Properties of Relations

Discrete Mathematics
Andrei Bulatov

Previous Lecture

- Cartesian product, the cardinality of Cartesian product
- Binary relations, higher arity relations
- Describing binary relations
  - list of pairs
  - matrix of relation
  - graph of relation

Properties of Binary Relations – Reflexivity

- From now on we consider only binary relations from a set A to the same set A. That is, such relations are subsets of $A \times A$.
- A binary relation $R \subseteq A \times A$ is said to be reflexive if $(a,a) \in R$ for all $a \in A$.

  $(a,b) \in R \subseteq \mathbb{Z} \times \mathbb{Z}$ if and only if $a \le b$

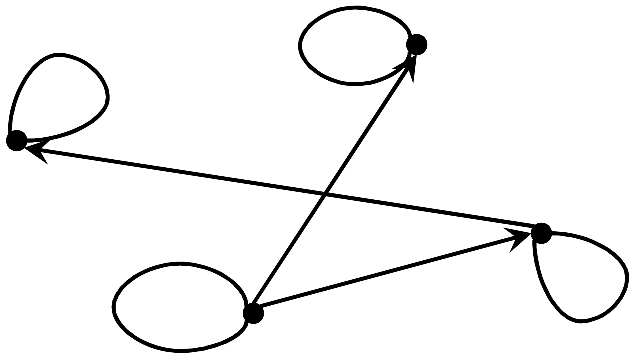
  This relation is reflexive, because $a \le a$ for all $a \in \mathbb{Z}$

Matrix:

$$\begin{pmatrix} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{pmatrix}$$

1's on the diagonal

Graph:



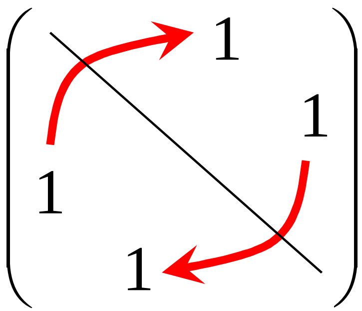
Loops at every vertex

Properties of Binary Relations – Symmetricity

- A binary relation $R \subseteq A \times A$ is said to be symmetric if, for any $a,b \in A$, if $(a,b) \in R$ then $(b,a) \in R$.

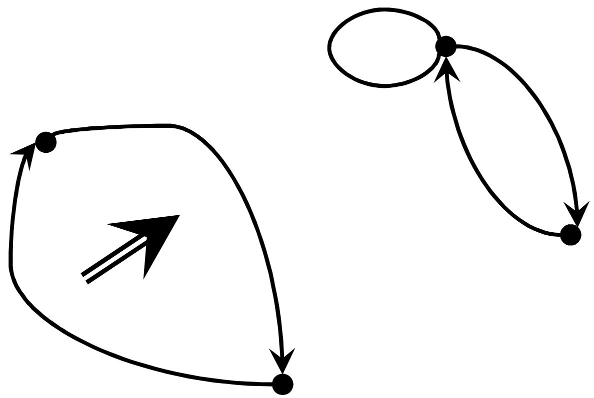
The relation Brotherhood (`x is a brother of y') on the set of men is symmetric, because if a is a brother of b then b is a brother of a

Matrix:



Matrix is symmetric w.r.t. the diagonal

Graph:



Graph is symmetric

Properties of Binary Relations – Transitivity

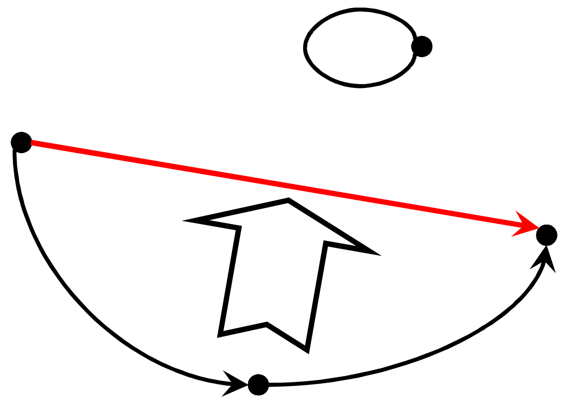
- A binary relation $R \subseteq A \times A$ is said to be **transitive** if, for any $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

The relation Div (`integer x divides y’) is transitive, because if a divides b and b divides c, then a divides c

Matrix:

?

