Foundations
Probability: Multivariate Models

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Exercise 3.1. [Uncorrelated does not imply independent]
Let $X \sim U(-1, 1)$ and $Y = X^2$. Clearly Y is dependent on X (in fact, Y is uniquely determined by X). However, show that $\rho(X, Y) = 0$. Hint: if $X \sim U(a, b)$ then $E[X] = (a + b)/2$ and V[X] = $(b - a)^2/12$.

Solution.

$$E[X] = \frac{a+b}{2} = 0, \qquad V[X] = \frac{(b-a)^2}{12} = \frac{1}{3}$$

$$\rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{V[X]V[Y]}} \tag{1}$$

$$Cov[X,Y] = E[XY] - E[X]E[Y]$$

$$Cov[X,Y] = E[X^3] - E[X]E[X^2] \tag{2}$$

using $E[g[X]] = \int_a^b p(X)\, g(X)\, dx$ and the fact that the density for a uniformly distributed variable $= \frac{1}{b-a}$ which is equal to $\frac{1}{2}$ for X:

$$E[X^2] = \int_{-1}^{1} \frac{1}{2} X^2\, dx = \frac{X^3}{6}|_{-1}^{1} = \frac{1}{3}$$

$$E[X^3] = \int_{-1}^{1} \frac{1}{2} X^3\, dx = \frac{X^4}{8}|_{-1}^{1} = 0$$

Substituting in 2:

$$Cov[X,Y] = 0 - 0 * \frac{1}{3} = 0$$

$$V[Y] = E[Y^2] - (E[Y])^2$$

$$E[Y^2] = E[X^4] = \int_{-1}^{1} \frac{1}{2} X^4\, dx = \frac{X^5}{10}|_{-1}^{1} = \frac{1}{5}$$

$$V[Y] = \frac{1}{5} - (\frac{1}{3})^2 = \frac{4}{45}$$

Substituting in 1: $\rho(X,Y) = \frac{0}{\sqrt{\frac{1}{2} * \frac{4}{45}}} = 0$

□

Exercise 3.3. [Correlation coefficient for linearly related variables is ± 1]
Show that, if $Y = aX + b$ for some parameters $a > 0$ and b, then $\rho(X, Y) = 1$. Similarly show that if $a < 0$, then $\rho(X, Y) = -1$.

Solution.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}}$$

Substituting $Y = aX + b$ and $V(aX + b) = a^2V(X)$:

$$\rho(X,Y) = \frac{Cov(X, aX+b)}{\sqrt{V(x)a^2V(x)}}$$

$$\rho(X,Y) = \frac{Cov(X, aX+b)}{|a|V(x)} \tag{3}$$

$$Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$$

$$= E[(X-\mu_x)(aX+b-a\mu_x-b)] = aE[X-\mu_x] = aV(X)$$

Substituting in 3:

$$\rho(X,Y) = \frac{aV(X)}{|a|V(X)} = \frac{a}{|a|}$$

if $a > 0$ then $\rho(X,Y) = 1$

if $a < 0$ then $\rho(X,Y) = -1$

$\square$