Logic Inference

Discrete Mathematics
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Previous Lecture

- Laws of logic
- Expressions for implication, biconditional, exclusive or
- Valid and invalid arguments
- Logic inference

General Definition of Inference

- The general form of an argument in symbolic form is

$$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow q$$

premises conclusion

- The argument is valid if whenever each of the premises is true the conclusion is also true
- The argument is valid if and only if the following compound statement is a tautology

$$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow q$$

Rules of Inference

- Verifying if a complicated statement is a tautology is nearly impossible, even for computer. Fortunately, general arguments can be replaced with small collection of simple ones, rules of inference.

- *Modus ponens*

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

``If you have a current password, then you can log onto the network.
You have a current password.
Therefore, you can log onto the network.''

Rule of Syllogism

-
$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

- The corresponding tautology $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

``If you send me an e-mail, then I'll finish writing the program.
If I finish writing the program, then I'll go to sleep early.''

p - `you will send me an e-mail'
q - `I will finish writing the program'
r - `I will go to sleep early'

``Therefore, if you send me an e-mail, then I'll go to sleep early''

Modus Tollens

-
$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

- The corresponding tautology $((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$

``If today is Friday, then tomorrow I'll go skiing''.
``I will not go skiing tomorrow''.

p - `today is Friday'
q - `I will go skiing tomorrow'

``Therefore, today is not Friday''

Rule of Disjunctive Syllogism

- $p \vee q$
  $\neg p$
  $\therefore q$
- The corresponding tautology $((p \vee q) \wedge \neg p) \rightarrow q$

``I'll go skiing this weekend.
I will not go skiing on Saturday.''

p - `I will go skiing on Saturday'
q - `I will go skiing on Sunday'

``Therefore, I will go skiing on Sunday''

Rule for Proof by Cases

- $p \rightarrow r$
  $q \rightarrow r$
  $\therefore (p \vee q) \rightarrow r$
- The corresponding tautology $((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$

``If today is Saturday, then I'll go skiing.
If today is Sunday, then I'll go skiing.

p - `today is Saturday'
q - `today is Sunday'
r - `I'll go skiing'

``Therefore, if today is Saturday or Sunday, then I will go skiing''

Rules of Contradiction, Simplification, and Amplification

- Rule of Contradiction *Reductio ad Absurdum*
  $\neg p \rightarrow F$
  $\therefore p$
  The corresponding tautology $(\neg p \rightarrow F) \rightarrow p$
- Rule of Simplification
  $p \wedge q$
  $\therefore p$
  The corresponding tautology $(p \wedge q) \rightarrow p$
- Rule of Amplification
  p
  $\therefore p \vee q$
  The corresponding tautology $p \rightarrow (p \vee q)$

Logic Inference

- The goal of an argument is to infer the required conclusion from given premises
- Formally, an argument is a sequence of statements, each of which is either a premises, or obtained from preceding statements by means of a rule of inference

Example

- Premises:
``It is not sunny this afternoon and it is colder than yesterday.
We will go swimming only if it is sunny.
If we do not go swimming, then we will take a canoe trip.
If we take a canoe trip, then we will be home by sunset.''
- Conclusion: ``We will be home by sunset.''
- Notation:
  p - it is sunny this afternoon
  q - it is colder than yesterday
  r - we will go swimming
  s - we will take a canoe trip
  t - we will be home by sunset
- Premises: $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$
- Conclusion: t

Example (cntd)

- We have $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$

Step	Reason
1. $\neg p \wedge q$	premise
2. $\neg p$	simplification
3. $r \rightarrow p$	premise
4. $\neg r$	modus tollens
5. $\neg r \rightarrow s$	premise
6. s	modus ponens
7. $s \rightarrow t$	premise
8. t	modus ponens

Logic Puzzles

- A prisoner must choose between two rooms each of which contains either a lady, or a tiger. If he chooses a room with a lady, he marries her, if he chooses a room with a tiger, he gets eaten by the tiger. The rooms have signs on them:

I	II
at least one of these rooms contains a lady	a tiger is in the other room

It is known that either both signs are true or both are false

Logic Puzzles (cntd)