Graph:

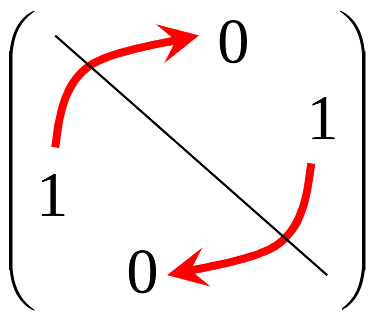


Properties of Binary Relations – Anti-Symmetricity

- A binary relation $R \subseteq A \times A$ is said to be anti-symmetric if, for any $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

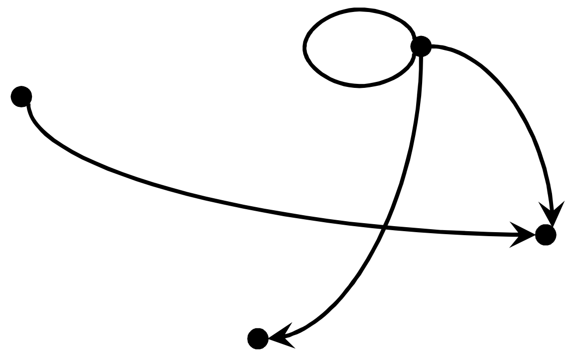
The relation Motherhood (`x is the mother of y') is anti-symmetric, because if a is the mother of b then b is not the mother of a

Matrix:



Matrix is anti-symmetric w.r.t. the diagonal

Graph:



There are no edges going towards each other

Examples

	reflexive	symmetric	transitive	anti-symmetric
Brotherhood x is a brother of y				
Neighborhood x is a neighbor of y				
$x \leq y$				
x,y are intergers and x divides y				

Orders and Equivalences

Discrete Mathematics

Andrei Bulatov

Properties of binary relations

- **Reflexivity**

  A binary relation $R \subseteq A \times A$ is said to be reflexive if $(a,a) \in R$ for all $a \in A$.

- **Symmetricity**

  A binary relation $R \subseteq A \times A$ is said to be symmetric if, for any $a,b \in A$, if $(a,b) \in R$ then $(b,a) \in R$.

- **Transitivity**

  A binary relation $R \subseteq A \times A$ is said to be transitive if, for any $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

- **Anti-symmetricity**

  A binary relation $R \subseteq A \times A$ is said to be anti-symmetric if, for any $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

Equivalence relations

- A binary relation R on a set A is said to be an **equivalence relations** if it is reflexive, symmetric, and transitive.
- Let $R \subseteq \text{People} \times \text{People}$. Pair $(a,b) \in R$ if and only if a and b are of the same age.
- Let $S \subseteq \text{Animals} \times \text{Animals}$. Pair $(a,b) \in S$ if and only if a and b belong to the same species.
- Equivalence classes.

  Take $a \in A$. The set $C(a) = \{ b \mid (a,b) \in R\}$ is called the **equivalence class** of a.
- For example, C(my father) is the set of all 72 year old people.

Equivalence Classes

- **Lemma**.
  (1) For any $a \in A$, the class $C(a) \neq \varnothing$
  (2) If $C(a) \neq C(b)$ then $C(a) \cap C(b) = \varnothing$
  (3) $A = \bigcup_{a \in A} C(a)$
- **Proof**
  (1) R is reflexive, therefore, $(a,a) \in R$. Hence $a \in C(a) \neq \varnothing$
  (2) Suppose $c \in C(a) \cap C(b)$.
  Thus we prove by contrapositive.
  We need to show that $C(a) = C(b)$
  For that we prove that any $x \in C(a)$ belongs to $C(b)$ as well, and vice versa, every $y \in C(b)$ belongs to $C(a)$

Equivalence Classes (cntd)

- First we show that $(a,b) \in R$

  Since $c \in C(a) \cap C(b)$, we have $(a,c), (b,c) \in R$.

  By symmetricity, $(a,c), (c,b) \in R$. Then, by transitivity, $(a,b) \in R$.

  Take $x \in C(b)$. We have $(b,x) \in R$. By transitivity, $(a,x) \in R$.

