This means that the two complex numbers π_1 and π_2 are linearly independent over $\mathbb{R}$ so that

$$\tilde{E} = \frac{\mathbb{C}}{Z\pi_1 + Z\pi_2}$$

is a compact complex torus. The relation (2.21) is called the *second Riemann relation.*

Thus (2.18) gives a well-defined holomorphic immersion

$$E \to \tilde{E}. \tag{2.22}$$

Since E is compact, this mapping must be a finite covering space mapping. In fact, it is immediate to see that the mapping (2.22) induces an isomorphism

$$H_1(E;\, \mathbb{Z}) \cong H_1(\tilde{E};\, \mathbb{Z})$$

or, what is the same, an isomorphism of fundamental groups. So by covering-space theory, the map (2.22) is itself an isomorphism

$$E \xrightarrow{\cong} \frac{\mathbb{C}}{\mathbb{Z}\pi_1 + \mathbb{Z}\pi_2}. \tag{2.23}$$

We shall later encounter the inverse isomorphism to (2.23). In particular, the composition of this inverse with the coordinate function x on E [see (2.18)] has special significance in the development of the subject, since it has a particularly nice explicit expression. It is called the *Weierstrass p-function.*

2.10 The Picard–Fuchs Equation

The same differential

$$\omega = \frac{dx}{y} = \frac{dx}{[x(x-1)(x-\lambda)]^{1/2}}$$

which gave the mapping (2.18) has another very surprising use. Namely, the numbers

$$\begin{aligned} \pi_1(\lambda) &= 2\int_0^1 \omega, \\ \pi_2(\lambda) &= 2\int_1^\lambda \omega, \end{aligned} \tag{2.24}$$

called the *periods* of E, depend on λ. It is not hard to see that they are, in fact, holomorphic functions of λ. For instance $\pi_2(\lambda)$ can also be obtained by integrating ω around the path γ, as shown in Figure 2.15, so that as λ moves slightly, the path of integration does not change and we can differentiate $\pi_2(\lambda)$ by differentiation under the integral sign. Let

$$\pi(\lambda) = \pi_1(\lambda) + \pi_2(\lambda) = 2\int_0^\lambda \omega.$$

$\pi(\lambda)$ can be obtained by integrating ω over the path (see Figure 2.16). As λ approaches 0, this integral becomes

$$\begin{aligned}\int_\gamma \frac{dx}{x(x-1)^{1/2}} &= \operatorname{residue}_0 \frac{1}{x(x-1)^{1/2}} \\ &= 2\pi i(-i) = 2\pi.\end{aligned}$$

Thus $\pi(\lambda)/2\pi$ has a power-series expansion around $\lambda = 0$. We wish to compute this expansion explicitly. To do this, we need to study the differential equation satisfied by the $\pi_i(\lambda)$, the so-called *Picard–Fuchs equation.*

And so we must compute the derivatives of $\pi_i(\lambda)$ with respect to λ; that is, we must differentiate under the integral sign; we must compute

$$\frac{\partial}{\partial\lambda}[x^{-1/2}(x-1)^{-1/2}(x-\lambda)^{-1/2}] = \tfrac{1}{2}x^{-1/2}(x-1)^{-1/2}(x-\lambda)^{-3/2},$$

$$\frac{\partial^2}{\partial\lambda^2}[x^{-1/2}(x-1)^{-1/2}(x-\lambda)^{-1/2}] = \tfrac{3}{4}x^{-1/2}(x-1)^{-1/2}(x-\lambda)^{-5/2}.$$

Now those readers familiar with the theory of deRham cohomology will recognize that there must be a relation between the three differentials

$$\omega, \frac{\partial\omega}{\partial\lambda}, \frac{\partial^2\omega}{\partial\lambda^2},$$

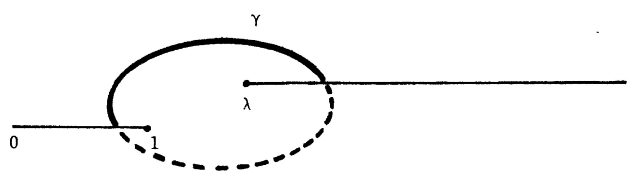

Figure 2.15. One of many equivalent paths of integration for $\pi_2(\lambda)$.

