Binomial Coefficients

Discrete Mathematics
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Previous Lecture

- Combinations
  $$\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}$$
- Combinations with repetitions
  $C(n + r – 1, r – 1)$

A Binomial

- A binomial is simply the sum of two terms, such as $x + y$
- We are to determine the expansion of $(x+y)^n$
- Let us start with $(x+y)^3$

$$(x+y)^3 = (x+y)\cdot(x+y)\cdot(x+y)$$

Every term in the expansion is obtained as the product of a term from the first binomial, a term from the second binomial, and a term from the third binomial

$$= xxx + xxy + xyx + xyy + yxx + yyx + yxy + yyy$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Each of the terms xxy, xyx, and yxx is obtained by selecting y from one of the 3 binomials. Therefore, the coefficient 3 of x^2y is, actually, the number of 1-combinations from a set with 3 elements

The Binomial Theorem

- **Theorem**.
  Let x and y be variables, and let n be a nonnegative integer. Then
  $$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$
- Proof.
  The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for $j = 0,1,2,\dots, n$.
  To count the number of terms of the form $x^{n-j}y^j$, note that to obtain such a term it is necessary to choose j y's from the n binomials (so that the other $n – j$ terms in the product are x's).
  Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{j}$
  Q.E.D.

Examples

- Expand $(x+y)^4$
  $$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$
  $$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
- Expand $(x+2y)^5$
  $$(x+2y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4(2y) + \binom{5}{2}x^3(2y)^2 + \binom{5}{3}x^2(2y)^3$$
  $$+ \binom{5}{4}x(2y)^4 + \binom{5}{5}(2y)^5$$
  $$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$$

Properties of Binomial Coefficients

- For any nonnegative integer n and any r with $0 \le r \le n$
  $$\binom{n}{r} = \binom{n}{n-r}$$
- Indeed, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$

Properties of Binomial Coefficients (cntd)

- For any nonnegative integer n
  $$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n}=2^n$$
- Proof 1: By the Binomial Theorem
  $$2^n=(1+1)^n=\binom{n}{0}1^n+\binom{n}{1}1^{n-1}\cdot 1+\binom{n}{2}1^{n-2}\cdot 1^2+\dots+\binom{n}{n-1}1\cdot 1^{n-1}+\binom{n}{n}1^n$$
  $$=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n}$$
- Proof 2: Recall that $\binom{n}{r}$ is the number of r-element subsets of a set with n elements. Therefore, the sum on the left side is the number of all subsets of an n-element set. We know this number equals 2^n

Pascal's Identity

- For any nonnegative integer n and any r with $0 \le r \le n$
  $$\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$$

Blaise Pascal

- Proof: As we know $\binom{n+1}{r}$ is the number of r-element subsets of an (n + 1)-element set. Take a set T with n + 1 elements.
  Pick an element $a \in T$ and set $S = T - \{a\}$.
  Every r- element subset of T either does not contain a and hence is an r-element subset of S (there are $\binom{n}{r}$ subsets of this type),
  or it contains a and the remaining elements form an (r – 1)-element subset of S (there are $\binom{n}{r-1}$ subsets of this type)

Pascal's Triangle

- Pascal's identity and the simple observation that $\binom{n}{0}=\binom{n}{n}=1$ allow us to give an inductive definition of binomial coefficients. It is convenient to arrange them into a triangle

$$\binom{5}{3}=\binom{4}{2}+\binom{4}{3}$$

$\binom{0}{0}$

$\binom{1}{0}\ \binom{1}{1}$

$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$

$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$

$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$

$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$

$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Exercises

- Determine the coefficient of x^9y^3 in the expansion of $(2x-3y)^{12}$
- With n a positive integer, evaluate the sum
  $$\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+\dots+2^n\binom{n}{n}$$

Computing Binomial Coefficients

- Computing factorials and binomial coefficients can be very difficult. Fortunately, there are many ways to simplify the computation
- Stirling formula: $n! \approx \sqrt{2\pi n}\,\frac{n^n}{e^n}$, where $e = 2.718281828459\dots$
- Gamma function: $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$
  $\Gamma(n + 1) = n!$
- Computing binomial coefficients:
  $$\binom{n}{r}=\frac{n!}{r!(n-r)!}\approx\frac{\sqrt{2\pi n}\,\frac{n^n}{e^n}}{\sqrt{2\pi r}\,\frac{r^r}{e^r}\cdot\sqrt{2\pi(n-r)}\,\frac{(n-r)^{n-r}}{e^{n-r}}}$$
  $$=\sqrt{\frac{n}{r(n-r)}}\left(\frac{n}{r}\right)^r\left(\frac{n}{n-r}\right)^{n-r}$$

Homework

Exercises from the Book:

No. 22a, 29, 30 (page 25)

No. 5a, 7, 10 (page 34)