Transform Techniques & PDE

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1 Partial Differential Equations

1.1 Formation of PDE

Notations used:

$$p = \frac{\partial \mathbf{z}}{\partial x},\ q = \frac{\partial \mathbf{z}}{\partial y}$$

$$r = \frac{\partial^2 \mathbf{z}}{\partial x^2},\ s = \frac{\partial^2 \mathbf{z}}{\partial x \partial y},\ t = \frac{\partial^2 \mathbf{z}}{\partial y^2}$$

Type I: Elimination of arbitrary constants

- If the number of arbitrary constants = number of independent variables, first order PDE is enough.
- If the number of arbitrary constants > number of independent variables, higher orders are possible.

Type II: Elimination of arbitrary functions

$$nf(x) = \mathcal{O}(PDE)$$

$$\text{Number of functions} = \text{Order of PDE}$$

Procedure:

(a) Find the order of the required PDE by the number of arbitrary constants and independent variables (or) by the number of arbitrary functions.

(b) Differentiate the function partially with respect to the independent variables to find the value of the arbitrary constants (or) arbitrary functions.

(c) Replace the values with the exact notations.

(d) Substitute in the given equation.

Note: The goal is just to find a PDE for the given equation, by eliminating the arbitrary constants (or) arbitrary functions. If substitution doesn't work out, relate the different values from (b), to find the required PDE.

Type III: Eliminating composite functions

If the equation is of the form $F\big(u(x,y), v(x,y)\big) = 0$, the direct solution is,

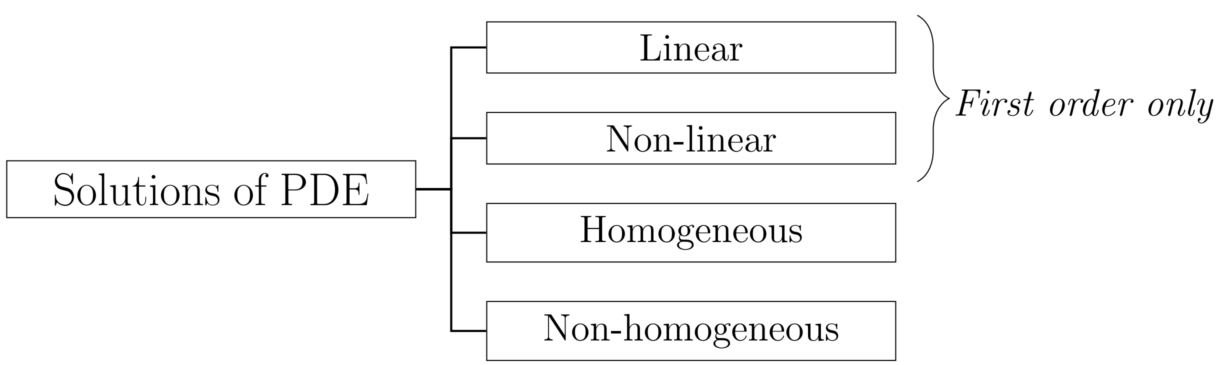
$$pP + qQ = R$$

$$P = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix} \quad Q = \begin{vmatrix} u_x & v_x \\ u_z & v_z \end{vmatrix}$$

$$R = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$$

1.2 Solutions of PDE

A solution can either be made of arbitrary constants or arbitrary functions. Both are valid for the same PDE.



Linear PDE:

Characteristics:

- It is of the **first** degree.
- It has no products of the dependent variable.
- Power of its dependent variable should be **1**.

Type I: Direct Method

The solution is obtained by directly integrating the respective PDEs. For e.g. (using notations q, s),

$$s = \frac{x}{y}$$

$$q = \int s \cdot dx = \frac{x^2}{2y} + f(y)$$

$$\int q \cdot dy = \frac{x^2}{2}\, log_e(y) + \int f(y) \cdot dy + g(x)$$

$$\implies \mathbf{z} = \frac{x^2}{2}\, log_e(y) + \phi(y) + g(x)$$

Note:

(a) In this method, you can begin your integration by using any of the independent variables (i.e) x or y. That's because most of our problems treat PDEs to be homogeneous.

(b) $\mathbf{z}$ is a $f(x, y)$. When you integrate $\mathbf{z}$ with respect to x, the integration constant could be a function of y. So, the constants that come out during integration, should be replaced by new functions $f(y)$, $g(x)$, etc.

