MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Predicates and Quantifiers

1. **Determine the truth value of each of these statements if the universe of each variable consists of (i) all real numbers, (ii) all integers.**

   (a) $\forall x\,(x > 0 \to \exists y\,(\frac{\sqrt{x}}{y} = 3))$**;**

   (b) $\exists x \forall y\,((2x - y = 4) \wedge (3x + y = 5))$**.**

   (a) This statement is true in the set of real numbers. Given x we need to find, provided $x > 0$, a value y such that $\frac{\sqrt{x}}{y} = 3$. Clearly, y can be chosen to be $\frac{\sqrt{x}}{3}$.

   The statement is false in universe of integers. Indeed, take $x = 5$. Then $x > 0$, and therefore there has to be an integer y such that $\frac{\sqrt{5}}{y} = 3$, which is impossible.

   (b) This statement is false for both universes. It suffices to show that for any x_0 there is y_0 such that $2x_0 - y_0 \neq 4$ or $3x_0 + y_0 \neq 5$. Take $y_0 = 2x_0 - 3$. Then $2x_0 - y_0 = 3 \neq 4$.

2. **Use predicates and quantifiers to express this statement**

   "Some students in this class grew up in the same town as exactly one other student in this class."

   There are several ways to express this. Here is one of them.
   Let the predicates be:
   $C(x)$, 'student x is in this class',
   $T(x, y)$, 'student x grew up in the same town as student y'

   Take a student x. The condition that there is another student in this class who grew up in the same town we can express as

   $$\exists y \Big(C(y) \wedge (x \neq y) \wedge T(x, y) \Big).$$

   The condition that there is exactly one such student can be expressed by using the statement above combined with a statement that says that any student who grew up in the same tawn with x is either x himself, or y:

   $$\exists y \Big(C(y) \wedge (x \neq y) \wedge T(x, y) \wedge \big(\forall z (T(x, z) \to ((z = x) \vee (z = y))\big) \Big).$$

   Finally, we just need to say that such x exists:

   $$\exists x \left(C(x) \wedge \exists y \Big(C(y) \wedge (x \neq y) \wedge T(x, y) \wedge \big(\forall z (T(x, z) \to ((z = x) \vee (z = y))\big) \Big) \right).$$

3. **Find a counterexample, if possible, to this universally quantified statement, where the universe for all variables consists of all integers**

   $$\forall x \forall y\,(3x - 4y \geq x).$$

   $x = 1$, $y = -1$ is a counterexample.

4. **Rewrite the following statement so that negations appear only within predicates (that is, no negation is outside a quantifier or an expression involving logical connectives)**

   $$\neg \exists x\,((\forall y \exists z\, P(x, y, z)) \to (\exists z \forall y\,(R(x, y, z) \wedge S(z, y)))).$$

   Use logic equivalences

$$\begin{aligned}
&\neg\exists x\ ((\forall y\exists z\ P(x,y,z)) \to (\exists z\forall y\ (R(x,y,z)\wedge S(z,y)))) \\
&\iff \forall x\ \neg((\forall y\exists z\ P(x,y,z)) \to (\exists z\forall y\ (R(x,y,z)\wedge S(z,y)))) && \text{law for negation and quantifiers} \\
&\iff \exists x\ \neg\Big(\neg\big((\forall y\exists z\ P(x,y,z)) \vee (\exists z\forall y\ (R(x,y,z)\wedge S(z,y)))\big)\Big) && \text{expression for implication} \\
&\iff \exists x\ \big((\forall y\exists z\ P(x,y,z)) \wedge \neg(\exists z\forall y\ (R(x,y,z)\wedge S(z,y)))\big) && \text{DeMorgan's law and double negation} \\
&\iff \exists x\ \big((\forall y\exists z\ P(x,y,z)) \wedge (\forall z\exists y\ \neg(R(x,y,z)\wedge S(z,y)))\big) && \text{negation of quantifiers} \\
&\iff \exists x\ \big((\forall y\exists z\ P(x,y,z)) \wedge (\forall z\exists y\ (\neg R(x,y,z)\vee \neg S(z,y)))\big) \\
& && \text{DeMorgan's law and double negation}
\end{aligned}$$

5. **Find a universe for variables x, y, and z for which the statement**

$$\forall x\forall y((x \neq y) \to \forall z((z = x) \vee (z = y)))$$

is true and another universe in which it is false.

Observe that the statement in fact claims that the universe contains at most two different elements (for any two different elements, any third one equals one of them). Thus the statement is true in $\{1, 2\}$ and false in $\{1, 2, 3\}$. Indeed, no matter which elements x, y we choose from $\{1, 2\}$, if they are different, that is, $\{x, y\} = \{1, 2\}$, then any z must be equal to either 1 or 2. For $\{1, 2, 3\}$ we can find a counterexample. Let $x = 1, y = 2$. Then $x \neq y$, so the premise of the implication is true. Now, take $z = 3$. Then $z \neq x$ and $z \neq y$, meaning the conclusion, as well as the implication, is false.

6. **Show that quantified statements**

$$\exists x(P(x) \oplus Q(x)) \quad \textbf{and} \quad (\exists x P(x)) \oplus (\exists x Q(x))$$

are not logically equivalent.

