Relations

Discrete Mathematics
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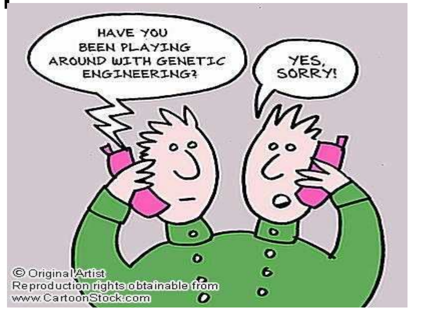
Previous Lecture

- Venn diagrams
- Operations of
  - Intersection
  - Union
  - Symmetric difference
  - Complement
  - Difference

Relations

- `**Relation**',   the connection between things or people
- Between people, family relations



  `to be brothers'   x is a brother of y

  `to be older'   x is older than y

  `to be parents'   x and y are parents of z

- Between things, numerical relations

  `to be greater than' $x < y$ on the set of real numbers

  `to be divisible by'   x is divisible by y on the set of integers

- Between things and people, legal relations

  `to be an owner'   x is an owner of y

Cartesian Product

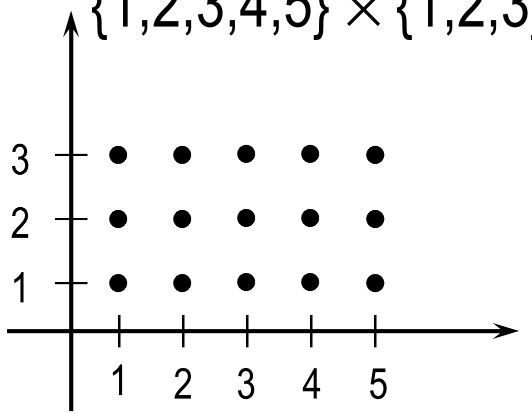
- The Cartesian product of sets A and B, denoted by $A \times B$, is the set of all ordered pairs of elements from A and B.

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

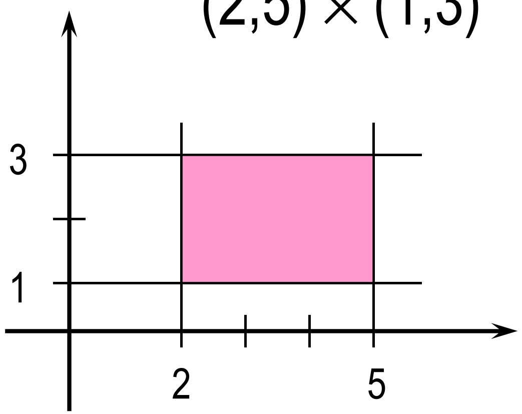
- The elements of the Cartesian product are ordered pairs. In particular, $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

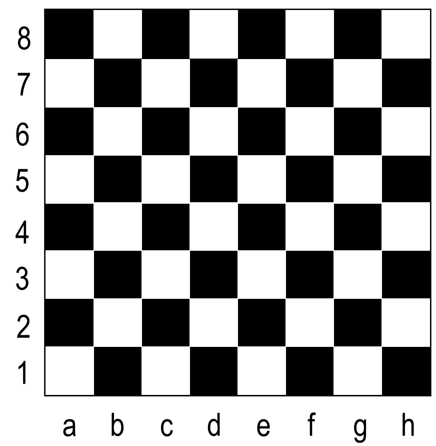
- If sets are thought of as `1-dimensional' objects, Cartesian products are 2-dimensional

$\{1,2,3,4,5\} \times \{1,2,3\}$



$(2,5) \times (1,3)$





Cartesian Product of More Than Two Sets

- Instead of ordered pairs we may consider ordered **triples**, or, more general, **k-tuples**.

  (a,b,c), an ordered triple

  (a,b,c,d), an ordered quadruple

  $(a_1, a_2, \ldots, a_k)$ a k-tuple

- Triples, quadruples, and k-tuples are elements of Cartesian products of 3, 4, and k sets, respectively

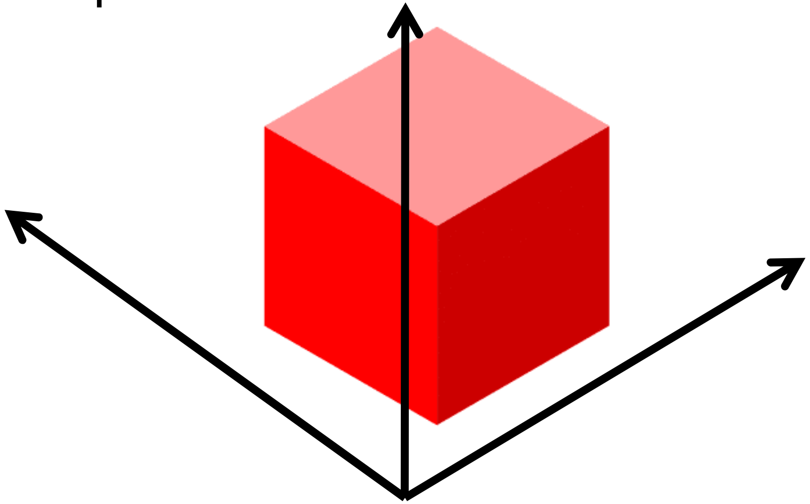
  $A \times B \times C = \{ (a,b,c) \mid a \in A,\ b \in B,\ c \in C\}$

