)) & \text{if } -. \end{cases}$$

These representations are all unitary, and we verify below, using Fourier analysis, that all but $\mathscr{P}^{-,0}$ are irreducible. Unitary equivalences $\mathscr{P}^{+,iv} \cong \mathscr{P}^{+,-iv}$ and $\mathscr{P}^{-,iv} \cong \mathscr{P}^{-,-iv}$ are implemented by operators that will be constructed in Chapter VII. All these representations are induced from the upper triangular subgroup in a sense that is to be made precise in Chapter VII.

(3) Complementary series $\mathscr{C}^u$, $0 < u < 1$. Here the Hilbert space is

$$\left\{ f : \mathbb{R} \to \mathbb{C} \,\Big|\, \|f\|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)\overline{f(y)}\, dx\, dy}{|x-y|^{1-u}} < \infty \right\}$$

and the action is

$$\mathscr{C}^u \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = |-bx+d|^{-1-u} f\left(\frac{ax-c}{-bx+d}\right).$$

These representations are irreducible unitary and will be discussed further in Chapter XVI. As with SL(2, $\mathbb{C}$) they arise from certain nonunitary principal series (defined below) by redefining the inner product.

(4) Others. There is the trivial representation, and there are two "limits of discrete series," $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$. The group action with $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$ is like that in the discrete series, but the norm is given by

$$\|f\|^2 = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx.$$

The reduction of $\mathscr{P}^{-,0}$ is given by

$$\mathscr{P}^{-,0} \cong \mathscr{D}_1^+ \oplus \mathscr{D}_1^-.$$

The representations $\mathscr{D}_1^+$ and $\mathscr{D}_1^-$ are not square integrable.

Proposition 2.7. For $n \geq 1$, $\mathscr{D}_n^+$ and $\mathscr{D}_n^-$ are irreducible unitary representations of SL(2, $\mathbb{R}$). For all pairs $(\pm, iv)$, $\mathscr{P}^{\pm,iv}$ is a unitary representation, and it is irreducible except for the case of $(-, 0)$. Moreover, $\mathscr{P}^{+,iv}$ is unitarily equivalent with $\mathscr{P}^{+,-iv}$, and $\mathscr{P}^{-,iv}$ is unitarily equivalent with $\mathscr{P}^{-,-iv}$.

Proof of irreducibility of $\mathscr{D}_n^+$. If U is a nonzero closed invariant subspace, then we can find f in U with $f(i) \neq 0$, and the average

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \mathscr{D}_n^+ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} f\, d\theta \tag{2.15}$$

will be in U, also. We can evaluate (2.15) at z as

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\mathscr{D}_n^+\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}f(z)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}(-z\sin\theta+\cos\theta)^{-n}f\left(\frac{z\cos\theta+\sin\theta}{-z\sin\theta+\cos\theta}\right)d\theta$$

$$= \frac{1}{2\pi i}\oint_{|\zeta|=1}\zeta^{-n}\left(-\frac{z}{2i}(\zeta-\zeta^{-1})+\frac{1}{2}(\zeta+\zeta^{-1})\right)^{-n}$$

$$\times f\left(\frac{\frac{z}{2}(\zeta+\zeta^{-1})+\frac{1}{2i}(\zeta-\zeta^{-1})}{\frac{-z}{2i}(\zeta-\zeta^{-1})+\frac{1}{2}(\zeta+\zeta^{-1})}\right)\frac{d\zeta}{\zeta} \qquad (\text{with } \zeta=e^{i\theta})$$

$$= \frac{1}{2\pi i}\oint_{|\zeta|=1}(2i)^n(z+i+\zeta^2(-z+i))^{-n}f\left(\frac{i(z+i)+\zeta^2(iz+1)}{z+i+\zeta^2(-z+i)}\right)\frac{d\zeta}{\zeta}.$$

As a function of ζ the integrand is analytic for $|\zeta|\le 1$ except at $\zeta=0$, where it has a simple pole. By the Cauchy Integral Formula the integral is

$$= (2i)^n f(i)(z+i)^{-n}.$$

Thus $(z+i)^{-n}$ is in U. If U is not the whole space, then $(z+i)^{-n}$ is in $U^\perp$ similarly, and we have a contradiction. Hence $\mathscr{D}_n^+$ is irreducible.

