Differential Geometry: Fibre Bundles and Connections

Course Project for PHY 442: General Relativity

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Abstract

In this project we will develop the theory of fibre bundles, starting from defining what a fibre bundle is from scratch, and then developing further structures and examples, including sections and vector fields. We attempt rigorously define connections on bundles, by defining an Ehresmann connection which splits the tangent space of the total space of the fibre bundle into vertical and horizontal subspaces. Then we consider specific cases of this very general definition of a connection. We look at linear connections, covariant derivatives, as well as developing some theory of group actions to consider principal fibre bundles. This allows us to look at the very important case of principal fibre bundle connections and in turn connections on associated bundles. We then look at how these concepts can come up in physics in the form of wavefunctions and gauge symmetries.

Contents

1 Introduction

The importance of differential geometry cannot be overstated when it comes to general relativity. It is impossible even to define what a vector is in spacetime without considering curves that lie on some manifold. Further, an essential part of general relativity is concerned with parallel transport of vectors. The whole concept of geodesics (and hence motion in general relativity) is built around this. We also often talk of the curvature of space time. The notions of parallel transport and curvature can only be defined elegantly in terms of something called a connection. In this report, we aim to develop further differential geometry language that will allow us to approach connections from a more abstract angle. We will consider fibre bundles, something that can be thought of as a generalization of a product space. In addition, we will develop further structure on these, at the end aiming to develop a rigorous theory of connections on fibre bundles. We will also consider some physical examples to see how we would be able to apply this theory onto physical situations.

2 Fibre Bundles

A fibre bundle can be defined in the following way:

Definition 2.1 (Fibre Bundle [1]). Let M and F be topological spaces. A fibre bundle over M with fibre F is a topological space E, along with a surjective continuous map $\pi : E \to M$, with the property that for each $x \in M$, there exists a neighbourhood U of x such that there exists a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$, that is called the local trivialization over E over U. Additionally we want the following condition to be true, for $x \in U$,

$$\pi(\pi^{-1}(x)) = \pi_1(\phi(\pi^{-1}(x))),$$

where π_1 is the canonical projection map from $U \times F$ to U. This can also be represented as the following diagram commuting:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

Or in simple terms, a fibre bundle is a space that can be locally represented as the product of a base space and a fibre that is identical at every point, along with a map from the bundle to the 'base space'. In this report, we will consider smooth fibre bundles. These are fibre bundles in which M, F and E are smooth manifolds, π is a smooth function and ϕ can be chosen to be a diffeomorphism instead of a homeomorphism. Fibre bundles are sometimes represented diagramatically [2]:

$$\begin{tikzcd}
F \arrow[r] & E \arrow[d, "\pi"] \\
& M
\end{tikzcd}$$

A fibre bundle is trivial, if there is a global trivialization i.e, if E isomorphic to a product space $M \times F$.

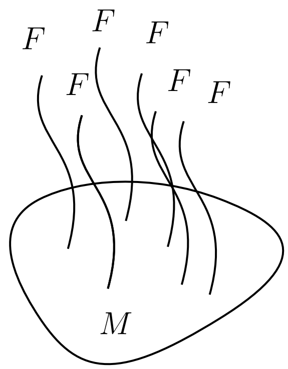

Figure 1: A pictorial representation of a fibre bundle

2.1 Tangent Bundle and Vector Bundles

The prototypical example of a fibre bundle is the tangent bundle to a smooth manifold. This is useful as it allows us to treat the bundle as a whole as a manifold. This is defined as follows:

Definition 2.2 (Tangent Bundle [2])**.** Given a smooth manifold M, the tangent bundle TM is the disjoin union of all the tangent spaces to M, i.e.

$$TM := \coprod_{p \in M} T_P M$$

along with a projection map $\pi : TM \to M$ that maps $X \in TM$ to the unique $p \in M$ such that $X \in T_p M$.

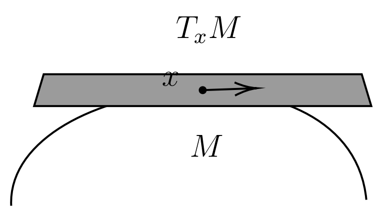

Figure 2: The tangent space of a manifold at point x

We can clearly see that the tangent bundle TM is a fibre bundle whose fibre bundles are isomorphic to $\mathbb{R}^{dim(M)}$. The tangent bundle is a specific example of what is called a vector bundle. A **vector bundle** is a fibre bundle whose fibres are vector spaces. Vector bundles provide us with a rich source of examples for fibre bundles. Some other examples of vector bundles include the cotangent bundle and tensor bundles on a manifold M, which are defined in a similar way to the tangent bundles.

