CNF and Resolution

Discrete Mathematics
Andrei Bulatov

Previous Lecture

- Rules of substitution
- Logic inference
- Inference and tautologies
- Rules of inference

Conjunctive Normal Form

- A literal is a primitive statement (propositional variable) or its negation

  $p,\ \neg p,\ q,\ \neg q$

- A clause is a disjunction of one or more literals

  $p \vee q,\ p \vee \neg q \vee r,\ \neg q,\ \neg s \vee s \vee \neg r \vee \neg q$

- A statement is said to be a Conjunctive Normal Form (CNF) if it is a conjunction of clauses

  $p \wedge (p \vee \neg q) \wedge (\neg r \vee \neg p)$
  $p \wedge q \wedge (\neg r \vee \neg p)$
  $(\neg r \vee q) \wedge (p \vee \neg q \vee \neg s \vee r) \wedge (\neg r \vee \neg p)$
  $\neg r$

CNF Theorem

- **Theorem**
  Every statement is logically equivalent to a certain CNF.

Proof (sketch)

Let Φ be a (compound) statement.

Step 1. Express all logic connectives in Φ through negation, conjunction, and disjunction. Let Ψ be the obtained statement.

Step 2. Using DeMorgan's laws move all the negations in Ψ to individual primitive statements. Let Θ denote the updated statement

Step 3. Using distributive laws transform Θ into a CNF.

Example

Find a CNF logically equivalent to $(p \to q) \to r$

Step 1. $\neg(\neg p \vee q) \vee r$

Step 2. $(p \wedge \neg q) \vee r$

Step 3. $(p \vee r) \wedge (\neg q \vee r)$

Rule of Resolution

- $$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore\ q \vee r \end{array}$$
  $q \vee r$ is called resolvent

- The corresponding tautology $((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$

  ``Jasmine is skiing or it is not snowing.
  It is snowing or Bart is playing hockey.''

  p - `it is snowing'
  q - `Jasmine is skiing'
  r - `Bart is playing hockey'

  ``Therefore, Jasmine is skiing or Bart is playing hockey''

Computerized Logic Inference

- Convert the premises into CNF
- Convert the *negation* of the conclusion into CNF
- Consider the collection consisting of all the clauses that occur in the obtained CNFs
- Use the rule of resolution to obtain the empty clause ($\varnothing$). If it is possible, then the argument is valid. Otherwise, it is not.

Why empty clause?

The only way to produce the empty clause is to apply the resolution rule to a pair of clauses of the form p and $\neg p$. Therefore, the collection of clauses is contradictory. In other words, for any choice of truth values for the primitive statements, if premises are true, the conclusion cannot be false.

Logic Puzzles

- A prisoner must choose between two rooms each of which contains either a lady, or a tiger. If he chooses a room with a lady, he marries her, if he chooses a room with a tiger, he gets eaten by the tiger. The rooms have signs on them:

I	II
at least one of these rooms contains a lady	a tiger is in the other room

It is known that either both signs are true or both are false

$(p \vee q) \to \neg p, \quad \neg p \to (p \vee q)$

Example

- A lady and a tiger
- Premises: $\neg p \to (p \vee q), \; (p \vee q) \to \neg p$
- Negation of the conclusion: $\neg q$
- Clauses:
  $\neg p, \; p \vee q, \; \neg q$
- Argument:

$p \vee q$	premise	$\varnothing$	resolvent
$\neg q$	premise		
p	resolvent		
$\neg p$	premise		

Predicates and Quantifiers

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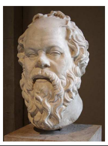
What Propositional Logic Cannot Do

- We saw that some declarative sentences are not statements without specifying the value of `indeterminates'

  ``$x + 2$ is an even number"
  ``If $x + 1 > 0$, then $x > 0$"
  ``A man has a brother"

- Some valid arguments cannot be expressed with all our machinery of tautologies, equivalences, and rules of inference

  Every man is mortal.
  Socrates is a man.
  $\therefore$ Socrates is mortal



Open Statements or Predicates

- Sentences like `$x$ is greater than 3' or `person x has a brother' are not true or false unless the variable is assigned some particular value.
- Sentence `x is greater than 3' consists of 2 parts.
  - The first part, x, is called the variable or the subject of the sentence.
  - The second part – the predicate, `is greater than 3' – refers to a property the subject can have.
- Sentences that have such structure are called open statements or predicates
- We write $P(x)$ to denote a predicate with variable x

Unary, Binary, and so on

`x is greater than 3'
`x is my brother'
`x is a human being'
} contain only 1 variable, unary predicates P(x)

`x is greater than y'
`x is the mother of y'
`car x has colour y'
} contain 2 variables, binary predicates Q(x,y)

`x divides y + z'
`x sits between y and z'
`x is a son of y and z'
} contain 3 variables, ternary predicates R(x,y,z)

Assigning a Value

- When a variable is assigned a value, the predicates turns into a statement, whose truth value can be evaluated.

