Chapter 8

Differential Galois Theory

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1. Introduction

Perhaps the easiest description of differential Galois theory is that it is about algebraic dependence relations between solutions of linear differential equations. To clarify this statement, let us consider three examples. First consider the differential equation

$$(1.1) \qquad z(1-z)y'' + (\frac{1}{2} - \frac{7}{6}z)y' + \frac{11}{3600}y = 0$$

It is not at all obvious from its appearance that the solutions of Eq.(1.1) are algebraic functions over $\mathbb{C}(z)$. That is, any solution of (1.1) satisfies a polynomial equation with coefficients in $\mathbb{C}(z)$ (As always, $\mathbb{C}(z)$ stands for the set of rational functions with complex coefficients). As a second example consider

$$(1.2) \quad (z^4-34z^3+z^2)y'''+(6z^3-153z^2+3z)y''+(z^2-112z+1)y'+(z-5)y=0$$

Again it may come as a surprise that any three independent solutions of (1.2) satisfy a homogeneous quadratic relation with coefficients in $\mathbb{C}$ (Beukers and Peters 1984). The third example is of a different kind. We know that the general solution of $\frac{d}{dz}y + f \cdot y = g$, $f, g \in \mathbb{C}(z)$ reads

$$y = \left(\int g \exp(\textstyle\int f dz) + C\right) \exp(-\textstyle\int f dz).$$

One might wonder whether or not the general solution of a second order linear differential equation can be written in a similar way, i.e. as a function involving only the coefficients of the differential equation, integrations and exponentiations. It will turn out that the answer is in general 'no' (see Theorem 2.4.3). The main point to our present story is that phenomena and questions such as the ones above were at the origin of differential Galois theory. By the end of the 19-th century such questions were studied by many people of whom we mention Halphén, (Fano 1900), (Picard 1898), (Vessiot 1892). One of the tools which slowly emerged was what we now call a differential Galois group. Here we give an informal description of what a differential Galois group is. For the precise definitions we refer to Section 2.1. Consider a linear differential equation

$$p_n(z)\partial^n y + \cdots + p_1(z)\partial y + p_0(z)y = 0,\ p_i(z) \in \mathbb{C}(z), \tag{1.3}$$

where $\partial = d/dz$. Let $y_1, \ldots, y_n$ be a basis of solutions and consider the field F obtained by adjoining to $\mathbb{C}(z)$ all functions $\partial^i y_j$ ($i = 0, \ldots, n-1; j = 1, \ldots, n$). Notice that for any $g = (g_{ij}) \in \mathrm{GL}(n, \mathbb{C})$ (=invertible $n \times n$ matrices) the set of functions $\sum_{j=1}^n g_{ij}y_j (i = 1, \ldots, n)$ is again a basis of solutions. Let $\mathcal{J}$ be the set of all $Q \in \mathbb{C}(z)[X_1^{(0)}, \ldots, X_n^{(0)}, X_1^{(1)}, \ldots, X_n^{(n-1)}]$ such that $Q(\ldots, \partial^j y_i, \ldots) = 0$ (we substituted $\partial^j y_i$ for $X_i^{(j)}$). The group

$$\left\{ (g_{ik}) \in GL(n, \mathbb{C}) |\ Q(\ldots, \partial^j(\sum_{k=1}^n g_{ik}y_k), \ldots) = 0 \text{ for all } Q \in \mathcal{J} \right\}$$

will be called the differential Galois group of (1.1) and we shall denote it by $\mathrm{Gal}_\partial(F/\mathbb{C}(z))$. As we see, it respects all relations over $\mathbb{C}(z)$ which exist between the functions y_i and their derivatives and thus it plays a rôle as a kind of bookkeeping system of algebraic relations. In particular, if the $\partial^j y_i$ ($j = 0, \ldots, n-1; i = 1, \ldots, n$) are algebraically independent over $\mathbb{C}(z)$, i.e. no Q's exist, we have $\mathrm{Gal}_\partial(F/\mathbb{C}(z)) = \mathrm{GL}(n, \mathbb{C})$ and the differential Galois group is maximal. On the other extreme, it will turn out that $\mathrm{Gal}_\partial(F/\mathbb{C}(z))$ is finite if and only if all solutions of (1.1) are algebraic over $\mathbb{C}(z)$. In that case $\mathrm{Gal}_\partial(F/\mathbb{C}(z))$ is actually isomorphic to the ordinary Galois group of the corresponding finite extension of $\mathbb{C}(z)$. Moreover, it turns out that a differential Galois group is a linear algebraic group, the standard example of a Lie group over $\mathbb{C}$ (see Section 3 for more precise definitions). Unfortunately, the study of linear algebraic groups was only at a very primitive stage in the 19-th century and could not be of any assistance. Nevertheless it did become clear that the differential Galois group is an important tool in algebraic dependence questions. Then, at the beginning of the 20-th century the study of these questions became more or less obsolete. It might be interesting to philosophize on the reasons for this silence. What matters for our story is that after preparatory work of (Ritt 1932), E.R.Kolchin published a paper in 1948 which marks the birth of modern differential Galois theory (Kolchin 1948). In this paper, and other papers as well, Kolchin took up the work of the 19-th century mathematicians and addressed questions such as existence and uniqueness of Picard-Vessiot extensions, and stressed the need for an approach which is entirely algebraic. Some years later I. Kaplansky (Kaplansky 1957) wrote a small booklet explaining the basics of the ideas of Ritt and Kolchin. For a quick and very pleasant introduction to differential Galois theory Kaplansky's book was practically the only reference up till now. In the meantime Kolchin had developed and generalized his ideas to a very large extent including systems of partial differential equations. This work culminated in two books (Kolchin 1973), (Kolchin 1985). Unfortunately, these books are very hard to read for a beginner. Remarkably enough, the development of differential Galois theory still remained in the hands of a small group of people until only a few years ago. It was then that Kolchin's ideas attracted attention from other fields of mathematics. One of the fields which could have benefited from differential Galois theory many years ago was transcendental number theory. Siegel (Siegel 1929) discovered that there exists a large class of functions, satisfying linear differential equations, for which one can establish algebraic independence results of their values at certain points provided that the functions themselves are algebraically independent over $\mathbb{C}(z)$. The latter problem could have been approached via the determination of differential Galois groups. However, apart from a few remarks, the first papers in which this connection is made explicitly are (Kolchin 1968) and later (Beukers, Brownawell and Heckman 1988). Very recently, N. M. Katz (Katz 1990) wrote a book on exponential sums and differential Galois theory, in which important parallels are drawn between l-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and differential Galois groups. Another such parallel can be drawn with the Mumford-Tate group of an Abelian variety. More generally, such parallels become clear if one views objects as differential equations, l-adic representations, etc. as examples of Tannakean categories (Deligne 1987). However, in this article we shall not go that far but restrict ourselves to giving a basic introduction to Galois theory of ordinary differential equations with some applications. We shall introduce differential Galois groups and their basic properties in Section 2. We formulate definitions and theorems in a fairly general form using differential fields of characteristic zero and algebraically closed constant field. However, the reader who is not interested in such generalities, is welcome to read $\mathbb{C}(z)$ or $\mathbb{C}((z))$ (that is the field of formal Laurent series in z) any time he or she meets the word differential field. The derivation ∂ then becomes ordinary differentiation and the field of constants is $\mathbb{C}$ in this case. In Section 3 we provide some background on linear algebraic groups and refer to the existing literature for proofs and more general definitions. Finally, in Section 4 we illustrate techniques for the computation of differential Galois groups by computing them for the generalized hypergeometric equation in one variable. The mathematically inclined reader may regret the absence of proofs for the main theorems of differential Galois theory and other theorems as well. However, in this article we have not attempted to be complete, but only to give proofs at those places where we thought they might be instructive. For some easily accessible proofs of the main theorems we refer to (Levelt 1990) in combination with Kaplansky's book. Finally we mention the survey (Singer 1989) and the book (Pommaret 1983). In the latter book the author takes up a theory for partial differential equations and gives some applications to physics, among which a claim for a new approach to gauge field theory.

2. Differential Fields and Their Galois Groups

2.1 Basic notions

The coefficients of our linear differential equation are chosen from a differential field. A *differential field* is a field F equipped with a map $\partial : F \to F$ satisfying the rules $\partial(f+g) = \partial f + \partial g$ and $\partial(fg) = f\partial g + g\partial f$. The map ∂ is called a *derivation* and we use the notation (F, ∂) for the differential field. The examples we shall mainly consider are $(\mathbb{C}(z), d/dz)$ and $(\mathbb{C}((z)), d/dz)$. Here, $\mathbb{C}((z))$ stands for the (not necessarily converging) Laurent series in z, and d/dz denotes ordinary differentiation with respect to z. A trivial and very uninteresting example is when F is any field and $\partial f = 0$ for all $f \in F$. The *constant field* of a differential field (F, ∂) is the subfield consisting of all elements of F whose derivation is zero. Notation: C_F or, if no confusion can arise, just C. In the remainder of this article **we shall assume that the characteristic of C is 0, and that C is algebraically closed.** For example, $C = \mathbb{C}$. Two differential fields (F_1, ∂_1) and (F_2, ∂_2) are said *to be differentially isomorphic* if there exists a field isomorphism $\phi : F_1 \to F_2$ such that $\phi \circ \partial_1 = \partial_2 \circ \phi$. The map ϕ is called a *differential isomorphism.* A differential isomorphism of a field to itself is called a *differential automorphism.* A differential field $(\mathcal{F}, \partial')$ is called a *differential extension* of (F, ∂) if $F \subset \mathcal{F}$ and ∂' restricted to F coincides with ∂. Usually, ∂' is again denoted by ∂, and we shall adopt this habit. For example, $(\mathbb{C}((z)), d/dz)$ is a differential extension of $(\mathbb{C}(z), d/dz)$. Conversely, (F, ∂) is called a *differential subfield* of $(\mathcal{F}, \partial)$. Let $u_1, \ldots, u_r \in \mathcal{F}$. The smallest differential subfield of $(\mathcal{F}, \partial)$ containing F and the elements $u_1, \ldots, u_r$ is denoted by $F < u_1, \ldots, u_r >$. It is actually obtained by adjoining to F the elements $u_1, \ldots, u_r$ together with all their derivatives. Let (F, ∂) be a differential field and consider the linear differential equation

$$(2.1.1) \qquad \mathcal{L}y = 0, \quad \mathcal{L} = \partial^n + f_1\partial^{n-1} + \cdots + f_n, \quad f_i \in F \quad (i = 1, \ldots, n)\,.$$

Usually, the solutions of (2.1.1) do not lie in F. So we look for differential extensions of (F, ∂) containing the solutions.

