The Knizhnik–Zamolodchikov Equations

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ABSTRACT This article is an outgrowth of a series of three one-hour lectures on the Knizhnik–Zamolodchikov equations presented at the 1998 Adelaide Summer School on "Differential Equations in Geometry and Physics." This article does not constitute a comprehensive account of all that is known about the KZ equations but, rather, is an introduction to some of the main results intended to motivate the reader to further study. The three main sections can be read, to a large extent, independently.

1 Monodromy of the KZ equations

In this first section we introduce the Knizhnik–Zamolodchikov equations [KZ84] and give an explicit characterization of the associated monodromy representation of the braid group, the so-called Drinfeld–Kohno theorem [Ko87, Ko88, Dr89a, Dr89b, Dr90]. Excellent accounts of the Drinfeld–Kohno theorem, including all the missing proofs and background on quantum groups, can be found in the textbooks [SS93, CP94, Ka95] or the recent lecture notes [EFK98].

1.1 The Knizhnik–Zamolodchikov equations

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Fix a symmetric invariant 2-tensor Ω on $\mathfrak{g}$ and a set of representations $\rho_i : \mathfrak{g} \longrightarrow \mathrm{End}(V_i)$, $i = 1, \dots, n$. We denote by Ω_{ij} the endomorphism of $W = V_1 \otimes \cdots \otimes V_n$ which acts as $(\rho_i \otimes \rho_j)(\Omega)$ on the ith and jth factors of W and as the identity on the others.

Let Y_n be the complex variety

$$Y_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}. \tag{1.1.1}$$

The Knizhnik–Zamolodchikov equations are the system of equations

$$\frac{\partial \phi}{\partial z_i} = h \sum_{\substack{j \\ j \neq i}} \frac{\Omega_{ij}}{z_i - z_j}\, \phi\,, \tag{1.1.2}$$

acting on functions $\phi : Y_n \longrightarrow W$.

When $V_i \cong V$ for all $i = 1, \dots, n$, we can use the symmetry of Ω_{ij} to rewrite (1.1.2) as the differential system

$$d\phi = h \sum_{1 \le i < j \le n} \frac{\Omega_{ij}}{z_i - z_j}(dz_i - dz_j)\phi \tag{1.1.3}$$

or as $\nabla\phi = 0$ where $\nabla = d + \Gamma$ with

$$\Gamma = -h\sum_{i<j} \frac{\Omega_{ij}}{z_i - z_j}(dz_i - dz_j) = -h\sum_{i<j} \Omega_{ij}\ d\log(z_i - z_j)\,. \tag{1.1.4}$$

We will refer to (1.1.2) as the KZ_n system and to ∇ as the KZ_n connection.

Remark. For concreteness, suppose $\mathfrak{g}$ is semisimple and x_a an orthonormal basis of $\mathfrak{g}$ (with respect to the Cartan–Killing form). Then

$$C = \sum_a x_a x_a \tag{1.1.5}$$

is an element in the center of $U(\mathfrak{g})$. We may take

$$\Omega = \tfrac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C) = \sum_a x_a \otimes x_a\,, \tag{1.1.6}$$

where the comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is defined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x\,, \qquad x \in \mathfrak{g}\,. \tag{1.1.7}$$

The invariance of Ω is expressed as

$$[\Delta(x), \Omega] = 0\,, \qquad \text{for all } x \in \mathfrak{g}\,, \tag{1.1.8}$$

and follows easily from the properties of C, e.g.,

$$[\Delta(x), \Delta(C)] = \Delta([x, C]) = 0\,. \tag{1.1.9}$$

Explicitly, for $\mathfrak{g} \cong \mathfrak{sl}_2$ defined by

$$[e, f] = h\,, \qquad [h, e] = 2e\,, \qquad [h, f] = -2f\,, \tag{1.1.10}$$

we can take

$$\Omega = e \otimes f + f \otimes e + \tfrac{1}{2} h \otimes h\,. \tag{1.1.11}$$

1.2 Connections, holonomy and monodromy

Consider more generally a vector bundle $p : E \to X$ over a complex analytic variety X of dimension n. Denote the fibre over the point $x \in X$ by $F_x = p^{-1}(x)$. A *connection* ∇ on E is a linear map from the space of sections $\Gamma(X, E)$ of E into the space $\Omega^1(X, E)$ of differential 1-forms on X with values in E such that

$$\nabla(f\phi) = (df)\phi + f\nabla(\phi)\,, \tag{1.2.1}$$

for any $f \in \mathcal{O}(X)$ and $\phi \in \Gamma(X, E)$. Locally we can write

$$\nabla = d + \Gamma\,, \tag{1.2.2}$$

with $\Gamma \in \Omega^1(X, \mathrm{End}(E))$. A section $\phi \in \Gamma(X, E)$ is called *horizontal* for the connection ∇ if $\nabla\phi = 0$, i.e., if locally ϕ is a solution of the system

$$d\phi = -\Gamma\phi\,. \tag{1.2.3}$$

Thus, solutions to the KZ_n-equations are horizontal sections of the trivial vector bundle $Y_n \times W$ over Y_n with respect to the KZ_n connection ∇ of (1.1.4).

Now, let $\gamma : [0, 1] \to X$ be a smooth path in X from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$. The pullback $\gamma^*\Gamma \equiv A(s)ds$ defines a matrix of differential 1-forms on the interval $[0, 1]$. By the theory of ordinary differential equations, there exists a unique smooth map T_γ from $[0, 1]$ into $\mathrm{Aut}(E)$ such that $T_\gamma(0) = 1$ and $w(s) = T_\gamma(s)w(0)$ is a solution of the differential equation

$$\frac{dw(s)}{ds} = A(s)w(s)\,. \tag{1.2.4}$$

The automorphism $T_\gamma \equiv T_\gamma(1)$ defines a linear isomorphism

$$T_\gamma \;:\; F_{x_0} \;\to\; F_{x_1}\,, \tag{1.2.5}$$

called the *parallel transport* along the path γ. If γ' is a path from x_1 to x_2 we may consider the composed path $\gamma\gamma'$. The uniqueness theorem on systems of first order linear differential equations implies

$$T_{\gamma\gamma'} = T_{\gamma'} \circ T_\gamma\,. \tag{1.2.6}$$

The *holonomy group* at x_0 is defined as the subgroup of $\mathrm{Aut}(F_{x_0})$ generated by T_γ for all loops based at $x_0 \in X$. In general the holonomy group depends on both the local as well as on the global structure of X. However,

Theorem 1.1. *Given a connection ∇, we have $T_\gamma = T_{\gamma'}$ for any pair (γ, γ') of homotopic paths in X iff the connection is flat, i.e., iff the curvature $K = \nabla^2 = d\Gamma + \Gamma \wedge \Gamma$ vanishes.*

Remark. A vector bundle over X with a flat connection ∇ is called a *local system over X*.

Thus, given flat connection ∇ and a point $x_0 \in X$, we have a homomorphism of groups

$$T \; : \; \pi_1(X, x_0) \; \to \; \operatorname{Aut}(F_{x_0}) \,. \tag{1.2.7}$$

This is called the *monodromy representation* of the fundamental group acting on the fibre.

Specializing the above result to a flat connection ∇ of a trivial vector bundle $Y_n \times W$ over the complex variety

$$Y_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\} \,, \tag{1.2.8}$$

we obtain a representation of the *pure braid group* $P_n = \pi_1(Y_n)$ on the vector space W. The symmetric group S_n acts freely on Y_n by $(z_1, \ldots, z_n)\sigma = (z_{\sigma(1)}, \ldots, z_{\sigma(n)})$. The pure braid group P_n is a subgroup of the *full braid group* $B_n = \pi_1(X_n)$ where X_n is the coset space Y_n/S_n. We have the following description of B_n.

