INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 8, 743–770 (1974)

DISCRETE (LEGENDRE) ORTHOGONAL POLYNOMIALS—A SURVEY

C. P. NEUMAN† AND D. I. SCHONBACH‡

Carnegie–Mellon University, Department of Electrical Engineering, Pittsburgh, Pennsylvania, U.S.A.

SUMMARY

The *discrete* (*Legendre*) *orthogonal polynomials* (DLOP's) are useful for approximation purposes. This set of mth degree polynomials $\{P_m(K, N)\}$ are orthogonal with unity weight over a uniform discrete interval and are completely determined by the normalization $P_m(0, N) = 1$. The authors are employing these polynomials as assumed modes in engineering applications of weighted residual methods. Since extensive material on these discrete orthogonal polynomials, and their properties, is not readily available, this paper is designed to unify and summarize the presently available information on the DLOP's and related polynomials. In so doing, many new properties have been derived. These properties, along with sketches of their derivation, are included. Also presented are a representation of the DLOP's as a product of vectors and matrices, and an efficient computational scheme for generating these polynomials.

INTRODUCTION

Problems in applied science and engineering commonly involve representing experimental data and approximating functions by smooth curves. Linear combinations of functions are frequently employed in the solution of these approximation problems. The functions oftentimes employed are polynomials. They are, by far, the simplest to understand and manipulate. Computationally, unlike, for example, the exponentials and trigonometric functions, evaluation of polynomials does not require numerous subprogram calls. Although the higher-order polynomials exhibit 'roly-poly' behaviour, sets of polynomials are complete; hence, arbitrary approximation accuracy can, in principle, be obtained. With polynomials, estimates or bounded expressions for the truncation error can generally be derived. Furthermore, polynomials are closed under both translation of the independent variable and scaling. Although this discussion is not intended to advocate the universality of polynomials for approximation purposes, they are indeed a practical set of functions for a wide variety of applications.

Since practical problems often involve the least-squares curve fitting of experimental data, a set of *orthogonal* polynomials is advantageous from the computational viewpoint. The technical literature abounds with extensive material on orthogonal polynomials of a continuous independent variable (e.g. Reference 12). Less attention, however, has been devoted to orthogonal polynomials of a discrete independent variable. In this case, the independent variable can assume only a finite number of values in the interval. The orthogonality of a set of discrete *orthogonal polynomials* $\{T_m(x)\}$ for the n points x_i with associated weights w_i is defined by

$$\sum_{i=1}^{n} w_i T_m(x_i) T_l(x_i) = 0 \quad \text{for } l \neq m \tag{1}$$

† Associate Professor. The work of this author was supported in part by the National Science Foundation Grant GJ-1075.

‡ Graduate Student, Hertz Foundation Fellow.

Received 3 September 1973
Revised 22 January 1974

Equation (1) completely determines the polynomials $\{T_m(x)\}$ up to an arbitrary multiplicative factor.

The discrete polynomials surveyed in this paper are orthogonal with unity weight over the *uniform* discrete interval $[\alpha, \beta]$. The $(N+1)$ uniformly spaced points in this interval are located at

$$x_i = \alpha + ih \quad \text{for } i = 0, 1, 2 \ldots, N,$$

where $h = (\beta - \alpha)/N$. The multiplicative factor is chosen to specify

$$T_m(\alpha) = 1 \quad \text{for } m = 0, 1, 2, \ldots, N$$

As noted in the section 'Definitions', these polynomials resemble the shifted Legendre polynomials, hence, they have been termed the *discrete* (*Legendre*) *orthogonal polynomials* (DLOP's). A variation of these polynomials was first considered by Chebyshev[6] in 1858 and later by Gram[4] in 1915, and since then they have been investigated by others. These DLOP's are a special case of the Hahn polynomials.[7]

Curve fitting (e.g. interpolation and regression) applications of discrete orthogonal polynomials are documented in References 1, 3 and 5. The advantages of applying the DLOP's to the problem of least-squares curve-fitting are illustrated in Appendix III. An error analysis of least-squares curve-fitting using discrete orthogonal polynomials is provided in Reference 10. In addition to curve fitting, the authors are employing the DLOP's as assumed modes in engineering applications of weighted residual methods.[2,11] Other potential areas of application include data filtering and smoothing.

Upon surveying the literature, it became apparent to the authors that extensive material on these discrete orthogonal polynomials and their properties is not readily available. The preparation of this paper has been motivated by this need for unifying and tabulating the presently available information on the DLOP's and related polynomials. In so doing, many new properties have been derived. These properties along with sketches of their derivation, are included in this paper. Also presented are a representation of the DLOP's as a product of vectors and matrices and an efficient computational scheme for generating these polynomials.

DEFINITIONS

The discrete (Legendre) orthogonal polynomials $\{P_m(K, N)\}$, where $m = 0, 1, 2, \ldots, N$ is the degree of the polynomial, are defined to be orthogonal with unity weight over the discrete interval $K = 0, 1, 2, \ldots, N$; that is,

$$\sum_{K=0}^{N} P_m(K, N) P_l(K, N) = 0 \quad \text{for } m \neq l \tag{2}$$

The DLOP's are completely determined by the normalization

$$P_m(0, N) = 1 \quad \text{for all } m \tag{3}$$

The Rodrigues-type formula

$$P_m(K, N) = \frac{(-1)^m}{m!\,N^{(m)}} \Delta^m \{K^{(m)}(K-N-1)^{(m)}\} \tag{4}$$

for the DLOP's can be derived[4] from (2) and (3). In (4),

$$\Delta^m g(K) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} g(K+m-j) \tag{5}$$

is the mth forward difference of $g(\cdot)$ with respect to K, and

$$K^{(m)} = K(K-1)(K-2)\dots(K-m+1) \tag{6}$$

is the mth *fading factorial* of K.

Application of (5) and (6) in conjunction with (4) yields the explicit expression:[4]

$$P_m(K,N) = \sum_{j=0}^{m} (-1)^j \binom{m}{j}\binom{m+j}{j}\frac{K^{(j)}}{N^{(j)}} = \sum_{j=0}^{m} l(m,j)\frac{K^{(j)}}{N^{(j)}} \tag{7}$$

for $m = 0, 1, 2, \dots, N$. In particular, the first five DLOP's are:

$$P_0(K,N) = 1$$

$$P_1(K,N) = 1-2\frac{K}{N}$$

$$P_2(K,N) = 1-6\frac{K}{N}+6\frac{K^{(2)}}{N^{(2)}}$$

$$P_3(K,N) = 1-12\frac{K}{N}+30\frac{K^{(2)}}{N^{(2)}}-20\frac{K^{(3)}}{N^{(3)}}$$

$$P_4(K,N) = 1-20\frac{K}{N}+90\frac{K^{(2)}}{N^{(2)}}-140\frac{K^{(3)}}{N^{(3)}}+70\frac{K^{(4)}}{N^{(4)}}$$

It is observed that the numeric coefficients of the DLOP's are (apart from sign) precisely the same as those of the shifted Legendre polynomials on the interval [0, 1]; hence the name DLOP's. The coefficients $l(m,j)$ of the first twenty-one polynomials are tabulated in Appendix I.

Equation (7) expresses $P_m(K,N)$ in terms of fading factorials of K. $P_m(K,N)$, expressed in terms of the powers of K, is

$$P_m(K,N) = \sum_{l=0}^{m}\left\{\sum_{j=l}^{m}(-1)^j\binom{m+j}{j}\binom{m}{j}\frac{1}{N^{(j)}}S_j^{(l)}\right\}K^l,$$

where the $S_j^{(l)}$ are the Stirling numbers of the first kind.

PROPERTIES

The purpose of this section is to tabulate the properties of the DLOP's. A number of these properties, and their derivation, are available in References 6 and 7.

More than half of these properties, however, are new and have been derived by the authors. In particular, the difference (in N) equations are all new. They are derived in Appendix II. Derivations of some of the other new properties are sketched in this section only when deemed pertinent.

Throughout this section, the ranges of K, m, and N are suppressed. They can be inferred from the particular equation. Specifically, $P_m(K,N)$ is defined for $K = 0, 1, 2, \dots, N$ and $m = 0, 1, 2, \dots, N$. The initial values for the difference and recursion equations have also been omitted and can be inferred from these equations.

Orthogonality relation

$$\sum_{K=0}^{N} P_m(K,N)P_l(K,N) = \begin{cases} 0; & m \neq l \\ \dfrac{1}{(2m+1)} \dfrac{(N+m+1)^{(m+1)}}{N^{(m)}}; & m = l \end{cases}$$

This orthogonality property is valuable from both the analytical as well as the computational point of view. For example, a function $f(K)$ of the discrete variable $K = 0, 1, 2, \ldots, N$ may be expanded in a generalized Fourier series of DLOP's:

$$f(K) = a_0 P_0(K,N) + a_1 P_1(K,N) + a_2 P_2(K,N) + \ldots,$$

where the Fourier coefficients a_m are specified by

$$a_m = (2m+1)\frac{N^{(m)}}{(N+m+1)^{(m+1)}} \sum_{K=0}^{N} f(K)P_m(K,N), \quad \text{for } m = 0, 1, 2, \ldots$$

Dual orthogonality relation

$$\sum_{m=0}^{N} \frac{(2m+1)N^{(m)}}{(N+m+1)^{(m+1)}} P_m(K,N)P_l(K,N) = \begin{cases} 0; & K \neq l \\ 1; & K = l \end{cases}$$

Symmetry property

$$P_m(N-K,N) = (-1)^m P_m(K,N) \tag{8}$$

Boundary values

$$P_m(0,N) = 1$$

$$P_m(N,N) = (-1)^m$$

Central values

The following central values have been derived using the second-order recurrence equation and the first-order difference (in K) recurrence equations in conjunction with the symmetry property (8):

