Problems to Week 9 Tutorial — MACM 101 (Fall 2014)

1. For each of the following functions, determine whether it is one-to-one and determine its range.

   (a) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$;

   (b) $f\colon \mathbb{Q} \to \mathbb{Q}$, $f(x) = 2x + 1$;

   (c) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^3 - x$;

2. Let $f\colon A \to B$ where $A = X \cup Y$ with $X \cap Y = \emptyset$. If $f|_X$ and $f|_Y$ are one-to-one, does it follow that f is one-to-one?

3. For each of the following functions $f\colon \mathbb{Z} \to \mathbb{Z}$, determine whether the function is onto. If the function is not onto, determine the range of f.

   (a) $f(x) = 2x - 3$;

   (b) $f(x) = x^2 + x$.

4. Prove each of the following for all $n \geq 1$ using the principal of mathematical induction.

   (a)
   $$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

   (b)
   $$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

   (c)
   $$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

5. *(Only for those familiar with complex numbers)* Prove DeMoivre's theorem
   $$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

6. Prove that for all natural numbers n if $n > 3$ then $2^n < n!$.