2.2 Mobius Strips and Cylinders

An instructive example, often included as part of a first introduction to fibre bundles is the example of a Mobius strip, viewed as a fibre bundle over the circle S^1 [3]. This can be

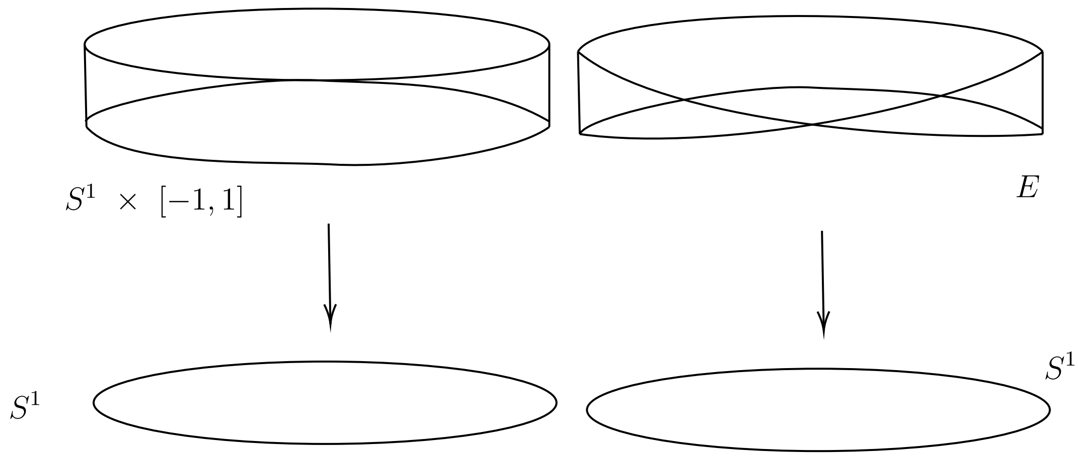

Figure 3: Comparison of a cylinder and a Mobius strip

contrasted with a cylinder, which is another fibre bundle with the same base space, and even the same fibre! However, the cylinder is a simple product of a base space S^1 with a fibre that can be thought of as a line interval $[-1, 1]$. The Mobius strip is a non trivial bundle. This gives us the intuition that fibre bundles can be thought of in some sense as 'twisted' products. An illustration of this can be found in Fig. 3.

2.3 Bundle Morphisms

We can define maps between two fibre bundles that have additional structure than just maps between the underlying sets. These maps are called bundle morphisms and can be defined as follows:

Definition 2.3 (Bundle Morphism [2])**.** Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be bundles and let $u : E \to E'$ and $v : M \to M'$. Then the pair (u, v) is called a bundle morphism if the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{v} & M' \end{array}$$

or in other terms $\pi' \circ u = v \circ \pi$.

Additionally, we can also define isomorphisms of bundles, which we take to be the map that shows two bundles are the 'same' as bundles.

Definition 2.4 (Bundle Isomorphism [2])**.** Two bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ are said to be isomorphic if there exist bundle morphisms (u, v) and (u^{-1}, v^{-1}) if satisfying the following commutative diagram:

$$\begin{array}{ccc} E & \underset{u^{-1}}{\overset{u}{\rightleftarrows}} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \underset{v^{-1}}{\overset{v}{\rightleftarrows}} & M' \end{array}$$

Such a (u, v) is called a bundle isomorphism, and we write $E \xrightarrow{\pi} M \cong_{\text{bdl}} E' \xrightarrow{\pi} M'$.

A bundle $E \xrightarrow{\pi} M$ is **locally isomorphic** to $E' \xrightarrow{\pi'} M'$ if at every point p, there is some neighbourhood of p, such that the restriction of the bundle to that neighbourhood is isomorphic to the bundle $E' \xrightarrow{\pi'} M'$.

