Digital Image Processing

Lecture # 8
Frequency Domain Enhancement

Image Enhancement in Frequency Domain

Joseph Fourier (1768 – 1830)

- Most famous for his work "*La Théorie Analitique de la Chaleur*" published in 1822
- Translated into English in 1878: "*The Analytic Theory of Heat*"

Nobody paid much attention when the work was first published
One of the most important mathematical theories in modern engineering

Background

- Any function that periodically repeats itself can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (Fourier series).

- Even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighting function (Fourier transform).
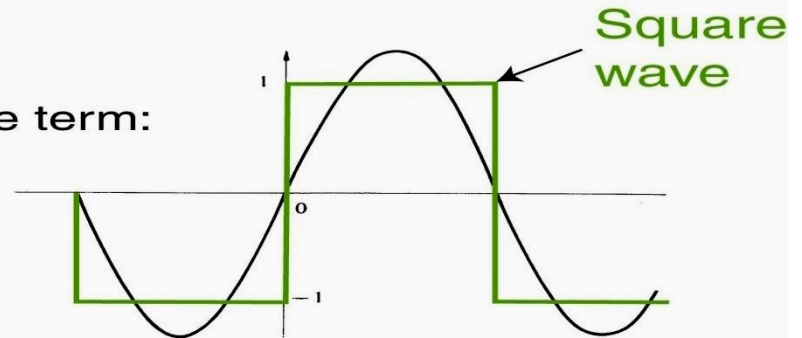
The big idea …

Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*
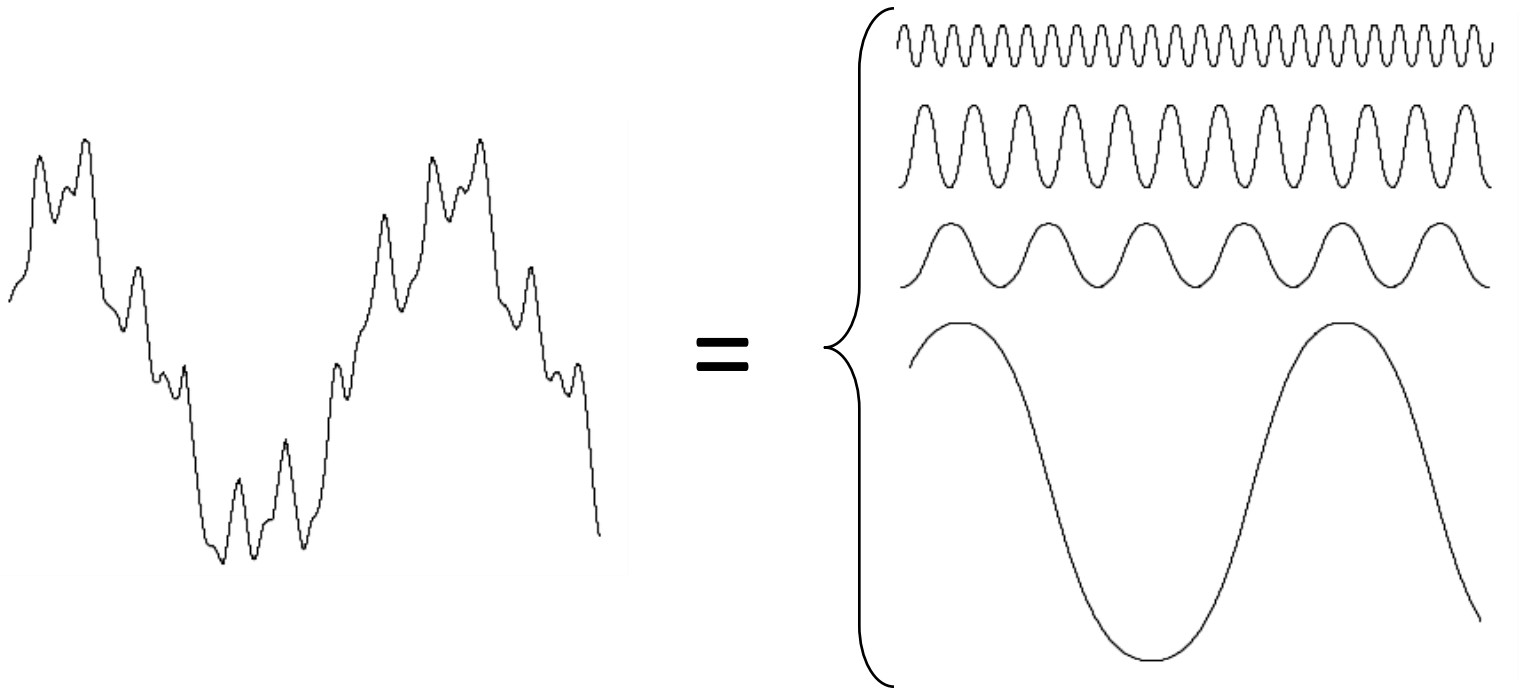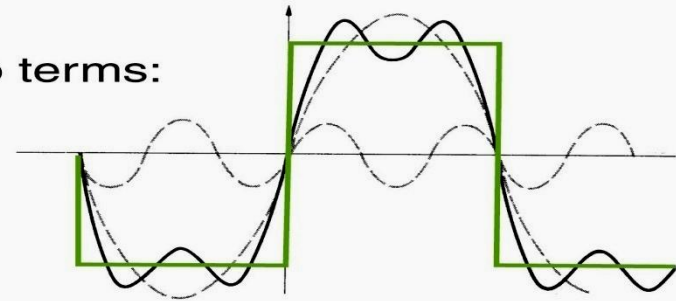
The big idea…

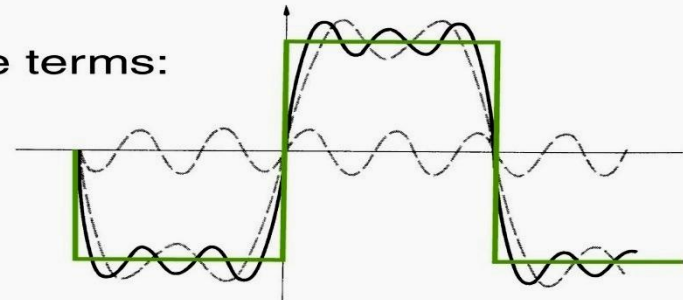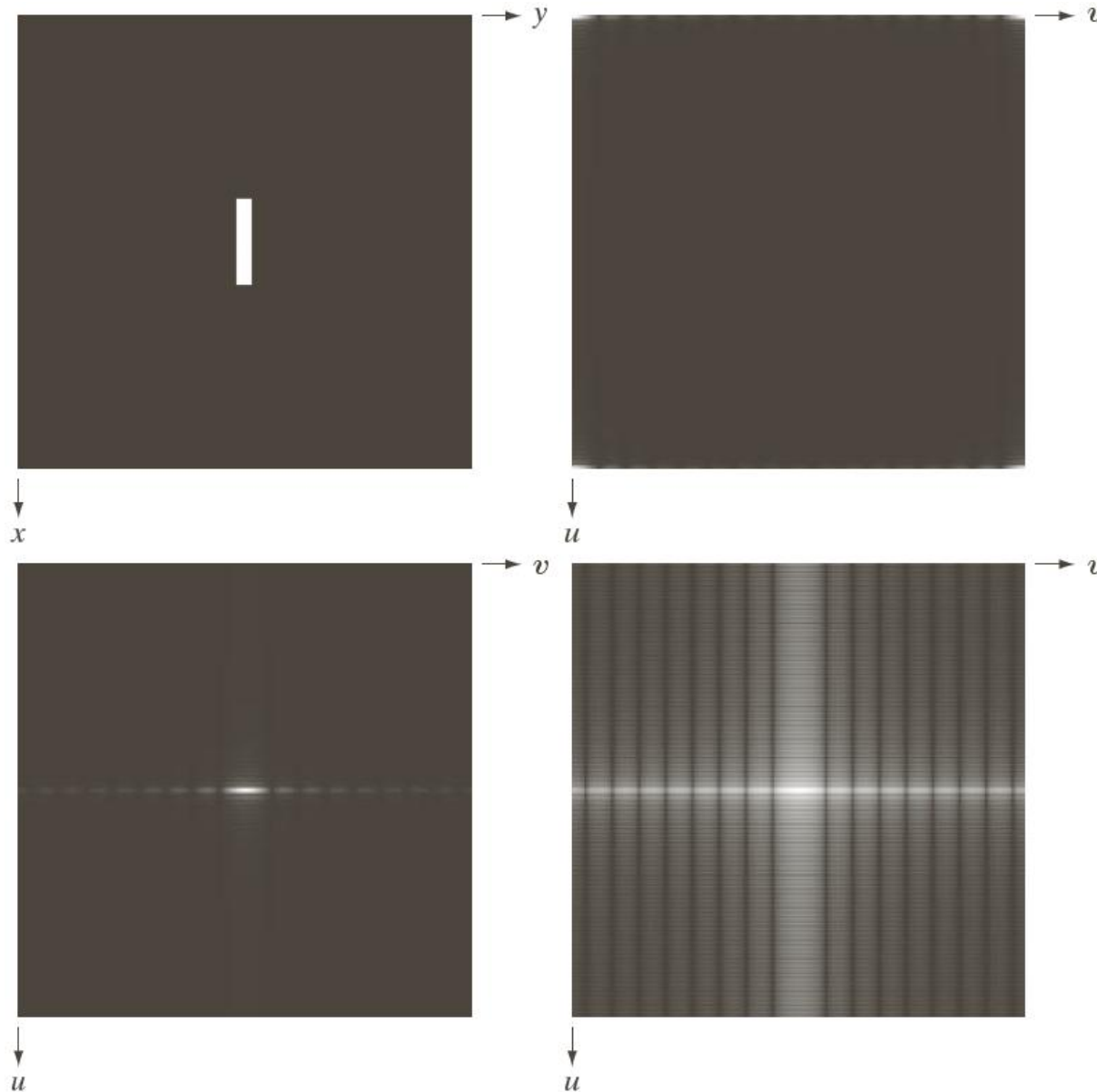Approximating a square wave as the sum of sine waves