(c) $\phi(y)$ denotes that you've already integrated $f(y)$, and it should be distinguished from its parent - $\phi(y)$. And, $g(x)$ has appeared as a result of integrating $f(y)$ due to the same reason mentioned at (b).

(d) Cases where the function itself depends on its own derivative, such as $\mathbf{z} = t$ *(notation)*, can be solved by using a characteristic function (same as in ODE), which includes all possible constants (so, you don't have to insert a constant after each integration).

Just integrate, find the values of the functions, and substitute in C.F.

Type II: Lagrange's method for Quasi-linear PDE:

In such PDEs, the highest order derivatives are linear, with coefficients existing as a function of lower order derivatives of dependent variables.

$$\mathbf{z}xp + \mathbf{z}yq = \mathbf{z}xy \text{ is an example.}$$

Procedure:

(a) Check whether the PDE is of first degree.

(b) Given equation should be of the quasi-linear form $pP + qQ = R$. If it's not, then make it!

(c) Substitute in the equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Rearrange the equation (variable-separable method) i.e., x terms to the side of dx, y terms to dy, etc., and integrate them.

Note: Independent variables should never be treated as constants during this integration. For e.g., you cannot have a y term along with the x terms as a product, and treat them as a constant during the integration with respect to x.

(d) Find "two solutions" by equating any two of the equations (any way), and integrating them as a whole.

(e) Substitute the constants obtained from integration (which are a function of the independent variables) into the equation,

$$F(C_1, C_2) = 0$$

Note: Both the solutions should be independent of each other. For e.g., $C_1 = 3C_2$ is not a valid solution.

If it fails at (c): Some composite functions cannot be eliminated by variable-separable method, when you

(i) Choose three constants l, m and n, such that

$$lP + mQ + nR = 0$$

(ii) Substitute l, m and n in the equation, and integrate with respect to the independent variables, to obtain an expression (say, U).

$$\int l \cdot dx + \int m \cdot dy + \int n \cdot dz = 0$$

(iii) Now, use another set of unique values for l, m and n satisfying (i), and follow the same steps to obtain another expression (say V).

(iv) Substitute in the equation,

$$F(u, v) = 0$$

1.3 Non-linear PDE:

Characteristics:

- Degree higher than 1.
- Product of p and q, or with the dependent variable.

Procedure for solving different types:

Type 1: $F(p, q) = 0$

- Complete solution:

  1) Write complete solution:

  $$\mathbf{z} = ax + by + c$$

  2) Substitute (a, b) for (p, q) in the given function, and find $b = f(a)$ by equating $f(a, b) = 0$.
  3) Substitute $b = f(a)$ in the complete solution,

  $$\mathbf{z} = ax + f(a)y + c$$

  .

- General solution:

  1) Substitute $c = \phi(a)$ in the complete solution,

  $$\mathbf{z} = ax + f(a)y + \phi(a)$$

  .

2) Differentiate partially with respect to a,

$$x + f'(a)y + \phi'(a) = 0$$

.

3) **Don't forget** to write, "By eliminating a from (1) and (2), we obtain the general solution".

- Singular solution:
  Partially differentiate complete solution with respect to c, and you'll get $0 = 1$. But since $0 \neq 1$, there's **no singular solution**.

Type 2: $F(\mathbf{z}, p, q) = 0$

- Complete solution:

1) Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \ \frac{\partial u}{\partial y} = a$$

Since $u = f(x, y)$, $\mathbf{z}$ can be $f(u)$.

$$\implies p = \frac{\partial \mathbf{z}}{\partial x} = \frac{d\mathbf{z}}{du} \cdot \frac{\partial u}{\partial x} = \frac{d\mathbf{z}}{du}$$

$$\implies q = \frac{\partial \mathbf{z}}{\partial y} = \frac{d\mathbf{z}}{du} \cdot \frac{\partial u}{\partial y} = a\frac{d\mathbf{z}}{du}$$

2) Substitute the "new" p, q in $F(\mathbf{z}, p, q) = 0$.
3) Rearrange the equation (variable-separable method), and integrate it with the respective $(u, \mathbf{z})$, to get the corresponding $f(u, \mathbf{z})$.
4) Substitute $u = x + ay$ in the equation, and rearrange it, to obtain the complete solution in the form $\mathbf{z} = f(x, y)$

.

