Foundations
Probability: Multivariate Models

Wafaa Mohammed

Exercise 3.3. [Correlation coefficient for linearly related variables is ± 1]
Show that, if $Y = aX + b$ for some parameters $a > 0$ and b, then $\rho(X, Y) = 1$. Similarly show that if $a < 0$, then $\rho(X, Y) = -1$.

Solution.

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{V(X)V(Y)}}$$

Substituting $Y = aX + b$ and $V(aX + b) = a^2 V(X)$:

$$\rho(X, Y) = \frac{Cov(X, aX + b)}{\sqrt{V(x)a^2V(x)}}$$

$$\rho(X, Y) = \frac{Cov(X, aX + b)}{|a|V(x)} \tag{1}$$

$$Cov(X, Y) = E[(X - \mu_x)(Y - \mu_y)]$$

$$= E[(X - \mu_x)(aX + b - a\mu_x - b)] = aE[X - \mu_x] = aV(X)$$

Substituting in 1:

$$\rho(X, Y) = \frac{aV(X)}{|a|V(X)} = \frac{a}{|a|}$$

if $a > 0$ then $\rho(X, Y) = 1$

if $a < 0$ then $\rho(X, Y) = -1$

$\square$