Theorem 1.2. *The braid group B_n on n strands is the group with generators σ_i, $i = 1, \ldots, n-1$, and relations*

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \,, && \textit{if } |i-j| > 1 \,, \\ \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j \,, && \textit{if } |i-j| = 1 \,. \end{aligned} \tag{1.2.9}$$

Remark. Choosing the basepoint $x_0 = (1, 2, \ldots, n) \in X_n$ we can pictorially represent the generators $\sigma_i \in B_n$ as in Figure 1.1.

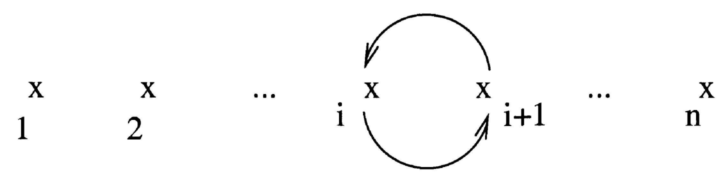

Figure 1.1. The generator $\sigma_i \in B_n$.

In order to extend the monodromy representation (1.2.7) to a representation of the full braid group B_n we need a left action of S_n on the vector space W. Then there exists a right action of S_n on the trivial vector bundle $Y_n \times W$ by

$$(z_1, \ldots, z_n; w)\sigma = (z_{\sigma(1)}, \ldots, z_{\sigma(n)}; \sigma^{-1}w) \,. \tag{1.2.10}$$

The quotient space $E = (Y_n \times W)/S_n$ becomes a (nontrivial) vector bundle over X_n. If the differential system (1.1.2) is invariant under the action of S_n the flat connection ∇ descends to a flat connection on E and we obtain a monodromy representation of the fundamental group of X_n, i.e., of the full braid group B_n.

1.3 Braid group representation associated to $\mathbf{KZ}_n$

Let $r(z)$ be a holomorphic function on the open set $\mathbb{C}\backslash\{0\}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. Let $\rho_i : \mathfrak{g} \longrightarrow \mathrm{End}(V_i)$, $i = 1, \ldots, n$, be a family of representations of $\mathfrak{g}$. As before we denote by r_{ij} the endomorphism of $W = V_1 \otimes \cdots \otimes V_n$ acting as $(\rho_i \otimes \rho_j)r$ on the i-th and j-th factor and as the identity on the others. Consider the (trivial) vector bundle $Y_n \times W$ over Y_n with connection $\nabla = d + \Gamma$ where

$$\Gamma = -h \sum_{i<j} r_{ij}(z_i - z_j)(dz_i - dz_j)\,. \tag{1.3.1}$$

Proposition 1.3. *The connection ∇ defined by (1.3.1) is flat iff $r(z)$ satisfies the classical Yang–Baxter equation*

$$\begin{aligned}[r_{12}(z_1 - z_2), r_{13}(z_1 - z_3)] + [r_{12}(z_1 - z_2), r_{23}(z_2 - z_3)]& \\ + [r_{13}(z_1 - z_3), r_{23}(z_2 - z_3)] &= 0\,,\end{aligned} \tag{1.3.2}$$

for all $(z_1, z_2, z_3) \in Y_3$.

Proof. Clearly $d\Gamma = 0$, so we have to show $\Gamma \wedge \Gamma = 0$. See, e.g., [CP94, p. 539]. $\square$

Lemma 1.4. *Let Ω be a symmetric invariant 2-tensor on $\mathfrak{g}$ and define*

$$r(z) = \frac{\Omega}{z}\,. \tag{1.3.3}$$

Then $r(z)$ satisfies the classical Yang-Baxter equation iff the following conditions are satisfied:

$$[\Omega_{12}, \Omega_{13} + \Omega_{23}] = [\Omega_{23}, \Omega_{12} + \Omega_{13}] = 0\,. \tag{1.3.4}$$

Remark. Suppose, more generally, we have a family Ω_{ij}, $1 \leq i < j \leq n$, of endomorphisms of $W = V_1 \otimes \cdots \otimes V_n$. Then, the connection defined by (1.3.1) is flat if the following conditions, known as the *infinitesimal braid group conditions*, are satisfied

$$\begin{aligned} [\Omega_{ij}, \Omega_{kl}] &= 0\,, && i,j,k,l \text{ distinct}\,, \\ [\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = [\Omega_{jk}, \Omega_{ij} + \Omega_{ik}] &= 0\,, && i,j,k \quad \text{distinct}\,. \end{aligned} \tag{1.3.5}$$

Note, in particular, that (1.3.5) implies

$$[\widetilde{\Omega}, \Omega_{ij}] = 0\,, \tag{1.3.6}$$

where

$$\widetilde{\Omega} = \sum_{i<j} \Omega_{ij}\,. \tag{1.3.7}$$

Theorem 1.5. *The* KZ_n*-connection is flat.*

Proof. It remains to show that Ω_{ij} defined in Section 1.1. satisfies the equations (1.3.4). These follow immediately from the invariance of Ω (equation (1.1.8)). Indeed

$$\begin{aligned}
[\Omega_{12}, \Omega_{13} + \Omega_{23}] &= [\Omega_{12}, (\Delta \otimes 1)(\Omega)] = 0\,, \\
[\Omega_{23}, \Omega_{12} + \Omega_{13}] &= [\Omega_{23}, (1 \otimes \Delta)(\Omega)] = 0\,.
\end{aligned} \tag{1.3.8}$$

□

We conclude that the KZ-equation (KZ_n) defines a flat connection on the trivial vector bundle $Y_n \times W$ over Y_n and, consequently, determines a monodromy representation

$$\rho_n^{\mathrm{KZ}} \ :\ \pi_1(Y_n) \ \longrightarrow\ \mathrm{Aut}(W) \tag{1.3.9}$$

of the pure braid group $P_n = \pi_1(Y_n)$.

Now suppose that $\rho_i = \rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ for all i, i.e., $W = V^{\otimes n}$. We can define a left action of S_n on $W \cong V^{\otimes n}$ by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}\,, \tag{1.3.10}$$

for $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$. It is clear that the KZ_n system is invariant under the action of S_n. Thus, in this case, we obtain a monodromy representation

$$\rho_n^{\mathrm{KZ}} \ :\ B_n = \pi_1(X_n) \ \longrightarrow\ \mathrm{Aut}(V^{\otimes n}) \tag{1.3.11}$$

of the full braid group B_n.

1.4 The Drinfeld–Kohno theorem

The Drinfeld–Kohno theorem gives an explicit description of the monodromy representation (1.3.11). Let us discuss the special case of KZ_2 first. We take the basepoint $x_0 = (1,2) \in X_2$. The generator $\sigma_1 \in B_2$ can be represented by the loop

$$\gamma(s) = (z_1(s), z_2(s)) = (\tfrac{1}{2}(3 - e^{i\pi s}), \tfrac{1}{2}(3 + e^{i\pi s}))\,, \tag{1.4.1}$$

such that $\gamma(0) = (1,2)$, $\gamma(1) = (2,1)$. Pulling back Γ along this loop yields

$$\gamma^*\Gamma = A(s)ds = i\pi h\Omega ds\,. \tag{1.4.2}$$

The differential equation (1.2.4) is solved by

$$w(s) = e^{i\pi h s\Omega} w(0)\,, \tag{1.4.3}$$

and thus

$$\rho_2^{\mathrm{KZ}}(\sigma_1)(v_1 \otimes v_2) = \tau\left(e^{i\pi h\Omega}(v_1 \otimes v_2)\right) , \tag{1.4.4}$$

where we have used the identification $(2,1; v_1 \otimes v_2) = (1,2; v_2 \otimes v_1)$ in $(Y_2 \times V^{\otimes 2})/S_2$ and we have introduced the interchange operator $\tau : V \otimes V \to V \otimes V$ by $\tau(v_1 \otimes v_2) = (v_2 \otimes v_1)$.