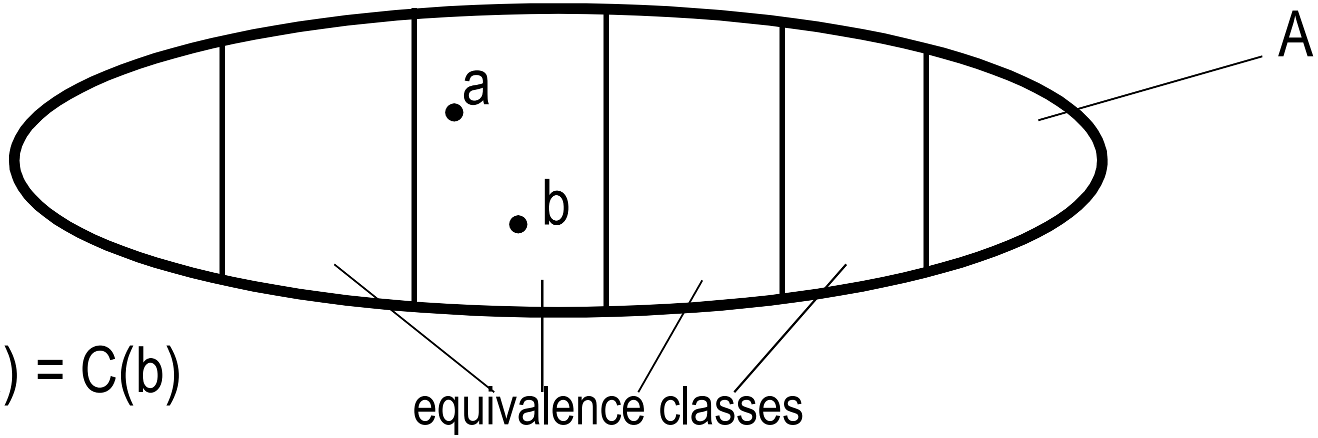
  Hence, $x \in C(a)$. Thus $C(b) \subseteq C(a)$. $C(a) \subseteq C(b)$ is similar.

(3) is obvious, because $a \in C(a)$.

Q.E.D.

Partitions

- Thus the equivalence classes divide up the set A into disjoint subsets.



C(a) = C(b)

- A collection of subsets $M_1, \ldots, M_n$ of a set A is called a partition if the following conditions hold.

  (1) Every $M_i \neq \varnothing$

  (2) If $M_i \neq M_j$ then $M_i \cap M_j = \varnothing$

  (3) $A = \bigcup_{i=1}^{n} M_i$

Partitions and Equivalence Relations

- Lemma shows that the equivalence classes constitute a partition of the set. Actually, a stronger statement is true
- **Theorem**. Let A be a set.

  (1) If R is an equivalence relation on A, then its equivalence classes form a partition of A.

  (2) If $M_1,\dots,M_n$ is a partition of the set A, then the relation R defined as follows: $(a,b) \in R$ if and only if $a,b \in M_i$ for some M_i, is an equivalence relation on A.

- Proof

  (1) Follows from Lemma

  (2) Homework

Congruences

- Let k be an integer. Integers a,b are **congruent modulo k**, denoted $a \equiv b \pmod{k}$, if their reminders when they are divided by k are equal, or, equivalently, if k divides $a - b$.

… -3, 0, 3, 6, … are congruent modulo 3,
and so are …, -4, -1, 2, 5, … and …, -5, -2, 1, 4, …

- The relation $\equiv \pmod{k}$, \`to be congruent modulo k' is
  - reflexive, because k divides $a - a = 0$
  - symmetric, because if k divides $a - b$ then it also divides $b - a$
  - transitive, because if k divides $a - b$ and $b - c$, then it also divides $a - c = (a - b) + (b - c)$
- $\equiv \pmod{k}$, is an equivalence relation with equivalence classes $\{ a \mid \text{there is } b \text{ with } a = bk + c\}$
- Arithmetic on these classes is called **modular arithmetic**

Orders

- A relation R on a set A is called a (partial) order if it is reflexive, transitive and anti-symmetric.

- Examples:
  - $a \leq b$ on the set of real numbers
  - $(a,b) \in$ Div if and only if a divides b

- Diagram of a partial order.

  Due to anti-symmetricity, all the elements of A are ranked with respect to the order R, that is b is ranked higher than a if $(a,b) \in R$.

  Due to transitivity, we do not need to know all pairs (a,b) from the relation, but only those, in which b is just higher than a.

