Properties of Relations

Discrete Mathematics
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Previous Lecture

- Cartesian product, the cardinality of Cartesian product
- Binary relations, higher arity relations

Binary Relations

- A binary relation from set A to set B is any subset of $A \times B$.
  If $A = B$ then we say that the relation is on the set A

`x is a brother of y'	$\subseteq$	$\text{People} \times \text{People}$
`x is older than y'	$\subseteq$	$\text{People} \times \text{People}$
`x is an owner of y'	$\subseteq$	$\text{People} \times \text{Properties}$
`x < y'	$\subseteq$	$\mathbb{R} \times \mathbb{R}$
`x divides y'	$\subseteq$	$\mathbb{Z} \times \mathbb{Z}$

More Relations (cntd)

- Binary relations can be generalized to subsets of Cartesian products of more than two sets.
- Any subset of the Cartesian product of 3 sets is called a ternary relation

  \`x and y are parents of z' is a subset of

  People $\times$ People $\times$ People
- Any subset of the Cartesian product of k sets is called a k-ary relation

  $\{ (a_1, a_2, \ldots, a_k) \mid a_1 + a_2 + \ldots + a_k = 3 \}$ is a subset of $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$

Sets, Relations, and Predicates

- Observe that sets, relations and predicates are essentially the same object.

Unary predicate	Set
$P(x)$	$A = \{ x \mid P(x) \}$
Binary predicate	Binary relation
$P(x,y)$	$R = \{ (x,y) \mid P(x,y) \}$
Ternary predicate	Ternary relation
$P(x,y,z)$	$R = \{ (x,y,z) \mid P(x,y,z) \}$

Relational Databases

- A relational database is a collection of tables like

No.	Name	Student ID	Supervisor	Thesis title
1.	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
…	…	…	…	…

A table consists of a schema and an instance …

The instance of this table is a 5-ary relation, a subset of the Cartesian product

$$\mathbb{Z}^+ \times \text{Names} \times \text{8-strings\_of\_digits} \times \text{Names} \times \text{Meaningful\_Sentences}$$

Describing Binary Relations

- A list of pairs.

  Among 6 people, Mark, Jerry, John, Randy, Aaron, and Ralph, Mark and Randy are brothers, and also John, Aaron and Ralph are brothers

  A = {Mark, Jerry, John, Randy, Aaron, Ralph }

  Brotherhood = { (x,y) | x is a brother of y}

  = { (Mark,Randy), (Randy,Mark), (John,Aaron), (Aaron,John), (John,Ralph), (Ralph,John), (Aaron,Ralph), (Ralph,Aaron) }

Describing Binary Relations (cntd)

- Matrix of a relation.

  Matrix of a relation $R \subseteq A \times B$ is a rectangular table, rows of which are labeled with elements of A (in any but fixed order), and columns are labeled with elements of B. We write 1 in the intersection of row a and column b if and only if $(a,b) \in R$; otherwise we write 0.

Brotherhood

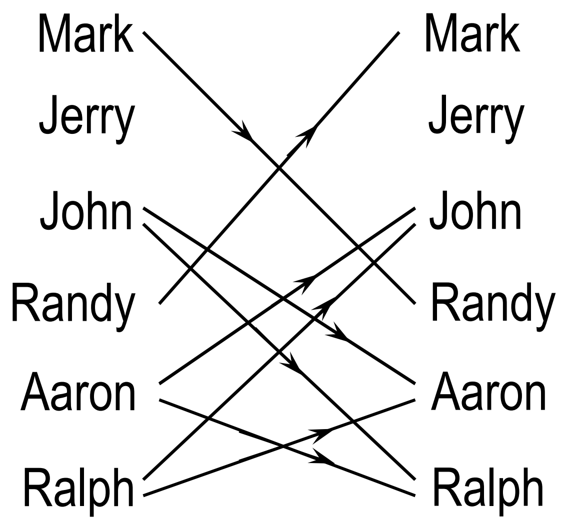
$$
\begin{array}{r c}
 & \begin{array}{cccccc} \text{Mark} & \text{Jerry} & \text{John} & \text{Randy} & \text{Aaron} & \text{Ralph} \end{array} \\
\begin{array}{r} \text{Mark} \\ \text{Jerry} \\ \text{John} \\ \text{Randy} \\ \text{Aaron} \\ \text{Ralph} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