For KZ_n, $n \geq 2$, in general the monodromy is hard to determine explicitly. In the case where V is the vector representation of $\mathfrak{sl}_m$ it can be shown that the monodromy representation is isomorphic to the so-called Pimsner–Popa–Temperley-Lieb representation of the braid group. In fact, in this case the representation factors through the Iwahori–Hecke algebra $H_n(q)$ where $q = e^{-i\pi h}$. The general case is described by the Drinfeld–Kohno theorem.

The element $R = e^{i\pi h\Omega} \in \mathrm{End}(V^{\otimes 2})$ obviously satisfies

$$\Delta(x)R = R\Delta(x) , \tag{1.4.5}$$

and thus is similar to the R-matrix of a quantum group. In fact, each solution $R \in \mathrm{End}(V^{\otimes 2})$ to the Yang–Baxter Equation (YBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.4.6}$$

gives rise to a representation of the braid group

$$\rho_n^R \ : \ B_n \ \to \ \mathrm{Aut}(V^{\otimes n}) \tag{1.4.7}$$

by

$$\rho_n^R(\sigma_i) = \tau_{i\,i+1} R_{i\,i+1} , \tag{1.4.8}$$

where $\tau_{i\,i+1}$ is the transposition of the i-th and $i+1$-st entry. Indeed, the YBE (1.4.6) implies (cf. (1.2.9))

$$\rho_n^R(\sigma_i)\rho_n^R(\sigma_j)\rho_n^R(\sigma_i) = \rho_n^R(\sigma_j)\rho_n^R(\sigma_i)\rho_n^R(\sigma_j) , \qquad |i-j| = 1 \tag{1.4.9}$$

while trivially

$$\rho_n^R(\sigma_i)\rho_n^R(\sigma_j) = \rho_n^R(\sigma_j)\rho_n^R(\sigma_i) , \qquad |i-j| > 1 .$$

The Drinfeld–Kohno theorem states

Theorem 1.6. *Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra, R the universal R-matrix of the quantized universal enveloping algebra $U_h(\mathfrak{g})$ and Ω as in (1.1.6). Then the braid group representations ρ_n^{KZ} and ρ_n^R are equivalent for any $n > 1$ and $\mathfrak{g}$-module V. That is, there exists a $u \in \mathrm{Aut}_{\mathbb{C}[\![h]\!]}(V^{\otimes n}[\![h]\!])$ such that*

$$\rho_n^{KZ}(\sigma) = u\rho_n^R(\sigma)u^{-1} , \tag{1.4.10}$$

for all $\sigma \in B_n$.

2 Solutions by hypergeometric integrals

In the second section we will discuss solutions of the Knizhnik–Zamolodchikov equations in terms of hypergeometric integrals. For special cases, these solutions were found in [CF87, DJMM89]; the general case is due to Schechtman and Varchenko [SV89, SV90a, SV90b, SV90c] who also showed a relation between the integration contours, determined by the homology of a certain local system, and representations of a certain quantum group. The homology of the local system is discussed in length in the monograph [Va95] and a more detailed account of all the material in this lecture can be found, again, in the textbook [EFK98].

2.1 Two simplifications

In this subsection we will discuss two simplifications of the KZ_n equation which will be useful in finding explicit solutions.

Recall the KZ_n equations

$$\kappa \frac{\partial}{\partial z_i} \phi = \sum_{\substack{j \\ j \neq i}} \frac{\Omega_{ij}}{z_i - z_j} \phi \,, \tag{2.1.1}$$

where $\phi : Y_n \to W$. Here $W = V_1 \otimes \cdots \otimes V_n$ where $\rho_i : \mathfrak{g} \to \mathrm{End}(V_i)$ is a family of representations of a finite-dimensional Lie algebra $\mathfrak{g}$, Ω is a symmetric, invariant 2-tensor on $\mathfrak{g}$ and the domain Y_n is defined in (1.1.1).

The first simplification occurs because of the $\mathfrak{g}$-invariance of the KZ_n equation. This can be expressed as

$$[\Delta^{(n-1)}(x), \nabla] = 0 \,, \qquad \forall x \in \mathfrak{g} \,, \tag{2.1.2}$$

where ∇ is the KZ_n connection and $\Delta^{(n)}$ the n-fold iterated comultiplication. This means we can look for solutions characterized by some $\mathfrak{g}$-invariant conditions. For instance, we might consider solutions in specific weight spaces

$$W \;\cong\; \bigoplus_{\lambda} W^{\lambda} \,, \tag{2.1.3}$$

or solutions on the space of singular vectors

$$W^{\mathfrak{n}_+} = \{ v \in W \mid \Delta^{(n-1)}(x) \cdot v = 0 \,, \forall x \in \mathfrak{n}_+ \} \,, \tag{2.1.4}$$

or a combination of both. Here we have chosen a Cartan subalgebra of $\mathfrak{g}$ and a set of positive roots such that $\mathfrak{b}_+$ is the corresponding Borel subalgebra and $\mathfrak{n}_+$ the nilpotent part. In fact, if W is completely reducible it is sufficient to look for solutions with values in $W^{\mathfrak{n}_+}$ since the others can be obtained through the $\mathfrak{g}$-action.

For definiteness, consider $\mathfrak{g} = \mathfrak{sl}_2$. Let $M_\mu \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\mu$ denote the Verma module of spin $\mu/2$, i.e., the module with highest weight vector v such that

$$h \cdot v = \mu v\,, \qquad e \cdot v = 0\,, \tag{2.1.5}$$

and with basis

$$f^k \cdot v\,, \qquad k \in \mathbb{Z}_{\geq 0}\,. \tag{2.1.6}$$

Note that

$$h(f^k \cdot v) = (\mu - 2k)(f^k \cdot v)\,. \tag{2.1.7}$$

Thus, we have a weight space decomposition

$$W = M_{\mu_1} \otimes \cdots \otimes M_{\mu_n} = \bigoplus_{m \geq 0} W^{\sum \mu_i - 2m}\,. \tag{2.1.8}$$

We will refer to m as the level of the subspace $W^{\sum \mu_i - 2m} \subset W$. Clearly, $W^{\sum \mu_i - 2m}$ has a basis

$$f^{m_1} \cdot v_1 \otimes \cdots \otimes f^{m_n} \cdot v_n \;\equiv f_1^{m_1} \ldots f_n^{m_n} \cdot v\,, \tag{2.1.9}$$

with $\sum_i m_i = m$, and where we have denoted the action of f on M_{μ_i} by f_i and $v = v_1 \otimes \cdots \otimes v_n$. Thus

$$\dim(W^{\sum \mu_i - 2m}) = p_n(m) = \binom{n+m-1}{m}\,, \tag{2.1.10}$$

where $p_n(m)$ is the number of partitions of m into n parts. Similarly, for generic μ_i,

$$\dim((W^{\mathfrak{n}_+})^{\sum \mu_i - 2m}) = p_{n-1}(m) = \binom{n+m-2}{m}\,. \tag{2.1.11}$$

As an example, consider KZ_n for $m = 0$. The subspace $W^{\sum \mu_i} \subset W$ is 1-dimensional and spanned by $v = v_1 \otimes \cdots \otimes v_n$. Now, by (1.1.11),

$$\Omega_{ij} \cdot v = \frac{\mu_i \mu_j}{2} v\,, \tag{2.1.12}$$

such that $\phi(z_1, \ldots, z_n) = \phi_0(z_1, \ldots, z_n) v$ with

$$\phi_0(z_1, \ldots, z_n) = \prod_{i<j} (z_i - z_j)^{\frac{\mu_i \mu_j}{2\kappa}}\,, \tag{2.1.13}$$

is a solution of KZ_n on $W^{\sum \mu_i}$.