2.4 The Frame Bundle

An important example of fibre bundles (and as we'll see later principal fibre bundles) is the frame bundle on a manifold M. A linear frame b at a point $x \in M$ is an ordered set $(b_1, \ldots, b_m)$ of basis vectors for the tangent space T_xM [4]. The total space $\mathbf{B}(M)$ is defined to be the set of all frames at all points in M, along with a projection map $\pi : \mathbf{B}(M) \to M$ that takes a frame to the point x of M to which it is attached.

Now, we can further develop the structure of this bundle by noticing that given some coordinate chart on a neighbourhood U of M, we have a natural coordinate basis and any other basis $(b_1, \ldots, b_m)$ can be expanded as [4]

$$b_i = g_i^j \frac{\partial}{\partial x^j}$$

where the elements g_i^j are the elements of some invertible matrix of rank $dim(M)$. However, this is exactly the definition of the Lie group $GL(dim(M), \mathbb{R})$, and we can define a one to one map between the fibres of $\mathbf{B}(M)$ and $GL(dim(M), \mathbb{R})$. So we can identify the fibre F with a Lie group G, that can act on the frame bundle, as well as on other bundles related to the frame bundle such as the tangent bundle. On the tangent bundle for example this would take the form of coordinate transformations. We will generalize this in the form of group actions, principal fibre bundles and associated bundles, so that we can finally, rigorously define a tensor as something that transforms like a tensor.

3 Smooth Sections of Fibre Bundles

One of the most important ideas in fibre bundle theory with regards to physics is the idea of a smooth 'cross-section' of a fibre bundle [4]. In some sense this is the assignment of a member of a fibre bundle to each point of the manifold. For example, a vector field could be thought of as assigning a tangent vector to each point of the manifold. However, such a definition is imprecise and can be vague on matters such as how do you make sure that this is assignment is smooth. So we define a section of a fibre bundle as follows.

Definition 3.1 (Section of a bundle [4])**.** A section of a bundle $E \xrightarrow{\pi} M$ is a map $s : M \to E$ such that

$$\pi \circ s = id_M$$

or in other words the image of each point $x \in M$ lies in the fibre attached to the same point in M.

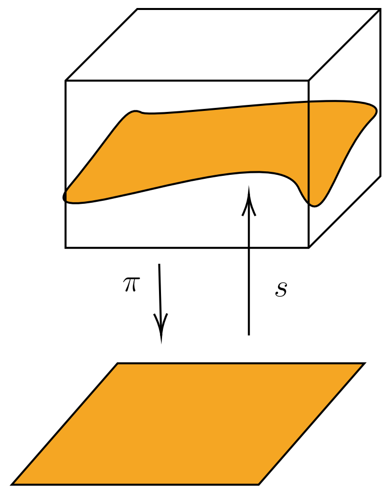

Figure 4: Pictorial representation of a section of a fibre bundle

Notice that in this definition s is a map from the manifold to the total space rather than the fibre. For a product bundle or a trivial bundle, these two definitions coincide, but for fibre bundles in general, we can have much more interesting behaviour.

This is also sometimes called a **global section** of the bundle. The set of smooth sections of a bundle are often denoted as $\Gamma(M, E)$ or even simply $\Gamma(E)$. Not all fibre bundles will allow a global section. If we restrict ourselves to $U \subset M$, we can consider a **local section**, defined only on U [3], which can be denoted as $\Gamma(U, E)$. Typically we will consider **smooth sections** of a bundle where s is restricted to be a smooth map.

3.1 Vector and Tensor Fields

Using the definition of a smooth section, as well as of the tangent bundle we can easily define vector fields on a manifold as follows.

Definition 3.2 (Vector field [2])**.** Let M be a smooth manifold, and let $TM \xrightarrow{\pi} M$ be its tangent bundle. A vector field on M is a smooth section s of the tangent bundle.

$$\begin{tikzcd} TM \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "s"] \end{tikzcd}$$

