Operations on Sets

Discrete Mathematics
Andrei Bulatov

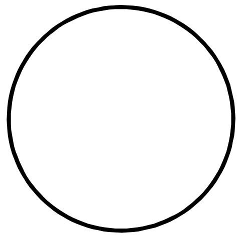
Previous Lecture

- Sets and elements
- Subsets, proper subsets, empty sets
- Universe
- Cardinality
- Power set

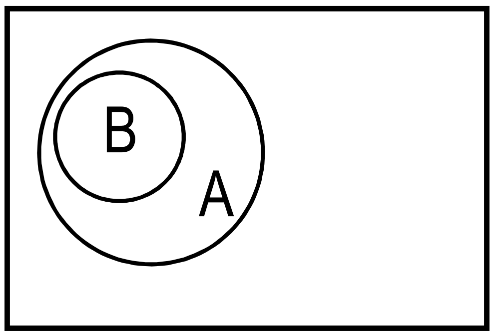
Venn Diagrams

- Often it is convenient to visualize various relations between sets. We use Venn diagrams for that.

universe



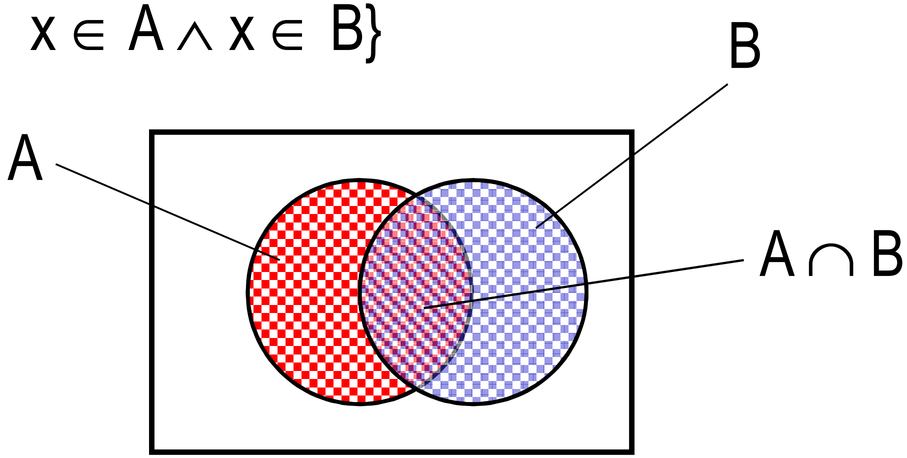
set



B is a subset of A

Intersection

- The intersection of sets A and B, denoted by $A \cap B$, is the set that contains those elements that belong to both A and B.
- $A \cap B = \{ x \mid x \in A \wedge x \in B\}$

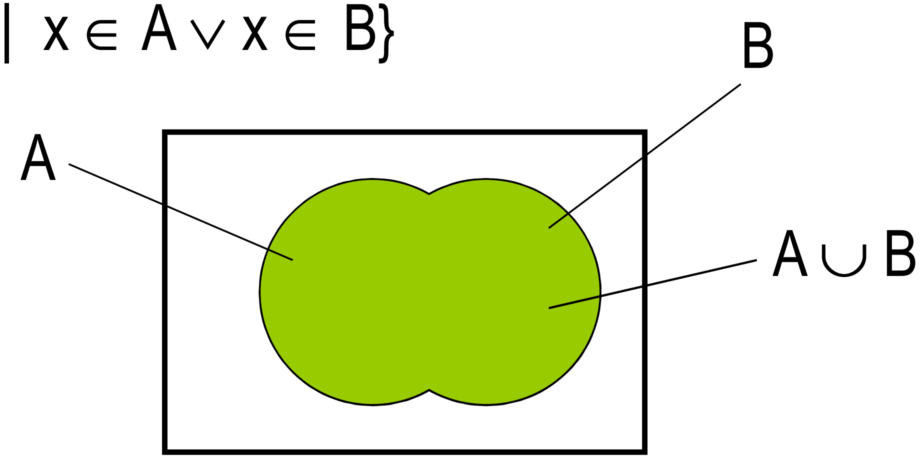


- Examples

$\{1,3,5,7\} \cap \{2,3,4,5,6\} = \{3,5\}$
$\{\text{Jan.},\text{Feb.},\text{Dec.}\} \cap \{\text{Jan.},\text{Feb.},\text{Mar.}\} = \{\text{Jan.},\text{Feb.}\}$
$\{x \mid \exists y \ \ x=2y\} \cap \{x \mid \exists y \ \ x=3y\} = \{x \mid \exists y \ \ x=6y\}$
$\mathbb{Z} \cap \mathbb{Q}^+ = \mathbb{Z}^+$

Union

- The union of sets A and B, denoted by $A \cup B$, is the set that contains those elements that are in A or in B.
- $A \cup B = \{ x \mid x \in A \vee x \in B \}$