Frequencies in Images

a b
c d

FIGURE 4.24
(a) Image.
(b) Spectrum
showing bright spots
in the four corners.
(c) Centered
spectrum. (d) Result
showing increased
detail after a log
transformation. The
zero crossings of the
spectrum are closer in
the vertical direction
because the rectangle
in (a) is longer in that
direction. The
coordinate
convention used
throughout the book
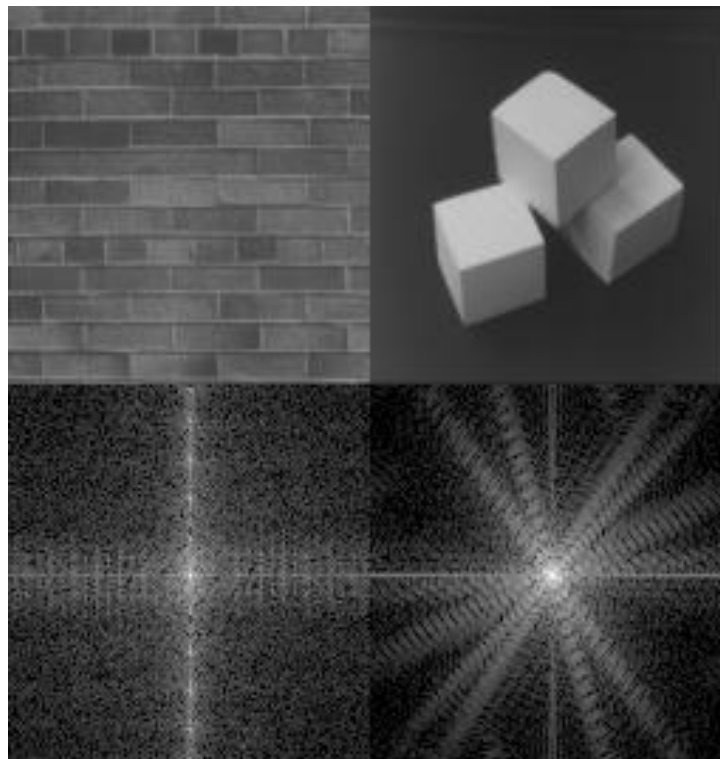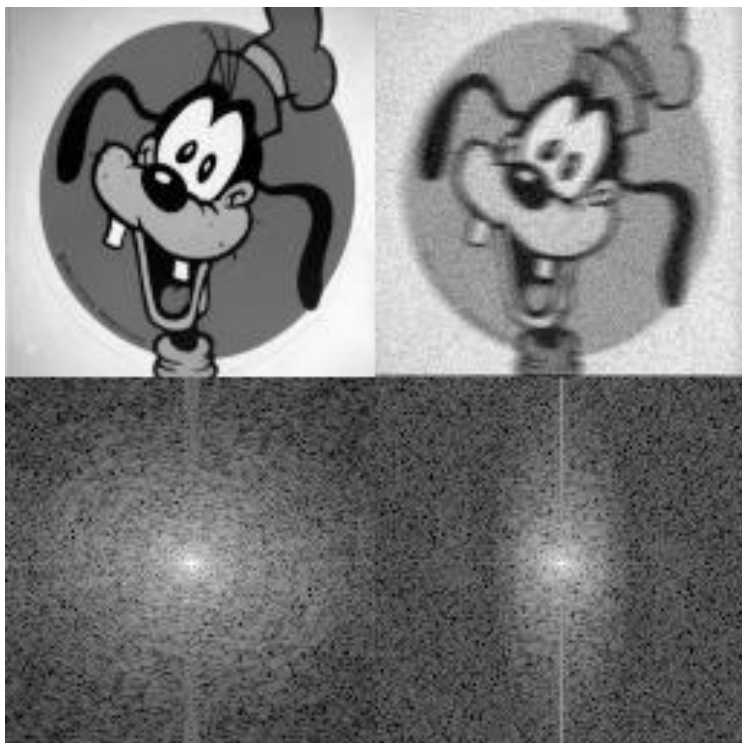places the origin of
the spatial and
frequency domains at
the top left.
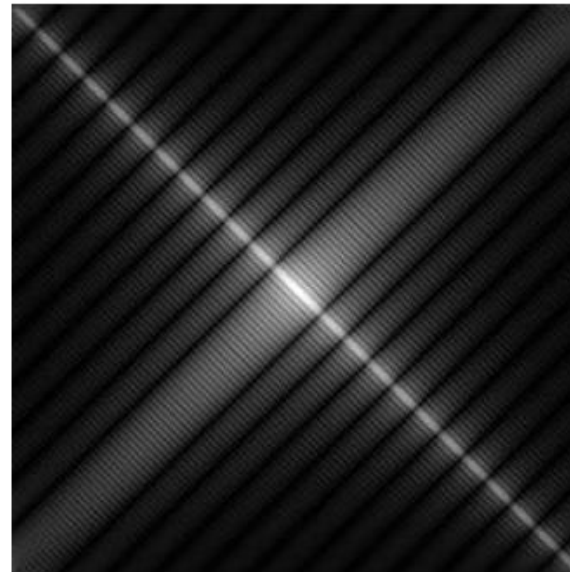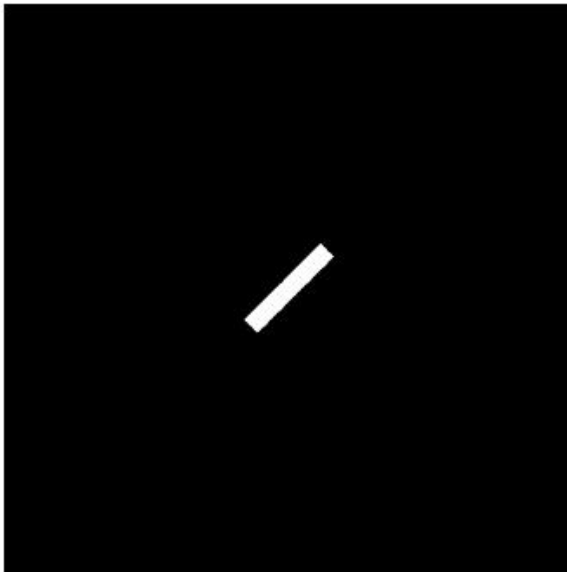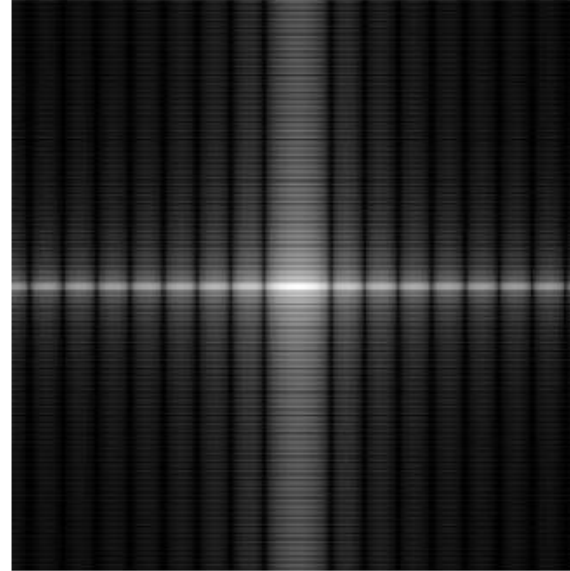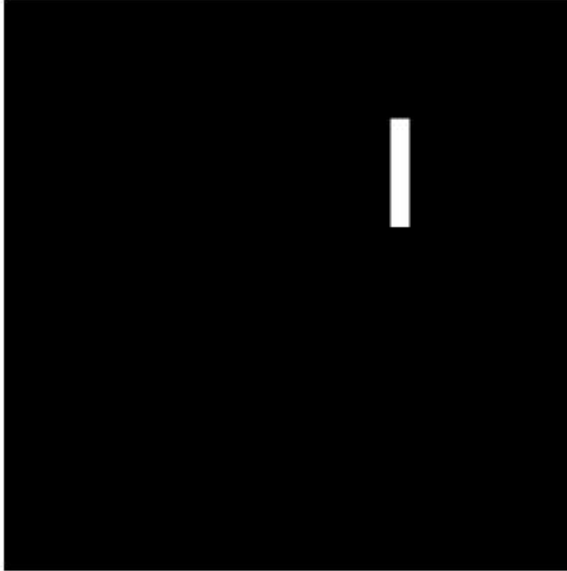
FIGURE 4.25
(a) The rectangle
in Fig. 4.24(a)
translated,
and (b) the
corresponding
spectrum.
(c) Rotated
rectangle,
and (d) the
corresponding
spectrum. The
spectrum
corresponding to
the translated
rectangle is
identical to the
spectrum
corresponding to
the original image
in Fig. 4.24(a).

Frequencies in Images

Basic 2D FT

$$x(n_1, n_2) = e^{j(\omega_1' n_1 + \omega_2' n_2)} \longrightarrow \boxed{h(n_1, n_2)} \longrightarrow y(n_1, n_2) = ?$$

$$(\omega_1', \omega_2')$$
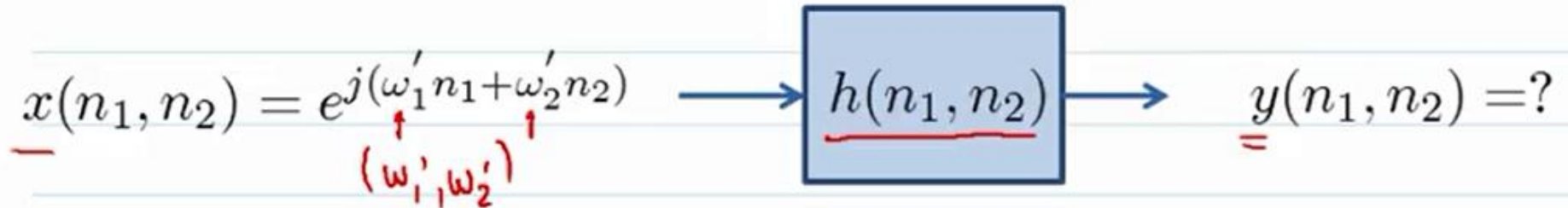
$$y(n_1, n_2) = x(n_1, n_2) ** h(n_1, n_2)$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{j\omega_1'(n_1-k_1)} e^{j\omega_2'(n_2-k_2)} h(k_1, k_2)$$

$$= \underbrace{e^{j\omega_1' n_1} e^{j\omega_2' n_2}}_{} \underbrace{\sum_{k_1} \sum_{k_2} h(k_1, k_2) e^{-j\omega_1' k_1} e^{-j\omega_2' k_2}}_{}$$
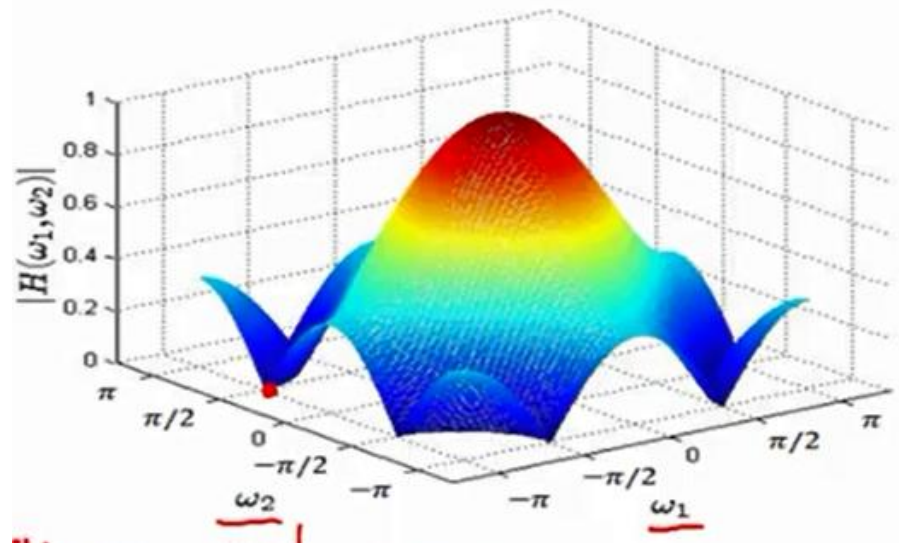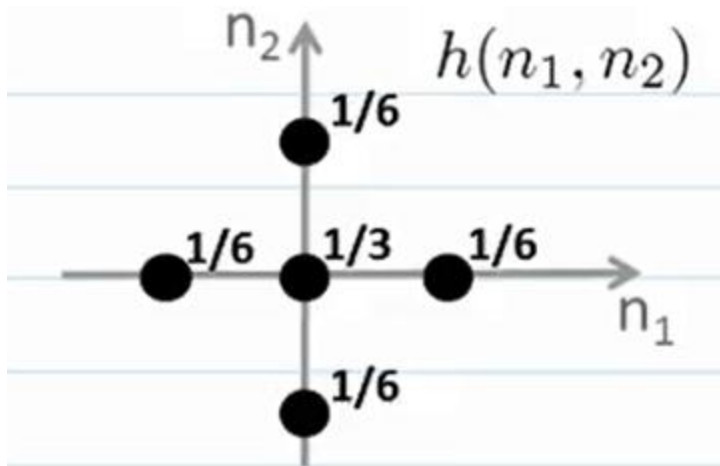
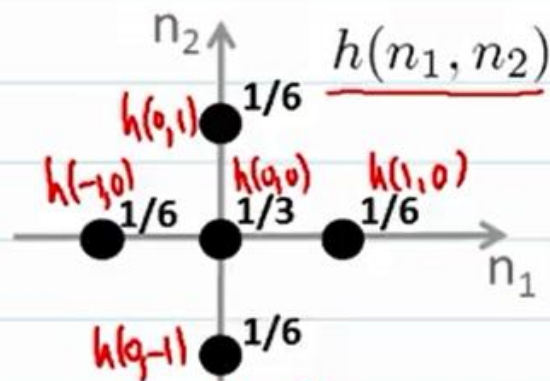$$H(\omega_1', \omega_2') \triangleq \text{frequency response}$$

2D FT

$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$$
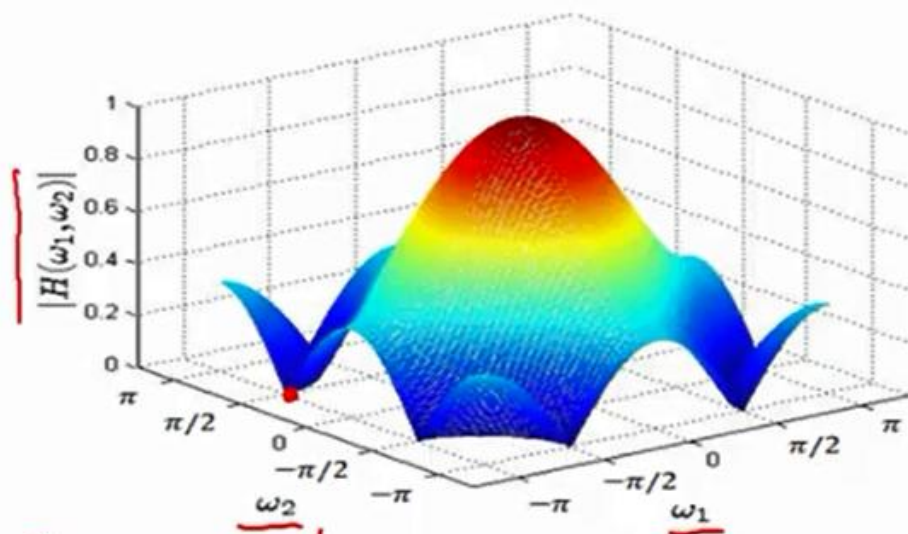
Example

