Predicates and Quantifiers II

Discrete Mathematics
Andrei Bulatov

Previous Lecture

- Predicates
- Assigning values, universe, truth values

Relational Databases

- A relational database is a collection of tables like

No.	Name	Student ID	Supervisor	Thesis title
1.	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
…	…	…	…	…

A table consists of a schema and an instance. A schema is a collection of attributes, where each attribute has an associated universe of possible values. An instance is a collection of rows, where each row is a mapping that associates with each attribute of the schema a value in its universe.

- Every table is a predicate that is true on the rows of the instance and false otherwise.

Quantifiers

- One way to obtain a statement from a predicate is to assign all its variables some values

- Another way to do that is to use expressions like

  `For every …'

  `There is … such that …'

  `A … can be found …'

  `Any … is …'

`Every man is mortal'

`There is x such that x is greater than 3'

`There is a person who is my father'

`For any x, $x^2 \geq 0$ '

quantification

Universal Quantifiers

- Abbreviates constructions like

  For all …
  For any …
  Every …
  Each …

- Asserts that a predicate is true for all values from the universe

  \`Every man is mortal’
  \`All lions are fierce’
  \`For any x, $x^2 \geq 0$ '

- Notation: $\forall$
- $\forall$x P(x) means that for every value a from the universe P(a) is true

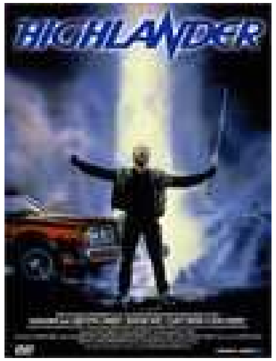
Universal Quantifiers (cntd)

`For any x, $x^2 \geq 0$ ' true!

`Every car is red' false! my car is not red

- $\forall$x P(x) is false if and only if there is at least one value a from the universe such that P(a) is false
- Such a value a is called a counterexample

Thus to disprove that `Every man is mortal' it suffices to recall the movie `Highlander'



Existential Quantifiers

- Abbreviates constructions like

  For some …
  For at least one …

  There is …
  There exists …

- Asserts that a predicate is true for at least one value from the universe

  \`There is a living king'
  \`Some people are fierce'
  \`There is x such that $x^2 \geq 10$'

- Notation: $\exists$

- $\exists$x P(x) means that there is a value a from the universe such that P(a) is true

Existential Quantifiers (cntd)

`There is a red car' true! my friend's car is red

`For some x, $x^2 < 0$' false!

- $\exists x$ P(x) is false if and only if for all a from the universe P(a) is false
- Disproving an existential statement is difficult!

Quantifiers and Negations

- Summarizing

	true	false
$\forall x\ P(x)$	For every value a from the universe $P(a)$ is true	There is a counterexample – a value a from the universe such that $P(a)$ is false
$\exists x\ P(x)$	There is a value a from the universe such that $P(a)$ is true	For all values a from the universe $P(a)$ is false

- Observe that

$\forall x\ P(x)$ is false if and only if $\exists x\ \neg P(x)$ is true

$\exists x\ P(x)$ is false if and only if $\forall x\ \neg P(x)$ is true

Example

- What is the negation of each of the following statements?

Statement		Negation
All lions are fierce	$\forall x\ P(x)$	There is a peaceful lion
Everyone has two legs	$\forall x\ P(x)$	There is a person having more than two legs, one leg, or no legs at all
Some people like coffee	$\exists x\ P(x)$	All people hate coffee
There is a lady in one of these rooms (Some rooms contain a lady)	$\exists x\ P(x)$	There is a tiger in every room

Multiple quantifiers

- Often predicates have more than one variable. In this case we need more than one quantifier.

$P(x,y)$ = ``car x has colour y"

$\forall x \forall y \; P(x,y)$	``every car is painted all colours
$\exists x \exists y \; P(x,y)$	``there is a car that is painted some colour
$\forall x \exists y \; P(x,y)$	``every car is painted some colour
$\exists x \forall y \; P(x,y)$	``there is a car that is painted all colours

Open and Bound Variables

- In the statement

  ``car x has some colour'' $\exists y\, P(x,y)$

  variables x and y play completely different roles.

- Variable y is **bound** by the existential quantifier. Effectively it disappeared from the statement.

- Variable x is not bound, it is **free**.