Proof of irreducibility of $\mathscr{P}^{\pm,iv}$ except for $\mathscr{P}^{-,0}$. We proceed as in Proposition 2.6. Let B be a bounded linear operator on $L^2(\mathbb{R})$ commuting with $\mathscr{P}^{\pm,iv}(g)$ for all g. An argument completely analogous to that in Proposition 2.6 shows that

$$\widehat{Bf}(\xi) = m(\xi)\hat{f}(\xi) \qquad \text{for } f\in L^2(\mathbb{R}),$$

where m is a linear combination of 1 and signum(ξ). (We still get (2.10), but $a^{-2}\xi_0$ gives only a half-line; hence signum(ξ) is not immediately excluded as a possibility for m.) To complete the proof, we shall show that the subspace U of $L^2(\mathbb{R})$ of functions whose Fourier transforms vanish on the right half-line is not stable under $\mathscr{P}^{\pm,iv}$ (except in the case of $\mathscr{P}^{-,0}$).

This subspace U is the space of boundary values of analytic functions $F(z)$ in the upper half-plane with $\sup_{y>0}\int_{-\infty}^{\infty}|F(x+iy)|^2\,dx<\infty$. Then $(x+i)^{-1}$ is in U, being the boundary value of $F(z)=(z+i)^{-1}$. Suppose that

$$\mathscr{P}^{\pm,iv}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(x+i)^{-1} = \begin{cases}|x|^{-1-iv}i^{-1}x(x+i)^{-1} & \text{if } +\\ (\operatorname{sgn} x)|x|^{-1-iv}i^{-1}x(x+i)^{-1} & \text{if } -\end{cases}$$

is in U. Then

$$\begin{cases}(\operatorname{sgn} x)\,|x|^{-iv} & \text{if } +\\ |x|^{-iv} & \text{if } -\end{cases}$$

is the nontangential boundary value of an analytic function F in the upper half-plane. Denote the principal branch of $\log z$ in $\operatorname{Im} z > 0$ by $\operatorname{Log} z$ and form $F(z) - e^{-iv \operatorname{Log} z}$. This has boundary value 0 on the right half-line but not on the left half-line (since we have excluded $(\pm, iv) = (-, 0)$), and this behavior cannot occur for an analytic function, by a theorem of Privalov. Thus U is not stable under $\mathscr{P}^{\pm, iv}$, and the irreducibility follows.

The representations listed in (1), (2), (3), and (4) above are the only irreducible unitary representations of $SL(2, \mathbb{R})$, up to equivalence. The only equivalences among them are those listed in Proposition 2.7. This completeness will be proved in Chapter XVI.

The full **nonunitary principal series** of $SL(2, \mathbb{R})$ is a family of representations indexed by pairs $(\pm, w)$ with $w = u + iv$ in $\mathbb{C}$. The representation $\mathscr{P}^{\pm, w}$ is given by

$$\mathscr{P}^{\pm, w}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \begin{cases} |-bx + d|^{-1-w} f((ax - c)/(-bx + d)) & \text{if } + \\ \operatorname{sgn}(-bx + d)|-bx + d|^{-1-w} f((ax - c)/(-bx + d)) & \text{if } -, \end{cases} \tag{2.16}$$

and the Hilbert space is L^2 of $\mathbb{R}$ with respect to $(1 + x^2)^{\operatorname{Re} w}\, dx$. Again $\mathscr{P}^{\pm, w}$ is not unitary except for w imaginary. However, $\mathscr{P}^{+, u}$ can be renormed for $0 < u < 1$ so as to become unitary, and we obtain the complementary series $\mathscr{C}^u$.

The nonunitary principal series contains all the irreducible finite-dimensional representations of $SL(2, \mathbb{R})$ as subrepresentations. To see this, we rewrite Φ_n in (2.1) by replacing z_1 by 1 and z_2 by x. The polynomials are no longer homogeneous but are of degree $\leq n$ in x. The action becomes

$$\Phi_n\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(x) = (-bx + d)^n P\left(\frac{ax - c}{-bx + d}\right). \tag{2.17}$$

Comparing (2.16) and (2.17), we see that

$$\Phi_n \subseteq \begin{cases} \mathscr{P}^{+, -(n+1)} & \text{if } n \text{ even} \\ \mathscr{P}^{-, -(n+1)} & \text{if } n \text{ odd}. \end{cases} \tag{2.18}$$

We can see some additional reducibility in $\mathscr{P}^{\pm, w}$ on a formal level by specializing the parameter w and by passing from z in the upper half-plane for discrete series to x on the real axis. We obtain continuous (not bicontinuous) inclusions

$$\mathscr{D}_n^+ \oplus \mathscr{D}_n^- \subseteq \begin{cases} \mathscr{P}^{+, n-1} & \text{if } n \text{ even} \\ \mathscr{P}^{-, n-1} & \text{if } n \text{ odd}. \end{cases} \tag{2.19}$$

There is no other reducibility for $\mathscr{P}^{\pm, w}$. The quotient by a finite-dimensional Φ is essentially the sum of two $\mathscr{D}$'s, and vice versa. All of these facts about reducibility are a little easier to see rigorously in a different realization of the representations, which we take up in the next section.