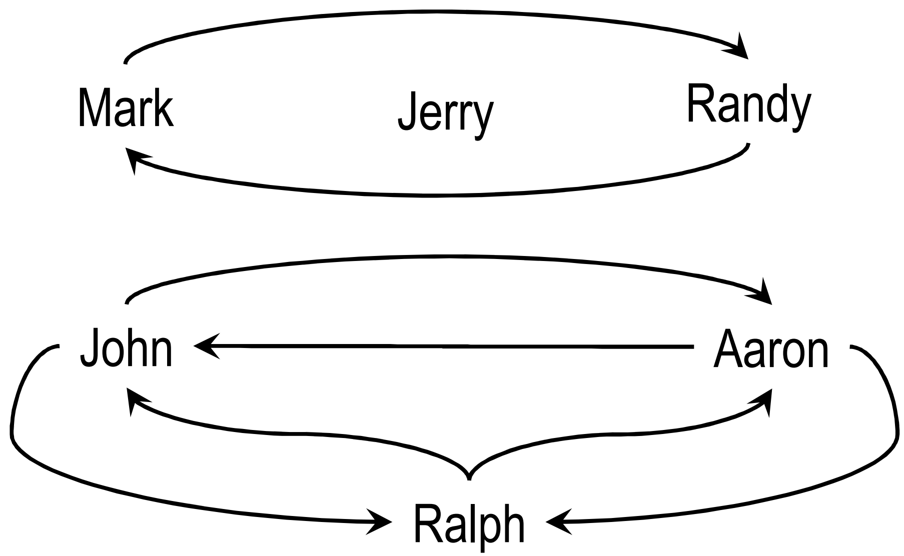
Describing Binary Relations (cntd)

- Graph of a relation

Graph of a relation $R \subseteq A \times B$ consists of two sets of vertices labeled by elements of A and B. A vertex a is connected to a vertex b with an edge (arc) if and only if $(a,b) \in R$.

If $A = B$ then we may use only one set of vertices





Cartesian Product, Intersection and Union

- **Theorem.** For any sets A,B,C

  (1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

  (2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

  (3) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

  (4) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

- Proof (of (2))

$$
\begin{aligned}
A \times (B \cup C) &= \{ (a,b) \mid a \in A \wedge b \in B \cup C \} \\
&= \{ (a,b) \mid a \in A \wedge (b \in B \vee b \in C) \} \\
&= \{ (a,b) \mid (a \in A \wedge b \in B) \vee (a \in A \wedge b \in C) \} \\
&= \{ (a,b) \mid a \in A \wedge b \in B\} \cup \{ (a,b) \mid a \in A \wedge b \in C \} \\
&= (A \times B) \cup (A \times C)
\end{aligned}
$$

Q.E.D.

Properties of Binary Relations – Reflexivity

- From now on we consider only binary relations from a set A to the same set A. That is, such relations are subsets of $A \times A$.
- A binary relation $R \subseteq A \times A$ is said to be reflexive if $(a,a) \in R$ for all $a \in A$.

  $(a,b) \in R \subseteq \mathbb{Z} \times \mathbb{Z}$ if and only if $a \le b$

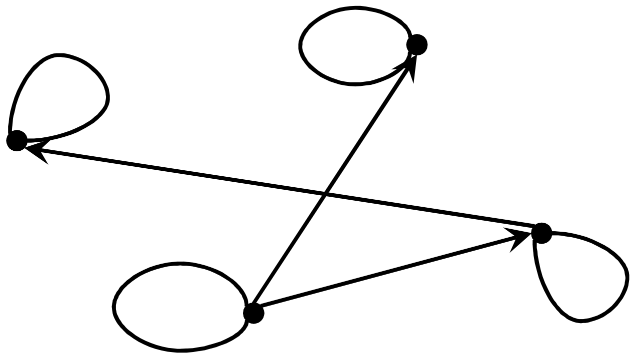
  This relation is reflexive, because $a \le a$ for all $a \in \mathbb{Z}$

Matrix:

$$\begin{pmatrix} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{pmatrix}$$

1's on the diagonal

Graph:



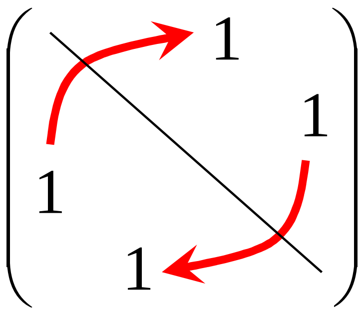
Loops at every vertex

Properties of Binary Relations – Symmetricity

- A binary relation $R \subseteq A \times A$ is said to be symmetric if, for any $a,b \in A$, if $(a,b) \in R$ then $(b,a) \in R$.