For N *even*:

$$P_m(\tfrac{1}{2}N, N) = \begin{cases} 1; & m = 0 \\ 0; & m = 1, 3, 5, \ldots \\ (-1)^{m/2} \dbinom{\frac{N+m}{2}}{\frac{m}{2}} \dbinom{\frac{N}{2}}{\frac{m}{2}} \Big/ \dbinom{N}{m}; & m = 2, 4, 6, \ldots \end{cases}$$

$$\Delta P_m(\tfrac{1}{2}N, N) = \begin{cases} 0; & m = 0 \\ (-1)^{(m+1)/2}\dfrac{2m}{N}\dbinom{\frac{N+m+1}{2}}{\frac{m}{2}}\dbinom{\frac{N}{2}}{\frac{m-1}{2}}\Big/\dbinom{N}{m-1}; & m = 1, 3, 5, \ldots \\ (-1)^{(m+2)/2}\dfrac{2(m+1)}{N+2}\dbinom{\frac{N+m}{2}}{\frac{m}{2}}\dbinom{\frac{N-2}{2}}{\frac{m}{2}}\Big/\dbinom{N}{m}; & m = 2, 4, 6, \ldots \end{cases}$$

For N *odd*:

$$P_m\left(\frac{N-1}{2}, N\right) = \begin{cases} 1; & m = 0 \\ \dfrac{1}{N}; & m = 1 \\ (-1)^{m/2}\dbinom{\frac{N+m+1}{2}}{\frac{m}{2}}\dbinom{\frac{N+1}{2}}{\frac{m}{2}}\Big/\dbinom{N+1}{m}; & m = 2, 4, 6, \ldots \\ (-1)^{(m-1)/2}\dbinom{\frac{N+m}{2}}{\frac{m-1}{2}}\dbinom{\frac{N-1}{2}}{\frac{m-1}{2}}\Big/\dbinom{N}{m}; & m = 3, 5, 7 \ldots \end{cases}$$

$$P_m\left(\frac{N+1}{2}, N\right) = (-1)^m P_m\left(\frac{N-1}{2}, N\right)$$

$$\Delta P_m\left(\frac{N-1}{2}, N\right) = \begin{cases} 0; & m = 0, 2, 4, \ldots \\ -\dfrac{2}{N}; & m = 1 \\ 2(-1)^{(m+1)/2}\dbinom{\frac{N+m}{2}}{\frac{m-1}{2}}\dbinom{\frac{N-1}{2}}{\frac{m-1}{2}}\Big/\dbinom{N}{m}; & m = 3, 5, 7, \ldots \end{cases}$$

Maximum value

From (16), the maximum value of $|P_m(K, N)|$ occurs for $m = N$ and for

$$K = \begin{cases} \dfrac{m}{2}; & m \text{ even} \\ \dfrac{m+1}{2}; & m \text{ odd} \end{cases}$$

From the explicit expression (7),

$$P_m(K, m) = (-1)^K \binom{m}{K}$$

and

$$P_m(K, m+1) = P_m(K, m) + P_m(K-1, m)$$

Thereby,

$$|P_m(K, N)| \leqslant \begin{cases} \dbinom{m}{\frac{m}{2}}; & m \text{ even} \\[2ex] \dbinom{m}{\frac{m-1}{2}}; & m \text{ odd} \end{cases}$$

Furthermore,

$$\max_{0 \leqslant K \leqslant N} \left| P_m(K, N) \right| \leqslant \max_{0 \leqslant K \leqslant m} \left| P_m(K, m) \right|$$

and

$$\max_{\{0 \leqslant K \leqslant N+1\}} \left| P_m(K, N+1) \right| \leqslant \max_{\{0 \leqslant K \leqslant N\}} \left| P_m(K, N) \right|$$

for $m \leqslant N$.

Summation formulas

$$\sum_{K=0}^{N} K^{(l)} P_m(K, N) = \begin{cases} 0; & l = 0, 1, \ldots, (m-1) \\[1ex] (-1)^m \dfrac{(m!)^2 (N+m+1)^{(m+1)}}{(2m+1)!}; & l = m \\[2ex] (-1)^m \dfrac{(l!)^2 (N+m+1)^{(m+1)} (N-m)^{(l-m)}}{(l+m+1)!(l-m)!}; & l = m, m+1, \ldots \end{cases}$$

$$\sum_{K=0}^{N} K^{(l)} P_m(K, N) = \begin{cases} 0; & l = 0, 1, \ldots, (m-1) \\[1ex] (-1)^m \dfrac{(m!)^2 (N+m+1)^{(m+1)}}{(2m+1)!}; & l = m \\[2ex] (-1)^m (N+m+1)^{(m+1)} \displaystyle\sum_{j=m}^{l} \dfrac{(j!)^2 (N-m)^{(j-m)}}{(j+m+1)!(l-m)!} S_l^{(j)}; & l = m, m+1, \ldots \end{cases}$$

where the $S_l^{(j)}$ are the Stirling numbers of the second kind.

$$\sum_{K=0}^{N} P_m(K, N) = \begin{cases} N+1; & m = 0 \\ 0; & m = 1, 2, \ldots \end{cases}$$

$$\sum_{K=0}^{N} K P_l(K, N) P_m(K, N) = \begin{cases} -\dfrac{m}{2(2m-1)(2m+1)} \dfrac{(N+m+1)^{(m+1)}}{N^{(m-1)}}; & l = (m-1) = 0, 1, 2, \ldots \\[2ex] \dfrac{1}{2(2m+1)} \dfrac{(N+m+1)^{(m+2)}}{N^{(m)}}; & l = m = 0, 1, 2, \ldots \\[2ex] -\dfrac{m+1}{2(2m+1)(2m+3)} \dfrac{(N+m+2)^{(m+2)}}{N^{(m)}}; & l = (m+1) = 1, 2, 3, \ldots \\[1ex] 0; & \text{otherwise} \end{cases}$$

$$\sum_{K=0}^{N} \binom{N}{K} \binom{\varepsilon}{K} P_m(K, N) = \binom{N+\varepsilon}{\varepsilon} P_m(\varepsilon, N+\varepsilon),$$

where ε is arbitrary.

$$\sum_{m=0}^{N} \binom{N}{m} P_m(K,N) t^m = (t+1)^{N-K} \sum_{l=0}^{N} \binom{N+K}{l} \binom{K}{l} \left(\frac{t}{t+l}\right)^l$$

Second-order recurrence equation

$$(m+1)(N-m)P_{m+1}(K,N) = (2m+1)(N-2K)P_m(K,N) - m(N+m+1)P_{m-1}(K,N) \tag{9}$$

for $m = 1, 2, 3, \ldots, (N-1)$, with the initial values

$$\left.\begin{aligned} P_0(K,N) &= 1 \\ P_1(K,N) &= (N-2K)/N \end{aligned}\right\} \tag{10}$$

Second-order difference (in K) equation

$$(K+2)(N-K-1)\Delta^2 P_m(K,N) + [N-2(K+1)+m(m+1)]\Delta P_m(K,N) + m(m+1)P_m(K,N) = 0$$

for $K = 0, 1, 2, \ldots, (N-2)$, with the initial values

$$\left.\begin{aligned} P_m(0,N) &= 1 \\ \Delta P_m(0,N) &= -m(m+1)/N \end{aligned}\right\}$$

$$(K+1)(N-K)P_m(K+1,N) - [(K+1)(N-K) + K(N-K+1) + m(m+1)]P_m(K,N) + K(N-K+1)P_m(K-1,N) = 0$$

for $K = 1, 2, 3, \ldots, (N-1)$, with the initial values

$$\left.\begin{aligned} P_m(0,N) &= 1 \\ P_m(1,N) &= 1 - m(m+1)/N \end{aligned}\right\}$$

Second-order difference (in N) equation

$$N(N+1)(N-K+1)P_m(K,N+1) - N[(N+1)(2N-2K+1) - m(m+1)]P_m(K,N) + (N-m)(N-m+1)(N-K)P_m(K,N-1) = 0 \tag{11}$$

for $N = m+1, m+2, m+3, \ldots$, with the initial values

$$\left.\begin{aligned} P_m(K,m) &= (-1)^K \binom{m}{K} \\ P_m(K,m+1) &= (-1)^K \binom{m}{K} \left[1 - \frac{K}{m+1-K}\right] \end{aligned}\right\}$$

First-order difference (in K)—recurrence equations

$$2(m+1)P_m(K,N) = -(N-m)\,\Delta P_{m+1}(K,N) + (N-2K-m-2)\,\Delta P_m(K,N)$$

$$2mP_m(K,N) = -(N-2K+m-1)\,\Delta P_m(K,N) + (N+m+1)\,\Delta P_{m-1}(K,N)$$

$$2(K+1)(N-K)\,\Delta P_m(K,N) = (m+1)(N-m)P_{m+1}(K,N) - (m+1)(N-2K+m)P_m(K,N)$$

$$2(K+1)(N-K)\,\Delta P_m(K,N) = m(N-2K-m-1)P_m(K,N) - m(N+m+1)P_{m-1}(K,N)$$

$$2K(N-K+1)P_m(K-1,N) = -(m+1)(N-m)P_{m+1}(K,N) + [2K(N-K+1)+(m+1)(N-2K-m)]P_m(K,N)$$

$$2K(N-K+1)P_m(K-1,N) = [K(N-K+1)-m(N-2K+m+1)]P_m(K,N) + m(N+m+1)P_{m-1}(K,N)$$

$$2(K+1)(N-K)P_m(K+1,N) = (m+1)(N-m)P_{m+1}(K,N) + [2(K+1)(N-K)-(m+1)(N-2K+m)]P_m(K,N)$$

$$2(K+1)(N-K)P_m(K+1,N) = [2(K+1)(N-K)+m(N-2K-m-1)]P_m(K,N) - m(N+m+1)P_{m-1}(K,N)$$

First-order difference (in N)—recurrence equations

$$2(N+m+1)(N-K)P_m(K,N-1) = (m+1)NP_{m+1}(K,N)+(2N-2K+m+1)NP_m(K,N) \tag{12}$$

$$2(N-m)(N-K)P_m(K,N-1) = (2N-2K-m)NP_m(K,N)-mNP_{m-1}(K,N) \tag{13}$$

$$2(N+1)(N-K+1)P_m(K,N+1) = -(N-m)(m+1)P_{m+1}(K,N) + (N+m+2)(2N-2K-m+1)P_m(K,N) \tag{14}$$

$$2(N+1)(N-K+1)P_m(K,N+1) = (N-m+1)(2N-2K+m+2)P_m(K,N) + m(N+m+1)P_{m-1}(K,N) \tag{15}$$