§6. Use of SU(1, 1)

A number of facts about representations of SL(2, $\mathbb{R}$) are easier to see when the restriction of the representation to K is more transparent. For this purpose it is more convenient to use the group SU(1, 1), which is conjugate to SL(2, $\mathbb{R}$) within SL(2, $\mathbb{C}$):

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \mathrm{SU}(1,1) \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} = \mathrm{SL}(2, \mathbb{R}). \tag{2.20}$$

Here
$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \, |\alpha|^2 - |\beta|^2 = 1 \right\}$$

acts by linear fractional transformations on the unit disc in the same way that SL(2, $\mathbb{R}$) acts on the upper half plane.

We use z for an upper half plane variable and ζ for a disc variable. The two are related by

$$z = \frac{\zeta + i}{i\zeta + 1} \quad \text{and} \quad \zeta = \frac{z - i}{-iz + 1}.$$

To change a representation from SL(2, $\mathbb{R}$) acting on functions $f(z)$ to SU(1, 1) acting on functions $F(\zeta)$, we use a change of variables and an extra factor

$$F(\zeta) = Tf(\zeta) = m\left(\frac{1}{\sqrt{2}}(i\zeta + 1)\right) f(z(\zeta))$$

$$f(z) = T^{-1}F(z) = m\left(\frac{1}{\sqrt{2}}(-iz + 1)\right) F(\zeta(z)).$$

Here the extra factor m is a homomorphism of $\mathbb{C}^\times$ into $\mathbb{C}^\times$ that is at our disposal and can depend on the representation; m is called a **multiplier.** If R is a representation of SL(2, $\mathbb{R}$) acting on functions $f(z)$, the corresponding representation of SU(1, 1) (which we denote also by R) acting on functions $F(\zeta)$ is

$$R\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} F(\zeta) = T\left(R\left[\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \right] (T^{-1}F) \right)(\zeta).$$

Assuming that R is given on SL(2, $\mathbb{R}$) by

$$R\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = s(-bz + d) f\left(\frac{az - c}{-bz + d} \right),$$

we find that R is given on SU(1, 1) by

$$R\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}F(\zeta) = \frac{m(-\beta\zeta + \bar{\alpha})}{m((i\alpha - \beta)\zeta + (-i\bar{\beta} + \bar{\alpha}))}\, s\left(\frac{(i\alpha - \beta)\zeta + (-i\bar{\beta} + \bar{\alpha})}{i\zeta + 1}\right)$$
$$\times\, m(i\zeta + 1)F\left(\frac{\alpha\zeta - \bar{\beta}}{-\beta\zeta + \bar{\alpha}}\right). \tag{2.21}$$

For $\mathscr{D}_n^+$ we choose $m = s$. Then $\mathscr{D}_n^+$, as a representation of SU(1, 1), acts on analytic functions on the disc by

$$\mathscr{D}_n^+\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}F(\zeta) = (-\beta\zeta + \bar{\alpha})^{-n}F\left(\frac{\alpha\zeta - \bar{\beta}}{-\beta\zeta + \bar{\alpha}}\right),$$

and the norm, except for a constant factor, is given by

$$\|F\|^2 = \begin{cases} \displaystyle\int_{|\zeta|<1} |F(\zeta)|^2(1 - |\zeta|^2)^{n-2}\, d\zeta & \text{for } n \geq 2 \\ \displaystyle\sup_{0 \leq r < 1} \int_0^{2\pi} |F(re^{i\theta})|^2\, d\theta & \text{for } n = 1. \end{cases}$$

Under our conjugation (2.20), the maximal compact subgroup becomes

$$K = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\right\}.$$

It is clear that the functions $\{\zeta^N, N \geq 0\}$ form an orthogonal basis for the representation space of $\mathscr{D}_n^+$, and each ζ^N is an eigenfunction of $\mathscr{D}_n^+(K)$:

$$\mathscr{D}_n^+\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\zeta^N = e^{(n+2N)i\theta}\zeta^N.$$

It is instructive to re-examine the proof of irreducibility of $\mathscr{D}_n^+$ in this realization.

