Mathematical Induction

Discrete Mathematics
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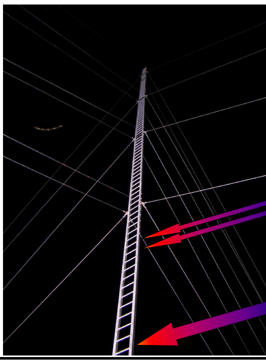
Principle of Mathematical Induction



Climbing an infinite ladder

Can we reach every step of it, if

For all k, standing on the rung k we can step on the rung k + 1

We can reach the first rung

Principle of Mathematical Induction

- Principle of mathematical induction:
  To prove that a statement that asserts that some property P(n) is true for all positive integers n, we complete two steps
  Basis step: We verify that P(1) is true.
  Inductive step: We show that the conditional statement
  P(k) → P(k + 1) is true for all positive integers k
- Symbolically, the statement
  $(P(1) \wedge \forall k\,(P(k) \to P(k+1))) \to \forall n\, P(n)$
- How do we do this?
  P(1) is usually an easy property
  To prove the conditional statement, we assume that P(k) is true (it is called inductive hypothesis) and show that under this assumption P(k + 1) is also true

The Domino Effect

- Show that all dominos fall:
- Basis Step: The first domino falls
- Inductive step: Whenever a domino falls, its next neighbor will also fall

Summation

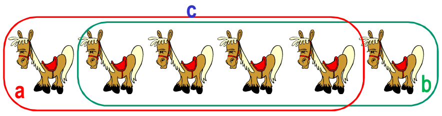
- Prove that the sum of the first n natural numbers equals $\frac{n(n+1)}{2}$
  that is $1+2+3+\cdots+n=\frac{n(n+1)}{2}$
- P(n): `the sum of the first n natural numbers …
- Basis step: P(1) means $1=\frac{1(1+1)}{2}$
- Inductive step: Make the inductive hypothesis, P(k) is true, i.e.
  $$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$
  Prove P(k + 1): $1+2+3+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$
  $$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$
  $$=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

More Summation

- Prove that $1+2+2^2+\cdots+2^n=2^{n+1}-1$
- Let P(n) be the statement `$1+2+2^2+\cdots+2^n=2^{n+1}-1$' for the integer n
- Basis step: P(0) is true, as $2^0=1=2^{0+1}-1$
- Inductive step: We assume the inductive hypothesis
  $$1+2+2^2+\cdots+2^k=2^{k+1}-1$$
  and prove P(k + 1), that is
  $$1+2+2^2+\cdots+2^k+2^{k+1}=2^{(k+1)+1}-1=2^{k+2}-1$$
  We have $1+2+2^2+\cdots+2^k+2^{k+1}=(2^{k+1}-1)+2^{k+1}$
  $$=2\cdot 2^{k+1}-1$$
  $$=2^{k+2}-1$$

No Horse of Different Color

- Prove that in any herd of n horses all horses are of the same color



- Basis step: n =1, clear
- Inductive step: Inductive hypothesis: Suppose in any herd of k horses all horses have the same color.
- Take a herd of k + 1 horses, and pick horses a and b there

  Take two `subherds' containing k horses each such that one contains a, but not b, and the other contains b, but not a

  The intersection contains some horse c. By inductive hypothesis a has the same color as c and b has the same color as c

The Cardinality of the Power Set

- We have proved that, for any finite set A, it is true that $| P(A) | = 2^{|A|}$
- Let Q(n) denote the statement `an n-element set has 2^n subsets'
- Basis step: Q(0), and empty set has only one subset, empty
- Inductive step. We make the inductive hypothesis, a k-element set A has 2^k subsets

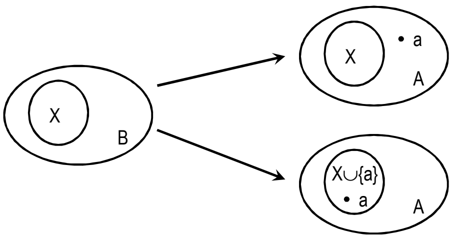
  We have to prove Q(k + 1), that is if a set A contains k + 1 elements, then $| P(A) | = 2^{k+1}$

  Fix an element $a \in A$, and set $B = A – \{a\}$.

