Functions

Discrete Mathematics
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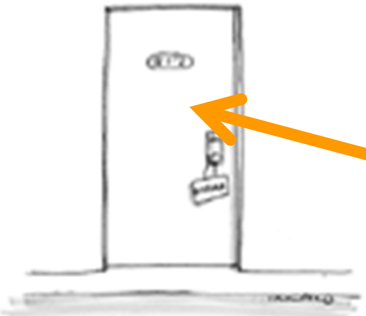
Previous Lecture

- Equivalence relations
- Partitions
- Partial orders
- Diagrams of partial orders

Functions

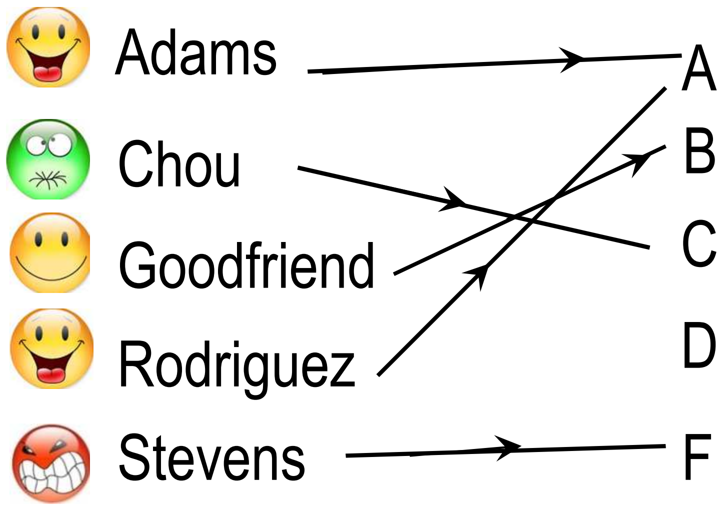
- In many instances we assign to each element of a set a particular element of a second set.
- For example , assign rooms to people in a hotel



- Or we may assign a grade to each student from a class
- What we get is a set of pairs (Person, Door) or (Student, Grade), that is, a relation, but a very particular one

Functions (cntd)

- A relation R from A to B is called a **function** from A to B, if for every $a \in A$ there is exactly one $b \in B$ such that $(a,b) \in R$. (Also mappings, transformations)



