Theorems and Proofs II

Discrete Mathematics
Andrei Bulatov

Previous Lecture

- Axioms and theorems
- Rules of inference for quantified statements

Rule of Universal Specification (cntd)

- If an open statement becomes true for all values of the universe, then it is true for each specific individual value from that universe

$$\frac{\forall x\, P(x)}{\therefore\ P(c)}$$

- Example

Premises: $\forall x\,(P(x) \to Q(x))$, $P(\text{Socrates})$

Step	Reason
1. $\forall x\,(P(x) \to Q(x))$,	premise
2. $P(\text{Socrates}) \to Q(\text{Socrates})$,	rule of universal specification
3. $P(\text{Socrates})$	premise
4. $Q(\text{Socrates})$	modus ponens

Rule of Universal Generalization

- Let us prove a theorem:

  If $2x - 6 = 0$ then $x = 3$.

- Proof

  Take any number c such that $2c - 6 = 0$. Then $2c = 6$, and, finally $c = 3$. As c is an arbitrary number this proves the theorem.

  Q.E.D

- Look at the first and the last steps.
  - In the first step instead of the variable we start to consider its generic value, that is a value that does not have any specific property that may not have any other value in the universe
  - In the last step having proved the statement for the generic value we conclude that the universal statement is also true

Rule of Universal Generalization (cntd)

- If an open statement $P(x)$ is proved to be true when x is assigned by any arbitrary chosen (generic) value from the universe, then the statement $\forall x\, P(x)$ is also true.
- Example: ``If $2x - 6 = 0$ then $x = 3$."
- Notation: $P(x)$ - ``$2x - 6 = 0$", $Q(x)$ - ``$2x = 6$", $R(x)$ - ``$x = 3$"
- Premises: $\forall x\, (P(x) \rightarrow Q(x))$, $\forall x\, (Q(x) \rightarrow R(x))$
- Conclusion: $\forall x\, (P(x) \rightarrow R(x))$,

Step	Reason
1. $\forall x\, (P(x) \rightarrow Q(x))$, $\forall x\, (Q(x) \rightarrow R(x))$	premises
2. $P(c) \rightarrow Q(c)$, $Q(c) \rightarrow R(c)$,	rule of univ. specification
3. $P(c) \rightarrow R(c)$	rule of syllogism
4. $\forall x\, (P(x) \rightarrow Q(x))$	rule of univ. generalization

Existential Rules

- Rule of Existential Specification.

  If $\exists x\, P(x)$ is true in a given universe, then there is value a in this universe with $P(a)$ true.

- Rule of Existential Generalization.

  If $P(a)$ is true for some value a in a given universe, then $\exists x\, P(x)$ is true in this universe.

Methods of Proving – Direct Proofs

- Direct proofs are used when we need to proof statements like

$$\forall x\,(P(x) \rightarrow Q(x))$$

- Main steps

Our goal is to prove that $P(a) \rightarrow Q(a)$ is a tautology for a generic value a.

1. Assume that $P(a)$ is true
2. Using axioms, previous theorems etc. prove that $Q(a)$ is true
3. Conclude that $P(a) \rightarrow Q(a)$ is true
4. Use the rule of universal generalization to infer

