Foundations
Probability: Univariate Models

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Exercise 2.3. [Conditional independence iff joint factorizes]
In the text we said $X \perp Y|Z$ iff

$$p(x, y|z) = p(x|z)p(y|z)$$

for all x, y, z such that $p(z) > 0$. Now prove the following alternative definition: $X \perp Y|Z$ iff there exist functions g and h such that

$$p(x, y|z) = g(x, z)h(y, z)$$

for all x, y, z such that $p(z) > 0$.

Solution. let's start from

$$p(x, y|z) = g(x, z)h(y, z) \tag{1}$$

and prove that it leads to the first definition.

Marginalizing X:

$$\sum_y P(x, y|z) = \sum_y g(x, z)h(y, z)$$

$$P(x|z) = g(x, z) \sum_y h(y, z)$$

$$g(x, z) = \frac{P(x|z)}{\sum_y h(y, z)} \tag{2}$$

Marginalizing Y:

$$\sum_x P(x, y|z) = \sum_x g(x, z)h(y, z)$$

$$P(y|z) = h(y, z) \sum_x g(x, z)$$

$$h(y, z) = \frac{P(y|z)}{\sum_x g(x, z)} \tag{3}$$

substituting 2 and 3 in 1:

$$P(x,y|z) = \frac{P(x|z)}{\sum_y h(y,z)} \frac{P(y|z)}{\sum_x g(x,z)} \tag{4}$$

since the total conditional probability over all values of X and Y should sum up to 1:

$$\sum_{x,y} p(x,y|z) = \sum_{x,y} g(x,z)h(y,z) = \sum_x g(x,z) \sum_y h(y,z) = 1$$

then, the denominator in 4 is equal to 1. hence:

$p(x,y|z) = p(x|z)p(y|z)$ which is the first definition.

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