We use f,g,h to denote functions

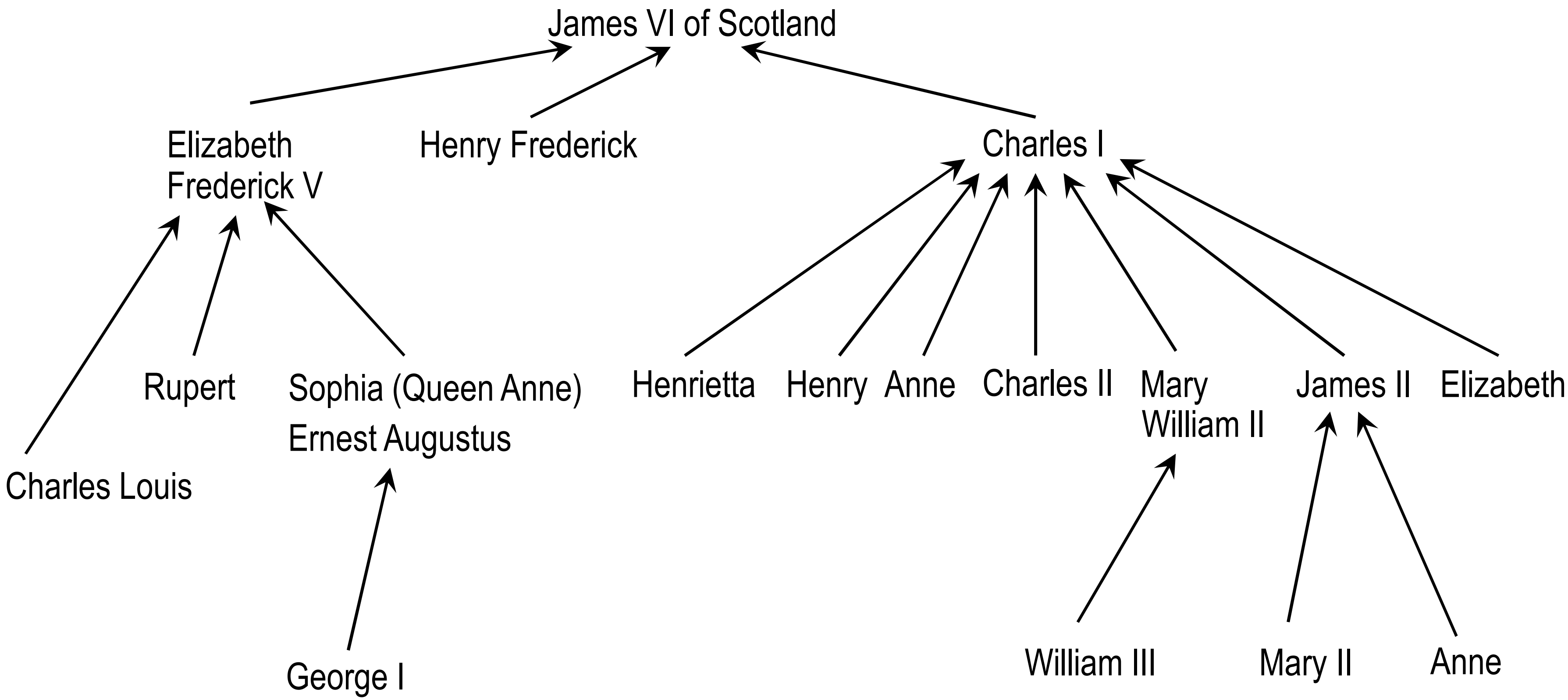
$f: A \rightarrow B$ $f(a) = b$

$f(\text{Rodriguez}) = A$

Example

- Consider the function from the set People to People:

  f(a) = b if b is the father of a.



(http://www.royal.gov.uk)

Domain and Codomain

- Let $f: A \to B$ be a function from A to B. Then A is called the **domain** of f, and B is called the **codomain** of f.

domain

codomain

- If $f(a) = b$, then b is called the **image** of a, and a is called the **preimage** of b.

- Also we say that f maps A to B

Domain and Codomain (cntd)

- The range of f is the set of all images of elements of A

  $\text{range}(f) = \{ b \in B \mid \exists a \in A \ \ f(a) = b \}$

- In our example:

  domain = { Adams, Chou, Goodfriend, Rodriguez, Stevens }

  codomain = { A, B, C, D, F }

  range = { A, B, C, F }

Restrictions and Extensions

- Let $f: A \to B$ be a function and $C \subseteq A$. The set

$$f(C) = \{\, b \in B \mid b = f(a) \text{ for some } a \in C \,\}$$

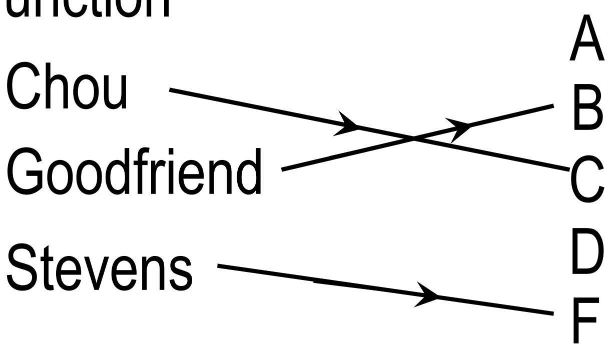
is called the **image** of C.

- Example: $f(\{\text{Adams, Rodriguez}\}) = \{A\}$

  $f(\{\text{Chou, Goodfriend, Stevens}\}) = \{B,C,F\}$

- Let $f: A \to B$ be a function and $C \subseteq A$. A function $f|_C : C \to B$ is called a **restriction** of f to C if $f|_C(a) = f(a)$ for all $a \in C$

- Example: Let C = {Chou, Goodfriend, Stevens}. Then the function



is a restriction of f

Restrictions and Extensions (cntd)

- Let $C \subseteq A$ and $f: C \to B$. Any function $g: A \to B$ such that $g(a) = f(a)$ for all $a \in C$ is called an extension of f.
- Let $A = \mathbb{R}$, $B = \mathbb{Z}$, $C = \mathbb{Z}$, and $f: \mathbb{Z} \to \mathbb{Z}$ is defined as follows:

  $$f(a) = a$$

  Let g be the floor function:

  $g(x) = \lfloor x \rfloor$ = the greatest integer less than or equal to x

  Clearly, $g: \mathbb{R} \to \mathbb{Z}$, and $g(a) = a = f(a)$ for any integer a.