- Examples

  {Mon,Tue,Wed,Thu,Fri} ∪ {Sat,Sun} = {Mon,Tue,Wed,Thu,Fri,Sat,Sun}

  $\{1,3,5,7\} \cup \{2,3,4,5,6\} = \{1,2,3,4,5,6,7\}$

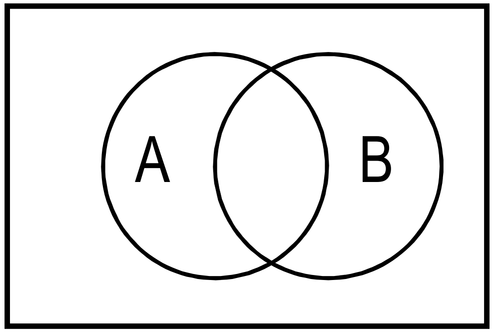
Disjoint Sets and Principle of Inclusion-Exclusion

- Sets A and B are said to be disjoint if $A \cap B = \varnothing$.

  Sets {Mon,Tue,Wed,Thu,Fri} and {Sat,Sun} are disjoint.

- Principle of inclusion-exclusion. For any finite sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

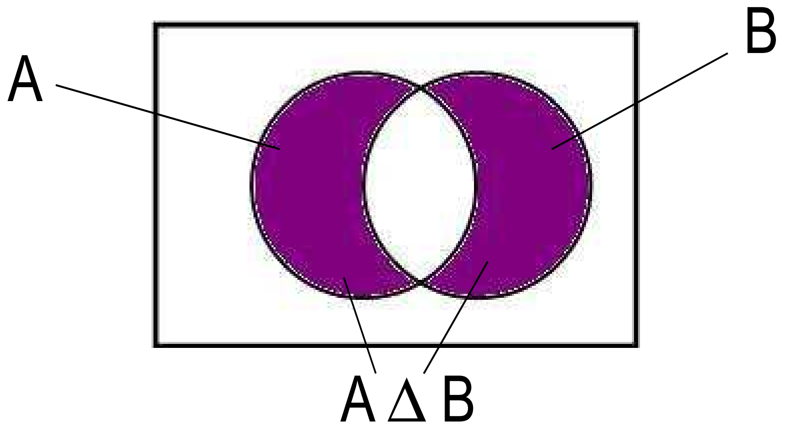


To count elements in $A \cup B$ we first count elements of A, then elements of B. Elements of $A \cap B$ are counted twice, so, we subtract the number of such elements

- If A and B are disjoint, then $|A \cup B| = |A| + |B|$

Symmetric Difference

- The symmetric difference of sets A and B, denoted by $A \Delta B$, is the set that contains those elements that are either in A or in B, but not in both.
- $A \Delta B = \{ x \mid x \in A \oplus x \in B\}$



- Example
  {Jan.,Feb.,Mar.} Δ {Dec.,Jan.,Feb.} = {Dec.,Mar.}

Disjoint Sets and Symmetric Difference

- **Theorem**. Sets A and B are disjoint if and only if

$$A \cup B = A \Delta B$$

- Proof.

Notice first that $A \Delta B \subseteq A \cup B$.
Suppose A and B are disjoint. To prove the equality, it suffices to show that $A \cup B \subseteq A \Delta B$.
Take $x \in A \cup B$. It belongs to A or B, but $x \notin A \cap B$, as the intersection is empty. Therefore, $x \in A \Delta B$.

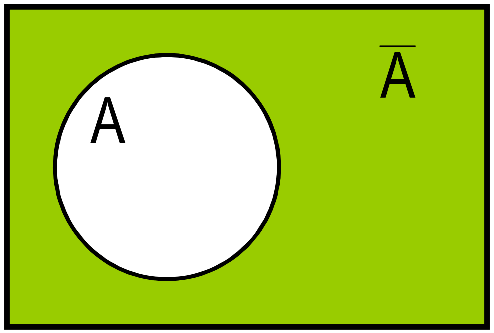
We prove by contraposition. Assume $A \cap B \neq \varnothing$. Say $x \in A \cap B$
Then $x \in A \cup B$.
However, from $x \in A \cap B$ we conclude that $x \notin A \Delta B$.
Therefore, $A \cup B \neq A \Delta B$.

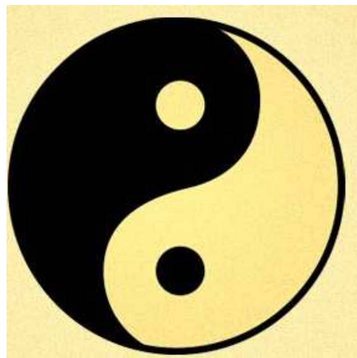
Q.E.D.