  The set B contains k elements, hence $| P(B) | = 2^k$

  Every subset X of B corresponds to two subsets of A

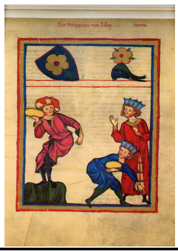
The Cardinality of the Power Set (cntd)



Therefore, $| P(A) | = 2 \cdot | P(B) | = 2 \cdot 2^k = 2^{k+1}$

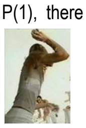
Odd Pie Fights

- An odd number of people stand in a yard at mutually distinct distances. At the same time each person throws a pie at their nearest neighbor, hitting this person. Show that there is at least one survivor, that is, at least one person who is not hit by a pie.



Odd Pie Fights (cntd)

- Let P(n) denote the statement `there is a survivor in the odd pie fight with 2n + 1 people'
- Basis step: P(1), there are 3 people



  Of the three people, suppose that the closest pair is A and B, and C is the third person. Since distances between people are different, the distances between A and C, and B and C are greater than that between A and B.

  Therefore, A and B throws pies at each other, and C survives.

Odd Pie Fights (cntd)

- Inductive step: Suppose that P(k) is true, that is, in the pie fight with 2k + 1 people there is a survivor.
- Consider the fight with 2(k + 1) + 1 people.

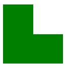
  Let A and B be the closest pair of people in this group of 2k + 3 people. Then they throw pies at each other.

  If someone else throws a pie at one of them, then for the remaining 2k + 1 people there are only 2k pies, and one of them survives.

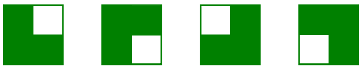
  Otherwise the remaining 2k + 1 people throw pies at each other, playing the pie fight with 2k + 1 people. By the inductive hypothesis, there is a survivor in such a fight.

Triomino

- Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using triominos



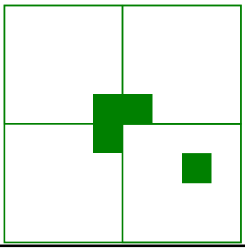
- P(n) denotes the statement above
- Basis step: P(1) is true, as 2×2 checkerboards with one square removed have one of the following shapes



Triomino (cntd)

- Inductive step: Suppose that P(k) is true that is every $2^k \times 2^k$ checkerboard with one square removed can be tiled with triominos.

  We have to prove P(k + 1), that is, every $2^{k+1} \times 2^{k+1}$ checkerboard without one square can be tiled.



Split the big checkerboard into 4 half-size checkerboards

Put one triomino as shown in the picture.

We have 4 $2^k \times 2^k$ checkerboards, each without one square. By the induction hypothesis, they can be tiled.

Principle of Strong Induction

- Sometimes mathematical induction is not enough. We can use the principle of strong induction.
- To prove that P(n) is true for all positive integers n, we complete two steps:
- Basis step: Verify that P(1) is true.
- Inductive step: Show that the statement

  $[\, P(1) \wedge P(2) \wedge \ldots \wedge P(k)\,] \rightarrow P(k + 1)$

  is true for all positive integers k.

Game with Matches

- Two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game.

- Show that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Strategy for the Second Player

- Let P(n) denote the statement \`the second player wins when there are initially n matches in each pile'.
- Basis step: P(1) is true, because in this case there is only one match in each pile, and the first player has only one choice, removing one match from one pile. Then the second player removes the match from the other pile and wins.
- Inductive step: Suppose that P(j) is true for all j with $1 \le j \le k$.

  We prove that P(k + 1) is true, that is, that the second player wins when each pile contain k + 1 matches.

  Suppose that the first player removes r matches from one pile leaving k + 1 – r matches there.

  By removing the same number of matches from the other pile the second player creates the situation of two piles with k + 1 – r matches in each. Apply the inductive hypothesis.

Homework

Exercises from the Book:

No. 3, 4a, 7b (page 244)