Logic Equivalence

Discrete Mathematics
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Previous Lecture

- Quantifiers
- Free and bound variables
- Multiple quantifiers and logic connectives

Quantifiers and Compound Statements

- Quantifiers can be used together with logic connectives

``Every car is either red or blue”

P(x) - ``car x is red” Q(x) - ``car x is blue”

$\forall x\, (P(x) \vee Q(x))$

``Everyone who knows a current password can logon onto the network”

P(x) - ``x knows a current password”

Q(x) - ``x can logon onto the network”

$\forall x\, (P(x) \rightarrow Q(x))$

Quantifiers and Compound Statements (cntd)

- Logic connectives can be put between quantified statements

  ``Every car is blue, or there is a red car''

  $P(x)$ - ``car $x$ is blue'' $Q(x)$ - ``car x is red''

  $(\forall x\ P(x)) \vee (\exists x\ Q(x))$

  ``For every number there is a smaller one, or there is the least number''

  We use predicate $x \le y$

  $(\forall x\ \exists y\ (y \le x)) \vee (\exists x\ \forall y\ (x \le y))$

Definitions

- Predicates and quantifiers are often used to give definitions

  ``The mother of x is the female parent of x”

  $P(x,y)$ - ``$y$ is a parent of $x$”  $Q(x)$ - ``x is a female”

  $M(x,y)$ is defined as $P(x,y) \wedge Q(y)$ ``y is the mother of x”

$$M(x,y) \leftrightarrow P(x,y) \wedge Q(y)$$

``x and y are brothers”

$$B(x,y) \leftrightarrow \neg Q(x) \wedge \neg Q(y) \wedge \exists z\, (P(x,z) \wedge P(y,z))$$

Definitions (cntd)

- A cow is a big rectangular animal with horns and four legs in the corners

Definition of a limit:

A number A is a limit of a sequence $\{a_n\}$ if for any number ε > 0 there is N such that for any $n > N$ we have $|a_n - A| < \varepsilon$

$$\forall \varepsilon\, \exists N\, \forall n\ ((\varepsilon > 0) \wedge (n > N) \rightarrow (|a_n - A| < \varepsilon))$$

Rules

- Predicates and quantifiers are (implicitly) present in rules and laws

``Everyone having income more that $20000 must file a tax report''

P(x) - ``x has income more than $20000''

Q(x) - ``x must file a tax report''

$\forall x\, (P(x) \rightarrow Q(x))$

Theorems

- Every theorem involves predicates and quantifiers

``For every statement there is an equivalent CNF''

C(x) - ``x is a CNF'' $\forall x\, \exists y\, (C(y) \wedge (x \Leftrightarrow y))$

``A parallelogram is a rectangle if all its angles are equal''

R(x) - ``parallelogram x is a rectangle''

A(x) - ``all angles of x are equal''

$\forall x\, (A(x) \rightarrow R(x))$

Universe and Interpretation

- A logic statement is meaningless

It only makes sense if we specify a universe and a particular meaning of the predicate

$\forall x \, P(x)$

Interpretation

universe: animals
P(x): x has horns

universe: cars
P(x): x is red

universe: numbers
P(x): x is even

Logical Equivalence of Predicates

- Recall that two compound statements Φ and Ψ are logically equivalent $(\Phi \Leftrightarrow \Psi)$ if and only if $\Phi \leftrightarrow \Psi$ is a tautology.

- For predicates:

  Two predicates P(x) and Q(x) are logically equivalent in a given universe if and only if, for any value a from the universe statements P(a) and Q(a) are equivalent

  if and only if the statement $\forall x\, (P(x) \leftrightarrow Q(x))$ is true in the given universe

``A parallelogram is a rectangle if and only if all its angles are equal"

P(x) - ``x is a rectangle" Q(x) - ``all angles of x are equal"

$P(x) \Leftrightarrow Q(x)$ in the universe of parallelograms

Logical Equivalence of Quantified Statements

- Two quantified statements are said to be logically equivalent if they are equivalent **for any** given universe.
- Consider statements

$$\exists x\,(P(x) \land Q(x)) \quad \text{and} \quad (\exists x\, P(x)) \land (\exists x\, Q(x))$$

We prove that they are NOT logically equivalent.

Logical Equivalence of Quantified Statements (cntd)

- We have to find a universe, in which they are not equivalent.

  Let the universe consist of integers,

  $P(x)$ means $x > 5$, and $Q(x)$ means $x < 3$.