  Thus, g is an extension of f

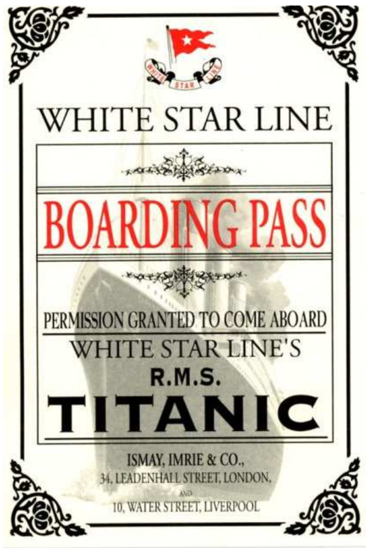
Describing Functions

- Function is a relation, therefore we can use all methods of describing relations. Although the graph and the matrix are not very economical.

  { (Adams,A), (Chou,C), (Goodfriend,B), (Rodriguez,A), (Stevens,F)}

- Function table

Student	Grade
Adams	A
Chou	C
Goodfriend	B
Rodriguez	A
Stevens	F



Describing Functions (cntd)

- Numerical functions can be computed using a formula

  $\mathrm{f}(x) = x^2$

  range(f) = { 0, 4, 9, …} – non-negative integers that are perfect squares

- The most general way is to use some algorithm to compute a function

  `The letter grade is A, if the numerical mark is in between 100 and 85; the letter grade is B, if …

  Functions in programming languages:

  int **floor**(float real) {…} in Java

  function floor(x: **real**): **integer** in Pascal

One-to-One Functions

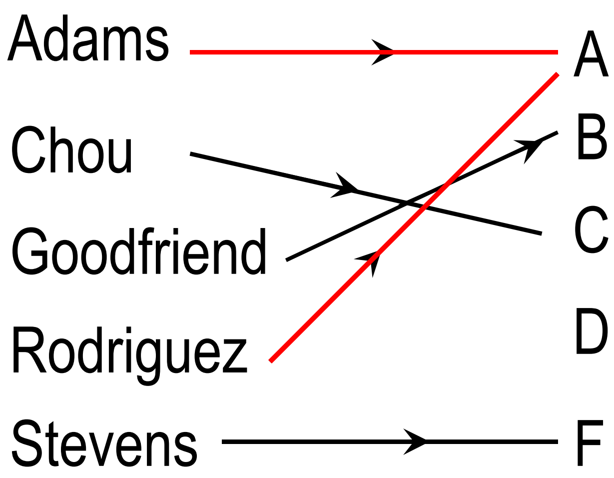
- A function f is said to be one-to-one, or injective, if and only if $f(a) = f(b)$ implies $a = b$.

  In other words no two elements are mapped into the same image.

  Contrapositive: if $a \neq b$ then $f(a) \neq f(b)$.

  Symbolically: $\forall a \, \forall b \, (f(a) = f(b) \rightarrow a = b)$

- Is this function injective?

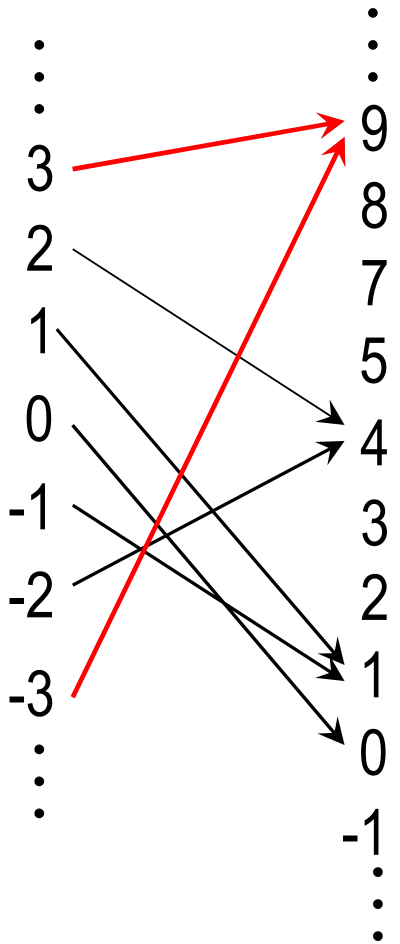


No!

