MACM 101 — Discrete Mathematics I

Outline Solutions to Midterm 1

1. **Can an equivalence class be empty? Explain.**

   No. By definition if R is an equivalence relation on a set A, an equivalence class associated with element $a \in A$ is given by $C(a) = \{b \in A \mid (a,b) \in R\}$. In particular, due to the reflexivity of R, $(a,a) \in R$, and so $a \in C(a)$.

2. **Let f be a bijection. Is f^{-1} a bijection as well? Explain.**

   Let $f : A \to B$. The inverse function of f is the function that assigns to an element $b \in B$ the uniqte element $a \in A$ such that $f(a) = b$.

   Method 1.
   The inverse function $f^{-1} : B \to A$ is defined by $f^{-1}(b) = a$ if and only if $f(a) = b$. If $f^{-1}(b_1) = f^{-1}(b_2) = a$, then $f(a) = b_1$ and $f(a) = b_2$, implying $b_1 = b_2$. That is f^{-1} is injective. For any $a \in A$ we have $f^{-1}(b) = a$, where $b = f(a)$. Thus f^{-1} is surjective.

   Method 2.
   Since $f(a) = b$ if and only if $f^{-1}(b) = a$, function f^{-1} has an inverse function, f. Therefore, it is a bijection.

3. **Is the statement**

   $$\forall x \forall y\, ((x < y) \to \exists z((x < z) \wedge (z < y)))$$

   true on (a) the set $\mathbb{Z}$ of all integers? (b) the set $\mathbb{Q}$ of all rational numbers? Note that $<$ is interpreted as the relation 'less than', the natural strict order.

   This statement claims that for any two numbers x, y such that $x < y$ there is a third number, z, such that $x < z < y$.

   This is obviously false on $\mathbb{Z}$; For instance $2 < 3$, but there is no integer z such that $2 < z < 3$.

   The statement is true on $\mathbb{Q}$. Indeed, take any $x, y \in \mathbb{Q}$ such that $x < y$. Then

   $$x < \frac{x+y}{2} < y.$$

4. **What is $\aleph_1$ in set theory?**

   It is the cardinality of the set $\mathbb{R}$ of all real numbers, or the cardinality of the set of all subsets of the set $\mathbb{Z}$ of all integers.

5. **Is the following argument valid or invalid? If it is invalid, explain why. If it is valid, give a formal inference.**
   "No plant can walk. All cedars are plants. Therefore, no cedar can walk".

   The argument is valid.
   Express these statements symbolically. Let the predicates be:

   $M(x)$, 'x can walk',
   $V(x)$, 'x is a plant,
   $G(x)$, 'x is a cedar

   Then the premises are translated as: $\forall x\, (V(x) \to \neg M(x))$, $\forall x\, (G(x) \to V(x))$.
   And the conclusion: $\forall x\, (G(x) \to \neg M(x))$.

Steps	Reason
1. $\forall x\, (V(x) \to \neg M(x))$	premise

2. $V(a) \to \neg M(a))$ rule of universal specification, a is generic
3. $\forall x\, (G(x) \to \neg V(x))$ premise
4. $G(a) \to \neg V(a))$ rule of universal specification, a is generic
5. $G(a) \to \neg M(a))$ rule of syllogism
6. $\forall x\, (G(x) \to \neg M(x))$ rule of universal generalization

6. **Prove that for any sets A, B, and C**

$$(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C).$$

Draw Venn diagrams of both sides of the equality.

Use the laws of set theory to transform both sides of the equality.

$(A \Delta B) \cap C$

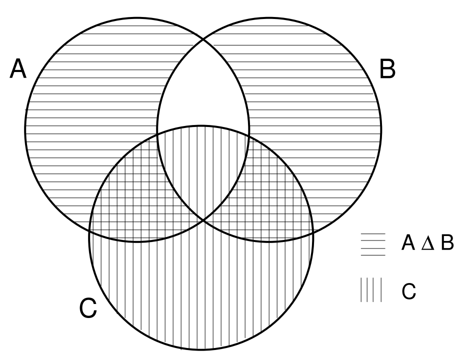
$= ((A \cup B) - (A \cap B)) \cap C$	expression for symmetric difference
$= ((A \cup B) \cap \overline{(A \cap B)}) \cap C)$	expression for difference
$= (A \cup B) \cap (\overline{A} \cup \overline{B}) \cap C$	DeMorgan's law.

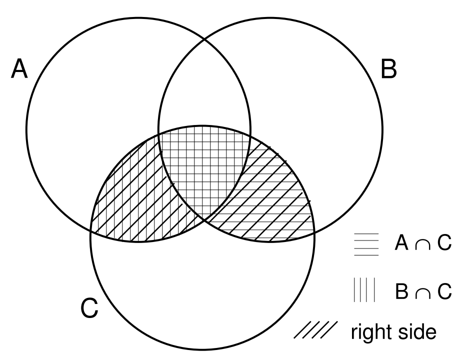
For the right part we have

$(A \cap C) \Delta (B \cap C)$

$= ((A \cap C) \cup (B \cap C)) - ((A \cap C) \cap (B \cap C))$	expression for symmetric difference
$= ((A \cap C) \cup (B \cap C)) - (A \cap B \cap C)$	associative and idempotence laws
$= ((A \cap C) \cup (B \cap C)) \cap \overline{(A \cap B \cap C)}$	expression for difference
$= ((A \cap C) \cup (B \cap C)) \cap (\overline{A} \cup \overline{B} \cup \overline{C})$	DeMorgan's law
$= (A \cup B) \cap C \cap (\overline{A} \cup \overline{B} \cup \overline{C})$	distributive law
$= (A \cup B) \cap \big((C \cap (\overline{A} \cup \overline{B})) \cup (C \cap \overline{C})\big)$	distributive law
$= (A \cup B) \cap \big((C \cap (\overline{A} \cup \overline{B})) \cup \emptyset\big)$	complement law
$= (A \cup B) \cap C \cap (\overline{A} \cup \overline{B})$	identity law.

In both cases we obtain the same set, therefore, the equality is true.





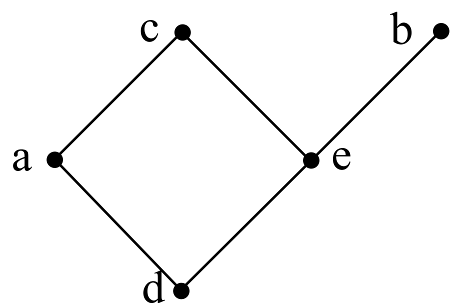
7. **What is the difference in the Induction Hypothesis in regular and strong induction?**

Suppose we are proving statement $P(n)$. The Induction Hypothesis in regular induction is $P(k)$, the statement P for one value of the numerical parameter. The Induction Hypothesis in strong induction is $P(1) \wedge P(2) \wedge \ldots \wedge P(k)$, the statement P for all values of the numerical parameter less than or equal k.

8. **Let a partial order be given by the following matrix:** [15]

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

(You do not need to check that this is an order.) Draw the diagram of this order. Determine all pairs of incomparable elements.



Incomparable elements are (a, e), (a, b), and (b, c).