The second simplification is a reduction of the n variables to $n-2$ variables. Note

$$\kappa \sum_i \frac{\partial}{\partial z_i}\phi = \sum_{i\neq j} \frac{\Omega_{ij}}{z_i - z_j}\phi = 0\,, \tag{2.1.14}$$

where we have used the symmetry of Ω_{ij}. This equation expresses the translation invariance of KZ_n. Changing to coordinates

$$\begin{aligned} w_1 &= z_1 + \cdots + z_n\,, \\ w_i &= z_i - z_1\,, \qquad\qquad i = 2,\ldots,n\,, \end{aligned} \tag{2.1.15}$$

gives

$$\phi(z_1,\ldots,z_n) = \widetilde{\phi}(w_2,\ldots,w_n)\,, \tag{2.1.16}$$

where $\widetilde{\phi}$ satisfies

$$\kappa\frac{\partial}{\partial w_i}\widetilde{\phi} = \left(\frac{\Omega_{i1}}{w_i} + \sum_{j\neq 1,i}\frac{\Omega_{ij}}{w_i - w_j}\right)\widetilde{\phi}\,. \tag{2.1.17}$$

Next, note that

$$\kappa \sum_i w_i \frac{\partial}{\partial w_i}\widetilde{\phi} = \widetilde{\Omega}\widetilde{\phi}\,, \tag{2.1.18}$$

which expresses the scale invariance of KZ_n. Here

$$\widetilde{\Omega} = \sum_{i<j}\Omega_{ij}\,. \tag{2.1.19}$$

This means that

$$\widetilde{\phi}(u^{-1}w_2,\ldots,u^{-1}w_n) = u^{-\widetilde{\Omega}/\kappa}\widetilde{\phi}(w_2,\ldots,w_n)\,. \tag{2.1.20}$$

Using (1.3.6) and defining new coordinates

$$\begin{aligned} u &= w_n\,, \\ u_i &= u^{-1}w_i = \frac{z_i - z_1}{z_n - z_1}\,, \qquad i = 2,\ldots,n-1\,, \end{aligned} \tag{2.1.21}$$

shows that

$$\widetilde{\phi}(w_2,\ldots,u) = u^{\widetilde{\Omega}/\kappa}\widetilde{\phi}(u_2,\ldots,u_{n-1},1) \;\equiv\; u^{\widetilde{\Omega}/\kappa}\psi(u_2,\ldots,u_{n-1})\,, \tag{2.1.22}$$

where ψ satisfies

$$\kappa \frac{\partial}{\partial u_i}\psi = \left(\frac{\Omega_{i1}}{u_i} + \sum_{\substack{2 \le j \le n-1 \\ j \ne i}} \frac{\Omega_{ij}}{u_i - u_j} + \frac{\Omega_{in}}{u_i - 1} \right) \psi\,, \qquad i = 2, \ldots, n-1\,. \tag{2.1.23}$$

In particular, the above analysis immediately gives the following solution of KZ_2

$$\phi(z_1, z_2) = (z_2 - z_1)^{\Omega_{12}/\kappa}(u_1 \otimes u_2)\,, \tag{2.1.24}$$

for arbitrary vectors $u_i \in V_i$. The monodromy (1.4.4) follows easily from this solution.

2.2 The KZ_3 equations for level $m = 1$

For a less trivial example consider the KZ_3 equations for Verma modules of $\mathfrak{sl}_2$ at level $m = 1$. Using the results of Section 2.1, we can write

$$\phi(z_1, z_2, z_3) = (z_3 - z_1)^{\frac{(\Omega_{12}+\Omega_{13}+\Omega_{23})}{\kappa}} \psi\left(\frac{z_2 - z_1}{z_3 - z_1}\right)\,, \tag{2.2.1}$$

where $\psi(x)$ satisfies

$$\kappa \frac{d\psi}{dx} = \left(\frac{\Omega_{12}}{x} + \frac{\Omega_{23}}{x-1}\right)\psi\,. \tag{2.2.2}$$

For $\mu_i \neq 0$ the level $m = 1$ subspace $(W^{\mathfrak{n}_+})^{\sum \mu_i - 2}$ is spanned by

$$\begin{aligned} w_1 &= (\mu_2 f_1 - \mu_1 f_2)\cdot v\,, \\ w_2 &= (\mu_3 f_2 - \mu_2 f_3)\cdot v\,, \end{aligned} \tag{2.2.3}$$

where $v = v_1 \otimes v_2 \otimes v_3$ is the highest weight vector of W. A straighforward calculation gives

$$\begin{aligned} \Omega_{12}\cdot w_1 &= (\tfrac{1}{2}\mu_1\mu_2 - \mu_1\mu_2)w_1\,, \\ \Omega_{12}\cdot w_2 &= \mu_3 w_1 + \tfrac{1}{2}\mu_1\mu_2 w_2\,, \\ \Omega_{23}\cdot w_1 &= \tfrac{1}{2}\mu_2\mu_3 w_1 + \mu_1 w_2\,, \\ \Omega_{23}\cdot w_2 &= (\tfrac{1}{2}\mu_2\mu_3 - \mu_2 - \mu_3)w_2\,, \\ \Omega_{13}\cdot w_1 &= \tfrac{1}{2}(\mu_2 - 2)\mu_3 w_1 - \mu_1 w_2\,, \\ \Omega_{13}\cdot w_2 &= -\mu_3 w_1 + \tfrac{1}{2}\mu_1(\mu_3 - 2)w_2\,, \end{aligned} \tag{2.2.4}$$

such that, in particular,

$$\widetilde{\Omega}\cdot w_i = \left(\tfrac{1}{2}(\mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3) - (\mu_1 + \mu_2 + \mu_3)\right) w_i\,. \tag{2.2.5}$$

Substituting these results in (2.2.2) one finds

$$\psi(x) = x^{\frac{\mu_1\mu_2-2\mu_1-2\mu_2}{2\kappa}}(1-x)^{\frac{\mu_2\mu_3}{\kappa}}\left(F(x)w_1 + x\frac{\kappa}{\mu_3}F'(x)w_2\right), \tag{2.2.6}$$

where $F(x)$ is a solution of the Gauss hypergeometric equation

$$x(1-x)F'' + (\gamma - (\alpha+\beta+1)x)F' - \alpha\beta F = 0\,, \tag{2.2.7}$$

with

$$\alpha = \frac{\mu_3}{\kappa}, \qquad \beta = -\frac{\mu_1}{\kappa}, \qquad \gamma = 1 - \frac{\mu_1+\mu_2}{\kappa}. \tag{2.2.8}$$

