Problems to Week 10 Tutorial — MACM 101 (Fall 2014)

1. Prove that $6 \cdot 7^n - 2 \cdot 3^n$ is divisible by 4 for any $n \geq 1$.

2. A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n-1$ moves are required to assemble a puzzle with n pieces.

3. Give a recursive definition of $P_m(n)$, the product of the integer m and the non-negative integer n.

4. A complete binary tree is a graph defined through the following recursive definition.
   Basis step: A single vertex is a complete binary tree.
   Inductive step: If T_1 and T_2 are disjoint complete binary trees with roots r_1, r_2, respectively, then the graph formed by starting with a root r, and adding an edge from r to each of the vertices r_1, r_2 is also a complete binary tree.
   Draw all complete binary trees that can be obtain by applying the inductive step at most 3 times (complete binary trees of level 3).
   Use structural induction to show that $n(T) \geq 2h(T) + 1$, where $n(T)$ denotes the number of vertices in T, and $h(T)$ denotes the height of T.

5. The board of directors of a pharmaceutical corporation has 10 members. An upcoming stockholders' meeting is scheduled to approve a new slate of company officers (chosen from the 10 board members).

   (a) How many different slates consisting of a president, vice president, secretary, and treasurer can the board present to the stockholders for their approval?

   (b) Three members of the board are physicians. How many slates from part (a) have (i) a physician nominated for presidency? (ii) exactly one physician appearing on the slate? (iii) at least one physician appearing on the slate?

6. How many arrangements are there of all the letters in SOCIOLOGICAL?

7. In how many different orders can five runners finish a race if no ties allowed?

8. A club has 25 members

   (a) How many ways are there to choose four members of the club to serve on an executive committee?

   (b) How many ways are there to choose a president, vice president, secretary, and tresurer of the club, where no person can hold more than one position?

9. In how many ways can 12 different books be distributed among four children so that (a) each child gets three books? (b) the two oldest children get four books each and the two youngest get two books each?