Foundations
Statistics

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Exercise 4.1. [MLE for the univariate Gaussian]
Show that the MLE for a univariate Gaussian is given by

$$\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} y_n$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(y_n - \hat{\mu})^2$$

Solution. 1. $\hat{\mu}$:
The MLE can be found be found by solving $\frac{d}{d\mu}NLL = 0$

$$NLL = -log(\sum_{n=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-1}{2\sigma^2}(y_n-\mu)^2})$$

$$\frac{d}{d\mu}NLL = \frac{d}{d\mu}[-log(\sum_{n=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-1}{2\sigma^2}(y_n-\mu)^2})]$$

$$= -\frac{d}{d\mu}log(\sum_{n=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2}}) - \frac{d}{d\mu}log(\sum_{n=1}^{N}e^{\frac{-1}{2\sigma^2}(y_n-\mu)^2)})$$

$$-\frac{d}{d\mu}\sum_{n=1}^{N}\frac{1}{2\sigma^2}(y_n-\mu)^2$$

$$\frac{1}{\sigma^2}\sum_{n=1}^{N}(y_n-\mu) = 0 \tag{1}$$

for 1 to be zero:

$$\sum_{n=1}^{N}(y_n-\mu) = 0$$

$$\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} y_n$$

2. $\hat{\sigma}^2$:
The MLE can be found be found by solving $\frac{d}{d\sigma^2}NLL = 0$

$$NLL = -log(\sum_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y_n-\mu)^2})$$

$$\frac{d}{d\sigma^2}NLL = \frac{d}{d\sigma^2}[-log(\sum_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y_n-\mu)^2})]$$

$$= -\frac{d}{d\sigma^2}log(\sum_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}}) - \frac{d}{d\sigma^2}log(\sum_{n=1}^{N} e^{\frac{-1}{2\sigma^2}(y_n-\mu)^2})$$

$$= \frac{d}{d\sigma^2}[\frac{N}{2}log(2\pi) + \frac{N}{2}log(\sigma^2)] - \frac{d}{d\sigma^2}\sum_{n=1}^{N} \frac{-1}{2\sigma^2}(y_n-\mu)^2$$

$$= \frac{N}{2\sigma^2} - \frac{1}{2(\sigma^2)^2}\sum_{n=1}^{N}(y_n-\mu)^2 = 0$$

$$\frac{1}{2\sigma^2}[N - \frac{1}{\sigma^2}\sum_{n=1}^{N}(y_n-\mu)^2] = 0 \tag{2}$$

for 2 to be zero:

$$N - \frac{1}{\sigma^2}\sum_{n=1}^{N}(y_n-\mu)^2 = 0$$

$$N\hat{\sigma^2} = \sum_{n=1}^{N}(y_n-\hat{\mu})^2$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(y_n-\hat{\mu})^2$$

□