Laws of Logic

Discrete Mathematics
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Previous Lecture

- Truth tables
- Tautologies and contradictions
- Logic equivalences

Logic Equivalences

- Compound statements Φ and Ψ are said to be logically equivalent if the statement Φ is true (false) if and only if Ψ is true (respectively, false)

or

- The truth tables of Φ and Ψ are equal

or

- For any choice of truth values of the primitive statements (propositional variables) of Φ and Ψ, formulas Φ and Ψ have the same truth value

- If Φ and Ψ are logically equivalent, we write

$$\Phi \Leftrightarrow \Psi$$

Why Logic Equivalences

- To simplify compound statements

  ``If you are a computer science major or a freshman and you are not a computer science major or you are granted access to the Internet, then you are a freshman or have access to the Internet''

- To convert complicated compound statements to certain `normal form' that is easier to handle

  Conjunctive Normal Form   CNF

Example Equivalences

- Implication and its contrapositive

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

- All tautologies are equivalent to T
- All contradictions are equivalent to F

Equivalences and Tautologies

- **Theorem** Compound statements Φ and Ψ are logically equivalent if and only if $\Phi \leftrightarrow \Psi$ is a tautology.

Proof

Suppose that $\Phi \Leftrightarrow \Psi$. Then these statements have equal truth tables

p	q	…	Φ	Ψ	$\Phi \leftrightarrow \Psi$
…	…	…	…	…	1
0	1	…	1	1	1
…	…	…	…	…	…
1	0	…	0	0	1
…	…	…	…	…	1

Equivalences and Tautologies (cntd)

Suppose now that $\Phi \leftrightarrow \Psi$ is a tautology. This means that for any choice of the truth values of Φ and Ψ, $\Phi \leftrightarrow \Psi$ is true.

If Φ is true, then Ψ must also be true.

If Φ is false, then to make the formula $\Phi \leftrightarrow \Psi$ true Ψ must also be false.

Q.E.D.

Laws of Logic

- Double negation

$\neg\neg p \Leftrightarrow p$

p	$\neg p$	$\neg\neg p$
0	1	0
1	0	1

Laws of Logic (cntd)

- DeMorgan's laws

$\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

$\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg (p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Example

- Construct the negation of

  `Miguel has a cell phone and he has a laptop’

  `Heather will go to the concert or Steve will go to the concert’

`Algebraic' Laws of Logic

- $p \wedge q \Leftrightarrow q \wedge p$
  $p \vee q \Leftrightarrow q \vee p$
  Commutative laws

- $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
  $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
  Associative laws

- $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
  $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
  Distributive laws

- $p \wedge p \Leftrightarrow p$
  $p \vee p \Leftrightarrow p$
  Idempotent laws

`Logic' Laws of Logic

- $p \wedge T \Leftrightarrow p$
  $p \vee F \Leftrightarrow p$
  Identity laws

- $p \wedge \neg p \Leftrightarrow F$
  $p \vee \neg p \Leftrightarrow T$
  Inverse laws
  the law of contradiction
  the law of excluded middle

- $p \wedge F \Leftrightarrow F$
  $p \vee T \Leftrightarrow T$
  Domination laws

- $p \wedge (p \vee q) \Leftrightarrow p$
  $p \vee (p \wedge q) \Leftrightarrow p$
  Absorption laws

Example

- Simplify the statement

$$\neg(q \vee r) \vee \neg(\neg q \vee p) \vee r \vee p$$

Expressing Connectives

- Some connectives can be expressed through others
  - $p \oplus q \iff \neg(p \leftrightarrow q)$
  - $p \leftrightarrow q \iff (p \rightarrow q) \land (q \rightarrow p)$
  - $p \rightarrow q \iff \neg p \lor q$

- **Theorem** Every compound statement is logically equivalent to a statement that uses only conjunction, disjunction, and negation

Example

``If you are a computer science major or a freshman and you are not a computer science major or you are granted access to the Internet, then you are a freshman or have access to the Internet”

p - `you can access the Internet from campus’

q - `you are a computer science major’

r - `you are a freshman’

Example

- Simplify the statement

$$(p \vee q) \leftrightarrow (p \rightarrow q)$$

First Law of Substitution

- Suppose that the compound statement Φ is a tautology. If p is a primitive statement that appears in Φ and we replace each occurrence of p by the same statement q, then the resulting compound statement Ψ is also a tautology.

- Let $\Phi = (p \rightarrow q) \vee (q \rightarrow p)$, and we substitute p by $p \vee (s \oplus r)$

  Therefore $((p \vee (s \oplus r)) \rightarrow q) \vee (q \rightarrow (p \vee (s \oplus r))$ is a tautology

Second Law of Substitution

- Let Φ be a compound statement, p an arbitrary (not necessarily primitive!) statement that appears in Φ, and let q be a statement such that $p \Leftrightarrow q$. If we replace one or more occurrences of p by q, then for the resulting compound statement Ψ we have $\Phi \Leftrightarrow \Psi$.

- Let $\Phi = (p \rightarrow q) \vee (q \rightarrow p)$, and we substitute the first occurrence of p by $p \vee (p \wedge q)$.
  Recall that $p \Leftrightarrow p \vee (p \wedge q)$ by Absorption Law.

  Therefore

$$(p \rightarrow q) \vee (q \rightarrow p) \quad \Leftrightarrow \quad ((p \vee (p \wedge q)) \rightarrow q) \vee (q \rightarrow p).$$

Homework

Exercises from the Book:

No. 1ai, 2, 6a, 6b, 14a (page 66)

- Express conjunction and disjunction through implication and negation (*)