First-order partial difference (in K and N) equations

$$NP_m(K,N) = (N-K)P_m(K,N-1)+KP_m(K-1,N-1) \tag{16}$$

$$KNP_m(K,N) = -(N-K+1)NP_m(K-1,N) + (N-m)(N+m+1)P_m(K-1,N-1) \tag{17}$$

$$[(N-m)(N+m+1)-K^2]NP_m(K,N) = (N-m)(N+m+1)(N-K)P_m(K,N-1) + (N-K+1)KNP_m(K-1,N) \tag{18}$$

$$(N-K)NP_m(K,N) = -N(N+1)P_m(K+1,N) + (N-m)(N+m+1)P_m(K,N-1) \tag{19}$$

$$[(N-m)(N+m+1)-(N-K)^2]NP_m(K,N) = N(N+1)(N-K)P_m(K+1,N) + (N-m)(N+m+1)KP_m(K-1,N-1) \tag{20}$$

First-order partial difference (in K and N)—recurrence equations

$$2(N+m+1)KP_m(K-1,N-1) = -(m+1)NP_{m+1}(K,N)+(2K+m+1)NP_m(K,N) \tag{21}$$

$$2(N-m)KP_m(K-1,N-1) = (2K-m)NP_m(K,N)+mNP_{m-1}(K,N) \tag{22}$$

$$2(N+1)(K+1)P_m(K+1,N+1) = (N-m)(m+1)P_{m+1}(K,N) + (N+m+2)(2K-m+1)P_m(K,N) \tag{23}$$

$$2(N+1)(K+1)P_m(K+1,N+1) = (N-m+1)(2K+m+2)P_m(K,N) - m(N+m+1)P_{m-1}(K,N) \tag{24}$$

Inverse polynomials

The following formula expresses the mth fading factorial in terms of the DLOP's:

$$K^{(m)} = \sum_{l=0}^{m} \left[(-1)^l \frac{(2l+1)(m!)^2 N^{(m)}}{(m-l)!(m+l+1)!} \right] P_l(K, N)$$

In particular, the first five inverse DLOP's are:

$$\begin{aligned}
1 &= P_0 \\
K &= \tfrac{1}{2}NP_0 - \tfrac{1}{2}NP_1 \\
K^{(2)} &= \tfrac{1}{3}N^{(2)}P_0 - \tfrac{1}{2}N^{(2)}P_1 + \tfrac{1}{6}N^{(2)}P_2 \\
K^{(3)} &= \tfrac{1}{4}N^{(3)}P_0 - \tfrac{9}{20}N^{(3)}P_1 + \tfrac{1}{4}N^{(3)}P_2 - \tfrac{1}{20}N^{(3)}P_3 \\
K^{(4)} &= \tfrac{1}{5}N^{(4)}P_0 - \tfrac{2}{5}N^{(4)}P_1 + \tfrac{2}{7}N^{(4)}P_2 - \tfrac{1}{10}N^{(4)}P_3 + \tfrac{1}{70}N^{(4)}P_4
\end{aligned}$$

The mth power of K, expressed in terms of the DLOP's, is

$$K^m = \sum_{l=0}^{m} \left\{ \sum_{j=l}^{m} (-1)^l \frac{(2l+1)(j!)^2 N^{(j)}}{(m-j)!(m+j+1)!} S_m^{(j)} \right\} P_l(K, N),$$

where the $s_m^{(j)}$ are the Stirling numbers of the second kind.

Relationships to classical functions

The relationships between the DLOP's and some of the classical functions are summarized in Table I.[7,8]

Table I. Relationships of the DLOP's to the classical functions

Function	Symbol	Relation
Hahn polynomial	Q_m	$P_m(K, N) = Q_m(K; 0, 0, N+1)$
Hypergeometric function	${}_3F_2$	$P_m(K, N) = {}_3F_2(-m, m+1, -K; 1, -N; 1)$
	${}_2F_1$	${}_2F_1\left(N+1, \frac{1}{2}; K+2; -1\right) = 2\frac{K+2}{N+1} \sum_{m=0}^{\infty} mP_m(K, N)$
Legendre polynomial	$\mathscr{P}_m$	$\mathscr{P}_m(x) = \left(\frac{1}{2}\right)^N \sum_{K=0}^{N} \binom{N}{K} (1-x)^K (1+x)^{N-K} P_m(K, N)$
Shifted Legendre polynomial	$\mathscr{P}_m^*$	$\mathscr{P}_m^*(x) = \sum_{K=0}^{N} \binom{N}{K} (1-X)^K X^{N-K} P_m(K, N)$
		$\mathscr{P}_m^*(x) = (-1)^m \lim_{N\to\infty} P_m(Nx, N)$
		$\frac{d^i}{dx^i}\mathscr{P}_m^*(0) = (-1)^m N^{(i)} \Delta^i P_m(0, N)$
Generalized Laguerre polynomial	L_n^α	$m!\binom{N}{K} P_m(K, N) = \int_0^\infty t^m e^{-t} L_m^0(t) L_{N-K}^{m+K+1}(t)\, dt$
Jacobi polynomial	$\mathscr{P}_K^{(\alpha,\beta)}$	$\mathscr{P}_K^{(N,0)}(1-2x) = (-1)^K x^{2K-N} \sum_{m=0}^{N} \binom{N}{m} (x-1)^m P_m(K, N)$

MATRIX REPRESENTATIONS

Attention now turns to computational aspects of generating and evaluating the DLOP's. To this end, this section is devoted to developing a representation for the DLOP's as a product of vectors and matrices. The properties of these matrices are summarized along with techniques for their generation. These matrix representations lead to advantageous computational schemes.

To facilitate the presentation, it is convenient to define the m vectors

$$\boldsymbol{\theta}(K) = [1, K, K^{(2)}, K^{(3)}, \ldots, K^{(m-1)}]^{\mathrm{T}}$$
$$\boldsymbol{\phi}(K) = [1, K, K^2, K^3, \ldots, K^{m-1}]^{\mathrm{T}}$$
$$\tilde{\mathbf{P}}(K, N) = [P_0(K, N), P_1(K, N), P_2(K, N), \ldots, P_{m-1}(K, N)]^{\mathrm{T}}$$

and the $(m \times m)$ diagonal matrices

$$\mathbf{D}(N) = \text{diagonal}\left\{\frac{1}{N^{(j-1)}}\right\}$$
$$\mathbf{D}^{-1}(N) = \text{diagonal}\,\{N^{(j-1)}\}$$

These definitions provide the following relationships:

$$\boldsymbol{\theta}(K) = \mathbf{S}\boldsymbol{\phi}(K)$$
$$\tilde{\mathbf{P}}(K, N) = \mathbf{LD}(N)\boldsymbol{\theta}(K)$$
$$\tilde{\mathbf{P}}(K, N) = \mathbf{LD}(N)\mathbf{S}\boldsymbol{\phi}(K)$$
$$\boldsymbol{\theta}(K) = \mathbf{D}^{-1}(N)\mathbf{L}^{-1}\tilde{\mathbf{P}}(K, N)$$
$$\tilde{\mathbf{P}}(K+l, N) = \mathbf{LD}(N)\mathbf{A}^l\boldsymbol{\theta}(K) \tag{25}$$
$$\Delta^l[\tilde{\mathbf{P}}(K, N)] = \mathbf{LD}(N)\mathbf{B}^l\boldsymbol{\theta}(K), \tag{26}$$

where

$$\mathbf{A} = \mathbf{I} + \mathbf{B}, \tag{27}$$

$\mathbf{I}$ is the $(m \times m)$ identity matrix, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{L}$, and $\mathbf{S}$ are $(m \times m)$ constant matrices. In the sequel, these matrices are defined and techniques for their generation outlined.

The lower triangular matrix

$$\mathbf{L} = \begin{bmatrix} 1 & & & & & \\ 1 & -2 & & & & \\ 1 & -6 & 6 & & & \mathbf{0} \\ 1 & -12 & 30 & -20 & & \\ 1 & -20 & 90 & -140 & 70 & \\ 1 & -30 & 210 & -560 & 630 & -252 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

contains the coefficients of the DLOP's (and the shifted Legendre polynomials). The elements of $\mathbf{L}$ are

$$l_{ij} = \begin{cases} (-1)^{j-1}\binom{i-1}{j-1}\binom{i+j-2}{j-1}; & \begin{cases} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, i \end{cases} \\ 0; & \text{otherwise} \end{cases} \tag{28}$$

These elements have the following properties:

$$l_{ij} = (-1)^{j-1}\frac{(i+j-2)!}{(i-j)![(j-1)!]^2}; \qquad \begin{cases} i = 1,2,3,\ldots,m \\ j = 1,2,3,\ldots,i \end{cases}$$

$$l_{i1} = 1; \qquad i = 1,2,3,\ldots,m$$

$$l_{ii} = (-1)^{i-1}\binom{2i-2}{i-1}; \qquad i = 1,2,3,\ldots,m$$

$$\sum_{j=1}^{i} l_{ij} = (-1)^{i-1}; \qquad i = 1,2,3,\ldots,m$$

$$l_{ij} = \frac{i+j-2}{(i-j)}l_{i-1,j}; \qquad \begin{cases} i = 2,3,4,\ldots,m \\ j = 2,3,4,\ldots,(i-1) \end{cases}$$

$$l_{ij} = \frac{1}{(i-1)}\left\{(2i-3)(l_{i-1,j} - 2l_{i-1,j-1}) - (i-2)l_{i-2,j}\right\}; \qquad \begin{cases} i = 3,4,5,\ldots,m \\ j = 2,3,\ldots,i \end{cases}$$

$$l_{ii} = -2\frac{2i-3}{(i-1)}l_{i-1,i-1}; \qquad i = 2,3,4,\ldots,m$$

$$l_{i,2} = -i(i-1); \qquad i = 1,2,3,\ldots,m$$

$$l_{ij} = 0; \qquad \begin{cases} i = 1,2,3,\ldots,m \\ j = i+1, i+2, i+3,\ldots,m \end{cases}$$

The elements of L can be generated columnwise through the recursion

$$\left.\begin{array}{ll} l_{i1} = 1; & i = 1,2,3,\ldots,m \\ l_{ij} = \dfrac{(i+j-2)}{(i-j)}l_{i-1,j}; & \begin{cases} i = 2,3,4,\ldots,m \\ j = 2,3,4,\ldots,(i-1) \end{cases} \\ l_{ii} = -2\dfrac{(2i-3)}{(i-1)}l_{i-1,i-1}; & i = 2,3,4,\ldots,m \end{array}\right\}$$

An alternative systematic procedure for calculating the elements l_{ij} is revealed by examining Pascal's triangle:

$$\begin{array}{l}
D \searrow \\
\qquad\qquad\qquad\qquad 1 \\
\qquad\qquad\qquad 1 \quad 1 \\
\qquad\qquad\quad 1 \quad 2 \quad 1 \\
R \rightarrow \qquad 1 \quad 3 \quad 3 \quad 1 \\
\qquad\quad 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
\qquad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
\quad 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}$$

For example, the non-zero elements of the fourth row of **L** can be obtained, except for the alternating sign, by multiplying the elements of the *fourth row* R of Pascal's triangle (1 3 3 1) by the elements of diagonal D. This diagonal (1 4 10 20) extends from the left-most element of R to the *fourth central element* of Pascal's triangle. By following this recipe, the fourth row of **L** is, apart from sign,

$$\begin{array}{rccccc} R: & (& 1 & 3 & 3 & 1 \;) \\ & & \circledast & \circledast & \circledast & \circledast \\ D: & (& 1 & 4 & 10 & 20 \;) \\ & & \downarrow & \downarrow & \downarrow & \downarrow \\ & (& 1 & 12 & 30 & 20 \;) \end{array}$$

Since the nth row of the triangle of Pascal is composed of the n binomial coefficients

$$\binom{n-1}{0}, \binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1},$$

this procedure is a consequence of (28).