P(x) = `x is greater than 3'
x=2 → P(2) = `2 is greater than 3' false
x=4 → P(4) = `4 is greater than 3' true

Q(x,y) = `car x has colour y'
x=my car, y=red → Q(my car,red) = `my car is red' false
x=my car, y=grey → Q(my car,grey) = `my car is grey' true

Universe

- We cannot assign a variable of a predicate ANY value. We need to obtain a meaningful statement!

`x is greater than 3'
23 0 a cat a big red truck

- Every variable of a predicate is associated with a universe or universe of discourse, and its values are taken from this universe

`x is greater than 3'	x is a number
`x is my brother'	x is a human
`x is an animal'	x is a ???
`car x has colour y'	x is a car y is a colour

Relational Databases

- A relational database is a collection of tables like

No.	Name	Student ID	Supervisor	Thesis title
1.	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
…	…	…	…	…

A table consists of a schema and an instance. A schema is a collection of attributes, where each attribute has an associated universe of possible values. An instance is a collection of rows, where each row is a mapping that associates with each attribute of the schema a value in its universe.

- Every table is a predicate that is true on the rows of the instance and false otherwise.

Quantifiers

- One way to obtain a statement from a predicate is to assign all its variables some values
- Another way to do that is to use expressions like
  - `For every …'
  - `There is … such that …'
  - `A … can be found …'
  - `Any … is …'

`Every man is mortal'
`There is x such that x is greater than 3'
`There is a person who is my father'
`For any x, $x^2 \geq 0$'

quantification

Universal Quantifiers

- Abbreviates constructions like
  - For all …
  - For any …
  - Every …
  - Each …
- Asserts that a predicate is true for all values from the universe
  - `Every man is mortal'
  - `All lions are fierce'
  - `For any x, $x^2 \geq 0$'
- Notation: $\forall$
- $\forall x$ P(x) means that for every value a from the universe P(a) is true

Universal Quantifiers (cntd)

\`For any x, $x^2 \geq 0$'	true!
\`Every car is red'	false! my car is not red

- $\forall$x P(x) is false if and only if there is at least one value a from the universe such that P(a) is false
- Such a value a is called a counterexample

Thus to disprove that \`Every man is mortal' it suffices to recall the movie \`Highlander'

Existential Quantifiers

- Abbreviates constructions like
  - For some …
  - For at least one …
  - There is …
  - There exists …
- Asserts that a predicate is true for at least one value from the universe
  - \`There is a living king'
  - \`Some people are fierce'
  - \`There is x such that $x^2 \geq 10$'
- Notation: $\exists$
- $\exists$x P(x) means that there is a value a from the universe such that P(a) is true

Existential Quantifiers (cntd)

\`There is a red car'	true! my friend's car is red
\`For some x, $x^2 < 0$'	false!

- $\exists$x P(x) is false if and only if for all a from the universe P(a) is false
- Disproving an existential statement is difficult!

Quantifiers and Negations

- Summarizing

	true	false
$\forall$x P(x)	For every value a from the universe P(a) is true	There is a counterexample – a value a from the universe such that P(a) is false
$\exists$x P(x)	There is a value a from the universe such that P(a) is true	For all values a from the universe P(a) is false

- Observe that
  - $\forall$x P(x) is false if and only if $\exists$x $\neg$P(x) is true
  - $\exists$x P(x) is false if and only if $\forall$x $\neg$P(x) is true

Example

- What is the negation of each of the following statements?

Statement		Negation
All lions are fierce	$\forall$x P(x)	There is a peaceful lion
Everyone has two legs	$\forall$x P(x)	There is a person having more than two legs, one leg, or no legs at all
Some people like coffee	$\exists$x P(x)	All people hate coffee
There is a lady in one of these rooms (Some rooms contain a lady)	$\exists$x P(x)	There is a tiger in every room

Homework

Exercises from the Book:

No. 1, 2, 4acij, 9a(i,iv), 12(vii,viii) (page 100-102)