$$\forall x\,(P(x) \rightarrow Q(x))$$

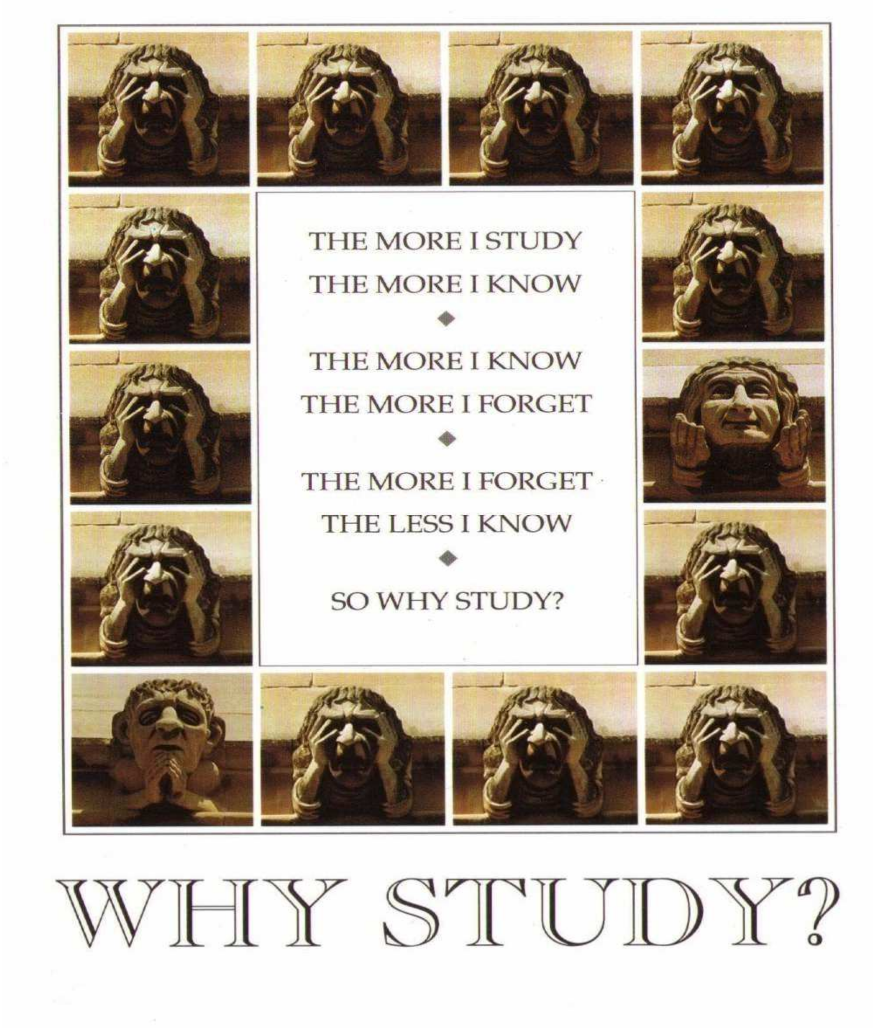
Methods of Proving – Direct Proofs

- Example: ``If $2x - 6 = 0$ then $x = 3$.”
- Notation: $P(x)$ - ``$2x - 6 = 0$”, $Q(x)$ - ``$2x = 6$”, $R(x)$ - ``$x = 3$”
- Need to prove: $\forall x\,(P(x) \rightarrow R(x))$
- Previous knowledge: $\forall x\,(P(x) \rightarrow Q(x))$, $\forall x\,(Q(x) \rightarrow R(x))$

Step	Reason
1. $P(c)$	assumption
1. $\forall x\,(P(x) \rightarrow Q(x))$, $\forall x\,(Q(x) \rightarrow R(x))$	premises
2. $P(c) \rightarrow Q(c)$, $Q(c) \rightarrow R(c)$,	rule of univ. specification
3. $R(c)$	Modus Ponens
4. $\forall x\,(P(x) \rightarrow R(x))$	rule of univ. generalization

Example

- **Theorem**. Everyone who studies more knows less.



Example

- **Proof**. Everyone who studies more knows less.
- $S(x)$ x studies more $M(x)$ x knows more

  $F(x)$ x forgets more $L(x)$ x knows less
- Premises: $\forall x\,(S(x) \rightarrow M(x)),\ \forall x\,(M(x) \rightarrow F(x)),\ \forall x\,(F(x) \rightarrow L(x))$
- Theorem: $\forall x\,(S(x) \rightarrow L(x))$

Step	Reason
1. $S(c)$	assumption
2. $\forall x\,(S(x) \rightarrow M(x)),\ \forall x\,(M(x) \rightarrow F(x)),\ \forall x\,(F(x) \rightarrow L(x))$	premises
3. $S(c) \rightarrow M(c),\ M(c) \rightarrow F(c),\ F(c) \rightarrow L(c)$	rule of univ. spec.
4. $L(c)$	Modus Ponens
5. $\forall x\,(S(x) \rightarrow L(x))$	rule of univ. gen.

Methods of Proving – Proof by Contraposition

- Sometimes direct proofs do not work
- Definition: n is even if and only if there is k such that $n = 2k$
- Prove that if $3n + 2$ is even, then n is also even

  That is $\forall x\,(E(3x + 2) \to E(x))$
- Let us try the direct approach:

  As for the generic value n the number $3n + 2$ is even, for some k we have $3n + 2 = 2k$. Therefore $3n = 2(k + 1)$.

  Now what?
- What if instead of $\forall x\,(E(3x + 2) \to E(x))$ we prove the contrapositive, $\forall x\,(\neg E(x) \to \neg E(3x + 2))$?

Methods of Proving – Proof by Contraposition (cntd)

- So assume that n is odd, that is there is k such that $n = 2k + 1$.

- Then $3n + 2 = 3\cdot(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1$.

  That is $3n + 2$ is odd.

- We have proved that $\neg$E$(3n + 2)$ is true, and therefore the contraposition $\forall x\ (\neg\mathrm{E}(x) \to \neg\mathrm{E}(3x + 2))$ is true.

