Foundations
Probability: Univariate Models

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Exercise 2.5. [Expected value of the minimum of two rv's]
Suppose X, Y are two points sampled independently and uniformly at random from the interval [0, 1]. What is the expected location of the leftmost point?

Solution. The leftmost point is the smaller of the two: $L = min(X, Y)$ and the goal is to find the expected value of L, $(E(L))$.

The CDF of L is:

$$P(l) = P(L \leq l) = P(min(X, Y) \leq l)$$
$$P(l) = 1 - P(X > l \; and \; Y > l)$$

using independence:

$$P(l) = 1 - P(X > l)P(Y > l) \tag{1}$$

since X,Y are uniformly distributed:

$$P(X > l) = P(Y > l) = 1 - l$$

substituting in 1:

$$P(l) = 1 - (1 - l)^2$$

The PDF of L is the derivative of the CDF:

$$p(l) = \frac{d}{dl}P(l) = \frac{d}{dl}(1 - (1 - l)^2) = 2(1 - l)$$

The expected value of L is:

$$E(L) = \int_0^1 l \, p(l) \, dl$$
$$= \int_0^1 2l \, (1 - l) \, dl$$
$$= \int_0^1 (2l - 2l^2) \, dl$$
$$= (l^2 - \frac{2}{3}l^3) \, |_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

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