MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Sets and Relations

1. **Using laws of set theory show that**

$$\overline{(A \cup B) \cap C} = (\overline{A} \cup \overline{C}) \cap (\overline{B} \cup \overline{C}).$$

$$\begin{array}{lll} \overline{(A \cup B) \cap C} & & \\ \quad = \overline{(A \cap C) \cup (B \cap C)} & & \text{distributive law} \\ \quad = \ (\overline{A \cap C}) \cap (\overline{B \cap C}) & & \text{DeMorgan's law} \\ \quad = \ (\overline{A} \cup (\overline{C}) \cap (\overline{B} \cup \overline{C}) & & \text{DeMorgan's law} \end{array}$$

2. **Let A, B, and C be sets. Show that**

$$(\overline{A} \cup B) \cap (\overline{C} - A) = \overline{C} - A.$$

Draw Venn diagrams for both expressions.

Method 1. From one of the problems of Tutorial 5 we know that $A - B = A \cap \overline{B}$. Then

$$\begin{array}{lll} (\overline{A} \cup B) \cap (\overline{C} - A) & & \\ \quad = \ (\overline{A} \cup B) \cap (\overline{C} \cap \overline{A}) & & \text{expression for difference} \\ \quad = \ (\overline{A} \cup B) \cap \overline{A} \cap \overline{C} & & \text{associative and commutative law} \\ \quad = \ \overline{A} \cap \overline{C} & & \text{absorption law} \\ \quad = \ \overline{C} - A & & \text{expression for difference} \end{array}$$

Method 2. First, prove that $(\overline{A} \cup B) \cap (\overline{C} - A) \subseteq \overline{C} - A$. This is obvious, since the left hand side is intersection of the right hand side with another set.

Next we prove that $\overline{C} - A \subseteq (\overline{A} \cup B) \cap (\overline{C} - A)$. Take $a \in \overline{C} - A$. We need to show that $a \in \overline{A} \cup B$ and that $a \in \overline{C} - A$. The latter is obvious. Then a does not belong to A and $a \in \overline{C}$. In particular, $a \in \overline{A}$, and therefore $a \in \overline{A} \cup B$. For Venn diagram see Fig. 1.

3. **What can you say about the sets A and B if we know that $\overline{A \cap B} = \overline{B}$? Explain.**

Since $\overline{A \cap B}$ contains the elements that do not belong to at least one the sets A, B, if $\overline{A \cap B} = \overline{B}$, then every element of $\overline{A}$ belongs to $\overline{A \cap B}$, and, hence, belongs to $\overline{B}$. Thus, $\overline{A} \subseteq \overline{B}$, and so $B \subseteq A$. Conversely, if $B \subseteq A$, then $A \cap B = B$, and therefore $\overline{A \cap B} = \overline{B}$.

4. **Show that for any sets A, B, and C**

$$(A \triangle B) \triangle C = A \triangle (B \triangle C).$$

We show that both $(A \triangle B) \triangle C$ and $A \triangle (B \triangle C)$ contain exactly the elements that belong to odd number of sets A, B, C, that is, $x \in (A \triangle B) \triangle C$ ($x \in A \triangle (B \triangle C)$) if and only if x belongs to all A, B, C, or it belongs to exactly one of the sets A, B, C.

Consider first $(A \triangle B) \triangle C$. Element x belongs to this set either if $x \in A \triangle B$ and $x \notin C$, or if $x \notin A \triangle B$ and $x \in C$. In the first case x belongs to exactly one of A and B, and does not belong to C. In the latter case there are two options. First, $x \notin A$ and $x \notin B$. In this case x only belongs to C. Second, $x \in A \cap B$; in this case x belongs to all three sets. The statement for $A \triangle (B \triangle C)$ is quite similar.