The equation (2.2.7) has two independent solutions. The one satisfying $F(0)=1$ is often denoted by ${}_2F_1(\alpha,\beta;\gamma;x)$ and has a series expansion (for $|x|<1$)

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{n\geq 0}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{x^n}{n!}, \tag{2.2.9}$$

where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$. It can be analytically continued to a multi-valued function on $\mathbb{C}\backslash\{0,1\}$ as shown, e.g., by its defining equation (2.2.7). The other solution is given by

$$x^{1-\gamma}\ {}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;x)\,. \tag{2.2.10}$$

For $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0$ we have the following integral formula due to Euler:

$${}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}dt\,, \tag{2.2.11}$$

i.e., in terms of the parameters μ_i, κ, we find a solution given by

$$\int_0^1 t^{-1-\frac{\mu_1}{\kappa}}(1-t)^{-\frac{\mu_2}{\kappa}}(1-xt)^{-\frac{\mu_3}{\kappa}}dt\,, \tag{2.2.12}$$

valid for

$$\mathrm{Re}\left(\frac{\mu_1}{\kappa}\right) < 0\,, \qquad \mathrm{Re}\left(\frac{\mu_2}{\kappa}\right) < 1\,. \tag{2.2.13}$$

To find an integral expression for the solution for arbitrary parameters we have to lift the integration interval $[0,1]$ in (2.2.12) to a contour C in the punctured complex plane $\mathbb{C}\backslash\{0,1,x^{-1}\}$. The integrand of (2.2.12), however, has various branch cuts so the contour is required to be such that the integrand has a continuous branch along C.

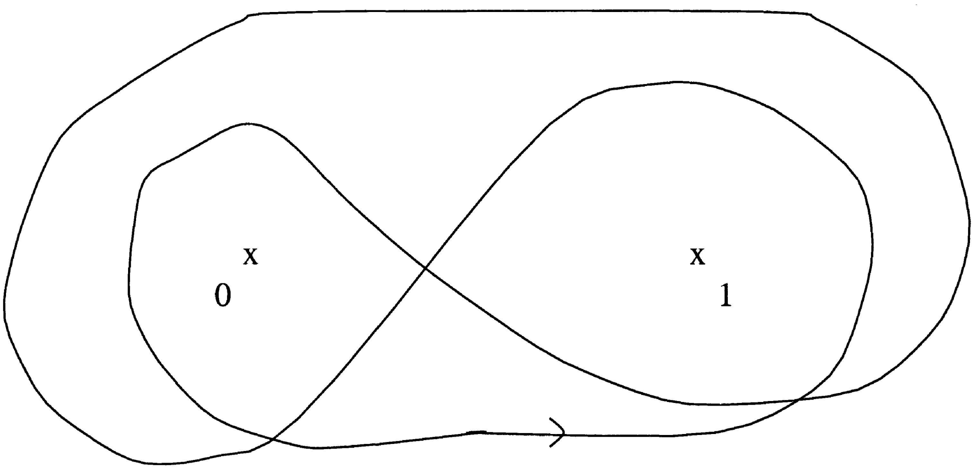

Figure 2.1. The Pochhammer contour.

The standard contour (for generic parameters) is the so-called Pochhammer contour of Figure 2.1. This contour encircles each of the branch points $\{0,1\}$ twice, in opposite directions, hence defines a continuous branch of the integrand.

In the region (2.2.13), the Pochhammer contour can be squeezed onto the real axis and we obtain that the function

$$\psi_C(x) = \int_C t^{-1-\frac{\mu_1}{\kappa}}(1-t)^{-\frac{\mu_2}{\kappa}}(1-xt)^{-\frac{\mu_3}{\kappa}}\,dt\,, \tag{2.2.14}$$

equals (2.2.12) up to a factor

$$(1-e^{-\frac{2\pi i\mu_1}{\kappa}})(1-e^{-\frac{2\pi i\mu_2}{\kappa}})\,. \tag{2.2.15}$$

Collecting the results, and reinstating the original variables, finally yields the following solutions for KZ_3 at $m=1$

$$\phi_C(\mathbf{z}) = \prod_{i<j}(z_i-z_j)^{\frac{\mu_i\mu_j}{2\kappa}} \sum_{r=1}^{3}\left(\int_C \left(\prod_{i=1}^{3}(t-z_i)^{-\frac{\mu_i}{\kappa}}\right)(t-z_r)^{-1}dt\right) f_r\cdot v \tag{2.2.16}$$

for suitable integration contours C.

2.3 Solution of KZ_n, arbitrary m

The solution of the $\mathfrak{sl}_2$ KZ_n equations for arbitrary $m\geq 0$ is a generalization of (2.2.16). We will just state the result. First of all, instead of one t, we need m integration variables $\mathbf{t}=(t_1,\ldots,t_m)$. For fixed $\mathbf{z}=(z_1,\ldots,z_n)$, let

$$\psi_m(\mathbf{z},\mathbf{t}) = \prod_{i<j}(z_i-z_j)^{\frac{\mu_i\mu_j}{2\kappa}} \prod_{p,j}(t_p-z_j)^{-\frac{\mu_j}{\kappa}} \prod_{p<n}(t_p-t_n)^{\frac{2}{\kappa}} \tag{2.3.1}$$

be the multivalued function defined over

$$Y_{\mathbf{z},m} = \{(t_1,\dots,t_m) \in \mathbb{C}^m \mid t_p \neq z_i,\, t_p \neq t_q\}\,. \tag{2.3.2}$$

For any partition $\mathbf{m} = (m_1,\dots,m_n)$ of m, i.e., $m = \sum_j m_j$, let

$$\rho_{\mathbf{m}}(\mathbf{z},\mathbf{t}) = \sum_{\sigma\in\Sigma_{\mathbf{m}}} \prod_i (t_n - z_{\sigma(i)})^{-1}\,, \tag{2.3.3}$$

where $\Sigma_{\mathbf{m}}$ is the set of all maps $\sigma : \{1,\dots,m\} \to \{1,\dots,n\}$ such that $\#\{\sigma^{-1}(j)\} = m_j$. Then, for a properly defined (m-dimensional) contour C, the function

$$\phi_C(\mathbf{z}) = \sum_{\mathbf{m}} \left(\int_C \psi_m(\mathbf{z},\mathbf{t})\rho_{\mathbf{m}}(\mathbf{z},\mathbf{t}) dt_1 \wedge \cdots \wedge dt_m \right) f_1^{m_1} \dots f_n^{m_n} \cdot v\,, \tag{2.3.4}$$

is a solution of KZ_n.

2.4 Quantum groups and the homology of local systems

In this section we discuss how to classify the allowed contours in (2.3.4), and how this is related to a problem in the representation theory of quantum groups. This section is going to be rather sketchy. For details, which are far beyond the scope of these lectures, one is advised to consult, in particular, [EFK98, Va95].

First of all, suppose we have a local system E over a complex variety X of dimension n, i.e., a vector bundle $E \to X$ with a flat connection ∇. The connection $\nabla : \Omega^0(X,E) \to \Omega^1(X,E)$ can be extended to a map $d_\nabla : \Omega^k(X,E) \to \Omega^{k+1}(X,E)$ by

$$d_\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s d\omega\,, \tag{2.4.1}$$

for $s \in \Gamma(X,E)$ and $\omega \in \Omega^k(X)$, such that $d_\nabla^2 = 0$. Thus we have a complex

$$0 \to \Omega^0(X,E) \to \Omega^1(X,E) \to \dots \to \Omega^n(X,E) \to 0\,. \tag{2.4.2}$$

The cohomology of this complex, $H^\bullet_{\mathrm{DR}}(X,E)$, is called the de-Rham cohomology of the local system E. Under good circumstances there is a dual homology theory, $H_\bullet(X,E)$, such that the pairing (integration of forms)

$$H_\bullet(X,E) \times H^\bullet_{\mathrm{DR}}(X,E) \to \mathbb{C}\,, \tag{2.4.3}$$

is defined and nondegenerate.