As noted previous the set of all vector fields can be represented as $\Gamma(TM)$. Additionally, an equivalent way to look vector fields is as derivations of smooth functions on M ($\mathcal{C}^\infty(M)$). That is, an $\mathbb{R}$-linear map that satisfies the Leibniz rule (for $f, g \in \mathcal{C}^\infty(M)$) [2]

$$s(fg) = gs(f) + fs(g).$$

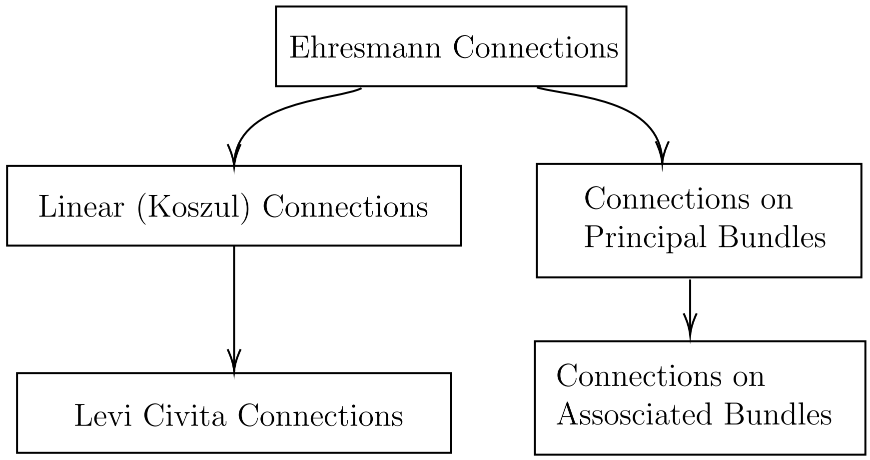

Figure 5: Schematic Diagram of Connections on Fibre Bundles

4 Connections on Fibre Bundles

4.1 Ehresmann Connections

The most general for of connections that we will consider are Ehresmann connections. As we can see in Fig. 5, connections on principal bundles and linear connections, including the Levi Civita connection are special cases of this. This type of connection can defined in terms of a vertical bundle VE, which is a sub-bundle of the tangent bundle of the total space TE. However, to define this rigorously, we need to introduce the concept of special maps called push-forwards and pull-backs that allow us take tangent vectors from one manifold to tangent vectors of another manifold.

4.1.1 Push-forward and pull-back

Definition 4.1 (Push-forward [2])**.** Let M and N be smooth manifolds, and $\phi : M \to N$ be a smooth map. The push-forward of ϕ at $p \in M$ is the linear map

$$\begin{aligned}(\phi_*)_p :& T_pM \xrightarrow{\sim} T_{\phi(p)}N \\ & X \to (\phi_*)_p(X)(f) \coloneqq X(f \circ \phi)\end{aligned}$$

where $f \in \mathcal{C}^\infty(N)$ is an arbitrary test function on which a tangent vector in $T_{\phi(p)N}$ can act on.

In some sense, the push forward can be thought of as the derivative of the map ϕ. It takes a tangent vector from M to a tangent vector on N.

If $\gamma : \mathbb{R} \to M$ is a smooth curve on M and ϕ is smooth, then $\phi \circ \gamma : \mathcal{R} \to N$ is also a smooth curve. We can then show that if we push forward the tangent vector $X_{\gamma,p}$ of the curve γ at point p, we will get a tangent vector to $\phi \circ \gamma$ at $\phi(p)$ or in more succinctly $X_{\phi\circ\gamma,\phi(p)}$.

Proof. If we can show this for an arbitrary function $f \in \mathcal{C}^\infty(N)$, then we can prove the two are equal:

$$\begin{aligned}(\phi_*)_p(X_{\gamma,p})(f) &= X_{\gamma,p}(f\circ\phi)\\ &= \frac{d(f\circ\phi)(\gamma(\lambda))}{d\lambda}\\ &= \frac{d(f(\phi\circ\gamma(\lambda)))}{d\lambda}\\ &= X_{\phi\circ\gamma,\phi(p)}\end{aligned}$$

□

Notice that, the push-forward induces a bundle map from the tangent bundle of M to the tangent bundle of N, which is called the tangent map and often denoted with ϕ_* and satisfies the following commutative diagram:

$$\begin{array}{ccc} TM & \xrightarrow{\phi_*} & TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{\phi} & N \end{array}$$

It is also useful to define the related notion of a pull-back of a smooth map.

Definition 4.2 (Pull-back [2]). Let $\phi : M \to N$ be a smooth map between smooth manifolds. The push-forward of ϕ at $p \in M$ is the linear map

$$\begin{aligned}(\phi^*)_p : &T^*_{\phi(p)}N \xrightarrow{\sim} T^*_pM\\ &\omega \to (\phi^*)_p(\omega)\end{aligned}$$

such that

$$(\phi^*)_p(\omega(X)) = \omega((\phi_*)_p(X)).$$

4.1.2 Vertical and Horizontal Subspaces

Now, we have the tools to define the vertical and horizontal subspaces of as smooth fibre bundle $E \xrightarrow{\pi} M$.