For $\mathscr{P}^{\pm,w}$, we specialize to $z = x$ real and to $\xi = e^{i\psi}$ on the unit circle. In (2.21), s is evaluated at a real point, hence need be defined only on $\mathbb{R}^\times$. We can take m to be any extension of s to $\mathbb{C}^\times$. In any event the norm becomes a multiple of the L^2 norm on the circle. For $\mathscr{P}^{+,w}$, we can choose $m = |\cdot|^{-1-w}$ and then the formula is

$$\mathscr{P}^{+,w}\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}F(e^{i\psi}) = |-\beta e^{i\psi} + \bar{\alpha}|^{-1-w}F\left(\frac{\alpha e^{i\psi} - \bar{\beta}}{-\beta e^{i\psi} + \bar{\alpha}}\right).$$

The functions $e^{iN\psi}$ form an orthogonal basis of the representation space, and

$$\mathscr{P}^{+,w}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}e^{iN\psi} = e^{2Ni\theta}e^{iN\psi}.$$

Differentiating the representation formally along the one-parameter group $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, in a way that will be made precise starting in Chapter III, we can see the irreducibility of $\mathscr{P}^{+,iv}$. (A closed invariant subspace must be generated by exponentials, by the Peter-Weyl Theorem applied to $\mathscr{P}^{+,iv}(K)$, and the differentiated action from $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ allows us to deduce the presence of all the exponentials from the presence of one of them.)

A different way of realizing $\mathscr{P}^{+,w}$ comes by choosing

$$m(z) = |z|^{-1-w}\left(\frac{z}{|z|}\right)^{-2N}.$$

Then the formula for $\mathscr{P}^{+,w}$ is

$$\mathscr{P}^{+,w}\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} F(e^{i\psi}) = |-\beta e^{i\psi} + \bar{\alpha}|^{-1-w+2N}(-\beta e^{i\psi} + \bar{\alpha})^{-2N} F\left(\frac{\alpha e^{i\psi} - \bar{\beta}}{-\beta e^{i\psi} + \bar{\alpha}}\right).$$

Taking $w = 2N - 1$, we see that the F's that extend to analytic functions on the closed disc form a non-closed invariant subspace that is equivalent with the non-closed subspace of functions for $\mathscr{D}_{2N}^{+}$ that are analytic on the closed disc. From this fact we easily deduce the continuous inclusion $\mathscr{D}_{2N}^{+} \subseteq \mathscr{P}^{+,2N-1}$.

Similar remarks apply to realizing $\mathscr{P}^{-,w}$ and to deducing the continuous inclusion of $\mathscr{D}_{2N+1}^{+}$ in $\mathscr{P}^{-,2N}$.

§7. Plancherel Formula

For a compact group, (1.10) gives the Plancherel formula as

$$\int_G |f(x)|^2\,dx = \sum_{\Phi} d_{\Phi}\|\Phi(f)\|_{\mathrm{HS}}^2, \qquad f \in L^2(G).$$

From Theorem 2.4 and the unitary trick, it follows that the representations Φ_n of (2.1) are the only irreducible unitary representations of SU(2), up to equivalence. Thus the Plancherel formula for SU(2) reads

$$\int_{\mathrm{SU}(2)} |f(x)|^2\,dx = \sum_{n=0}^{\infty} (n+1)\|\Phi_n(f)\|_{\mathrm{HS}}^2, \qquad f \in L^2(\mathrm{SU}(2)). \quad (2.22)$$

In Chapter XI we shall prove a Fourier inversion formula for SU(2):

$$h(1) = \sum_{n=0}^{\infty} (n+1)\,\mathrm{Tr}(\Phi_n(h)), \qquad h \in C^{\infty}(\mathrm{SU}(2)). \quad (2.23)$$

The Plancherel formula follows from the Fourier inversion formula by taking $h = f^* * f$ and doing a passage to the limit, since $\|A\|_{\mathrm{HS}}^2 = \mathrm{Tr}(A^*A)$.