The inverse matrix

$$\mathbf{L}^{-1} = \begin{bmatrix} 1 & & & & & & \\ \frac{1}{2} & -\frac{1}{2} & & & & & \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & & & \mathbf{0} & \\ \frac{1}{4} & -\frac{9}{20} & \frac{1}{4} & -\frac{1}{20} & & & \\ \frac{1}{5} & -\frac{2}{5} & \frac{2}{7} & -\frac{1}{10} & \frac{1}{70} & & \\ \frac{1}{6} & -\frac{5}{14} & \frac{25}{84} & -\frac{5}{36} & \frac{1}{28} & -\frac{1}{252} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

contains the coefficients of the inverse DLOP's. The elements

$$l_{ij}^{-1} = \begin{cases} (-1)^{j-1}(2j-1)\dfrac{[(i-1)!]^2}{(i-j)!(i+j-1)!}; & \begin{cases} i = 1, 2, 3, \ldots, m \\ j = 1, 2, 3, \ldots, i \end{cases} \\ 0; & \text{otherwise} \end{cases}$$

of $\mathbf{L}^{-1}$ can be generated columnwise through the recursion

$$\left.\begin{array}{ll} l_{ii}^{-1} = 1/i; & i = 1, 2, 3, \ldots, m \\ l_{ij}^{-1} = \dfrac{i^2}{i(i-1)-j(j-1)} l_{i-1,j}^{-1}; & \begin{cases} i = 2, 3, 4, \ldots, m \\ j = 2, 3, 4, \ldots, m \end{cases} \\ l_{ii}^{-1} = \dfrac{(i-1)}{2(2i-3)} l_{i-1,i-1}^{-1}; & i = 2, 3, 4, \ldots, m \end{array}\right\}$$

The lower triangular matrix

$$\mathbf{S} = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & -1 & 1 & & & & & \\ 0 & 2 & -3 & 1 & & & & \mathbf{0} \\ 0 & -6 & 11 & -6 & 1 & & & \\ 0 & 24 & -50 & 35 & -10 & 1 & & \\ 0 & -120 & 274 & -225 & 85 & -15 & 1 & \\ 0 & 720 & -1764 & 1624 & -735 & 175 & -21 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

contains the *Stirling numbers of the first kind.* The elements of $\mathbf{S}$ can be generated recursively through:

$$\left.\begin{array}{ll} s_{ii} = 1; & i = 1, 2, 3, \ldots, m \\ s_{i1} = 0; & i = 2, 3, 4, \ldots, m \\ s_{ij} = s_{i-1,j-1} - (i-2)s_{i-1,j}; & \begin{cases} i = 3, 4, 5, \ldots, m \\ j = 2, 3, 4, \ldots, (i-1) \end{cases} \end{array}\right\}$$

The inverse matrix

$$\mathbf{S}^{-1} = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 1 & 1 & & & & & \\ 0 & 1 & 3 & 1 & & & & \\ 0 & 1 & 7 & 6 & 1 & & & \\ 0 & 1 & 15 & 25 & 10 & 1 & & \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \\ 0 & 1 & 63 & 301 & 350 & 140 & 21 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

contains the *Stirling numbers of the second kind.* These numbers can be generated recursively through:

$$\left.\begin{array}{ll} s_{ii}^{-1} = 1; & i = 1, 2, 3, \ldots, m \\ s_{i1}^{-1} = 0; & i = 2, 3, 4, \ldots, m \\ s_{i2}^{-1} = 1; & i = 2, 3, 4, \ldots, m \\ s_{ij}^{-1} = s_{i-1,j-1}^{-1} + (j-1)s_{i-1,j}^{-1}; & \begin{cases} i = 4, 5, 6, \ldots, m \\ j = 3, 4, 5, \ldots, (i-1) \end{cases} \end{array}\right\}$$

The matrix $\mathbf{B}$ has non-zero elements on only the first subdiagonal:

$$b_{ij} = \begin{cases} j; & \begin{cases} j = 1, 2, \ldots, (m-1) \\ i = j+1 \end{cases} \\ 0; & \text{otherwise} \end{cases} \tag{29}$$

Computation of the lth forward difference (26) of $\tilde{\mathbf{P}}(K, N)$ requires $\mathbf{B}^l$. From (29), $\mathbf{B}^l$ has non-zero elements on only the lth subdiagonal, and $\mathbf{B}^l = 0$ for $l \geqslant m$. The elements

$$b^l_{ij} = \begin{cases} (i-1)^{(l)}; & \begin{cases} j = 1, 2, \ldots, (m-l) \\ i = j+l \end{cases} \\ 0; & \text{otherwise} \end{cases}$$

of $\mathbf{B}^l$ can be generated recursively via:

$$\left.\begin{aligned} b^l_{l+1,1} &= l! \\ b^l_{i+1,j+1} &= \left(\frac{i}{j}\right) b^l_{ij}; \qquad \begin{cases} j = 1, 2, 3, \ldots, (m-l) \\ i = j+l \end{cases} \end{aligned}\right\}$$

The matrix $\mathbf{A}$ has non-zero elements on only the main diagonal and the first subdiagonal:

$$a_{ij} = \begin{cases} 1; & i = j = 1, 2, 3, \ldots, m \\ j; & i = (j+1) = 2, 3, \ldots, m \\ 0; & \text{otherwise} \end{cases}$$

The lth increment (25) of $\tilde{\mathbf{P}}(K, N)$ requires the banded matrix

$$\mathbf{A}^l = \begin{bmatrix} 1 & & & & & & & 0 \\ l & 1 & & & & & & \\ l^{(2)} & 2l & 1 & & & & & \\ l^{(3)} & 3l^{(2)} & 3l & 1 & & & & \\ l^{(4)} & 4l^{(3)} & 6l^{(2)} & 4l & 1 & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ l^{(l)} & & & & & & & \\ & (l+1)l^{(l)} & & \ddots & & & & \\ 0 & & & & & & & \\ & & \binom{m-1}{l} l^{(l)} \cdots & \binom{m-1}{4} l^{(4)} & \binom{m-1}{3} l^{(3)} & \binom{m-1}{2} l^{(2)} & (m-1)l & 1 \end{bmatrix}$$

The elements of .

$$\mathbf{A}^l = (\mathbf{I}+\mathbf{B})^l = \sum_{i=0}^{l} \binom{l}{i} \mathbf{B}^i$$

are

$$a_{ij}^{l} = \begin{cases} 0; & i < j \\ \binom{i-1}{i-j} l^{(i-j)}; & j \leqslant i \leqslant j+l \\ 0; & j+l < i \end{cases} \qquad \begin{cases} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, m \end{cases}$$

and can be generated diagonally through the recursion:

$$\left.\begin{array}{ll} a_{ii}^{l} = 1\,; & i = 1, 2, 3, \ldots, m \\ a_{i+1,1}^{l} = (l-i+2)a_{i1}^{l}\,; & i = 1, 2, 3, \ldots, l \\ a_{i+1,j+1}^{l} = \left(\dfrac{i}{j}\right)a_{ij}^{l}; & \begin{cases} i = 2, 3, 4, \ldots, m \\ j = 1, 2, 3, \ldots, m \\ (i-j) = 1, 2, 3, \ldots, l \end{cases} \end{array}\right\}$$

ZEROS

Although explicit formulas for the zeros of the DLOP's have not been found, a number of facts and properties of the zeros are available. At the outset, it is noted that the zeros of $P_m(K, N)$ are functions of *both* m and N.

This section delineates nine statements that are useful in locating the roots of the DLOP's. Most of these statements are analogous to their continuous counterparts for the classical orthogonal polynomials.[9,12]

Application of Rolle's theorem, in conjunction with the Rodrigues-type formula (4), leads to the following:

(i) The m zeros of $P_m(.\,, N)$ are all real, distinct, and lie in the open interval $(0, N)$; that is, $0 < r_1 < r_2, \ldots, r_{m-1} < r_m < N$, where $P_m(r_i, N) = 0$ for $i = 1, 2, \ldots, m$. In general, the zeros do not assume integral values.

The next two statements are a consequence of the symmetry property (8).

(ii) If r is a zero of $P_m(.\,, N)$, then $(N-r)$ is also a zero.

(iii) $N/2$ is a zero of the *odd* mth degree DLOP.

The next three statements describe the subintervals in which the zeros are located.

(iv) If r is a zero of $P_m(.\,, N)$, then $P_m(r-1, N)\,.\,P_m(r+1, N) < 0$.

(v) Any two consecutive zeros of $P_m(.\,, N)$ are more than one unit apart; that is,

$$r_i + 1 < r_{i+1} \quad \text{for } i = 1, 2, \ldots, (m-1)$$

(vi) The closed interval $[K, K+1]$ for any K in $[0, N-1]$ contains at most one zero of $P_m(.\,, N)$.

