MACM 101 — Discrete Mathematics I

Outline Solutions to Exercises on Functions and Induction

1. **Is the function $f\colon \mathbb{R}^+ \to \mathbb{R}$ defined as $f(x) = \sqrt{x} + x + 2$ one-to-one?**

   It is one-to-one. To see this, take two numbers $x, y \in \mathbb{R}^+$ such that $f(x) = f(y)$ and show that $x = y$. We have

   $$\begin{aligned} f(x) &= f(y) \\ \sqrt{x} + x + 2 &= \sqrt{y} + y + 2 \\ (y - x) + (\sqrt{y} - \sqrt{x}) &= 0 \\ (\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x}) + (\sqrt{y} - \sqrt{x}) &= 0 \\ (\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x} + 1) &= 0. \end{aligned}$$

   Then either $\sqrt{y} - \sqrt{x} = 0$ implying $x = y$, or $\sqrt{y} + \sqrt{x} + 1 = 0$, which is impossible for nonnegative real numbers x, y.

2. **Determine whether or not the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto, if $f((m, n)) = m^2 + n$.**

   It is onto. To show that we have to find, given an integer b, a pair (m, n) such that $m^2 + n = b$. Clearly, we can choose $m = 0$ and $n = b$.

3. **Let $f(x) = ax + b$ and $g(x) = cx^2 + dx$, where a, b, c, and d are constants. Compute $f \circ g$ and $g \circ f$. Determine for which constants a, b, c, and d it is true that $f \circ g = g \circ f$.**

   First, compute $f \circ g$ and $g \circ f$:

   $$\begin{aligned} f \circ g(x) &= f(g(x)) = a \cdot g(x) + b = a(cx^2 + dx) + b = acx^2 + adx + b \\ g \circ f(x) &= g(f(x)) = c \cdot (f(x))^2 + d \cdot f(x) = c(ax + b)^2 + d(ax + b) = a^2bx^2 + (2abc + ad)x + b^2c + bd. \end{aligned}$$

   Now, let us express the condition $f \circ g = g \circ f$:

   $$\begin{aligned} (f \circ g)(x) &= (g \circ f)(x) \\ acx^2 + adx + b &= a^2bx^2 + (2abc + ad)x + b^2c + bd. \end{aligned}$$

   Two polynomials are equal for all values of x if and only their coefficients are equal. Therefore, the equality $f \circ g = g \circ f$ holds if and only if the constants a, b, and c satisfy the equality

   $$ac = a^2b, \qquad 2abc = 0, \qquad b = b^2c + bd.$$

   Then, the second condition implies one of a, b, c must be 0. If $a = 0$ then from the third condition we obtain $b(bc + d - 1) = 0$. Therefore, either $b = 0$, c, d are any; or bc are any and $d = -bc + 1$. If $b = 0$ then either $a = 0$ and c, d are any; or $a = \frac{c}{b}$ and $d = -bc + 1$, b is any. Finally if $c = 0$ then ether $b = 0$ and c, d are any; or $a = 0$, $b \neq 0$ is any, and $d = 1$.

4. **If $f \circ g$ is one-to-one, does it follow that g is one-to-one?**

   Yes, it does. Suppose a, b from the domain of g (and therefore from that of $f \circ g$) are such that $g(a) = g(b)$. Then

   $$f \circ g(a) = f(g(a)) = f(g(b)) = f \circ g(b).$$

   Since $f \circ g$ is one-to-one, this implies $a = b$. Thus for any a, b such that $g(a) = g(b)$ we have $a = b$, which means g is one-to-one.

5. **Show that the function $f\colon \mathbb{R}-\{-1\}\to\mathbb{R}-\{2\}$ defined by**

$$f(x)=\frac{4x+3}{2x+2}$$

is a bijection, and find the inverse function.