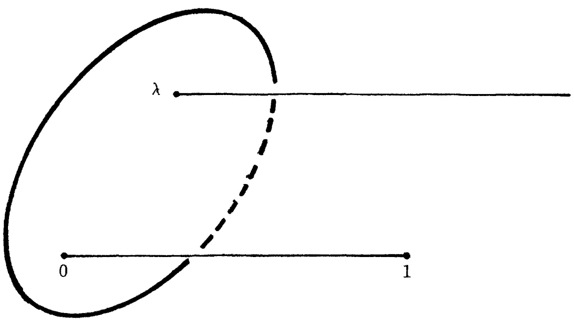

Figure 2.16. One of many equivalent paths of integration for $\pi(\lambda)$.

that is, some linear combination of these (with coefficients functions of λ) must be an exact differential. So the corresponding linear combination of

$$\pi_i, \ \frac{d\pi_i}{d\lambda}, \ \frac{d^2\pi_i}{d\lambda^2}$$

must be zero.

But let's just compute. If we fix λ and differentiate with respect to the variable x; we get

$$\begin{aligned} d\frac{x^{1/2}(x-1)^{1/2}(x-\lambda)^{1/2}}{(x-\lambda)^2} &= \left[\frac{1}{2}x^{-1/2}(x-1)^{1/2}(x-\lambda)^{-3/2}\right. \\ &\quad +\frac{1}{2}x^{1/2}(x-1)^{-1/2}(x-\lambda)^{-3/2} \\ &\quad \left.-\frac{3}{2}x^{1/2}(x-1)^{1/2}(x-\lambda)^{-5/2}\right] dx \\ &= (x-1)\frac{\partial\omega}{\partial\lambda} + x\frac{\partial\omega}{\partial\lambda} - 2x(x-1)\frac{\partial^2\omega}{\partial\lambda^2} \\ &= [(x-\lambda)+(\lambda-1)]\frac{\partial\omega}{\partial\lambda} + [(x-\lambda)+\lambda]\frac{\partial\omega}{\partial\lambda} \\ &\quad -2[(x-\lambda)+\lambda][(x-\lambda)+(\lambda-1)]\frac{\partial^2\omega}{\partial\lambda^2} \end{aligned}$$

$$= \frac{1}{2}\omega + (\lambda - 1)\frac{\partial\omega}{\partial\lambda} + \frac{1}{2}\omega + \lambda\frac{\partial\omega}{\partial\lambda} - 2[(x-\lambda)+\lambda]$$

$$\times \left(\frac{3}{2}\frac{\partial\omega}{\partial\lambda} + (\lambda - 1)\frac{\partial^2\omega}{\partial\lambda^2}\right) = \omega + (2\lambda - 1)\frac{\partial\omega}{\partial\lambda} - \frac{3}{2}\omega$$

$$-3(\lambda - 1)\frac{\partial\omega}{\partial\lambda} - 3\lambda\frac{\partial\omega}{\partial\lambda} - 2\lambda(\lambda - 1)\frac{\partial^2\omega}{\partial\lambda^2}$$

$$= -\frac{1}{2}\omega - (4\lambda - 2)\frac{\partial\omega}{\partial\lambda} - 2\lambda(\lambda - 1)\frac{\partial^2\omega}{\partial\lambda^2}.$$

Integrating both sides around our cycles, we get the Picard–Fuchs equation:

$$0 = \frac{1}{4}\pi_i + (2\lambda - 1)\frac{d\pi_i}{d\lambda} + \lambda(\lambda - 1)\frac{d^2\pi_i}{d\lambda^2}. \tag{2.25}$$

Now we will appeal to the elementary theory of ordinary differential equations with regular singular points.† The indicial polynomial of (2.25) is

$$q(r) = r(r-1) + r = r^2.$$

This means that the vector space of solutions of (2.25) near $\lambda = 0$ is generated by

$\sigma_1(\lambda)$ holomorphic and nonvanishing at 0,

$(\lambda\sigma_2(\lambda) + (\log\lambda)\sigma_1(\lambda))$ where σ_2 is holomorphic and nonvanishing at 0.

Thus it must be that if we normalize so that

$$\sigma_1(0) = 1,$$

then

$$\sigma_1(\lambda) = \frac{\pi(\lambda)}{2\pi}.$$

† See E. Coddington, *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, N.J.: Prentice-Hall, 1961), Chapter Four, for more details.