The following two statements characterize the interlacing of the zeros of two different order DLOP's.

(vii) Let $r_0 \triangleq 0$ and $r_{m+1} \triangleq N$. Then each open interval (r_{i-1}, r_i), for $i = 1, 2, 3, \ldots, (m+1)$, contains exactly one zero of $P_{m+1}(.\,, N)$. That is, if $s_1 < s_2 < \ldots < s_m < s_{m+1}$ are the $(m+1)$ zeros of $P_{m+1}(.\,, N)$, then $r_{i-1} < s_i < r_i$ for $i = 1, 2, \ldots, (m+1)$.

(viii) Between any two consecutive zeros of $P_m(.\,, N)$ there lies at least one zero of $P_l(.\,, N)$, provided that $0 \leqslant m < l$; that is, $P_l(s, N) = 0$ for at least one $s \in (r_{i-1}, r_i)$ for $i = 1, 2, \ldots, (m+1)$.

The final statement, which follows from the sixth line of Table I, relates the zeros of the DLOP's to those of the shifted Legendre polynomials.

(ix) As the number of points (N) in the interval tends to infinity, the zeros of the DLOP's divided by N approach the zeros of the shifted Legendre polynomials. Symbolically, if $\theta_1 < \theta_2 < \ldots < \theta_{m-1} < \theta_m$ are the m zeros of the mth order shifted Legendre polynomial, $\mathscr{P}_m^*(x)$, then

$$\lim_{N\to\infty} P_m(\theta_i N, N) = \mathscr{P}_m^*(\theta_i) = 0 \quad \text{for } i = 1, 2, 3, \ldots, m$$

GENERATION SCHEME

The explicit expression

$$\begin{aligned} P_m(K, N) &= \sum_{j=0}^{m} (-1)^j \binom{m}{j}\binom{m+j}{j}\frac{K^{(j)}}{N^{(j)}} \\ &= \sum_{j=0}^{m} l(m, j)\frac{K^{(j)}}{N^{(j)}} \end{aligned} \tag{7}$$

where

$$l(m, j) \triangleq (-1)^j \binom{m}{j}\binom{m+j}{j},$$

suggests a direct approach for generating the DLOP's. The coefficients $l(m, j)$ which are independent of both K and N, can be computed once and then stored. In fact, the $l(m, j)$ are tabulated in Appendix I for $m = 0, 1, 2, \ldots, 20$.

This seemingly direct approach, however, requires $\sim\frac{1}{2}M^2$ real multiplications and additions plus $\sim M$ real divisions. The same task, however, can be accomplished with considerably less computational effort by using the second-order recurrence relation

$$(m+1)(N-m)P_{m+1}(K, N) = (2m+1)(N-2K)P_m(K, N) - m(N+m+1)P_{m-1}(K, N) \tag{9}$$

for $m = 1, 2, 3, \ldots$ with initial values

$$\left.\begin{aligned} P_0(K, N) &= 1 \\ P_1(K, N) &= (N-2K)/N \end{aligned}\right\} \tag{10}$$

To develop an iteration scheme for the coefficients in (9), let

$$\left.\begin{aligned} c_0(m) &= m(N+m+1) \\ c_1(m) &= (2m+1)(N-2K) \\ c_2(m) &= (m+1)(N-m) \end{aligned}\right\} \tag{30}$$

With these coefficients, the recurrence relation (9) can be rewritten as

$$P_{m+1}(K, N) = \{c_1(m)P_m(K, N) - c_0(m)P_{m-1}(K, N)\}/c_2(m)$$

The coefficients c_0, c_1, and c_2 can be computed recursively via:

$$\left.\begin{aligned} c_0(m+1) - c_0(m) &= (m+1)(N+m+2) - m(N+m+1) \\ &= N + 2(m+1) \\ c_1(m+1) - c_1(m) &= (2m+3)(N-2K) - (2m+1)(N-2K) \\ &= 2(N-2K) \\ c_2(m+1) - c_2(m) &= (m+2)(N-m-1) - (m+1)(N-m) \\ &= N - 2(m+1) \end{aligned}\right\} \tag{31}$$

for $m = 0, 1, 2, \ldots$ with

$$\left.\begin{aligned} c_0(0) &= 0 \\ c_1(0) &= (N-2K) \\ c_2(0) &= N \end{aligned}\right\}$$

Finally, it is convenient to label the right-hand sides of (31) as follows:

$$\left.\begin{aligned} \delta_0(m+1) &= N+2(m+1) \\ \delta_1 &= 2(N-2K) \\ \delta_2(m+1) &= N-2(m+1) \end{aligned}\right\}$$

δ_1 is independent of m. δ_0 and δ_2 obey the recursions

$$\left.\begin{aligned} \delta_0(m+1) &= \delta_0(m)+2 \\ \delta_2(m+1) &= \delta_2(m)-2 \end{aligned}\right\},$$

for $m = 0, 1, 2, \ldots$ with

$$\left.\begin{aligned} \delta_0(0) &= N \\ \delta_2(0) &= N \end{aligned}\right\}$$

Consequently, the computational scheme for generating the first M DLOP's is:

Initialization:

$$\begin{aligned} c_0 &= 0 & \delta_1 &= c_1+c_1 \\ c_1 &= N-K-K & \delta_2 &= N \\ c_2 &= N & P_0 &= 1 \\ \delta_0 &= N & P_1 &= c_1/N \end{aligned}$$

Iteration for $m = 1, 2, \ldots, (M-2)$:

$$\begin{aligned} \delta_0 &= \delta_0+2 \\ \delta_2 &= \delta_2-2 \\ c_0 &= c_0+\delta_0 \\ c_1 &= c_1+\delta_1 \\ c_2 &= c_2+\delta_2 \\ P_{m+1} &= (c_1P_m-c_0P_{m-1})/c_2 \end{aligned}$$

Since K, m, and N are integers, the coefficients δ_0, δ_1, δ_2, and c_0, c_1, c_2 can be calculated in integer mode. The total number of arithmetic operations required to execute this computational scheme is:

$(m-1)$ real divisions

$(2m-4)$ real multiplications

$(m+2)$ real subtractions

$(4m-7)$ integer additions

m integer subtractions

Since this scheme requires $\sim 2M$ real multiplications and $\sim M$ real divisions, both *linear* in M, it provides a considerable improvement over using the explicit expression (7) for generating the DLOP's.

DLOP's ON THE INTERVAL $[\alpha, \beta]$

In the sections 'Definitions' through 'Generation scheme', attention has centred on discrete polynomials that are orthogonal, with unity weight, over the interval $K = 0, 1, 2, \ldots, N$. Attention now turns to delineating a set of polynomials $\{U_m(x)\}$ that are orthogonal, with unity weight, over the uniformly divided interval

$$x = \alpha, \alpha+h, \alpha+2h, \ldots, \alpha+Nh = \beta$$

By representing the independent variable x in the form

$$x = \alpha + Kh,$$

where $h = (\beta-\alpha)/N$, the DLOP's can be converted into the $\{U_m(x)\}$ and *vice versa* through the change of independent variable

$$K = \frac{x-\alpha}{h} \leftrightarrow x = \frac{\alpha(N-K)+\beta K}{N} \tag{32}$$

Specifically,

$$U_m(x) = P_m\left(\frac{x-\alpha}{h}, N\right)$$

and

$$P_m(K, N) = U_m\left[\frac{\alpha(N-K)+\beta K}{N}\right]$$

The properties of these *generalized DLOP's* $\{U_m(x)\}$ can be obtained from the counterpart properties of the DLOP's summarized in the sections 'Definitions'–'Generation scheme' through the change of independent variable (32). From (7), the explicit expression for the generalized DLOP's is

$$U_m(x) = \sum_{j=0}^{m} \left[(-1)^j \binom{m}{j} \binom{m+j}{j}\right] \frac{(x-\alpha)_h^{(j)}}{h^j N^{(j)}}, \tag{33}$$

where $(a)_h^{(j)} = a(a-h)(a-2h)\ldots[a-(j-1)h]$ is the *generalized jth fading factorial* of a. In particular, the first four generalized DLOP's are

$$U_0(x) = 1$$

$$U_1(x) = 1 - 2\frac{(x-\alpha)}{hN}$$

$$U_2(x) = 1 - 6\frac{(x-\alpha)}{hN} + 6\frac{(x-\alpha)(x-\alpha-h)}{h^2 N(N-1)}$$

$$U_3(x) = 1 - 12\frac{(x-\alpha)}{hN} + 30\frac{(x-\alpha)(x-\alpha-h)}{h^2 N(N-1)} - 20\frac{(x-\alpha)(x-\alpha-h)(x-\alpha-2h)}{h^3 N(N-1)(N-2)}$$

Thus, the DLOP's provide a set of orthogonal functions, with unity weight, for any uniformly divided interval $[\alpha, \beta]$.

APPENDIX I

Coefficients of the first twenty-one DLOP's

The purpose of this Appendix is to tabulate the coefficients of the first twenty-one discrete (Legendre) orthogonal polynomials (7). The polynomials may be written in the form

$$P_m(K, N) = \sum_{j=0}^{m} l(m, j)\frac{K^{(j)}}{N^{(j)}},$$

for $0 \leqslant K \leqslant N$, where $K^{(j)} = K(K-1)(K-2)\ldots(K-j+1)$ is the jth fading factorial of K. The coefficients $l(m, j)$, for $j = 0, 1, 2, \ldots, m$, and $m = 0, 1, 2, \ldots, 20$, (which are also the coefficients of the shifted Legendre polynomials on the interval [0, 1]) are listed in Table II.