Note, first of all, that f is a function from $\mathbb{R}-\{3\}$, meaning that for any $a\in\mathbb{R}-\{3\}$ the number $f(a)$ is defined. Clearly, $f(-1)$ is not defined. Next we prove that f is one-to-one by showing that if $f(a)=f(b)$ for some $a,b\in\mathbb{R}-\{-1\}$ then $a=b$:

$$\begin{aligned} f(a) &= f(b)\\ \frac{4a+3}{2a+2} &= \frac{4b+3}{2b+2}\\ (4a+3)(2b+2) &= (2a+2)(4b+3)\\ 8ab+8a+6b+6 &= 8ab+6a+8b+6\\ a &= b. \end{aligned}$$

Thus, f is one-to-one.

To show that f is onto we take any $b\in\mathbb{R}-\{2\}$, and find $a\in\mathbb{R}-\{-1\}$ such that $f(a)=b$.

$$\begin{aligned} f(a) &= b\\ \frac{4a+3}{2a+2} &= b\\ 4a+3 &= 2ab+2b\\ a &= \frac{3-2b}{2b-4}. \end{aligned}$$

Thus, the required number a exists, and f is onto. Moreover, this computation gives us the inverse function:

$$f^{-1}(x)=\frac{3-2x}{2x-4}.$$

6. **The kth harmonic number is defined to be**

$$H_k=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}.$$

Prove that harmonic numbers satisfy the equality

$$H_1+H_2+\cdots+H_n=(n+1)H_n-n$$

for all $n\in\mathbb{N}$.

We use induction. Let $P(n)$ denote this equality for the integer n.
Basis case. $P(1)$ means the equality $H_1=1=(1+1)\cdot 1-1=(1+1)H_1-1$, which is obviously true.
Inductive step. Suppose that $P(k)$ is true, that is,

$$H_1+H_2+\cdots+H_k=(k+1)H_k-k.$$

We have to prove $P(k+1)$:

$$H_1+H_2+\cdots+H_k+H_{k+1}=((k+1)+1)H_{k+1}-(k+1).$$

Observe that $H_{k+1}=H_k+\frac{1}{k+1}$. We have

$$\begin{aligned} &H_1+H_2+\cdots+H_k+H_{k+1}\\ &\quad= (k+1)H_k-k+H_{k+1}\\ &\quad= (k+1)(H_k+\frac{1}{k+1})-1-k+H_{k+1}\\ &\quad= (k+1)H_{k+1}-(k+1). \end{aligned}$$

7. **Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined by the rule: $F_1 = F_2 = 1$ and $F_k = F_{k-2} + F_{k-1}$ for $k > 2$. Lucas numbers $L_1, L_2, L_3, \ldots$ are defined in a similar way by the rule: $L_1 = 1$, $L_2 = 3$ and $L_k = L_{k-2} + L_{k-1}$ for $k > 2$.**
   Show that Fibonacci and Lucas numvers satisfy the following equality for all $n \geq 2$

$$L_n = F_{n-1} + F_{n+1}.$$

   We use induction. Let $P(n)$ denote this equality for the integer n.
   Basis case. $P(2)$ and $P(3)$ mean the equalities $L_2 = 3 = 1 + 2 = F_1 + F_3$ and $L_3 = 4 = 1 + 3 = F_2 + F_4$.
   Inductive step. Suppose that $P(k-1)$ and $P(k)$ is true, that is,

$$L_{k-1} = F_{k-2} + F_k, \qquad L_k = F_{k-1} + F_{k+1}.$$

   We have to prove $P(k+1)$:

$$L_{k+1} = F_k + F_{k+2}.$$

   Using the definitions of L_n and F_n, and the induction hypothesis

$$\begin{aligned} L_{k+1} &= L_{k-1} + L_k \\ &= (F_{k-2} + F_k) + (F_{k-1} + F_{k+1}) \\ &= (F_{k-2} + F_{k-1}) + (F_k + F_{k+1}) \\ &= F_k + F_{k+2}. \end{aligned}$$

8. **Prove that any amount of more that 7 cents can be represented by 3- and 5-cent coins. (Assume 3-cent coins exist.)**