Let's solve explicitly for the power-series expansion

$$\sum_{n=0}^{\infty} a_n \lambda^n$$

of σ_1. First,

$$\sigma_1'(\lambda) = \sum_{n \geq 0} (n+1) a_{n+1} \lambda^n,$$

$$\sigma_1''(\lambda) = \sum_{n \geq 0} (n+2)(n+1) a_{n+2} \lambda^n.$$

So from (2.25) we obtain

$$\sum_{n \geq 0} [\lambda(\lambda - 1)(n+2)(n+1)a_{n+2} + (2\lambda - 1)(n+1)a_{n+1} + \tfrac{1}{4}a_n]\lambda^n = 0.$$

Rewriting we get

$$\sum_{n \geq 0} [(n + \tfrac{1}{2})^2 a_n - (n+1)^2 a_{n+1}]\lambda^n = 0.$$

Since $a_0 = 1$ and we now know that

$$a_{n+1} = \left[\frac{n + (1/2)}{n+1}\right]^2 a_n,$$

we obtain

$$a_n = \binom{-1/2}{n}^2. \tag{2.26}$$

Notice that we have obtained along the way that near $\lambda = 0$

$$\pi_1(\lambda) \sim \log \lambda.$$

To sum up, we have a very specific power series

$$\sum_{n=0}^{\infty} \binom{-1/2}{n}^2 \lambda^n,$$

which gives us "the" solution to the Picard–Fuchs equation that stays bounded near the singular point $\lambda = 0$. It is no accident that the coefficients in this power series are rational, as we shall soon see.

2.11 Rational Points on Cubics Over $\mathbb{F}_p$

So far everything we have done with the differential ω and the function $\pi(\lambda)$ lies in the realm of analysis or geometry. But surprisingly enough these computations have number-theoretic applications. For this we must consider cubic curves over the finite fields

$$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z},$$

where p is an odd prime number. Given $\lambda \in \mathbb{Z}$, we reduce λ mod p and let C_λ denote the solution set of

$$y^2 = x(x-1)(x-\lambda)$$

in $\mathbb{F}_p \times \mathbb{F}_p$. The question we will ask is,

What is the cardinality of C_λ?

Or, equivalently,

For how many $x \in \mathbb{F}_p$ is it true that $x(x-1)(x-\lambda) \in \mathbb{F}_p^2$?

Now since the multiplicative group $(\mathbb{F}_p - \{0\})$ is cyclic, $x(x-1) \times (x-\lambda) \in \mathbb{F}_p^2$ if and only if

$$[x(x-1)(x-\lambda)]^{(p-1)/2} \equiv 1 \quad \text{or} \quad x = 0, 1, \lambda.$$

In all other cases $[x(x-1)(x-\lambda)]^{(p-1)/2} \equiv -1$. We can summarize all this in the very neat formula

$$(\text{number of points in } C_\lambda) \equiv \sum_{x \in \mathbb{F}_p} \{1 + [x(x-1)(x-\lambda)]^{(p-1)/2}\} \tag{2.27}$$

modulo p. We next wish to simplify the right-hand side of (2.27). For this we use the (character) formulas

$$\begin{aligned} \sum_{x \in \mathbb{F}_p} x^k &\equiv 0 \qquad \text{if } (p-1) \nmid k, \\ \sum_{x \in \mathbb{F}_p} x^k &\equiv -1 \qquad \text{if } (p-1) \mid k. \end{aligned} \tag{2.28}$$

These are easily proved by noting that

$$\sum x^k = y^k \sum x^k$$

for any $y \neq 0$. Now suppose we write

$$[x(x-1)(x-\lambda)]^{(p-1)/2} \tag{2.29}$$

as a polynomial in x. Then by (2.28) the only term in (2.29) which will contribute to

$$\sum_{x \in \mathbb{F}_p} [x(x-1)(x-\lambda)]^{(p-1)/2}$$

is the term involving x^{p-1}. The coefficient of this term is the coefficient of $x^{(p-1)/2}$ in the polynomial expansion of

$$[(x-1)(x-\lambda)]^{(p-1)/2}.$$

But we can multiply this last expression out explicitly:

$$\left[\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}(-1)^k x^{(p-1)/2-k}\right]\left[\sum_{l=0}^{(p-1)/2}\binom{(p-1)/2}{l}(-1)^l\lambda^l x^{(p-1)/2-l}\right]$$

