Foundations
Probability: Univariate Models

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Exercise 2.1. [Conditional independence]
a. Let $H \in 1, ..., K$ be a discrete random variable, and let e_1 and e_2 be the observed values of two other random variables E_1 and E_2. Suppose we wish to calculate the vector

$$\overrightarrow{P}(H|e_1, e_2) = (P(H = 1|e_1, e_2), ..., P(H = K|e_1, e_2))$$

Which of the following sets of numbers are sufficient for the calculation?

i. $P(e_1, e_2), P(H), P(e_1|H), P(e_2|H)$

ii. $P(e_1, e_2), P(H), P(e_1, e_2|H)$

iii. $P(e_1|H), P(e_2|H), P(H)$

b. Now suppose we now assume $E_1 \perp E_2|H$ (i.e., E_1 and E_2 are conditionally independent given H). Which of the above 3 sets are sufficient now?

Show your calculations as well as giving the final result. Hint: use Bayes rule.

Solution. a. $\overrightarrow{P}(H|e_1, e_2) = \frac{P(H)P(e_1,e_2|H)}{P(e_1,e_2)}$, set ii will be sufficient.

b. in case of conditional independence, $P(e_1, e_2|H) = P(e_1|h)P(e_2|h)$, so set i will be sufficient in this case.

□

Exercise 2.3. [Conditional independence iff joint factorizes]
In the text we said $X \perp Y|Z$ iff

$$p(x, y|z) = p(x|z)p(y|z)$$

for all x, y, z such that $p(z) > 0$. Now prove the following alternative definition: $X \perp Y|Z$ iff there exist functions g and h such that

$$p(x, y|z) = g(x, z)h(y, z)$$

for all x, y, z such that $p(z) > 0$.

Solution. let's start from

$$p(x,y|z) = g(x,z)h(y,z) \tag{1}$$

and prove that it leads to the first definition.

Marginalizing X:

$$\sum_y P(x,y|z) = \sum_y g(x,z)h(y,z)$$

$$P(x|z) = g(x,z)\sum_y h(y,z)$$

$$g(x,z) = \frac{P(x|z)}{\sum_y h(y,z)} \tag{2}$$

Marginalizing Y:

$$\sum_x P(x,y|z) = \sum_x g(x,z)h(y,z)$$

$$P(y|z) = h(y,z)\sum_x g(x,z)$$

$$h(y,z) = \frac{P(y|z)}{\sum_x g(x,z)} \tag{3}$$

substituting 2 and 3 in 1:

$$P(x,y|z) = \frac{P(x|z)}{\sum_y h(y,z)}\frac{P(y|z)}{\sum_x g(x,z)} \tag{4}$$

since the total conditional probability over all values of X and Y should sum up to 1:

$$\sum_{x,y} p(x,y|z) = \sum_{x,y} g(x,z)h(y,z) = \sum_x g(x,z)\sum_y h(y,z) = 1$$

then, the denominator in 4 is equal to 1. hence:

$p(x,y|z) = p(x|z)p(y|z)$ which is the first definition.

□

Exercise 2.5. [Expected value of the minimum of two rv's]
Suppose X, Y are two points sampled independently and uniformly at random from the interval [0, 1]. What is the expected location of the leftmost point?

Solution. The leftmost point is the smaller of the two: $L = min(X, Y)$ and the goal is to find the expected value of L, $(E(L))$.

The CDF of L is:

$$P(l) = P(L \leq l) = P(min(X, Y) \leq l)$$

$$P(l) = 1 - P(X > l \; and \; Y > l)$$

using independence:

$$P(l) = 1 - P(X > l)P(Y > l) \tag{5}$$

since X,Y are uniformly distributed:

$$P(X > l) = P(Y > l) = 1 - l$$

substituting in 5:

$$P(l) = 1 - (1 - l)^2$$

The PDF of L is the derivative of the CDF:

$$p(l) = \frac{d}{dl}P(l) = \frac{d}{dl}(1 - (1 - l)^2) = 2(1 - l)$$

The expected value of L is:

$$E(L) = \int_0^1 l\, p(l)\, dl$$

$$= \int_0^1 2l\,(1 - l)\, dl$$

$$= \int_0^1 (2l - 2l^2)\, dl$$

$$= (l^2 - \frac{2}{3}l^3)\,|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

□

Exercise 2.7. [Deriving the inverse gamma density]
Let $X \sim Ga(a, b)$, and $Y = 1/X$. Derive the distribution of Y.