$h(n_1, n_2)$

$$H(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$= h(0,0) + h(-1,0) e^{j\omega_1} + h(1,0) e^{-j\omega_1} + h(0,-1) e^{j\omega_2} + h(0,1) e^{-j\omega_2}$$

$$= \frac{1}{3} + \frac{1}{6} e^{j\omega_1} + \frac{1}{6} e^{-j\omega_1} + \frac{1}{6} e^{+j\omega_2} + \frac{1}{6} e^{-j\omega_2}$$

$$= \frac{1}{3} + \frac{1}{6} 2 \cos\omega_1 + \frac{1}{6} \cdot 2 \cos\omega_2$$

$$= \frac{1}{3}(1 + \cos\omega_1 + \cos\omega_2)$$
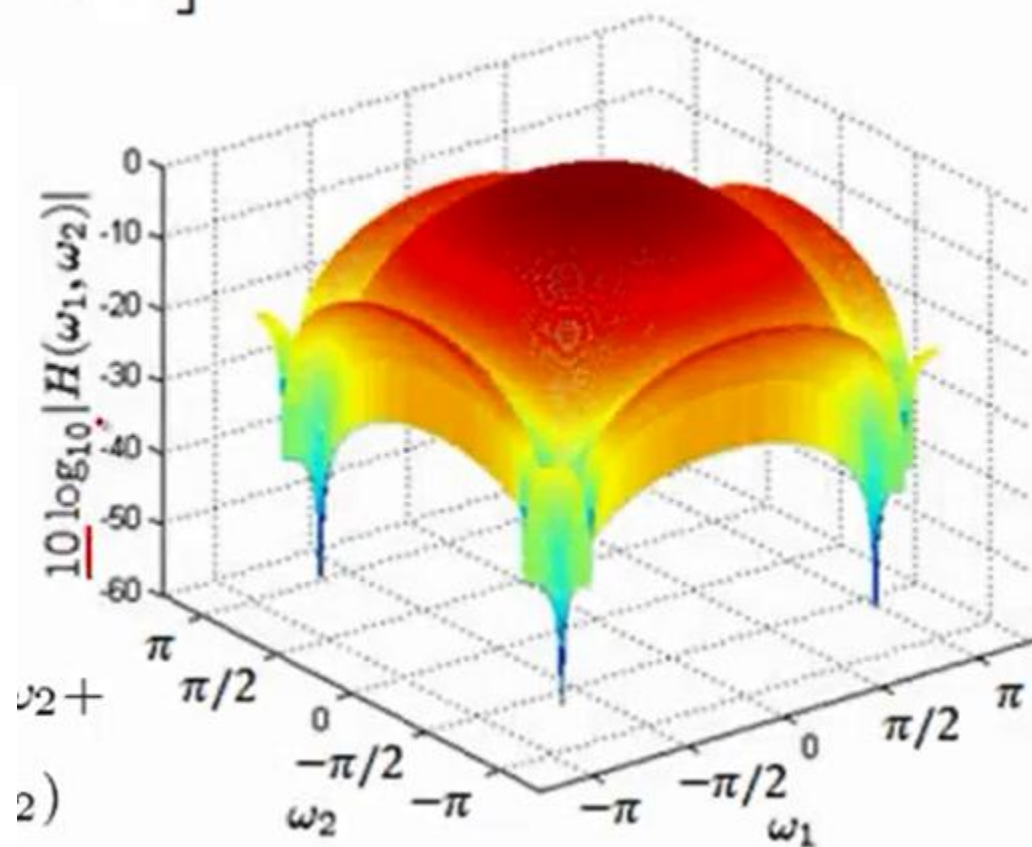
$$-1 = 1 \cdot e^{\pm j\pi}$$

$$H(0,0) = 1$$
$$H(-\pi, \pi/2) = 0$$

Example

$$h(n_1, n_2) = \begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$

$$0.124 \cdot 2 \cdot \cos w_1$$

$$h(-1,1) \qquad h(1,1)$$
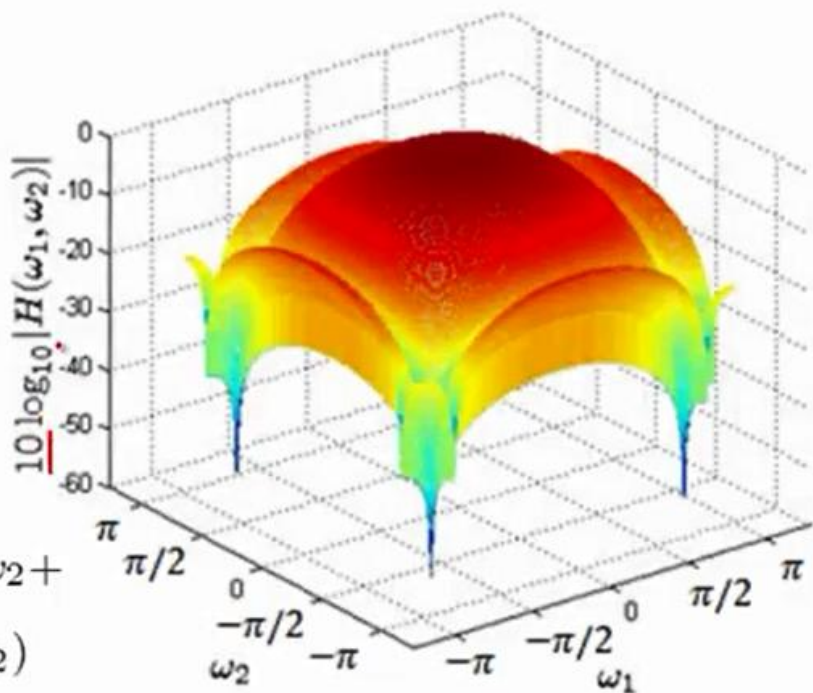
$$h(n_1, n_2) = \begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix} \quad h(0,0)$$

$$0.075 \cdot 2 \cdot \cos(w_1 + w_2)$$

$$H(\omega_1, \omega_2) = 0.204 + 0.124 \cdot 2 \cdot \cos\omega_1 + 0.124 \cdot 2 \cdot \cos\omega_2 +$$

$$0.075 \cdot 2 \cdot \cos(\omega_1 + \omega_2) + 0.075 \cdot 2 \cdot \cos(\omega_1 - \omega_2)$$

Example

$$h(n_1, n_2) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$
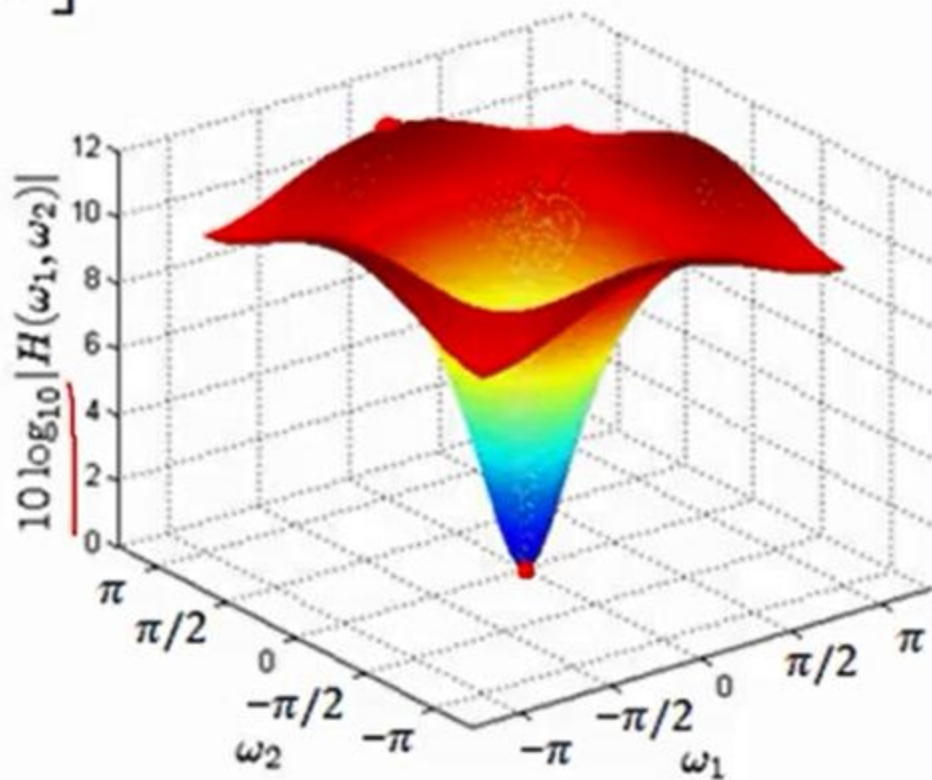
$$h(n_1, n_2) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & \boxed{9} & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

$h(0,0)$

$$H(\omega_1, \omega_2) = 9 - 2 \cdot cos\omega_1 - 2 \cdot cos\omega_2 -$$
$$2 \cdot cos(\omega_1 + \omega_2) - 2 \cdot cos(\omega_1 - \omega_2)$$
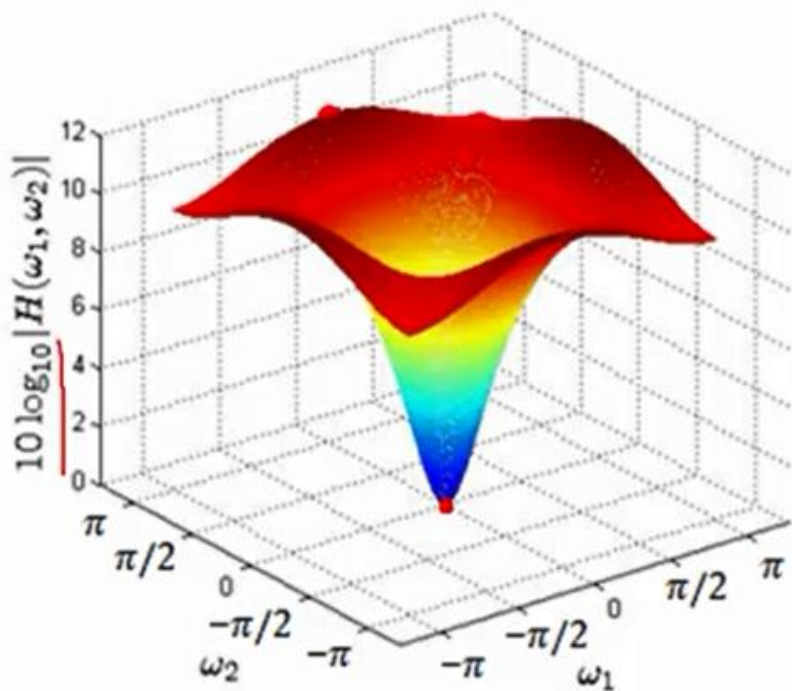
$H(0,0) = 1 \rightarrow \log H(0,0) = 0$

$H(0,\pi) = 13, \quad H(\pi,\pi) = 9$

2D FT to 2D DFT

$$X(\omega_1, \omega_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}$$

$$X(k_1, k_2) = X(\omega_1, \omega_2)\big|_{\omega_1 = \frac{2\pi}{N_1}k_1, \, \omega_2 = \frac{2\pi}{N_2}k_2} \qquad \begin{cases} k_1 = 0, \ldots, N_1 - 1 \\ k_2 = 0, \ldots, N_2 - 1 \end{cases}$$

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2)e^{-j\frac{2\pi}{N_1}n_1 k_1}e^{-j\frac{2\pi}{N_2}n_2 k_2} \qquad \begin{cases} k_1 = 0, \ldots, N_1 - 1 \\ k_2 = 0, \ldots, N_2 - 1 \end{cases}$$

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2)e^{j\frac{2\pi}{N_1}n_1 k_1}e^{j\frac{2\pi}{N_2}n_2 k_2} \qquad \begin{cases} n_1 = 0, \ldots, N_1 - 1 \\ n_2 = 0, \ldots, N_2 - 1 \end{cases}$$

The Discrete Fourier Transform (DFT)

The *Discrete Fourier Transform* of $f(x, y)$, for $x = 0, 1, 2...M-1$ and $y = 0,1,2...N-1$, denoted by $F(u, v)$, is given by the equation:
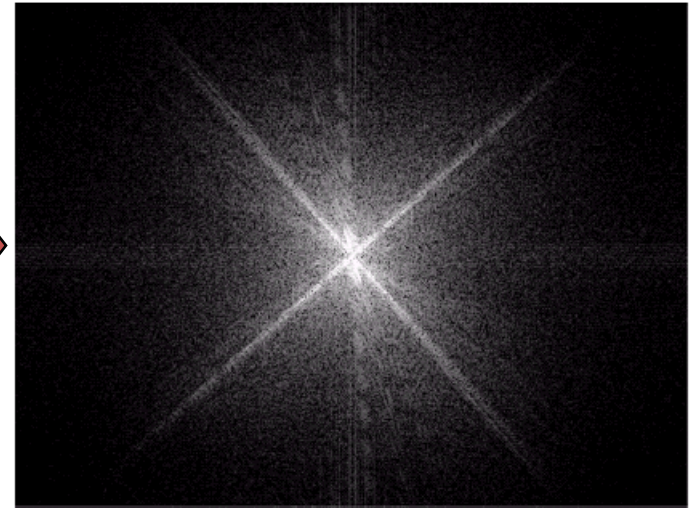
$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

for $u = 0, 1, 2...M-1$ and $v = 0, 1, 2...N-1$.