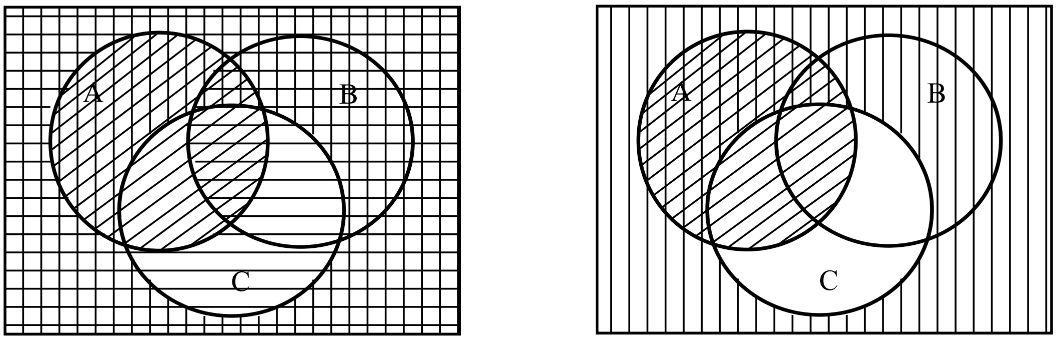


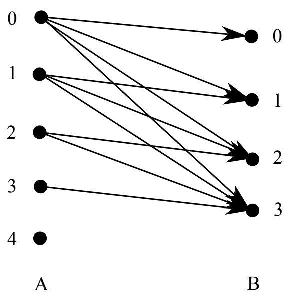
Figure 1: On the left: Vertical hatching represents $\overline{C}$, horizontal hatching represents $\overline{A} \cup B$, diagonal hatching represents A. Therefore $\overline{C} - A$ is the area with only vertical hatching, without diagonal one; and $(\overline{A} \cup B) \cap (\overline{C} - A)$ is represented by the area with both vertical and horisontal hatching, but without diagonal hatching. On the right: Vertical hatching represents $\overline{C}$, diagonal hatching represents A. Therefore $\overline{C} - A$ is the area with only vertical hatching, without diagonal one

5. **Make a list of pairs, construct the matrix, and draw the graph of the relation R from the set $A = \{0, 1, 2, 3, 4\}$ to the set $B = \{0, 1, 2, 3\}$ such that $(a, b) \in R$ if and only if $a - b < 1$.**

   The set of pairs $R = \{(0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$; the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

   and the graph



6. **Prove that**

$$C \times B \times (A \cap C) = (C \times B \times A) \cap (C \times B \times C).$$

   Method 1. We have

$$\begin{aligned}
&C \times B \times (A \cap C) \\
&\quad = \{(a, b, c) \mid (a \in C) \wedge (b \in B) \wedge (c \in A \cap C)\} \\
&\quad = \{(a, b, c) \mid (a \in C) \wedge (b \in B) \wedge (c \in A \wedge c \in C)\} \\
&\quad = \{(a, b, c) \mid ((a \in C) \wedge (b \in B) \wedge (c \in A)) \wedge ((a \in C) \wedge (b \in B) \wedge (c \in C))\} \\
&\quad = \{(a, b, c) \mid (a \in C) \wedge (b \in B) \wedge (c \in A)\} \cap \{(a, b, c) \mid (a \in C) \wedge (b \in B) \wedge (c \in C)\} \\
&\quad = (C \times B \times A) \cap (C \times B \times C).
\end{aligned}$$

Method 2. We show that $C \times B \times (A \cap C) \subseteq (C \times B \times A) \cap (C \times B \times C)$, and that $(C \times B \times A) \cap (C \times B \times C) \subseteq C \times B \times (A \cap C)$.

$C \times B \times (A \cap C) \subseteq (C \times B \times A) \cap (C \times B \times C)$. Take an element (a, b, c) from $C \times B \times (A \cap C)$. Then $c \in A \cap C$, and hence $c \in A$ and $c \in C$. Since $a \in C, b \in B$, we have $(a, b, c) \in C \times B \times A$, and we have $(a, b, c) \in C \times B \times C$. Thus $(a, b, c) \in (C \times B \times A) \cap (C \times B \times C)$.

$(C \times B \times A) \cap (C \times B \times C) \subseteq C \times B \times (A \cap C)$. Take an element (a, b, c) from $(C \times B \times A) \cap (C \times B \times C)$. Then $(a, b, c) \in C \times B \times A$ and $(a, b, c) \in C \times B \times C$. This implies $a \in C, b \in B$; and also it implies that $c \in A \cap C$. Thus, $(a, b, c) \in C \times B \times (A \cap C)$.

7. **Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$, that is elements of this relation are pairs of pairs of integers, such that $((a, b), (c, d)) \in R$ if and only if $a - d = c - b$. Show that R is an equivalence relation.**

   We should prove that R is reflexive, symmetric, and transitive.

   Reflexivity. For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, we have $a - b = a - b$.

   Symmetricity. Let $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$ be such that $a - d = c - b$. Then $c - b = a - d$, and therefore $((c, d), (a, b)) \in R$.

   Transitivity. Let $((a, b), (c, d)), ((c, d), (e, f)) \in R$, that is, $a - d = c - b$ and $c - f = e - d$. Adding this two equations up we get $a - d + c - f = c - b + e - d$, hence, $a - f = e - b$ implying $((a, b), (e, f)) \in R$.

8. **Relation R is given by matrix**

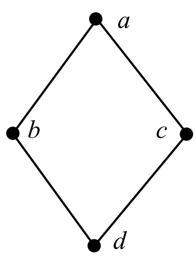
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

   Is R an order? If yes, what its minimal, maximal, least, and greatest elements are?

   We need to check that this relation is reflexive, transitive, and anti-symmetric. Reflexivity can be easily seen from the matrix: every entry on the diagonal equals 1. Anti-symmetricity also can be seen from the matrix: for every off-diagonal entry that is equal to 1 the symmetric entry equals 0.