The groups SL(2, ℂ) and SL(2, ℝ) also each have a Plancherel formula and a Fourier inversion formula for $C_{\mathrm{com}}^\infty(G)$, but no longer involving only a discrete sum. We give only the Fourier inversion formulas, since they again immediately imply corresponding Plancherel formulas. For SL(2, ℂ), the formula is

$$h(1) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{Tr}(\mathscr{P}^{n,iv}(h))(n^2 + v^2)\,dv, \qquad h \in C_{\mathrm{com}}^\infty(\mathrm{SL}(2, \mathbb{C})) \quad (2.24)$$

for a suitable normalization of Haar measure. For SL(2, ℝ), the formula is

$$\begin{aligned} h(1) = \int_{-\infty}^{\infty} \mathrm{Tr}(\mathscr{P}^{+,iv}(h))v \tanh\left(\frac{\pi v}{2}\right) dv + \int_{-\infty}^{\infty} \mathrm{Tr}(\mathscr{P}^{-,iv}(h))v \coth\left(\frac{\pi v}{2}\right) dv \\ + \sum_{n=2}^{\infty} 4(n-1)\,\mathrm{Tr}(\mathscr{D}_n^+(h) + \mathscr{D}_n^-(h)), \qquad h \in C_{\mathrm{com}}^\infty(\mathrm{SL}(2, \mathbb{R})) \quad (2.25)\end{aligned}$$

for a suitable normalization of Haar measure. These formulas are quite a bit more subtle than (2.23); the very existence of the indicated traces is not immediately evident. Proofs of the formulas will be given in Chapter XI, §§2–3.

§8. Problems

1. (a) Prove that $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$.
   (b) Prove that $\mathfrak{su}(2)$ is simple as a Lie algebra.
   (c) Prove that $\mathfrak{su}(2)$ is not isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.
2. Prove that $\mathfrak{sl}(2, \mathbb{R})$ is isomorphic to $\mathfrak{so}(2, 1)$. More specifically let $C(u, v)$ be the bilinear form on $\mathbb{R}^3$ given by

$$C\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = u_1 v_1 + u_2 v_2 - u_3 v_3,$$

   and realize $\mathfrak{so}(2, 1)$ concretely as the Lie algebra of linear maps of $\mathbb{R}^3$ into itself that are skew-symmetric relative to C:

$$\mathfrak{so}(2, 1) = \{L \in \mathrm{End}(\mathbb{R}^3) \mid C(Lu, v) + C(u, Lv) = 0, \qquad \text{for all } u, v \in \mathbb{R}^3\}.$$

   Exhibit this $\mathfrak{so}(2, 1)$ concretely as isomorphic with $\mathfrak{sl}(2, \mathbb{R})$.
3. This problem leads by steps to a direct proof, without structure theory, that $\mathfrak{sl}(2, \mathbb{C})$ is the only three-dimensional simple Lie algebra $\mathfrak{g}$ over ℂ, up to isomorphism.
   (a) Show from $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ that $\mathrm{Tr}(\mathrm{ad}\, X) = 0$ for all X.

(b) Using Engel's Theorem (A.17), choose X_0 such that ad X_0 is not nilpotent. Show from (a) and linear algebra that ad X_0 is diagonable.

(c) Show that a suitable multiple X of X_0 is a member of a basis $\{X, Y, Z\}$ of $\mathfrak{g}$ in which ad X has the matrix realization

$$\operatorname{ad} X = \begin{array}{c c} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{array}.$$

(d) Writing $[Y, Z]$ in terms of the basis $\{X, Y, Z\}$ and applying the Jacobi identity to $(\operatorname{ad} X)[Y, Z]$, show that $X \leftrightarrow h$, $Y \leftrightarrow ce$ leads to an isomorphism of $\mathfrak{g}$ with $\mathfrak{sl}(2, \mathbb{C})$.

4. Following steps as in Problem 3, prove that $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3)$ are the only three-dimensional simple Lie algebras $\mathfrak{g}$ over $\mathbb{R}$, up to isomorphism. [Hints: (a) is still okay. Define X_0 as in (b), and show that the nonzero eigenvalues of ad X_0 are two in number, are distinct, have sum 0, and have product real. For (c), find a basis $\{X, Y, Z\}$ with $X = c_0 X_0$ such that ad X has matrix realization either

$$\operatorname{ad} X = \begin{array}{c c} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{array} \quad \text{or} \quad \operatorname{ad} X = \begin{array}{c c} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}.$$

For (d), show that the first case leads to an isomorphism with $\mathfrak{sl}(2, \mathbb{R})$ and the second case leads to an isomorphism with $\mathfrak{so}(3)$.]