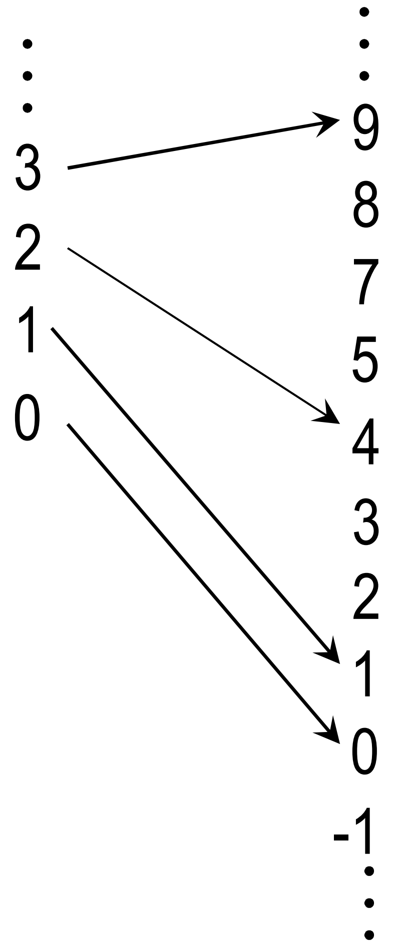
One-to-One Functions (cntd)

- Let's consider the function $f(x) = x^2$ on $\mathbb{Z}$

  Is it injective?



No!



Yes! on $\mathbb{N}$

Onto Functions

- A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called a surjection if it is onto.

  Symbolically: $\forall b \, \exists a \, (f(a) = b)$

- Examples:

  - $f(x) = x + 1$

    Yes, because for any $b \in \mathbb{Z}$ there is $a \in \mathbb{Z}$ such that $a + 1 = b$

  - $f(x) = x^2$ on $\mathbb{Z}$

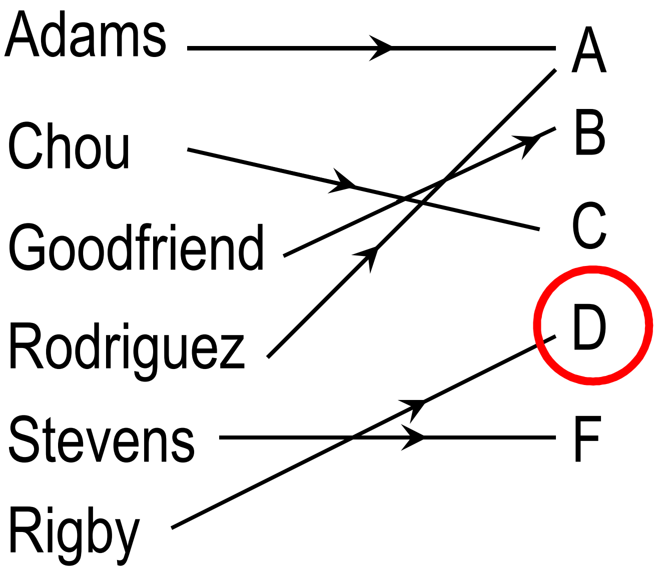
    No, because $\sqrt{2}$ is not an integer

  - What about the same function on $\mathbb{R}^+$

    Yes, the square root of a real number is a real number

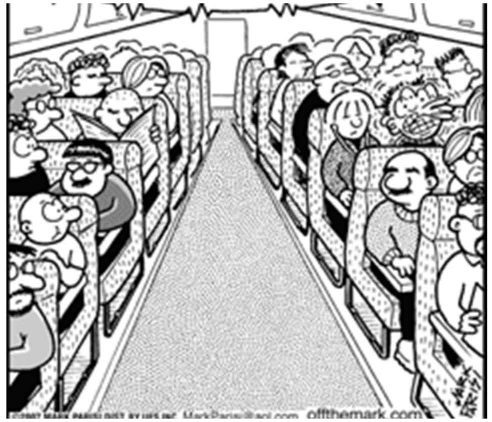
Onto Functions (cntd)

- More examples



No!

Yes!