Definition 4.3 (Vertical Bundle [5]). Let $E \xrightarrow{\pi} M$ be a smooth fibre bundle. The vertical bundle is the kernel of the tangent map π_*, or compactly

$$VE := \ker(\pi_*).$$

This vertical bundle is a sub-bundle of the tangent bundle TE [5], and at each point $e \in E$, the fibre V_eE of the vertical bundle is a subspace of the tangents space T_eE. This leads directly to the main definition we have been working towards:

Definition 4.4 (Ehresmann Connection [5]). An Ehresmann connection on a fibre bundle $E \xrightarrow{\pi} M$ is a choice of a complementary sub-bundle HE to VE, such that the tangent bundle can be written as a direct sum $TE = HE \oplus VE$. That is to say that at each point $e \in E$, we can write the tangent space $T_eE = V_eE \oplus H_eE$.

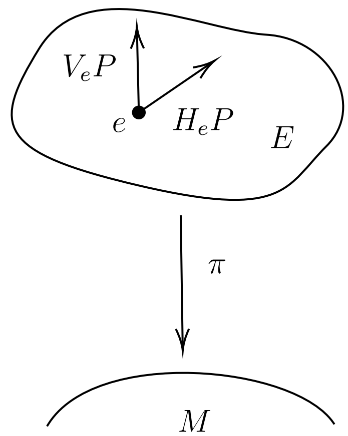

Figure 6: Pictorial representation of vertical and horizontal subspaces

Notice that the vertical bundle is uniquely defined in terms of the projection π but there can be many choices of the horizontal bundle HE.

We can also give an equivalent definition in terms of a connection one-form. However, we have to be careful as this one-form is not a one-form in the traditional sense of the word [5]. However it is vector-valued one form Φ (or more simply put it maps elements from TE to TE) such that $\Phi \circ \Phi = \Phi$ and $\Phi(\eta) = \eta$ for all $\eta \in TE$. So it can be considered a projection from TE to VE and the kernel of the specific map would be our horizontal subspace HE. So a choice of HE is equivalent to a choice of the connection one-form [5].

From here, notions such as parallel transport and curvature can be defined on the fibre bundle. However, these connections are not necessarily compatible with the symmetries of our physical system and for this we need to introduce further concepts such as principal bundles and connections on them.

4.2 Linear Connections on a Vector Bundle

One very special case of an Ehresmann connection is a linear connection on a vector bundle. Suppose we have a smooth vector bundle E. Then a linear connection can be defined in the following way

Definition 4.5 (Linear (Koszul) connections [6])**.** An Ehresmann connection on a smooth vector bundle E over a manifold M is said to be linear if H_e depends linearly on $e \in E_x$ for each $x \in M$. More precisely, consider a map $m_\lambda : E \to E$, such that fibre-wise, every element is multiplied by $\lambda \in \mathbb{R}$. Then the push-forward $(m_\lambda)_*$ will map from TE to TE. The connection is linear if $H_{\lambda e}E = (m_\lambda)_* H_e E$.

Additionally, it can be shown that the vertical subspace V_vE can be identified with $E_{\pi(v)}$ (the fibre at $\pi(v)$) , since for a vector space, the tangent space can be identified with the vector space itself. So, we can view the map Φ as a map $\Phi : TE \to E$ instead [6].

We can define parallel transport using something called a horizontal lift. This can be used to define a covariant derivative ∇ on TM, and conversely a covariant derivative specifies a linear Ehresmann connection by defining the horizontal bundle HE.

The covariant derivative of a section $s : M \to E$ along a tangent vector $X \in T_xM$ can be thought of as a map $\nabla_X : \Gamma(E) \to E_x$ such that

$$\nabla_X s = \nabla s(X) = \Phi \circ s_*(X).$$

Since at every point $\nabla s(x)X \in E_x$ this implies that $\nabla s(x) : T_xM \to E_x$, which tells us (along with the linearity) that $\nabla s \in \Gamma(\mathrm{Hom}(TM, E))$. Additionally $\mathrm{Hom}(TM, E)$ can be identified with $T^*M \otimes E$.