We need to find a counter-example. Let the universe be the set of all cars, $P(x)$ means that x is red, and $Q(x)$ means that x is not red. Then the first statement says that there is a car that is either red or it is not red. This is clearly true. The second statement claims that either there is a red car, or there is a car which is not red. Since both parts of the exclsive OR are true, the second statement is false.

7. **Determine whether the following argument is valid or invalid and explain why.**

'Everyone enrolled in the university has lived in a dormitory.'
'Mia has never lived in a dormitory.'
'Therefore, Mia is not enrolled in the university.'

Express these statements symbolically. Let the predicates be:

$U(x)$, 'x enrolled in the university',
$D(x)$, 'x has lived in a dormitory'

Then the premises are translated as: $\forall x\ (U(x) \to D(x))$, $\neg D(\text{Mia})$.

And the conclusion: $\neg F(\text{my pet bird})$.
Then the argument may look as follows:

Steps	Reason
1. $\forall x\ (U(x) \to D(x))$	premise
2. $(U(\text{Mia}) \to D(\text{Mia}))$	rule of universal specification
3. $\neg D(\text{Mia})$	premise
4. $\neg U(\text{Mia})$	modus tollens.

Thus the argument is valid.

8. **Given premises:**

   'All hummingbirds are richly colored.'
   'No large birds live on honey.'
   'Birds that do not live on honey are dull in color.'

   infer the conclusion

   'Hummingbirds are small.'

   Let the predicates be:

   $H(x)$, 'x is a hummingbird',
   $C(x)$, 'x is richly colored,
   $S(x)$, 'x is small',
   $L(x)$, 'x lives on honey'

   Then the premises are translated as: $\forall x\ (H(x) \to R(x))$, $\forall x\ \neg(\neg S(x) \wedge L(x))$, $\forall x\ (\neg L(x) \to \neg R(x))$.

   And the conclusion: $\forall x\ (H(x) \to S(x)$.

Steps	Reason
1. $\forall x\ (H(x) \to R(x))$	premise
2. $H(a) \to R(a)$	rule of universal specification to Step 1, where a is a generic element
3. $\forall x\ (\neg L(x) \to \neg R(x))$	premise
4. $\neg L(a) \to \neg R(a)$	rule of universal specification to Step 3, where a is a generic element
5. $R(a) \to L(a)$	contrapositive to Step 4
6. $H(a) \to L(a)$	rule of syllogism to Steps 2 and 5
7. $\forall x\ \neg(\neg S(x) \wedge L(x))$	premise
8. $\neg(\neg S(a) \wedge L(a))$	rule of universal specification to Step 7, where a is a generic element
9. $L(a) \to S(a)$	DeMorgan's law and expression for implication to Step 8
10. $H(a) \to S(a)$	rule of syllogism to Steps 6 and 9
11. $\forall x\ (H(x) \to S(x))$	rule of universal generalisation to Step 10.

9. **What is wrong with this proof?**

   Theorem. 3 is less than 1.

   Proof. Every integer number is either less than 1 or greater than 1 or equals 1. Let c be an arbitrary integer number. Therefore, it is less than 1 or greater than 1 or equals 1. Suppose it is less than 1. By the rule of universal generalization, if an arbitrary number is less than 1, every number is less than 1. Therefore, 3 is less than 1.

   Express these statements symbolically. Let the predicates be:

   $L(x)$, 'x is less than 1',
   $G(x)$, 'x is greater than 1'
   $E(x)$, 'x equals 1'[1mm] Then the theorem translated as: $L(3)$.

   Express the proof in formal terms. First we start off with the statement $\forall x\ (L(x) \vee G(x) \vee E(x))$, which is true in the universe of integers. Then we use the rule of universal specification to obtain $L(n) \vee G(n) \vee E(n)$ for a generic value n of the variable. Next we say that since the disjunction is true, one of the conditions is true, and assume that the first one is true. Having $L(n)$ we use the rule of universal generalization to conclude $\forall x\ L(x)$, and then the rule of universal specification to obtain $L(3)$.

   Thus, the premise used: $\forall x\ (L(x) \vee G(x) \vee E(x))$ (the true statement in the proof)

   And the conclusion: $L(n)$.
   Then the argument may look as follows:

Steps	Reason

1. $\forall x\ (L(x) \vee G(x) \vee E(x))$ premise
2. $(L(n) \vee G(n) \vee E(n))$ rule of universal specification
3. $L(n)$ some rule of inference
4. $\forall x L(x)$ rule of universal generalization
5. $L(3)$ rule of universal specification.

Note that the rule of inference used in Step 3 is not known to us, and it looks like

$$\frac{p \vee q \vee r}{\triangle\ \ p}$$

If it is a valid rule of inference, the following statement must be a tautology

$$(p \vee q) \rightarrow p.$$

However, it is not a tautology, since when $p = 0, q = 1$ the statement is false. Thus the argument is invalid.

Less formally, the mistake made in the proof is a mishandling the generic value n. For a generic value we cannot assume any property that is not shared by every element in the universe. Therefore $L(n) \vee G(n) \vee E(n)$ is a property we can assume for the generic value n, while $L(n)$ is not. We can consider cases proving some statements. For instance, we want to prove statement $Q(n)$ for a generic value n. Then we can first assume n is even, and prove $Q(n)$ in this case, then assume n is odd and prove $Q(n)$ in this case. Note that the rule of universal generalization we can only apply when $Q(n)$ is proved in both cases. This means $Q(n)$ is true for any element in the universe, no matter, if it is even or odd.