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A modified projection method with a new direction for solving variational inequalities

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ABSTRACT

In this paper, we propose a new projection method for solving variational inequality problems, which can be viewed as an improvement of the method of Li et al. [M. Li, L.Z. Liao, X.M. Yuan, A modified projection method for co-coercive variational inequality, European Journal of Operational Research 189 (2008) 310–323], by adopting a new direction. Under the same assumptions as those in Li et al. (2008), we establish the global convergence of the proposed algorithm. Some preliminary computational results are reported, which illustrated that the new method is more efficient than the method of Li et al. (2008).

1. Introduction

Let Ω be a nonempty closed convex subset of R^n, and let $F(\cdot)$ be a mapping from R^n into itself. The variational inequality problem, denoted by $\mathrm{VI}(F,\Omega)$, is to find a vector $u^* \in \Omega$, such that

$$F(u^*)^T(u - u^*) \geqslant 0, \quad \forall u \in \Omega. \tag{1}$$

Problem $\mathrm{VI}(F,\Omega)$ includes nonlinear complementarity problems (when $\Omega = R^n_+$) and system of nonlinear equation (when $\Omega = R^n$); it has many applications in the fields such as mathematical programming, network economics, transportation research, game theory, and regional sciences; see the excellent monograph of Facchinei and Pang [2] and the references therein.

The projection methods for variational inequalities generate the next iteration according to the recursion:

$$u^{k+1} = P_\Omega[u^k - \sigma_k g(u^k)], \tag{2}$$

where $P_\Omega(\cdot)$ denotes the orthogonal projection map onto Ω, σ_k is a suitable step size and $g(u^k)$ is called profitable direction (see Definition 2.2). This type of methods is one of the simplest numerical methods for solving variational inequality problems, especially when the projection onto Ω is easy to implement.

Among the projection type methods, the most simple one is the method proposed by Goldstein [3], and Levitin and Polyak [4], which just uses $F(\cdot)$ as the profitable direction. This projection method generally updates the iterations according to the following formula: given an initial point $u^0 \in R^n$,

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^k)], \tag{3}$$

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under suitable assumptions, e.g., that F is Lipschitz continuous with a constant $L > 0$

$$\|F(x) - F(y)\| \leqslant L\|x - y\|, \tag{4}$$

strong monotone with a constant modulus $\alpha > 0$

$$(x - y)^T(F(x) - F(y)) \geqslant \alpha\|x - y\|^2, \tag{5}$$

and the step size β_k satisfies

$$0 < \beta_L \leqslant \beta_k \leqslant \beta_U < \frac{2\alpha}{L^2}, \tag{6}$$

this projection method is globally convergent.

The efficiency of Goldstein–Levitin–Polyak's projection method depends heavily on the choice of the step size β_k. If β_k is too large, it may not converge while a too small choice of β_k will cause the convergence speed too slow. To overcome this difficulty in choosing the step size, He et al. [5] and Han and Sun [6] proposed some strategies of choosing β_k self-adaptively.

The projection methods proposed by He et al. [5] and Han and Sun [6] are efficient and convenient to implement. Nonetheless, the convergence conditions, i.e., the strong monotonicity and Lipschitz continuity of the underlying mapping F, are too restrictive. To remove such restrictive conditions in convergence, some modified projection methods were proved. Notably, He [7], Sun [8] and Solodov and Tseng [9] gave the following projection and contraction method

$$u^{k+1} = u^k - \gamma\rho(u^k, \beta_k)g(u^k, \beta_k), \tag{7}$$

with a new profitable direction

$$g(u, \beta) = e(u, \beta) - \beta[F(u) - F(u - e(u, \beta))], \tag{8}$$

where

$$\rho(u, \beta) = e(u, \beta)^T \frac{g(u, \beta)}{\|g(u, \beta)\|^2}, \tag{9}$$

is the step size, and

$$e(u, \beta) = u - P_\Omega[u - \beta F(u)], \tag{10}$$

is the residual function. Under the condition that the underlying mapping F is monotone, the method converges globally for suitably chosen parameter β_k. Han and Lo [10] construct a new profitable direction

$$d(u, \beta) = e(u, \beta) + \beta F(u - e(u, \beta)),$$

and based on this direction, they gave a new projection method for solving $\mathrm{VI}(F, \Omega)$.

Most recently, Li et al. [1] present a new projection method for solving variational inequalities. This projection method updates the iterations according to the following formula: given an initial point $u^0 \in R^n$, let

$$\bar{u}^k = P_\Omega[u^k - \theta\alpha_k^* e(u^k, \beta_k)], \tag{11}$$

$$u^{k+1} = P_\Omega[u^k - \gamma\tau_k^*(u^k - \bar{u}^k)], \tag{12}$$

where

$$\tau_k^* = \frac{\Upsilon(\theta\alpha_k^*) + \|u^k - \bar{u}^k\|^2}{2\|u^k - \bar{u}^k\|^2}, \quad \alpha_k^* = \left(1 - \frac{\beta_k}{4c}\right), \tag{13}$$

and $\theta \in (0, 2)$ and $\gamma \in (0, 2)$ are two constants. Under the condition that the underlying mapping F is co-coercive (see Definition 2.3), they proved that both $e(u^k, \beta_k)$ and $u^k - \bar{u}^k$ are profitable definitions and the method can be viewed as a prediction–correction method in the sense that it first makes a prediction $\bar{u}^k$ and then updates the iterate by using a correction step (12).