It can be shown that this covariant derivative satisfies the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s.$$

4.2.1 Connection Coefficients

Now, we can consider the local expression of the connection, something that we are more familiar with. Let U be an open set around which we consider a local trivialization $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ (since E is a vector bundle). We can identify, within U, sections of the vector bundle s, with maps $U \to \mathbb{R}^m$. This gives us a natural way to define a covariant derivative that sends s to ds where $ds(x) : T_xM \to \mathbb{R}^m$.

According to Ref. [6], any other connection ∇ can only differ by a bilinear map $\Gamma_\phi : TM|_U \times \mathbb{R}^m \to \mathbb{R}^m$. So, any connection can be written as

$$\nabla s(x)X = ds(x)X + \Gamma_\phi(X, s(x))$$

Now, consider coordinates $(x^1, \ldots, x^n)$ on U. This gives us a basis $(\partial_1, \ldots, \partial_n)$ of TM. Additionally, the trivialization ϕ gives us a local basis $(e_{(1)}, \ldots, e_{(m)})$. Then the connection coefficients $\Gamma^a_{ib} : U \to \mathbb{R}^m$ are defined by

$$\Gamma_\phi(\partial_i, e_{(b)}) = \Gamma^a_{ib} e_{(a)}$$

4.2.2 Levi Civita Connections

A **Levi Civita connection** is a special type of linear connection on the tangent bundle TM of a manifold, that is torsion free and preserves the metric, called the Levi Civita connection. The fact that every pseudo Riemannian manifold has a unique Levi Civita connection is called the **fundamental theorem of (pseudo) Reimannian geometry** [7]. The connection coefficients of the Levi Civita connection are known as Christoffel Symbols.

4.3 Principal Fibre Bundles

A fibre bundle may also be associated with a structure group G, which can be thought of as the set of homeomorphisms of the fibre F onto itself. This is more precisely defined in terms of actions of a Lie group.

Given a manifold M and a Lie group $(G, \bullet)$, we can define the action of lie group on the manifold.

Definition 4.6 (Left G-action [2])**.** Let $(G, \bullet)$ be a Lie group and let M be a smooth manifold. A smooth map

$$\begin{aligned}\triangleright :& G \times M \to M \\ &(g, p) \to g \triangleright p\end{aligned}$$

satisfying

i) $\forall p \in M : e \triangleright p = p$;

ii) $\forall g_1, g_2 \in G : \forall p \in M : (g_1 \bullet g_2) \triangleright p = g_1 \triangleright (g_2 \triangleright p)$,

is called a left Lie group action, or left G-action, on M.

Definition 4.7 (Right G-action [2])**.** Similarly, a right G-action on M is a smooth map

$$\begin{aligned}\triangleleft :& M \times G \to M \\ &(p, g) \to p \triangleleft g\end{aligned}$$

satisfying

i) $\forall p \in M : p \triangleleft e = p$;

ii) $\forall g_1, g_2 \in G : \forall p \in M : p \triangleleft (g_1 \bullet g_2) = (p \triangleleft g_1) \triangleleft g_2$.

To consider the different types of group actions we can, we have to define a few specific subsets and subgroups of the group G and manifold M.

The *orbit* of a point in the manifold is all of the points that you can get to by applying group actions. Or in more precise terms:

Definition 4.8 (Orbit [2])**.** Let $\triangleright : G \times M \to M$ be a left G-action. For each $p \in M$, we define the orbit of p as the set

$$G_p \coloneqq \{q \in M \mid \exists g \in G : q = g \triangleright p\}$$

We can define a relation between elements of the manifold $\equiv$ such that $p \sim q$ if an only if $\exists g \in G : q = g \triangleright p$. Or if two elements are part of the same orbit. It can be shown that this is an equivalence relation [2]. So, the set M can be split into quotient group $M/\sim$ with respect to the equivalence relation $\sim$. This is often written as M/G and called the orbit space of M. More specifically,

$$M/G \coloneqq M/\sim \; = \{G_p \mid p \in M\}.$$

We also define a subgroup of G called the stabilizer of an element $p \in M$. This is basically any group element whose action leaves p the same.

