Project 5 :
Thermometer in Varying Temperature Stream

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Analyzing the model :

We look at our model of the response of the thermometer :

$$X_s - Y_s = \tau \frac{dY_s}{dt} \tag{1}$$

τ is a scalar constant, $Y_s,\ X_s$ are both function of t, such that $X_s(t) = x - x_s$, $Y_s(t) = y - y_s$, where x is the temperature of the fluid in the stream, and y is the response of the thermometer and $x_s,\ y_s$ are both constants.

Solving the model using specific data :

We are given $x(t) = x_s - Asin(\omega t)$, using our knowledge that $X_s = x(t) - x_s$, it follow that :

$$X_s = Asin(\omega t) \tag{2}$$

substituting the value of X_s in our model:

$$\tau Y_s^{'} + Y_s = Asin(\omega t) \tag{3}$$

We can see that we have something that looks like a first order O.D.E, we try to solve the system by taking the Laplace transform of (3) :

$$\mathcal{L}\{\tau Y_s^{'} + Y_s\} = \mathcal{L}\{Asin(\omega t)\} \implies \tau\mathcal{L}\{Y_s^{'}\} + \mathcal{L}\{Y_s\} = A\mathcal{L}\{sin(\omega t)\} \tag{4}$$

Before approaching the terms in (4), we recall some important transforms :

$$\mathcal{L}\{y'\} = sY + y(0) \tag{5}$$

$$\mathcal{L}\{sin(\omega t)\} = \frac{w}{s^2 + w^2} \tag{6}$$

Now we are ready to approach the terms in (4), for convention we express the Laplace transform of $Y_s(t)$ as $Y(s)$:

$$\mathcal{L}\{Y_s^{'}\} = sY - Y_s(0) \tag{7}$$

$$\mathcal{L}\{Asin(\omega t)\} = \frac{A\omega}{s^2 + w^2} \tag{8}$$

Since at $t = 0$ the thermometer is at equilibrium, the response of the thermometer is zero, thus we have the following initial condition:

$$Y_s(0) = 0 \implies Y(0) = 0 \tag{9}$$

We use the important transformation and our initial condition and substitute in (4) :

$$\tau s Y + Y = \frac{A\omega}{s^2 + \omega^2} \implies \boxed{Y = \frac{A\omega}{s^2 + \omega^2} \cdot \frac{1/\tau}{s + 1\ /\tau}} \tag{10}$$

To solve for $Y_s(t)$ we have to take the inverse Laplace transform of our result in (10).

$$Y = \mathcal{L}^{\text{-}1}\{\frac{A\omega}{s^2 + \omega^2} \cdot \frac{1/\tau}{s + 1\ /\tau}\} \implies \frac{A}{\tau}\mathcal{L}^{\text{-}1}\{\frac{\omega}{s^2 + \omega^2} \cdot \frac{1}{s + 1\ /\tau}\} \tag{11}$$

we see that we have a product of two functions which, on their own, can be computed directly from a transformation table. Before delving into the problem, we recall the definition of a convolution of two functions.

Definition 1 *Let $f(t)$, $g(t)$ be two functions defined on $\mathbb{R}$ whose Laplace transforms are $F(s)$ and $G(s)$ and let $u(t)$ be an arbitrary variable. We say that the convolution of f and g denoted by $f \star g$ is a function of t such that :*

$$f \star g = \mathcal{L}^{\text{-}1}\{F(s)\ G(s)\} = \int_0^t f(u)g(t-u)du \tag{12}$$

Now we tackle the term in (11) by setting $F(s) = \frac{\omega}{s^2+\omega^2}$, $G(s) = e^{\frac{1}{s+1\ /\tau}}$, it follows from the transformation table that :

$$f(t) = \sin(\omega t),\ g(t) = e^{\frac{-1}{\tau}t} \tag{13}$$

Looking at the our term in (11), we can easily see that our function in (11) is exactly $F(s)\ G(s)$. Using our definition in (12), it follows that:

$$\mathcal{L}^{\text{-}1}\{\frac{\omega}{s^2 + \omega^2} \cdot \frac{1}{s + 1\ /\tau}\} = \int_0^t sin(wu)e^{\frac{-1}{\tau}(t-u)}du \tag{14}$$

Using integration by parts, it is easy to show that:

$$\mathcal{L}^{\text{-}1}\{\frac{\omega}{s^2 + \omega^2} \cdot \frac{1}{s + 1\ /\tau}\} = \int_0^t sin(wu)e^{\frac{-1}{\tau}(t-u)}du = \frac{\tau^2\omega e^{\frac{-t}{\tau}}}{\tau^2\omega^2 + 1} - \frac{\tau^2\omega cos(\omega t)}{\tau^2\omega^2 + 1} + \frac{\tau sin(\omega t)}{\tau^2\omega^2 + 1} \tag{15}$$

Now substituting in (11) :

$$Y_s = \frac{A}{\tau}(\frac{\tau^2\omega e^{\frac{-t}{\tau}}}{\tau^2\omega^2 + 1} - \frac{\tau^2\omega cos(\omega t)}{\tau^2\omega^2 + 1} + \frac{\tau sin(\omega t)}{\tau^2\omega^2 + 1}) = \boxed{\frac{A\tau\omega e^{\frac{-t}{\tau}}}{\tau^2\omega^2 + 1} - \frac{A\tau\omega cos(\omega t)}{\tau^2\omega^2 + 1} + \frac{A sin(\omega t)}{\tau^2\omega^2 + 1}} \tag{16}$$