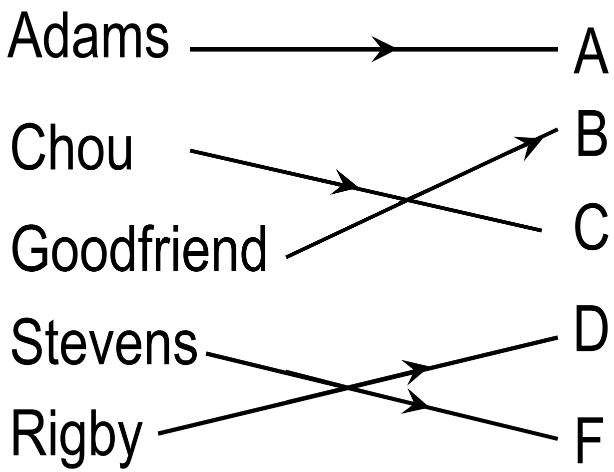


This mapping was onto!

- f(a) = b if b is the father of a

Bijections

- A function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.

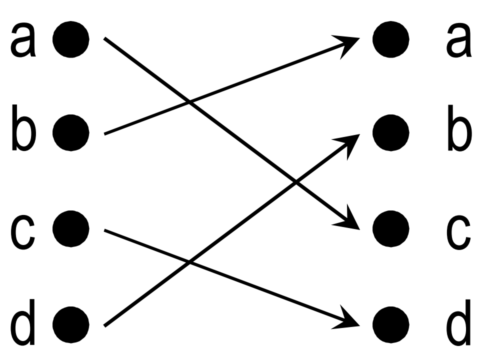


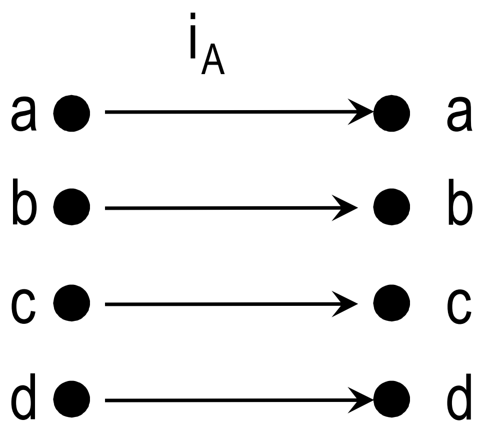
- If there is a bijection from a set A to a set B, then these sets in a certain sense are equal or identical.

Bijections (cntd)

- Numerical functions:
  - $f(x) = x + 1$ is a bijection on $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, but not on $\mathbb{N}$
  - $f(x) = x^2$ is a bijection on $\mathbb{R}^+$, but is not on any other numerical set

- A bijection from a set A to the same set A is called a **permutation** of A



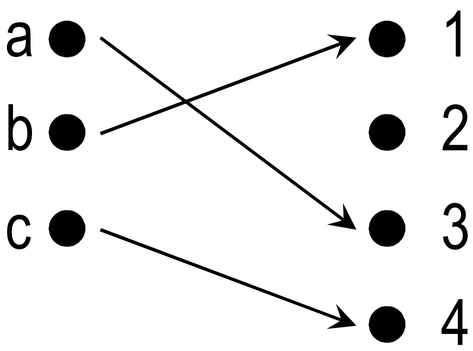


- The **identity function** on a set A is the function $i_A : A \to A$, where

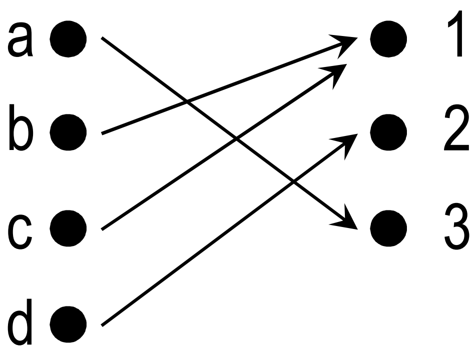
$$i_A(x) = x$$

Functions and Properties

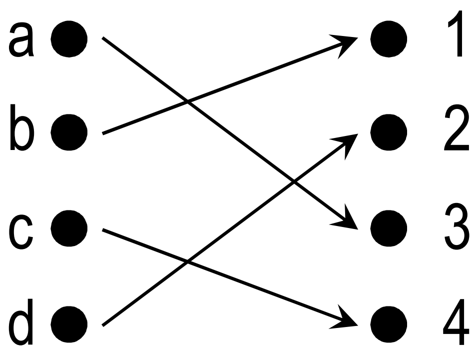
- Examples of different types of correspondences