has as coefficient of $x^{(p-1)/2}$ the sum

$$\begin{aligned}(-1)^{(p-1)/2}\sum_{k+l=(p-1)/2}\binom{(p-1)/2}{k}\binom{(p-1)/2}{l}\lambda^l \\ &= (-1)^{(p-1)/2}\sum_{r=0}^{(p-1)/2}\binom{(p-1)/2}{r}^2\lambda^r \\ &\equiv (-1)^{(p-1)/2}\sum_{r=0}^{(p-1)/2}\binom{-1/2}{r}^2\lambda^r\end{aligned}$$

modulo p. This last congruence needs a little explanation. The integer

$$\frac{1}{r!}\frac{p-1}{2}\cdot\frac{p-3}{2}\cdot\dots\cdot\frac{p-(2r-1)}{2}$$

represents the same element of $\mathbb{F}_p$ as does

$$a(p-i)(p-3)\cdot\dots\cdot[p-(2r-1)], \tag{2.30}$$

where a is any integer such that

$$ar!\,2^r \equiv 1 \mod p.$$

But then

$$(2.30) \equiv a\cdot(-1)(-3)\cdot\dots\cdot[-(2r-1)] \bmod p,$$

which gives the desired congruence.

Now if $r \geq (p+1)/2$, then

$$\binom{-1/2}{r} = \frac{(-1)(-3)\cdots(-p)\cdots}{r!\,2^r} \equiv 0,$$

so the formula for the cardinality modulo p of C_λ is

$$(-1)(-1)^{(p-1)/2}\sum_{r=0}^{\infty}\binom{-1/2}{r}^2\lambda^r.$$

This is the *same* formula as the one for the period function $\pi(\lambda)$, the solution to the Picard–Fuchs equation which is holomorphic at 0! Of course, this fact is not accidental—the reason for it is a rather deep one, discovered only in recent years by Y. Manin. To close this chapter, we will try to give some idea of Manin's result.

2.12 Manin's Result: The Unity of Mathematics

The fundamental ingredient in this discussion will be the algebra-geometric version of the Lefschetz fixed-point theorem from topology. The topological version says that if f is a differentiable mapping

$$f: M \to M,$$

where M is a compact differentiable manifold such that the graph of f meets the diagonal transversely in $M \times M$, then the Lefschetz number

$$L(f) = \sum_{p \in M} \sigma_p(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{trace}[f^*: H^n(M; \mathbb{C}) \to H^n(M; \mathbb{C})],$$

where

$$\sigma_p = \begin{cases} 0 & \text{if } f(p) \neq p, \\ \begin{Bmatrix} +1 \\ -1 \end{Bmatrix} & \text{if (graph } f\text{) meets (diagonal) with} \begin{Bmatrix}\text{positive} \\ \text{negative}\end{Bmatrix} \text{orientation.}\dagger \end{cases}$$

Now transversality at p means, in local coordinates, that the mapping

$$(\text{identity} - f)$$

has maximal rank at p. So

$$\sigma(p) = \operatorname{sign} \det[I - J_p(f)],$$

where $J_p(f)$ is the Jacobian matrix of f at p. Now if A is a complex-valued matrix in triangular form

$$A = \begin{bmatrix} a_1 & & \\ & \ddots & * \\ 0 & & a_n \end{bmatrix},$$

then

$$\det[I - A] = \sum_{r=0}^{n} (-1)^r \sum_{j_1 < \cdots < j_r} a_{j_1} \cdot \cdots \cdot a_{j_r} = \sum_{r=0}^{n} (-1)^r \operatorname{trace}(\Lambda^r A),$$

where $\Lambda^r A$ denotes the linear endomorphism which A induces on the rth

† See, for example, Section 7, Chapter Four, of E. Spanier's *Algebraic Topology* (New York: McGraw-Hill, 1966).

exterior power of $\mathbb{C}^n$. Thus we can rewrite the Lefschetz fixed-point formula as

$$L(f) = \sum_{r,p}(-1)^r \frac{\operatorname{trace} \Lambda^r J_p(f)}{|\det[1 - J_p(f)]|} = \sum_{n=0}^{\infty}(-1)^n \operatorname{trace}(f^*|_{H^n(M)}). \tag{2.31}$$

The main point to notice is that we have two formulas for $L(f)$, one given by *local* invariants and the other by *global* ones.

