Propositional Logic

Discrete Mathematics

Andrei Bulatov

What is Logic?

"Computer science is continuation of logic by other means"

Georg Gottlob

"Contrariwise", continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic"

Lewis Carroll, *Through the Looking Glass*

"Logic celebrates the unity of a pathological masculine self-identity that cannot listen and recognizes only negation but not difference"

Andrea Nye, *Words of Power*

Use of Logic

- In mathematics and rhetoric:

    - Give precise meaning to statements.
    - Distinguish between valid and invalid arguments.
    - Provide rules of `correct' reasoning.

    Natural language can be very ambiguous

    `If you do your homework, then you'll get to watch the game.'

    $\wr\wr$

    `If you don't do your homework, then you will not get to watch ...'

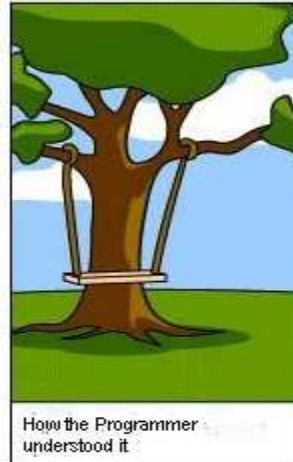   `You do your homework, or you'll fail the exam.'

    =

    `If you don't do your homework, then you'll fail the exam.'

Use of Logic (cntd)

- In computing:
  - Derive new data / knowledge from existing facts
  - Design of computer circuits.
  - Construction of computer programs.
  - Verification of correctness of programs and circuit design.
  - Specification

What the customer really needed

How the Programmer understood it

What the customer got

Statements (propositions)

● Propositional logic deals with statements and their truth values

● A statement is a declarative sentence that can be true or false

● Truth values are TRUTH (T or 1) and FALSE (F or 0).

● Examples:
- $1 + 1 = 2$ (statement, T)

- The moon is made of cheese   (statement, F)

- Go home!   (not statement, imperative)

- What a beautiful garden!   (not statement, exclamation)

- $y + 1 = 2$ (not statement, uncertain)

- The God exists   (statement, ?)

Compound Statements

- Simplest statements are called primitive statements

- We cannot decide the truth value of a primitive statement. This is not what logic does.

  We shall use propositional variables to denote primitive statements, *p, q, r*, …

- Instead we combine primitive statements by means of logic connectives into compound statements or formulas and look how the truth value of a compound statement depends on the truth values of the primitive statements it includes.
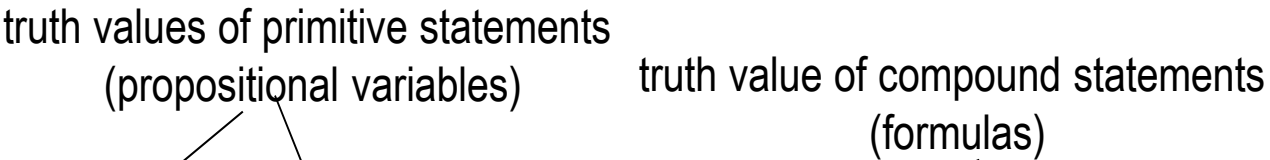
  We will denote compound statements by Φ, Ψ, …

Logic Connectives

- negation (not, $\neg$)   `It is not true that at least one politician was honest'

- conjunction (and, $\wedge$) `In this room there is a lady, and in the other room there is a tiger'

- disjunction (or, $\vee$)   `Margaret Mitchell wrote `Gone with the Wind or I am going home'

- implication (if…, then…, $\rightarrow$)   `If there is a tiger in this room, then there is no lady there'

- exclusive or (either …, or …, $\oplus$)   `There is either a tiger in this room, or a lady'

- biconditional (equivalence) (if and only if, $\leftrightarrow$)
     `There is a lady in this room if and only if there is a tiger in the other room'

Truth tables

Truth table is a way to specify the exact dependence of the truth value of a compound statement through the values of primitive statements involved

truth values of primitive statements
(propositional variables)

truth value of compound statements
(formulas)

p	q	Φ
0	0	0
0	1	1
…	…	…

Truth Tables of Connectives (negation and conjunction)

● Negation

p	¬p
F (0)	T (1)
T (1)	F (0)

unary connective

`Today is Friday' $\Rightarrow$
`Today is not Friday'

● Conjunction

p	q	p ∧ q
0	0	0
0	1	0
1	0	0
1	1	1

binary connective

`Today is Friday'
`It is raining' $\Rightarrow$

`Today is Friday and it is raining'

Truth Tables of Connectives (disjunction)

● Disjunction – inclusive `or'

p	q	p ∨ q
0	0	0
0	1	1
1	0	1
1	1	1

`Students who have taken calculus can take this course'   ⟹
`Students who have taken computing can take this course'

`Students who have taken calculus or computing can take this course'

Be careful with `or' constructions in natural languages!

`You do your homework, or you'll fail the exam.'