- General solution:

1) Substitute $c = \phi(a)$ in the complete solution.
2) Differentiate partially with respect to a.
3) **Don't forget** to write, "By eliminating a from expressions (1) and (2), we obtain the general solution".

- Singular solution:
Partially differentiate complete solution with respect to c, and you'll get $0 = 1$. But since $0 \neq 1$, there's **no singular solution**.

Type 3: $F(x, p) = f(y, q)$

- Complete solution:

1) Rearrange the expression to the required form, and equate the functions to constant a,

$$F(x, p) = f(y, q) = a$$

2) Obtain (p, q) from the above as,

$$p = \phi(x, a),\ \ q = \psi(y, a)$$

3) Substitute (p, q) in the equation $d\mathbf{z} = pdx + qdy$, and integrate it to get $\mathbf{z}$.

$$\int d\mathbf{z} = \int p \cdot dx + \int q \cdot dy$$

- General solution:

  1) Substitute $c = \phi(a)$ in the complete solution.
  2) Differentiate partially with respect to a.
  3) **Don't forget** to write, "By eliminating a from expressions (1) and (2), we obtain the general solution".

- Singular solution:
  Partially differentiate complete solution with respect to c, and you'll get $0 = 1$. But since $0 \neq 1$, there's **no singular solution**.

Type 4: $\mathbf{z} = px + qy + f(p, q)$

- Complete solution:
  Substitute $p = a$, $q = b$ in the given equation to get the complete solution,

$$\mathbf{z} = ax + by + f(a, b)$$

- General solution:

    1) Substitute $b = \phi(a)$ in the complete solution,

    $$\mathbf{z} = ax + \phi(a)y + f\big(a, \phi(a)\big)$$

    2) Differentiate partially with respect to a,

    $$x + \phi'(a)y + f'\big(a, \phi(a)\big) = 0$$

    3) **Don't forget** to write, "By eliminating a from expressions (1) and (2), we obtain the general solution".

- Singular solution: (Type 4 *has* singular solution)

    1) Partially differentiate complete solution with respect to (a, b), to obtain two equations.

    2) Do any substitution possible, to eliminate the constants, and arrive at a generalized solution, as singular solution doesn't have constants!

Reduction to Standard form:

- For equations of the form $F(x^m \mathbf{z}^k p, y^n \mathbf{z}^k q) = 0$, substitute

$$X = \begin{cases} x^{1-m}, \ m \neq 1 \\ log(x), \ m = 1 \end{cases} \quad Y = \begin{cases} y^{1-n}, \ n \neq 1 \\ log(y), \ n = 1 \end{cases}$$

$$\mathbf{Z} = \begin{cases} \mathbf{z}^{k+1}, \ k \neq -1 \\ log(\mathbf{z}), \ k = 1 \end{cases}$$

- Updated notations are,

$$P = \frac{\partial \mathbf{Z}}{\partial X} = \frac{\partial \mathbf{Z}}{\partial \mathbf{z}} \cdot \frac{\partial \mathbf{z}}{\partial x} \cdot \frac{\partial x}{\partial X}$$

$$Q = \frac{\partial \mathbf{Z}}{\partial Y} = \frac{\partial \mathbf{Z}}{\partial \mathbf{z}} \cdot \frac{\partial \mathbf{z}}{\partial y} \cdot \frac{\partial y}{\partial Y}$$

- Substitute P and Q in the given equation, and solve the non-linear PDE, by finding the type.

Note: For the complete, general, and singular solutions, don't forget to replug the original x, y, p, q.

1.4 Solving linear homogeneous n$^{\text{th}}$ order PDE with constant coefficients:

Equation of the form, ($a_0, a_1, a_2, \ldots\ a_n =$ constants)

$$a_0\frac{\partial^n \mathbf{z}}{\partial x^n} + a_1\frac{\partial^n \mathbf{z}}{\partial x^{n-1}\partial y} + a_2\frac{\partial^n \mathbf{z}}{\partial x^{n-2}\partial^2 y} + \ldots + a_n\frac{\partial^n \mathbf{z}}{\partial y} = f(x, y)$$

Solution is $\mathbf{z} = \text{C.F} + \text{P.I}$

Characteristic function:

(i) Express in symbolic form,

$$D = \frac{\partial}{\partial x},\ D' = \frac{\partial}{\partial y}$$

(ii) Put $D' = 1$, $D = m$, and equate $f(m) = 0$, to find the roots m_1 and m_2.