Now if M is a compact complex manifold, let Γ be the global sections functor, and let

$$\Gamma\mathscr{A}^{0,0} \xrightarrow{\bar\partial} \Gamma\mathscr{A}^{0,1} \xrightarrow{\bar\partial} \Gamma\mathscr{A}^{0,2} \xrightarrow{\bar\partial} \cdots \tag{2.32}$$

be the Dolbeault complex on M, that is

$$\mathscr{A}^{0,q} = \text{sheaf of } C^\infty \ (0, q)\text{-forms on } M.$$

(For more details on this, see Gunning and Rossi [2].) The cohomology groups of this complex are denoted

$$H^q(M; \mathcal{O}) \quad \text{or} \quad H^{0,q}(M).$$

If M is a Kähler manifold, then $H^q(M; \mathcal{O})$ is a direct summand of the complex-valued deRham cohomology. Suppose now our mapping

$$F: M \to M$$

is holomorphic. Then f induces a morphism on the sequence (2.32), and the formula (2.31) continues to hold when the deRham complex is replaced by the complex (2.32). In that case formula (2.31) reads

$$\sum_{r,p}(-1)^r \frac{\operatorname{trace} \Lambda^r J''_p(f)}{|\det[1 - J_p(f)]|} = \sum_{n=0}^{\infty}(-1)^n \operatorname{trace}(f^*|_{H^n(M,\mathcal{O})}),$$

where $J''_p(f)$ is the restriction of the cotangent space mapping $J_p(f)$ to the subspace of type (0, 1). But as before

$$\sum_r (-1)^r \operatorname{trace} \Lambda^r J''_p(f) = \det[1 - J''_p(f)],$$

and if $J'_p(f)$ denotes the (1, 0)-part of $J_p(f)$, then

$$\det[1 - J_p(f)] = \det[1 - J'_p(f)] \cdot \det[1 - J''_p(f)].$$

Thus our formula becomes

$$\sum (-1)^n \operatorname{trace}(f^*|_{H^n(M,\mathcal{O})}) = \sum_{p \text{ fixed}} \frac{1}{\det[1 - J'_p(f)]}. \tag{2.33}$$

Now the marvelous fact is that formula (2.33) continues to hold in a purely algebraic context. The reader who is unfamiliar with sheaf cohomology in algebraic geometry over an arbitrary algebraically closed field k is urged to read on in any case. The results used will be the formal analogues of the corresponding theorems over $\mathbb{C}$, and that formal analogy will probably carry the reader through. In any case, the beauty of the result should motivate most interested readers to further study of these topics.

Suppose, for instance, that k is the algebraic closure of the field $\mathbb{F}_p$ of p elements, $\lambda \in (\mathbb{F}_p - \{0, 1\})$, and M is the solution set to

$$y^2 z = x(x - z)(x - \lambda z)$$

in kP_2 (one-dimensional subspaces of k^3). In other words, M is the solution set to

$$y^2 = x(x - 1)(x - \lambda)$$

in k^2 with the point at infinity thrown in. Let f be the Frobenius mapping

$$\begin{aligned} f\colon \quad M &\longrightarrow M \\ (x, y, z) &\longrightarrow (x^p, y^p, z^p). \end{aligned}$$

Now $d(x^p)/dx = px^{p-1} = 0$, and

$$H^n(M; \mathcal{O}) = 0, \qquad \text{if } n > 1 = \dim_k M$$

for the *sheaf* $\mathcal{O}$ of regular algebraic functions on M. Thus the formula (2.33) reads

$$1 - \operatorname{trace}(f^*|_{H^1(M;\mathcal{O})}) = \text{number of fixed points of } f. \tag{2.34}$$

But the fixed points of f are nothing more than the number of points of M which are represented by triples (x, y, z) of elements of $\mathbb{F}_p$, since

$$x = x^p \qquad \text{if and only if } x \in \mathbb{F}_p.$$

Since the point at infinity is one fixed point of f, the formula (2.34) becomes

$$-\operatorname{trace} f^*|_{H^1(M;\mathcal{O})} = \text{number of points in } C_\lambda.$$

So we will explain the connection between the formula for the cardinality of C_λ and the period function $\pi(\lambda)$ by explicitly computing the trace of f^* on $H^1(M; \mathcal{O})$. We will have to use the fact that if $q, q' \in M$, then

$$H^1(M; \mathcal{O}) \cong$$

$$\frac{(\text{algebraic functions with poles at } q \text{ and } q')}{(\text{functions with poles at } q) + (\text{functions with poles at } q')}. \tag{2.35}$$