Table II. Coefficients of the discrete (Legendre) orthogonal polynomials

$l(0,0)$ =	1
$l(1,0)$ =	1
$l(1,1)$ =	−2
$l(2,0)$ =	1
$l(2,1)$ =	−6
$l(2,2)$ =	6
$l(3,0)$ =	1
$l(3,1)$ =	−12
$l(3,2)$ =	30
$l(3,3)$ =	−20
$l(4,0)$ =	1
$l(4,1)$ =	−20
$l(4,2)$ =	90
$l(4,3)$ =	−140
$l(4,4)$ =	70
$l(5,0)$ =	1
$l(5,1)$ =	−30
$l(5,2)$ =	210
$l(5,3)$ =	−560
$l(5,4)$ =	630
$l(5,5)$ =	−252
$l(6,0)$ =	1
$l(6,1)$ =	−42
$l(6,2)$ =	420
$l(6,3)$ =	−1680
$l(6,4)$ =	3150
$l(6,5)$ =	−2772
$l(6,6)$ =	924
$l(7,0)$ =	1
$l(7,1)$ =	−56
$l(7,2)$ =	756
$l(7,3)$ =	−4200
$l(7,4)$ =	11550
$l(7,5)$ =	−16632
$l(7,6)$ =	12012
$l(7,7)$ =	−3432
$l(8,0)$ =	1
$l(8,1)$ =	−72
$l(8,2)$ =	1260
$l(8,3)$ =	−9240
$l(8,4)$ =	34650
$l(8,5)$ =	−72072
$l(8,6)$ =	84084
$l(8,7)$ =	−51480
$l(8,8)$ =	12870
$l(9,0)$ =	1
$l(9,1)$ =	−90
$l(9,2)$ =	1980
$l(9,3)$ =	−18480
$l(9,4)$ =	90090
$l(9,5)$ =	−252252
$l(9,6)$ =	420420
$l(9,7)$ =	−411840
$l(9,8)$ =	218790
$l(9,9)$ =	−48620
$l(10,0)$ =	1
$l(10,1)$ =	−110
$l(10,2)$ =	2970
$l(10,3)$ =	−34320
$l(10,4)$ =	210210
$l(10,5)$ =	−756756
$l(10,6)$ =	1681680
$l(10,7)$ =	−2333760
$l(10,8)$ =	1969110
$l(10,9)$ =	−923780
$l(10,10)$ =	184756
$l(11,0)$ =	1
$l(11,1)$ =	−132
$l(11,2)$ =	4290
$l(11,3)$ =	−60060
$l(11,4)$ =	450450
$l(11,5)$ =	−2018016
$l(11,6)$ =	5717712
$l(11,7)$ =	−10501920
$l(11,8)$ =	12471030
$l(11,9)$ =	−9237800
$l(11,10)$ =	3879876
$l(11,11)$ =	−705432

Table II. Coefficients of the discrete (Legendre) orthogonal polynomials—continued

l(12, 0) =	1
l(12, 1) =	−156
l(12, 2) =	6006
l(12, 3) =	−100100
l(12, 4) =	900900
l(12, 5) =	−4900896
l(12, 6) =	17153136
l(12, 7) =	−39907296
l(12, 8) =	62355150
l(12, 9) =	−64664600
l(12, 10) =	42678636
l(12, 11) =	−16224936
l(12, 12) =	2704156
l(13, 0) =	1
l(13, 1) =	−182
l(13, 2) =	8190
l(13, 3) =	−160160
l(13, 4) =	1701700
l(13, 5) =	−11027016
l(13, 6) =	46558512
l(13, 7) =	−133024320
l(13, 8) =	261891630
l(13, 9) =	−355655300
l(13, 10) =	327202876
l(13, 11) =	−194699232
l(13, 12) =	67603900
l(13, 13) =	−10400600
l(14, 0) =	1
l(14, 1) =	−210
l(14, 2) =	10920
l(14, 3) =	−247520
l(14, 4) =	3063060
l(14, 5) =	−23279256
l(14, 6) =	116396280
l(14, 7) =	−399072960
l(14, 8) =	960269310
l(14, 9) =	−1636014380
l(14, 10) =	1963217256
l(14, 11) =	−1622493600
l(14, 12) =	878850700
l(14, 13) =	−280816200
l(14, 14) =	40116600
l(15, 0) =	1
l(15, 1) =	−240
l(15, 2) =	14280
l(15, 3) =	−371280
l(15, 4) =	5290740
l(15, 5) =	−46558512
l(15, 6) =	271591320
l(15, 7) =	−1097450640
l(15, 8) =	3155170590
l(15, 9) =	−6544057520
l(15, 10) =	9816086280
l(15, 11) =	−10546208400
l(15, 12) =	7909656300
l(15, 13) =	−3931426800
l(15, 14) =	1163381400
l(15, 15) =	−155117520
l(16, 0) =	1
l(16, 1) =	−272
l(16, 2) =	18360
l(16, 3) =	−542640
l(16, 4) =	8817900
l(16, 5) =	−88884432
l(16, 6) =	597500904
l(16, 7) =	−2804596080
l(16, 8) =	9465511770
l(16, 9) =	−23371634000
l(16, 10) =	42536373880
l(16, 11) =	−56949525360
l(16, 12) =	55367594100
l(16, 13) =	−38003792400
l(16, 14) =	17450721000
l(16, 15) =	−4808643120
l(16, 16) =	601080390
l(17, 0) =	1
l(17, 1) =	−306
l(17, 2) =	23256
l(17, 3) =	−775200
l(17, 4) =	14244300
l(17, 5) =	−162954792
l(17, 6) =	1249320072
l(17, 7) =	−6731030592
l(17, 8) =	25293088250
l(17, 9) =	−75957810500
l(17, 10) =	164068870680
l(17, 11) =	−265764451680
l(17, 12) =	321132045780
l(17, 13) =	−285028443000
l(17, 14) =	180324117000
l(17, 15) =	−76938289920
l(17, 16) =	19835652870
l(17, 17) =	−2333606220
l(18, 0) =	1
l(18, 1) =	−342
l(18, 2) =	29070
l(18, 3) =	−1085280
l(18, 4) =	22383900
l(18, 5) =	−288304632
l(18, 6) =	2498640144
l(18, 7) =	−15297796800
l(18, 8) =	68362029450
l(18, 9) =	−227873431500
l(18, 10) =	574241047380
l(18, 11) =	−1101024156960
l(18, 12) =	1605660228900
l(18, 13) =	−1767176346600
l(18, 14) =	1442592936000
l(18, 15) =	−846321139120
l(18, 16) =	337206098790
l(18, 17) =	−81676217700
l(18, 18) =	9075135300
l(19, 0) =	1
l(19, 1) =	−380
l(19, 2) =	35910
l(19, 3) =	−1492260
l(19, 4) =	34321980
l(19, 5) =	−494236512
l(19, 6) =	4805077200
l(19, 7) =	−33145226400
l(19, 8) =	167797708650
l(19, 9) =	−638045608200
l(19, 10) =	1850332263780
l(19, 11) =	−4128840588600
l(19, 12) =	7110781013700
l(19, 13) =	−9424940515200
l(19, 14) =	9521113377600
l(19, 15) =	−7193730107520
l(19, 16) =	3934071152550
l(19, 17) =	−1470171918600
l(19, 18) =	335780006100
l(19, 19) =	−35345263800
l(20, 0) =	1
l(20, 1) =	−420
l(20, 2) =	43890
l(20, 3) =	−2018940
l(20, 4) =	51482970
l(20, 5) =	−823727520
l(20, 6) =	8923714800
l(20, 7) =	−68840085600
l(20, 8) =	391527986850
l(20, 9) =	−1682120239800
l(20, 10) =	5550996791340
l(20, 11) =	−14221562027400
l(20, 12) =	28443124054800
l(20, 13) =	−44431862428800
l(20, 14) =	53952975806400
l(20, 15) =	−50356110752640
l(20, 16) =	35406650372950
l(20, 17) =	−18132120329400
l(20, 18) =	6379820115900
l(20, 19) =	−1378465288200
l(20, 20) =	137846528820

APPENDIX II

Derivation of equations involving differences in N

The purpose of this Appendix is to outline the derivation of equations (11)–(24) involving differences in N. The development begins with the explicit expression for the DLOP's

$$P_m(K,N) = \sum_{l=0}^{m} (-1)^l \binom{m}{l}\binom{m+l}{l}\frac{K^{(l)}}{N^{(l)}} \tag{7}$$

Multiplying both sides of (7) by N yields

$$NP_m(K,N) = \sum_{l=0}^{m} (-1)^l \binom{m}{l}\binom{m+l}{l}\frac{K^{(l)}}{(N-1)^{(l-1)}} \tag{A1}$$

Then multiplying the RHS of (A1) by $1 = (N-l)/(N-l) = [(N-K)+(K-l)]/(N-l)$ and simplifying the result leads to

$$NP_m(K,N) = \sum_{l=0}^{m} \left\{(-1)^l \binom{m}{l}\binom{m+l}{l}\frac{1}{(N-1)^{(l)}}[(N-K)K^{(l)} + K(K-1)^{(l)}]\right\} \tag{A2}$$

Finally, upon applying (7) to rewrite the RHS,

$$NP_m(K,N) = (N-K)P_m(K,N-1) + KP_m(K-1,N-1) \tag{16}$$

Replacing N by $(N+1)$ in (16) and adding $2(N-K+1)$ times the result to the first-order difference (in K) recurrence equation

$$2K(N-K+1)P_m(K-1,N) = -(m+1)(N-m)P_{m+1}(K,N) + [2K(N-K+1) + (m+1)(N-2K-m)]P_m(K,N)$$

leads to

$$2(N+1)(N-K+1)P_m(K,N+1) = -(m+1)(N-m)P_{m+1}(K,N) + (N+m+2)(2N-2K-m+1)P_m(K,N) \tag{14}$$

Upon subtracting the second-order recurrence equation (9) from (14),

$$2(N+1)(N-K+1)P_m(K,N+1) = -(N-m+1)[2(N-K+1)+m]P_m(K,N) + m(N+m+1)P_{m-1}(K,N) \tag{15}$$

