Integers

Discrete Mathematics
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Integers

"*God made the integers; all else is the work of man*"

Leopold Kroenecker

Division

- Most of useful properties of integers are related to division
- If a and b are integers with $a \neq 0$, we say that a **divides** b if there is an integer c such that $b = ac$.
- When a divides b we say that a is a **divisor** (**factor**) of b, and that b is a **multiple** of a.
- The notation $a \mid b$ denotes that a divides b. We write $a \nmid b$ when a does not divide b
- Example. Let n and d be positive integers. How many positive integers not exceeding n are divisible by d?

  The numbers in question have the form dk, where k is a positive integer and $0 < dk \le n$. Therefore, $0 < k \le n/d$. Thus the answer is $\lfloor n/d \rfloor$

Properties of Divisibility

- Let a, b, and c be integers. Then
  (i) if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;
  (ii) if $a \mid b$, then $a \mid bc$ for all integers c;
  (iii) if $a \mid b$ and $b \mid c$, then $a \mid c$.
- Proof.

(i) Suppose $a \mid b$ and $a \mid c$. This means that there are k and m such that $b = ak$ and $c = am$.

Then $b + c = ak + am = a(k + m)$, and a divides $b + c$.

Properties of Divisibility (cntd)

- If a, b, and c are integers such that $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ whenever m and n are integers.
- Proof.

  By part (ii) it follows that $a \mid mb$ and $a \mid nc$.

  By part (i) it follows that $a \mid mb + nc$.

- If $a \mid b$ and $b \mid a$, then $a = \pm b$.
- Proof.

  Suppose that $a \mid b$ and $b \mid a$. Then $b = ak$ and $a = bm$ for some integers k and m.

  Therefore $a = bm = akm$, which is possible only if $k,m = \pm 1$.

The Division Algorithm

- **Theorem** (The division algorithm)

  Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \le r < d$, such that $a = dq + r$

- d is called the divisor, a is called the dividend, q is called the quotient, and r is called the remainder

- Examples:
  - Let $a = 101$ and $d = 11$
    Then $101 = 11 \cdot 9 + 2$
  - Let $a = -11$ and $d = 3$
    Then $-11 = 3 \cdot (-4) + 1$
  - Let $a = 3$ and $d = 11$
    Then $3 = 11 \cdot 0 + 3$

Representation of Integers

- In most cases we use decimal representation of integers. For example, 657 means

$$6 \cdot 100 + 5 \cdot 10 + 7 = 6 \cdot 10^2 + 5 \cdot 10^1 + 7 \cdot 10^0$$

- Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$$

where k is a nonnegative number, $a_k, a_{k-1}, \ldots, a_1, a_0$ are nonnegative integers less than b, and $a_k \neq 0$

- Such a representation of n is called the base b expansion of n, denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$

Binary Expansion

- Important case of a base is 2. The base 2 expansion is called the binary expansion of a number

$$n = a_k \cdot 2^k + a_{k-1} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0$$

- Find the binary expansion of 165

$$165 = 2 \cdot 82 + 1$$
$$82 = 2 \cdot 41 + 0$$
$$41 = 2 \cdot 20 + 1$$
$$20 = 2 \cdot 10 + 0$$
$$10 = 2 \cdot 5 + 0$$
$$5 = 2 \cdot 2 + 1$$
$$2 = 2 \cdot 1 + 0$$
$$1 = 2 \cdot 0 + 1$$

165	2	82	1 (a_0)
82	2	41	0 (a_1)
41	2	20	1 (a_2)
20	2	10	0 (a_3)
10	2	5	0 (a_4)
5	2	2	1 (a_5)
2	2	1	0 (a_6)
			1 (a_7)

$$165 = (10100101)_2$$

Hexadecimal Expansion

- Using A, B, C, D, E, F for 10, 11, 12, 13, 14, 15, respectively, find the base 16 expansion of 175627

Homework

Exercises from the Book

No. 2, 3, 4, 10, 12, 14 (page 603)