(2.1.1) Definition. *A differential extension $(\mathcal{F}, \partial)$ of (F, ∂) such that*

i. $C_{\mathcal{F}} = C_F$,

ii. *$\mathcal{F}$ contains n C_F-linear independent solutions $y_1, \ldots, y_n$ of* Eq.(2.1.1) *and* $\mathcal{F} = F < y_1, \ldots, y_n >$,

is called a Picard-Vessiot extension *of* Eq.(2.1.1).

(2.1.2) Theorem. (Kolchin). *Let (F, ∂) be a differential field. Assume, as we do throughout this article, that the characteristic of F is zero and that C_F is algebraically closed. Then to any linear differential equation there exists a Picard-Vessiot extension. Moreover, this extension is unique up to differential isomorphism.*

For a proof we refer to (Kolchin2 1948), (Kolchin 1973) or (Levelt 1990).

(2.1.3) *Remark.* Note the similarity between a Picard-Vessiot extension of a linear differential equation and the splitting field of a polynomial.

(2.1.4) Lemma. *Let (F,∂) be a differential field and $u_1,\dots,u_r \in F$. Then $u_1,\dots,u_r$ are linearly dependent over C_F if and only if the determinant*

$$W(u_1,\dots,u_r)=\begin{vmatrix} u_1 & u_2 & \cdots & u_r \\ \partial u_1 & \partial u_2 & \cdots & \partial u_r \\ \vdots & \vdots & \ddots & \vdots \\ \partial^{r-1}u_1 & \partial^{r-1}u_2 & \cdots & \partial^{r-1}u_r \end{vmatrix}$$

vanishes.

Proof. If $u_1,\dots,u_r$ are linearly dependent over C_F then so are the columns of $W(u_1,\dots,u_r)$, hence this determinant vanishes. Suppose conversely that $W(u_1,\dots,u_r)$ vanishes. By induction on r we shall prove that $u_1,\dots,u_r$ are linearly dependent over C_F. First notice the identity $W(vu_1,\dots,vu_r) = v^rW(u_1,\dots,u_r)$ for any $v \in F$. In particular, if we take $v = u_r^{-1}$ (and assuming $u_r \neq 0$), then

$$\begin{aligned} W(u_1,\dots,u_r)/u_r^r &= W(u_1/u_r,\dots,u_{r-1}/u_r,1) \\ &= (-1)^{r-1}W(\partial(u_1/u_r),\dots,\partial(u_{r-1}/u_r)). \end{aligned}$$

If $r = 1$, then our statement is obvious. Now suppose that $r > 1$. If $u_r = 0$ we have a linear dependence relation. So we assume $u_r \neq 0$. The vanishing of $W(u_1,\dots,u_r)$ then implies the vanishing of $W(\partial(u_1/u_r),\dots,\partial(u_{r-1}/u_r))$. By induction hypothesis there exist $a_1,a_2,\dots,a_{r-1} \in C_F$, not all zero, such that $a_1\partial(u_1/u_r) + \cdots + a_{r-1}\partial(u_{r-1}/u_r) = 0$, hence $a_1u_1 + \cdots + a_{r-1}u_{r-1} = a_ru_r$ for some $a_r \in C_F$, as asserted. □

(2.1.5) *Remark.* The determinant $W(u_1,\dots,u_r)$ is called the *Wronskian determinant* of $u_1,\dots,u_r$. If $y_1,\dots,y_n$ is an independent set of solutions of Eq.(2.1.1) one easily verifies that $\partial W = -f_1W$, where $W = W(y_1,\dots,y_n)$.

(2.1.6) Corollary. *Let $y_1,\dots,y_n$ be a set of solutions of* Eq.(2.1.1), *linearly independent over C_F. Then any solution y of* Eq.(2.1.1) *is a C_F-linear combination of $y_1,\dots,y_n$. In particular, the solutions of* Eq.(2.1.1) *form a linear vector space of dimension n over C_F.*

Proof. Since $y,y_1,\dots,y_n$ all satisfy the same linear differential equation of order n, we have $W(y,y_1,\dots,y_n) = 0$. Lemma 2.1.4 now implies that $y,y_1,\dots,y_n$ are linearly dependent over C_F, and our corollary follows. □

2.2 Galois theory

(2.2.1) Definition. *Let $(\mathcal{F}, \partial)$ be a Picard-Vessiot extension of* Eq.(2.1.1). *The* differential Galois group *of* Eq.(2.1.1) *(or of $\mathcal{F}/F$) is defined as the group of all differential automorphisms $\phi : \mathcal{F} \to \mathcal{F}$ such that $\phi f = f$ for all $f \in F$. Notation:* $\mathrm{Gal}_\partial(\mathcal{F}/F)$.

Let $\phi : \mathcal{F} \to \mathcal{F}$ be an element of $\mathrm{Gal}_\partial(\mathcal{F}/F)$. Let y be any solution of Eq.(2.1.1). Since ϕ fixes F, we have $0 = \phi(\mathcal{L}y) = \mathcal{L}(\phi y)$. Hence ϕy is again a solution of Eq.(2.1.1). In other words, elements of $\mathrm{Gal}_\partial(\mathcal{F}/F)$ act as C_F-linear maps on the n-dimensional vector space V of solutions of Eq.(2.1.1).

Conversely, it follows from the definitions that any linear map in $GL(V)$ which respects all polynomial relations over F between the solutions of equation (2.1.1) and their derivatives, is an element of $\mathrm{Gal}_\partial(\mathcal{F}/F)$. So we find, just like in the introduction, the following statement

(2.2.2) Lemma. *Let notations be as above, and let $\mathcal{J}$ be the set of polynomials $Q \in F[X_1^{(0)}, \ldots, X_n^{(0)}, X_1^{(1)}, \ldots, X_n^{(n-1)}]$ such that $Q(\ldots, \partial^j y_i, \ldots) = 0$ (we have substituted $\partial^j y_i$ for $X_i^{(j)}$). Then*

$$\mathrm{Gal}_\partial(\mathcal{F}/F) = \left\{ g_{ik} \in GL(n, C_F) \mid Q(\ldots, \partial^j(\textstyle\sum_{k=1}^n g_{ik} y_k), \ldots) = 0 \quad \forall Q \in \mathcal{J} \right\}.$$

(2.2.3) *Remark.* The determinant of $\phi \in \mathrm{Gal}_\partial(\mathcal{F}/F)$ can be read off from its action on $W(y_1, \ldots, y_n)$, since $W(\phi y_1, \ldots, \phi y_n) = \det \phi W(y_1, \ldots, y_n)$.

We now state the principal theorems on differential Galois groups. Again, for their proofs we refer to (Kolchin1 1948) or (Kaplansky 1957).

(2.2.4) Theorem. (Kolchin). *The differential Galois group of a Picard-Vessiot extension is a linear algebraic group over the field of constants. Its dimension equals the transcendence degree of the Picard-Vessiot extension.*

The reader who is not familiar with algebraic groups might consult Section 3 for a bare minimum of definitions, examples and results. The more ambitious reader might consult (Humphreys 1972 and 1975) or (Freudenthal and de Vries 1969).

The following result is known as the Galois correspondence for differential fields.

(2.2.5) Theorem. (Kolchin). *Let $(\mathcal{F}, \partial)$ be a Picard-Vessiot extension of (F, ∂) with differential Galois group $G = \mathrm{Gal}_\partial(\mathcal{F}/F)$. Then,*

i. If $f \in \mathcal{F}$ is such that $\phi f = f$ for all $\phi \in G$, then $f \in F$.

ii. *Let H be an algebraic subgroup of G such that $F = \{f \in \mathcal{F} | \phi f = f$ for all $\phi \in H\}$. Then $G = H$.*

iii. *There is a one-to-one correspondence between algebraic subgroups H of G and intermediate differential extensions M of F (i.e. $F \subset M \subset \mathcal{F}$) given by*

$$H = \mathrm{Gal}_\partial(\mathcal{F}/M) \qquad M = \{f \in \mathcal{F} | \phi f = f \ \forall \phi \in H\} .$$

iv. *Under the correspondence given in iii) a normal algebraic subgroup H of G corresponds to a Picard-Vessiot extension M of F and conversely. In such a situation we have $\mathrm{Gal}_\partial(M/F) = G/H$.*

Of particular interest among the subgroups of G is the connected component of the identity, G^0 (see Section 3). Its fixed field is an algebraic extension of F, since G^0 is a subgroup of finite index. A particular case is when G is finite. Then $\dim G$ is zero and, by Theorem 2.2.4, the extension $\mathcal{F}/F$ is algebraic. Clearly, the converse also holds.

2.3 Examples

In two of the examples below, we have considered differential equations over any differential field F. As we said before, the reader who does not like such generalities is welcome to substitute $\mathbb{C}(z)$ for F and $\mathbb{C}$ for C_F, or any other familiar fields.

(2.3.1) Example.

$$\partial y = ay, \qquad a \in F$$

Let $\mathcal{F}$ be the Picard-Vessiot extension and u a non-trivial solution. Clearly, any element ϕ of $G = \mathrm{Gal}_\partial(\mathcal{F}/F)$ acts as $\phi : u \to \lambda u$ for some $\lambda \in C_F^\times$. One easily checks that any algebraic subgroup of $C_F^\times$ is either $C_F^\times$ itself, or a finite cyclic subgroup of order m, say. In the latter case we see that $\phi : u^m \to u^m$ for any $\phi \in G$. Hence $u^m \in F$ and u is algebraic over F.

(2.3.2) Example.

$$\partial y = a \ , \ a \in F \ , \ a \neq 0$$

This is obviously not a homogeneous equation, so instead we consider $a\partial^2 y = (\partial a)(\partial y)$. Letting u be a solution of $\partial y = a$, we easily see that $1, u$ form a basis of solutions of our homogeneous equation. Let $\mathcal{F} = F(u)$ be the Picard-Vessiot extension. Let $\phi \in G = \mathrm{Gal}_\partial(\mathcal{F}/F)$. Then $\phi u = \alpha u + \beta$ for some $\alpha, \beta \in C_F$. Since $\partial(\phi u) = \partial(\alpha u + \beta) = \alpha \partial u = \alpha a$ and $\partial(\phi u) = \phi(\partial u) = \phi a = a$ we see that $\alpha a = a$. Hence $\alpha = 1$. Thus G is a subgroup of the additive group $G_a = \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} | \lambda \in C_F\}$. One easily checks that an algebraic subgroup of G_a is either G_a itself or the trivial group. The latter case corresponds to $u \in F$. Notice in particular, that if $u \notin F$ then G is one-dimensional and so, by Theorem 2.2.4, u is transcendental over F.

