Cardinality

Discrete Mathematics
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How to Count Elements in a Set

- How many elements are in a set?
- Easy for finite sets, just count the elements.
- What about infinite sets? Does it make sense at all to ask about the number of elements in an infinite set?
- Can we say that this infinite set is larger than that infinite set?
- Which set is larger: the set of all integers or the set of even integers?

  the set of all integers or the set of all rationals?

  the set of all integers or the set of all reals?

Cardinality and Bijections

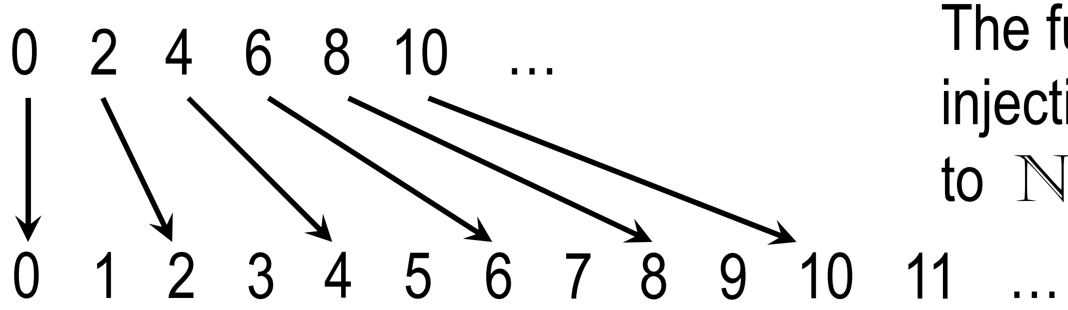
- Sets A and B (finite or infinite) have the same cardinality if and only if there is a bijection from A to B
- $|\mathbb{N}| = |2\mathbb{N}|$

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ 0 & 2 & 4 & 6 & 8 & 10 & & \dots \end{array}$$

The function $f\colon \mathbb{N} \to 2\mathbb{N}$, where $f(x) = 2x$, is a bijection

Comparing Cardinalities

- Let A and B be sets. We say that $|A| \le |B|$ if there is an injective function from A to B.



The function $f(x) = x$ is an injective function from $2\mathbb{N}$ to $\mathbb{N}$. Therefore $|2\mathbb{N}| \le |\mathbb{N}|$

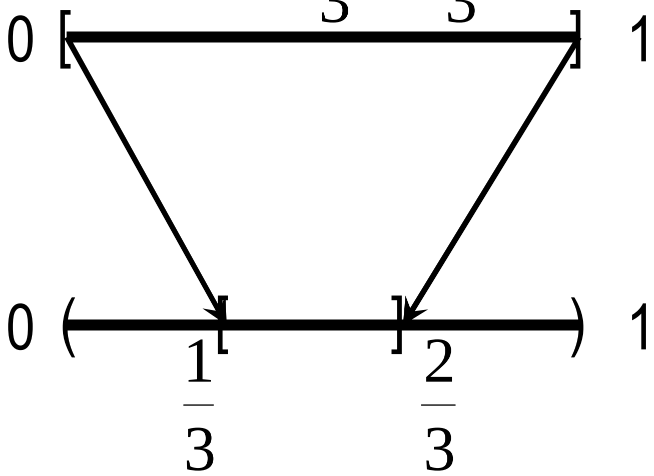
- If there is an injective function from A to B, but not from B to A, we say that $|A| < |B|$
- If there is an injective function from A to B and an injective function from B to A, then we say that A and B have the same cardinality
- Exercise: Prove that a bijection from A to B exists if and only if there are injective functions from A to B and from B to A.

Example

- Let A be the closed interval [0;1] (it includes the endpoints) and B – the open interval (0;1) (it does not include the endpoints)
- There are injective functions f and g from A to B and B to A, respectively.

$f: A \to B$

$f(x) = \frac{1}{3}x + \frac{1}{3}$



$f: B \to A$

$f(x) = x$

Countable and Uncountable

- A set A is said to be **countable** if $|A| \le |\mathbb{N}|$
- This is because an injective function from A to $\mathbb{N}$ can be viewed as assigning numbers to the elements of A, thus counting them
- Sets that are not countable are called **uncountable**
- Countable sets:

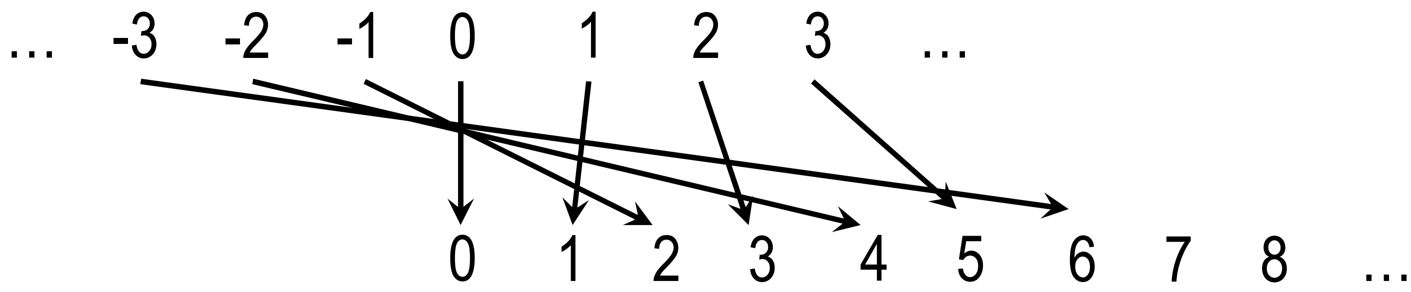
  finite sets

$$\begin{array}{ccc} a \bullet & \longrightarrow & \bullet\ 1 \\ b \bullet & \longrightarrow & \bullet\ 2 \\ c \bullet & \longrightarrow & \bullet\ 3 \\ & & \vdots \end{array}$$

  any subset of $\mathbb{N}$

More Countable Sets

- The set of all integers is countable



- In other words we can make a list of all integers

0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, …

- The cardinality of the set of all natural numbers is denoted by $\aleph_0$

More Countable Sets (cntd)