`Today is Friday or Saturday'

Truth Tables of Connectives (exclusive or)

- Exclusive `or'

  One of the statements is true but not both

p	q	p $\oplus$ q
0	0	0
0	1	1
1	0	1
1	1	0

  `You can follow the rules or be disqualified.'

  `Natalie will arrive today or Natalie will not arrive at all.'

Truth Tables of Connectives (implication)

🔴 Implication

p	q	p → q
0	0	1
0	1	1
1	0	0
1	1	1

Note that logical (*material*) implication does not assume any causal connection.

`**If** black is white, **then** we live in Antarctic.’

‘**If** pigs fly, **then** Paris is in France.’
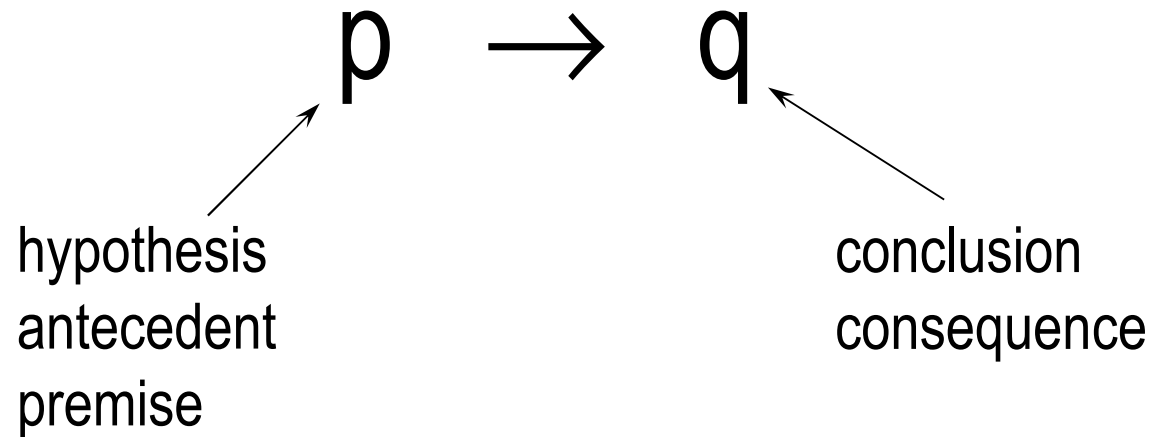
Implication as a promise

🔴 Implication can be thought of as a promise, and it is true if the promise is kept

`If` I am elected, **then** I will lower taxes'

- He is not elected and taxes are not lowered   promise kept!

- He is not elected and taxes are lowered   promise kept!

- He is elected, but (=and) taxes are not lowered   promise broken!

- He is elected and taxes are lowered   promise kept!

Playing with Implication

- Parts of implication

$$p \rightarrow q$$

hypothesis
antecedent
premise

conclusion
consequence

`if  p, **then**  q'          `q **unless** ¬p'   `a **sufficient con-
                                                        dition for** q is p'

`if  p, q'                    `p **implies** q'

`p  **is sufficient for**  q'    `p **only if** q'   `a **necessary con-
                                                        dition** for p is q'

`q  **if**  p'                   `q **whenever** p'

`q  **when**  p'                 `q **follows from** p'

Playing with Implication (cntd)

🔴 Converse, contrapositive, and inverse

$$p \rightarrow q$$ `The home team wins whenever it is raining'
(`If it is raining then the home team wins')

- **Converse** $$q \rightarrow p$$

    `If the home team wins, then it is raining'

- **Contrapositive** $$\neg q \rightarrow \neg p$$

    `If the home team does not win, then it is not raining'

- **Inverse** $$\neg p \rightarrow \neg q$$

    `If it is not raining, then the home team does not win'

Truth Tables of Connectives (biconditional)

● Biconditional or Equivalence

One of the statements is true if and only if the other is true

p	q	p $\leftrightarrow$ q
0	0	1
0	1	0
1	0	0
1	1	1

`You can take the flight if and only if you buy a ticket.'

Example

`You can access the Internet from campus if you are a computer science major or if you are not a freshman.'

p - `you can access the Internet from campus'

q - `you are a computer science major'

r - `you are a freshman'

Tautologies

- Tautology is a compound statement (formula) that is true for all combinations of truth values of its propositional variables

$$(p \rightarrow q) \vee (q \rightarrow p)$$

p	q	$(p \rightarrow q) \vee (q \rightarrow p)$
0	0	1
0	1	1
1	0	1
1	1	1

Contradictions

- Contradiction is a compound statement (formula) that is false for all combinations of truth values of its propositional variables

$$(p \oplus q) \wedge (p \oplus \neg q)$$

p	q	$(p \oplus q) \wedge (p \oplus \neg q)$
0	0	0
0	1	0
1	0	0
1	1	0

Homework

Exercises from the Book:

No. 1 ,3, 4, 8a, 8c (page 54)