Foundations
Statistics

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Exercise 4.2. [MAP estimation for 1D Gaussians]
(Source: Jaakkola.) Consider samples $x_1, ..., x_n$ from a Gaussian random variable with known variance σ^2 and unknown mean μ. We further assume a prior distribution (also Gaussian) over the mean, $\mu \sim \mathcal{N}(m, s_2)$, with fixed mean m and fixed variance s^2. Thus the only unknown is μ.

a. Calculate the MAP estimate $\hat{\mu}_{MAP}$. You can state the result without proof. Alternatively, with a lot more work, you can compute derivatives of the log posterior, set to zero and solve.
b. Show that as the number of samples n increase, the MAP estimate converges to the maximum likelihood estimate.
c. Suppose n is small and fixed. What does the MAP estimator converge to if we increase the prior variance s^2?
d. Suppose n is small and fixed. What does the MAP estimator converge to if we decrease the prior variance s^2?

Solution. a.

$$\hat{\mu}_{MAP} = \frac{n\bar{x}}{n + \frac{\sigma^2}{s^2}} + \frac{\frac{\sigma^2}{s^2} m}{n + \frac{\sigma^2}{s^2}}$$

where $\bar{x} = \sum_{i=1}^{n} x_i$.

b.
The maximum likelihood estimate of the mean is the empirical mean $\hat{\mu}_{MLE} = \bar{x}$.
As $n \to \infty$, the fixed ratio $\frac{\sigma^2}{s^2}$ becomes negligible compared to it. Thus,

$$\hat{\mu}_{MAP} = \frac{n\bar{x}}{n} + 0 = \bar{x}$$

c.
As $s^2 \to \infty$, the ratio $\frac{\sigma^2}{s^2}$ converges to zero. Thus,

$$\hat{\mu}_{MAP} = \frac{n\bar{x}}{n + 0} + \frac{0}{n} = \bar{x}$$

The intuition behind this is that when the prior variance increases, the prior becomes flat (uninformative), and the posterior is dominated by the likelihood.

d.
As $s^2 \to 0$, the small value of n will be negligible compared to it. Thus,

$$\hat{\mu}_{MAP} = 0 + \frac{\frac{\sigma^2}{s^2} m}{\frac{\sigma^2}{s^2}} = m$$

The intuition behind this is that when the prior variance decreases, the prior places almost all its weight at $\mu = m$, so the posterior will be dominated by the prior.

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