# Common Divisors

Discrete Mathematics

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# Previous Lecture

- Representation of numbers
- Prime and composite numbers

# The Greatest Common Divisor

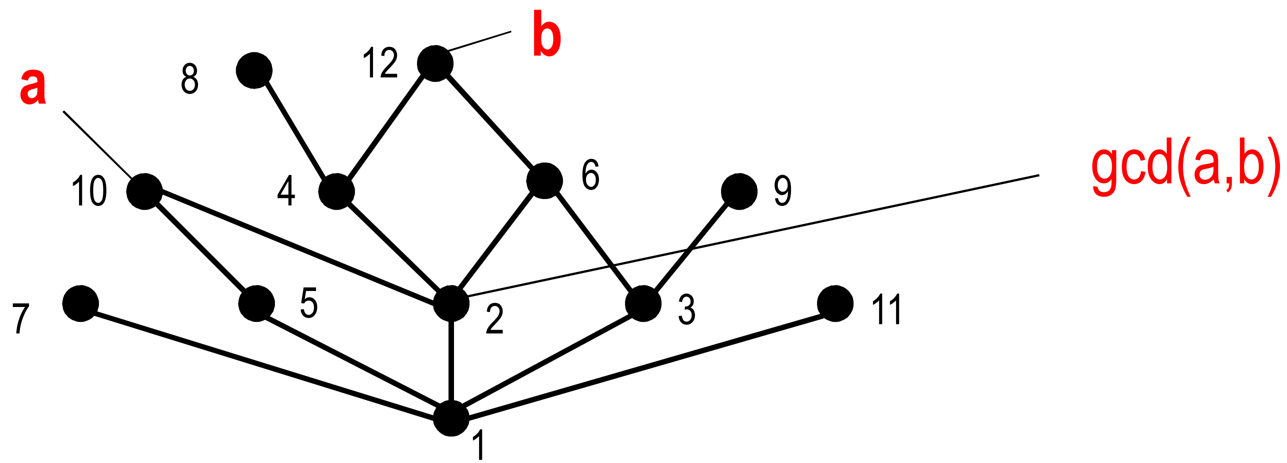
- For integers $a$ and $b$, a positive integer $c$ is said to be a **common divisor of $a$ and $b$** if $c \mid a$ and $c \mid b$
- Let $a, b$ be integers such that $a \neq 0$ or $b \neq 0$. Then a positive integer $c$ is called the **greatest common divisor of $a, b$** if
  - (a) $c \mid a$ and $c \mid b$ (that is $c$ is a common divisor of $a, b$)
  - (b) for any common divisor $d$ of $a$ and $b$, we have $d \mid c$

- What are the common divisors, and the greatest common divisor of 42 and 70?
- The greatest common divisor of $a$ and $b$ is denoted by $\gcd(a,b)$

# The Greatest Common Divisor (cntd)

- **Theorem**
  For any positive integers a and b, there is a unique positive integer c such that c is the greatest common divisor of a and b

- First try:
  Take the largest common divisor, in the sense of usual order
  Does not work: Why every other common divisor divides it?



# The Greatest Common Divisor (cntd)

- **Proof**.

  Given $a, b$, let $S = \{ as + bt \mid s,t \in \mathbb{Z},\ as + bt > 0 \}$.

  Since $S \neq \emptyset$, it has a least element $c$. We show that $c = \gcd(a,b)$

  We have $c = ax + by$ for some integers $x$ and $y$.

  If $d \mid a$ and $d \mid b$, then $d \mid ax + by = c$.

  If $c \nmid a$, we can use the division algorithm to find $a = qc + r$, where $q,r$ are integers and $0 < r < c$.

  Then $r = a - qc = a - q(ax + by) = a(1 - qx) + b(-qy) \in S$, a contradiction

  Therefore $c \mid a$, and by a similar argument $c \mid b$.

# The Greatest Common Divisor (cntd)

- **Proof**. (cntd)

  Finally, if $c$ and $d$ are greatest common divisors, then $c \mid d$ and $d \mid c$. Thus $c = d$.

  Q. E. D.

# Euclidean Algorithm: Small Example

- To warm up, let us find the greatest common divisor of 287 and 91

  $287 = 91 \cdot 3 + 14$

  Note that any common divisor of 287 and 91 is also a divisor of $14 = 287 - 91 \cdot 3$.

  Conversely, every common divisor of 91 and 14 is also a divisor of $287 = 91 \cdot 3 + 14$. Thus $\gcd(287,91) = \gcd(91,14)$.

  Next $91 = 14 \cdot 6 + 7$.

  By the same argument $\gcd(91,14) = \gcd(14,7)$.

  Finally, since $7 \mid 14$, $\gcd(14,7) = 7$.

  Thus, $\gcd(287,91) = 7$.

# Euclidean Algorithm: Key Property

- **Lemma**.

  Let $a = bq + r$, where $a, b, q$, and $r$ are integers. Then $\gcd(a,b) = \gcd(b,r)$

- Proof

  Let $d$ be a common divisor of $a$ and $b$. Then $d$ also divides $r = a - bq$. Thus, $d$ is a common divisor of $b$ and $r$.

  Now, let $d$ be a common divisor of $b$ and $r$. Then $d$ also divides $a = bq + r$.