DFT & Images
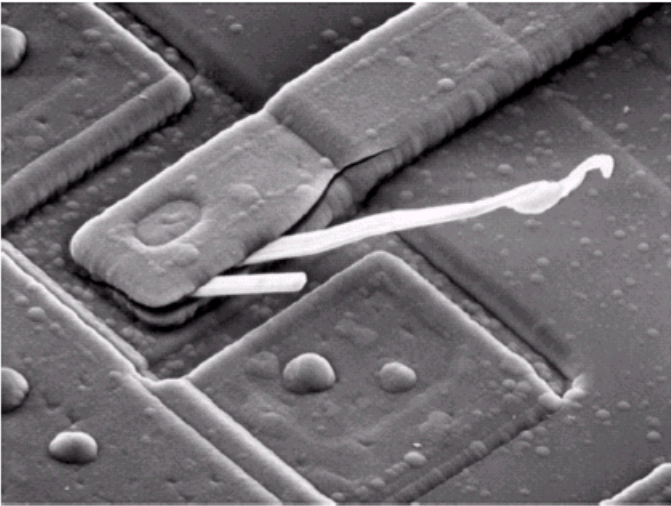
Scanning electron microscope image of an integrated circuit magnified ~2500 times

DFT
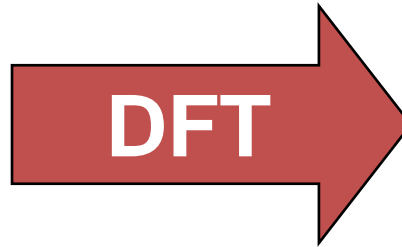
Fourier spectrum of the image

The Inverse DFT

It is really important to note that the Fourier transform is completely **reversible**
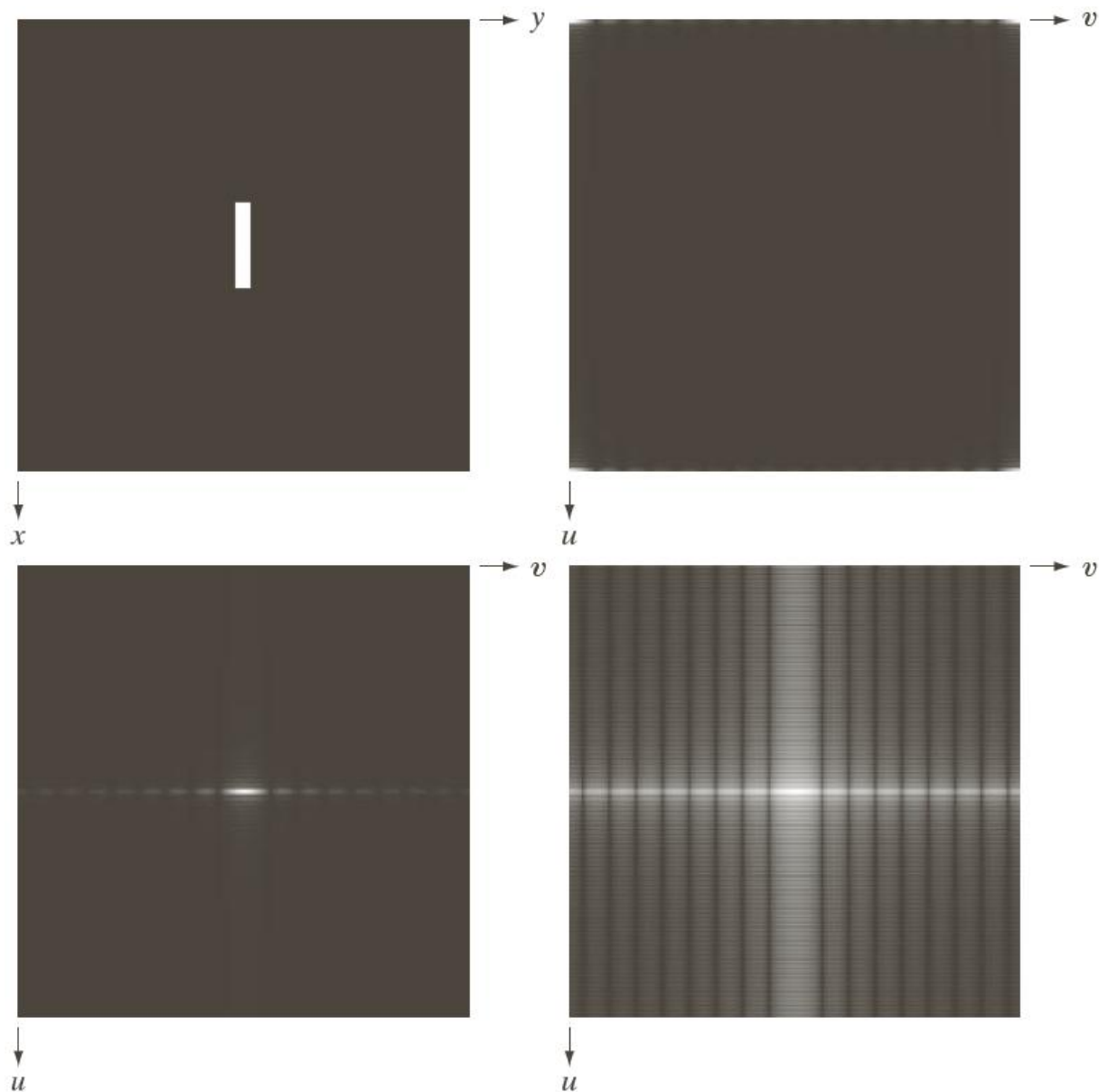
The inverse DFT is given by:

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M+vy/N)}$$

for x = 0, 1, 2...M-1 and y = 0, 1, 2...N-1

Frequencies in Images

a b
c d

FIGURE 4.24
(a) Image.
(b) Spectrum showing bright spots in the four corners.
(c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

Some Basic Frequency Domain Filters

Low Pass Filter
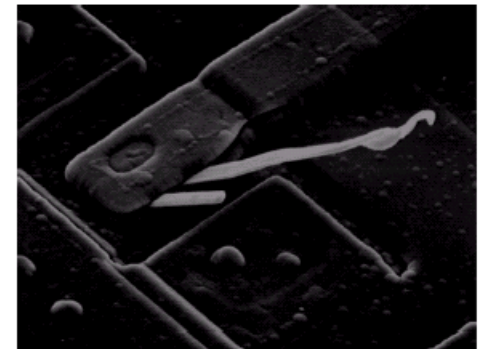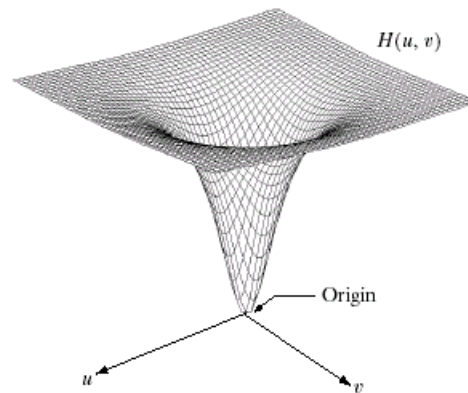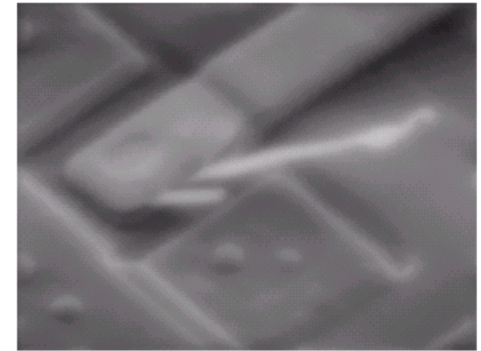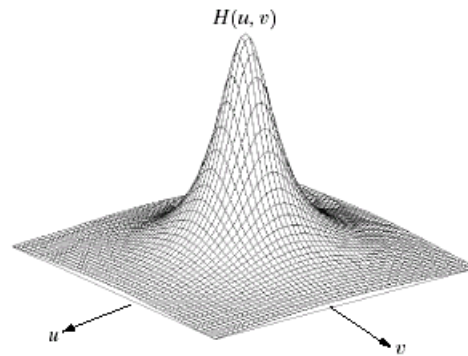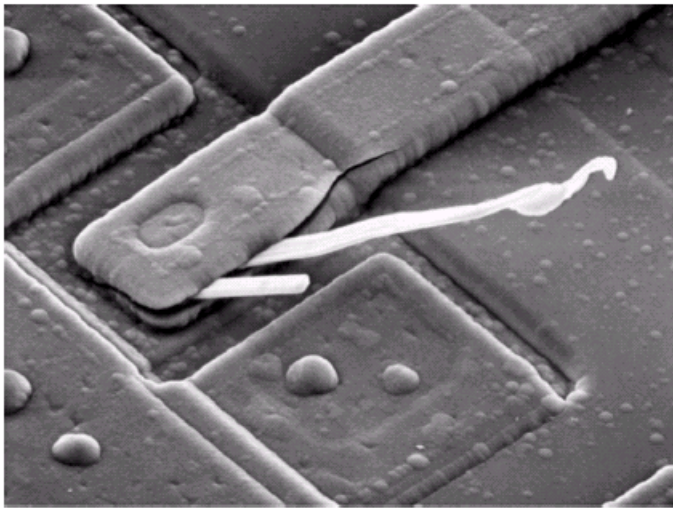
High Pass Filter
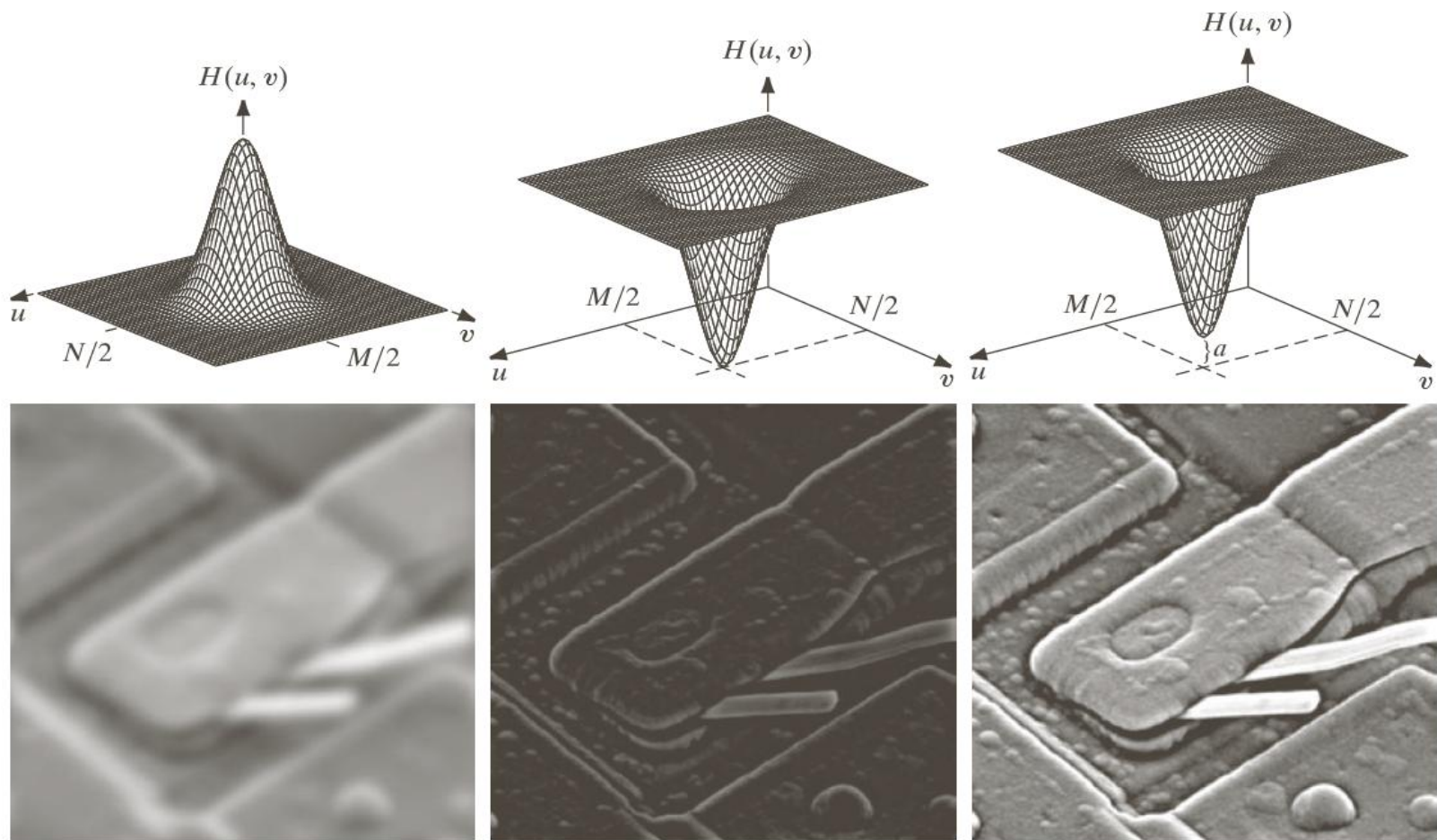
a b c
d e f

FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

Ideal Low Pass Filter

Simply cut off all high frequency components that are a specified distance D_0 from the origin
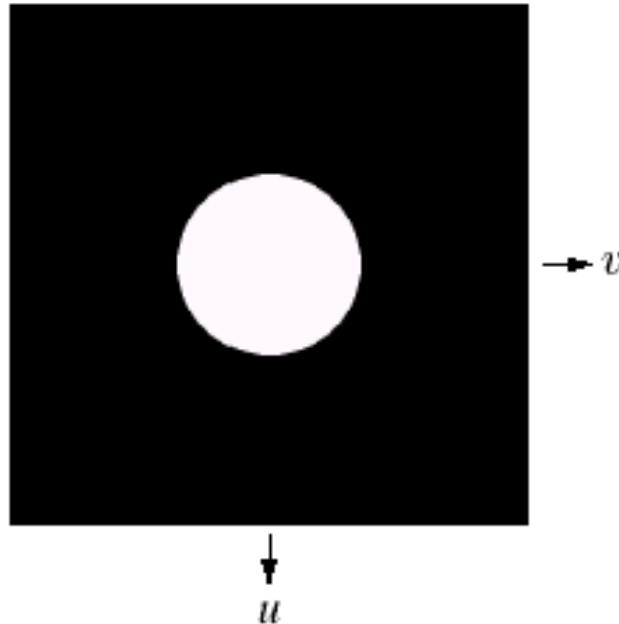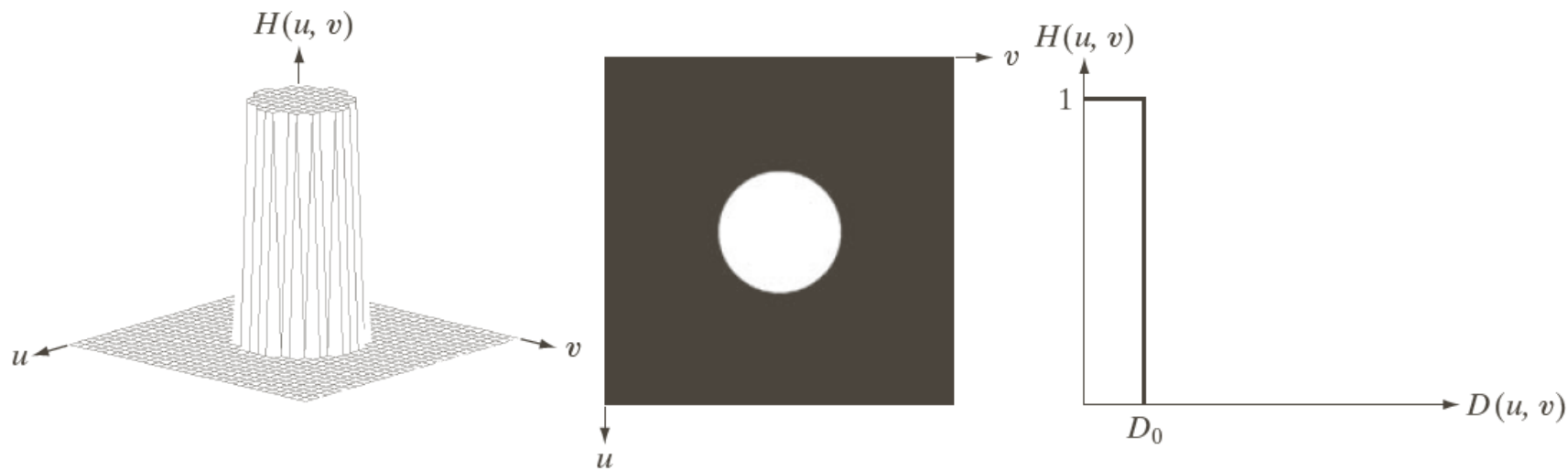
a b c

FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal Low Pass Filter (cont…)
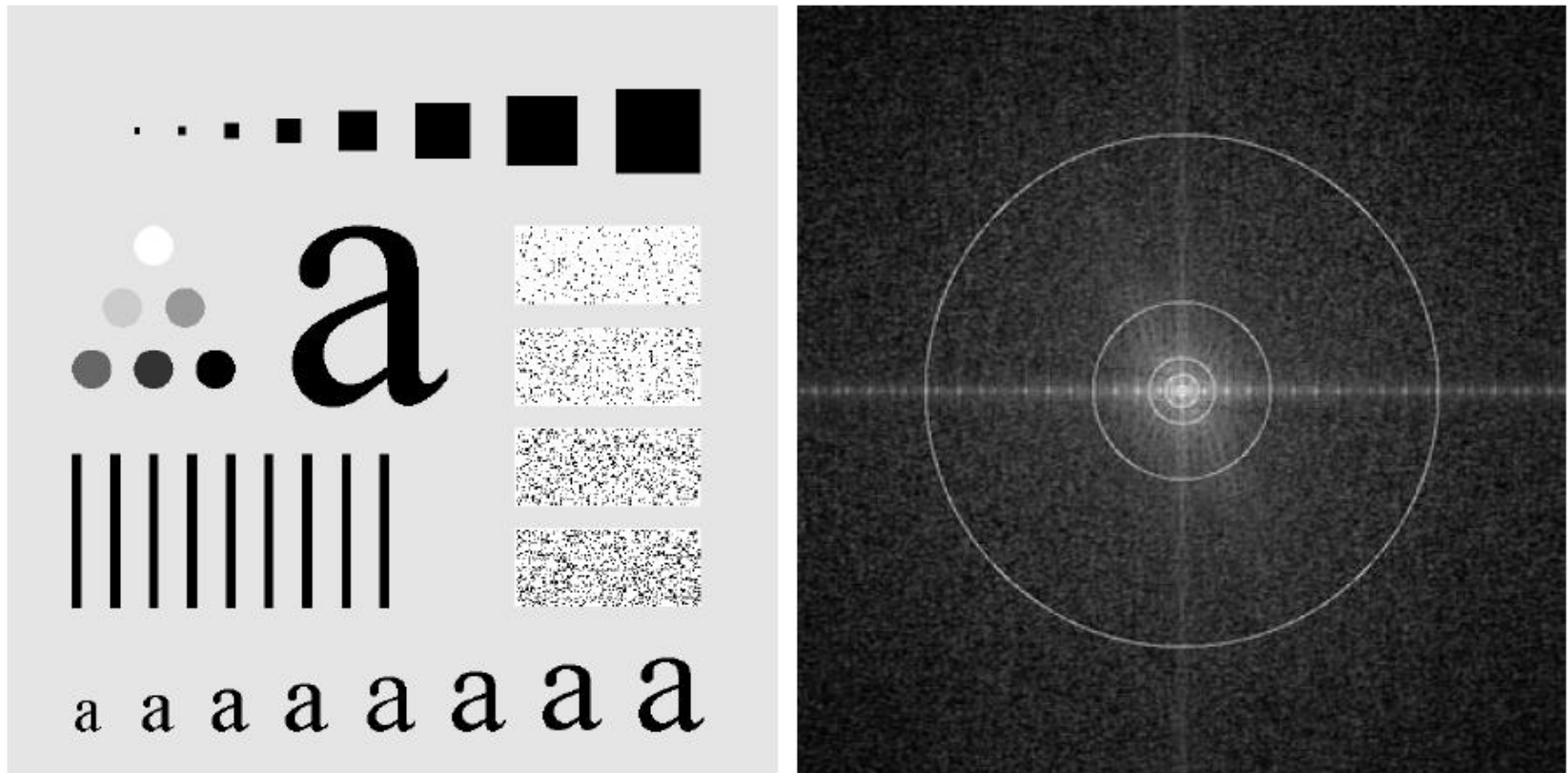
a b

FIGURE 4.41 (a) Test pattern of size 688×688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.

Results of ILPF
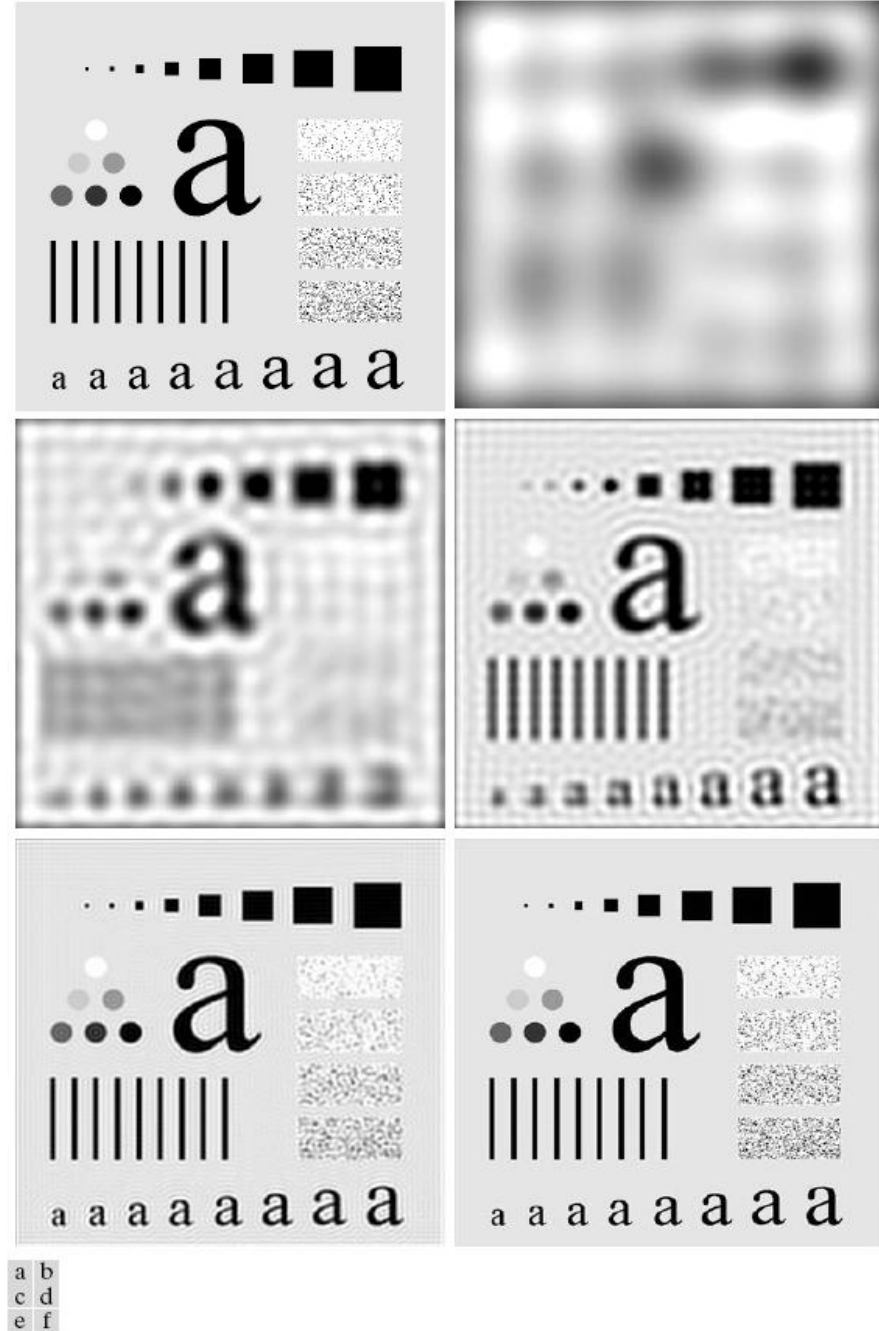
FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

Spatial representation of ILPF
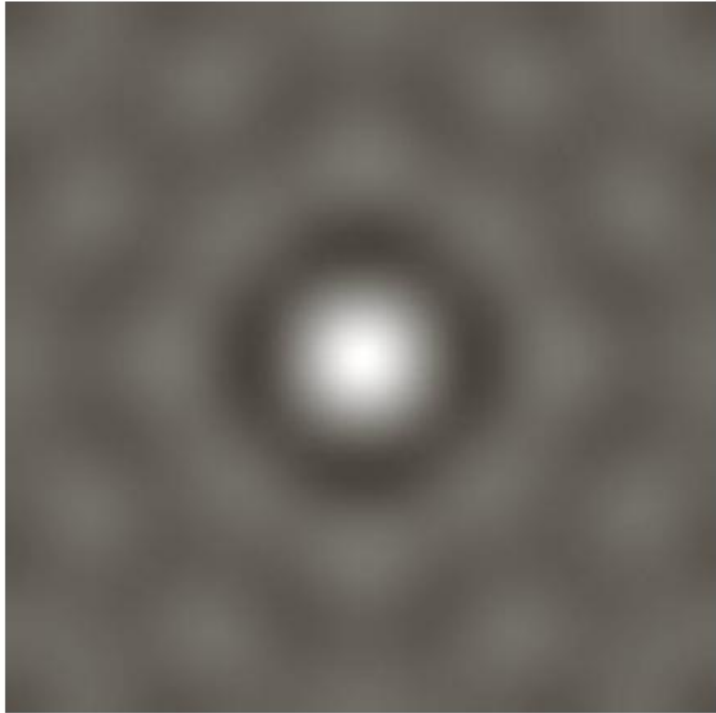
a b

FIGURE 4.43
(a) Representation in the spatial domain of an ILPF of radius 5 and size 1000×1000.
(b) Intensity profile of a horizontal line passing through the center of the image.