Complement

- Let A be a set and U a universe, $A \subseteq U$. The complement of A, denoted by $\overline{A}$, is the set that comprises all elements of U that do not belong to A.

$$\overline{A} = \{ x \mid x \in U \text{ and } x \notin A \} = \{ x \mid x \notin A \}$$





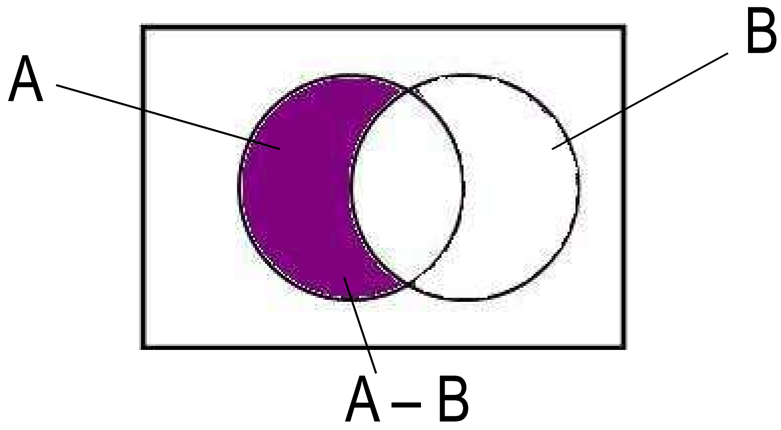
- Let the universe be the set of all integers, and $A = \{ x \mid \exists y \;\; x=2y \}$
  Then $\overline{A}$ is the set of all odd numbers

- The universe is the Latin alphabet, $A = \{ a,e,I,o,u,y \}$.
  Then $\overline{A} = \{ b,c,d,f,g,h,j,k,l,m,n,p,q,r,s,t,v,w,x,z \}$.

Difference

- The difference of sets A and B (or relative complement of B in A), denoted by $A - B$, is the set containing those elements that are in A, but not in B.

$$A - B = \{ x \mid x \in A \wedge x \notin B \}.$$



- $\{1,3,5\} - \{1,2,3\} = \{5\}$
- Clearly, $\overline{A} = U - A$.

Laws of Set Theory

- Similar to logic connectives and formulas, expressions built from set operations and sets also satisfy some laws.
- **Theorem**. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof. We will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$

Prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Take $x \in \overline{A \cap B}$.

By the definition, $x \notin A \cap B$. Therefore, $x \notin A$ or $x \notin B$.

Hence $x \in \overline{A}$ or $x \in \overline{B}$. Thus, $x \in \overline{A} \cup \overline{B}$

Now we prove that $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$. Take $x \in \overline{A} \cup \overline{B}$.

By definition, $x \in \overline{A}$ or $x \in \overline{B}$. Therefore, $x \notin A$ or $x \notin B$.

This implies $x \notin A \cap B$. And, finally, $x \in \overline{A \cap B}$.

Q.E.D.

Another Proof

- Another way to prove equalities for sets is to use the set builder construction and some logic.

$$
\begin{aligned}
\overline{A \cap B} &= \{ x \mid x \notin A \cap B \} && \text{by definition of complement} \\
&= \{ x \mid \neg(x \in A \cap B) \} && \text{by definition of does not belong symbol} \\
&= \{ x \mid \neg(x \in A \wedge x \in B) \} && \text{by definition of intersection} \\
&= \{ x \mid \neg(x \in A) \vee \neg(x \in B) \} && \text{by De Morgan's law} \\
&= \{ x \mid (x \notin A) \vee (x \notin B) \} && \text{by def. of does not belong symbol} \\
&= \{ x \mid (x \in \overline{A}) \vee (x \in \overline{B}) \} && \text{by definition of complement} \\
&= \{ x \mid x \in \overline{A} \cup \overline{B} \} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

Q.E.D.

Sets and Logic

- If we look closer at the second proof, we notice that there is a very tight connection between set operations and logic connectives

$\neg$ corresponds to complement $\overline{}$

$\vee$ corresponds to union $\cup$

$\wedge$ corresponds to intersection $\cap$

$\oplus$ corresponds to symmetric difference Δ

0 (false) corresponds to the empty set $\varnothing$

1 (truth) corresponds to the universe U

More Laws of Set Theory

$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws

More Laws of Set Theory (cntd)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Distributive laws

$\overline{A \cap B} = \overline{A} \cup \overline{B}$
$\overline{A \cup B} = \overline{A} \cap \overline{B}$

De Morgan's laws

$A \cap (A \cup B) = A$
$A \cup (A \cap B) = A$

Absorption laws

$A \cup \overline{A} = U$
$A \cap \overline{A} = \varnothing$

Complement laws

Homework

Exercises from the Book:

No. 1, 4, 6, 8bc, 16, 17bc (page 146 – 147)