Rules of Inference

Previous Lecture

- Logically equivalent statements
- Main logic equivalences
 - double negation
 - DeMorgan's laws
 - commutative, associative, and distributive laws
 - idempotent, identity, and domination laws
 - the law of contradiction and the law of excluded middle
 - absorption laws

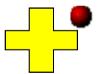
Example

Simplify the statement

$$\neg (q \lor r) \lor \neg (\neg q \lor p) \lor r \lor p$$

Expressing Connectives

- Some connectives can be expressed through others
 - $p \oplus q \Leftrightarrow \neg(p \leftrightarrow q)$
 - $p \leftrightarrow q \iff (p \rightarrow q) \land (q \rightarrow p)$



Theorem Every compound statement is logically equivalent to a statement that uses only conjunction, disjunction, and negation

Example

"If you are a computer science major or a freshman and you are not a computer science major or you are granted access to the Internet, then you are a freshman or have access to the Internet"

- p 'you can access the Internet from campus'
- q 'you are a computer science major'
- r 'you are a freshman'

Example

Simplify the statement

$$(p \lor q) \longleftrightarrow (p \to q)$$

First Law of Substitution

- Suppose that the compound statement Φ is a tautology. If p is a primitive statement that appears in Φ and we replace each occurrence of p by the same statement q, then the resulting compound statement Ψ is also a tautology.
 - Let $\Phi = (p \rightarrow q) \lor (q \rightarrow p)$, and we substitute p by $p \lor (s \oplus r)$

Therefore $((p \lor (s \oplus r)) \rightarrow q) \lor (q \rightarrow (p \lor (s \oplus r))$ is a tautology

Second Law of Substitution

Let Φ be a compound statement, p an arbitrary (not necessarily primitive!) statement that appears in Φ , and let q be a statement such that p \Leftrightarrow q. If we replace one or more occurrences of p by q, then for the resulting compound statement Ψ we have $\Phi \Leftrightarrow \Psi$.

Let Φ = (p → q) ∨ (q → p), and we substitute the first occurrence of p by p ∨ (p ∧ q).
Recall that p ⇔ p ∨ (p ∧ q) by Absorption Law.

Therefore

$$(p \rightarrow q) \lor (q \rightarrow p) \iff ((p \lor (p \land q)) \rightarrow q) \lor (q \rightarrow p).$$

Logic Inference

One of the main goals of logic is to distinguish valid and invalid arguments

What can we say about the following arguments:

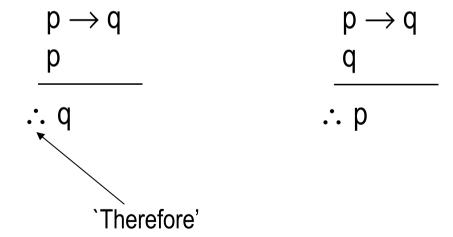
"If you have a current password, then you can log onto the network. You have a current password. Therefore, you can log onto the network."

and

"If you have a current password, then you can log onto the network. You can log onto the network. Therefore, you have a current password."

Logic Inference (cntd)

- Write these arguments in symbolic form
 - p you have a current password
 - q you can log onto the network



Inference and Tautologies

• Check that $\Phi = ((p \rightarrow q) \land p) \rightarrow q$ is a tautology

If $(p \rightarrow q) \land p$ is false, that is if one of $p \rightarrow q$ and p is false, then Φ is true

If $(p \to q) \land p$ is true, then both $p \to q$ and p are true. Since the implication $p \to q$ is true and p is true, q must also be true. Therefore Φ is true.

Therefore whatever values of p and q are, if $(p \rightarrow q) \land p$ is true, then q is also true.

The first example is a valid argument!

Inference and Tautologies

• Let us try $\Psi = ((p \rightarrow q) \land q) \rightarrow p$

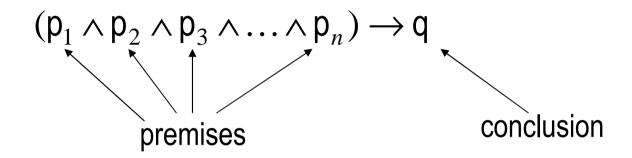
р	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	Ψ
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

In the case p = 0, q = 1 both conditions ($p \rightarrow q$ and q) are true, but p is false.

This is not a valid argument!

General Definition of Inference

The general form of an argument in symbolic form is



- The argument is valid if whenever each of the premises is true the conclusion is also true
- The argument is valid if and only if the following compound statement is a tautology

$$(p_1 \land p_2 \land p_3 \land ... \land p_n) \rightarrow q$$

Rules of Inference

Verifying if a complicated statement is a tautology is nearly impossible, even for computer. Fortunately, general arguments can be replaced with small collection of simple ones, rules of inference.

Modus ponens

$$\begin{array}{c}
 p \\
 \hline
 p \\
 \hline
 q
\end{array}$$

"If you have a current password, then you can log onto the network.

You have a current password.

Therefore, you can log onto the network."

Rule of Syllogism

 $\begin{array}{c}
p \to q \\
q \to r \\
\therefore p \to r
\end{array}$

• The corresponding tautology $((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$

"If you send me an e-mail, then I'll finish writing the program. If I finish writing the program, then I'll go to sleep early."

- p 'you will send me an e-mail'
- q 'I will finish writing the program'
- r 'I will go to sleep early'

"Therefore, if you send me an e-mail, then I'll go to sleep early"

Modus Tollens

p → q ¬q ∴ ¬p

• The corresponding tautology $((p \rightarrow q) \land \neg q) \rightarrow \neg p$

"If today is Friday, then tomorrow I'll go skiing".

"I will not go skiing tomorrow".

p - 'today is Friday'

q - 'I will go skiing tomorrow'

"Therefore, today is not Friday"

Rule of Disjunctive Syllogism

- p ∨ q <u>¬p</u> ∴ q
- The corresponding tautology $((p \lor q) \land \neg p) \rightarrow q$
 - "I'll go skiing this weekend.

 I will not go skiing on Saturday."
 - p 'I will go skiing on Saturday'
 - q 'I will go skiing on Sunday'

"Therefore, I will go skiing on Sunday"

Rule for Proof by Cases

 $\begin{array}{c}
p \to r \\
q \to r \\
\hline
 \therefore (p \lor q) \to r
\end{array}$

• The corresponding tautology $((p \rightarrow r) \land (q \rightarrow r)) \rightarrow ((p \lor q) \rightarrow r)$

"If today is Saturday, then I'll go skiing. If today is Sunday, then I'll go skiing.

- p 'today is Saturday'
- q 'today is Sunday'
- r `l'll go skiing'

"Therefore, if today is Saturday or Sunday, then I will go skiing"

Rules of Contradiction, Simplification, and Amplification

Rule of Contradiction

Reductio ad Absurdum

$$\frac{\neg p \to F}{}$$
The corresponding tautology $(\neg p \to F) \to p$
∴ p

Rule of Simplification

$$\frac{p \land q}{...p}$$
 The corresponding tautology $(p \land q) \rightarrow p$

Rule of Amplification

$$\frac{p}{\therefore p \vee q}$$
 The corresponding tautology $p \to (p \vee q)$

Homework

Exercises from the Book:

No. 1a, 3c, 4a, 5c, 9a (page 84)