Numerical Methods

WAFFLE'S CRAZY PEANUT

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1 Solution of Equations:

1.1 Fixed-point Iteration:

(i) Identify whether the given $f(x)$ is linear algebraic, non-linear algebraic, or transcendental equation.

(ii) Start finding $f(0)$, $f(1)$, ... until there's a change of sign in the value, which corresponds to the limit (a, b) within which the root lies.

(iii) Write the function in the form $x = \phi(x)$

(iv) Check the condition $|\phi'(a)| < 1$ and $|\phi'(b)| < 1$

(v) Now, find x_0, $x_1 = \phi(x_0)$, $x_2 = \phi(x_1)$, ... for values lying in the limit (a, b) and stop when the repetition of rounded values occurs.

Note: For infinite series, as there's no specific interval, find the roots directly by taking $x = f(x)$, neglecting higher powers, and iterating using step (v).

Keep in mind: Root-finding will be easier if iterations begin with the value nearer to a or b, based on whether $|f(a)|$ or $|f(b)|$ is closer to zero.

If $f(a)$ is closer to zero, then the root is closer to a, and vice versa.

1.2 Newton-Raphson Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Procedure:

(i) Identify whether the given $f(x)$ is linear algebraic, non-linear algebraic, or transcendental equation.

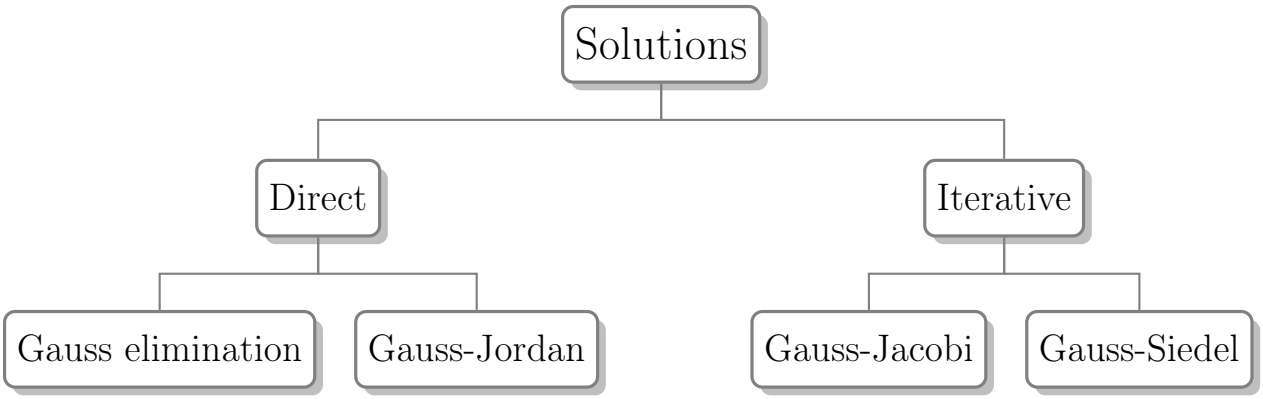
(ii) Start finding $f(0)$, $f(1)$, ... until there's a change of sign in the value, which corresponds to the limit (a, b) within which the root lies.

(iii) Now, find x_0, x_1, x_2... using the iterative formula, and proceed until repetition occurs.

Some formulas (can be derived):

- If $x = \frac{1}{N}$,
$$x_{n+1} = x_n(2 - Nx_n)$$
- If $x = \sqrt{N}$,
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$
- If $x = \frac{1}{\sqrt{N}}$,
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$
- If $x = N^{1/k}$,
$$x_{n+1} = \frac{1}{k}\left((k-1)x_n + \frac{N}{x_n^{k-1}}\right)$$

1.3 Solution of linear system of equations:



- **Direct:** In this method, the unknowns will be eliminated one by one, and hence the given system of equations will be transformed into an equivalent upper-triangular matrix.
- **Iterative:** Here, the unknowns are not eliminated, but are *approximated* by subsequent iterations, starting from an initial value of "zero" for the variables (x_0, y_0, z_0), and proceeding for the rounded-number-repetition.

1.4 Gauss elimination:

(i) Get the equations in matrix form $\mathrm{AX} = \mathrm{B}$

(ii) Write the augmented matrix $[\mathrm{A}, \mathrm{B}]$

(iii) Resolve the components in such a way that A becomes upper-triangular.

(iv) Solve the equations to obtain the unknowns.