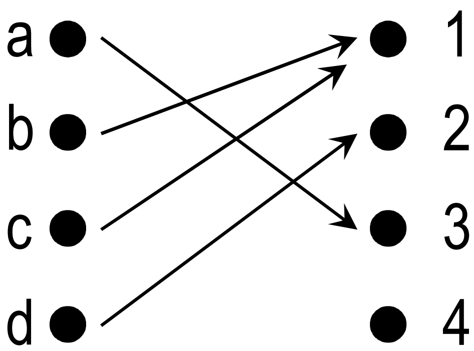
one-to-one, not onto



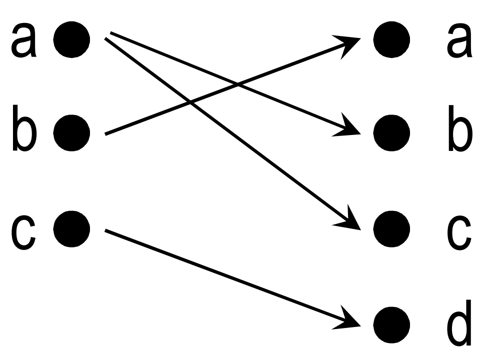
onto, not one-to-one



one-to-one and onto



neither one-to-one nor onto

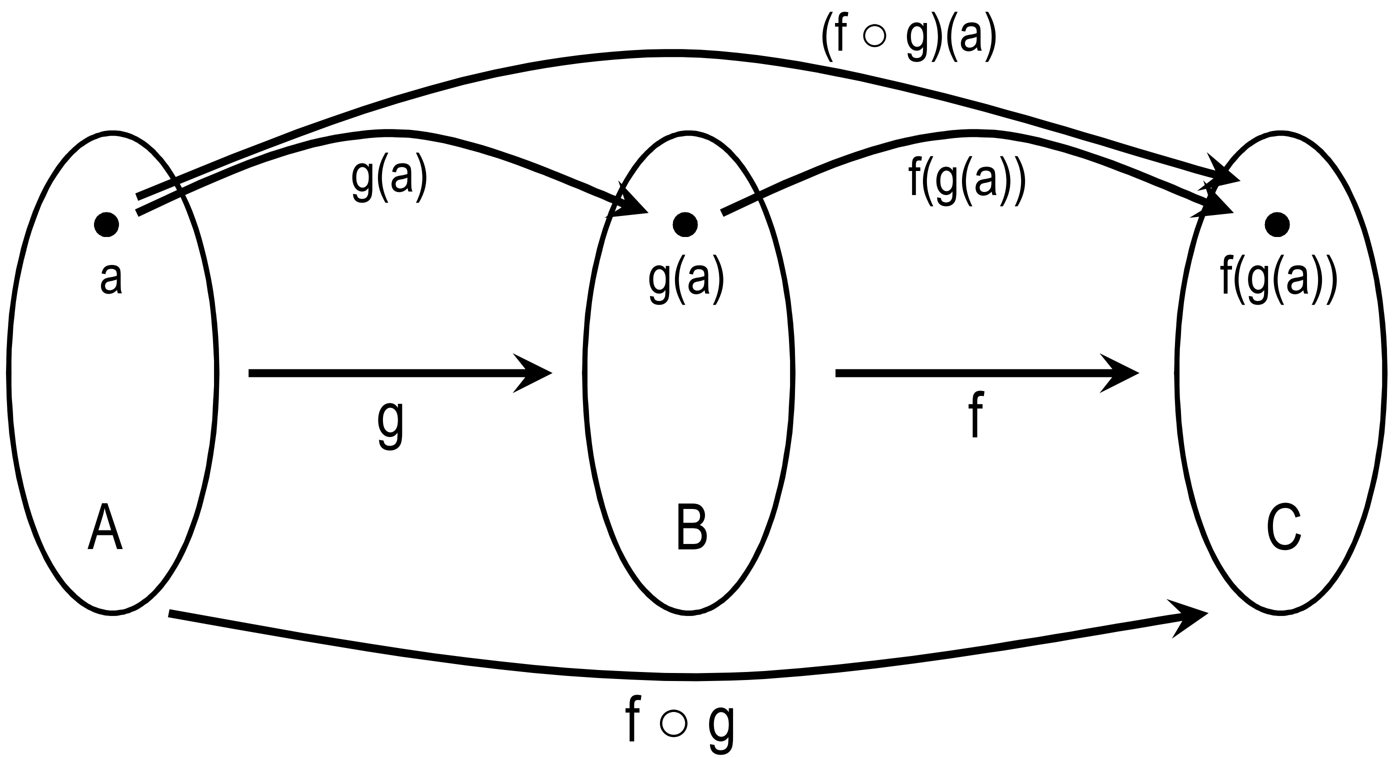


not a function

Composition of Functions

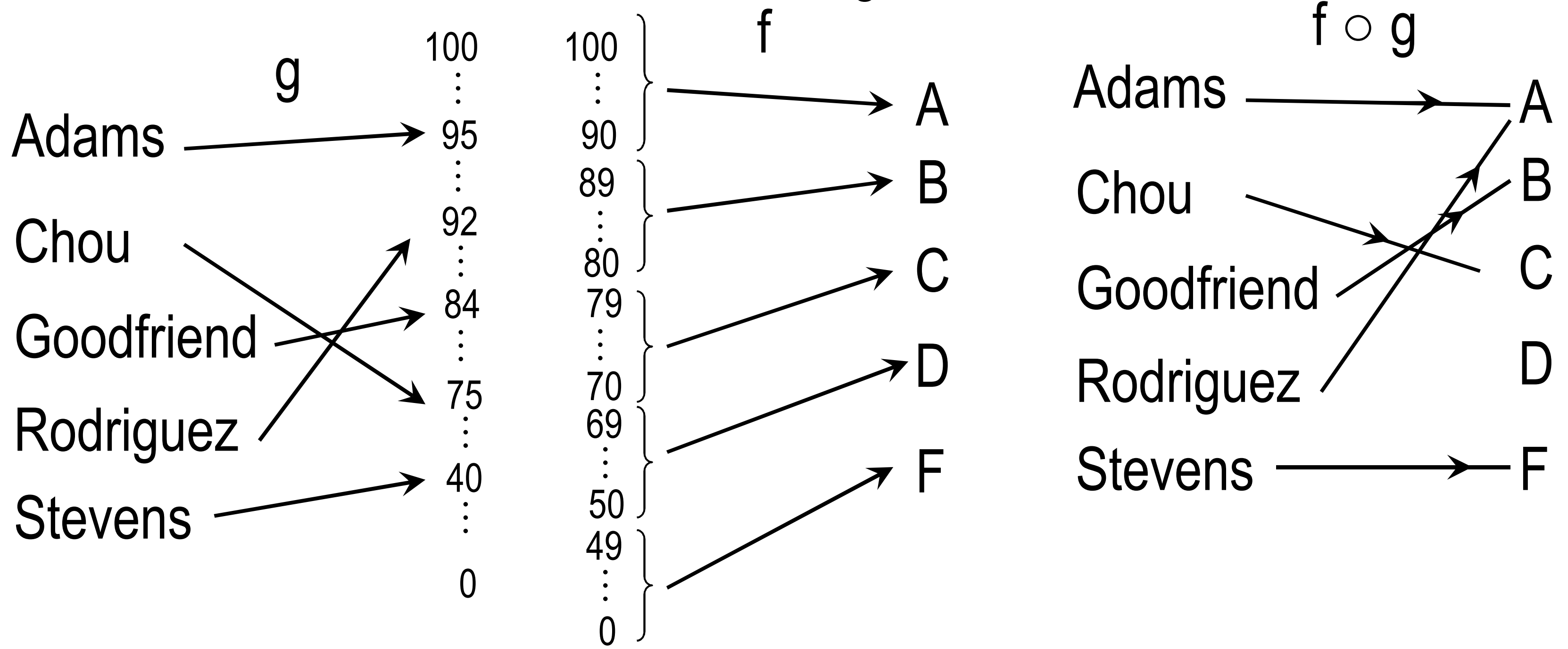
- Let g be a function from A to B and let f be a function from B to C. The composition of the functions f and g, denoted by $f \circ g$, is the function from A to C defined by

$$(f \circ g)(a) = f(g(a))$$



Composition of Functions (cntd)

- Suppose that the students first get numerical grades from 0 to 100 that are later converted into letter grade.



- Let f(a) = b mean `b is the father of a'.

  What is $f \circ f$?

