Project 3: Whales and Krill Modeling

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What we know :

1- at $B = 0$, The krill population grow at a 25 % rate

2- at $K = 0$, The Blue Whales population decrease by 5 %

3- 500 tons/acre of Krill increase the growth rate of blue whales by 2 % per year

4- A population of 150000 blue whales decrease Krill growth rate by 10 % per year

We look at our pair of O.D.Es :

$$\frac{dB}{dt} = -0.05B + (\frac{0.02}{500})BK \tag{1}$$

$$\frac{dK}{dt} = 0.25K - (\frac{0.10}{150000})BK \tag{2}$$

<u>Part II Solutions :</u>

1. Since the units of B, K are *whale*, *ton acre*$^{-1}$, then by rules of Dimension Analysis the units of their derivatives with respect time are *whale year*$^{-1}$ and *ton acre*$^{-1}$ *year*$^{-1}$

2. Looking at the right side of each of the O.D.Es, we can notice we have a system of differential equations of two entities B,K in the form :

$$B'(t) = C_1B + C_2BK$$
$$K'(t) = C_3K + C_4BK$$

Generalization:

lets assume we have an eco-system of 2 populations $X_1(t), X_2(t)$, the governing system of O.D.Es according to this model:

$$\text{rate of change of species } X_1 = \text{ change in } X_1(\text{at } X_2 = 0) + \frac{\text{change in } X_1'(\text{at } X_2 = K_0)}{K_0} \quad (3)$$

Using our generalization :

- Since the whales decrease by a rate 0.05 when the krill population is zero and a population of 500 tons/acre of Krill increase the rate of growth of whales by 2 % $\implies$ the first term $-0.05B$ is negative since the population decrease, the second term is positive since the growth rate of whales increase by 2% when K = 500. and hence the first governing O.D.E is :

$$\frac{dB}{dt} = -0.05B + (\frac{0.02}{500})BK$$

- Since the Krill population increase by 25 % when the whale population is zero, and a population of 150000 whale decrease the growth rate of krill by 10 % per year $\implies$ the first term is $0.025K$ is positive since Krill grow at a 25% rate when the blue whales are non-existent, and the second term is negative since the population of whales will decrease the Krill growth rate by 10%, hence the second governing O.D.E is:

$$\frac{dK}{dt} = 0.25K - (\frac{0.10}{150000})BK$$

3. Using our generalization once again :

- the term $(\frac{0.02}{500})BK$ is controlling the change in whale population as the Krill population changes, where 0.02 is the increase in the rate of change whales corresponding to a density of 500 tons/acre of Krill

- the term $-(\frac{0.10}{150000})BK$ is controlling the change in the Krill population as the Whale population changes, where 0.10 is the decrease in rate of growth of the Krill population corresponding to a whale population of 150000.

4. Using Pandas and IPython :

Figure 1: doing the first 3 steps of Euler's method, $h = 1$, $0 \leq t \leq 3$

```
[34]: import numpy as np
      import pandas as pd
      f = lambda B, K: (-0.05*B) + ((0.02/500)*B*K)
      g = lambda B, K: (0.25)*K - ((0.1/150000)*B*K)
      B0 = 75000
      K0 = 150
      h = 1
```

```
[35]: t = [0,1,2,3]
```

```
[36]: B = np.zeros(4)
      K = np.zeros(4)
```

```
[37]: B[0] = B0
      K[0] = K0
```

```
[38]: for i in range(len(t)-1):
          B[i+1] = B[i] + (h*f(B[i], K[i]))
          K[i+1] = K[i] + (h*g(B[i], K[i]))
```

```
[41]: df = pd.DataFrame(
      {
      "B" : B,
      "K" : K}
      )
```

```
[42]: df
```

[42]:

	B	K
0	75000.000000	150.000000
1	71700.000000	180.000000
2	68631.240000	216.396000
3	65793.739032	260.593983

Part III Solutions :

5. Using the same script but with little tweaks :

Figure 2: tweaking the data set and the initial conditions, $0 \leq t \leq 100$

In [14]:
```
import numpy as np
import pandas as pd
f = lambda B, K: (-0.05*B) + ((0.02/500)*B*K)
g = lambda B, K: (0.25*K) - ((0.1/150000)*B*K)
B0 = 75000
K0 = 150
h = 1
```

In [22]:
```
t = np.arange(101)
B = np.zeros(101)
K = np.zeros(101)
```

In [23]:
```
B[0] = B0
K[0] = K0
```

In [26]:
```
for i in range(len(t)-1):
    B[i+1] = B[i] + (h*f(B[i],K[i]))
    K[i+1] = K[i] + (h*g(B[i],K[i]))
```

In [31]:
```
df = pd.DataFrame(
{
 "B": B,
 "K": K
}
)
```

In [32]:
```
df
```

Out[32]:

	B	K
0	75000.000000	150.000000
1	71700.000000	180.000000
2	68631.240000	216.396000
3	65793.739032	260.593983
4	63189.870181	314.312177
...	...	...
96	57676.924526	0.015260
97	54793.113505	0.018488
98	52053.498349	0.022434
99	49450.870143	0.027264
100	46978.380566	0.033182

101 rows × 2 columns

Figure 3: Plotting the whale population and the krill population for $0 \leq t \leq 100$

```
In [40]: import matplotlib.pyplot as plt
         plt.plot(t,B,'b',label='Whale')
         plt.plot(t,K,'g',label ='Krill')
         plt.grid()
         plt.legend(loc='upper right')
         plt.show()
```