This comes out of the exact sheaf sequence

$$0 \to \mathcal{O} \to \mathcal{O}(\infty \cdot q) + \mathcal{O}(\infty \cdot q') \to \mathcal{O}(\infty \cdot q + \infty \cdot q') \to 0,$$

where

$$\mathcal{O}(n \cdot q) = \text{sheaf of algebraic functions on } M \text{ with } q \text{ as the only pole and no worse than order } (-n) \text{ at } q,$$

$$\mathcal{O}(\infty \cdot q) = \lim_{n \to \infty} \mathcal{O}(n \cdot q).$$

Also, using formal power series, we can define differentials on M, and as in the complex case, the k-vector space of everywhere-regular differentials is one dimensional with generator written locally at q

$$\omega = dx + \sum_{r \geq 1} a_r(\lambda)[x - x(q)]^r \, dx. \tag{2.36}$$

Also just as before,

$$1 + \sum_{r \geq 1} a_r(\lambda)[x - x(q)]^r$$

satisfies the Picard–Fuchs equation, that is,

$$\begin{aligned} &\left(\lambda(\lambda - 1)\frac{\partial^2}{\partial \lambda^2} + (2\lambda - 1)\frac{\partial}{\partial \lambda} + \frac{1}{4}\right)\left(1 + \sum_{r \geq 1} a_r(\lambda)[x - x(q)]^r\right) \\ &\qquad = \frac{d}{dx}\left(\text{series expansion for } \frac{x^{1/2}(x-1)^{1/2}(x-\lambda)^{1/2}}{(x-\lambda)^2}\right). \end{aligned} \tag{2.37}$$

Furthermore, referring again to (2.35), we see that the pairing

$$\begin{array}{ccccc} H^1(M; \mathcal{O}) & \times & \left\{\begin{matrix}\text{regular} \\ \text{differentials}\end{matrix}\right\} & \longrightarrow & k \\ h & \times & \omega & \longmapsto & \operatorname{res}_q(h\omega) \end{array} \tag{2.38}$$

is nondegenerate, so, in particular,

$$\dim_k H^1(M; \mathcal{O}) = 1.$$

This is the algebraic version of the *Serre duality theorem*. Using these facts we are in a position to compute

$$\operatorname{trace}[f^*: H^1(M; \mathcal{O}) \to H^1(M; \mathcal{O})].$$

Namely, the algebraic *Riemann–Roch theorem* (see Chapter Three) will always assure us of the existence of an algebraic function h whose only poles are q and q' and which has a simple pole at q. We write

$$h = \frac{1}{x - x(q)} + \sum_{l \geq 0} b_l[x - x(q)]^l.$$

Then the map f sends $h(x)$ to

$$h(x^p) = \frac{1}{[x - x(q)]^p} + \sum_{l \geq 0} b_l [x - x(q)]^{pl}$$

as long as q is chosen to lie in C_λ. So the trace of f^* must be the coefficient of

$$\frac{1}{[x - x(q)]}$$

in the series expansion of $h(x^p)\omega$. That is, referring to (2.36), we have

$$(\operatorname{trace} f^*) = a_{p-1}(\lambda). \tag{2.39}$$

But now by (2.37)

$$\left(\lambda(\lambda - 1)\frac{\partial^2}{\partial\lambda^2} + (2\lambda - 1)\frac{\partial}{\partial\lambda} + \frac{1}{4}\right)a_{p-1}(\lambda)[x - x(q)]^{p-1}$$
$$= \frac{d}{dx}\{c(\lambda)[x - x(q)]^p\} = 0,$$

so that $a_{p-1}(\lambda)$ *satisfies the Picard–Fuchs equation*! Also $a_{p-1}(\lambda)$ is univalued around $\lambda = 0$, so to compute its series expansion we make the formal computations leading to (2.26), and we obtain

$$a_{p-1}(\lambda) = c\sum_{r=0}^{\infty}\binom{-1/2}{r}^2\lambda^r.$$

To evaluate c then we need only compute the number of points of C_λ for one value of λ (which we have already done). Thus

$$\begin{aligned}\operatorname{trace} f^*|_{H^1(M;\mathcal{O})} &= -\text{cardinality of } C_\lambda \\ &= (-1)^{(p-1)/2}\sum_{r=0}^{\infty}\binom{-1/2}{r}^2\lambda^r \\ &= (-1)^{(p-1)/2}\sum_{r=0}^{(p-1)/2}\binom{-1/2}{r}^2\lambda^r.\end{aligned}$$