  Then $(\exists x\, P(x)) \wedge (\exists x\, Q(x))$ claims that ``there is a number greater than 5, and there is a number less than 3''. This is true, as 6 witnesses the first claim, and 2 - the second claim.

  $\exists x\, (P(x) \wedge Q(x))$ means that some (the same!) number is greater than 5 and less than 3, which is impossible.

Logical Equivalence of Quantified Statements

- Consider statements

  $\exists x\, (P(x) \vee Q(x))$ and $(\exists x\, P(x)) \vee (\exists x\, Q(x))$

  We prove that they are logically equivalent.
- We had to prove that in ANY UNIVERSE and INTERPRETATION if the first statement is true then the second is true, and if the second is true then the first is true.
- Since we cannot consider every possible universe, we use the following trick:
- We consider an arbitrary universe
- This means that, although we look at a certain universe, we do not make any assumptions about it, except those that follow from the problem
- In our case: P(x) and Q(x) are interpreted somehow

Logical Equivalence of Quantified Statements (cntd)

- Suppose that $\exists x\,(P(x) \vee Q(x))$ is true in a certain universe. This means that there is a in the universe such that $P(a)$ is true or $Q(a)$ is true. If $P(a)$ is true then a witnesses that $\exists x\, P(x)$ is true. By the rule of amplification we conclude that $(\exists x\, P(x)) \vee (\exists x\, Q(x))$ is also true. The case when $Q(a)$ is true is similar.

- Now suppose that $(\exists x\, P(x)) \vee (\exists x\, Q(x))$ is true in a certain universe. This means that either $\exists x\, P(x)$ is true or $\exists x\, Q(x)$ is true (or both). If $\exists x\, P(x)$ is true then there is a value a in the universe such that $P(a)$ is true. By the rule of amplification we conclude that $P(a) \vee Q(a)$ is true. Therefore, a witnesses that $\exists x\,(P(x) \vee Q(x))$ is also true. The case when $\exists x\, Q(x)$ is true is similar.

Q.E.D.

More Equivalences

- $\forall x\,(P(x) \wedge Q(x)) \;\Leftrightarrow\; (\forall x\, P(x)) \wedge (\forall x\, Q(x))$

  Prove!

- $\forall x\,(P(x) \vee Q(x))$ is not equivalent to $(\forall x\, P(x)) \vee (\forall x\, Q(x))$

  Find a counter-example!

Much More Equivalences

- If $\Phi \Leftrightarrow \Psi$ is a pair of logically equivalent compound statements, and $\Phi(x)$, $\Psi(x)$ denote the open compound statements obtained from Φ and Ψ by replacing every propositional variable occurring in these statements $(p,q,r, \ldots)$ with open statements $(P(x),Q(x),R(x),\ldots)$. Then

$$\forall x\, \Phi(x) \Leftrightarrow \forall x\, \Psi(x) \quad \text{and} \quad \exists x\, \Phi(x) \Leftrightarrow \exists x\, \Psi(x)$$

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$ distributive law

P(x)

Q(x)

R(x)

$$\forall x\, (P(x) \wedge (Q(x) \vee R(x))) \Leftrightarrow \forall x\, ((P(x) \wedge Q(x)) \vee (P(x) \wedge R(x)))$$

Much More Equivalences (cntd)

- $\exists x\, \neg(P(x) \wedge Q(x)) \Leftrightarrow \exists x\, (\neg P(x) \vee \neg Q(x))$

  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ DeMorgan's law

- $\exists x\, (P(x) \vee (Q(x) \vee R(x))) \Leftrightarrow \exists x\, ((P(x) \vee Q(x)) \vee R(x))$

  $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ associativity

- $\forall x\, (P(x) \vee (P(x) \wedge Q(x))) \Leftrightarrow \forall x\, P(x)$

  $p \vee (p \wedge q) \Leftrightarrow p$ absorption law

- $\forall x\, (P(x) \vee \neg P(x)) \Leftrightarrow T$

Homework

Exercises from the Book:

No. 17ab, 18ac, 25a (page 100-102)

7b, 8b (page 116)

Represent in symbolic form

a definition “Jaywalk means to cross a roadway, not being a lane, at any place which is not within a crosswalk and which is less one block from an intersection at which traffic control signals are in operation”.

a rule “No driver of a vehicle shall drive such vehicle on, over, or across any fire hose laid on any street or private road, unless directed so to do by the person in charge of such hose or a police officer”