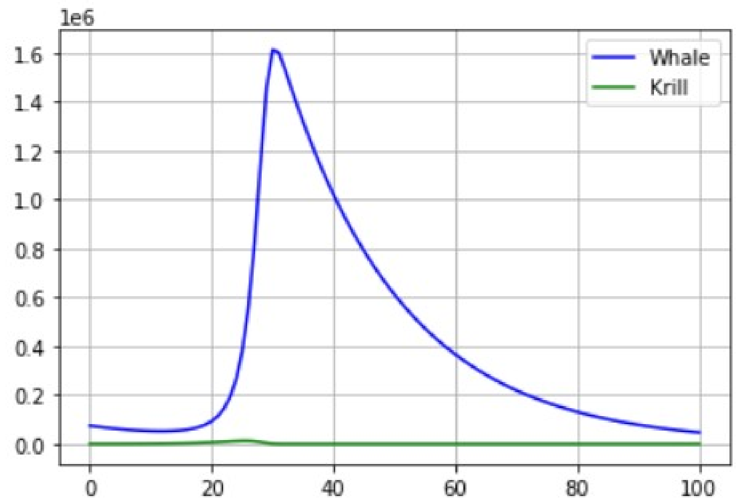
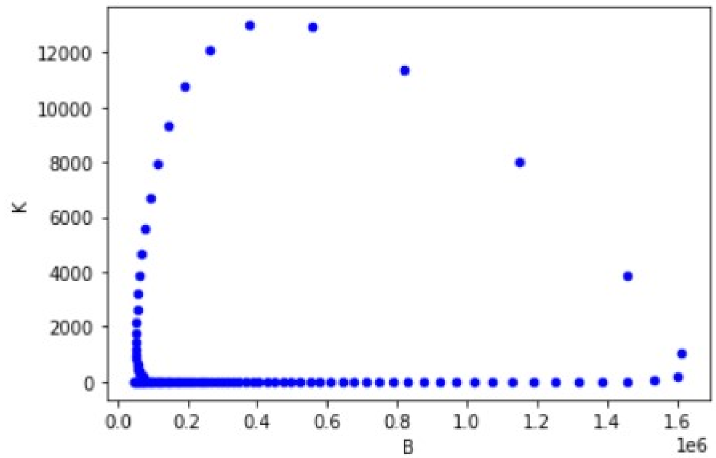


Figure 4: Scatter plot of the Whale vs Krill population

```
In [29]: df.plot.scatter(x='B',
                         y='K', c ='Blue')

Out[29]: <AxesSubplot:xlabel='B', ylabel='K'>
```



6. What happens after 100 years:

   if we look at figure 3, we can see that both B(t), K(t) have an asymptote at $y = 0$ as t tends to infinity, which implies that both species almost extinct over the course of 100 years. After $t \approx 30$ both the population of Krill and Whales start to decline. The Whale rate of growth decrease by a factor of $-0.05B$ and the rate of growth of Krill depends on the rate of growth of Whales. Similarly, the Krill growth rate decreases by a factor of $\frac{-0.1}{150000}$. That causes both population to decline at t grows until they nearly extinct at $t = 100$.

8. The Lotke-Volterra model of this species implies that the Krill population increase the growth of the Whale population, and the Whale population decreases the growth of Krill population. But the Whale population growth rate is decreased by a factor of $-0.05B$. Hence, as $t \to \infty$ both the population of Krill and Whales decrease until they go extinct.

9. The Whales growth rate depends on the initial population of Whales ($-0.05B$) and the population of the Krill $\frac{0.02}{500}BK$. We can ignore the constants, as they represent the same model but with different initial conditions (0.05, $\frac{0.02}{500}$).

   The Krill growth rate depends on the initial population of Krill ($0.25K$)and the whale population ($\frac{-0.1}{150000}BK$). we can Ignore the constants as the first case.

Commentary :

Using a numerical, graphical driven approach has proven to be beneficial in several cases where a general solution for the system does not exist or when the general solution is hard to compute.

Using a simulation for a Lotke-Volterra as the one shown in the project helps us understand the interaction between the two species, in our case the Whales and the Krill. However, this method can fail when the system fails to be approximated by numerical methods or when it has discontinuities.

I split my work into 3 stages:

1- Analyzing the Lotka-Volttera model
2- Numerical approximation of the system solution
3- Graphing the numerical solution

1- Analyzing the model :

I started by checking the system of O.D.Es and trying to understand the different terms and how they effect the population growth. Then, I tried to understand the interaction between the two species B, K.

2- Numerical Approximation:

I then wrote an IPython Jupyter-Notebook script that solves the system using Euler's method, setting $h = 1$ and varying t from 0 to 100. Then I used Pandas library to turn the data set into a table.

2- Graphing the solution:

I used matplotlib and numpy to plot a graph of the two function B(t), K(t) and varying t from 0 to 100. Then I used matploblib to plot a scatter plot of B on the x-axis and K on the y-axis.

Problems I faced:

- I had problems understanding the model and the interaction between the two species. For example, in the first equation $\frac{dB}{dt} = -0.05B + (\frac{0.02}{500})BK$ the rate of growth of Whales increases by a rate of $-0.05B$ and increase by a rate of $0.02/500BK$, so it seems like the rate of growth is not experiencing any fluctuations. However, looking at figure 3, we can see the population of Whales fluctuates reaching a maximum at $t \approx 30$.

- The units of the Krill population acres/ton decreases the range of values of K $(0 < K < 1200)$ which is negligible considering the range of B values $(0 < B < 1600000)$. That caused some visual problems with the graphs, as the values of K seems to behave like a straight line along the x-axis.

- I had to reference the documentation of Pandas several times to remember the syntax, which took time but was worthwhile.