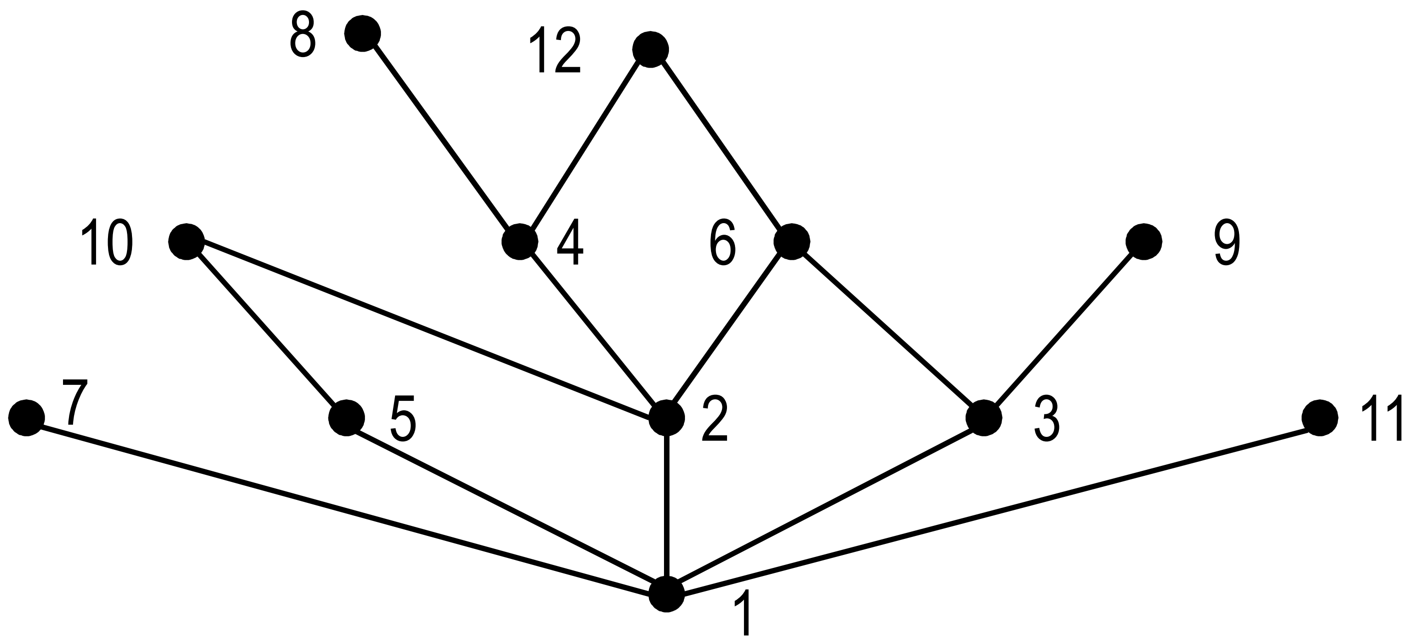
Diagram of a Partial Order

- Rules of drawing a diagram:

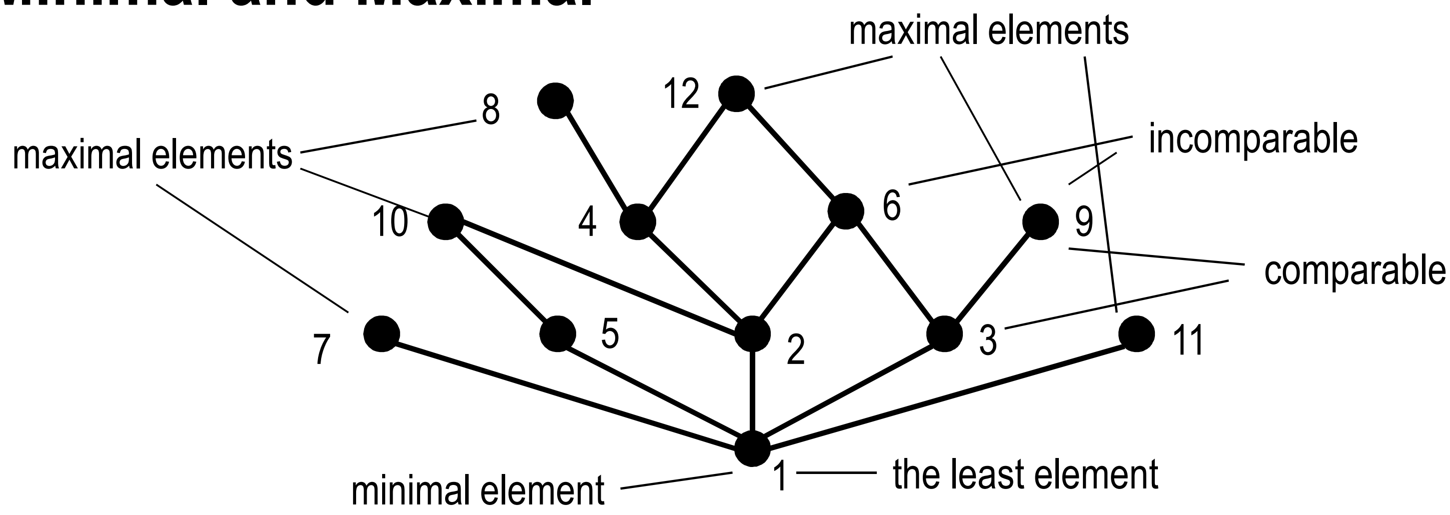
  if a is higher than b, put it higher

  connect every element only with elements that are just higher, so avoid triangles.