Definition 4.9 (Stabilizer [2])**.** Let $\triangleright : G \times M \to M$ be a left G-action. The stabilizer of $p \in M$ is

$$S_p \coloneqq \{g \in G \mid g \triangleright p = p\}.$$

The intersection of S_p for all $p \in M$ is known as the kernel of a G-action.

Definition 4.10 (Free and Transitive Actions [2])**.** A left G-action $\triangleright : G \times M \to M$ is said to be

i) free if for all $p \in M$, we have $S_p = \{e\}$;

ii) transitive if for all $p, q \in M$, there exists $g \in G$ such that $q = g \triangleright p$.

These notions can also similarly be defined for right actions of G on the manifold M.

We can now consider what a principal G-bundle is. Roughly speaking, this is a fibre bundle, equipped with a $G - action$, whose fibres are the same as the lie group G. However, we define this in a different way, as they need to be related in a specific way.

Definition 4.11 (G-bundle [4])**.** A bundle $E \xrightarrow{\pi} M$ is a G-bundle if E is a right G-space and if $E \xrightarrow{\pi} M$ is isomorphic to the bundle $E \xrightarrow{\rho} E/G$, where ρ is the usual projection map that takes you from an element of $e \in E$ to the equivalence class $[e]$.

$$\begin{array}{ccc} E & \xrightarrow{u} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ M & \xrightarrow{v} & E/G \end{array}$$

Notice that the fibres of the bundle will be the same as $\rho^{-1}([e]) = G_p$, the orbits of the G-action. In general, these are **not** homeomorphic to each other so this is not a fibre bundle but a more general object that is just called a bundle. However, if the G-action is free, the orbits will be homeomorphic (and diffeomorphic) to G, so we have a fibre bundle with fibres that 'look' like the Lie group G.

Definition 4.12 (Principal G-bundle [4])**.** A G-bundle $E \xrightarrow{\pi} M$ is a principal fibre budle if G acts freely on E. Then G is called the **structure group** of the bundle. Since all of the orbits are now homeomorphic, we have a fibre bundle with fibre G.

The frame bundle $\mathbf{B}(M)$ is an example of a principal G-bundle with structure group $G = GL(m, \mathbb{R})$.

Isomorphisms of principal bundle are defined in terms of **principal bundle morphisms**, which are bundle morphisms (u, v) such that the map u is compatible with G in a special way that is called **G-equivariant**

$$u(p \triangleleft g) = u(p) \blacktriangleleft g, \qquad (\forall p \in P, \forall g \in G)$$

where $\blacktriangleleft$ is the group action on the other bundle. An isomorphism of principal bundles is then defined as a principal bundle morphism that is also a bundle isomorphism. Now, we move onto an important theorem about principal bundles.

Theorem 4.1 (Sections of a principal bundle [4])**.** *A principal G-bundle $P \xrightarrow{\pi} M$ is trivial (isomorphic to $M \times G \xrightarrow{\pi_1} M$) if and only if it possesses a continuous global section.*

In general, given any principal bundle, there is a great variety of fibre bundles that can be associated with it in a precise manner. The general idea is that we can form a fibre bundle such that G can act on each fibre F and act as a group of transformations.

Definition 4.13 (Associated bundle [2])**.** Let $P \xrightarrow{\pi} M$ be a principal G-bundle and let F be a smooth manifold, equipped with a left G-action $\triangleright$. We define

i) $P_F \coloneqq (P \times F)/{\sim_G}$, where $\sim_G$ is the equivalence relation

$$(p, f) \sim_G (p', f') \iff \exists g \in G : p' = p \triangleleft g,\ f' = g^{-1} \triangleright f.$$

We denote the of points of P_F as $[p, f]$.

ii) The map

$$\begin{aligned} \pi_F : & P_F \to M \\ & [p, f] \to \pi(p) \end{aligned}$$

$P_F \xrightarrow{\pi_F} M$ is a fibre bundle over M with fibre F that is called an associated bundle to the principal bundle, with the association being through the left action of G on F.

Generally, any vector bundle can be thought of as an associated bundle to a principal bundle with $G = GL(V, \mathbb{R})$. Tangent and cotangent bundles on a manifold M can be thought of as associated bundles to the frame bundle $\mathbf{B}(M)$. There is an important theorem which highlights the utility of thinking about certain bundles to be associated bundles rather on their own as then you can think of cross sections as maps to the fibre F.