  Finally, we conclude that the theorem $\forall x\ (\mathrm{E}(3x + 2) \to \mathrm{E}(x))$ is also true.

Methods of Proving – Proof by Contraposition (cntd)

- Main steps

Our goal is to prove that $P(a) \to Q(a)$ is a tautology for a generic value a.

Instead we prove the contrapositive $\neg Q(a) \to \neg P(a)$

1. Assume that $\neg Q(a)$ is true
2. Using axioms, previous theorems etc. prove that $\neg P(a)$ is true
3. Conclude that $\neg Q(a) \to \neg P(a)$ is true
4. Conclude that $P(a) \to Q(a)$ is true
5. Use the rule of universal generalization to infer

$$\forall x\, (P(x) \to Q(x))$$

Methods of Proving – Proof by Contradiction

- Proofs by contradiction use the Rule of Contradiction

$$\frac{\neg p \rightarrow F}{\therefore\ p}$$

- Can be used to prove statements of any form

- Main steps
  1. Assume $\neg p$.
  2. Using axioms, previous theorems etc. infer a contradiction
  3. Conclude p.

- Usually the contradiction has the form $\exists x\,(Q(x) \wedge \neg Q(x))$

Example

- **Definition**: a barber is called strict if he shaves those and only those who do not shave themselves.
- **Theorem**. There is no strict barber.

  (All barbers are not strict.)
- **Proof**.
- Assume the contrary: a strict barber c exists
- Does he shave himself?
- If no ($\neg q$), then by the definition he must shave himself (q)
- If yes (q), then by definition he must not ($\neg q$)
- Either way we have $q \wedge \neg q$, a contradiction
- We conclude that a strict barber does not exist

Example (cntd)

Another Example

- Definition: a real number is said to be rational if it can be represented as a fraction $\frac{a}{b}$ where a,b are integers
- Prove that $\sqrt{2}$ is irrational
- Proof

Suppose that $\sqrt{2}$ is rational, that is there are integers a,b such that $\sqrt{2} = \frac{a}{b}$.

We may assume that a,b have no common divisor.

Squaring we obtain $a^2 = 2b^2$.

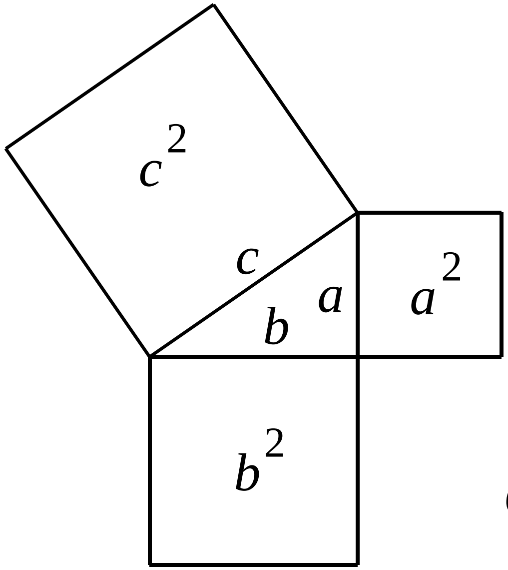
Since a^2 is even, a is also even, hence $a = 2c$ for some c.

Therefore $2b^2 = 4c^2$, and so $b^2 = 2c^2$.

Hence b is even.

We get that a and b have a common factor – 2. A contradiction.

Pythagoras

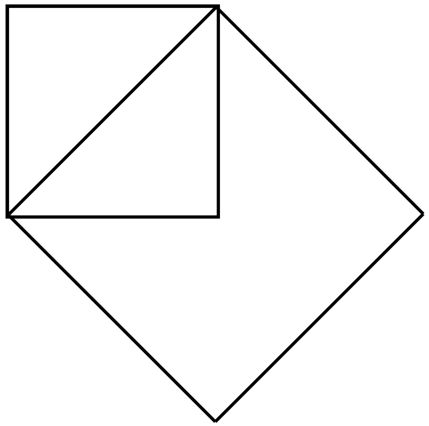


$$c^2 = a^2 + b^2$$

``Number is the ruler of forms and ideas, and the cause of gods and demons''

numbers = rational numbers

$\sqrt{2}$ does not belong to this world



Proving Existential Statements

- How to prove $\exists x\ P(x)$.
- Constructive proofs: find or construct a value a such that P(a) is true.

  Prove that there is a grey car…

  My car is grey!
- Pure proofs of existence:

  Assume that $\forall x\ \neg P(x)$.

  Using axioms, previous theorems etc. infer a contradiction

  Thus, this is a proof by contradiction.

Homework

Exercises from the Book:

No. 5, 9, 11, 13, 15, 17 (page 116-117)