Applying this to the case at hand, the multi-valued function $\psi_m(\mathbf{z},\mathbf{t})$ of (2.3.1), defines a 1-dimensional local system $\mathcal{S}$ over $Y_{\mathbf{z},m}$, i.e., a line-bundle over $Y_{\mathbf{z},m}$ with a flat connection ∇. The connection can be taken as

$$\nabla f = df + f d \log \psi_m . \tag{2.4.4}$$

The m-th homology of the local system $\mathcal{S}$ over $Y_{\mathbf{z},m}$ gives the possible integration contours in (2.3.4).

To illustrate the relation between the homology $H_\bullet(Y_{\mathbf{z},m},\mathcal{S})$ and quantum groups consider again our example of KZ_3 and $m=1$. The local system $\mathcal{S}$ on $Y_{\mathbf{z},1} = \mathbb{C}\backslash\{z_1,z_2,z_3\}$ determined by

$$\phi_1(\mathbf{z},t) = \prod_{1\le i<j\le 3} (z_i - z_j)^{\frac{\mu_i\mu_j}{2\kappa}} \prod_{1\le j\le 3} (t-z_j)^{-\frac{\mu_j}{\kappa}} \tag{2.4.5}$$

has monodromy around the points z_j given by $q^{-\mu_j}$ where we have defined $q = e^{\frac{2\pi i}{\kappa}}$.

To compute $H_1(Y_{\mathbf{z},1},\mathcal{S})$ consider a basis (b_0,s_0) of $\Omega_0(Y_{\mathbf{z},1},\mathcal{S})$ and a basis (b_i,s_i), $i=1,2,3$, of $\Omega_1(Y_{\mathbf{z},1},\mathcal{S})$ where b_0 is the 0-cycle and b_i, $i=1,2,3$, are the 1-cycles given in Figure 2.2, and the s_i are some fixed sections over b_i.

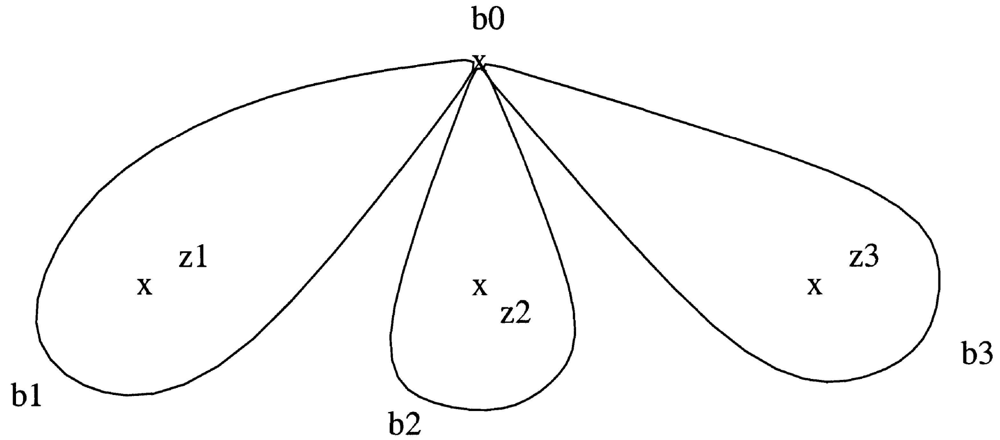

Figure 2.2. The integration cycles.

The sections s_i, $i=0,\dots,3$, can be chosen in such a way that the boundary operator

$$\partial \;:\; \Omega_1(Y_{\mathbf{z},1},\mathcal{S}) \;\to\; \Omega_0(Y_{\mathbf{z},1},\mathcal{S}) \tag{2.4.6}$$

takes the form

$$\begin{aligned}
\partial(b_1,s_1) &= (q^{\frac{\mu_1}{2}} - q^{-\frac{\mu_1}{2}})q^{\frac{(\mu_2+\mu_3)}{4}}(b_0,s_0)\,,\\
\partial(b_2,s_2) &= (q^{\frac{\mu_2}{2}} - q^{-\frac{\mu_2}{2}})q^{\frac{(-\mu_1+\mu_3)}{4}}(b_0,s_0)\,,\\
\partial(b_3,s_3) &= (q^{\frac{\mu_3}{2}} - q^{-\frac{\mu_3}{2}})q^{\frac{(-\mu_1-\mu_2)}{4}}(b_0,s_0)\,.
\end{aligned} \tag{2.4.7}$$

The homology $H_1(Y_{\mathbf{z},1}, \mathcal{S})$ follows trivially from (2.4.7).

The relation to quantum groups is as follows. Consider the quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ defined by the relations (these defining relations are related to the more conventional ones by a rescaling of f)

$$\begin{aligned}
[e,f] &= q^{\frac{h}{2}} - q^{-\frac{h}{2}}, \\
[h,e] &= 2e, \\
[h,f] &= -2f,
\end{aligned} \tag{2.4.8}$$

and a comultiplication $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ defined by

$$\begin{aligned}
\Delta(h) &= h \otimes 1 + 1 \otimes h, \\
\Delta(e) &= e \otimes q^{\frac{h}{4}} + q^{-\frac{h}{4}} \otimes e, \\
\Delta(f) &= f \otimes q^{\frac{h}{4}} + q^{-\frac{h}{4}} \otimes f.
\end{aligned} \tag{2.4.9}$$

Let $M_\mu(q)$ be the Verma module of $U_q(\mathfrak{sl}_2)$ of highest weight μ and $W(q) \equiv M_{\mu_1}(q) \otimes M_{\mu_2}(q) \otimes M_{\mu_3}(q)$ with highest weight vector $v \equiv v_1 \otimes v_2 \otimes v_3$. The $m = 1$ weight space $W(q)^{\sum \mu_i - 2}$ is spanned by $\{f_1 \cdot v, f_2 \cdot v, f_3 \cdot v\}$ and an easy calculation, using (2.4.8) and (2.4.9), shows that the vector $\sum_i I_i f_i \cdot v \in (W(q)^{\mathfrak{n}_+})^{\sum \mu_i - 2}$ provided

$$\begin{aligned}
(q^{\frac{\mu_1}{2}} - q^{-\frac{\mu_1}{2}}) q^{\frac{(\mu_2+\mu_3)}{4}} I_1 + (q^{\frac{\mu_2}{2}} - q^{-\frac{\mu_2}{2}}) q^{\frac{(-\mu_1+\mu_3)}{4}} I_2 \\
+ (q^{\frac{\mu_3}{2}} - q^{-\frac{\mu_3}{2}}) q^{\frac{(-\mu_1-\mu_2)}{4}} I_3 = 0.
\end{aligned} \tag{2.4.10}$$

In other words, the map $(b_0, s_0) \mapsto v$, $(b_i, s_i) \mapsto f_i \cdot v$ defines isomorphisms

$$\begin{aligned}
\Omega_0(Y_{\mathbf{z},1}, \mathcal{S}) &\cong W(q)^{\sum \mu_i}, \\
\Omega_1(Y_{\mathbf{z},1}, \mathcal{S}) &\cong W(q)^{\sum \mu_i - 2}, \\
H_1(Y_{\mathbf{z},1}, \mathcal{S}) &\cong (W(q)^{\mathfrak{n}_+})^{\sum \mu_i - 2}.
\end{aligned} \tag{2.4.11}$$

3 Operator approach

In this final section we discuss the operator approach to the Knizhnik–Zamolodchikov equations. This approach was developed by many physicists (see, in particular, [KZ84]). A mathematically rigorous treatment, in the case of $\mathfrak{g} = \mathfrak{sl}_2$, was first given in [TK88] (see also [FR92]). We will show that compositions of intertwiners between certain modules of the affine Lie algebra $\widehat{\mathfrak{g}}$ satisfy the KZ equations and will indicate how the free field realization of $\widehat{\mathfrak{g}}$ (see [Wa86] for $\mathfrak{g} = \mathfrak{sl}_2$ and, e.g., [FF89, FF90, BMP90a, BMP90b], in the general case) can be used to reproduce the explicit integral expressions of Section 2. Again, we refer to the lecture notes [EFK98] for more details.