- The set of positive rational numbers is countable
- Every rational number can be represented as a fraction p/q

  We do not insist that p and q do not have a common divisor

$$
\begin{array}{ccccc}
1/1 & 2/1 & 3/1 & 4/1 & \cdots \\
1/2 & 2/2 & 3/2 & 4/2 & \cdots \\
1/3 & 2/3 & 3/3 & 4/3 & \cdots \\
1/4 & 2/4 & 3/4 & 4/4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

- This gives an injection from $\mathbb{Q}^+$ to $\mathbb{N}$. The converse injection is

  f(x) = x + 1

The Smallest Infinite Set

- Theorem.
  If A is an infinite set, then $|A| \geq \aleph_0$

- Proof requires mathematical induction. Wait for a few days.

Uncountable Sets

- Can we make a list of all real numbers?
- Every real number can be represented as an infinite decimal fraction, like 3.14159265358979323846264338327950288419711…
- Suppose we have constructed a list of all real numbers

1. $a_{10}.a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}\dots$
2. $a_{20}.a_{21}a_{22}a_{23}a_{24}a_{25}a_{26}a_{27}\dots$
3. $a_{30}.a_{31}a_{32}a_{33}a_{34}a_{35}a_{36}a_{37}\dots$
4. $a_{40}.a_{41}a_{42}a_{43}a_{44}a_{45}a_{46}a_{47}\dots$
5. $a_{50}.a_{51}a_{52}a_{53}a_{54}a_{55}a_{56}a_{57}\dots$

$\vdots \qquad \vdots$

Here the a_{ij} are digits 0,1,2,…,9

Let

$$b_i = \begin{cases} 4, & \text{if } a_{ii} \ne 4, \\ 5 & \text{otherwise} \end{cases}$$

- It is not hard to see that the number $0.b_1b_2b_3b_4b_5b_6b_7\dots$ is not in this list

Cantor's Theorem

- Theorem (Cantor). For any set $|P(A)| > |A|$.

  Proof.

  Suppose that there is a bijection $f\colon A \to P(A)$.

  We find a set that does not belong to the range of f. A contradiction with the assumption that f is bijective.

  Consider the set $T = \{ a \in A \mid a \notin f(a) \}$

  If T is in the range of f, then there is $t \in A$ such that $f(t) = T$.

  Either $t \in T$ or $t \notin T$.

  If $t \in T$ then $t \in f(t)$, and we get $t \notin T$.

  If $t \notin T$ then $t \in T$.

  Q.E.D.

Cantor's Theorem (cntd)

- This method is called Cantor's diagonalization method
- The cardinality of P(A) is denoted by $2^{|A|}$
- Thus, we obtain an infinite series of infinite cardinals

$$|\mathbb{N}| = \aleph_0$$

$$2^{\aleph_0} = \aleph_1 \quad (= |\mathbb{R}|)$$

$$2^{\aleph_1} = \aleph_2$$

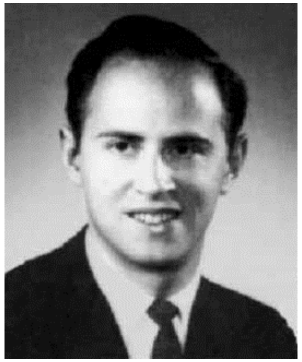
$$\vdots$$

Continuum Hypothesis

- We just proved that $\aleph_0 < |\mathbb{R}|$. Does there exist a set A such that $\aleph_0 < |A| < |\mathbb{R}|$?

- The negative answer to this question is known as the continuum hypothesis.

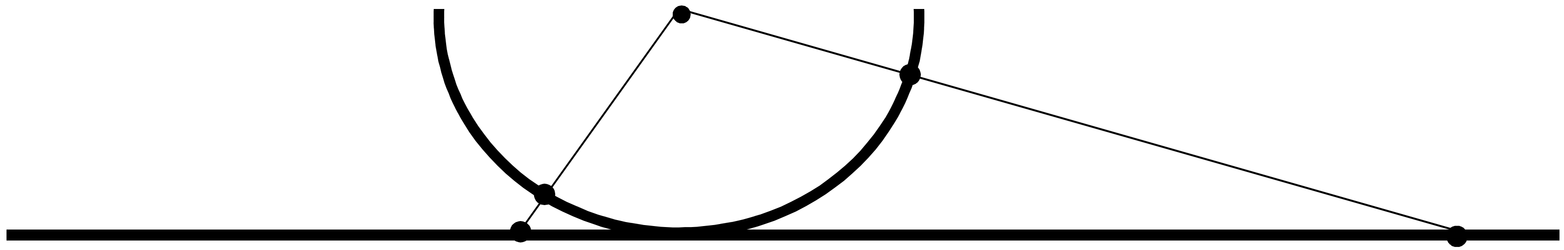
- Continuum hypothesis is the first problem in the list of Hilbert's problems



- Paul Cohen resolved the question in 1963. The answer is shocking: You can think either way.

More Uncountable Sets

- For any real numbers a, b, the open interval (a;b) has the same cardinality as $\mathbb{R}$



Homework

Exercises from the Book:

No. 1def, 2b, 4 (page A-32)

- Construct a bijective mapping between the closed interval [0;1] and the square $[0;1] \times [0;1]$