(2.3.3) Example.

$$zy'' + \frac{1}{2}y' - \frac{1}{4}y = 0 \quad \text{with } (F, \partial) = (\mathbb{C}(z), \frac{d}{dz})$$

A basis of solutions reads $y_1 = e^{\sqrt{z}}$, $y_2 = e^{-\sqrt{z}}$. Hence the Picard-Vessiot extension is $\mathcal{F} = \mathbb{C}(\sqrt{z}, e^{\sqrt{z}})$. Let $\phi \in \mathrm{Gal}_\partial(\mathcal{F}/F)$. Then $\phi y_1 = \alpha y_1 + \beta y_2$, $\phi y_2 = \gamma y_1 + \delta y_2$ for certain $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha\delta - \beta\gamma \neq 0$. Moreover, $y_1 y_2 = 1$. Hence $\phi(y_1)\phi(y_2) = 1$, which immediately implies $1 = (\alpha y_1 + \beta y_2)(\gamma y_1 + \delta y_2) = \alpha\gamma y_1^2 + (\beta\gamma + \alpha\delta)y_1 y_2 + \beta\delta y_2^2$. Hence $\alpha\gamma = \beta\delta = 0$, $\beta\gamma + \alpha\delta = 1$. An easy computation now shows that

$$G \subset \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix} \middle| \; \lambda \in \mathbb{C}^\times \right\} \tag{2.3.1}$$

Since $e^{\sqrt{z}}$ is transcendental over $\mathbb{C}(z)$, we have dim G=1. Moreover, via Galois correspondence, the sequence of fields $\mathbb{C}(z) \subset \mathbb{C}(\sqrt{z}) \subset \mathcal{F}$ corresponds to the sequence of algebraic groups $G \supset G_1 \supset \{1\}$, where G_1 has index 2 in G. With all this information it is now a simple exercise to show that the inclusion sign in Eq.(2.3.1) is actually an equality sign. The connected component of the identity is precisely the group G_1, which equals $\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^\times\}$.

(2.3.4) Example.

$$y'' + \frac{1}{z}y' + y = 0, \quad \text{with } (F, \partial) = (\mathbb{C}(z), \frac{d}{dz})$$

This is the Bessel-equation of order 0. A basis of solutions is formed by $J_0(z)$ and $Y_0(z) = J_0(z)\log z + f(z)$, where $f(z)$ is some power series in z. Both $J_0(z)$ and $f(z)$ have infinite radius of convergence. Let $\mathcal{F}$ be the Picard-Vessiot extension and G its differential Galois group . One easily verifies that $W(J_0, Y_0) = 1/z \in \mathbb{C}(z)$. Hence $G \subset SL(2, \mathbb{C})$. Secondly, G acts irreducibly on the space of solutions. This can be seen as follows. Suppose G acts reducibly, that is, there exists a solution y such that $\phi : y \to \lambda(\phi)y$ for any $\phi \in G$. This means that y'/y is fixed under G, hence $y'/y \in \mathbb{C}(z)$. This is certainly not possible if y contains $\log z$. Hence we can take $y = J_0(z)$, and $J_0'/J_0 \in \mathbb{C}(z)$. Again, this is impossible since J_0 is known to have infinitely many zeros and J_0'/J_0 would have infinitely many poles. So G acts irreducibly. Thirdly, $J_0(z)$ is transcendental over $\mathbb{C}(z)$ and $Y_0(z)$ is transcendental over $\mathbb{C}(z, J_0(z))$ for the very simple reason that $\log z$ is transcendental over the field of Laurent series in z. Hence the transcendence degree of $\mathcal{F}/\mathbb{C}(z)$ is at least two, implying that dim$G \geq 2$. It is a nice exercise to verify that an algebraic group $G \subset SL(2, \mathbb{C})$, acting irreducibly and of dimension ≥ 2 is actually equal to $SL(2, \mathbb{C})$. An alternative for the third argument is the following consideration. Let z describe a closed loop around the origin. After analytic continuation along this loop, we find that $J_0 \to J_0$ and $Y_0 \to Y_0 + 2\pi i J_0$. Hence G contains the element $\tau = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$. It is again a nice exercise to show that an algebraic group

$G \subset SL(2,\mathbb{C})$, acting irreducibly and which contains τ, is precisely $SL(2,\mathbb{C})$ itself.

2.4 Applications

(2.4.1) Theorem. *Let $\mathcal{F}/F$ be a Picard-Vessiot extension corresponding to Eq.(2.1.1). Let V be its C_F-vector space of solutions and $G = \mathrm{Gal}_\partial(\mathcal{F}/F)$ the differential Galois group . Then the following statements are equivalent:*

i. *There is a non-trivial linear subspace $W \subset V$ which is stable under G.*
ii. *The operator $\mathcal{L}$ factors as $\mathcal{L}_1\mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear differential operators with coefficients in F and order strictly less than n.*

Moreover, $y \in W \Leftrightarrow \mathcal{L}_2 y = 0$.

Proof. ii)$\Rightarrow$i). Let $W \subset V$ be the space of solutions of $\mathcal{L}_2 y = 0$. Since $0 = \phi(\mathcal{L}_2 y) = \mathcal{L}_2(\phi y)$ for any $\phi \in G$, the space W is clearly stable under G. i)$\Rightarrow$ii). Let $v_1, \ldots, v_r$ be a basis of W. Consider the r-th order linear differential equation $\mathcal{L}_2 y = W(v_1, \ldots, v_r, y)/W(v_1, \ldots, v_r)$. Notice that the leading coefficient of $\mathcal{L}_2$ is 1 and the other coefficients are determinants in $v_1, \ldots, v_r$ and their derivatives divided by $W(v_1, \ldots, v_r)$. It is easy to see that the coefficients of $\mathcal{L}_2$ are fixed under G hence, by Theorem 2.2.5(i), they lie in F. Notice that $\mathcal{L}_2 v_i = 0$ for $i = 1, 2, \ldots, r$. We also have $\mathcal{L}v_i = 0$ for $i = 1, 2, \ldots, r$. By division with remainder of differential operators we find $\mathcal{L}_1$ and $\mathcal{L}_3$ such that $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2 + \mathcal{L}_3$, where $\mathcal{L}_3$ has order less than r, However, we have automatically $\mathcal{L}_3 v_i = 0$ for $i = 1, 2, \ldots, r$. Since $v_1, \ldots, v_r$ are linearly independent, this implies $\mathcal{L}_3 = 0$. □

We shall call Eq.(2.1.1) *irreducible* over F if $\mathcal{L}$ does not factor over F. So Theorem 2.4.1 implies that Eq.(2.1.1) is irreducible if and only its differential Galois group acts irreducibly on the space of solutions.

(2.4.2) Corollary. *Let notations be as in the previous theorem. Then G^0, the component of the identity in G, acts irreducibly on V if and only if $\mathcal{L}$ does not factor over any finite extension of F.*

From ordinary Galois theory we know that the zeros of a polynomial P can be determined by repeatedly taking roots if and only if the Galois group of the splitting field of P is solvable. One of the nice applications of differential Galois theory is an analogue of this theorem for differential equations. A differential extension L of F is called a *Liouville extension* if there exists a chain of extensions $F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = L$ such that $F_{i+1} = F_i < u_i >$ $(i = 0, \ldots, r-1)$, where u_i is a solution of an equation of the form $\partial y = a_i y$ or $\partial y = a_i$, $a_i \in F_i$. In other words, Liouville extensions arise by repeatedly solving first order differential equations (or, as some people say, by quadratures). From the next theorem it follows for example, that the Bessel equation of order zero (see Example 2.3.4) cannot be solved by quadratures.

(2.4.3) Theorem. *Let $\mathcal{L}y = 0$ be a linear differential equation over F and $\mathcal{F}/F$ its Picard-Vessiot extension. Then $\mathcal{F}$ is a Liouville extension if and only if* $\mathrm{Gal}_\partial(\mathcal{F}/F)$ *is solvable in the sense of algebraic groups.*

Proof. Suppose L/F is a Liouville extension. Then, via the differential Galois correspondence, the chain $F = F_0 \subset F_1 \subset \cdots \subset F_r = L$ corresponds to the chain of algebraic subgroups $\mathrm{Gal}_\partial(\mathcal{F}/F) = G_0 \supset G_1 \supset \cdots \supset G_n = \{\mathrm{id}\}$, and we have that G_{i+1} is normal in G_i for every $i, 0 \le i < r$ because F_{i+1}/F_i is a Picard-Vessiot extension. According to Examples 2.3.1 and 2.3.2 the differential Galois group of F_{i+1}/F_i is Abelian, hence G_i/G_{i+1} is Abelian. Thus we see that $\mathrm{Gal}_\partial(\mathcal{F}/F)$ is solvable. Now suppose that $G = \mathrm{Gal}_\partial(\mathcal{F}/F)$ is solvable. Then, by the Lie-Kolchin theorem, we can find a basis of solutions $u_1, \ldots, u_n$ of $\mathcal{L}y = 0$ such that G acts on $u_1, \ldots, u_n$ by upper triangular matrices. In other words, to every $g \in G$ there exist $g_{ij} \in C_F$ such that $g : u_i \to g_{ii}u_i + g_{i,i+1}u_{i+1} + \cdots + g_{in}u_n$. In particular, G acts by multiplication with elements from C_F on u_n. Hence, $\partial u_n/u_n$ is fixed under G and thus, $\partial u_n = au_n$ for some $a \in F$. Put $v_i = \partial(u_i/u_n)$ $(i = 1, 2, \ldots, n-1)$. Note that we can recover the u_i from the v_i by integration and multiplication by u_n. Moreover, the group G acts on $v_1, \ldots, v_{n-1}$ again by upper triangular matrices. Hence we can repeat the argument and find that $\partial v_{n-1} = bv_{n-1}$ for some $b \in F$ and G acts by upper triangular matrices on $\partial(v_i/v_{n-1})$ $(i = 1, \ldots, n-2)$. Repeating this argument n times we find the proof of our assertion. □

In what follows we shall often be interested in linear differential equations modulo some equivalence.