  $A \times B \times C \times D = \{ (a,b,c,d) \mid a \in A,\ b \in B,\ c \in C,\ d \in D\}$

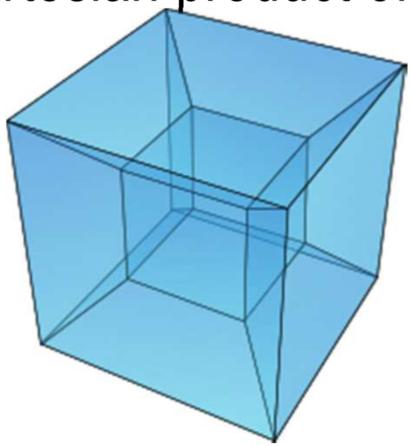
  $A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k\}$

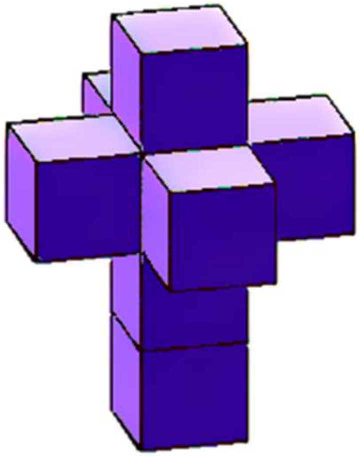
Cartesian Product of More Than Two Sets

- Cartesian product of 3 sets can be viewed as a `3-dimensional' set



- Cartesian product of more than 3 sets is a multi-dimensional set





Cardinality of Cartesian Product

- **Theorem.**

  $$|A \times B| = |A| \cdot |B|$$

  $$|A_1 \times A_2 \times \ldots \times A_k| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_k|$$

- Proof

  When creating an ordered pair (a,b), to each of the $|A|$ elements of A we can add any of the $|B|$ elements of B. Totally, we have $|A| \cdot |B|$ ordered pairs.

Q.E.D.

Binary Relations

- A binary relation from set A to set B is any subset of $A \times B$.
  If $A = B$ then we say that the relation is on the set A

`x is a brother of y' $\subseteq$ $People \times People$

`x is older than y' $\subseteq$ $People \times People$

`x is an owner of y' $\subseteq$ $People \times Properties$

`x < y' $\subseteq$ $\mathbb{R} \times \mathbb{R}$

`x divides y' $\subseteq$ $\mathbb{Z} \times \mathbb{Z}$

More Relations (cntd)

- Binary relations can be generalized to subsets of Cartesian products of more than two sets.
- Any subset of the Cartesian product of 3 sets is called a ternary relation

  `x and y are parents of z' is a subset of

  People $\times$ People $\times$ People
- Any subset of the Cartesian product of k sets is called a k-ary relation

  $\{ (a_1, a_2, \ldots, a_k) \mid a_1 + a_2 + \ldots + a_k = 3 \}$ is a subset of $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$

Sets, Relations, and Predicates

- Observe that sets, relations and predicates are essentially the same object.

Unary predicate	Set
$P(x)$	$A = \{ x \mid P(x) \}$
Binary predicate	Binary relation
$P(x,y)$	$R = \{ (x,y) \mid P(x,y) \}$
Ternary predicate	Ternary relation
$P(x,y,z)$	$R = \{ (x,y,z) \mid P(x,y,z) \}$

Relational Databases

- A relational database is a collection of tables like

No.	Name	Student ID	Supervisor	Thesis title
1.	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
…	…	…	…	…

A table consists of a schema and an instance …

The instance of this table is a 5-ary relation, a subset of the Cartesian product

$$\mathbb{Z}^+ \times \text{Names} \times \text{8-strings\_of\_digits} \times \text{Names} \times \text{Meaningful\_Sentences}$$

Describing Binary Relations

- A list of pairs.

  Among 6 people, Mark, Jerry, John, Randy, Aaron, and Ralph, Mark and Randy are brothers, and also John, Aaron and Ralph are brothers

  A = {Mark, Jerry, John, Randy, Aaron, Ralph }

  Brotherhood = { (x,y) | x is a brother of y}

  = { (Mark,Randy), (Randy,Mark), (John,Aaron), (Aaron,John), (John,Ralph), (Ralph,John), (Aaron,Ralph), (Ralph,Aaron) }

Describing Binary Relations (cntd)

- Matrix of a relation.