   Denote by $P(n)$ the statement that n can be represented by 3- and 5-cent coins; that is, $n = 3m + 5\ell$ for some $k, \ell \geq 0$.
   Basis step. To prove that $P(8)$ is true just notice that $8 = 3 \cdot 1 + 5 \cdot 1$, that is $m = \ell = 1$ in this case.
   Inductive step. Suppose that $P(k)$ is true, that is, $k = 3m + 5\ell$ for some $m, \ell \geq 0$.
   We need to prove $P(k+1)$, that is, to find $m', \ell' \geq 0$ such that $k + 1 = 3m' + 5\ell'$.
   If $\ell > 0$, we can set $m' = m + 2$ and $\ell' = \ell - 1$. Then

$$3m' + 5\ell' = 3(m + 2) + 5(\ell - 1) = 3m + 5\ell + 1 = k + 1.$$

   If $\ell = 0$ then $k = 3m$, that is, k is divisible by 3. This means $k \geq 9$ and so $m \geq 3$. Then we can set $m' = m - 3$ and $\ell' = \ell + 2$:

$$3m' + 5\ell' = 3(m - 3) + 5(\ell + 2) = 3m + 5\ell + 1 = k + 1.$$

9. **The game of Chomp is played by two players. In this game, cookies are laid out on a rectangular grid. The cookie in the top left position is poisoned. The two players take turns making moves; at each move, a player is required to eat a remaining cookie, together with all cookies to the right and/or below (that is all the remaining cookies in the rectangle, in which the first cookie eaten is the top left corner). The loser is the player who has no choice but to eat the poisoned cookie. Prove that if the board is square (and bigger than 1×1) then the first player has a winning strategy.**

   Let the players are denoted by A and B, and A goes first. In his first move A eats the cookie that lies on the diagonal of the grid just next to the poisoned one. This way he leaves only one row and one column of cookies (with the poisoned cookie in the intersection). We will show using the principle of strong induction that in this configuration A wins. If the size of the grid is n, we denote this statement by $P(n)$.
   Basis step. Prove $P(2)$, that is for a 2×2 grid. Then B can eat either the non-poisoned cookie in the first row, or the non-poisoned cookie in the first column. In either case A can eat the remaining non-poisoned cookie leaving B no choice.

Inductive step. Suppose that $P(i)$ is true for all $i \leq k$, that is, in the configuration when cookies are arranged in one row and one column of length i with the poisoned one in the intersection, A wins.Consider the position with a row and a column of length $k+1$. Then B can eat several cookies either from the row, or from the column. Suppose he eats $\ell > 0$ cookies from the row. So, after his move the position consists of a row of cookies of length $k+1-\ell$ and a column of of length $k+1$. Then A responds eating ℓ cookies from the column, and in the new position we have a row and a column of cookies of length $k+1-\ell$. By the inductive hypothesis $P(k+1-\ell)$ is true, and therefore A wins in this position.

10. **A complete binary tree is a graph defined through the following recursive definition.**
Basis step: A single vertex is a complete binary tree.
Inductive step: If T_1 and T_2 are disjoint complete binary trees with roots r_1, r_2, respectively, the the graph formed by starting with a root r, and adding an edge from r to each of the vertices r_1, r_2 is also a complete binary tree.
Prove that a complete binary tree has odd number of vertices.

Let $P(T)$ denote the statement that the complete binary tree T has odd number of vertices. This number we will denote by $n(T)$.
Basis step. Let T contains only one vertex. Then $n(T) = 1$ and it is odd. Thus $P(T)$ is true.
Inductive step. Let T is built from trees T_1, T_2 as described in the definition and suppose that $P(T_1)$ and $P(T_2)$ are true. This means $n(T_1), n(T_2)$ are odd, in other words, $n(T_1) = 2k+1$, $n(T_2) = 2\ell+1$ for some integers $k, \ell \geq 0$. Then

$$n(T) = n(T_1) + n(T_2) + 1 = 2k+1+2\ell+1+1 = 2(k+\ell+1)+1,$$

that is, $n(T)$ is odd, and $P(T)$ is true, as required.