(2.4.4) Definition. *Two differential equations $\mathcal{L}_1 y = 0$ and $\mathcal{L}_2 y = 0$ of order n are called* equivalent *over F if there exists a linear differential operator $\mathcal{L}$ such that $\mathcal{L}u_1, \ldots, \mathcal{L}u_n$ is a basis of solutions of $\mathcal{L}_2 y = 0$ whenever $u_1, \ldots, u_n$ is a basis of solutions of $\mathcal{L}_1 y = 0$.*

(2.4.5) *Remark.* It is not hard to show that if such an $\mathcal{L}$ exists, then there exists an inverse differential operator $\mathcal{L}'$ which maps bases of solutions of $\mathcal{L}_2 y = 0$ into bases of solutions of $\mathcal{L}_1 y = 0$. Namely, if we carry out a 'left' version of the Euclidean algorithm to $\mathcal{L}$ and $\mathcal{L}_1$, we can find linear differential operators $\mathcal{L}'$, $\mathcal{L}_3$ over F such that $\mathcal{L}'\mathcal{L} + \mathcal{L}_3\mathcal{L}_1 = 1$. Notice, that for any solution y of $\mathcal{L}_1 y = 0$ we have $\mathcal{L}'\mathcal{L}y = y$. Hence $\mathcal{L}'$ is the desired inverse.

2.5 Monodromy and local Galois groups

In this part we shall make some comments on the case of differential equations over $\mathbb{C}(z)$, which is the most common one in mathematics and physics. For a proper understanding of this Subsection some knowledge of the concepts of monodromy, (ir)regular singularity, local exponents is desirable. Since this is not the place to introduce them, we refer to the introductory books (Poole 1936) (Ince 1926) (Hille 1976). Let $\mathcal{L}y = 0$ be a linear differential equation over $\mathbb{C}(z)$. Let $y_1, \ldots, y_n$ be a basis of solutions around a regular point z_0. They are given by converging power series in $z - z_0$ and can be continued analytically along any path avoiding the singularities of the differential equation. After analytic continuation along a closed loop Γ beginning and ending in z_0, the functions $y_1, \ldots, y_n$ undergo a linear substitution, called the *monodromy substitution* corresponding to Γ. It is obvious that such a monodromy substitution is an element of the differential Galois group of the differential equation. Actually, for Fuchsian equations (having only regular singularities) more is true.

(2.5.1) Theorem. *Let $\mathcal{L}y = 0$ be a Fuchsian equation of order n over $\mathbb{C}(z)$ and $\mathcal{M}$ the group generated by the monodromy substitutions acting on the space of solutions. Let G be the differential Galois group of $\mathcal{L}y = 0$ and let $\overline{\mathcal{M}}$ be the Zariski-closure of $\mathcal{M}$ in $GL(n, \mathbb{C})$. Then $G = \overline{\mathcal{M}}$.*

Proof. First note that $\mathcal{M} \subset G$ and hence $\overline{\mathcal{M}} \subset G$. Thus it suffices to show that the field which is fixed by $\mathcal{M}$ is precisely $\mathbb{C}(z)$. However, this follows already from Riemann. Any function $f(z)$ of z, having trivial monodromy and such that at any $z_0 \in \mathbb{C} \cup \{\infty\}$ there exists $n \in \mathbb{N}$ such that $(z - z_0)^n f(z)$ is bounded near z_0 (denote $z - z_0 = 1/z$ if $z_0 = \infty$) is necessarily rational. The boundedness follows from the fact that we have only regular singularities. □

From the fact that $\mathcal{M} \subset G$ one often obtains elements of G which determine the possibilities for G to a large extent. Consider Example 2.3.4 of the Bessel equation. There we had found the monodromy element τ around $z = 0$. Together with the fact that G acts irreducibly in this case, this already yielded $SL(2, \mathbb{C}) \subset G$. An elegant way to study a differential equation over $\mathbb{C}(z)$ locally at a point z_0 is to consider it as a differential equation over $\mathbb{C}((z - z_0))$. Here, $\mathbb{C}((z - z_0))$ denotes the field of (not necessarily converging) Laurent expansions in $z - z_0$, and we replace $z - z_0$ by $1/z$ if $z_0 = \infty$. Let $\mathcal{F}$ be the Picard-Vessiot extension of $\mathcal{L}y = 0$ considered as a linear differential equation over $\mathbb{C}(z)$, and let $\mathcal{F}'$ be the smallest differential field containing both $\mathbb{C}((z - z_0))$ and $\mathcal{F}$. It is not hard to show that $\mathrm{Gal}_\partial(\mathcal{F}'/\mathbb{C}((z - z_0))) \subset \mathrm{Gal}_\partial(\mathcal{F}/\mathbb{C}(z))$. We note that $\mathcal{F}'$ can only be strictly larger than $\mathbb{C}((z - z_0))$ if z_0 is a singularity of $\mathcal{L}y = 0$. So it makes sense to look only locally at singular points. The nice thing is, that for linear differential equations over $\mathbb{C}((z - z_0))$ there is a complete classification theory, developed by many people of whom we mention Fuchs, Frobenius, (Turrittin 1955), (Levelt 1975) and very recently (Babbitt and Varadarajan 1989). In the following theorem we restrict ourselves to the case $z_0 = 0$, the general case being entirely similar.

(2.5.2) Theorem. *Consider an n-th order linear differential equation with coefficients in $\mathbb{C}((z))$. Then there exist $d \in \mathbb{N}$, a diagonal matrix $P(X)$ with entries in $\mathbb{C}[X]$, a constant matrix A in Jordan normal form and a vector $(f_1, \ldots, f_n)$ with entries in $\mathbb{C}((z))$ such that*

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = e^{P(z^{1/d})} z^A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

yields a basis of solutions $y_1, \ldots, y_n$ of our differential equation.

(2.5.3) *Remark.* The notation z^A stands for the matrix obtained by expanding $z^A = e^{A \log z}$ via the Taylor series for e^x. For example, if $A = \operatorname{diag}(a_1, \ldots, a_n)$ then $z^A = (z^{a_1}, \ldots, z^{a_n})$ and if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $z^A = \begin{pmatrix} 1 & \log z \\ 0 & 1 \end{pmatrix}$.

(2.5.4) *Remark.* If $A = \operatorname{diag}(a_1, \ldots, a_n)$ then the theorem states that there exists a basis of solutions of the form $\exp(P_i(z^{1/d}))z^{a_i} f_i(z)$, $f_i(z) \in \mathbb{C}((z))$ $(i = 1, \ldots, n)$. Moreover, if $z = 0$ is a regular singularity, then all entries $P_i(X)$ of $P(X)$ are identically zero.

(2.5.5) *Remark.* Notice that the Picard-Vessiot extension $\mathcal{F}'$ of our differential equation can be obtained by adjoining to $\mathbb{C}((z))$ the elements $z^{1/d}, \exp(P_i(z^{1/d}))$ and the entries of z^A. A simple consideration shows that $\operatorname{diag}(e^{t_1}, \ldots, e^{t_n})$ is contained in $\mathrm{Gal}_\partial(\mathcal{F}'/\mathbb{C}((z)))$ for any n-tuple $t_1, \ldots, t_n \in \mathbb{C}$ satisfying $\sum_{i=1}^n a_i t_i = 0$ whenever $\sum_{n=1}^n a_i P_i(z^{1/d}) = 0$ with $a_1, \ldots, a_n \in \mathbb{Z}$. This element acts on the solutions y_i via $y_i \to e^{t_i} y_i$ $(i = 1, \ldots, n)$.

As an example take the Bessel equation again (Example 2.3.4). At $z = 0$, a regular singularity, we have two independent solutions, namely the Bessel function $J_0(z)$ and the function $Y_0(z) = \log z J_0(z) + f(z)$, where $f(z)$ is a power series in z with infinite radius of convergence (see (Erdélyi et al. 1953)). Clearly, $\mathrm{Gal}_\partial(\mathcal{F}'/\mathbb{C}((z))) = \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \mid \lambda \in \mathbb{C}\}$. At $z = \infty$, an irregular singularity, we have the solutions $\exp(iz)f(1/z)$ and $\exp(-iz)f(-1/z)$, where $f(t)$ is an asymptotic expansion in t (see (Erdélyi et al. 1953). Clearly, $\mathrm{Gal}_\partial(\mathcal{F}'/\mathbb{C}((z))) = \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^\times\}$. Thus we find that $\mathrm{Gal}_\partial(\mathcal{F}/\mathbb{C}(z))$ contains a unipotent subgroup and a semisimple subgroup. It must be noted that the irreducibility of the action of $\mathrm{Gal}_\partial(\mathcal{F}/\mathbb{C}(z))$ cannot be proved by such local considerations. This is a global property which must be decided in other ways (see Example 2.3.4 and also Section 4). It turns out that in many cases the existence of a special element, found by local considerations, and the fact that $\mathrm{Gal}_\partial(\mathcal{F}/\mathbb{C}(z))^0$ acts irreducibly, found by other means, largely determine the group $\mathrm{Gal}_\partial(\mathcal{F}/\mathbb{C}(z))$. This is the basic principle used in Section 4, where we determine Galois groups of hypergeometric differential equations. In Section 3 the required tools from linear algebraic groups are provided. Finally we mention that instead of considering linear differential equations over $\mathbb{C}((z-z_0))$ we can consider them over the smaller field $\mathbb{C}_{an}((z-z_0))$, the field of locally converging Laurent series in $z-z_0$. In the neighbourhood of regular singularities this does not yield anything new. If z_0 is an irregular singularity however, it makes a big difference. Considerations over $\mathbb{C}_{an}((z-z_0))$ give a finer classification and by studying the so-called *Stokes phenomenon* which then arises and one can sometimes obtain more information on the differential Galois group (see (Martinet and Ramis 1989) or (Duval and Mitschi 1989)). Moreover, Stokes's matrices form an important ingredient in understanding the local to global behaviour of differential Galois groups of equations having irregular singularities.

3. Linear Algebraic Groups

3.1 Definitions and examples

In order to understand and be able to work with differential Galois groups, some knowledge of linear algebraic groups is indispensable. In fact, it is the strong classification theory of linear algebraic groups and the familiarity of their representations which lends its power to the study of linear differential equations. In this Section we collect some definitions and theorems on linear algebraic groups which will be useful in the explicit determination of differential Galois groups. Readers who are interested in a systematic account of linear algebraic groups and their representations should consult (Humphreys 1972 and 1975) or (Springer 1981). In this Section we let k be an algebraically closed field of characteristic zero, for example, $k=\mathbb{C}$.