Solution.

$$Y = 1/X \qquad X = 1/Y$$

using the change of variables formula:

$$p(y) = p(x)\,|\frac{dx}{dy}|$$

$$p(y) = p(1/y)\,|\frac{d1/y}{dy}|$$

$$p(y) = \frac{b^a}{\Gamma(a)}(\frac{1}{y})^{a-1}\,e^{\frac{-b}{y}}\,(\frac{1}{y^2})$$

$$p(y) = \frac{b^a}{\Gamma(a)}y^{-(a+1)}\,e^{\frac{-b}{y}}$$

Which is the inverse gamma distribution.

□

Exercise 2.9. [Bayes rule for medical diagnosis]
After your yearly checkup, the doctor has bad news and good news. The bad news is that you tested positive for a serious disease, and that the test is 99% accurate (i.e., the probability of testing positive given that you have the disease is 0.99, as is the probability of testing negative given that you don't have the disease). The good news is that this is a rare disease, striking only one in 10,000 people. What are the chances that you actually have the disease? (Show your calculations as well as giving the final result.)

Solution. Let X represent having the disease and Y represent testing positive.

Using Bayes rule:

$$P(X|Y) = \frac{P(X)\,P(Y|X)}{P(Y)}$$

$$= \frac{P(X)\,P(Y|X)}{P(X)\,P(Y|X) + P(X^{‘})\,P(Y|X^{‘})}$$

$P(X)$ = the probability of having the disease = $\frac{1}{10{,}000}$ = 0.0001

$P(X^{‘})$ = the probability of not having the disease = $1 - 0.0001 = 0.9999$

$P(Y|X)$ = the probability of testing positive when having the disease = 0.99

$P(Y|X^{‘})$ = the probability of testing positive when not having the disease $= 1 - 0.99 = 0.01$

$P(X|Y)$ = the probability of having the disease when testing positive

$$P(X|Y) = \frac{0.0001 * 0.99}{0.0001 * 0.99 + 0.9999 * 0.01} = 0.0098$$

The chances of having the disease when testing positive are 0.98%. $\square$

Exercise 2.11. [Probabilities are sensitive to the form of the question that was used to generate the answer]
(Source: Minka.) My neighbor has two children. Assuming that the gender of a child is like a coin flip, it is most likely, a priori, that my neighbor has one boy and one girl, with probability 1/2. The other possibilities—two boys or two girls—have probabilities 1/4 and 1/4.

a. Suppose I ask him whether he has any boys, and he says yes. What is the probability that one child is a girl?

b. Suppose instead that I happen to see one of his children run by, and it is a boy. What is the probability that the other child is a girl?

Solution. let G = girl and B = boy and the two children are 1 and 2.

Assuming the gender of a child is like a coin flip, we have 4 possible outcomes: $\{B_1B_2, B_1G_2, G_1G_2, G_1B_2\}$ each with a probability of 1/4.

a. Given the event that neighbor has answered yes to having a boy, the possible outcomes are: $\{B_1B_2, B_1G_2, G_1B_2\}$ since having two girls is no longer a possibility. In this case, the probability that one child is a girl (doesn't matter which child) is:

$$= \frac{P(G_1B_2) + P(B_1G_2)}{P(G_1B_2) + P(B_1G_2) + P(B_1B_2)}$$
$$= \frac{2}{3}$$

b. In the event of seeing one child and it is a boy, let's assume that you saw child 1 (the same holds if you saw child 2, just flip the numbers), the possible outcomes are: $\{B_1B_2, B_1G_2\}$. In this case, the probability that the other child is a girl is:

$$= \frac{P(B_1G2)}{P(B_1G_2) + p(B_1B_2)}$$

$$= \frac{1}{2}$$

$\square$