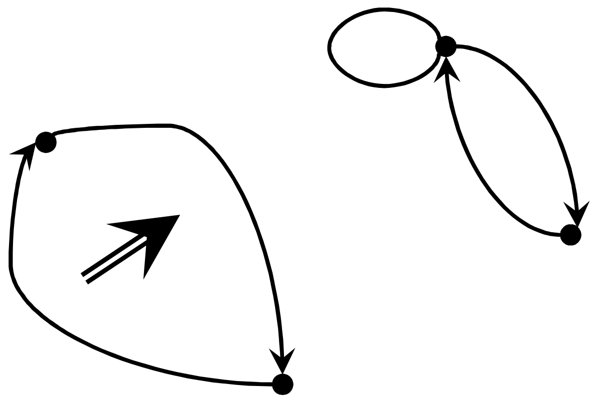
The relation Brotherhood (`x is a brother of y') on the set of men is symmetric, because if a is a brother of b then b is a brother of a

Matrix:



Matrix is symmetric w.r.t. the diagonal

Graph:



Graph is symmetric

Properties of Binary Relations – Transitivity

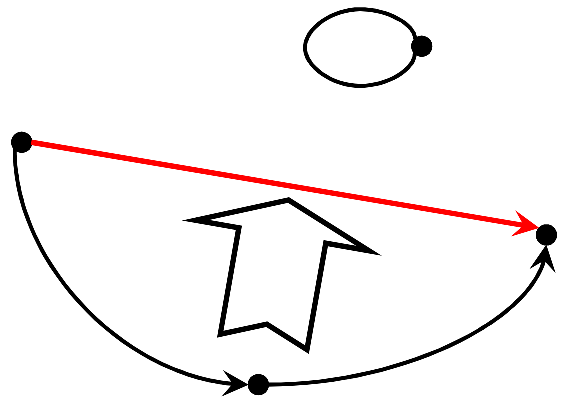
- A binary relation $R \subseteq A \times A$ is said to be **transitive** if, for any $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

The relation Div (`integer x divides y') is transitive, because if a divides b and b divides c, then a divides c

Matrix:

?

Graph:

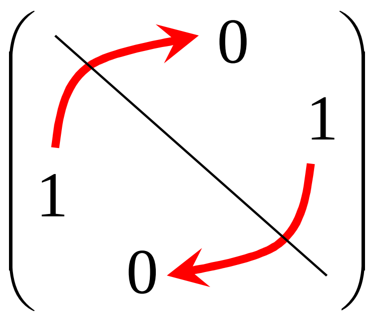


Properties of Binary Relations – Anti-Symmetricity

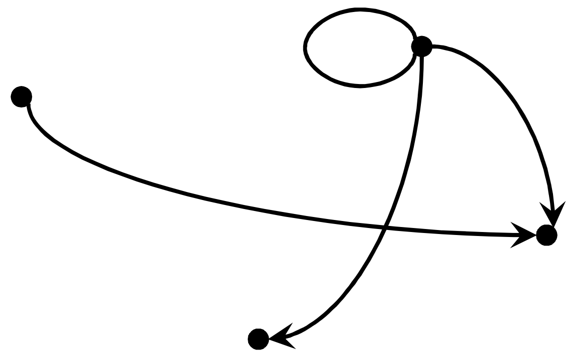
- A binary relation $R \subseteq A \times A$ is said to be anti-symmetric if, for any $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

The relation Motherhood (`x is the mother of y') is anti-symmetric, because if a is the mother of b then b is not the mother of a

Matrix:



Graph:



Matrix is anti-symmetric w.r.t. the diagonal

There are no edges going towards each other

Examples

	reflexive	symmetric	transitive	anti-symmetric
Brotherhood x is a brother of y				
Neighborhood x is a neighbor of y				
$x \leq y$				
x,y are intergers and x divides y				