(3.1.1) Definition. *A* linear algebraic group *over k is a subgroup $G\subset GL(n,k)$ with the property that there exist polynomials $p_r\in k[X_{11},\ldots,X_{nn}]$ $(r=1,\ldots,m)$ in the n^2 variables X_{ij} $(i,j=1,\ldots,n)$ such that $G=\{(g_{ij})\in GL(n,k)\mid p_r(g_{11},\ldots,g_{nn})=0$ for $r=1,\ldots,m\}$.*

Examples:

i. The special linear group $SL(n,k)=\{g\in GL(n,k)\mid \det g=1\}$.
ii. The orthogonal group $O(n,k)=\{g\in GL(n,k)\mid {}^tgg=Id\}$ where ${}^tg$ denotes the transpose of g.
iii. The special orthogonal group $SO(n,k)=O(n,k)\cap SL(n,k)$.
iv. The symplectic group $Sp(2n,k)=\{g\in SL(2n,k)\mid {}^tgJg=J\}$ in even dimension, where J is any non degenerate anti-symmetric $2n\times 2n$ matrix.
v. The exceptional group $G_2=\{g\in SL(7,k)\mid F(gx,gy,gz)=F(x,y,z)\}$ where F is a sufficiently general antisymmetric trilinear form.
vi. The group of upper triangular matrices $U=\{g\in GL(n,k)\mid g_{ij}=0$ for all $i<j\}$.

vii. The unipotent group $U_1 = \{g \in U(n,k) \mid g_{ii} = 1 \text{ for all } i\}$. In fact, any algebraic subgroup of U_1 is called unipotent.

Notice that the groups mentioned above are actually Lie groups over k. This follows very quickly from the definition of linear algebraic groups. Of course, over $\mathbb{R}$ for example, there exist other Lie groups such as $SU(n,\mathbb{C})$. It should be emphasized here that this Lie group cannot be realized as the set of complex solutions of polynomial equations, since the definition of $SU(n,\mathbb{C})$ involves complex conjugation, which is not an algebraic operation over $\mathbb{C}$. It follows from the definition that a linear algebraic group is an algebraic variety. It may or may not be irreducible. To avoid confusion with 'irreducible' in the sense of representation theory we shall reserve, by slight abuse of terminology, the word '*connected*' for 'irreducible' in the sense of algebraic geometry. The connected component of our algebraic group which contains the identity element will be called the *component of the identity* and is denoted by G^0. The group G^0 is a normal algebraic subgroup of G of finite index and the connected components of G are precisely the cosets with respect to G^0. Moreover, any algebraic subgroup of G of finite index automatically contains G^0. An example is $SO(n,k)$ which is the component of the identity in $O(n,k)$. The coset decomposition consists of the determinant 1 and -1 matrices. Another example occurs in Example 2.3.3.

(3.1.2) Definition. *An m-dimensional* rational representation *of a linear algebraic group $G \subset GL(n,k)$ is a homomorphism $\rho : G \to GL(m,k)$ with the property that there exist polynomials $\rho_{ij} \in k[X_{11},\ldots,X_{nn}]$ $(i,j = 1,\ldots,m)$ and an $r \in \mathbb{Z}$ such that $(\rho(g))_{ij} = \det(g)^r \rho_{ij}(g_{11},\ldots,g_{nn})$. In other words, up to a possible common factor of the form $\det(g)^r$ $(r \in \mathbb{Z})$, the homomorphism ρ is given by polynomials.*

Notice that if $G \subset SL(n,k)$, any rational representation of G is given by polynomials, since $\det(g) = 1$ for all $g \in G$. We shall assume that the reader is familiar with elementary concepts such as irreducible representations, equivalence of representations, invariant subspace, etc. Let $\rho : G \to GL(m,k)$ be a rational representation, which may even be the standard inclusion $G \subset GL(n,k)$. Then we have the dual representation $\rho^d : g \to {}^t\rho(g^{-1})$, again in dimension m. Let $\rho' : G \to GL(m',k)$ be another rational representation. Then the direct sum representation $\rho \oplus \rho'$ is the $m+m'$ dimensional representation obtained by simply writing

$$(\rho \oplus \rho')(g) = \begin{pmatrix} \rho(g) & \\ & \rho'(g) \end{pmatrix}.$$

The mm' dimensional tensor representation $\rho \otimes \rho'$ is obtained as follows. Replace each entry $\rho(g)_{ij}$ $(i,j = 1,\ldots,m)$ in $\rho(g)$ by the $m' \times m'$ matrix $\rho(g)_{ij} \times \rho'(g)$. This yields an $mm' \times mm'$ matrix which we call $(\rho \otimes \rho')(g)$. One easily checks that this yields a rational representation. By repeating these constructions and taking subrepresentations of them, we obtain a wealth of new representations. For example, the r-th symmetric product and the r-th exterior power of a representation ρ are subrepresentations of $\rho \otimes \rho \otimes \cdots \otimes \rho$ (r times). The adjoint representation of G is a subrepresentation of $i \otimes i^d$, where $i : G \rightarrow GL(n,k)$ is a faithful representation. In this paper our main interest will be in reductive groups. Usually they are defined as algebraic groups whose unipotent radical is trivial. Here we prefer to use a definition which is more practical for our purposes.

(3.1.3) Definition. *Let $G \subset GL(n,k)$ be an algebraic group and $\rho : G \rightarrow GL(m,k)$ a faithful rational representation. We call G* reductive *if ρ is completely reducible, i.e. $k^m = V_1 \oplus \cdots \oplus V_r$, where the V_i are irreducible $\rho(G)$-invariant subspaces.*

That this definition does not depend on ρ is shown by the following theorem.

(3.1.4) Theorem. *Any rational representation of a reductive group is completely reducible.*

A proof that our definition is equivalent to the usual definition, can be found in (Beukers, Brownawell and Heckman 1988, Appendix). It relies heavily on the Lie-Kolchin theorem. It is clear that Theorem 3.1.4 greatly simplifies the study of representations of reductive groups. A reductive group is called *semi-simple* if its centre is finite. In particular, a reductive group $G \subset SL(n,k)$ which acts irreducibly on k^n is automatically semi-simple since, by Schur's lemma, elements of the centre of G are scalar, of which there exist only finitely many in $SL(n,k)$.

3.2 Theorems

The following statements give us very easy criteria to recognize algebraic groups from the occurrence of certain typical elements. They are taken from (Katz 1990), where the Lie algebra versions are given. In all theorems of this Subsection we let $G \subset GL(V)$ be a reductive, connected algebraic group acting irreducibly on the finite dimensional vector space V. The notation $C \cdot G$ stands for the group obtained by taking all products of elements of some suitable scalar group C with elements from G. The notation $\mathrm{diag}(d_1, \ldots, d_n)$ will stand for the $n \times n$ matrix having the elements $d_1, \ldots, d_n$ on the diagonal and zeros at all off-diagonal places.

(3.2.1) Theorem. (O. Gabber) *Let $D \subset GL(V)$ be a group of diagonal matrices such that $dGd^{-1} = G$ for all $d \in D$. Consider the diagonal group T consisting of all* $\mathrm{diag}(t_1, \ldots, t_n) \in GL(V)$ *such that $t_i t_j = t_k t_l$ whenever $d_i d_j = d_k d_l$ for all* $\mathrm{diag}(d_1, \ldots, d_n) \in D$. *Then $T^0 \subset C \cdot G$.*

The proof of this theorem is based on the fact that derivations of semi-simple Lie algebras are inner. The following three theorems rely heavily on the classification theory of semi-simple Lie algebras. Theorem 3.2.2 is a collection of results found by Gabber, Kostant, Zarhin, Kazhdan-Margulis, Beukers and Heckman and undoubtedly many other people, while Theorems 3.2.3 and 3.2.4 were found by O. Gabber with the purpose of determining differential Galois groups.

(3.2.2) Theorem. *Let $D \subset GL(V)$ be such that $dGd^{-1} = G$, $\forall d \in D$. Then*

i. *if D contains $\mathrm{diag}(\lambda, 1, \dots, 1)$ with $\lambda \neq \pm 1$, then $G = C \cdot SL(V)$ (i.e. $G = SL(V)$ or $G = GL(V)$).*

ii. *if D contains $\mathrm{diag}(-1, 1, \dots, 1)$ then $G = C \cdot SL(V)$ or $G = C \cdot SO(V)$.*

iii. *if D contains an element with 1 on all diagonal places and 0 at all other places with precisely one exception, then $G = C \cdot SL(V)$ or $G = C \cdot Sp(V)$.*

iv. *if $D = \{\mathrm{diag}(\lambda, \lambda^{-1}, 1, \dots, 1) \mid \lambda \neq 0\}$ then $G = C \cdot SL(V)$ or $G = C \cdot Sp(V)$ or $G = C \cdot SO(V)$.*

(3.2.3) Theorem. (O. Gabber) *Let $D \subset GL(V)$ be such that $dGd^{-1} = G$, $\forall d \in D$. Then*

i. *if $D = \{\mathrm{diag}(\lambda, \lambda, \lambda^{-1}, \lambda^{-1}, 1, \dots, 1) \mid \lambda \neq 0\}$, then we have the following possibilities:*

$G = C \cdot SL(V)$ or $C \cdot SO(V)$ or $C \cdot Sp(V)$.
$G = C \cdot (SL(2) \times (SL(k)$ or $SO(k)$ or $Sp(k)))$ with $k = \dim V - 2$ and in standard tensor representation.
$G = C \cdot G_2$ in 7-dimensional standard representation.
$G = C \cdot SO(7)$ in 8-dimensional spin representation.
$G = C \cdot SL(3)$ in 8-dimensional adjoint representation.
$G = C \cdot (SL(3) \times SL(3))$ in 9-dimensional standard tensor representation.

ii. *if $D = \{\mathrm{diag}(\lambda, \mu, \lambda\mu, \lambda^{-1}, \mu^{-1}, (\lambda\mu)^{-1}, 1, \dots, 1)\}$, then we have the same possibilities as in the previous case, except that in the second possibility only $k = 4$ is allowed.*

(3.2.4) Theorem. (O. Gabber) *If $\dim V$ is a prime p, then G has the following possibilities, $C \cdot SL(V)$, $C \cdot SO(V)$, $C \cdot G_2$ (only when $p = 7$), $C \cdot \mathrm{symm}^{p-1}(SL(2, k))$, where symm^{p-1} stands for the $(p-1)$-the symmetric power of the standard representation of $SL(n, k)$.*

Notice that all the above criteria require that G be a connected group acting irreducibly on V. Without either of these requirements the theorems are false. When determining the differential Galois group of a given linear differential equation in practice, it is usually not too hard to decide that the differential Galois group acts irreducibly (see Theorem 2.4.1). However, connectedness is harder to check, and very often G is not even connected. So one usually studies G^0, the component of the identity. Now the problem is to check that G^0 acts irreducibly on V. In some cases, like the hypergeometric equation (see Section 4) this is doable, but in other cases it may be a very tedious job. The following theorem partly avoids these problems at the cost of determination of irreducibility of higher order differential equations.

