Rules of Inference

Discrete Mathematics

Andrei Bulatov

Previous Lecture

- Logically equivalent statements
- Main logic equivalences
  - double negation
  - DeMorgan’s laws
  - commutative, associative, and distributive laws
  - idempotent, identity, and domination laws
  - the law of contradiction and the law of excluded middle
  - absorption laws

Example

- Simplify the statement

$$\neg(q \vee r) \vee \neg(\neg q \vee p) \vee r \vee p$$

Expressing Connectives

- Some connectives can be expressed through others
  - $p \oplus q \quad \Leftrightarrow \quad \neg(p \leftrightarrow q)$
  - $p \leftrightarrow q \quad \Leftrightarrow \quad (p \rightarrow q) \wedge (q \rightarrow p)$
  - $p \rightarrow q \quad \Leftrightarrow \quad \neg p \vee q$

- **Theorem** Every compound statement is logically equivalent to a statement that uses only conjunction, disjunction, and negation

Example

``If you are a computer science major or a freshman and you are not a computer science major or you are granted access to the Internet, then you are a freshman or have access to the Internet''

p - `you can access the Internet from campus'

q - `you are a computer science major'

r - `you are a freshman'

Example

- Simplify the statement

$$(p \vee q) \leftrightarrow (p \rightarrow q)$$

First Law of Substitution

- Suppose that the compound statement Φ is a tautology. If p is a primitive statement that appears in Φ and we replace each occurrence of p by the same statement q, then the resulting compound statement Ψ is also a tautology.

- Let $\Phi = (p \to q) \vee (q \to p)$, and we substitute p by $p \vee (s \oplus r)$

  Therefore $((p \vee (s \oplus r)) \to q) \vee (q \to (p \vee (s \oplus r))$ is a tautology

Second Law of Substitution

- Let Φ be a compound statement, p an arbitrary (not necessarily primitive!) statement that appears in Φ, and let q be a statement such that $p \Leftrightarrow q$. If we replace one or more occurrences of p by q, then for the resulting compound statement Ψ we have $\Phi \Leftrightarrow \Psi$.

- Let $\Phi = (p \rightarrow q) \vee (q \rightarrow p)$, and we substitute the first occurrence of p by $p \vee (p \wedge q)$.
  Recall that $p \Leftrightarrow p \vee (p \wedge q)$ by Absorption Law.

  Therefore

$$(p \rightarrow q) \vee (q \rightarrow p) \quad \Leftrightarrow \quad ((p \vee (p \wedge q)) \rightarrow q) \vee (q \rightarrow p).$$

Logic Inference

- One of the main goals of logic is to distinguish valid and invalid arguments

What can we say about the following arguments:

``If you have a current password, then you can log onto the network. You have a current password. Therefore, you can log onto the network.”

and

``If you have a current password, then you can log onto the network. You can log onto the network. Therefore, you have a current password.”

Logic Inference (cntd)

- Write these arguments in symbolic form

  p - you have a current password

  q - you can log onto the network

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array} \qquad\qquad \begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore\ p \end{array}$$

\`Therefore'

Inference and Tautologies

- Check that $\Phi = ((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology

  If $(p \rightarrow q) \wedge p$ is false, that is if one of $p \rightarrow q$ and p is false, then Φ is true

  If $(p \rightarrow q) \wedge p$ is true, then both $p \rightarrow q$ and p are true. Since the implication $p \rightarrow q$ is true and p is true, q must also be true. Therefore Φ is true.

Therefore whatever values of p and q are, if $(p \rightarrow q) \wedge p$ is true, then q is also true.

The first example is a valid argument!

Inference and Tautologies

- Let us try $\Psi = ((p \to q) \land q) \to p$

p	q	$p \to q$	$(p \to q) \land q$	Ψ
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

In the case $p = 0$, $q = 1$ both conditions ($p \to q$ and q) are true, but p is false.
This is not a valid argument!

General Definition of Inference

- The general form of an argument in symbolic form is

$$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow q$$

premises conclusion

- The argument is valid if whenever each of the premises is true the conclusion is also true

- The argument is valid if and only if the following compound statement is a tautology

$$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \rightarrow q$$

Rules of Inference

- Verifying if a complicated statement is a tautology is nearly impossible, even for computer. Fortunately, general arguments can be replaced with small collection of simple ones, rules of inference.

- *Modus ponens*

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

``If you have a current password, then you can log onto the network.
You have a current password.
Therefore, you can log onto the network.''

Rule of Syllogism

- $$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$$

- The corresponding tautology $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

  ``If you send me an e-mail, then I'll finish writing the program.
  If I finish writing the program, then I'll go to sleep early.''

  p - `you will send me an e-mail'

  q - `I will finish writing the program'

  r - `I will go to sleep early'

  ``Therefore, if you send me an e-mail, then I'll go to sleep early''

Modus Tollens

- $$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

- The corresponding tautology $((p \rightarrow q) \land \neg q) \rightarrow \neg p$

``If today is Friday, then tomorrow I'll go skiing''.
``I will not go skiing tomorrow''.

p - `today is Friday'

q - `I will go skiing tomorrow'

``Therefore, today is not Friday''

Rule of Disjunctive Syllogism

- $$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

- The corresponding tautology $((p \lor q) \land \neg p) \to q$

``I'll go skiing this weekend.
I will not go skiing on Saturday.''

p - `I will go skiing on Saturday'

q - `I will go skiing on Sunday'

``Therefore, I will go skiing on Sunday''

Rule for Proof by Cases

- $$\begin{array}{l} p \rightarrow r \\ q \rightarrow r \\ \hline \therefore (p \vee q) \rightarrow r \end{array}$$

- The corresponding tautology $((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$

``If today is Saturday, then I'll go skiing.
If today is Sunday, then I'll go skiing.

p - `today is Saturday'

q - `today is Sunday'

r - `I'll go skiing'

``Therefore, if today is Saturday or Sunday, then I will go skiing''

Rules of Contradiction, Simplification, and Amplification

- Rule of Contradiction *Reductio ad Absurdum*

$$\begin{array}{l} \neg p \to F \\ \hline \therefore\ p \end{array}$$

The corresponding tautology $(\neg p \to F) \to p$

- Rule of Simplification

$$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array}$$

The corresponding tautology $(p \wedge q) \to p$

- Rule of Amplification

$$\begin{array}{l} p \\ \hline \therefore\ p \vee q \end{array}$$

The corresponding tautology $p \to (p \vee q)$

Homework

Exercises from the Book:

No. 1a, 3c, 4a, 5c, 9a (page 84)