MACM 101 — Discrete Mathematics I

Outline Solutions to Midterm 1

1. **How to prove that a statement** $\forall x\ P(x)$ **where** $P(x)$ **is a predicate is false?**

   It suffices to find a counterexample, a value a from the universe such that $P(a)$ is false.

2. **Construct a truth table** $(\neg r \wedge (p \oplus q)) \leftrightarrow r$

p	q	r	Φ
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

3. **Prove that the Rule of Syllogism is a valid argument.**

   The Rule of Syllogism is the following rule:

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \\ \triangle\ \ p \to r \end{array}$$

   We start with constructing the corresponding tautology

$$((p \to q) \wedge (q \to r)) \to (p \to r).$$

   Now it only remains to show that this statement is indeed a tautology. This can be done by constructing a truth table, or by simplifying the statement.

$$\begin{array}{lll}
\multicolumn{2}{l}{((p \to q) \wedge (q \to r)) \to (p \to r)} & \\
\iff & \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r) & \text{expression for implications} \\
\iff & (p \wedge \neg q) \vee (q \wedge \neg r)) \vee \neg p \vee r & \text{DeMorgan's laws} \\
\iff & ((p \vee \neg p) \wedge (\neg q \vee \neg p)) \vee ((q \vee r) \wedge (\neg r \vee r)) & \text{associative, commutative, and distributive laws} \\
\iff & (\neg q \vee \neg p) \vee (q \vee r) & \text{law of excluded middle and domination laws} \\
\iff & \neg q \vee q \vee \neg p \vee r & \text{associativity and commutativity laws} \\
\iff & T & \text{law of excluded middle and identity law}
\end{array}$$

4. **State the Absorption laws of set theory.**

   For any sets A, B
   $A \cup (A \cap B) = A$,
   $A \cap (A \cup B) = A$.

5. **Show that** $((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$ **and** $(p \to q) \oplus r$ **are not logically equivalent.**

   Method 1. Observe that for $p = q = 0$, $r = 1$ the first expression is true, while the second is false.

   Method 2. Construct the truth tables.

6. **Define the difference of two sets, give the set builder construction for it, draw Venn diagram.**

   The difference of sets A and B (or relative complement of B in A), denoted by $A - B$, is the set containing those elements that are in A, but not in B.

$$A - B = \{x | x \in A \wedge x \notin B\}.$$

7. **Explain the Rule of Universal Specification. How is it used in direct proofs?**

   If statement $\forall\, xP(x)$ is true, then $P(c)$ is true for any value c from the universe. Symbolically

$$\frac{\forall x P(x)}{\triangle\;\; P(c)}$$

   The rule of universal specification is often used as the first step in direct proofs to translate a universal statement, usually a premise, into the subject area. In this case c is a generic value, that is, a value that supposed to be from the universe, but that is noe assumed to have any properties except those posessed by all elements in the universe.

8. **Given premises:**
   "If I am sleeping, then I am snoring."
   "If I am not sleeping, then I will wake up tired."
   "If I wake up tired, then I will feel bad all day."
   infer the conclusion
   "If I am not snoring, then I will feel bad all day."

   Let the primitive statements be:

   l, 'I am sleeping'
   s, 'I am snoring'
   t, 'I will wake up tired'
   b, 'I will feel bad all day'

   Then the premises are translated as: $l \to s$, $\neg l \to t$, $t \to b$.

   And the conclusion: $\neg s \to b$.

Steps	Reason
1. $l \to s$	premise
2. $\neg s \to \neg l$	contrapositive of Step 1
3. $\neg l \to t$	premise
4. $\neg s \to t$	rule of syllogism to Steps 2 and 3
5. $t \to b$	premise
6. $\neg s \to b$	rule of syllogism to Steps 4 and 5.