

Relations

Discrete Mathematics
Andrei Bulatov

Discrete Mathematics - Relations

13-2

Previous Lecture

- Venn diagrams
- Operations of
 - Intersection
 - Union
 - Symmetric difference
 - Complement
 - Difference

Discrete Mathematics - Relations

13-3

Relations

- 'Relation', the connection between things or people
- Between people, family relations
 - 'to be brothers' x is a brother of y
 - 'to be older' x is older than y
 - 'to be parents' x and y are parents of z
- Between things, numerical relations
 - 'to be greater than' $x < y$ on the set of real numbers
 - 'to be divisible by' x is divisible by y on the set of integers
- Between things and people, legal relations
 - 'to be an owner' x is an owner of y

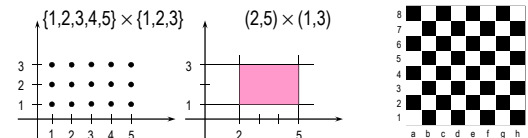


Discrete Mathematics - Relations

13-4

Cartesian Product

- The **Cartesian product** of sets A and B , denoted by $A \times B$, is the set of all **ordered pairs** of elements from A and B .
 $A \times B = \{ (a,b) \mid a \in A, b \in B \}$
- The elements of the Cartesian product are ordered pairs. In particular, $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.
- If sets are thought of as '1-dimensional' objects, Cartesian products are 2-dimensional



Discrete Mathematics - Relations

13-5

Cartesian Product of More Than Two Sets

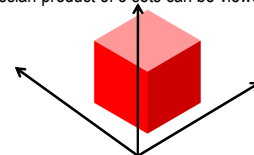
- Instead of ordered pairs we may consider ordered **triples**, or, more general, **k-tuples**.
 (a,b,c) , an ordered triple
 (a,b,c,d) , an ordered quadruple
 (a_1, a_2, \dots, a_k) a k -tuple
- Triples, quadruples, and k -tuples are elements of Cartesian products of 3, 4, and k sets, respectively
 $A \times B \times C = \{ (a,b,c) \mid a \in A, b \in B, c \in C \}$
 $A \times B \times C \times D = \{ (a,b,c,d) \mid a \in A, b \in B, c \in C, d \in D \}$
 $A_1 \times A_2 \times \dots \times A_k = \{ (a_1, a_2, \dots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_k \in A_k \}$

Discrete Mathematics - Relations

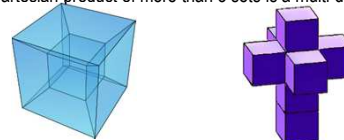
13-6

Cartesian Product of More Than Two Sets

- Cartesian product of 3 sets can be viewed as a '3-dimensional' set



- Cartesian product of more than 3 sets is a multi-dimensional set



Cardinality of Cartesian Product

- Theorem.**

$$|A \times B| = |A| \cdot |B|$$

$$|A_1 \times A_2 \times \dots \times A_k| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_k|$$

- Proof**

When creating an ordered pair (a,b) , to each of the $|A|$ elements of A we can add any of the $|B|$ elements of B . Totally, we have $|A| \cdot |B|$ ordered pairs.

Q.E.D.

Binary Relations

- A **binary relation** from set A to set B is any subset of $A \times B$. If $A = B$ then we say that the relation is **on** the set A .

' x is a brother of y ' \subseteq People \times People

' x is older than y ' \subseteq People \times People

' x is an owner of y ' \subseteq People \times Properties

' $x < y$ ' \subseteq $\mathbb{R} \times \mathbb{R}$

' x divides y ' \subseteq $\mathbb{Z} \times \mathbb{Z}$



More Relations (cntd)

- Binary relations can be generalized to subsets of Cartesian products of more than two sets.
- Any subset of the Cartesian product of 3 sets is called a **ternary relation**.
' x and y are parents of z ' is a subset of
People \times People \times People
- Any subset of the Cartesian product of k sets is called a **k -ary relation**.