  Therefore the pairs $a,b$ and $b,r$ have the same common divisors. Hence, $\gcd(a,b) = \gcd(b,r)$.

# Euclidean Algorithm: The Algorithm

- Let $a$ and $b$ be positive integers with $a \geq b$. Set $r_0 = a$ and $r_1 = b$ Successively apply the division algorithm until the remainder is $0$

$$\begin{array}{ll} r_0 = r_1 q_1 + r_2 & 0 \leq r_2 < r_1 \\ r_1 = r_2 q_2 + r_3 & 0 \leq r_3 < r_2 \\ \vdots & \\ r_{k-2} = r_{k-1} q_{k-1} + r_k & 0 \leq r_k < r_{k-1} \\ r_{k-1} = r_k q_k & \end{array}$$

- Eventually, the remainder is zero, because the sequence of remainders $a = r_0 > r_1 > r_2 > \ldots \geq 0$ cannot contain more than $a$ elements.
- Furthermore, $\gcd(a,b) = \gcd(r_0, r_1) = \cdots = \gcd(r_{k-2}, r_{k-1})$

$$= \gcd(r_{k-1}, r_k) = \gcd(r_k, 0) = r_k$$

- Hence $\gcd(a,b)$ is the last nonzero remainder in the sequence

# Greatest Common Divisor

- **Theorem.**

  If a, b are integers and d is their greatest common divisor, then there are integers u, v such that d = au + bv.

- **Proof.**

  We use the Euclidean algorithm and the notation $a = r_0,\ b = r_1,\ d = r_k$

  $$\begin{aligned} r_0 &= r_1 q_1 + r_2 \\ &\vdots \\ r_{k-3} &= r_{k-2} q_{k-2} + r_{k-1} \\ r_{k-2} &= r_{k-1} q_{k-1} + r_k \\ r_{k-1} &= r_k q_k \end{aligned}$$

  We have

  $$\begin{aligned} d = r_k &= r_{k-2} - r_{k-1} q_{k-1} \\ &= r_{k-2} - (r_{k-3} - r_{k-2} q_{k-2}) q_{k-1} \\ &= (r_{k-4} - r_{k-3} q_{k-3}) - (r_{k-3} - (r_{k-4} - r_{k-3} q_{k-3}) q_{k-2}) q_{k-1} \\ &\vdots \\ &= r_0 u + r_1 v = au + bv \end{aligned}$$

# Example

- Find $d = \gcd(821,123)$ and integers $u$ and $v$ such that

  $d = 821u + 123v$

# More Primes

- Prime numbers have some very special properties with respect to division
- **Properties of primes**.

(1) If $a,b$ are integers and $p$ is prime such that $p \mid ab$ then $p \mid a$ or $p \mid b$.

(2) Let $a_i$ be an integer for $1 \le i \le n$, and $p$ is prime and $p \mid a_1 a_2 \ldots a_n$ then $p \mid a_i$ for some $1 \le i \le n$

# The Fundamental Theorem of Arithmetic

- **Theorem**.

  Every integer $n > 1$ can be represented as a product of primes uniquely, up to the order of the primes.

- **Proof**.
- Existence

  By contradiction. Suppose that there is an $n > 1$ that cannot be represented as a product of primes, and let $m$ be the smallest such number.

  $m$ is not prime, therefore $m = st$ for some $s$ and $t$

  But then $s$ and $t$ can be written as products of primes, because $s < m$ and $t < m$.

  Therefore $m$ is a product of primes

# Example

- Find the prime factorization of 980,220

# Least Common Multiple

- A positive integer $c$ is called a common multiple of integers $a$ and $b$ if $a \mid c$ and $b \mid c$

- The number $c$ is called the least common multiple of $a$ and $b$, denoted $\mathrm{lcm}(a,b)$ if it is a common multiple and for any common multiple $d$ we have $c \mid d$

- **Theorem.**

  For any integers $a$ and $b$, the least common multiple exists.

# Least Common Multiple (cntd)

- Find lcm(231,455).

- **Theorem**

  For any integers $a$ and $b$ we have $ab = \text{lcm}(a,b) \cdot \text{gcd}(a,b)$

# Relatively Prime

- Numbers a and b such that gcd(a,b) = 1 are called relatively prime
- How many relatively prime numbers are there?
- Euler's totient function $\varphi(n)$ is the number of numbers k such that $0 < k < n$ and n and k are relatively prime.
- If p is prime then every $k < p$ is relatively prime with p. Hence, $\varphi(p) = p - 1$.
- **Lemma.**

  If a and b are relatively prime then $\varphi(ab) = \varphi(a) \cdot \varphi(b)$
- **Corollary.**

  If $n = p_1^{s_1} p_2^{s_2} \dots p_u^{s_u}$ is the prime factorization of n, then

$$\varphi(n) = n \cdot \left(1 - \frac{1}{p_1}\right)^{s_1} \left(1 - \frac{1}{p_2}\right)^{s_2} \cdots \left(1 - \frac{1}{p_u}\right)^{s_u}$$

# Homework

Exercises from the Book:

No. 1ab, 4, 5, 10, 15 (page 237)

No. 1ab, 5, 7, 9 (page 241)