Replacing m by $(m-1)$ and N by $(N-1)$ in (14) and N by $(N-1)$ in (15) yields, respectively,

$$2N(N-K)P_{m-1}(K,N) = -m(N-m)P_m(K,N-1) + (N+m)(2N-2K-m)P_{m-1}(K,N-1) \tag{3}$$

and

$$2N(N-K)P_m(K,N) = -(N-m)[2(N-K)+m]P_m(K,N-1) + m(N+m)P_{m-1}(K,N-1) \tag{4}$$

Then subtracting m times (A3) from $(2N-2K-m)$ times (A4), thus eliminating the $P_{m-1}(K,N-1)$ term, and simplifying the result provides

$$2(N-m)(N-K)P_m(K,N-1) = N(2N-2K-m)P_m(K,N) - NmP_{m-1}(K,N) \tag{13}$$

Subtracting N times (9) from $(N+m+1)$ times (13) and simplifying,

$$2(N+m+1)(N-K)P_m(K,N-1) = (m+1)NP_{m+1}(K,N) + (2N-2K+m+1)P_m(K,N) \tag{12}$$

Substituting the symmetry property (8) into (12)–(15) yields

$$2(N+m+1)(N-K)(-1)^m P_m(N-K-1, N-1) = (m+1)N(-1)^{m+1}P_{m+1}(N-K, N) + (2N-2K+m+1)(-1)^m P_m(N-K, N) \quad (A5)$$

$$2(N-m)(N-K)(-1)^m P_m(N-K-1, N-1) = N(2N-2K-m)(-1)^m P_m(N-K, N) \quad (A6)$$

$$2(N+1)(N-K+1)(-1)^m P_m(N-K+1, N+1) = -(m+1)(N-m)(-1)^{m+1}P_{m+1}(N-K, N) + (N+m+2)(2N-2K-m+1)(-1)^m P_m(N-K, N) \quad (A7)$$

$$2(N+1)(N-K+1)(-1)^m P_m(N-K+1, N+1) = (N-m+1)[2(N-K+1)+m] \times (-1)^m P_m(K-K, N) + m(N+m+1)(-1)^{m-1}P_{m-1}(N-K, N) \quad (A8)$$

Multiplying both sides of (A5)–(A8) by $(-1)^m$ and then replacing K by $(N-K)$ leads to the first-order partial difference-recurrence equations

$$2(N+m+1)KP_m(K-1, N-1) = -(m+1)NP_{m+1}(K, N) + (2K+m+1)NP_m(K, N) \quad (21)$$

$$2(N-m)KP_m(K-1, N-1) = (2K-m)NP_m(K, N) + mNP_{m-1}(K, N) \quad (22)$$

$$2(N+1)(K+1)P_m(K+1, N+1) = (m+1)(N-m)P_{m+1}(K, N) + (N+m+2)(2K-m+1)P_m(K, N) \quad (23)$$

$$2(N+1)(K+1)P_m(K+1, N+1) = (N-m+1)(2K+m+2)P_m(K, N) - m(N+m+1)P_{m-1}(K, N) \quad (24)$$

Adding (23) to (14) and replacing K by $(K-1)$ results in

$$KNP_m(K, N) = -(N-K+1)NP_m(K-1, N) + (N-m)(N+m+1)P_m(K-1, N-1) \quad (17)$$

Upon adding $(N-m)(N+m+1)$ times (16) to $(-K)$ times (17) and simplifying,

$$[(N-m)(N+m+1)-K^2]NP_m(K, N) = (N-m)(N+m+1)(N-K)P_m(K, N-1) + (N-K+1)KNP_m(K-1, N) \quad (18)$$

Substituting (8) into both (17) and (18) and then multiplying both sides by $(-1)^m$ yields

$$KNP_m(N-K, N) = -(N-K+1)NP_m(N-K+1, N) + (N-m)(N+m+1)P_m(N-K, N-1) \quad (A9)$$

$$N[(N-m)(N+m+1)-K^2]P_m(N-K, N) = (N-m)(N+m+1)(N-K)P_m(N-K-1, N-1) + (N-K+1)KNP_m(N-K+1, N) \quad (A10)$$

Then replacing K by $(N-K)$ in (A9) and (A10) results in

$$(N-K)NP_m(K, N) = -N(N+1)P_m(K+1, N) + (N-m)(N+m+1)P_m(K, N-1) \quad (19)$$

$$[(N-m)(N+m+1)-(N-K)^2]NP_m(K, N) = N(N+1)(N-K)P_m(K+1, N) + (N-m)(N+m+1)KP_m(K-1, N-1) \quad (20)$$

Finally, to derive the second-order difference (in N) equation (11), add $(N+m+1)$ times (13) to N times (15) and simplify. There results

$$N(N+1)(N-K+1)P_m(K, N+1) - N[(N+1)(2N-2K+1)-m(m+1)]P_m(K, N) + (N-m)(N-m+1)(N-K)P_m(K, N-1) = 0 \quad (11)$$

APPENDIX III

A least-squares curve-fitting example

To illustrate application of the DLOP's, this Appendix is devoted to the problem of approximating a function $f(\cdot)$ by a linear combination of modes $\{\phi_j(\cdot)|j = 0, 1, 2, \ldots\}$. It is assumed that $f(\cdot)$ is observed at $(N+1)$ *uniformly* spaced abscissae.

Without loss of generality, the domain of the independent variable K is assumed to be $K = 0, 1, 2, \ldots, N$. The sequence of $(N+1)$-data points of $f(\cdot)$ is denoted by $\{x(K)|K = 0, 1, 2, \ldots, N\}$. The objective is to find the *least-squares* approximation, $z(K)$, of $x(K)$ by a linear combination of $(M+1)$-modes:

$$x(K) \simeq z(K) = \sum_{j=0}^{M} c_j^{(M)}\phi_j(K) = \mathbf{c}^{\mathrm{T}}\mathbf{\Phi}(K) \tag{A11}$$

In (A11), $\mathbf{c}$ and $\mathbf{\Phi}(K)$ are $(M+1)$-vectors. The number of modes $(M+1)$ is determined by specifying the 'goodness-of-fit' desired. In general, the modal coefficients $c_j^{(M)}$ depend on the order of approximation M.

The modal coefficients are found by minimizing the sum-of-errors-squared

$$\begin{aligned} J(M+1) &= \sum_{K=0}^{N} [x(K)-z(K)]^2 = \sum_{K=0}^{N} [x(K) - \sum_{j=0}^{M} c_j^{(M)}\phi_j(K)]^2 \\ &= \sum_{K=0}^{N} [x(K) - \mathbf{c}^{\mathrm{T}}\mathbf{\Phi}(K)]^2 \end{aligned} \tag{A12}$$

between $x(K)$ and $z(K)$. Setting the partial-derivatives of $J(M+1)$ with respect to the modal coefficients equal to zero yields the regression equations:

$$\sum_{j=0}^{M}\left\{\sum_{K=0}^{N} \phi_i(K)\phi_j(K)\right\}c_j^{(M)} = \sum_{K=0}^{N} x(K)\phi_i(K), \quad \text{for } i = 0, 1, 2, \ldots, M \tag{A13}$$

For convenience, equation (A13), which provides $(M+1)$-equations in the $(M+1)$-modal coefficients $c_j^{(M)}$, is rewritten in the matrix form of equation (A14), shown overleaf.

In (A14), $\mathbf{A} \triangleq \sum_{K=0}^{N}[\mathbf{\Phi}(K)\mathbf{\Phi}^{\mathrm{T}}(K)]$ is an $[(M+1)\times(M+1)]$ symmetric matrix, $\mathbf{b} \triangleq \sum_{K=0}^{N} x(K)\mathbf{\Phi}(K)$, and $\mathbf{c}$ are $(M+1)$-vectors. Gauss elimination, or a similar technique, is commonly employed to solve (A14) for the modal coefficients. This approach requires $\sim(M+1)^3$ arithmetic operations, in addition to the calculation of the $[(M+1)(M+2)/2]$ elements of $\mathbf{A}$ and the $(M+1)$ elements of $\mathbf{b}$. Furthermore, the modal matrix $\mathbf{A}$ is oftentimes ill-conditioned numerically.

Most of this computational effort, however, can be eliminated by employing orthogonal functions (e.g. the DLOP's) as the assumed modes. Thereby, let

$$\phi_j(K) = P_j(K, N), \quad \text{for } j = 0, 1, 2, \ldots, M$$

The *orthogonality relation* of the DLOP's (section 'Properties') indicates that the modal matrix $\mathbf{A}$ is *diagonal* with elements

$$a_{jj} = \sum_{K=0}^{N} P_j^2(K, N) = \frac{1}{(2j+1)}\frac{(N+j+1)^{(j+1)}}{N^{(j)}}, \quad \text{for } j = 0, 1, 2, \ldots, M,$$

$$\underbrace{\begin{bmatrix} \sum_{K=0}^{N} \phi_0^2(K) & \sum_{K=0}^{N} \phi_0(K)\phi_1(K) & \cdots & \sum_{K=0}^{N} \phi_0(K)\phi_M(K) \\ \sum_{K=0}^{N} \phi_1(K)\phi_0(K) & \sum_{K=0}^{N} \phi_1^2(K) & \cdots & \sum_{K=0}^{N} \phi_1(K)\phi_M(K) \\ \vdots & & \ddots \; \sum_{K=0}^{N} \phi_j^2(K) \; \ddots & \vdots \\ \sum_{K=0}^{N} \phi_M(K)\phi_0(K) & \sum_{K=0}^{N} \phi_M(K)\phi_1(K) & \cdots & \sum_{K=0}^{N} \phi_M^2(K) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_0^{(M)} \\ c_1^{(M)} \\ \vdots \\ c_j^{(M)} \\ \vdots \\ c_M^{(M)} \end{bmatrix}}_{\mathbf{c}} = \underbrace{\begin{bmatrix} \sum_{K=0}^{N} x(K)\phi_0(K) \\ \sum_{K=0}^{N} x(K)\phi_1(K) \\ \vdots \\ \sum_{K=0}^{N} x(K)\phi_j(K) \\ \vdots \\ \sum_{K=0}^{N} x(K)\phi_M(K) \end{bmatrix}}_{\mathbf{b}} \tag{A14}$$

$$\mathbf{A} \quad \mathbf{c} = \mathbf{b}$$

which are *independent* of M. Thereby, the modal coefficients c_j, as specified by (A14), are:

$$c_j = (2j+1)\frac{N^{(j)}}{(N+j+1)^{(j+1)}}\sum_{K=0}^{N} x(K)P_j(K, N), \quad \text{for } j = 0, 1, 2, \ldots, M \tag{A15}$$

It is emphasized that the jth coefficient c_j depends solely upon the jth mode and is independent of M. Hence, to improve the approximation (A11) by employing $(M+1)$ instead of M modes requires calculation of c_M alone—c_0 through c_{M-1} remain *unchanged*. This feature is a key advantage in utilizing *orthogonal* modes. Incrementing the order of the approximation in the case of non-orthogonal modes demands the complete solution of the $(M+1)$-regression equations (A14) for *all* of the $(M+1)$-modal coefficients.