$\{(a_1, a_2, \dots, a_k) \mid a_1 + a_2 + \dots + a_k = 3\}$ is a subset of $\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$

Sets, Relations, and Predicates

- Observe that sets, relations and predicates are essentially the same object.

Unary predicate
 $P(x)$

Set
 $A = \{x \mid P(x)\}$

Binary predicate
 $P(x,y)$

Binary relation
 $R = \{(x,y) \mid P(x,y)\}$

Ternary predicate
 $P(x,y,z)$

Ternary relation
 $R = \{(x,y,z) \mid P(x,y,z)\}$

Relational Databases

- A **relational database** is a collection of **tables** like

No.	Name	Student ID	Supervisor	Thesis title
1.	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
...

A table consists of a schema and an instance ...

The instance of this table is a 5-ary relation, a subset of the Cartesian product

$\mathbb{Z}^+ \times \text{Names} \times 8\text{-strings_of_digits} \times \text{Names} \times \text{Meaningful_Sentences}$

Describing Binary Relations

- A list of pairs.
Among 6 people, Mark, Jerry, John, Randy, Aaron, and Ralph, Mark and Randy are brothers, and also John, Aaron and Ralph are brothers
 $A = \{\text{Mark, Jerry, John, Randy, Aaron, Ralph}\}$
Brotherhood = $\{(x,y) \mid x \text{ is a brother of } y\}$
= $\{(\text{Mark,Randy}), (\text{Randy,Mark}), (\text{John,Aaron}), (\text{Aaron,John}), (\text{John,Ralph}), (\text{Ralph,John}), (\text{Aaron,Ralph}), (\text{Ralph,Aaron})\}$

Describing Binary Relations (cntd)

- Matrix of a relation.

Matrix of a relation $R \subseteq A \times B$ is a rectangular table, rows of which are labeled with elements of A (in any but fixed order), and columns are labeled with elements of B . We write 1 in the intersection of row a and column b if and only if $(a,b) \in R$; otherwise we write 0.

Brotherhood

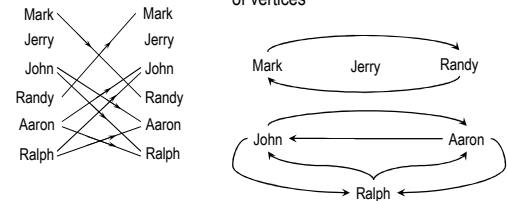
	Mark	Jerry	John	Randy	Aaron	Ralph
Mark	0	0	0	1	0	0
Jerry	0	0	0	0	0	0
John	0	0	0	0	1	1
Randy	1	0	0	0	0	0
Aaron	0	0	1	0	0	1
Ralph	0	0	1	0	1	0

Describing Binary Relations (cntd)

- Graph of a relation

Graph of a relation $R \subseteq A \times B$ consists of two sets of vertices labeled by elements of A and B . A vertex a is connected to a vertex b with an edge (arc) if and only if $(a,b) \in R$.

If $A = B$ then we may use only one set of vertices



Cartesian Product, Intersection and Union

- Theorem.** For any sets A, B, C

- (1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- (2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- (3) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
- (4) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

- Proof (of (2))

$$\begin{aligned}
 A \times (B \cup C) &= \{(a,b) \mid a \in A \wedge b \in B \cup C\} \\
 &= \{(a,b) \mid a \in A \wedge (b \in B \vee b \in C)\} \\
 &= \{(a,b) \mid (a \in A \wedge b \in B) \vee (a \in A \wedge b \in C)\} \\
 &= \{(a,b) \mid a \in A \wedge b \in B\} \cup \{(a,b) \mid a \in A \wedge b \in C\} \\
 &= (A \times B) \cup (A \times C)
 \end{aligned}$$

Q.E.D.

Homework

Exercises from the Book:

No. 1, 4, 5a (page 252)

- Prove part (3) of the theorem on slide 13-14