5. (a) Exhibit a continuous homomorphism of SU(2) onto SO(3) with kernel $\{\pm I\}$. [Hint: The map

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

carries the unit sphere S^2 in $\mathbb{R}^3$ one-one onto $\mathbb{C} \cup \{\infty\}$, on which SU(2) acts by linear fractional transformations:

$$w = g(z) = \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}} \qquad \text{for } g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \text{ in SU(2).}$$

Reinterpret these linear fractional transformations back on S^2 as the restrictions to S^2 of rotations of $\mathbb{R}^3$.]

(b) Which of the representations Φ_n of SU(2) given by (2.1) yield well-defined representations of SO(3)? Give an explicit realization of an irreducible representation of SO(3) of dimension 5.

6. The representation π of Theorem 2.4 is realized concretely on a space of polynomials by the formula (2.1). In this realization what polynomial corresponds to v_i? (The answer should be stated up to a multiplicative constant independent of i.)
7. Prove that the discrete series representation $\mathscr{D}_n^+$ of SL(2, $\mathbb{R}$) remains irreducible when restricted to the upper triangular subgroup of SL(2, $\mathbb{R}$).
8. With the complementary series $\mathscr{C}^u$ of SL(2, $\mathbb{R}$), $0 < u < 1$, realized as in §5, show that every f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ has finite norm.
9. What subgroup of SU(1, 1) plays the role of the diagonal subgroup of SL(2, $\mathbb{R}$) when SU(1, 1) and SL(2, $\mathbb{R}$) are identified as in §6?

Problems 10 to 13 rederive some classical formulas for the Mellin transform in $\mathbb{R}^2$ and interpret the results as a decomposition of the representation in Problem 10 of Chapter I as an integeral of principal series representations of SL(2, $\mathbb{R}$).

10. Fix a real number v, and let H_v^+ be the Hilbert space of functions

$$\{F:\mathbb{R}^2 \to \mathbb{C} \,|\, F(tx, ty) = |t|^{-1-iv}F(x, y) \quad \text{for } t \in \mathbb{R}\}$$

of finite norm

$$\|F\|_v^2 = \frac{1}{2\pi}\int_0^{2\pi} |F(\cos\theta, \sin\theta)|^2\, d\theta.$$

Let SL(2, $\mathbb{R}$) act on $\mathbb{R}^2$ as left multiplication on column vectors, and let $P^{+,iv}$ be the unitary representation of SL(2, $\mathbb{R}$) on H_v^+ given by

$$P^{+,iv}(g)F(x, y) = F(g^{-1}(x, y)).$$

Prove that the map of H_v^+ to $L^2(\mathbb{R})$ given by $F \to f$ with $f(x) = F(1, x)$ is a unitary equivalence (except for a constant factor) of $P^{+,iv}$ and $\mathscr{P}^{+,iv}$.
11. Starting from the analogous Hilbert space H_v^- obtained from

$$\{F:\mathbb{R}^2 \to \mathbb{C} | F(tx, ty) = (\operatorname{sgn} t)|t|^{-1-iv}F(x, y) \quad \text{for } t \in \mathbb{R}\},$$

construct a unitary representation $P^{-,iv}$ of SL(2, $\mathbb{R}$) on H_v^-, and show it is unitarily equivalent with $\mathscr{P}^{-,iv}$.
12. For g in SL(2, $\mathbb{R}$) and F in $L^2(\mathbb{R}^2)$, define $\Phi(g)F(x, y) = F(g^{-1}(x, y))$. On the dense subspace $C^\infty_{\text{com}}(\mathbb{R}^2 - \{0\})$, define

$$\{F:\mathbb{R}^2\, E_vF(x, y) = \int_0^\infty F(tx, ty)t^{iv}\, dt$$

Prove that E_v carries this dense subspace in equivariant fashion onto a dense subspace of

$$H_v^+ \oplus H_v^- = \{F:\mathbb{R}^2 \to \mathbb{C} \,|\, F(tx, ty) = t^{-1-iv}F(x, y) \qquad \text{for } t > 0\}.$$

13. Applying the Fourier inversion formula and Plancherel formula to the function $h(s) = e^s F(e^s x, e^s y)$, $-\infty < s < \infty$, establish the formulas

$$F(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (E_v F)(x, y)\, dv$$

and

$$\|F\|^2 = \int_{-\infty}^{\infty} \|E_v F\|_v^2\, dv$$

for F in $C_{\mathrm{com}}^{\infty}(\mathbb{R}^2 - \{0\})$.