- For real and distinct roots,

$$\text{C.F} = f_1(y + m_1x) + f_2(y + m_2x)$$

- For real and equal roots,

$$\text{C.F} = f_1(y + mx) + xf_2(y + mx)$$

Particular Integral:

$$\text{P.I} = \frac{F(x, y)}{f(D, D')}$$

- $F(x, y) = e^{ax+by}$

$$D = a,\ D' = b$$

$$\text{P.I} = e^{ax+by} \cdot \frac{1}{f(a, b)}$$

- $F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$

$$D^2 = -a^2,\ D'^2 = -b^2,\ DD' = -ab$$

$$\text{P.I} = F(x, y) \cdot \frac{1}{f(-a^2, -ab, -b^2)}$$

- $F(x,y) = x^m y^n$
  1) Take a higher-order term out of the denominator, and bring the remaining up! (inverse),
  $$\text{P.I} = \frac{F(x,y)}{1 + f(D,D')}$$
  $$\implies \text{P.I} = (1 + f(D,D'))^{-1} \cdot F(x,y)$$
  2) Expand using binomial expansion (stopping with PDE's highest order as n, which is mostly '2'),
  $$(1+x)^{-1} = 1 - x + x^2$$
  3) Multiply the expressions, and differentiate $F(x,y)$ if it's D, and integrate if it's $\frac{1}{D}$

- $F(x,y) = \phi(x,y)$
  1) Factorize the denominator as $(D, m_1D')(D, m_2D')$
  2) Substitute $y = c - m_1x$ in the numerator, and integrate it with respect to x, while neglecting (D, m_1D') in the denominator.
  3) Loop (2), until the denominator vanishes.

  For e.g.,
  $$\text{P.I} = \frac{(y-1)e^x}{D^2 - DD' - 2D'^2}$$

$$= \frac{(y-1)e^x}{(D-2D')(D+D')}$$

Sub. $y = c - 2x$,

$$= \frac{1}{D+D'} \int (c-2x-1)e^x \cdot dx$$

$$= \frac{1}{D+D'}(y+1)e^x$$

Sub. $y = c + x$,

$$= \int (c+x+1)e^x \cdot dx \ = ye^x$$

1.5 Solving linear non-homogeneous $\mathbf{n^{th}}$ order PDE with constant coefficients:

(i) Factorize the mixture of D, D', D^2, D'^2, DD' to one of the forms,

- $(D - m_1D' - c_1)(D - m_2D' - c_2)$
- $(D' - m_1D - c_1)(D' - m_2D - c_2)$

(ii) Find the coefficients m and c.

(iii) Substitute in the respective C.F,

- $e^{c_1x} f_1(y + mx) + e^{c_2x} f_2(y + mx)$
- $e^{c_1y} f_1(x + my) + e^{c_2y} f_2(x + my)$

(iv) Finding P.I remains the same.

1.6 Second-order PDE:

General form:

$$\left(AD^2 + BDD' + CD'^2\right)\mathbf{z} = F(x, y)$$

where A, B, C are $f(x, y)$

- $B^2 - 4AC < 0 \to$ elliptic
- $B^2 - 4AC = 0 \to$ parabolic
- $B^2 - 4AC > 0 \to$ hyperbolic

1.7 Integral surface passing through the given curve:

Procedure:

(i) Solve the PDE by *any* method (pg. no. 8 → 16)

(ii) Substitute the solutions in the given curves, and relate them in any way to get the "constant-less" general equation (similar to singular solution).

This equation gives a curve that passes through both of the given curves.

2 Fourier Series

2.1 Dirichlet's Condition:

The given function $f(x)$

- is finite, periodic, single-valued
- has finite number of discontinuities
- has the utmost finite number of maximas & minimas

2.2 General Fourier Series:

For a periodic function defined over $(c, c+2l)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where a_0, a_n & b_n are Euler's integrals

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) \cdot dx$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) \cdot dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot dx$$

If the given interval is of the form $(-l, l)$ - either as a whole (or) split into multiple intervals for defining a discontinuous function, check whether the function is odd or even.

Note: For discontinuous functions, be sure to switch the inequalities when 'minus' sign is encountered.