1.5 Gauss-Jordan:

(iii) The difference here is that the components should be resolved in such a way that A becomes a diagonal matrix.

1.6 Gauss-Jacobi:

For a system of equations of the form,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Assuming diagonal-dominance,

$$|a_1| > |b_1| + |c_1|$$
$$|b_2| > |a_2| + |c_2|$$
$$|c_3| > |a_3| + |b_3|$$

We get the rule,

$$x_{n+1} = \frac{1}{a_1}(d_1 - b_1y_n - c_1z_n)$$
$$y_{n+1} = \frac{1}{b_2}(d_2 - a_2x_n - c_2z_n)$$
$$z_{n+1} = \frac{1}{c_3}(d_3 - a_3x_n - b_3y_n)$$

Starting with $x(0) = 0,\ y(0) = 0,\ z(0) = 0$, the iteration is carried out until the values start repeating with the required degree of accuracy.

1.7 Gauss-Siedel:

The "faster" iteration rule is (substitute all the known values),

$$x_{n+1} = \frac{1}{a_1}(d_1 - b_1y_n - c_1z_n)$$
$$y_{n+1} = \frac{1}{b_2}(d_2 - a_2x_{n+1} - c_2z_n)$$
$$z_{n+1} = \frac{1}{c_3}(d_3 - a_3x_{n+1} - b_3y_{n+1})$$

1.8 Matrix Inversion by Gauss-Jordan:

(i) Write the augmented matrix $[\mathrm{A}, \mathrm{I}]$

(ii) Resolve the components in such a way that A becomes an identity matrix.

(iii) The inverse is the other partition of the matrix.

1.9 Eigen values using Power method:

(i) If the given square matrix A is of order n, then choose an Eigen vector χ_0 of order $n \times 1$.

(ii) Find $\mathrm{A}\chi_0$ and take the largest number out. This number is λ_1 and the remaining $n \times 1$ matrix is χ_1

(iii) Proceed for rounded-number-repetition, which gives the final values for the largest Eigen value λ, and the corresponding Eigen vector χ.

(iv) To get the least Eigen value, find $\mathrm{B} = \mathrm{A} - \lambda\mathrm{I}$ and choose an Eigen vector ψ_0

(v) Follow the same step, now taking the least value out. The final value obtained is the Eigen value for B. To get the Eigen value for A, add both the Eigen values. This is the least Eigen value of A.

(vi) Use other identities to find the last Eigen value,

$$\text{Sum of Eigen values} = \text{Sum of Diagonal elements}$$

$$\text{Product of Eigen values} = |\mathrm{A}|$$

1.10 Eigen values using Jacobian method:

(i)

2 Interpolation:

2.1 Lagrange's Interpolation: (unequal)

$$y = f(x) = \frac{(x-x_1)(x-x_2)...\ (x-x_n)}{(x_0-x_1)(x_0-x_2)...\ (x_0-x_n)}\ y_0$$
$$+\ \frac{(x-x_0)(x-x_2)...\ (x-x_n)}{(x_1-x_0)(x_1-x_2)...\ (x_1-x_n)}\ y_1$$
$$+\ ...\ +\ \frac{(x-x_0)(x-x_1)...\ (x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...\ (x_n-x_{n-1})}\ y_n$$

2.2 Newton's Divided Differences: (unequal)

$$\Delta_{x_1} f(x_0) = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

(i) Generate Δ, Δ^2,... Δ^n for the given table values of x.

(ii) Use the interpolation formula to get the polynomial,

$$y = f(x_0) + (x - x_0)\ f(x_0, x_1) + (x - x_0)(x - x_1)\ f(x_0, x_1, x_2) + ...$$

2.3 Cubic Spline: (equal)

(i) Find the interval h, number of intervals n, so that i takes the values from 1, 2,... $(n-1)$, and $M_0 = 0$, $M_n = 0$.

(ii) Substitute the values of i in the formula one by one,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\ \left(y_{i-1} - 2y_i + y_{i+1}\right)$$

(iii) Solve the equations for values of M_i.

(iv) Take i =1, 2,... n, and substitute for each interval in the formula,

$$y = S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right]$$
$$+\frac{1}{h}\left[(x_i - x)(y_{i-1} - \frac{h^2}{6}M_{i-1})\right]$$
$$+\frac{1}{h}\left[(x - x_{i-1})(y_i - \frac{h^2}{6}M_i)\right]$$

2.4 Newton's Forward & Backward Difference:

(i)