- Another example:

  ``x is the least number''  $Q(x,y)$ = ``x is less than y''

  $\forall y\ Q(x,y)$ or $\forall y\ (x \le y)$

  ``x is the greatest number'' $\forall y\ Q(y,x)$ or $\forall y\ (y \le x)$

Defining New Predicates

- Quantifying some variables helps creating new predicates

  ``car x has some colour'' $Q(x) = \exists y\, P(x,y)$

- Let $P(x,y)$ mean ``lion x likes y''

  $Q(y) = \forall x\ P(x,y)$ ``every lion likes y'' $Q(\text{meat})$ is true

  $Q(\text{apples})$ is false

  $R(x) = \forall y\ P(x,y)$ ``lion x likes everything'' $R(x)$ is always false

  (say it using quantifiers: $\forall x\, \neg R(x)$ or $\forall x\, \neg(\forall y\ P(x,y))$)

  $S(x) = \exists y\ P(x,y)$ ``lion x has favorite food''

Quantifiers and Compound Statements

- Quantifiers can be used together with logic connectives

``Every car is either red or blue”

P(x) - ``car x is red” Q(x) - ``car x is blue”

$\forall x\,(P(x) \vee Q(x))$

``Everyone who knows a current password can logon onto the network”

P(x) - ``x knows a current password”

Q(x) - ``x can logon onto the network”

$\forall x\,(P(x) \rightarrow Q(x))$

Quantifiers and Compound Statements (cntd)

- Logic connectives can be put between quantified statements

``Every car is blue, or there is a red car''

$P(x)$ - ``car $x$ is blue''   $Q(x)$ - ``car x is red''

$(\forall x\, P(x)) \vee (\exists x\, Q(x))$

``For every number there is a smaller one, or there is the least number''

We use predicate $x \le y$

$(\forall x\, \exists y\, (y \le x)) \vee (\exists x\, \forall y\, (x \le y))$

Definitions

- Predicates and quantifiers are often used to give definitions

  ``The mother of x is the female parent of x”

  $P(x,y)$ - ``$y$ is a parent of $x$”  $Q(x)$ - ``x is a female”

  $M(x,y)$ is defined as $P(x,y) \wedge Q(y)$ ``y is the mother of x”

$$M(x,y) \leftrightarrow P(x,y) \wedge Q(y)$$

  ``x and y are brothers”

$$B(x,y) \leftrightarrow \neg Q(x) \wedge \neg Q(y) \wedge \exists z\,(P(x,z) \wedge P(y,z))$$

Definitions (cntd)

- A cow is a big rectangular animal with horns and four legs in the corners

Definition of a limit:

A number A is a limit of a sequence $\{a_n\}$ if for any number ε > 0 there is N such that for any $n > N$ we have $|a_n - A| < \varepsilon$

$$\forall \varepsilon \exists N \forall n \ ((\varepsilon > 0) \wedge (n > N) \rightarrow (|a_n - A| < \varepsilon))$$

Rules

- Predicates and quantifiers are (implicitly) present in rules and laws

``Everyone having income more that $20000 must file a tax report”

P(x) - ``x has income more than $20000”

Q(x) - ``x must file a tax report”

$\forall x\ (P(x) \rightarrow Q(x))$

Theorems

- Every theorem involves predicates and quantifiers

``For every statement there is an equivalent CNF''

C(x) - ``x is a CNF'' $\forall x\ \exists y\ (C(y) \wedge (x \Leftrightarrow y))$

``A parallelogram is a rectangle if all its angles are equal''

R(x) - ``parallelogram x is a rectangle''

A(x) - ``all angles of x are equal''

$\forall x\ (A(x) \rightarrow R(x))$

Universe and Interpretation

- A logic statement is meaningless

It only makes sense if we specify a universe and a particular meaning of the predicate

$\forall x\, P(x)$

Interpretation

universe: animals
P(x): x has horns

universe: cars
P(x): x is red

universe: numbers
P(x): x is even

Homework

Exercises from the Book:

No. 17ab, 18ac, 25a (page 100-102)

7b, 8b (page 116)

Represent in symbolic form

a definition “Jaywalk means to cross a roadway, not being a lane, at any place which is not within a crosswalk and which is less one block from an intersection at which traffic control signals are in operation”.

a rule “No driver of a vehicle shall drive such vehicle on, over, or across any fire hose laid on any street or private road, unless directed so to do by the person in charge of such hose or a police officer”