3.1 Affine Lie algebras

Let $\mathfrak{g}$ be a simple, finite dimensional, Lie algebra. Fix a symmetric, invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$ such that the length squared of a long root equals 2, let $\{x_a\}$ be an orthonormal basis of $\mathfrak{g}$ and, finally, let $\mathrm{h}^\vee$ denote the dual Coxeter number of $\mathfrak{g}$.

The affine Lie algebra $\widehat{\mathfrak{g}}$ is the (unique) central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, namely

$$\widehat{\mathfrak{g}} \cong (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k \,. \tag{3.1.1}$$

Let us denote the generator $x \otimes t^n$, $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, of $\widehat{\mathfrak{g}}$ by $x[n]$. The defining commutation relations of $\widehat{\mathfrak{g}}$ are

$$[x[m], y[n]] = [x, y][m+n] + km(x, y)\delta_{m+n,0}\,, \qquad x, y \in \mathfrak{g}\,, m, n \in \mathbb{Z}\,. \tag{3.1.2}$$

We will denote by

$$\widetilde{\mathfrak{g}} \cong \widehat{\mathfrak{g}} \oplus \mathbb{C}d\,, \tag{3.1.3}$$

the extension of $\widehat{\mathfrak{g}}$ by the derivation d

$$[d, x[n]] = nx[n]\,. \tag{3.1.4}$$

We consider two kinds of representations of $\widehat{\mathfrak{g}}$ associated to a representation $\rho : \mathfrak{g} \to \operatorname{End} V$. The evaluation representation $\widehat{\rho} : \widehat{\mathfrak{g}} \to \operatorname{End} V(z)$ is defined by

$$\widehat{\rho}(x[n]) = \rho(x)z^n\,, \qquad \widehat{\rho}(k) = 0\,. \tag{3.1.5}$$

Furthermore, each $\rho : \mathfrak{g} \to \operatorname{End} V$ can be viewed as a representation of $\widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[t]$ by taking the trivial action of $\mathfrak{g} \otimes t\mathbb{C}[t]$. We then have the induced representation

$$V_k = \operatorname{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}} V\,, \tag{3.1.6}$$

for any $k \in \mathbb{C}$ (which is identified with the action of the central element $k \in \widehat{\mathfrak{g}}$).

If V is an irreducible $\mathfrak{g}$-module then, obviously, the evaluation representation $V(z)$ is an irreducible $\widehat{\mathfrak{g}}$-module, while the induced module, V_k, will be irreducible for generic values of k.

3.2 The Virasoro algebra

The Virasoro algebra, Vir, is the algebra with generators $L[m]$, $m \in \mathbb{Z}$, and central element c with defining commutation relations

$$[L[m], L[n]] = (m-n)L[m+n] + \frac{c}{12}m(m^2-1)\delta_{m+n,0}\,, \qquad m, n \in \mathbb{Z}\,. \tag{3.2.1}$$

The Virasoro algebra plays an important role in the theory of affine Lie algebras due to the following theorem.

Theorem 3.1. *For any $\mathfrak{g}$-module V the induced module $V_k = \mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}} V$ extends to a module of the semi-direct product* Vir $\ltimes \widehat{\mathfrak{g}}$ *by the so-called Sugawara construction*

$$L[m] = \frac{1}{2(k+\mathrm{h}^\vee)} \sum_{n\in\mathbb{Z}} \sum_a :x_a[n]x_a[m-n]:\,, \qquad c = \frac{k \dim \mathfrak{g}}{k+\mathrm{h}^\vee}\,, \tag{3.2.2}$$

where $:\ldots:$ *denotes the normal ordering defined by*

$$:x[m]y[n]: = \begin{cases} x[m]y[n]\,, & m \leq -1\,, \\ y[n]x[m]\,, & m \geq 0\,. \end{cases} \tag{3.2.3}$$

The semi-direct product structure is given by

$$[L[m], x[n]] = -nx[m+n]\,. \tag{3.2.4}$$

Note, in particular, that

$$[L[0], x[n]] = -nx[n]\,, \tag{3.2.5}$$

so that we can identify $d = -L[0]$.

Note, also, that for irreducible V and v in the highest weight component of $\mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}} V$, we have

$$L[0]\cdot v = \frac{1}{2(k+\mathrm{h}^\vee)} \sum_a x_a[0]x_a[0]\cdot v = \frac{C_V}{2(k+\mathrm{h}^\vee)}\, v\,, \tag{3.2.6}$$

where C_V is the eigenvalue of the quadratic Casimir operator on V. For future purposes we denote

$$\Delta_V = \frac{C_V}{2(k+\mathrm{h}^\vee)}\,. \tag{3.2.7}$$

Note that

$$V_k \cong \bigoplus_{n\geq 0} V_k[n]\,, \tag{3.2.8}$$

where

$$V_k[n] = \{v \in V_k \mid L[0]\cdot v = (n+\Delta_V)v\}\,. \tag{3.2.9}$$

Obviously, $V_k[0] \cong V$.

For both notational and calculational purposes it is useful to introduce formal power series with coefficients in $\widehat{\mathfrak{g}}$ and Vir, respectively,

$$
\begin{aligned}
x(z) &= \sum_{n\in\mathbb{Z}} x[n]z^{-n-1}\,, \\
L(z) &= \sum_{n\in\mathbb{Z}} L[n]z^{-n-2}\,,
\end{aligned}
\tag{3.2.10}
$$

in terms of which, e.g., (3.2.2) reads

$$
L(z) = \frac{1}{2(k+\mathrm{h}^\vee)} \sum_a \,:x_a(z)x_a(z):\, . \tag{3.2.11}
$$

3.3 Intertwiners and the KZ equations

Let L_λ be a finite dimensional irreducible highest weight module of $\mathfrak{g}$ with highest weight λ, and let

$$
L_{\lambda,k} = \mathrm{Ind}_{\mathfrak{g}_+}^{\widehat{\mathfrak{g}}}\, L_\lambda\,. \tag{3.3.1}
$$

The $\widehat{\mathfrak{g}}$-module $L_{\lambda,k}$ is irreducible for generic k. Let $\rho : \mathfrak{g} \to \mathrm{End}\, V$ be any representation of $\mathfrak{g}$.

