Propositional Logic II

Discrete Mathematics
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Previous Lecture

- Statements, primitive and compound
- Logic connectives:
  - negation $\neg$
  - conjunction $\wedge$
  - disjunction $\vee$
  - exclusive or $\oplus$
  - implication $\rightarrow$
  - biconditional $\leftrightarrow$
- Truth tables

Truth Tables of Connectives (implication)

- Implication

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Note that logical (*material*) implication does not assume any causal connection.
`**If** black is white, **then** we live in Antarctic.'
'**If** pigs fly, **then** Paris is in France.'

Playing with Implication (cntd)

- Converse, contrapositive, and inverse

$p \rightarrow q$ `The home team wins whenever it is raining'
(`If it is raining then the home team wins')

- Converse $q \rightarrow p$
  `If the home team wins, then it is raining'
- Contrapositive $\neg q \rightarrow \neg p$
  `If the home team does not win, then it is not raining'
- Inverse $\neg p \rightarrow \neg q$
  `If it is not raining, then the home team does not win'

Truth Tables of Connectives (biconditional)

- Biconditional or Equivalence
  One of the statements is true if and only if the other is true

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

`You can take the flight if and only if you buy a ticket.'

Example

`You can access the Internet from campus if you are a computer science major or if you are not a freshman.'

p - `you can access the Internet from campus'

q - `you are a computer science major'

r - `you are a freshman'

Tautologies

- Tautology is a compound statement (formula) that is true for all combinations of truth values of its propositional variables

$(p \to q) \lor (q \to p)$

p	q	$(p \to q) \lor (q \to p)$
0	0	1
0	1	1
1	0	1
1	1	1

"To be or not to be"

Contradictions

- Contradiction is a compound statement (formula) that is false for all combinations of truth values of its propositional variables

$(p \oplus q) \land (p \oplus \neg q)$

"Black is white and black is not white"

p	q	$(p \oplus q) \land (p \oplus \neg q)$
0	0	0
0	1	0
1	0	0
1	1	0

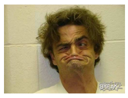
An Example

- Construct the truth table of the following compound statement

$p \to (q \lor \neg p)$

Another Example

- Write the following as propositional formulas and construct the truth tables of the resulting compound statements

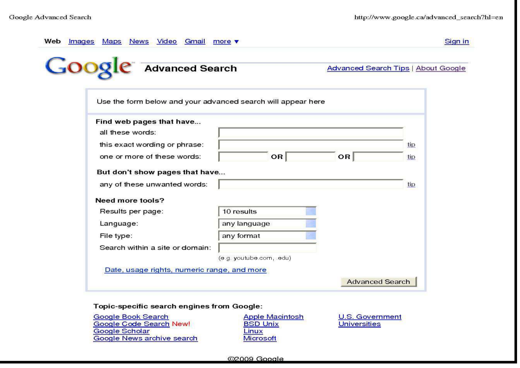
"An inhabitant of a castle in Transylvania is either sane or insane, and is a human or a vampire"

"If a person is an insane vampire then he believes only in false things and always lies"

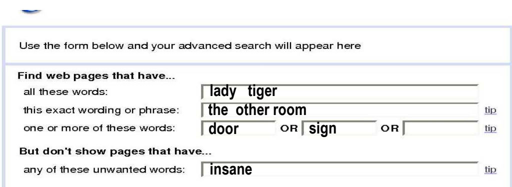
Web Search

- Find all pages that contain `discrete' AND `mathematics'
- Find all pages that contain `discrete' OR `mathematics'

Web Search



Web Search


Use the form below and your advanced search will appear here

Find web pages that have...
all these words: lady tiger
this exact wording or phrase: the other room tip
one or more of these words: door OR sign OR tip
But don't show pages that have...
any of these unwanted words: insane tip

- (lady $\wedge$ tiger) $\wedge$ (the other room) $\wedge$ (door $\vee$ sign) $\wedge$ $\neg$ insane

Logic Equivalences

- Compound statements Φ and Ψ are said to be logically equivalent if the statement Φ is true (false) if and only if Ψ is true (respectively, false)

or

- The truth tables of Φ and Ψ are equal

or

- For any choice of truth values of the primitive statements (propositional variables) of Φ and Ψ, formulas Φ and Ψ have the same truth value
- If Φ and Ψ are logically equivalent, we write

$$\Phi \Leftrightarrow \Psi$$

Why Logic Equivalences

- To simplify compound statements

``If you are a computer science major or a freshman and you are not a computer science major or you are granted access to the Internet, then you are a freshman or have access to the Internet"

- To convert complicated compound statements to certain `normal form' that is easier to handle

Conjunctive Normal Form CNF

Example Equivalences

- Implication and its contrapositive

p	q	$p \to q$	$\neg q \to \neg p$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

- All tautologies are equivalent to T
- All contradictions are equivalent to F

Equivalences and Tautologies

- **Theorem** Compound statements Φ and Ψ are logically equivalent if and only if $\Phi \leftrightarrow \Psi$ is a tautology.

Proof

Suppose that $\Phi \Leftrightarrow \Psi$. Then these statements have equal truth tables

p	q	…	Φ	Ψ	$\Phi \leftrightarrow \Psi$
…	…	…	…	…	1
0	1	…	1	1	1
…	…	…	…	…	…
1	0	…	0	0	1
…	…	…	…	…	1

Equivalences and Tautologies (cntd)

Suppose now that $\Phi \leftrightarrow \Psi$ is a tautology. This means that for any choice of the truth values of Φ and Ψ, $\Phi \leftrightarrow \Psi$ is true.

If Φ is true, then Ψ must also be true.

If Φ is false, then to make the formula $\Phi \leftrightarrow \Psi$ true Ψ must also be false.

Q.E.D.

Homework

Exercises from the Book:
No. 9, 13, 17 (*) (page 54)
No. 1a i,iii (page 66)