Logical equivalences in Predicate Logic

Two statements in predicate logic, A and B, are logically equivalent, $A \equiv B$, if they have the same truth value for all environments and interpretations.

$A \equiv B$ exactly when $[\![A]\!] \equiv [\![B]\!]$ for all environments and interpretations.

Example:
$\forall(P(x) \wedge Q(x)) \equiv (\forall x P(x)) \wedge (\forall x Q(x))$

However

$\forall x(P(x) \vee Q(x)) \not\equiv (\forall x P(x)) \vee (\forall x Q(x))$

$P(x)$ means "x is even"

$Q(x)$ means "x is odd"

$\neg(A \wedge B) \equiv (\neg A) \vee \neg B$

De Morgan's Law for Quantifiers:

$\neg\forall x A \equiv \exists x \neg A$

$\neg\exists x A \equiv \forall x \neg A$

Example
$\neg\forall x(Q(x) \wedge R(x, y)) \equiv \exists x \neg(Q(x) \wedge R(x, y))$

Nested Quantifiers

Example
"Everybody has a parent"

$\forall x \exists y P(x, y)$

"Everybody loves somebody"

$\exists y \forall x L(x, y)$

"Each veagle is not the smallest dog"

$\forall x(B(x) \to \exists z S(z, x))$

"There is only one number larger than x"

$\exists y(y > x \wedge \forall w(w > x \to y = w))$

Proofs

A <u>theorem</u> is a true statement that can be proved to be true. *A theorem that is not true is called a <u>joke</u>.*

Fermat's Last theorem
$z^n = x^n + y^n$ has integer solutions only when $n = 2$

A <u>proof</u> is a sequence of statements such that each statement:

1. is an assumption
2. is a previously proven statement
3. logically fallows previous statements in the proof

Definitions that can be used in proofs

- An integer x is <u>even</u> if there is an integer k such that $x = 2k$
- An integer x is <u>odd</u> if there is an integer k such that $x = 2k + 1$
- A number r is <u>rational</u> if there are integers p and q such that $q \neq$ and $r = \frac{p}{q}$