We consider $\widehat{\mathfrak{g}}$ intertwiners

$$
\Psi \;:\; L_{\lambda,k} \;\rightarrow\; L_{\lambda',k} \otimes V(z)\,, \tag{3.3.2}
$$

i.e., Ψ satisfies

$$
\Psi\, x[n] = (x[n] \otimes 1 + z^n\, 1 \otimes \rho(x))\; \Psi\,. \tag{3.3.3}
$$

We can also think of these intertwiners as maps

$$
\Psi(z) = \sum_{n\in\mathbb{Z}} \Psi[n]z^{-n} \;:\; L_{\lambda,k} \;\rightarrow\; L_{\lambda',k} \otimes V\,. \tag{3.3.4}
$$

Theorem 3.2. *Let $g : L_\lambda \to L_{\lambda'} \otimes V$ be a $\mathfrak{g}$-homomorphism. For generic k there exists a unique $\widehat{\mathfrak{g}}$ intertwiner $\Psi^g : L_{\lambda,k} \to L_{\lambda',k} \otimes V(z)$ such that on $w \in L_{\lambda,k}[0] \cong L_\lambda$ the degree zero component of $\Psi^g \cdot w$ is equal to $g \cdot w$.*

In fact, the slightly modified $\widehat{\mathfrak{g}}$-intertwiner

$$
\Phi(z) = z^{-\Delta}\Psi(z)\,, \tag{3.3.5}
$$

where

$$
\Delta \;\equiv\; \Delta_{L_\lambda} - \Delta_{L'_\lambda} + \Delta_V\,, \tag{3.3.6}
$$

will have nice intertwining properties with $L[0]$ as well (cf. (3.2.6) and (3.2.7)).

Lemma 3.3. *The intertwiners (3.3.5) satisfy the following differential equation*

$$(k+\mathrm{h}^\vee)\frac{d\Phi}{dz} = \sum_a :(x_a(z)\otimes\rho(x_a))\Phi(z):\ . \tag{3.3.7}$$

Proof. The left hand side equals $[L[0],\Phi(z)]$, up to a factor of z. Using the Sugawara expression (3.2.2) for $L[0]$ and the intertwining property (3.3.3) of $\Phi(z)$ we immediately arrive at (3.3.7). □

Now, suppose we have a family of representations $\rho_i : \mathfrak{g} \to \mathrm{End}(V_i)$, $i=1,\ldots,n$, dominant integral weights λ_i, $i=0,\ldots,n$, and intertwiners

$$\Phi_i(z_i) \ :\ L_{\lambda_i,k} \ \to\ L_{\lambda_{i-1},k}\otimes V_i\,. \tag{3.3.8}$$

Consider the composition

$$\Phi_1(z_1)\ldots\Phi_n(z_n)\ :\ L_{\lambda_n,k}\ \to\ L_{\lambda_0,k}\otimes(V_1\otimes\cdots\otimes V_n)\,. \tag{3.3.9}$$

Choose a $u_n\in L_{\lambda_n,k}[0]\cong L_{\lambda_n}$ and $u_0\in(L_{\lambda_0,k}[0])^*\cong(L_{\lambda_0})^*$. Then define $\phi: Y_n\to W\equiv V_1\otimes\cdots\otimes V_n$ by

$$\phi(z_1,\ldots,z_n) = \langle u_0, \Phi_1(z_1)\ldots\Phi_n(z_n)u_n\rangle\,. \tag{3.3.10}$$

Remark. Of course, a priori, $\phi(z_1,\ldots,z_n)$ only makes sense as a formal power series in $z_1,\ldots,z_n$. One can prove, however, that the right hand side of (3.3.10) converges for $|z_1|>\ldots>|z_n|$ and can be analytically continued to a (multi-valued) function on Y_n.

Theorem 3.4. *The map $\phi: Y_n\to W$ satisfies the equation*

$$(k+\mathrm{h}^\vee)\frac{\partial}{\partial z_i}\phi = \left(\sum_{\substack{1\leq j\leq n\\ j\neq i}}\frac{\Omega_{ij}}{z_i-z_j}+\frac{\Omega_{i\ n+1}}{z_i}\right)\phi\,, \tag{3.3.11}$$

where Ω is defined in (1.1.6).

Proof. By using equation (3.3.7) and bringing $x_a(z)$ to the right until it hits u_n, using the intertwining property of $\Phi_j(z_j)$. □

Note that (3.3.11) is precisely the KZ_{n+1} equation with $z_{n+1}=0$ and $\kappa=k+\mathrm{h}^\vee$ which, due to the translation invariance, is equivalent to KZ_{n+1} for arbitrary z_{n+1} (cf. (2.1.17)).

3.4 Free field realizations

Solutions of the KZ equations can be obtained by explicit computation of (3.3.10). To this end one first realizes both the affine algebra $\widehat{\mathfrak{g}}$ and the intertwiners (3.3.8) in terms of free fields (i.e., in terms of generators satisfying a Heisenberg algebra), after which the computation of (3.3.10) is essentially just a repeated application of Wick's theorem.

Recall that free field realizations in the finite dimensional context, i.e., realizations of the algebra by means of differential operators on polynomial rings, arise from the (right) action of $\mathfrak{g}$ on sections of a line bundle over the flag manifold $B_-\backslash G$ determined by a character $\chi_\lambda : B_- \to \mathbb{C}$ where B_- is a Borel subgroup of G. For instance, this way one obtains a realization of $\mathfrak{sl}_2$ on $\mathbb{C}[\![z]\!]$ given by

$$\begin{aligned} e &= \frac{d}{dz}\,, \\ h &= -2z\frac{d}{dz} + \mu\,, \\ f &= -z^2\frac{d}{dz} + \mu z\,. \end{aligned} \tag{3.4.1}$$

Obviously, this representation is ismorphic to M_μ under

$$z^k \;\mapsto\; f^k \cdot v\,. \tag{3.4.2}$$

Similarly, one can find free field realizations of $\widehat{\mathfrak{g}}$ by affinizing the construction above, i.e., by considering the action of $\widehat{\mathfrak{g}}$ on sections of a line bundle over a semi-infinite flag manifold. For $\widehat{\mathfrak{sl}_2}$ one finds

$$\begin{aligned} e(z) &= \beta(z)\,, \\ h(z) &= 2 :\gamma(z)\beta(z): +\alpha_+ i\partial\phi(z)\,, \\ f(z) &= - :\gamma(z)\gamma(z)\beta(z): -\alpha_+ i\gamma(z)\partial\phi(z) - k\partial\gamma(z)\,, \end{aligned} \tag{3.4.3}$$

where $\alpha_+ = \sqrt{2(k+2)}$, and

$$\begin{aligned} \beta(z) &= \sum_{n\in\mathbb{Z}} \beta[n]z^{-n-1}\,, \\ \gamma(z) &= \sum_{n\in\mathbb{Z}} \gamma[n]z^{-n}\,, \\ i\partial\phi(z) &= \sum_{n\in\mathbb{Z}} a[n]z^{-n-1}\,, \end{aligned} \tag{3.4.4}$$

with commutators

$$\begin{aligned} [\gamma[m], \beta[n]] &= \delta_{m+n,0}\,, \\ [a[m], a[n]] &= m\delta_{m+n,0}\,. \end{aligned} \tag{3.4.5}$$

The intertwiners arise similarly from the left action of $\mathfrak{b}_-$. We refrain from giving the explicit expressions and refer to, e.g., [BMP90a, BMP90b] for more details as well as for the quantum group structure underlying the set of intertwiners. As remarked above, having obtained a free field realization of the intertwiners, one can explicitly compute the compositions (3.3.10) and ultimately reproduce the integral expressions (2.3.4).

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