A second advantage of utilizing orthogonal modes emerges from evaluating the minimizing value of the sum-of-errors-squared (A12):

$$\begin{aligned} J(M+1) &= \sum_{K=0}^{N} x^2(K) - 2\mathbf{c}^{\mathrm{T}} \sum_{K=0}^{N} x(K)\mathbf{\Phi}(K) + \mathbf{c}^{T}\left\{\sum_{K=0}^{N} \mathbf{\Phi}(K)\mathbf{\Phi}^{\mathrm{T}}(K)\right\}\mathbf{c} \\ &= \sum_{K=0}^{N} x^2(K) - 2\mathbf{c}^{\mathrm{T}}\mathbf{b} + \mathbf{c}^{\mathrm{T}}\mathbf{A}\mathbf{c} \end{aligned} \tag{A16}$$

From (A14), $\mathbf{Ac} = \mathbf{b}$, the minimizing value of $J(M+1)$ is

$$\hat{J}(M+1) = \sum_{K=0}^{N} x^2(K) - \mathbf{c}^{\mathrm{T}}\mathbf{b}, \tag{A17}$$

which leads to the recursion

$$\hat{J}(M+1) = \hat{J}(M) - c_M^{(M)} b_M, \tag{A18}$$

where $\hat{J}(0) \triangleq \sum_{K=0}^{N} x^2(K)$ and $b_M = \sum_{K=0}^{N} x(K)\phi_M(K)$. Even though evaluation of (A13) may become numerically sensitive as M increases, this recursion does prove useful for determining the number of modes $(M+1)$ that are required to insure the goodness of fit desired.

The foregoing discussion is illustrated through the least-squares curve-fitting of

$$x(K) = f(K) + \text{noise}\,(K), \tag{A19}$$

for $K = 0, 1, 2, \ldots, N = 20$. In (A19), $f(K)$ is the fourth-degree polynomial

$$f(K) = 1 - \left(\frac{K}{N}\right) + 8\left(\frac{K}{N}\right)^2 - 7\left(\frac{K}{N}\right)^3 + 9\left(\frac{K}{N}\right)^4,$$

which for $N = 20$ becomes

$$f(K) = 1 - (5\times10^{-2})K + (2\times10^{-2})K^2 - (8{\cdot}75\times10^{-4})K^3 + (5{\cdot}625\times10^{-5})K^4,$$

and noise (K) is, for each K, a uniformly distributed random variable between $\pm 5\times10^{-4}$. The empirical data $x(K)$ and the polynomial $f(K)$ are presented in Table III.

For the DLOP modes, the calculated values of a_{jj}, b_j, c_j, and $\hat{J}(j+1)$ are listed in Table IV for $j = 0, 1, 2, \ldots, 9$. This Table indicates that little improvement in the minimum value of the sum-of-errors-squared is realized for $j > 4$. The corresponding *five* mode approximation of $f(K)$ is, therefore:

$$\begin{aligned} z(K) &= \sum_{j=1}^{5} c_j P_j(K) \\ &= 0{\cdot}99989 - 0{\cdot}99789\left(\frac{K}{N}\right) + 7{\cdot}9889\left(\frac{K}{N}\right)^2 - 6{\cdot}9821\left(\frac{K}{N}\right)^3 + 8{\cdot}9913\left(\frac{K}{N}\right)^4 \\ &= 0{\cdot}99989 - (4{\cdot}9894\times10^{-2})K + (1{\cdot}9972\times10^{-2})K^2 - (8{\cdot}7276\times10^{-4})K^3 + (5{\cdot}6196\times10^{-5})K^4. \end{aligned}$$

Table III. Curve-fitting data

K	Experimental $x(K)$	Exact $f(K)$
0	0·99972000	1·00000000
1	0·96914000	0·96918125
2	0·97422000	0·97390000
3	1·01123000	1·01093125
4	1·07828000	1·07840000
5	1·17541000	1·17578125
6	1·30369000	1·30390000
7	1·46507000	1·46493125
8	1·66208000	1·66240000
9	1·90104000	1·90118125
10	2·18757000	2·18750000
11	2·52859000	2·52893125
12	2·93471000	2·93440000
13	3·41410000	3·41418125
14	3·97967000	3·97990000
15	4·64486000	4·64453125
16	5·42243000	5·42240000
17	6·32939000	6·32918125
18	7·38188000	7·38190000
19	8·59895000	8·59893125
20	10·00033000	10·00000000

To indicate the increased computational effort required and the ill-conditioning that results when not employing orthogonal modes, the least-squares curve-fitting of (A19) is repeated for the non-orthogonal modes $\{(K/N)^j\}$ for $j = 0, 1, 2, \ldots, M$. For this set of modes, the (i, j) element of the modal matrix $\mathbf{A}$ is

$$a_{ij} = \sum_{K=0}^{N} \left(\frac{K}{N}\right)^i \left(\frac{K}{N}\right)^j = \frac{1}{N^{i+j}} \sum_{K=0}^{N} K^{i+j},$$

$$= \begin{cases} N+1, & \text{for } i = j = 0 \\ \dfrac{1}{i+j+1} \displaystyle\sum_{l=0}^{i+j} (-1)^l \binom{i+j+1}{l} B_l N^{1-l}, & \text{for } (i+j) \neq 0, \end{cases}$$

Table IV. DLOP solution data

j	a_{jj}	b_j	c_j	$\hat{J}(j+1)$
0	21·00000	6·99624(1)	3·33154	1·5036(2)
1	7·70000	−3·10299(1)	−4·02985	2·5314(1)
2	5·59263	1·16054(1)	2·07513	1·2310
3	5·32632	−2·50481	−4·70272(−1)	5·3088(−2)
4	6·09219	5·68699(−1)	9·33488(−2)	9·4258(−7)
5	8·09985	−1·25913(−3)	−1·55451(−4)	7·4685(−7)
6	12·33669	−4·32248(−4)	−3·50376(−5)	7·3171(−7)
7	21·38359	−1·01634(−3)	−4·75290(−5)	6·8340(−7)
8	42·08987	1·40778(−3)	3·34471(−5)	6·3631(−7)
9	94·14840	1·64992(−3)	1·75247(−5)	6·0740(−7)

Note: $\hat{J}(0) \triangleq \sum_{K=0}^{N=20} x^2(K) = 3{\cdot}8344(2)$

where B_l is the lth Bernoulli number.

For $M = 5$, the modal matrix is

$$\mathbf{A} = \begin{bmatrix} 21 & 10{\cdot}5 & 7{\cdot}175 & 5{\cdot}5125 & 4{\cdot}516663 \\ 10{\cdot}5 & 7{\cdot}175 & 5{\cdot}5125 & 4{\cdot}516663 & 3{\cdot}854156 \\ 7{\cdot}175 & 5{\cdot}5125 & 4{\cdot}516663 & 3{\cdot}854156 & 3{\cdot}382122 \\ 5{\cdot}5125 & 4{\cdot}516663 & 3{\cdot}854156 & 3{\cdot}382122 & 3{\cdot}029130 \\ 4{\cdot}516663 & 3{\cdot}854156 & 3{\cdot}382122 & 3{\cdot}029130 & 2{\cdot}755497 \end{bmatrix}$$

$$= 21 \begin{bmatrix} 1{\cdot}000000 & 0{\cdot}500000 & 0{\cdot}341667 & 0{\cdot}262500 & 0{\cdot}215079 \\ 0{\cdot}500000 & 0{\cdot}341667 & 0{\cdot}262500 & 0{\cdot}215079 & 0{\cdot}183531 \\ 0{\cdot}341667 & 0{\cdot}262500 & 0{\cdot}215079 & 0{\cdot}183531 & 0{\cdot}161053 \\ 0{\cdot}262500 & 0{\cdot}215079 & 0{\cdot}183531 & 0{\cdot}161053 & 0{\cdot}144244 \\ 0{\cdot}215079 & 0{\cdot}183531 & 0{\cdot}161053 & 0{\cdot}144244 & 0{\cdot}131214 \end{bmatrix}$$

It is observed that $\frac{1}{21}\mathbf{A}$ resembles the ill-conditioned Hilbert matrix, and is itself extremely ill-conditioned. To indicate this ill-conditioning, the determinant of $\frac{1}{21}\mathbf{A}$ is listed in Table V for $M = 0, 1, 2, \ldots, 9$. Thus, the solution of the regression equations (A14) for non-orthogonal modes, especially for large M, can become a delicate numerical procedure.

Table V. Determinant of the modal matrix **A**

M	Determinant ($\frac{1}{21}\mathbf{A}$)
0	1·00
1	9·17(−2)
2	6·12(−4)
3	2·84(−7)
4	8·87(−12)
5	1·82(−17)
6	2·38(−24)
7	1·92(−32)
8	9·14(−42)
9	2·46(−52)

REFERENCES

1. R. T. Birge and J. W. Weinberg, 'Least squares fitting of data by means of polynomials, *Rev. Mod. Phys.* **19**, 298–360 (1947).
2. B. A. Finlayson, *The Method of Weighted Residuals and Variational Principles*, Academic Press. New York, 1972.
3. P. G. Guest, *Numerical Methods of Curve Fitting*, Cambridge University Press, New York, 1961.
4. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.
5. C. Jordan, 'Approximation and graduation according to the principle of least squares by orthogonal polynomials, *Ann. Math. Statist.*, **3**, 257–357 (1932).

6. C. Jordan, *Calculus of Finite Differences*, Rottig and Remwalter, Sopron, Hungary, 1939.
7. S. Karlin and J. McGregor, 'The Hahn polynomials, formulas and an application, *Scripta Math.* **26**, 33–46 (1961).
8. P. A. Lee, 'An integral representation and some summation formulas for the Hahn polynomials', *SIAM J.* **19**, 266–72 (1970).
9. R. J. Levit, 'The zeroes of the Hahn polynomials', *SIAM Rev.*, **9**, 191–203 (1967).
10. A. Ralston, *A First Course in Numerical Analysis*, McGraw-Hill, New York, 1965.
11. D. I. Schonbach, 'Discrete weighted residual methods', *Ph.D. Dissertation*, Dept of Electrical Engng, Carnegie-Mellon Univ., Pittsburgh, Penn. (in preparation).
12. G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, New York, 1959.