Butterworth LPF
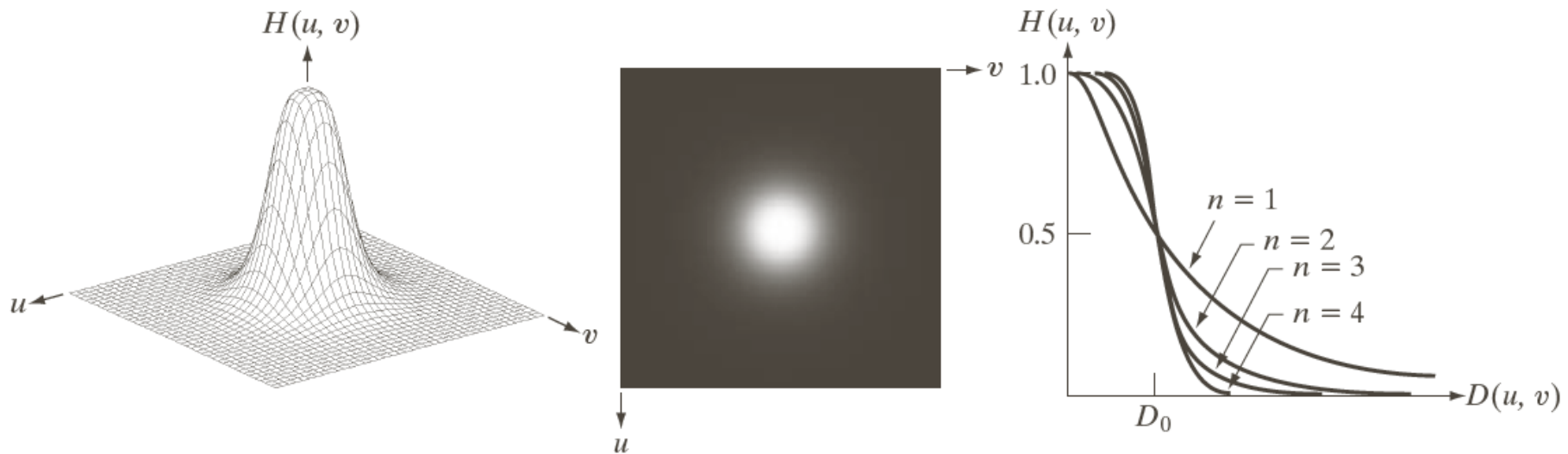
a b c

FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Results of Butterworth LPF

a b
c d
e f

FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

Spatial representation of butterworth LPF

a b c d
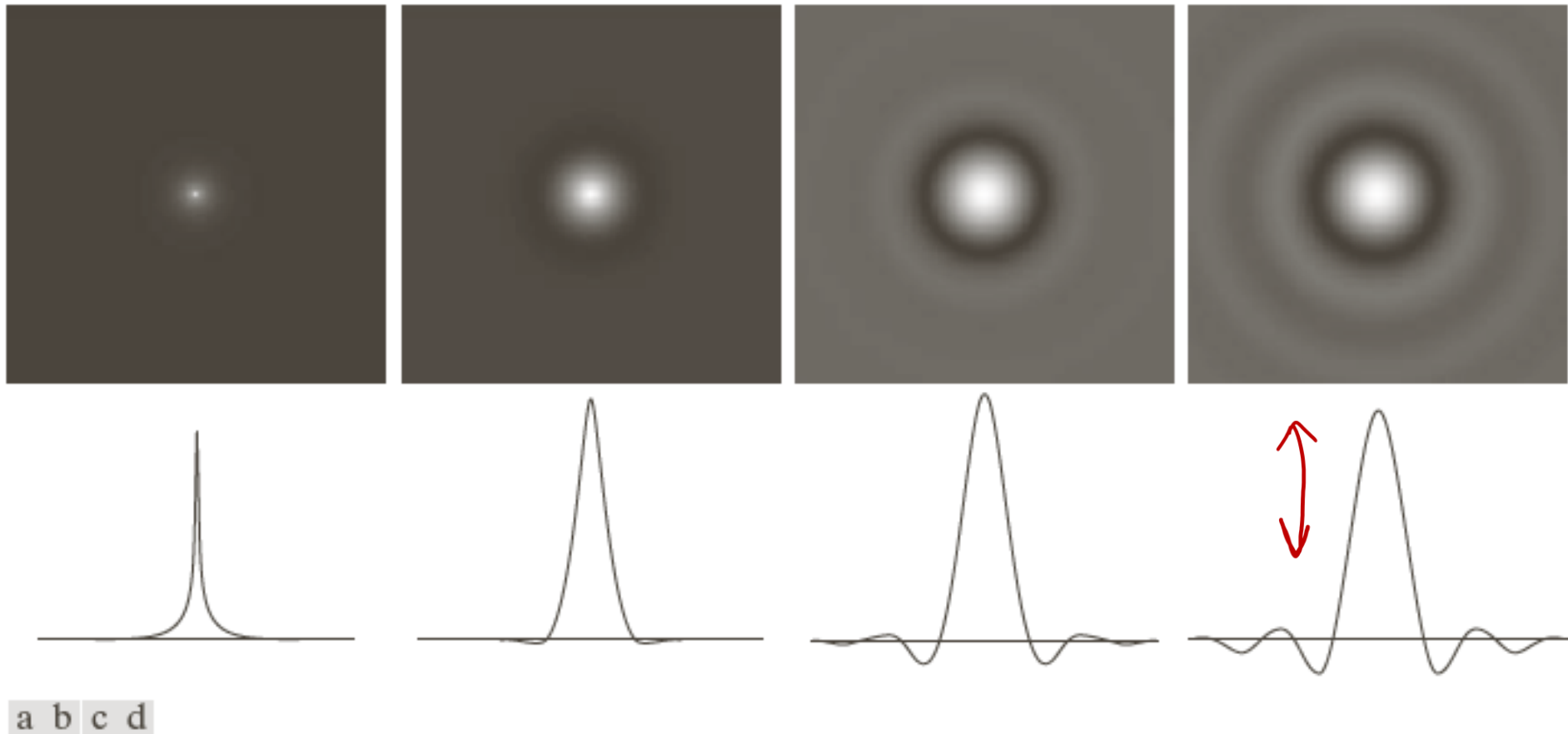
FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000×1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

