Binomial Coefficients

Discrete Mathematics
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Previous Lecture

- Combinations

$$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$

- Combinations with repetitions

  $C(n + r – 1, r – 1)$

A Binomial

- A binomial is simply the sum of two terms, such as x + y
- We are to determine the expansion of $(x+y)^n$
- Let us start with $(x+y)^3$

$$(x+y)^3 = (x+y)\cdot(x+y)\cdot(x+y)$$

Every term in the expansion is obtained as the product of a term from the first binomial, a term from the second binomial, and a term from the third binomial

$$= xxx + xxy + xyx + xyy + yxx + yyx + yxy + yyy$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Each of the terms xxy, xyx, and yxx is obtained by selecting y from one of the 3 binomials. Therefore, the coefficient 3 of x^2y is, actually, the number of 1-combinations from a set with 3 elements

The Binomial Theorem

- **Theorem.**

  Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- Proof.

  The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for j = 0,1,2,…, n.

  To count the number of terms of the form $x^{n-j}y^j$, note that to obtain such a term it is necessary to choose j y's from the n binomials (so that the other n – j terms in the product are x's).

  Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{j}$

Q.E.D.

Examples

- Expand $(x+y)^4$

$$
\begin{aligned}
(x+y)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 \\
&= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\end{aligned}
$$

- Expand $(x+2y)^5$

$$
\begin{aligned}
(x+2y)^5 &= \binom{5}{0}x^5 + \binom{5}{1}x^4(2y) + \binom{5}{2}x^3(2y)^2 + \binom{5}{3}x^2(2y)^3 \\
&\quad + \binom{5}{4}x(2y)^4 + \binom{5}{5}(2y)^5 \\
&= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5
\end{aligned}
$$

Properties of Binomial Coefficients

- For any nonnegative integer n and any r with $0 \le r \le n$

$$\binom{n}{r} = \binom{n}{n-r}$$

- Indeed, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$

Properties of Binomial Coefficients (cntd)

- For any nonnegative integer n

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

- Proof 1: By the Binomial Theorem

$$2^n=(1+1)^n=\binom{n}{0}1^n+\binom{n}{1}1^{n-1}\cdot 1+\binom{n}{2}1^{n-2}\cdot 1^2+\cdots+\binom{n}{n-1}1\cdot 1^{n-1}+\binom{n}{n}1^n$$
$$=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$$

- Proof 2: Recall that $\binom{n}{r}$ is the number of r-element subsets of a set with n elements. Therefore, the sum on the left side is the number of all subsets of an n-element set. We know this number equals 2^n

Pascal's Identity

- For any nonnegative integer n and any r with $0 \le r \le n$

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Blaise Pascal

- Proof: As we know $\binom{n+1}{r}$ is the number of r-element subsets of an (n + 1)-element set. Take a set T with n + 1 elements.

  Pick an element $a \in T$ and set $S = T - \{a\}$.

  Every r- element subset of T either does not contain a and hence is an r-element subset of S (there are $\binom{n}{r}$ subsets of this type),

  or it contains a and the remaining elements form an (r – 1)-element subset of S (there are $\binom{n}{r-1}$ subsets of this type)

Pascal's Triangle

- Pascal's identity and the simple observation that $\binom{n}{0} = \binom{n}{n} = 1$ allow us to give an inductive definition of binomial coefficients. It is convenient to arrange them into a triangle

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}
\end{array}$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}$$

Exercises

- Determine the coefficient of x^9y^3 in the expansion of $(2x-3y)^{12}$

- With n a positive integer, evaluate the sum

$$\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+\cdots+2^n\binom{n}{n}$$

Computing Binomial Coefficients

- Computing factorials and binomial coefficients can be very difficult. Fortunately, there are many ways to simplify the computation
- Stirling formula: $n! \approx \sqrt{2\pi n}\,\frac{n^n}{e^n}$, where e = 2.718281828459…
- Gamma function: $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$

  $\Gamma(n + 1) = n!$
- Computing binomial coefficients:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \approx \frac{\sqrt{2\pi n}\,\frac{n^n}{e^n}}{\sqrt{2\pi r}\,\frac{r^r}{e^r} \cdot \sqrt{2\pi(n-r)}\,\frac{(n-r)^{n-r}}{e^{n-r}}}$$

$$= \sqrt{\frac{n}{r(n-r)}}\left(\frac{n}{r}\right)^r \left(\frac{n}{n-r}\right)^{n-r}$$

Homework

Exercises from the Book:

No. 22a, 29, 30 (page 25)

No. 5a, 7, 10 (page 34)