(3.2.5) Theorem. *Let $G \subset GL(V)$ be a linear algebraic group such that G modulo its scalars is an infinite group. Then the following statements are equivalent,*

i. *G acts irreducibly on the symmetric square S^2V of V.*
ii. *$G = C \cdot SL(V)$ or $G = C \cdot Sp(V)$.*

A proof of this theorem can be found in (Beukers, Brownawell and Heckman 1988, Appendix).

4. Hypergeometric Differential Equations

4.1 Introductory remarks

Let $p, q \in \mathbb{N}$, $q \geq p$ and $\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q \in \mathbb{C}$. For the moment, take $\beta_q = 1$ and $\beta_i \notin \mathbb{Z}_{\leq 0}$ $(i = 1, \ldots, q)$. The generalized hypergeometric function in one variable z is defined by

$$(4.1.1) \qquad {}_pF_{q-1}\left(\begin{matrix}\alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_{q-1}\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} z^n$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ is the *Pochhammer symbol.* The adjective 'generalized' refers to the fact that it is a generalization of the classical case $q = 2$, which was already studied by Euler, Gauss and Riemann. See (Erdélyi et al. 1953), (Klein 1933) or (Gray 1986). Note, that if $\alpha_i \in \mathbb{Z}_{\leq 0}$ for some i, then ${}_pF_{q-1}$ is a polynomial. When $q > p$ the radius of convergence of (4.1.1) is infinite. When $p = q$ and ${}_pF_{q-1}$ is not a polynomial, the radius of convergence of (4.1.1) is one. When specializing α_i, β_j we obtain a large number of familiar functions which occur throughout mathematics and physics, particularly in the case when $q = 2$, $p \leq q$. We give some examples:

i.
$$ {}_0F_1\left(\begin{matrix}\cdot \\ 1\end{matrix}\middle| z^2\right) = J_0(2iz),$$

where J_0 is the Bessel function of order zero.

ii.
$$ {}_2F_1\left(\begin{matrix}1 & 1 \\ & 2\end{matrix}\middle| z\right) = -\frac{\log(1-z)}{z}\,.$$

iii.
$$ {}_2F_1\left(\begin{matrix}1 & \alpha \\ & 1\end{matrix}\middle| z\right) = (1-z)^{-\alpha}\,. $$

iv.
$$ {}_2F_1\left(\begin{matrix}-n & n+1 \\ & 1\end{matrix}\middle| \frac{1-x}{2}\right) = P_n(x), $$

where P_n is the Legendre polynomial of degree n.

v.
$$ {}_1F_1\left(\begin{matrix}-m \\ 1/2\end{matrix}\middle| x^2\right) = (-1)^m \frac{m!}{(2m)!} H_{2m}(x), $$

where $H_{2m}(x)$ is the Hermite polynomial of degree $2m$.

Let now α_i, β_j $(i = 1, \ldots, p\,; j = 1, \ldots, q)$ be arbitrary complex numbers. Throughout this Section we write $\partial = z(d/dz)$. Consider the following differential equation of order q,

$$ \prod_{j=1}^{q} (\partial + \beta_j - 1)F = z \prod_{i=1}^{p} (\partial + \alpha_i)F \tag{4.1.2} $$

which we call the *hypergeometric differential equation* . Let $f(z)$ be any solution of the form

$$ z^\lambda \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 = 1 \tag{4.1.3} $$

Substitution of (4.1.3) into (4.1.2) yields $\lambda = 1 - \beta_i$ for some i and the recursion relation

$$ a_{n+1} = \frac{\prod_{i=1}^{p}(n+\lambda+\alpha_i)}{\prod_{j=1}^{q}(n+\lambda+\beta_j)} a_n $$

We thus see that solution (4.1.3) is in fact

$$ z^{1-\beta_i}{}_pF_{q-1}\left(\begin{matrix}1+\alpha_1-\beta_i & \cdots & 1+\alpha_p-\beta_i \\ 1+\beta_1-\beta_i & \overset{\vee}{\cdots} & 1+\beta_q-\beta_i\end{matrix}\middle| z\right) \tag{4.1.4} $$

where $^\vee$ denotes suppression of the term $1 + \beta_i - \beta_i$. In particular, when $\beta_q = 1$ we see that

$$ {}_pF_{q-1}\left(\begin{matrix}\alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_{q-1}\end{matrix}\middle| z\right) $$

is a solution. Whenever $\beta_j - \beta_i \in \mathbb{Z}$ for some i, j, the function (4.1.4) may not be well defined since the coefficients may become infinite. In that case the functions (4.1.4) do not give a basis of solutions of (4.1.2) and it turns out, that solutions containing $\log z$ also show up. If, on the other hand, $\beta_i - \beta_j \notin \mathbb{Z}$ for all i, j, then the functions (4.1.4) do form a basis of solutions of (4.1.2). There are some crucial differences between the cases $p = q$ and $p < q$. As remarked already, the radius of convergence of the hypergeometric function with $p = q$ is generally 1, and with $p < q$ it is infinite. A reason for this shows up if we write out (4.1.2) explicitly. In the case $p = q$ the equation has three singularities $0, 1, \infty$ and it is a Fuchsian equation. A solution around $z = 0$ usually does not converge beyond 1 and so its radius of convergence is 1. In case $p < q$ there are only the singularities $0, \infty$ and ∞ is an irregular singularity. The monodromy in this case is not very interesting since it is the image of the fundamental group $\pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$, and thus cyclic. The differential Galois group will carry much more information in this case. When $p = q$ however, our equation is Fuchsian and according to Theorem 2.5.1 the monodromy group determines the differential Galois group . We also have the following theorem.

(4.1.1) Theorem. (Pochhammer) *Suppose $p = q$. Then there exist $p - 1$ linearly independent holomorphic solutions of* Eq.(4.1.2) *in a neighbourhood of $z = 1$.*

A proof can be found in (Beukers and Heckman 1989). In fact, Fuchsian hypergeometric equations are characterized by this property.

(4.1.2) Theorem. *Suppose we are given a Fuchsian differential equation over $\mathbb{C}(z)$ of order n which has the singularities $0, 1, \infty$ and no others. If this equation admits $n - 1$ linearly independent holomorphic solutions in a neighbourhood of $z = 1$ then it is equivalent over $\mathbb{C}(z)$ to an equation of the form Eq.(4.1.2) with $p = q = n$.*

A consequence of all the above is, that in the case $p = q$ the local monodromy matrix aroud $z = 1$ has $n - 1$ eigenvalues 1 with corresponding eigenvectors. Such a matrix is called a *pseudoreflection* (if the n-th eigenvalue would be -1 we would have had a true reflection). Its Jordan normal form is either $\text{diag}(\lambda, 1, \ldots, 1)$ with $\lambda \neq 1$ or a matrix with 1 on the diagonal places, a non zero element at the place $1, 2$ and zeros everywhere else. The fact that the local monodromy around $z = 1$ is generated by a pseudoreflection is crucial in the determination of the differential Galois group in case $p = q$. The rest of this Section will be devoted to a sketch of a systematic treatment of the differential Galois group of a hypergeometric differential equation .

4.2 Reducibility and imprimitivity

Recall (Subsection 2.4) that a differential equation is called irreducible over $\mathbb{C}(z)$ if its corresponding differential operator does not factor over $\mathbb{C}(z)$. If $\mathcal{F}/\mathbb{C}(z)$ is the corresponding Picard-Vessiot extension, then it follows from Theorem 2.4.1 that a differential equation is irreducible over $\mathbb{C}(z)$ if and only if its differential Galois group $\text{Gal}_\partial(\mathcal{F}/\mathbb{C}(z))$ acts irreducibly on the space of solutions.

(4.2.1) Lemma. Eq.(4.1.2) *is irreducible over $\mathbb{C}(z)$ if and only if $\alpha_i \neq \beta_j \pmod{\mathbb{Z}}$ for all i, j.*

Proof. Here we shall prove that $\alpha_i \neq \beta_j \pmod{\mathbb{Z}}$ for all i, j implies irreducibility. An elementary proof of the converse can be found in (Katz 1990) or (Beukers and Heckman 1989). Suppose Eq.(4.1.2) has a non-trivial factorization $\mathcal{L}_1\mathcal{L}_2 y = 0$. Usual theory shows that $\mathcal{L}_2 y = 0$ has a solution of the form $z^\lambda \sum_{k=0}^{\infty} a_k z^k$. It is automatically a solution of Eq.(4.1.2) and the arguments from Subsection 4.1 show that it is of the form (4.1.4). This implies the existence of a hypergeometric function with parameters, say, $\mu_1, \ldots, \mu_p; \nu_1, \ldots, \nu_q$ $\nu_q = 1$ and $\mu_i - \nu_j \neq \mathbb{Z}$ $\forall i, j$ which satisfies a linear differential equation of order $< q$. Writing a_n for the coefficients of the power series of this function, we deduce such a differential equation from the existence of a non trivial recurrence for the a_n of the form

(4.2.1)
$$A_k(n)a_{n+k} + A_{k-1}(n)a_{n+k-1} + \cdots + A_1(n)a_{n+1} + A_0(n)a_n = 0, \quad \forall n \geq 0,$$

where the $A_i(n)$ are polynomials in n of degree $< q$ and $a_n \neq 0$ $\forall n \geq 0$. Since $a_n = (\mu_1)_n \cdots (\mu_p)_n/(\nu_1)_n \cdots (\nu_q)_n$ for all n, the quotients a_{n+i}/a_n are rational functions in n and Eq.(4.2.1) is equivalent to

$$A_k(x)\frac{\prod_{i=1}^{p}(\mu_i + x)\cdots(\mu_i + x + k - 1)}{\prod_{j=1}^{q}(\nu_j + x)\cdots(\nu_j + x + k - 1)} + \cdots + A_0(x) = 0$$

Here, x is some arbitrary variable replacing n. Since $\mu_i - \nu_j \notin \mathbb{Z}$ for all i, j and $\deg A_k < q$, the left most term has for some j a pole of the form $x = 1 - k - \nu_j$ which none of the other terms possesses. Hence relation (4.2.1) cannot exist, which implies our assertion. □

For the following lemma, recall Definition 2.4.4 on the equivalence of linear differential equations .