   To verify transitivity let us name the elements of the set on which the relation is given by a, b, c, d (accordingly to the order of rows). Then the list of pairs of this relation is $R = \{(a, a), (b, a), (b, b), (c, a), (c, c), (d, a), (d, b), (d, c), (d, d)\}$. For every two pairs of the form $(x, y) \in R$ and $(y, z) \in R$ we must verify that $(x, z) \in R$. Note that x can be equal to y, or y can be equal to z, or even $x = y = z$. We proceed as follows. Take (a, a) for (x, y) then the second pair should start with a. There is only one such pair (a, a). We have $(a, a), (a, a) \in R$ and have to check that the pair $(x, z) \in R$; that is $(a, a) \in R$. It is true. Next take (b, a) for (x, y). Again the only option for (y, z) is (a, a), therefore (x, z) equals (b, a), and it is in R. Then we check (b, b) (and the matching pairs are $(b, a), (b, b)$), then (c, a), etc.

   Once we proved that R is an order we can draw its diagram (see next page). From the diagram we see that R



   has one minimal element, d, which is also the least element. R also has one maximal element a, which is also the greatest element.

9. **Let $A = \{1, 2, 3, 4\}$, and let R be a binary relation on $A \times A$ given by: $((a, b), (c, d)) \in R$ if and only if a divides c and b divides d. Show that R is an order and draw its diagram.**

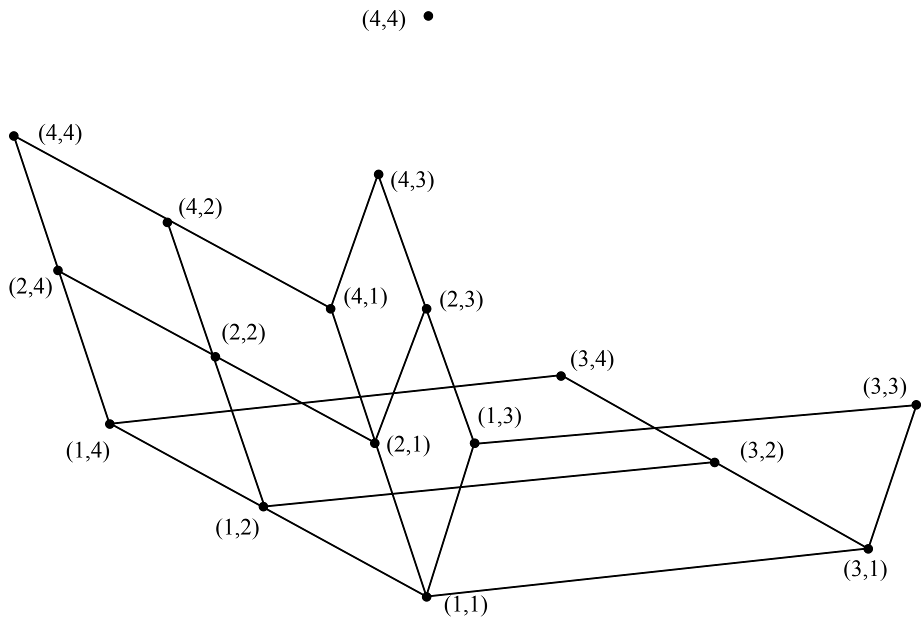
   Let us denote $B = A \times A$. This is a set of pairs: $B = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3)$
   Show that R is an order.

   Reflexivity. For any $(a, b) \in B$ we have $((a, b), (a, b)) \in R$ because a divides a and b divides b.

   Anti-Symmetricity. Suppose that $((a, b), (c, d)) \in R$ and $((c, d), (a, b)) \in R$. Then from the first pair we obtain a divides c and b divides d, while from the second one we get c divides a and d divides b. Now, since a divides c and c divides a, it follows that $a = c$. Similarly $b = d$. Thus $(a, b) = (c, d)$.

   Transitivity. Suppose that $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$. Then from the first pair we obtain a divides c and b divides d, while from the second one we get c divides e and d divides f. Now, since a divides c and c divides e, it follows that a divides e. Similarly b divides f. Thus $((a, b), (e, f)) \in R$.

   Finally the diagram of this order looks as follows:



10. **Give an example of a relation that is not reflexive, symmetric, not transitive, and not antisymmetric.**

    We give an artificial example simply by choosing a set of pairs that satisfies all the requirements. Let $A = \{a, b, c\}$ and set $R = \{(a, b), (b, a), (b, c), (c, b)\}$. This relation is not reflexive as none of the pairs $(a, a), (b, b), (c, c)$ belong to R. It is symmetric, because $(a, b) \in R$, and $(b, a) \in R$, and also $(b, c) \in R$, and $(c, b) \in R$. Relation R is not transitive, since $(a, b), (b, c) \in R$, but $(a, c) \notin R$. Finally, it is not anti-symmetric, because $(a, b), (b, a) \in R$, but $a \neq b$.