2.13 Some Remarks on Serre Duality

The usual form of Serre duality in the theory of complex manifolds is not the one we used to make this last computation. Instead we used the nondegeneracy of the pairing

$$H^i(M;\Omega^j) \otimes H^k(M;\Omega^l) \to H^m(M;\Omega^m), \tag{2.40}$$

where $(i + k) = (j + l) = m =$ complex dimension of the complex manifold M, and

$$\Omega^l = \text{sheaf of holomorphic } l \text{ forms on } M.$$

$H^*(M; \Omega^j)$ is computed from the exact sequence

$$\Gamma(\Omega^j \otimes \mathscr{A}^{0,0}) \xrightarrow{\bar\partial} \Gamma(\Omega^j \otimes \mathscr{A}^{0,1}) \xrightarrow{\bar\partial} \cdots,$$

which generalizes (2.32). The pairing (2.40) is given by

$$(\omega, \eta) \to \int_M \omega \wedge \eta,$$

where ω is a $\bar\partial$-closed (j, i) form and η is a $\bar\partial$-closed (l, k) form. Thus in the case which concerns us, the usual complex version reads

$$\begin{aligned} H^1(M; \mathscr{O}) \otimes H^0(M; \Omega^1) &\longrightarrow \mathbb{C}, \\ (\eta, \omega) &\longmapsto \int_M \eta \wedge \omega. \end{aligned} \tag{2.41}$$

To see that this is the same (in the complex case) as the pairing (2.38), consider the open covering

$$U_1 = (M - q), \qquad U_2 = (M - q')$$

of M. Then if f is a function with simple pole at q and only other pole q', the corresponding Čech one-cochain with coefficients in $\mathscr{O}$ is given by

$$f|_{U_1 \cap U_2} \in C^1(M; \mathscr{O}). \tag{2.42}$$

Next construct the double complex of Čech groups

$$\begin{array}{ccccccccc}
 & & f|_{U_1\cap U_2}\uparrow\delta & & f|_{U_1\cap U_2}\uparrow\delta & & \uparrow\partial & & \\
0 & \longrightarrow & C^1(M; \mathscr{O}) & \longrightarrow & C^1(M; \mathscr{A}^{0,0}) & \xrightarrow{\bar\partial} & C^1(M; \mathscr{A}^{0,1}) & \xrightarrow{\bar\partial} & \\
 & & \uparrow\delta & & (g, h)\uparrow\delta & & (\bar\partial g, \bar\partial g)\uparrow\delta & & \\
0 & \longrightarrow & C^0(M; \mathscr{O}) & \longrightarrow & C^0(M; \mathscr{A}^{0,0}) & \longrightarrow & C^0(M; \mathscr{A}^{0,1}) & \xrightarrow{\bar\partial} & \\
 & & \uparrow & & \uparrow & & (\bar\partial g)\uparrow & & \\
0 & & \mathbb{C} & \longrightarrow & \Gamma(\mathscr{A}^{0,0}) & \xrightarrow{\bar\partial} & \Gamma(\mathscr{A}^{0,1}) & \xrightarrow{\bar\partial} & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and follow the cochain (2.42) across to the corresponding representative $\bar{\partial} g$ in $\Gamma\mathscr{A}^{0,1}$. To do this, pick

$$g\begin{cases}\text{smooth at } q',\\ = (1/2)f \text{ away from a neighborhood of } q',\end{cases}$$

$$h\begin{cases}\text{smooth at } q,\\ = (-1/2)f \text{ away from a neighborhood of } q,\end{cases}$$

such that $g - h = f$ everywhere except at the points q and q'. Then

$$\bar{\partial} g = \bar{\partial} h,$$

and the equivalence of the pairings (2.38) and (2.41) follows from the equation

$$\int_q f\omega = \lim_{\text{radius}\to 0} \int_q g\omega = \int_M \omega \wedge dg = \int_M \omega \wedge \bar{\partial} g.$$

The point is that the (equivalent) form (2.38) of the pairing carries over to the general algebraic case, whereas the form (2.41) of the pairing makes no sense except when we are working over the complex numbers.