(4.2.2) Lemma. *Let $\mathcal{H}$ and $\mathcal{H}'$ be two irreducible (over $\mathbb{C}(z)$) hypergeometric equations with parameters $\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q$ and $\alpha'_1, \ldots, \alpha'_{p'}; \beta'_1, \ldots, \beta'_{q'}$ respectively. Then $\mathcal{H}$ and $\mathcal{H}'$ are equivalent over $\mathbb{C}(z)$ if and only if $p = p'$, $q = q'$ and, after renumbering if necessary, $\alpha_i \equiv \alpha'_i (\mathrm{mod}\ \mathbb{Z})$ $\beta_j \equiv \beta'_j (\mathrm{mod}\ \mathbb{Z})$ for $i = 1, \ldots, p; j = 1, \ldots, q$.*

Proof. We shall only show that $\alpha_i \equiv \alpha'_i (\mathrm{mod}\ \mathbb{Z})$, $\beta_j \equiv \beta'_j (\mathrm{mod}\ \mathbb{Z})$ for all i, j implies the equivalence. Let $V(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$ be the solution space of $\mathcal{H}$. Notice that $(\partial + \alpha_k)f \in V(\alpha_1, \ldots, \alpha_k + 1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$ for every $f \in V(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$. Moreover, $\partial + \alpha_k$ is a $\mathbb{C}$-linear map. Its kernel, which is nothing but the one dimensional space spanned by $z^{-\alpha_k}$, is contained in $V(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$ if and only if $\alpha_k = \beta_l - 1$ for some l. Hence, under our assumptions, the systems with parameters $\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q$ and $\alpha_1, \ldots, \alpha_k + 1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q$ are equivalent. An isomorphism of the solution spaces is given by the differential map $\partial + \alpha_k$. Similarly, the map $\partial + \beta_k - 1$ maps $V(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$ bijectively onto $V(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_k - 1, \ldots, \beta_q)$. So, by using the operators $\partial + \alpha_k$, $\partial + \beta_k - 1$ and their inverses one can shift the parameters α_i, β_j freely by integers. □

An important consequence of Lemma 4.2.2 is that the differential Galois group in the irreducible case depends only on the parameters mod $\mathbb{Z}$. From now on we shall assume that our hypergeometric differential equation is irreducible. We now go over to imprimitivity. Let V be a vector space and G a group acting on V in an irreducible way. We shall say that G is *imprimitive* on V if there exists a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_m$ $(m > 1)$ such that G permutes the V_i. If such a decomposition does not exist we call G *primitive* on V. Let V be the vector space of solutions of Eq.(4.1.2) and G its differential Galois group . We have already assumed that G acts irreducibly on V. There are two more or less obvious ways in which G is imprimitive on V. They are known as Kummer induction and (inverse) Belyi induction, using Katz' terminology. Kummer induction arises as follows. Consider Eq.(4.1.2) and replace z by $\zeta z^{1/d}/d^{p-q}$, where d is a natural number and ζ any d-th root of unity. We obtain

$$\mathcal{A}F = \zeta z^{1/d}\mathcal{B}F \tag{4.2.2}$$

where

$$\mathcal{A} = \prod_{i=1}^{q}(\partial + \frac{\beta_i - 1}{d}), \quad \mathcal{B} = \prod_{j=1}^{p}(\partial + \frac{\alpha_j}{d}).$$

One easily verifies the following operator equation,

$$z^{\frac{d-1}{d}}\left\{\sum_{t=0}^{d-1}(\mathcal{A}z^{-\frac{1}{d}})^{d-t-1}z^{-\frac{1}{d}}(z^{\frac{1}{d}}\mathcal{B})^t\right\}(\mathcal{A} - z^{\frac{1}{d}}\mathcal{B}) = z(z^{-\frac{1}{d}}\mathcal{A})^d - (z^{\frac{1}{d}}\mathcal{B})^d$$

Notice that

$$z(z^{-\frac{1}{d}}\mathcal{A})^d = \prod_{k=0}^{d-1}\prod_{i=1}^{q}\left(\partial + \frac{\beta_i - 1}{d} - \frac{k}{d}\right)$$

$$(z^{\frac{1}{d}}\mathcal{B})^d = z\prod_{k=0}^{d-1}\prod_{j=1}^{p}\left(\partial + \frac{\alpha_j}{d} + \frac{k}{d}\right)$$

Hence, Eq.(4.2.2) is a factor of the hypergeometric differential equation $\mathcal{H}$ with parameters

$$\frac{\alpha_1}{d}, \frac{\alpha_1 + 1}{d}, \ldots, \frac{\alpha_1 + d - 1}{d}, \frac{\alpha_2}{d}, \ldots, \frac{\alpha_p + d - 1}{d}; \frac{\beta_1}{d}, \frac{\beta_1 + 1}{d}, \ldots, \frac{\beta_q + d - 1}{d}.$$

More precisely, the direct sum of the solution spaces of Eq.(4.2.1) taken over all d-th roots of unity ζ, is precisely the solution space of $\mathcal{H}$. In particular, if $f_1(z), \ldots, f_q(z)$ is a basis of solutions of Eq.(4.2.1), then $f_i(\zeta z^{1/d}/d^{p-q})$ $(i = 1, \ldots, q, \quad \zeta^d = 1)$ is a basis of solutions of $\mathcal{H}$. Moreover, G, the differential Galois group of $\mathcal{H}$ permutes the solution spaces of Eq.(4.2.1) for different ζ, since G acts on $z^{1/d}$ by multiplication with d-th roots of unity. So, G is imprimitive. Notice that the parameter set of $\mathcal{H}$ has the property that modulo $\mathbb{Z}$ this set does not change if we add $1/d$ to all parameters.

(4.2.3) Definition. *A hypergeometric differential equation with the parameters* $\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q$ *is called* Kummer induced *if it is irreducible and if there exists a number* $d \in \mathbb{N}$ *such that modulo* $\mathbb{Z}$ *we have equality of the following sets*

$$\left\{\alpha_1 + \frac{1}{d}, \ldots, \alpha_p + \frac{1}{d}\right\} \equiv \{\alpha_1, \ldots, \alpha_p\} \pmod{\mathbb{Z}}$$

$$\left\{\beta_1 + \frac{1}{d}, \ldots, \beta_q + \frac{1}{d}\right\} \equiv \{\beta_1, \ldots, \beta_q\} \pmod{\mathbb{Z}}$$

By the remarks made above, its differential Galois group is imprimitive. A study of the differential Galois group can now be carried out by studying the differential Galois group of the hypergeometric equation from which it is induced. Belyi induction arises as follows. Let $a, b \in \mathbb{N}$ and define the algebraic function $t(z)$ of z by $z = \gamma t^a(1-t)^b$, where $\gamma = (1+b/a)^a(1+a/b)^b$ is chosen such that branching takes place only above $z = 0, 1, \infty$. Now take $\lambda, \mu \in \mathbb{C}$. Then $t^\lambda(1-t)^\mu$ satisfies the hypergeometric differential equation of order $q = p = a+b$ in z with parameters $\lambda, \lambda + 1/a, \ldots, \lambda + (a-1)/a, \mu, \mu + 1/b, \ldots, \mu + (b-1)/b; \nu, \nu + 1/(a+b), \ldots, \nu + (a+b-1)/(a+b)$, where $\nu = (a\lambda + b\mu)/(a+b)$. Inverse Belyi induction is a small variation on the above theme. With the same notations, let $z^{-1} = \gamma t^a(1-t)^b$, Then the function $t^\lambda(1-t)^\mu$ satisfies almost the same hypergeometric differential equation , the difference being that the α- and β-parameters are now interchanged.

(4.2.4) Definition. *A hypergeometric differential equation is called* Belyi induced *if it is irreducible and if there exist* $\lambda, \mu, \nu \in \mathbb{C}$ *and* $a, b \in \mathbb{N}$ *such that* $a\lambda + b\mu = (a+b)\nu$ *and the parameter set modulo* $\mathbb{Z}$ *is given by*

$$\lambda, \lambda + \frac{1}{a}, \ldots, \lambda + \frac{a-1}{a}, \mu, \ldots, \mu + \frac{b-1}{b}; \nu, \nu + \frac{1}{a+b}, \ldots, \nu + \frac{a+b-1}{a+b}.$$

The equation is called inverse Belyi induced *if, with the same notations, its parameter set modulo* $\mathbb{Z}$ *reads*

$$\nu, \nu + \frac{1}{a+b}, \ldots, \nu + \frac{a+b-1}{a+b}; \lambda, \lambda + \frac{1}{a}, \ldots, \lambda + \frac{a-1}{a}, \mu, \ldots, \mu + \frac{b-1}{b}$$

(4.2.5) Theorem. *Let G be the differential Galois group of* Eq.(4.1.2), *which we assume to be irreducible. Then G is imprimitive if and only if* Eq.(4.1.2) *is either Kummer induced or Belyi induced or inverse Belyi induced.*

This theorem is not stated as such in either (Katz 1990) or (Beukers and Heckman 1989). The proof is fairly technical, but its ingredients are contained in the two references mentioned.

4.3 The primitive case

(4.3.1) Theorem. *Let G be the differential Galois group of* Eq.(4.1.2), *which we assume to be irreducible. Suppose G is primitive. Then either G^0 acts irreducibly or G^0 consists of scalars, in which case we have either $G^0 \simeq \mathbb{C}^\times$ or $G^0 = \{id\}$.*

(4.3.2) *Remark.* From the proof of this theorem one sees that the latter cases can only occur when $p = q$. In such a case we have that G modulo its centre is finite. Theorem 4.3.3 will describe precisely when this happens.

Proof. First assume $p < q$. Suppose G^0 acts reducibly. The fixed field $K/\mathbb{C}(z)$ corresponding to G^0 is algebraic, and since $0, \infty$ are the only singularities, it must be of the form $K = \mathbb{C}(z^{1/e})$ for some $e \in \mathbb{N}$. But then G/G^0 is cyclic. Letting W be an irreducible invariant subspace of G^0, one sees that a maximal subset of distinct subspaces in $\{gW | \ g \in G\}$ yields a system of imprimitivity for G. Hence G is imprimitive, contrary to our assumption. Thus we conclude that G^0 acts irreducibly. Now assume $p = q$. This case is more tedious due to the occurrence of equations which have only algebraic solutions. We sketch the proof here, and rely on results from (Beukers and Heckman 1989). Let H be the monodromy group of Eq.(4.1.2) and let G be its Zariski closure in $GL(n)$. In H we have the so-called reflection subgroup H_r which is generated by the elements $\{hh_1h^{-1} | h \in H\}$, where h_1 is the local monodromy element around the point $z = 1$. According to our remarks in Subsection 4.1, h_1 is a pseudo reflection. Let h_0 be the monodromy element around $z = 0$. It is well known that H is generated by h_1, h_0. So, H/H_r is generated by h_0, i.e. H/H_r is cyclic. In (Beukers and Heckman 1989, Theorem 5.14) it is shown that if H_r is imprimitive, then H_r is finite. Since H/H_r is cyclic this implies that $G^0 = \{id\}$ or $\mathbb{C}^\times$. Now suppose that H_r is primitive. Then (Beukers and Heckman Prop.6.3) states that $\overline{H}_r^0$, component of the identity of the Zariski closure of H_r, is either trivial or irreducible on V. This proves our assertion. □

The following theorem enables one to recognize the cases of Theorem 4.3.1. We say that two sets of points $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ with $|a_i| = |b_j| = 1$ for $i, j = 1, \ldots, n$ *interlace* on the unit circle $|z| = 1$ if, following this circle clockwise, one meets the points of A and B alternately.