Gaussian LPF
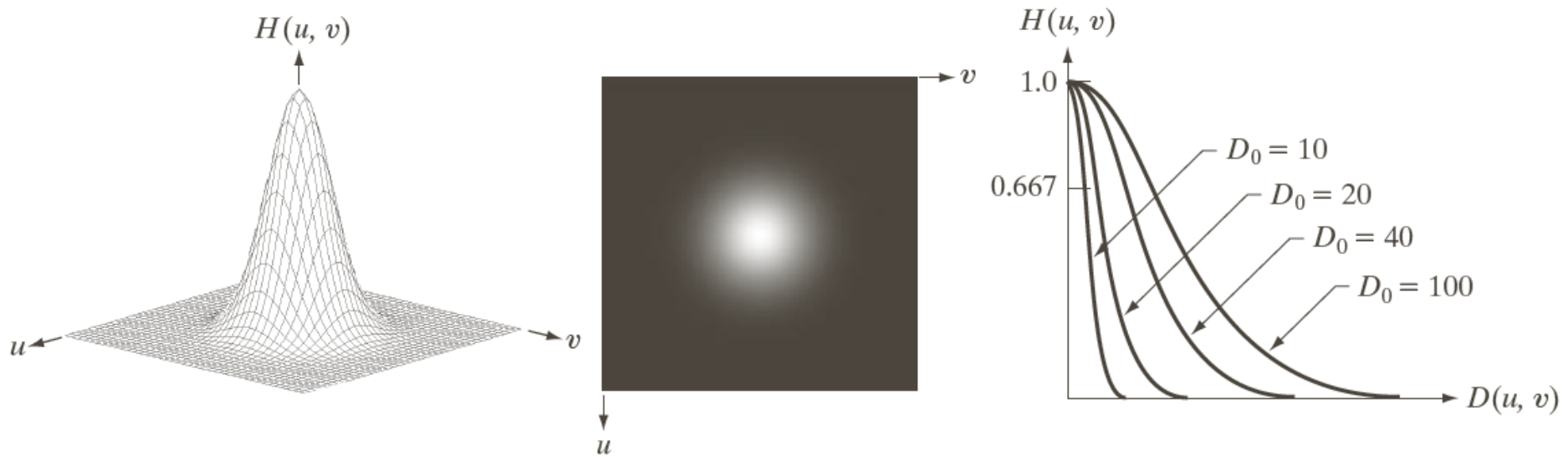
a b c

FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Results of Gaussian LPF

a b
c d
e f

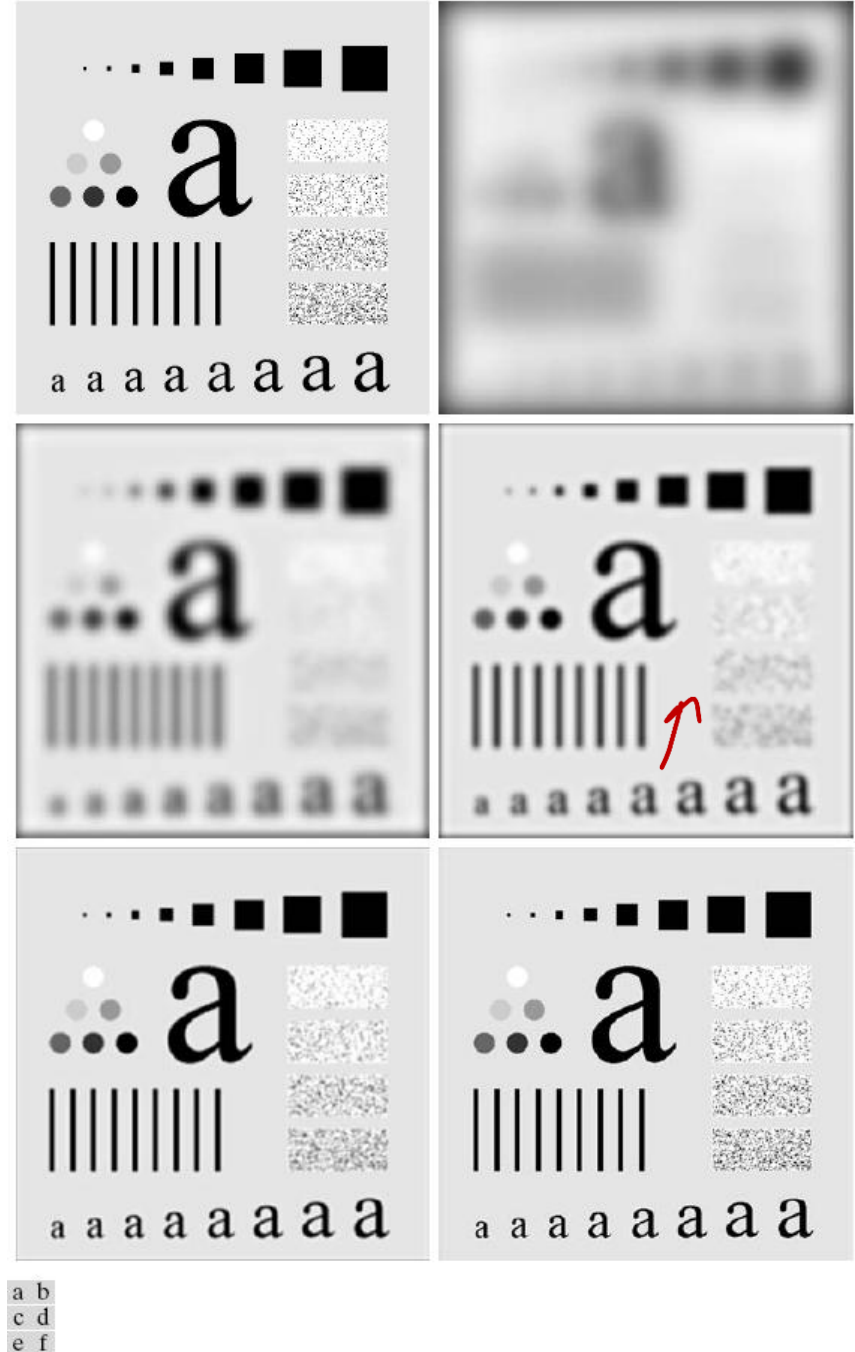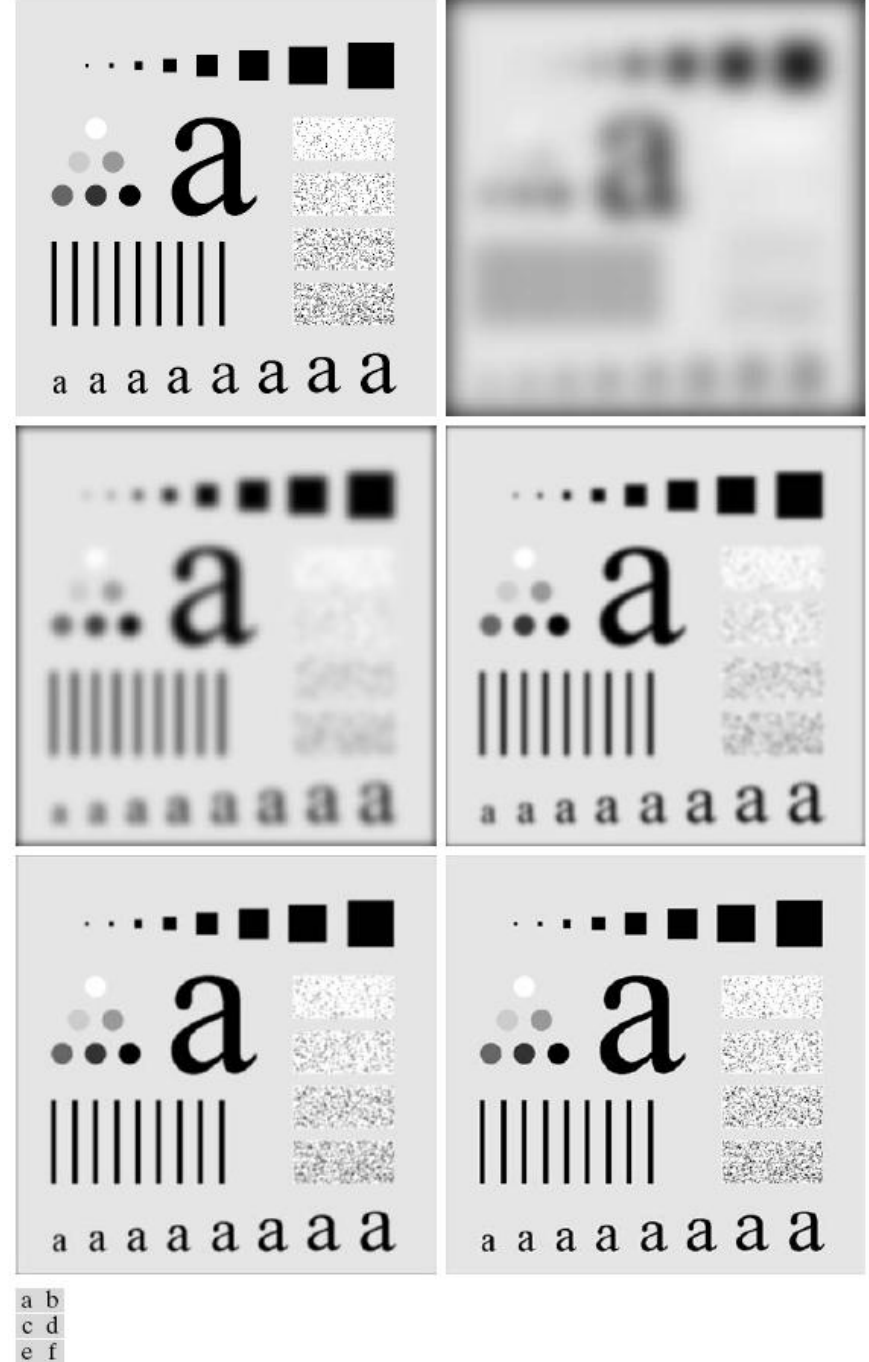**FIGURE 4.48** (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

Applications of LPFs

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.

a b

FIGURE 4.49
(a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

Applications of LPFs

a b c

FIGURE 4.50 (a) Original image (784 × 732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

Applications of LPFs
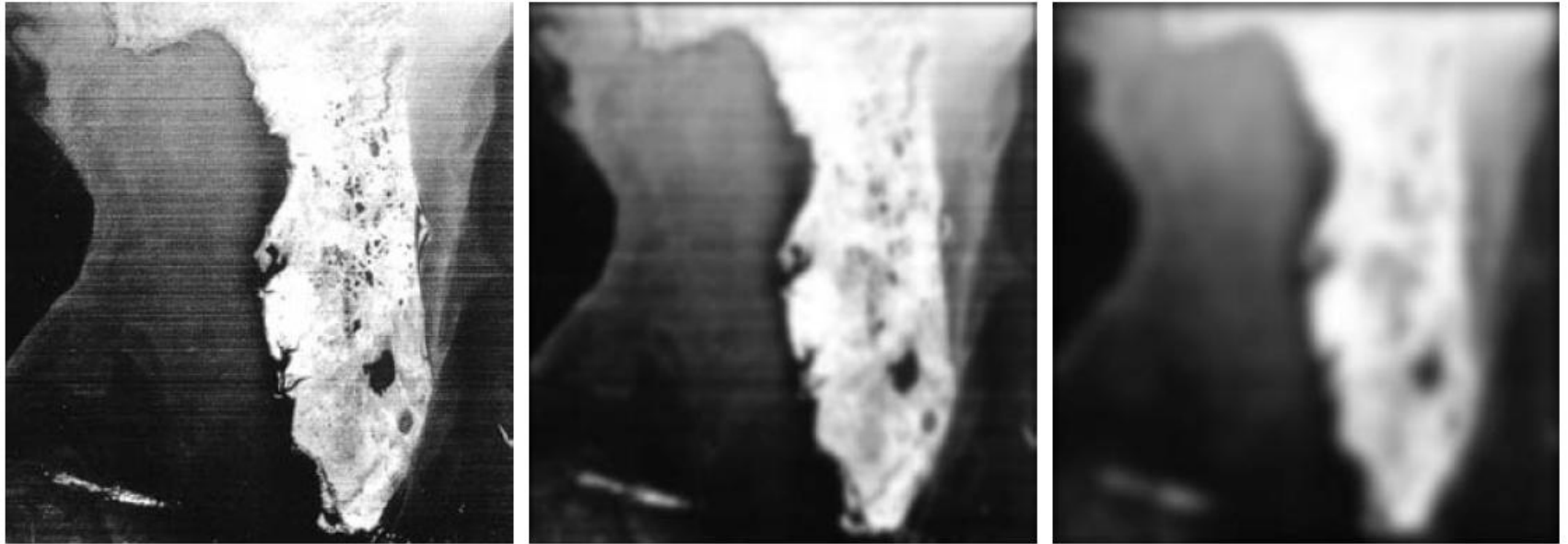
a b c

FIGURE 4.51 (a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)

LPF Summary

TABLE 4.4

Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{if } D(u, v) > D_0 \end{cases}$	$H(u, v) = \dfrac{1}{1 + [D(u, v)/D_0]^{2n}}$	$H(u, v) = e^{-D^2(u,v)/2D_0^2}$

Highpass Filter (HPFs)
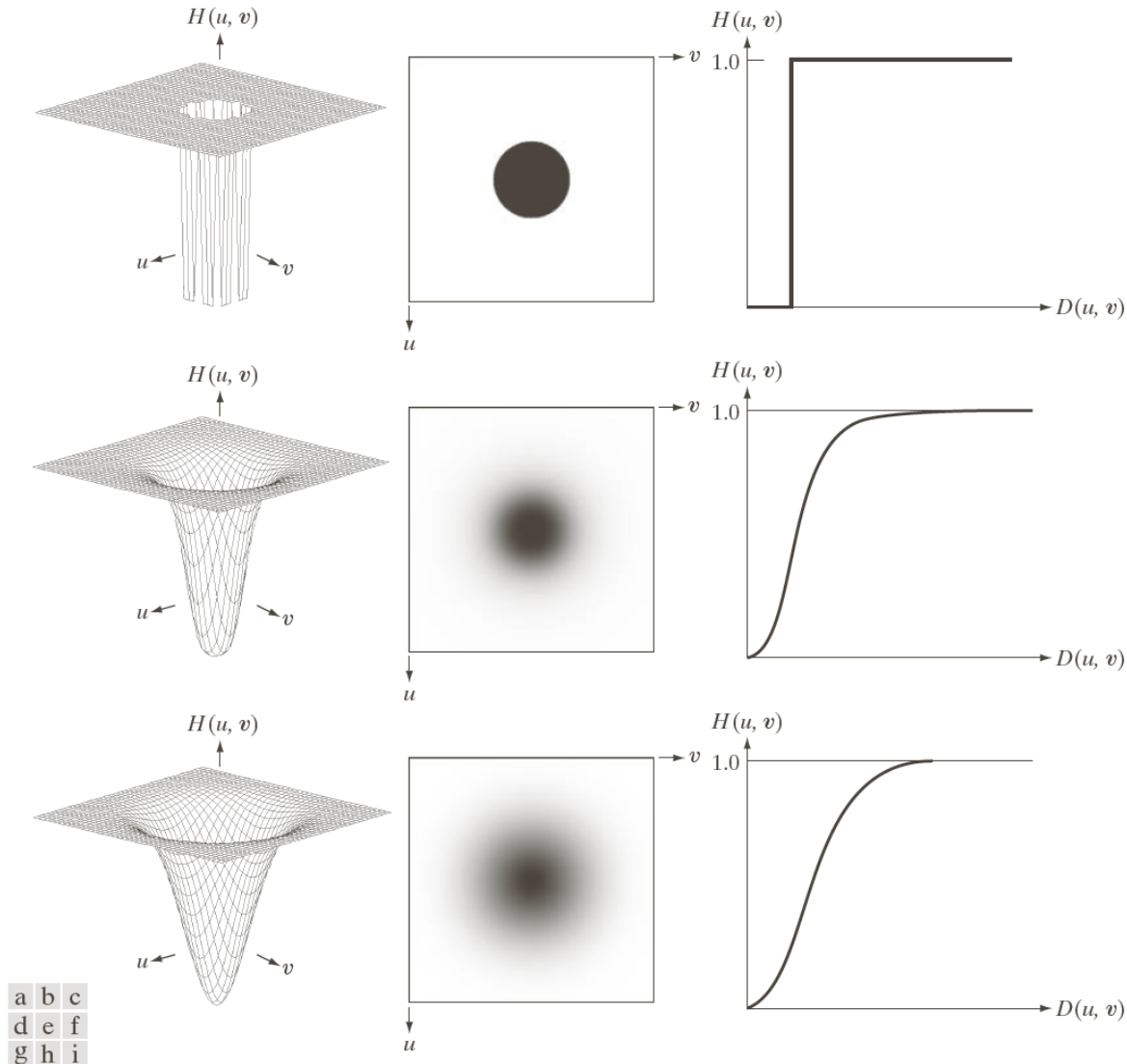
FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Spatial representation of IHPF

