Integers

Integers

"God made the integers; all else is the work of man" Leopold Kroenecker

Division

- Most of useful properties of integers are related to division
- If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that b = ac.
- When a divides b we say that a is a divisor (factor) of b, and that b is a multiple of a.
- The notation a | b denotes that a divides b. We write a ∤ b when a does not divide b
- Example. Let n and d be positive integers. How many positive integers not exceeding n are divisible by d?

The numbers in question have the form dk, where k is a positive integer and $0 < dk \le n$. Therefore, $0 < k \le n/d$. Thus the answer is $\lfloor n/d \rfloor$

Properties of Divisibility

- Let a, b, and c be integers. Then
 - (i) if a | b and a | c, then a | (b + c);
 - (ii) if a | b, then a | bc for all integers c;
 - (iii) if a | b and b | c, then a | c.
- Proof.
 - (i) Suppose a | b and a | c. This means that there are k and m such that b = ak and c = am.

Then b + c = ak + am = a(k + m), and a divides b + c.

Properties of Divisibility (cntd)

- If a, b, and c are integers such that a | b and a | c, then a | mb + nc whenever m and n are integers.
- Proof.

By part (ii) it follows that a | mb and a | nc.

By part (i) it follows that a | mb + nc.

- If $a \mid b$ and $b \mid a$, then $a = \pm b$.
- Proof.

Suppose that $a \mid b$ and $b \mid a$. Then b = ak and a = bm for some integers k and m.

Therefore a = bm = akm, which is possible only if $k,m = \pm 1$.

The Division Algorithm

- Theorem (The division algorithm)
 - Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \le r < d$, such that a = dq + r
- d is called the divisor, a is called the dividend, q is called the quotient, and r is called the remainder
- Examples:
 - Let a = 101 and d = 11Then $101 = 11 \cdot 9 + 2$
 - Let a = -11 and d = 3Then $-11 = 3 \cdot (-4) + 1$
 - Let a = 3 and d = 11Then $3 = 11 \cdot 0 + 3$

Representation of Integers

In most cases we use decimal representation of integers. For example, 657 means

$$6 \cdot 100 + 5 \cdot 10 + 7 = 6 \cdot 10^2 + 5 \cdot 10^1 + 7 \cdot 10^0$$

Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \dots + a_1 b + a_0$$

where k is a nonnegative number, $a_k, a_{k-1}, ..., a_1, a_0$ are nonnegative integers less than b, and $a_k \neq 0$

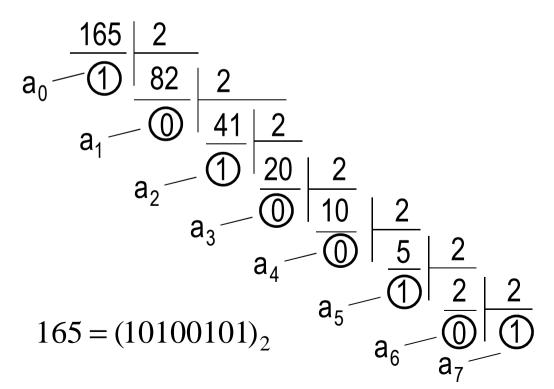
Such a representation of n is called the base b expansion of n, denoted by $(a_k a_{k-1} ... a_1 a_0)_b$

Binary Expansion

Important case of a base is 2. The base 2 expansion is called the binary expansion of a number

$$n = a_k \cdot 2^k + a_{k-1} \cdot 2^{k-1} + \dots + a_1 \cdot 2 + a_0$$

Find the binary expansion of 165



$$165 = 2 \cdot 82 + 1$$

$$82 = 2 \cdot 41 + 0$$

$$41 = 2 \cdot 20 + 1$$

$$20 = 2 \cdot 10 + 0$$

$$10 = 2 \cdot 5 + 0$$

$$5 = 2 \cdot 2 + 1$$

$$2 = 2 \cdot 1 + 0$$

$$1 = 2 \cdot 0 + 1$$

Hexadecimal Expansion

Using A, B, C, D, E, F for 10, 11, 12, 13, 14, 15, respectively, find the base 16 expansion of 175627

Homework

Exercises from the Book

No. 2, 3, 4, 10, 12, 14 (page 603)