- Notation:
  - p - the first room contains a lady
  - q - the second room contains a lady
- Premises:
  - $(p \vee q) \to \neg p$,
  - $\neg p \to (p \vee q)$

Logic Puzzles (cntd)

$(p \vee q) \to \neg p$,
$\neg p \to (p \vee q)$

- Argument

Step	Reason
1. $\neg p \to (p \vee q)$	premise
2. $p \vee p \vee q$	expression for implication
3. $p \vee q$	idempotent law
4. $(p \vee q) \to \neg p$	premise
5. $\neg p$	modus ponens
6. q	rule of disjunctive syllogism to 3 and 5

Conjunctive Normal Form

- A **literal** is a primitive statement (propositional variable) or its negation

  $p,\ \neg p,\ q,\ \neg q$

- A **clause** is a disjunction of one or more literals

  $p \vee q,\ p \vee \neg q \vee r,\ \neg q,\ \neg s \vee s \vee \neg r \vee \neg q$

- A statement is said to be a Conjunctive Normal Form (CNF) if it is a conjunction of clauses

  $p \wedge (p \vee \neg q) \wedge (\neg r \vee \neg p)$
  $p \wedge q \wedge (\neg r \vee \neg p)$
  $(\neg r \vee q) \wedge (p \vee \neg q \vee \neg s \vee r) \wedge (\neg r \vee \neg p)$
  $\neg r$

CNF Theorem

- **Theorem**

  Every statement is logically equivalent to a certain CNF.

  Proof (sketch)

  Let Φ be a (compound) statement.

  Step 1. Express all logic connectives in Φ through negation, conjunction, and disjunction. Let Ψ be the obtained statement.

  Step 2. Using DeMorgan's laws move all the negations in Ψ to individual primitive statements. Let Θ denote the updated statement

  Step 3. Using distributive laws transform Θ into a CNF.

Example

Find a CNF logically equivalent to $(p \to q) \to r$

Step 1. $\neg(\neg p \vee q) \vee r$

Step 2. $(p \wedge \neg q) \vee r$

Step 3. $(p \vee r) \wedge (\neg q \vee r)$

Rule of Resolution

- $p \vee q$
  $\neg p \vee r$ $\qquad$ $q \vee r$ is called resolvent
  $\therefore\ q \vee r$
- The corresponding tautology $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

``Jasmine is skiing or it is not snowing.
It is snowing or Bart is playing hockey."

p - `it is snowing'
q - `Jasmine is skiing'
r - `Bart is playing hockey'

``Therefore, Jasmine is skiing or Bart is playing hockey"

Computerized Logic Inference

- Convert the premises into CNF
- Convert the *negation* of the conclusion into CNF
- Consider the collection consisting of all the clauses that occur in the obtained CNFs
- Use the rule of resolution to obtain the empty clause ($\varnothing$). If it is possible, then the argument is valid. Otherwise, it is not.

Why empty clause?

The only way to produce the empty clause is to apply the resolution rule to a pair of clauses of the form p and $\neg p$. Therefore, the collection of clauses is contradictory. In other words, for any choice of truth values for the primitive statements, if premises are true, the conclusion cannot be false.

Example

- A lady and a tiger
- Premises: $\neg p \rightarrow (p \vee q)$, $(p \vee q) \rightarrow \neg p$
- Negation of the conclusion: $\neg q$
- Clauses:
  $\neg p$, $p \vee q$, $\neg q$
- Argument:

$p \vee q$	premise	$\varnothing$	resolvent
$\neg q$	premise		
p	resolvent		
$\neg p$	premise		

Homework

Exercises from the Book:
No. 5, 9a (page 84-85)

- Prove that resolution is a valid rule of inference
- Same arrangements as in the `A lady or a tiger' problem. This time if a lady is in Room I, then the sign on it is true, but if a tiger is in it, then the sign is false. If a lady is in Room II, then the sign on it is false, and if a tiger is in it, then the sign is true. Signs are

I	II
both rooms contain ladies	both rooms contain ladies