Primes

Discrete Mathematics

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Previous Lecture

- Integers
- Division
- Properties of divisibility
- The division algorithm

Primes

- Every integer n (except for 1 and -1) has at least 2 positive divisors, 1 and n (or -n).
- A positive number that does not have any other positive divisor is called prime
- Prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, …
- Mersenne numbers are the numbers of the form $M_n = 2^n - 1$
- There are many prime numbers among Mersenne numbers. The greatest known prime number is $M_{32582657} = 2^{32582657} - 1$
- The next candidate is $M_{M_{61}} = 2^{2305843009213693951} - 1$

- A positive number that is not prime is called composite

Composite Numbers

- Every composite number has a prime divisor.
- Proof.

Let S be the set of all composite numbers that do not have a prime divisor

Since $S \subseteq \mathbb{N}$, by the Well-Ordering Principle, it has a least element r.

As r is not prime, it has a divisor, therefore, $r = uv$ for some positive integers u and v.

$u < r$ and $v < r$. Therefore $u \notin S$, and u has a prime divisor p.

Since $p \mid u$ and $u \mid r$, we conclude that $p \mid r$, a contradiction.

How many prime numbers are there?

- **Theorem** (Euclid)
  There are infinitely many prime numbers.
- Proof.

  By contradiction. Suppose that $\{p_1, p_2, \ldots, p_k\}$ is the set of all prime numbers, and let $a = p_1 p_2 \ldots p_k + 1$

  Since a is greater than any member of the list, a is composite.

  By the previous statement, a has a prime divisor, that is for some p_j we have $p_j \mid a$

  Since $p_j \mid a$ and $p_j \mid p_1 p_2 \ldots p_k$ we have $p_j \mid a - p_1 p_2 \ldots p_k = 1$

  A contradiction.
- If n is a positive integer, then there are approximately $\dfrac{n}{\ln n}$ prime numbers not exceeding n

Open Problems about Primes

- Goldbach's Conjecture

  Every positive even number can be represented as the sum of two prime numbers.

  For example: 4 = 2 + 2, 8 = 5 + 3, 42 = 37 + 5

  Goldbach's conjecture is known to be true for even numbers up to $2 \cdot 10^{17}$

- The Twin Prime Conjecture

  Twin primes are primes that differ by 2, such as 3 and 5, 5 and 7, 11 and 13, etc.

  The Twin Prime Conjecture asserts that there are infinitely many twin primes.

  The record twin primes: $16{,}896{,}987{,}339{,}975 \cdot 2^{171{,}960} \pm 1$

The Greatest Common Divisor

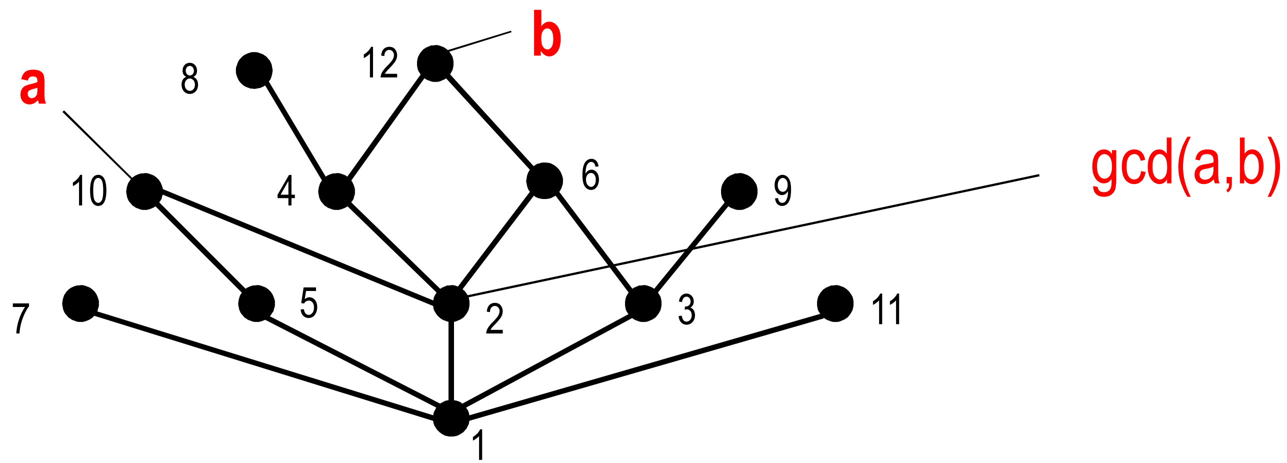
- For integers a and b, a positive integer c is said to be a common divisor of a and b if $c \mid a$ and $c \mid b$
- Let a, b be integers such that $a \neq 0$ or $b \neq 0$. Then a positive integer c is called the greatest common divisor of a, b if
  (a) $c \mid a$ and $c \mid b$ (that is c is a common divisor of a, b)
  (b) for any common divisor d of a and b, we have $d \mid c$

- What are the common divisors, and the greatest common divisor of 42 and 70?
- The greatest common divisor of a and b is denoted by $\gcd(a,b)$

The Greatest Common Divisor (cntd)

- **Theorem**
  For any positive integers a and b, there is a unique positive integer c such that c is the greatest common divisor of a and b

- First try:
  Take the largest common divisor, in the sense of usual order
  Does not work: Why every other common divisor divides it?



The Greatest Common Divisor (cntd)

- **Proof**.

  Given a, b, let $S = \{ as + bt \mid s,t \in \mathbb{Z},\ as + bt > 0 \}$.

  Since $S \neq \emptyset$, it has a least element c. We show that $c = \gcd(a,b)$

  We have $c = ax + by$ for some integers x and y.

  If $d \mid a$ and $d \mid b$, then $d \mid ax + by = c$.

  If $c \nmid a$, we can use the division algorithm to find $a = qc + r$, where q,r are integers and $0 < r < c$.

  Then $r = a – qc = a – q(ax + by) = a(1 – qx) + b(-qy) \in S$, a contradiction

  Therefore $c \mid a$, and by a similar argument $c \mid b$.

The Greatest Common Divisor (cntd)

- **Proof**. (cntd)

  Finally, if c and d are greatest common divisors, then $c \mid d$ and $d \mid c$. Thus $c = d$.

  Q. E. D.

Homework

Exercises from the Book

No. 2, 4, 12, 15, 16 (page 230)