MACM 101 — Discrete Mathematics I

Exercises on Functions and Induction. Due: Friday, November 7th (at the beginning of the class)

Reminder: the work you submit must be your own. Any collaboration and consulting outside resources must be explicitly mentioned on your submission.
Please, use a pen. 30 points will be taken off for pencil written work.

1. Is the function $f\colon \mathbb{R}^+ \to \mathbb{R}$ defined as $f(x) = \sqrt{x} + x + 2$ one-to-one?

2. Determine whether or not the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto, if $f((m,n)) = m^2 + n$.

3. Let $f(x) = ax + b$ and $g(x) = cx^2 + dx$, where a, b, c, and d are constants. Compute $f \circ g$ and $g \circ f$. Determine for which constants a, b, c, and d it is true that $f \circ g = g \circ f$.
(*Hint: Polynomials are equal as functions if and only if they have the same coefficients.*)

4. If $f \circ g$ is one-to-one, does it follow that g is one-to-one?

5. Show that the function $f\colon \mathbb{R} - \{-1\} \to \mathbb{R} - \{2\}$ defined by
$$f(x) = \frac{4x+3}{2x+2}$$
is a bijection, and find the inverse function.
(*Hint: Pay attention to the domain and codomain.*)

6. The kth harmonic number is defined to be
$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}.$$
Prove that harmonic numbers satisfy the equality
$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$$
for all $n \in \mathbb{N}$.

7. Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined by the rule: $F_1 = F_2 = 1$ and $F_k = F_{k-2} + F_{k-1}$ for $k > 2$.
   Lucas numbers $L_1, L_2, L_3, \ldots$ are defined in a similar way by the rule: $L_1 = 1$, $L_2 = 3$ and $L_k = L_{k-2} + L_{k-1}$ for $k > 2$.
   Show that Fibonacci and Lucas numvers satisfy the following equality for all $n \geq 2$
   $$L_n = F_{n-1} + F_{n+1}.$$

8. Prove that any amount of more that 7 cents can be represented by 3- and 5-cent coins. (Assume 3-cent coins exist.)

9. The game of Chomp is played by two players. In this game, cookies are laid out on a rectangular grid. The cookie in the top left position is poisoned. The two players take turns making moves; at each move, a player is required to eat a remaining cookie, together with all cookies to the right and/or below (that is all the remaining cookies in the rectangle, in which the first cookie eaten is the top left corner). The loser is the player who has no choice but to eat the poisoned cookie. Prove that if the board is square (and bigger than 1×1) then the first player has a winning strategy.

10. A complete binary tree is a graph defined through the following recursive definition.
    Basis step: A single vertex is a complete binary tree.
    Inductive step: If T_1 and T_2 are disjoint complete binary trees with roots r_1, r_2, respectively, the the graph formed by starting with a root r, and adding an edge from r to each of the vertices r_1, r_2 is also a complete binary tree.
    Prove that a complete binary tree has odd number of vertices.