  Matrix of a relation $R \subseteq A \times B$ is a rectangular table, rows of which are labeled with elements of A (in any but fixed order), and columns are labeled with elements of B. We write 1 in the intersection of row a and column b if and only if $(a,b) \in R$; otherwise we write 0.

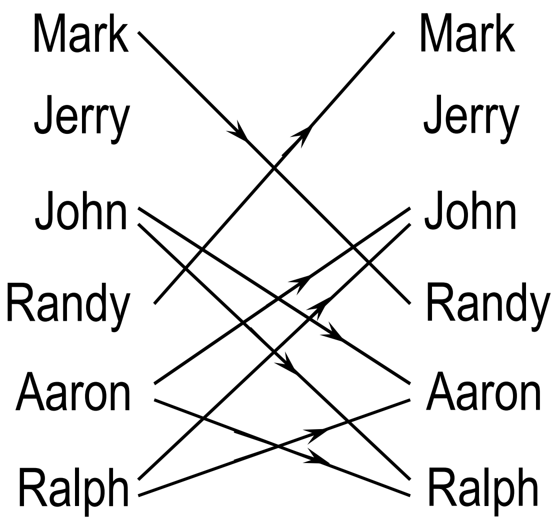
Brotherhood

$$
\begin{array}{r c}
 & \begin{array}{cccccc} \text{Mark} & \text{Jerry} & \text{John} & \text{Randy} & \text{Aaron} & \text{Ralph} \end{array} \\
\begin{array}{r} \text{Mark} \\ \text{Jerry} \\ \text{John} \\ \text{Randy} \\ \text{Aaron} \\ \text{Ralph} \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

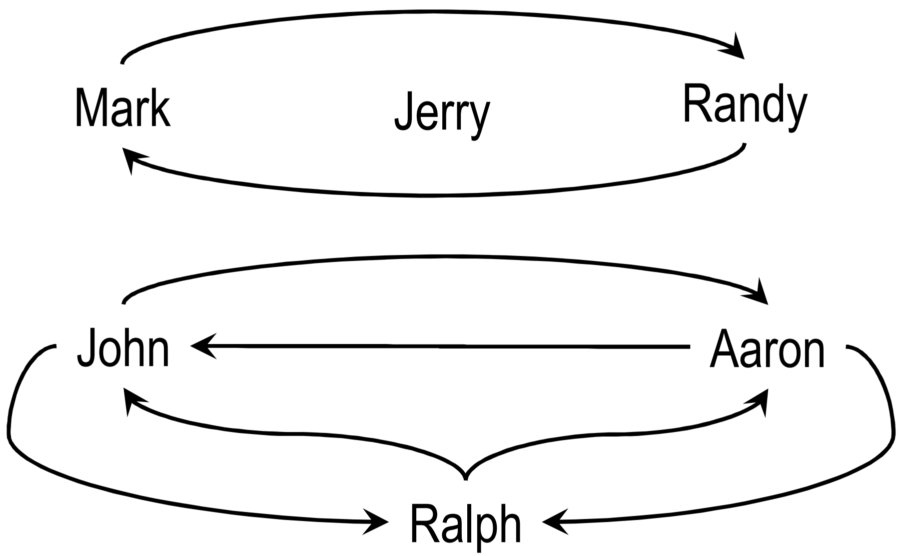
Describing Binary Relations (cntd)

- Graph of a relation

  Graph of a relation $R \subseteq A \times B$ consists of two sets of vertices labeled by elements of A and B. A vertex a is connected to a vertex b with an edge (arc) if and only if $(a,b) \in R$.



If $A = B$ then we may use only one set of vertices



Cartesian Product, Intersection and Union

- **Theorem.** For any sets A,B,C

  (1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

  (2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

  (3) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

  (4) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

- Proof (of (2))

$$
\begin{aligned}
A \times (B \cup C) &= \{ (a,b) \mid a \in A \wedge b \in B \cup C \} \\
&= \{ (a,b) \mid a \in A \wedge (b \in B \vee b \in C) \} \\
&= \{ (a,b) \mid (a \in A \wedge b \in B) \vee (a \in A \wedge b \in C) \} \\
&= \{ (a,b) \mid a \in A \wedge b \in B\} \cup \{ (a,b) \mid a \in A \wedge b \in C \} \\
&= (A \times B) \cup (A \times C)
\end{aligned}
$$

Q.E.D.

Homework

Exercises from the Book:

No. 1, 4, 5a (page 252)

- Prove part (3) of the theorem on slide 13-14