a b c
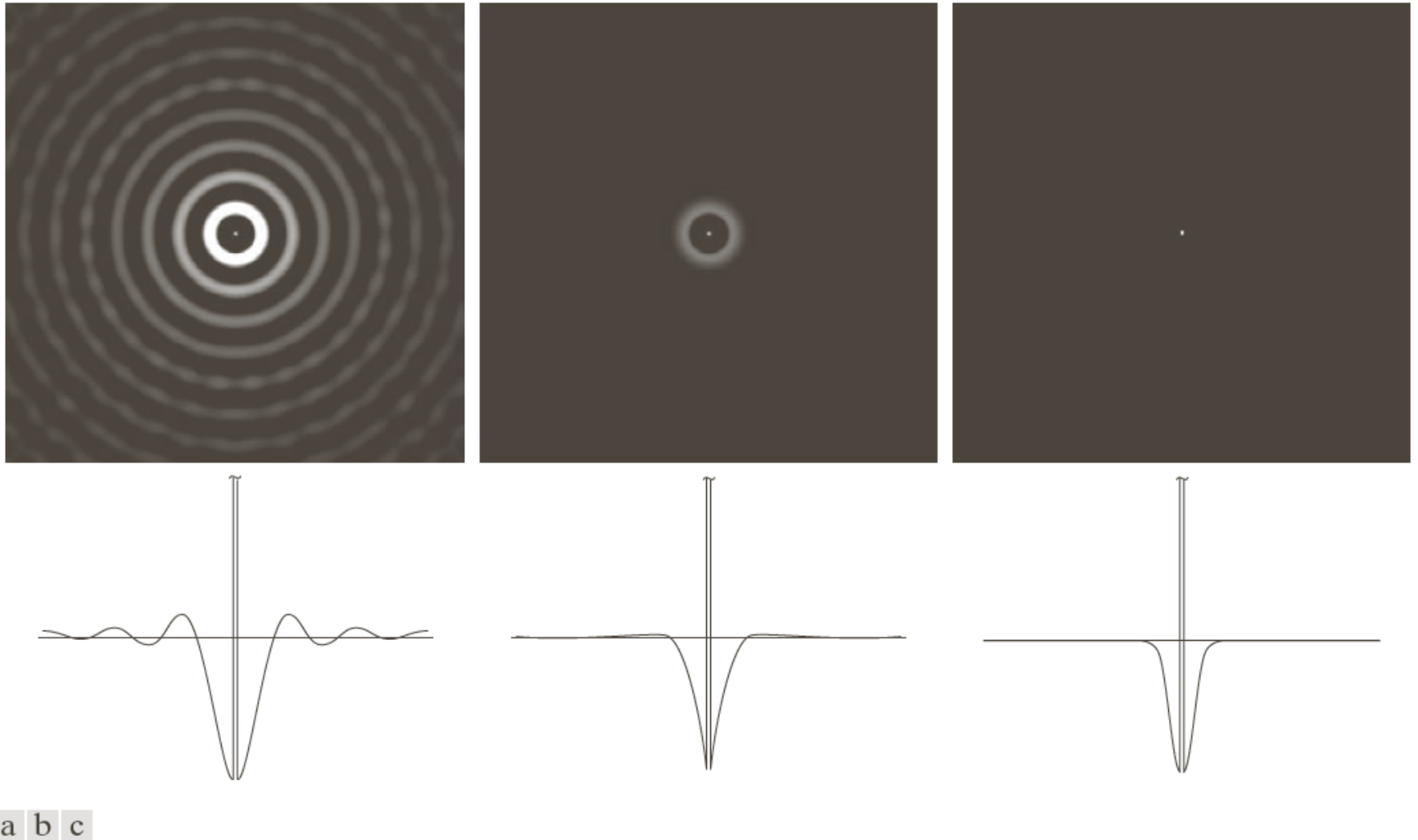
FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

Results of IHPF

a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60,$ and $160.$

Results of BHPF

a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60,$ and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

Results of GHPF

a b c

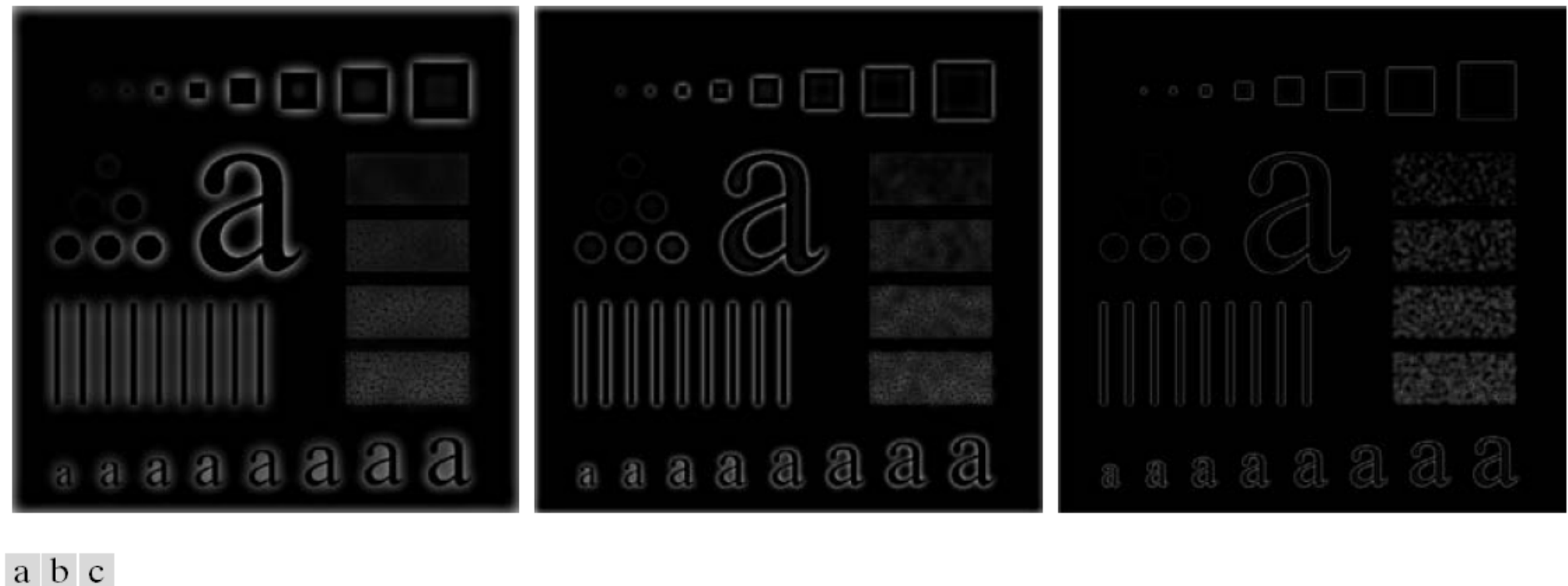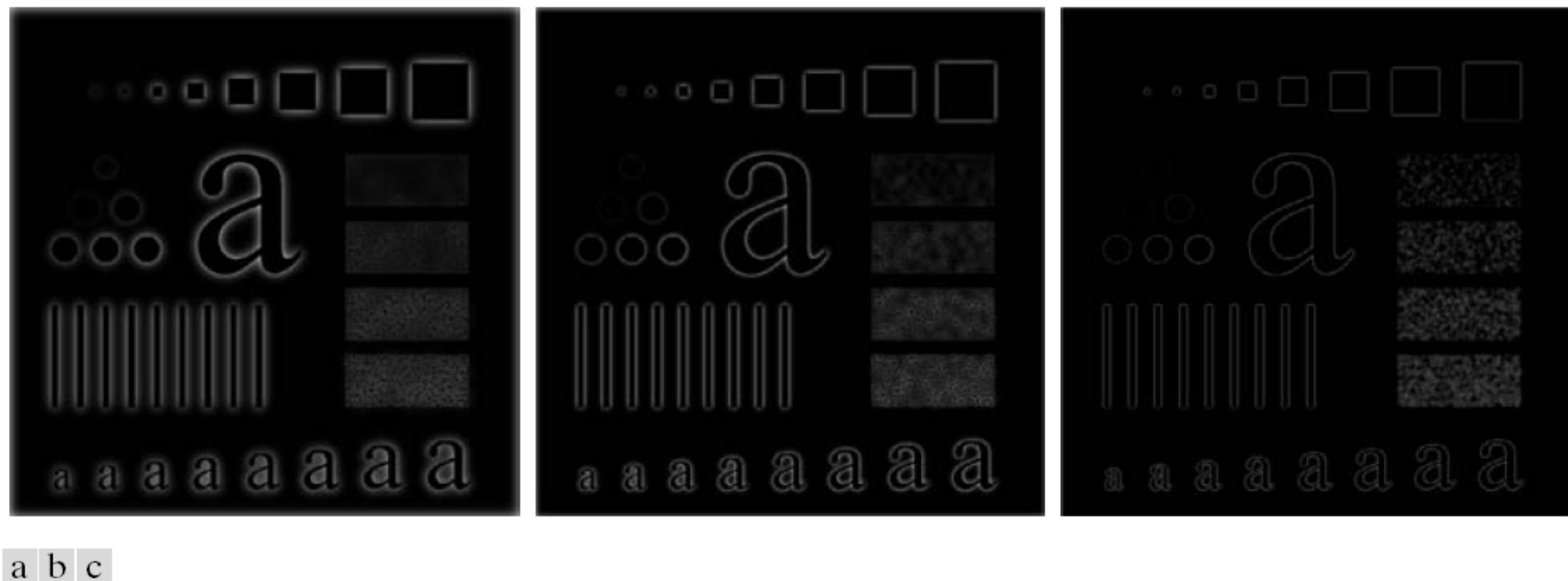**FIGURE 4.56** Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60,$ and $160,$ corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

Applications of HPFs

a b c

FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

Applications of HPFs

a b

FIGURE 4.58
(a) Original, blurry image.
(b) Image enhanced using the Laplacian in the frequency domain. Compare with Fig. 3.38(e).

Summary of HPFs

TABLE 4.5

Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

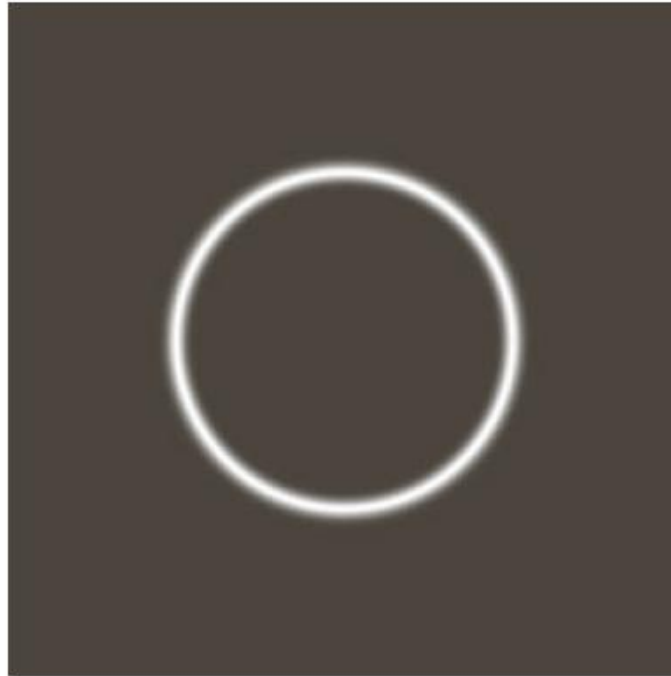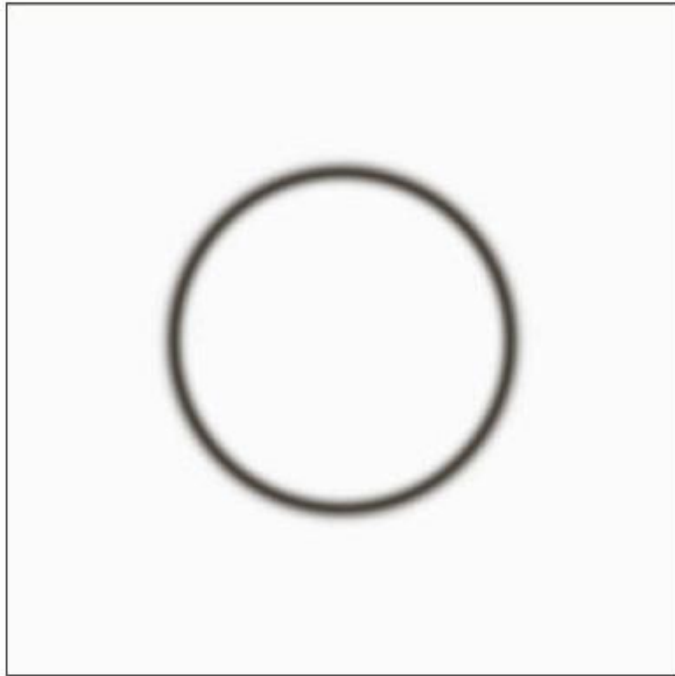| Ideal	Butterworth	Gaussian
$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{if } D(u, v) > D_0 \end{cases}$	$H(u, v) = \dfrac{1}{1 + [D_0/D(u, v)]^{2n}}$	$H(u, v) = 1 - e^{-D^2(u,v)/2D_0^2}$

Bandreject Filters

a b c

FIGURE 5.15 From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.

Bandreject and bandpass Filters

a b

FIGURE 4.63
(a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

Summary of Bandreject Filters

TABLE 4.6

Bandreject filters. W is the width of the band, D is the distance $D(u, v)$ from the center of the filter, D_0 is the cutoff frequency, and n is the order of the Butterworth filter. We show D instead of $D(u, v)$ to simplify the notation in the table.

Ideal	**Butterworth**	**Gaussian**
$H(u, v) = \begin{cases} 0 & \text{if } D_0 - \dfrac{W}{2} \leq D \leq D_0 + \dfrac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u, v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2} \right]^{2n}}$	$H(u, v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW} \right]^2}$

Readings from Book (3rd Edn.)

- Frequency Domain (Chapter-4)
- Frequency Filters (Chapter-4)

Acknowledgements

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- Computer Vision for Computer Graphics, Mark Borg