Theorem 4.2 (Sections as functions on the principal bundle [4])**.** *If $P_F \xrightarrow{\pi_F} M$ is an associated fibre bundle then its sections correspond to maps $\phi : P \to F$ that satisfy*

$$\phi(g \triangleright p) = \phi(p) \triangleleft g. \qquad (\forall p \in P, \forall g \in G)$$

The section s_ϕ corresponding to ϕ is defined by $s_\phi(x) \coloneqq [p, \phi(p)]$ where $p \in \pi^{-1}(\{x\})$.

4.4 Connections on a Principal Bundle

Recall that a general connection on a fibre bundle is the choice of a specific horizontal bundle that can also be equated to the choice of a vector valued one form Φ . We now consider a specific case of this.

Definition 4.14 (Principal connection [5])**.** An Ehresmann connection H on a principal bundle P is said to be a principal connection if it is G-equivariant in the sense that $H_{e \triangleleft g} = R_{g*} H_e$, where R_{g*} is the push forward of the right action of $R_g e = e \triangleleft g$.

Now, given the connection one form Φ we can also define a Lie algebra valued 1-form

$$\omega : P \to \mathfrak{g}.$$

In terms of local coordinates (and some local section s), we can define $A_\alpha = s_\alpha^* \omega$.

A connection on a principal bundle induces a connection on any associated bundle. In particular, if the associated bundle is linear, then the connection will be linear as well. Consequently, we can associate a covariant derivative and connection coefficients with it.

5 Applications to Physics

In this section, we will consider some applications of the formalism that we have built up.

5.1 The $\mathbb{C}$-Line Bundle and Quantum Mechanics

The wavefunction in quantum mechanics can be thought of as a section of a $\mathbb{C}$-line bundle over the manifold of physics space M [8]. Now, if we consider this bundle to be an associated bundle to the frame bundle. Then, a change of coordinates on the base manifold corresponds to a coordinate change in the frames. We can establish a connection on the frame bundle which induces a connection, and a covariant derivative on our $\mathbb{C}$-line bundle. Thus, now we can define our momentum operator as $P_\alpha = -i\nabla_a$, so that it will be invariant under general coordinate transformations. In Euclidean coordinates, this is not necessary, however if we

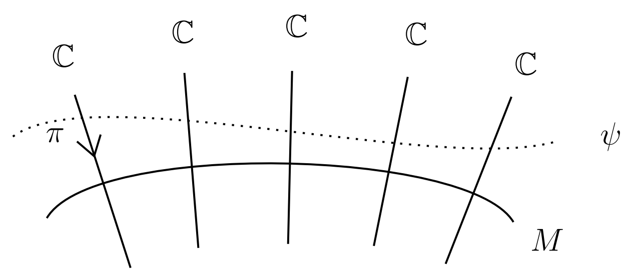

Figure 7: Wavefunction as a section of a $\mathbb{C}$-line bundle

want to do quantum mechanics on a curved space such as 2-sphere, we will have to consider the curvature of the space when constructing our canonical momentum operator.

5.2 Gauge Symmetries

In physics, we are often interested in studying the transformations of some field under symmetries of some group G. The technical details of this can be found Refs. [9] and [10], but a quick overview is as follows. A **gauge transformation** on a principal G-bundle is a bundle automoprhism (a morphism from a principal bundle to itself) that is compatible with the action of the gauge group G. Corresponding to the gauge transformation, there will be an associated effect on the associated vector bundles. Additionally, there will be a change in the connection one-form A_μ, which is called the **gauge boson field**. Usually, in quantum field theory, for example, the base manifold M is taken to be Minkowski space, and we consider the transformations of matter fields ψ under gauge transformations. These matter fields are considered to be sections of complex vector bundles associated with the principal G-bundle. We can then also consider the field strength or curvature related to this connection. A specific example of this is the $U(1)$-bundle, whose connection A_μ is in some sense the four vector potential of electrodynamics, and the curvature $F_{\mu\nu}$ is the electromagnetic field strength tensor.

6 Conclusion

Differential geometry and in particular, fibre bundles are a powerful tool for describing physics in an elegant geometric language. In this report, we built up the basic tools to be able to make general statements about important objects such as connections and curvature both in the context of general relativity, as well in physics as a whole. It is important to also recognize that a lot of this formalism is introduced in a coordinate free manner, which means that we are not relying on some specific coordinate system or chart on the manifold.

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