Composition of Numerical Functions

- Let $g(x) = x^2$ and $f(x) = x + 1$. Then
  $(f \circ g)(x) = f(g(x)) = g(x) + 1 = x^2 + 1$

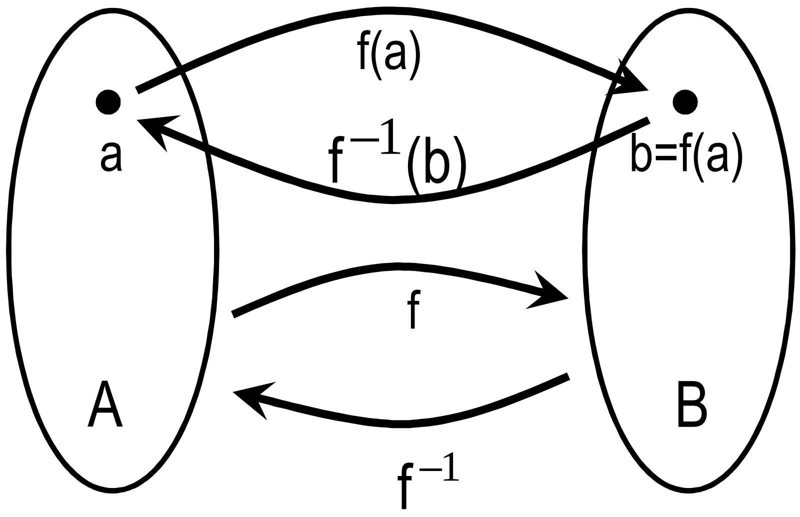
- Thus, to find the composition of numerical functions f and g given by formulas we have to substitute $g(x)$ instead of x in $f(x)$.

Inverse Functions

- Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

  The inverse function is denoted by f^{-1}.

  Thus, $f^{-1}(b) = a$ if and only if $f(a) = b$



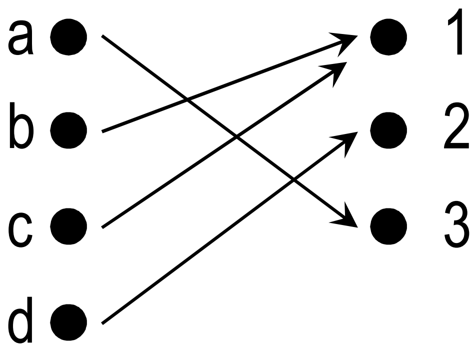
Note!
f^{-1} does not mean $\dfrac{1}{f(x)}$

$f \circ f^{-1} = i_B$

$f^{-1} \circ f = i_A$

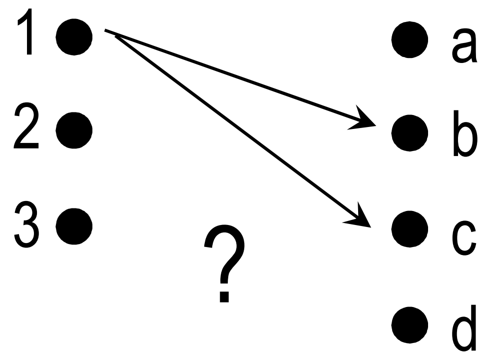
Inverse Functions (cntd)

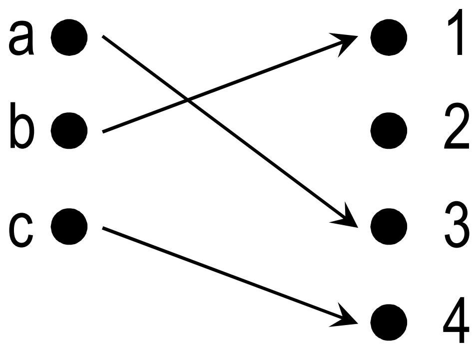
- If a function f is not a bijection, the inverse function does not exist. Why?
- If f is not a bijection, it is either not one-to-one, or not onto



onto, not one-to-one

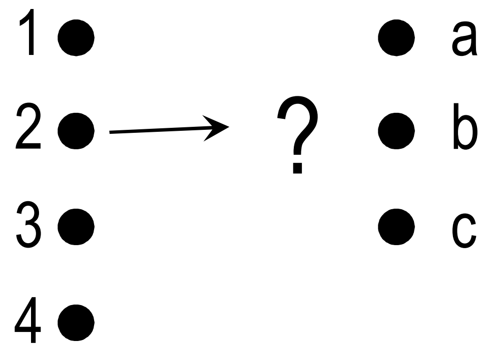
$f^{-1}(1) = ?$





one-to-one, not onto

$f^{-1}(2) = ?$



Homework

Exercises from the Book:

No. 1, 2, 6a,15, 16ace, 18 (page 258)