Problem 1

Determine if each of these functions $f : \{a, b, c, d\} \to \{a, b, c, d\}$ is or is not:

1) one-to-one (injective)

2) onto (surjective)

3) a one-to-one correspondence (bijective)

a. $f(a) = b, f(b) = a, f(c) = c, f(d) = d$
   1) It is injective as every input maps to a distinct output.
   2) It is surjective as every element in the codomain is mapped to.
   3) It is bijective as it is both injective and surjective.
b. $f(a) = b, f(b) = b, f(c) = d, f(d) = c$
   1) It is not injective as $f(a)$ and $f(b)$ both map to b.
   2) It is not surjective as a is not mapped.
   3) It is not bijective as it is neither injective or surjective.
c. $f(a) = d, f(b) = b, f(c) = c, f(d) = d$
   1) It is not injective as $f(a)$ and $f(d)$ both map to d.
   2) It is not surjective as a is not mapped to.
   3) It is not bijective as it is neither injective or surjective.

Problem 2

Determine if each of these functions $f : \mathbb{Z} \to \mathbb{Z}$ is or is not:

1) one-to-one (injective)

2) onto (surjective)

3) a one-to-one correspondence (bijective)

a. $f(x) = 2x + 1$
   1) It is injective as every input maps to a distinct output.
   2) It is not surjective as no even number is ever mapped to $f(x)$
   3) It is not bijective as it is not surjective.
b. $f(x) = x - |x|$
   1) It is not injective as $f(1) = 0$ and $f(2) = 0$.
   2) It is not surjective as any number beyond 0 as y approaches ∞ is not mapped to (i.e. 1).
   3) It is not bijective as it is neither injective or surjective.
c. $f(x) = x - 45$
   1) It is injective as every input maps to a distinct output.
   2) It is surjective as every element in the codomain is mapped to.
   3) It is bijective as it is both injective and surjective.
d. $f(x) = \lfloor 2^x \rfloor$
   1) It is not injective as $f(-2) = 0$ and $f(-1) = 0$.
   2) It is not surjective as as any number beyond 0 as y approaches $-\infty$ is not mapped to (i.e. −1).
   3) It is not bijective as it is not injective or surjective.
e. $f(x) = 1 - x$
   1) It is injective as every input maps to a distinct output.
   2) It is surjective as every element in the codomain is mapped to.
   3) It is bijective as it is both injective and surjective.
f. $f(x) = -(x^3 + x)$
   1) It is injective as every input maps to a distinct output.

2) It is surjective as every element in the codomain is mapped to.
3) It is bijective as it is both injective and surjective.

Problem 3

Determine whether each of these functions $f : \mathbb{R} \to \mathbb{R}$ is or is not:

1) one-to-one (injective)

2) onto (surjective)

3) a one-to-one correspondence (bijective)

a. $f(x) = 2x + 1$
   1) It is injective as every input maps to a distinct output.
   2) It is surjective as every element in the codomain is mapped to.
   3) It is bijective as it is both injective and surjective.
b. $f(x) = -3x + 4$
   1) It is injective as every input maps to a distinct output.
   2) It is surjective as every element in the codomain is mapped to.
   3) It is bijective as it is both injective and surjective.
c. $f(x) = -3x^2 + 7$
   1) It is not injective as $f(-1) = 4$ and $f(1) = 4$.
   2) It is surjective as any number beyond 7 as y approaches ∞ is not mapped to (i.e. 8).
   3) It is not bijective as it is neither injective or surjective.
d. $f(x) = \begin{cases} 0 \text{ if } x=0 \\ \frac{1}{x} \text{ if } x \neq 0 \end{cases}$
   1) It is injective as every input maps to a distinct output.
   2) It is surjective as every element in the codomain is mapped to.
   3) It is bijective as it is both injective and surjective.

Problem 4

Recall that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is:

a. one-to-one but not onto
   $f(x) = x + 1$ is a good example as it is injective (every input maps to a distinct output), however, it is not surjective as 0 is not mapped to.
b. onto but not one-to-one
   $f(x) = |x - 1|$ is a good example as it is surjective (every element in the codomain is mapped to), however, it is not injective as $f(0) = 1$ and $f(2) = 1$.
c. neither one-to-one nor onto
   $f(x) = x^2 + 1$ is a good example as it is not injective ($f(-1) = 2$ and $f(1) = 2$), and it is not surjective (0 is never mapped to).

(**Hint**: consider using the absolute value, floor, or ceiling functions for part b)

Problem 5

Find $f \circ g$ and $g \circ f$ where $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3x + 4$ and $g(x) = x^2$
$f \circ g = f(g(x)) = f(x^2) = 3(x^2) + 4 = 3x^2 + 4.$
$g \circ f = g(f(x)) = g(3x + 4) = (3x + 4)^2 = 9x^2 + 24x + 16$