Now, we compute the amplitude of the steady state term of our wave equation in (16) :

$$|Y_s| = \sqrt{(\frac{-A\tau\omega}{\tau^2\omega^2 + 1}^2) + (\frac{A}{\tau^2\omega^2 + 1}^2)} = A\sqrt{\frac{\tau^2\omega^2 + 1}{(\tau^2\omega^2 + 1)^2}} = \boxed{\frac{A}{\sqrt{\tau^2\omega^2 + 1}}} \tag{17}$$

Now we compute the phase angle of our steady state wave equation ϕ :

$$\tan(\phi) = \frac{\frac{-A\tau\omega}{\tau^2\omega^2+1}}{\frac{A}{\tau^2\omega^2+1}} = -\omega\tau \implies \phi = \arctan(-w\tau) \tag{18}$$

We can rewrite our wave equations as :

$$Y_s(t) = |Y_s|\; sin(\omega t + \phi) = \frac{A}{\sqrt{\tau^2\omega^2+1}} sin(\omega t + \arctan(-\omega\tau)) \tag{19}$$

To compute the response of the thermometer we just substitute for our constants: $\tau = 0.1$, $\omega = 20$, $A = 2$:

$$Y_s = \frac{2}{\sqrt{(0.1^2\; 20^2) + 1}} sin(20t + arctan(-20 \cdot 0.1)) \approx \boxed{0.896 sin(20t - 63.5)} \tag{20}$$

Solving the model with different assumptions :

We make the following assumptions :

- the mass of the mercury is $m = \frac{100}{3}\; lbF$
- the surface area of the the bulb is $A = 1 ft^2$
- the film coefficient is $h = 1\; hr ft^2 F$
- the frequency $f = 1$ cycle/second

Now we redo our calculations :

$$\tau = \frac{mC}{hA} = \frac{\frac{100}{3}\frac{3}{100}}{1} \tag{21}$$

$$\omega = 2\pi f = 2\pi \text{ radian/second} \tag{22}$$

Now we just substitute in (17) and (18) to compute the amplitude and the phase angle :

$$|Y_s| = \frac{1}{4\pi^2 + 1} \tag{23}$$

$$\phi = \arctan(-2\pi) = 0 \tag{24}$$

We substitute the values of $|Y_s|$, ϕ to compute Y_s :

$$Y_s = \frac{1}{4\pi^2 + 1} \sin(2\pi t) \tag{25}$$

Comparing the two models :

Now we plot the original model and our modified version $0 \leq t \leq 10$ with a step size of 0.1 :

```
[56]: import numpy as np
      import pandas as pd
      import math
      import matplotlib.pyplot as plt
```

```
[57]: y = lambda t : 0.896*(math.sin(20*t - 63.5))
      g = lambda t : 0.0247*math.sin(2*math.pi*t)
```

```
[58]: t_vals = np.arange(1,10,0.01)
      y_vals = [y(t) for t in t_vals]
      g_vals = [g(t) for t in t_vals]
```

```
[59]: plt.plot(t_vals, y_vals, '-b', label = 'original model')
      plt.plot(t_vals, g_vals, '-r', label = 'modified model')
      plt.legend(loc="upper left")
```

```
[59]: <matplotlib.legend.Legend at 0x28aee3e9cc0>
```

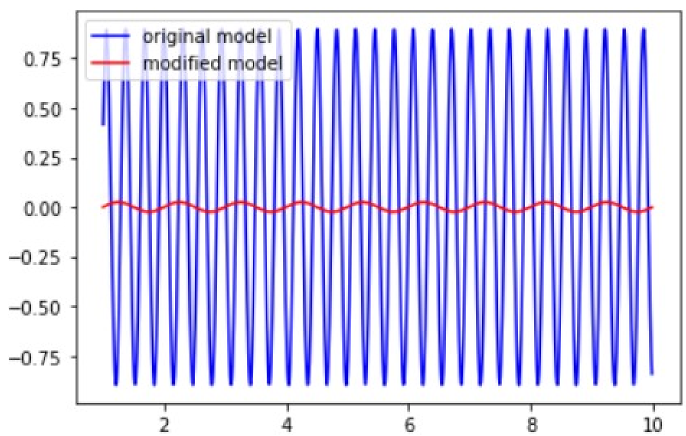

Figure 1: Plotting the two models in IPython

Observing the two models, we can see that our modified model has less intensity the the original model which is due to it having a smaller amplitude and a smaller frequency. The computation were almost identical to the original model.

Commentary :

This project consists of four main parts:

1. Analyzing the model :
   In this part I explained the constants and the variables; what they represent and how they relate to the model.

2. Finding the general solution to the model :
   In this part I used the Laplace transform method and the convolution property to find a general solution to the differential equation of our model.

3. Solving for the model :
   In this part is used our initial values and constants to find an exact solution to the model.

4. Solving the model with different assumptions : In this part I assumed the starting conditions and the values of the constants and solved for the exact solution of the model.

5. Comparing the two models :
   In this part I compared the two models by plotting them using IPython and Jupyter Notebook.

Problems I faced :

* Understanding the model was not easy, it took me some time to understand how the thermometer works and the nature of the functions.

* I didn't quite understand why we used the Laplace transform method while solving the O.D.E in (3) would have been an easier alternative.

* I avoided using partial fraction to solve for Y_s and used the convolution method instead hoping that I would save some time. But it turns out that using convolution is more tedious due to the integral in (14).

* At first I thought that the model was an equation of three variables x, y, t, which made the Laplace transform make more sense. But I solved (3) as regular O.D.E and it gave me the exact same answer as the Laplace transform method which made me realize that the function is only of two variables y, t.