(4.3.3) Theorem. *Suppose $p = q$. Then G modulo its centre is finite if and only if the following conditions hold,*

i. $\exists \delta \in \mathbb{C}$ *such that* $\mu_i = \alpha_i + \delta \in \mathbb{Q}$ *and* $\nu_j = \beta_j + \delta \in \mathbb{Q}$ *for all* i, j.

ii. *Letting N be the common denominator of μ_i, ν_j, the sets* $\{\exp(2\pi i h \mu_k) | \ k = 1, \ldots, q\}$ *and* $\{\exp(2\pi i h \nu_l | \ l = 1, \ldots, q\}$ *interlace on the unit circle $|z| = 1$ for all $h \in \mathbb{N}$ with $1 \le h \le N$,* $\gcd(h, N) = 1$.

In particular, if $\delta \in \mathbb{Q}$ then G is finite.

(4.3.4) *Remark.* In case $p = q = 2$ a complete list of all cases where G is finite was given by H.A.Schwarz (Schwarz 1873). For the case $p = q > 2$ a complete list was determined in (Beukers and Heckman 1989, Section 8). A proof of Theorem 4.3.3 can be found in the same reference in Section 4.

The following theorem gives a rough description of the differential Galois groups that occur for hypergeometric differential equations in case G^0 acts primitively (i.e. the general case as we know by now).The case $p \neq q$ was settled by O.Gabber and N.M.Katz in 1986 (Katz 1990) and the case $p = q$ by (Beukers and Heckman 1989).

(4.3.5) Theorem. *Let V be the solution space of* Eq.4.1.2 *and G its differential Galois group . Suppose G is primitive and that G^0 acts irreducibly on the space V. Then we have the following possibilities for G,*

If $p = q$, $\exp 2\pi i \sum_k(\alpha_k - \beta_k) \neq \pm 1$ *then* $G = C \cdot SL(V)$.
If $p = q$, $\exp 2\pi i \sum_k(\alpha_k - \beta_k) = 1$ *then* $G = C \cdot SL(V)$ *or* $C \cdot Sp(V)$.
If $p = q$, $\exp 2\pi i \sum_k(\alpha_k - \beta_k) = -1$ *then* $G = C \cdot SL(V)$ *or* $C \cdot SO(V)$.
If $q - p = 1$ *then* $G = GL(V)$.
If $q - p$ *is odd then* $G = C \cdot SL(V)$.
If $q-p$ *is positive and even, then* $G = C \cdot SL(V)$ *or* $C \cdot SO(V)$ *or* $C \cdot Sp(V)$ *or, in addition*
if $q = 7, p = 1$, $G = C \cdot G_2$
if $q = 8, p = 2$, $G = C \cdot SL(3)$, $SL(3)$ *in adjoint representation*
if $q = 8, p = 2$, $G = C \cdot (SL(2) \times SL(2) \times SL(2))$
if $q = 8, p = 2$, $G = C \cdot (SL(2) \times Sp(4))$
if $q = 8, p = 2$, $G = C \cdot (SL(2) \times SL(4))$
if $q = 9, p = 3$, $G = C \cdot (SL(3) \times SL(3))$.

Proof. Suppose $p = q$. We know from Section 4.1 that the local monodromy around the point $z = 1$ is generated by the pseudoreflection h (i.e. $h - Id$ has rank one) with special eigenvalue $\lambda = \exp 2\pi i \sum_k(\beta_k - \alpha_k)$. Clearly, the group hG^0h^{-1} is again connected and so $hG^0h^{-1} = G^0$, i.e. h normalizes G^0. We now obtain the first three assertions of our theorem by application of Theorem 3.2.3 to the normalizing element h and the group G^0. Suppose $q > p$. From classical references (Barnes 1906) or (Meijer 1946) we know that we have at ∞ a basis of formal solutions $y_1, \ldots, y_q$ given by

$$\begin{pmatrix} y_1 \\ \vdots \\ y_q \end{pmatrix} = \exp(P(z^{1/(q-p)}))z^A \begin{pmatrix} f_1 \\ \vdots \\ f_q \end{pmatrix},$$

where $P(X) = \operatorname{diag}(X, \zeta X, \ldots, \zeta^{q-p-1}X, 1, \ldots, 1)$, A is a constant $q \times q$ matrix in Jordan normal form and the f_i are formal Laurent expansions in $z^{-1/(q-p)}$. From the remarks made at the end of Section 2.5 we see that the local Galois group, and hence the group G, contains the torus D consisting of all

$\mathrm{diag}(d_1,\ldots,d_q)$ such that $d_i = 1$ for $i > q-p$ and $d_1^{a_1}\cdots d_{q-p}^{a_{q-p}} = 1$ whenever $\sum_{i=1}^{q-p} a_i\zeta^{i-1} = 0$, $a_i \in \mathbb{Z}$. Trivially, $\mathcal{D}$ normalizes G^0. We apply Gabber's Theorem 3.2.1. In order to do so, we must determine all $1 \le i,j,k,l \le q$ such that $d_id_j = d_kd_l$ for all elements of D. Since $d_i = 1$ for $i > q-p$ this means that we have to watch for non trivial relations of the form $d_id_j = d_kd_l$, $d_id_j = d_k$, $d_i = d_k$, $d_id_j = 1$, $d_i = 1$ with $1 \le i,j,k,l \le q-p$. From the definition of D this means that we must check for non trivial relations of the form $\zeta^i + \zeta^j = \zeta^k + \zeta^l$, $\zeta^i + \zeta^j = \zeta^k$ or $\zeta^i + \zeta^j = 0$. It is not hard to show that

i. $\zeta^i + \zeta^j = \zeta^k + \zeta^l \Rightarrow i = k, j = l$ or $q-p$ even, $i \equiv k + \frac{q-p}{2} \bmod q-p$, $j \equiv l + \frac{q-p}{2} \bmod q-p$ or the same possibilities with i and j interchanged.
ii. $\zeta^k = \zeta^i + \zeta^j \Rightarrow 6 \mid (p-q)$, $i \equiv k \pm \frac{q-p}{6} \pmod{q-p}$, $j \equiv k \mp \frac{q-p}{6} \pmod{q-p}$.
iii. $\zeta^i + \zeta^j = 0 \Rightarrow q-p$ even, $j \equiv i + \frac{q-p}{2} \pmod{q-p}$.

Thus, if $q-p$ is odd there are no restrictions and Theorem 3.2.1 implies that $\mathbb{C}\cdot G^0$ contains the torus

$$T = \{\mathrm{diag}(t_1,\ldots,t_{q-p},1,\ldots,1) \mid t_1\cdots t_{q-p} \ne 0\}.$$

In particular $\mathbb{C}\cdot G^0$ contains $T_1 = \{\mathrm{diag}(t,1,\ldots,1) | t \ne 0\}$ and Theorem 3.2.2 now implies $\mathbb{C}\cdot G^0 = GL(V)$. Hence $SL(V) \subset G$. Moreover, if $q-p = 1$, we see that $D = T_1$ and so $G = GL(V)$ in this case. Now suppose $q-p$ is even, but $6 \nmid (q-p)$. Then the relations are generated by $d_id_j = 1$, $i - j \equiv \frac{q-p}{2} \pmod{q-p}$ and Theorem 3.2.1 implies that $\mathbb{C}\cdot G^0$ contains

$$\{\mathrm{diag}(t_1,\ldots,t_{\frac{q-p}{2}},t_1^{-1},\ldots,t_{\frac{q-p}{2}}^{-1},1,\ldots,1) \mid t_1\cdots t_{\frac{q-p}{2}} \ne 0\}$$

In particular, $\mathbb{C}\cdot G^0$ contains the torus $\{\mathrm{diag}(t,1,\ldots,1,t^{-1},1,\ldots,1) \mid t \ne 0\}$ and Theorem 3.2.2 implies that $\mathbb{C}\cdot G^0$ is $GL(V)$, $\mathbb{C}\cdot Sp(V)$ or $\mathbb{C}\cdot SO(V)$. Finally, if $6 \mid (q-p)$ then, by the same arguments as above, we find that $\mathbb{C}\cdot G^0$ contains (after permutation of basis vectors, if necessary)

$$\{\mathrm{diag}(ts,t,s,(ts)^{-1},t^{-1},s^{-1},1,\ldots,1) \mid ts \ne 0\},$$

which is called the G_2-torus. Application of Theorem 3.2.3 yields the desired result. □

In the cases of Theorem 4.3.5 where we have a choice between $SL(V)$ and a smaller selfdual group, there is a very easy criterion to make this decision.

(4.3.6) Theorem. *Let V be the solution space of* Eq.4.1.2 *and G its differential Galois group . Then the following two statements are equivalent,*

i. *G is contained in $\mathbb{C}\cdot Sp(V)$ or $\mathbb{C}\cdot SO(V)$*
ii. *$\exists\delta$ such that $\{\alpha_1,\ldots,\alpha_p\} = \{\delta-\alpha_1,\ldots,\delta-\alpha_p\} \pmod{\mathbb{Z}}$ and $\{\beta_1,\ldots,\beta_q\} = \{\delta-\beta_1,\ldots,\delta-\beta_q\} \pmod{\mathbb{Z}}$.*

The proof, which is elementary, can be found in (Katz 1990). In this same book it is shown that there actually exist examples of hypergeometric differential equations having G_2 as differential Galois group .

(4.3.7) Theorem. (Gabber-Katz) *Let V be the solution space of* Eq.(4.1.2) *and G its differential Galois group . Then the following statements are equivalent,*

i. $G = C \cdot G_2$

ii. $q = 7, p = 1$ *and* $\exists \mu, \nu$ *with* $\mu \neq 0, \nu \neq 0, \mu + \nu \neq 0$ *such that after renumbering of indices, if necessary, the numbers* $\beta_1 - \alpha_1, \ldots, \beta_7 - \alpha_1$*, considered (mod* $\mathbb{Z}$*), equal* $\frac{1}{2}, \mu, \nu, -\mu, -\nu, \mu + \nu, -\mu - \nu$.

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