# Propositional Logic

Discrete Mathematics
Andrei Bulatov

---

## What is Logic?

“Computer science is continuation of logic by other means”
Georg Gottlob

“Contrariwise”, continued Tweedledee, “if it was so, it might be; and if it were so, it would be; but as it isn’t, it ain’t. That’s logic”
Lewis Carroll, *Through the Looking Glass*

“Logic celebrates the unity of a pathological masculine self-identity that cannot listen and recognizes only negation but not difference”
Andrea Nye, *Words of Power*

---

## Use of Logic

- In mathematics and rhetoric:
  - Give precise meaning to statements.
  - Distinguish between valid and invalid arguments.
  - Provide rules of `correct’ reasoning.

Natural language can be very ambiguous

`If you do your homework, then you’ll get to watch the game.’
≀
`If you don’t do your homework, then you will not get to watch ...’

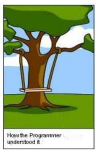
`You do your homework, or you’ll fail the exam.’
‖
`If you don’t do your homework, then you’ll fail the exam.’

---

## Use of Logic (cntd)

- In computing:
  - Derive new data / knowledge from existing facts
  - Design of computer circuits.
  - Construction of computer programs.
  - Verification of correctness of programs and circuit design.
  - Specification

What the customer really needed



How the Programmer understood it

What the customer got

---

## Statements (propositions)

- Propositional logic deals with statements and their truth values
- A statement is a declarative sentence that can be true or false
- Truth values are TRUTH (T or 1) and FALSE (F or 0).
- Examples:
  - $1 + 1 = 2$ (statement, T)
  - The moon is made of cheese (statement, F)
  - Go home! (not statement, imperative)
  - What a beautiful garden! (not statement, exclamation)
  - $y + 1 = 2$ (not statement, uncertain)
  - The God exists (statement, ?)

---

## Compound Statements

- Simplest statements are called primitive statements
- We cannot decide the truth value of a primitive statement. This is not what logic does.

  We shall use propositional variables to denote primitive statements, *p, q, r*, ...
- Instead we combine primitive statements by means of logic connectives into compound statements or formulas and look how the truth value of a compound statement depends on the truth values of the primitive statements it includes.

  We will denote compound statements by $\Phi$, $\Psi$, ...

## Logic Connectives

- negation (not, $\neg$) `It is not true that at least one politician was honest'
- conjunction (and, $\wedge$) `In this room there is a lady, and in the other room there is a tiger'
- disjunction (or, $\vee$) `Margaret Mitchell wrote `Gone with the Wind or I am going home'
- implication (if…, then…, $\rightarrow$) `If there is a tiger in this room, then there is no lady there'
- exclusive or (either …, or …, $\oplus$) `There is either a tiger in this room, or a lady'
- biconditional (equivalence) (if and only if, $\leftrightarrow$) `There is a lady in this room if and only if there is a tiger in the other room'

---

## Truth tables

Truth table is a way to specify the exact dependence of the truth value of a compound statement through the values of primitive statements involved

truth values of primitive statements (propositional variables)

truth value of compound statements (formulas)

| p | q | $\Phi$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| … | … | … |

---

## Truth Tables of Connectives (negation and conjunction)

- Negation

| p | $\neg p$ |
|---|---|
| F (0) | T (1) |
| T (1) | F (0) |

unary connective

`Today is Friday' $\Rightarrow$ `Today is not Friday'

- Conjunction

| p | q | $p \wedge q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

binary connective

`Today is Friday' `It is raining' $\Rightarrow$ `Today is Friday and it is raining'

---

## Truth Tables of Connectives (disjunction)

- Disjunction – inclusive `or'

| p | q | $p \vee q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

`Students who have taken calculus can take this course' `Students who have taken computing can take this course' $\Rightarrow$ `Students who have taken calculus or computing can take this course'

Be careful with `or' constructions in natural languages!

`You do your homework, or you'll fail the exam.'

`Today is Friday or Saturday'

---

## Truth Tables of Connectives (exclusive or)

- Exclusive `or'
  One of the statements is true but not both

| p | q | $p \oplus q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

`You can follow the rules or be disqualified.'

`Natalie will arrive today or Natalie will not arrive at all.'

---

## Truth Tables of Connectives (implication)

- Implication

| p | q | $p \rightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Note that logical (*material*) implication does not assume any causal connection.
`**If** black is white, **then** we live in Antarctic.'
'**If** pigs fly, **then** Paris is in France.'

## Implication as a promise

- Implication can be thought of as a promise, and it is true if the promise is kept

`**If** I am elected, **then** I will lower taxes’

| | |
|---|---|
| - He is not elected and taxes are not lowered | promise kept! |
| - He is not elected and taxes are lowered | promise kept! |
| - He is elected, but (=and) taxes are not lowered | promise broken! |
| - He is elected and taxes are lowered | promise kept! |

---

## Playing with Implication

- Parts of implication

$$p \rightarrow q$$

hypothesis
antecedent
premise

conclusion
consequence

`**if** p, **then** q’
`**if** p, q’
`p **is sufficient for** q’
`q **if** p’
`q **when** p’

`q **unless** ¬p’
`p **implies** q’
`p **only if** q’
`q **whenever** p’
`q **follows from** p’

`a **sufficient condition for** q is p’
`**a necessary condition for** p is q’

---

## Playing with Implication (cntd)

- Converse, contrapositive, and inverse

$p \rightarrow q$ `The home team wins whenever it is raining’
(`If it is raining then the home team wins’)

- Converse $q \rightarrow p$
  `If the home team wins, then it is raining’
- Contrapositive $\neg q \rightarrow \neg p$
  `If the home team does not win, then it is not raining’
- Inverse $\neg p \rightarrow \neg q$
  `If it is not raining, then the home team does not win’

---

## Truth Tables of Connectives (biconditional)

- Biconditional or Equivalence
  One of the statements is true if and only if the other is true

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

`You can take the flight if and only if you buy a ticket.’

---

## Example

`You can access the Internet from campus if you are a computer science major or if you are not a freshman.’

p - `you can access the Internet from campus’

q - `you are a computer science major’

r - `you are a freshman’

---

## Tautologies

- Tautology is a compound statement (formula) that is true for all combinations of truth values of its propositional variables

$$(p \rightarrow q) \vee (q \rightarrow p)$$

| $p$ | $q$ | $(p \rightarrow q) \vee (q \rightarrow p)$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

## Contradictions

- Contradiction is a compound statement (formula) that is false for all combinations of truth values of its propositional variables

$(p \oplus q) \wedge (p \oplus \neg q)$

| p | q | $(p \oplus q) \wedge (p \oplus \neg q)$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 0 |

---

## Homework

Exercises from the Book:

No. 1 ,3, 4, 8a, 8c (page 54)