MACM 101 — Discrete Mathematics I

Exercises on Predicates and Quantifiers. Due: Friday, October 10th (at the beginning of the class)

Reminder: the work you submit must be your own. Any collaboration and consulting outside resources must be explicitly mentioned on your submission.
Please, use a pen. 30 points will be taken off for pencil written work.

1. Determine the truth value of each of these statements if the universe of each variable consists of (i) all real numbers, (ii) all integers.

   (a) $\forall x\ (x > 0 \to \exists y\ (\frac{\sqrt{x}}{y} = 3))$;

   (b) $\exists x \forall y\ ((2x - y = 4) \wedge (3x + y = 5))$.

2. Use predicates and quantifiers to express this statement

   "Some students in this class grew up in the same town as exactly one other student in this class."

3. Find a counterexample, if possible, to this universally quantified statement, where the universe for all variables consists of all integers

   $$\forall x \forall y\ (3x - 4y \geq x).$$

4. Rewrite the following statement so that negations appear only within predicates (that is, no negation is outside a quantifier or an expression involving logical connectives)

   $$\neg \exists x\ ((\forall y \exists z\ P(x, y, z)) \to (\exists z \forall y\ (R(x, y, z) \wedge S(z, y)))).$$

5. Find a universe for variables x, y, and z for which the statement

   $$\forall x \forall y ((x \neq y) \to \forall z ((z = x) \vee (z = y)))$$

   is true and another universe in which it is false.

6. Show that quantified statements

$$\exists x(P(x) \oplus Q(x)) \quad \text{and} \quad (\exists x P(x)) \oplus (\exists x Q(x))$$

   are not logically equivalent.

7. Determine whether the following argument is valid or invalid and explain why.

   'Everyone enrolled in the university has lived in a dormitory.'
   'Mia has never lived in a dormitory.'
   'Therefore, Mia is not enrolled in the university.'

8. Given premises:

   'All hummingbirds are richly colored.'
   'No large birds live on honey.'
   'Birds that do not live on honey are dull in color.'

   infer the conclusion

   'Hummingbirds are small.'

9. What is wrong with this proof?

   Theorem. 3 is less than 1.

   Proof. Every integer number is either less than 1 or greater than 1 or equals 1. Let c be an arbitrary integer number. Therefore, it is less than 1 or greater than 1 or equals 1. Suppose it is less than 1. By the rule of universal generalization, if an arbitrary number is less than 1, every number is less than 1. Therefore, 3 is less than 1.