Case 1: Odd function $f(-x) \neq f(x)$

$$a_0 = 0,\ a_n = 0$$
$$b_n = \frac{2}{l}\int_0^l f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot dx$$
$$\implies f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Case 2: Even function $f(-x) = f(x)$

$$a_0 = \frac{2}{l}\int_0^l f(x) \cdot dx$$
$$a_n = \frac{2}{l}\int_0^l f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) \cdot dx$$
$$b_n = 0$$
$$\implies f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Case 3: Neither odd, nor even

$$f(x) \text{ remains the same}$$

2.3 Convergence:

If the given $f(x)$ converges at $x = \alpha$,

- which is a limit of the given function, then

$$f(x) = \frac{1}{2}\big[f(c) + f(c+2l)\big]$$

- which is a discontinuity, then

$$f(x) = \frac{1}{2}\big[f(x^-) + f(x^+)\big]$$

- and, if it's continuous, then $f(x) = \alpha$.

2.4 Half-range Expansion:

$f(x)$ over the interval (o, c) can be expanded into two distinct half-range series.

- **Half-range cosine series:**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^c f(x) \cdot dx$$

$$a_n = \frac{2}{l}\int_0^c f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot dx$$

- **Half-range sine series:**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^c f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot dx$$

2.5 Root Mean Square:

$$[f(x)]_{\text{rms}} = \sqrt{\frac{1}{b-a}\int_a^b f(x)^2\cdot dx}$$

2.6 Passeval's Identity:

If $f(x)$ has a Fourier series (i.e.) the function satisfies Dirichlet's condition, and has a period of $(-l, l)$, then

$$\frac{1}{l}\int_{-l}^{l}\big(f(x)\big)^2\cdot dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Corollary:

(i) If the interval is $(0, 2l)$

$$\int_0^{2l} \big(f(x)\big)^2 \cdot dx = l \left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right)$$

(ii) For half-range cosine series,

$$\int_0^{l} \big(f(x)\big)^2 \cdot dx = \frac{l}{2} \left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \right)$$

(iii) For half-range sine series,

$$\int_0^{l} \big(f(x)\big)^2 \cdot dx = \frac{l}{2} \left(\sum_{n=1}^{\infty} b_n^2 \right)$$

2.7 Complex form of Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{(a_n - i\ b_n)}{2}\ e^{i\frac{n\pi x}{l}} + \sum_{n=1}^{\infty} \frac{(a_n + i\ b_n)}{2i}\ e^{-i\frac{n\pi x}{l}}$$

(OR)

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n\ e^{i\frac{n\pi x}{l}} + \sum_{n=1}^{\infty} C_{-n}\ e^{-i\frac{n\pi x}{l}}$$

$$C_n = \frac{(a_n - i\ b_n)}{2}$$

$$C_n = \frac{1}{2l} \int_{-l}^{l} f(x) \cdot e^{-i\frac{n\pi x}{l}} \cdot dx$$

For values of $n = 0, 1,\ \& -1$, corresponding values of $C_0, C_n,\ \&\ C_{-n}$ can be generated. So,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\ e^{i\frac{n\pi x}{l}}$$

2.8 Harmonic Analysis: (numerical)

$$\text{Mean of } f(x) = \frac{1}{b-a} \int_{a}^{b} f(x) \cdot dx$$

$$\frac{a_0}{2} = \text{Mean of } f(x) \implies a_0 = \frac{2\sum y}{N}$$

$$a_n = \frac{2\sum y \cdot \cos\left(\frac{n\pi x}{l}\right)}{N}$$

$$b_n = \frac{2\sum y \cdot \sin\left(\frac{n\pi x}{l}\right)}{N}$$

Tabulate the values, and write the summed up $f(x)$, up to the n-values specified in the question.

3 Applications of PDE

3.1 Solving Linear PDE:

Most of the "linear PDE" can be solved by using variable-separable method.

Procedure:

(a) Write the assumption $Z = X(x) \cdot Y(y)$

(b) Using the assumption, reformulate the given PDE in terms of X and Y (i.e.) substitute, and partially differentiate with respect to independent variables.

(c) Separate X and Y terms, and equate them to some constant (say 'a').

(d) Individually solve the ODE by integrating the terms along with a.

(e) Substitute the obtained X and Y in the assumption (a) to get the required function Z.