Problems to Week 8 Tutorial — MACM 101 (Fall 2014)

1. Determine which of the following relations R on the set A are reflexive, symmetric, transitive, and anti-symmetric.

   (a) $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Draw the graph and the matrix of this relation.

   (b) A is the set of all students at SFU, and $(x, y) \in R$ means that the height of x differs from the hight of y by no more than one inch.

   (c) A is the set of ordered pairs of real numbers, that is, $A = \mathbb{R} \times \mathbb{R}$, and $((x_1, x_2), (y_1, y_2)) \in R$ if and only if $x_1 = y_1$ and $x_2 \leq y_2$.

2. Check that the following relations R on the set A are equivalence relations, find their equivalence classes, the number of equivalence classes, and determine which equivalence class the element z belongs to.

   (a) Let A be the set of all possible strings of 3 or 4 letters in alphabet $\{A, B, C, D\}$, let $z = BCAD$, and let $(x, y) \in R$ if and only if x and y have the same first letter and the same third letter.

   (b) Let A be the power set of $\{1, 2, 3, 4, 5\}$, let $z = \{1, 2, 3\}$, and let $(x, y) \in R$ if and only if $x \cap \{1, 3, 5\} = y \cap \{1, 3, 5\}$.

3. Prove that the following relation is an order on the Cartesian product $\{1, 2, 3\} \times \{1, 2, 3\}$ and draw its diagram:

   $$((x_1, y_1), (x_2, y_2)) \in R \quad \text{if and only if} \quad x_1 < x_2, \text{ or } x_1 = x_2 \text{ and } y_1 \leq y_2.$$

   (Such an order is called the *lexicographic order*.)

4. Prove that the following relation on the set of all nonempty subsets of $\{a, b, c, d\}$ is an order, draw its diagram, find all the maximal, minimal, least and greatest elements:

   $$(x, y) \in R \quad \text{if and only if} \quad x \subseteq y.$$

5. Determine whether or not the following relations are functions. If a relation is a function, find its range.

   (a) $\{(x, y) \mid x, y \in \mathbb{Z},\ y = x^2 + 7\}$, a relation from $\mathbb{Z}$ to $\mathbb{Z}$;

   (b) $\{(x, y) \mid x, y \in \mathbb{R},\ y^2 = x\}$, a relation from $\mathbb{R}$ to $\mathbb{R}$.

6. For each of the following functions, determine whether it is one-to-one and determine its range.

   (a) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$;

   (b) $f\colon \mathbb{Q} \to \mathbb{Q}$, $f(x) = 2x + 1$;

   (c) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^3 - x$;

7. Determine whether or not the following relations are functions. If a relation is a function, find its range.

   (a) $\{(x, y) \mid x, y \in \mathbb{Z},\ y = x^2 + 7\}$, a relation from $\mathbb{Z}$ to $\mathbb{Z}$;

   (b) $\{(x, y) \mid x, y \in \mathbb{R},\ y^2 = x\}$, a relation from $\mathbb{R}$ to $\mathbb{R}$.

8. For each of the following functions, determine whether it is one-to-one and determine its range.

   (a) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$;

   (b) $f\colon \mathbb{Q} \to \mathbb{Q}$, $f(x) = 2x + 1$;

   (c) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^3 - x$;

9. Let $f\colon A \to B$ where $A = X \cup Y$ with $X \cap Y = \emptyset$. If $f|_X$ and $f|_Y$ are one-to-one, does it follow that f is one-to-one?

10. For each of the following functions $f\colon \mathbb{Z} \to \mathbb{Z}$, determine whether the function is onto. If the function is not onto, determine the range of f.

    (a) $f(x) = 2x - 3$;

    (b) $f(x) = x^2 + x$.