- Relation of divisibility on {1,2,…,12}



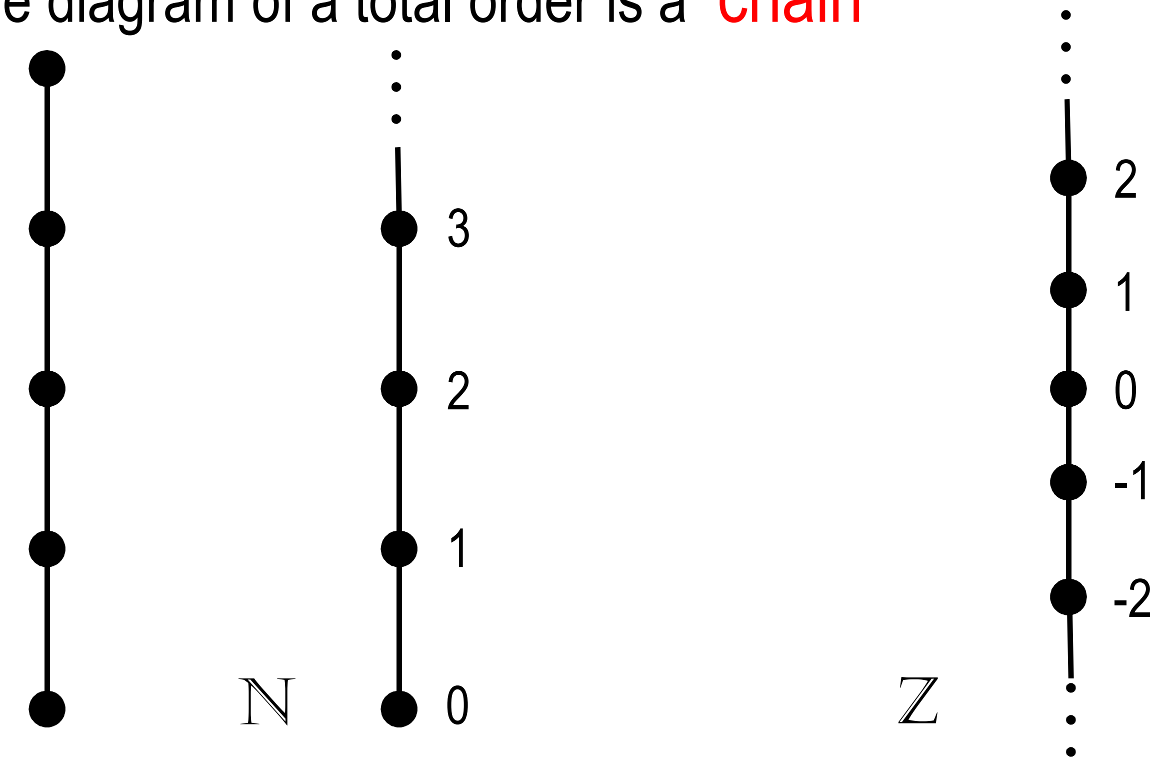
Minimal and Maximal



- Elements a,b are said to be comparable if $(a,b) \in R$ or $(b,a) \in R$
- Otherwise they are called incomparable
- Element a is minimal if for any b if $(b,a) \in R$ then $a = b$
- Element a is maximal if for any b if $(a,b) \in R$ then $a = b$
- Element a is called the least element if for any b, $(a,b) \in R$
- Element a is called the greatest element if for any b, $(b,a) \in R$

Total Order

- A partial order is said to be **total** if every two elements are comparable
- Sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are totally ordered with respect to $\leq$
- The diagram of a total order is a **chain**



Homework

- Are the following relations reflexive? symmetric? transitive? anti-symmetric?
    - Motherhood: `x is the mother of y'
    - Intersect: `straight lines x and y intersect'

- Show that the relation $\subseteq$ on the power set of a set is an order. Draw the diagram of this relation on the power set $P(\{a, b, c\})$.

- Which of the properties: reflexivity, symmetricity, transitivity, and anti-symmetricity, should be true for a relation expressing the idea of similarity (not necessarily identity)?