Finite Probability

Discrete Mathematics
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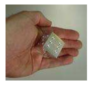
Experiments and Outcomes

- Experiment: Tossing a coin
  Outcomes: {heads, tails}

- Experiment: Rolling a dice
  Outcomes: {1,2,3,4,5,6}



- Experiment: Rolling two dice
  Outcomes: $\{1,\dots,6\} \times \{1,\dots,6\}$
  or $\{A \subseteq \{1,\dots,6\} : |A| \le 2\}$

- Experiment: Buying 3 lottery tickets (out of 100,000)
  Outcomes: 3-element subsets of $\{1,\dots,100000\}$

Sample Space and Events

- The set of all outcomes of an experiment is called the sample space
- Sometimes we are interested not in a single outcome, but an event that happens in several outcomes

Examples:
- Get heads at least 3 times when tossing 5 coins

- Win a prize in lottery

- Get 2 aces in a poker hand

Events

- Let S be the sample space of a certain experiment. An event is any subset of S

Examples:
- Experiment: Tossing 2 coins
  Sample space: $S = \{\text{heads,tails}\} \times \{\text{heads,tails}\}$
  Event: Get exactly 1 heads
  $A = \{(\text{heads,tails}),(\text{tails,heads})\}$
- Experiment: Rolling 2 dice
  Sample space: $S = \{1,\dots,6\} \times \{1,\dots,6\}$
  Event: The sum of the dice is 6
  $A = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

Probability

- In all our experiments each of the possible outcomes has the same likelihood of occurrence, or the same probability of occurrence
- If this is the case we can use the model of classic or finite probability
- Under the assumption of equal likelihood, let S be the sample space for an experiment. If $|S| = n$, $a \in S$, and $A \subseteq S$, then

$\Pr(\{a\}) = \Pr(a) = \frac{1}{n}$ the probability that a occurs

$\Pr(A) = \frac{|A|}{n}$ the probability that A occurs

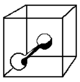
Examples

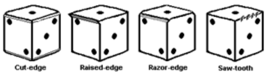
- The probability of getting heads in the coin tossing experiment
  Sample space: $S = \{\text{heads,tails}\}$, Event: $A = \{\text{heads}\}$,
  $\Pr(A) = \frac{|A|}{|S|} = \frac{1}{2}$
- The probability to get even number in the dice rolling experiment
  Sample space: $S = \{1,2,3,4,5,6\}$, Event: $A = \{2,4,6\}$
  $\Pr(A) = \frac{|A|}{|S|} = \frac{3}{6} = \frac{1}{2}$
- 100 tickets, numbered 1,2,3,…, 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). Find the probability that ticket 47 wins a prize while ticket 73 does not.

Equal Likelihood

- Equal likelihood of outcomes is a nontrivial property.
- It is not the case for flipping coins!
  See recent Persi Diaconis work

- One can make a crooked dice:
  loaded dice
  floaters
  tapping dice
  shapes
  bevels





Equal Likelihood (cntd)

- Subsets vs. permutation in dice rolling
- Do the events A = `we get 1 and 3' and B = `we get 2 and 2' have the same likelihood?
- Suppose we have two pairs of dice: colored and white

- Then A occurs if `red is 1, white is 3' and `red is 3, white is 1'
  B occurs only if `red is 2, white is 2'
  So A is twice more likely than B
- However, the probabilities do not depend on the color…

Equal Likelihood (cntd)

- Among 100 lottery tickets there is 1 winning ticket. I buy 2 tickets. Find the probability I win. (Suppose ticket #1 wins.)
- Method 1. Experiment: I buy 2 tickets (unordered)
  Sample space: S = {2-element subsets of {1,…,100} }
  Event: A = {1 belongs to my set}
  $$\Pr(A) = \frac{|A|}{|S|} = \frac{99}{C(100,2)} = \frac{1}{50}$$
- Method 2. Experiment: I buy a ticket, and after hesitating one more
  Sample space: S = {permutations of size 2 }
  Event: A = {permutations of size 2 containing 1}
  $$\Pr(A) = \frac{|A|}{|S|} = \frac{99+99}{P(100,2)} = \frac{2\cdot 99}{100\cdot 99} = \frac{1}{50}$$

More General Probability

- Sample space: Any set S
- Event: `Any' subset of S
- Probability: A measure, that is a function Pr: P(S) $\to$ [0,1], such that
  - $\Pr(\varnothing) = 0$
  - $\Pr(S) = 1$
  - $\Pr(A) \ge 0$ for all $A \subseteq S$
  - for any disjoint $A, B \subseteq S$, $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

More General Probability: Crooked Dice

- Suppose we made a loaded dice
  S = {1,2,3,4,5,6}
  Pr(1) = 1/16,
  Pr(2) = Pr(3) = Pr(4) = Pr(5) = 1/8
  Pr(6) = 7/16
  Pr({i,j,…,m}) = Pr(i) + Pr(j) + … + Pr(m)
- Find Pr({1,3,5})

More General Probability: Geometric Probability

- How to measure the area of an island?

- Draw a rectangle around the island and drop many random points
- Then $\dfrac{\text{area of the island}}{\text{area of the rectangle}} \approx \dfrac{\#\text{ of points within the island}}{\text{total }\#\text{ of points}}$
- Sample space: Points in the rectangle
  Events: Measurable sets of points
  Probability: The area of an event

Properties of Probability

Theorem

Let S be the sample space of a certain experiment, A,B events. Then

a) $\Pr(\overline{A}) = 1 - \Pr(A)$

b) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Proof

b) $\Pr(A \cup B) = \Pr(A - B) + \Pr(B - A) + \Pr(A \cap B)$ (as these sets are disjoint)

$= (\Pr(A - B) + \Pr(A \cap B)) + (\Pr(B - A) + \Pr(A \cap B)) - \Pr(A \cap B)$

$= \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Q. E. D.

Examples

- Two integers are selected, at random and without replacement, from {1,2,…,100}. What is the probability the integers are consecutive?
- If three integers are selected, at random and without replacement, from {1,2,…,100}, what is the probability their sum is even?

Homework

Exercises from the Book:

No. 1, 5, 9, 15 (page 156)

1, 4, 7 (page 164 – 165)