Since both $e(u^k, \beta_k)$ and $u^k - \bar{u}^k$ are profitable directions, it is easy to prove that any positive combination of them are still a profitable direction. That is,

$$\eta e(u^k, \beta_k) + \tau(u^k - \bar{u}^k), \tag{14}$$

with $\eta > 0,\ \tau > 0$ is also a profitable direction. Inspired by this, we construct a new projection method for solving $\mathrm{VI}(F, \Omega)$ by adopting the new profitable direction. We also propose a new technique for identifying optimal step size along the direction. Under the condition that F is co-coercive, the same condition used in [1], we prove that this modified method has the global convergence property. Our preliminary computational experience shows that the new algorithm is efficient for variational inequality problems.

The paper is organized as followings. In the next section, we give some useful preliminaries. In Section 3, we describe the method formally, and in Section 4, we show its global convergence. We report our preliminary computational results in Section 5 and give some final conclusions in the last section.

2. Preliminaries

Now, let us summarize some basic properties and concepts that will be used in the subsequent sections.

First, we denote $\|x\| = \sqrt{x^T x}$ as the Euclidean norm. Let Ω be a nonempty closed convex subset of R^n and Ω^* be the solution set of VI(F,Ω). Throughout the paper, we assume that Ω^* is nonempty. Let $P_\Omega(\cdot)$ denote the projection mapping from R^n onto Ω, i.e.,

$$P_\Omega(v) = \arg\min\{\|v-u\| \mid u \in \Omega\}.$$

From the above definition, it follows that the projection mapping $P_\Omega(\cdot)$ has the following two properties:

$$\{v - P_\Omega(v)\}^T\{w - P_\Omega(v)\} \leqslant 0, \quad \forall v \in R^n,\ \forall w \in \Omega, \tag{15}$$

and

$$(v-w)^T\{P_\Omega(v) - P_\Omega(w)\} \geqslant \|P_\Omega(v) - P_\Omega(w)\|^2, \quad \forall v \in R^n,\ \forall w \in R^n. \tag{16}$$

Consequently, we have

$$\|P_\Omega(v) - P_\Omega(w)\| \leqslant \|v-w\|, \quad \forall v \in R^n,\ \forall w \in R^n, \tag{17}$$

$$\|P_\Omega(v) - u\|^2 \leqslant \|v-u\|^2 - \|v - P_\Omega(v)\|^2, \quad \forall u \in \Omega. \tag{18}$$

Lemma 2.1. [11]. *Let $\beta > 0$, then u^* solves* VI(F,Ω) *if and only if*

$$u^* = P_\Omega[u^* - \beta F(u^*)].$$

Then solving VI(F,Ω) is equivalent to finding a zero point of $e(u,\beta)$.

The following two lemmas give the properties of $\|e(u,\beta)\|$ which are needed in our later convergence analysis.

Lemma 2.2. *For all $u \in R^n$ and $\tilde\beta \geqslant \beta > 0$, it holds that*

$$\|e(u,\tilde\beta)\| \geqslant \|e(u,\beta)\|. \tag{19}$$

Proof. See [12]. □

Lemma 2.3. *For any $u \in \Omega$ and $\tilde\beta \geqslant \beta > 0$, we have*

$$\frac{\|e(u,\beta)\|}{\beta} \geqslant \frac{\|e(u,\tilde\beta)\|}{\tilde\beta}. \tag{20}$$

Proof. See [13]. □

Definition 2.1. Let $c_0 > 0$ be a constant and $\varphi(u) : R^n \to R$ be a continuous function. We call $\varphi(u)$ an error measure function of VI(F,Ω) on Ω (or R^n) if it satisfies

$$\varphi(u) \geqslant c_0\|e(u,\beta)\|^2, \quad \forall u \in \Omega \text{ (or } R^n),$$

and

$$\varphi(u) = 0 \Longleftrightarrow e(u,\beta) = 0.$$

Definition 2.2. Let $\Pi(u)$ be a function from R^n into itself. We call $\Pi(u)$ a profitable direction of VI(F,Ω) if

$$(u-u^*)^T\Pi(u) \geqslant \varphi(u), \quad \forall u \in \Omega,$$

where $\varphi(u)$ is an error measure function and u^* is an arbitrary solution of VI(F,Ω).

In fact, from Definitions 2.1 and 2.2, we get that if u is not a solution of VI(F,Ω),

$$\left(\nabla\left(\frac{1}{2}\|u-u^*\|^2\right)\right)^T \Pi(u) \geqslant \varphi(u) \geqslant c_0\|e(u,\beta)\|^2 > 0.$$

In other words, $-\Pi(u)$ is a descent direction of the function $\frac{1}{2}\|u-u^*\|^2$, although u^* is unknown.

Definition 2.3. Let F be a mapping from a set $\Omega \subseteq R^n \to R^n$, then

(a) F is said to be monotone on Ω, if

$$(u-v)^T(F(u)-F(v)) \geqslant 0, \quad \forall u,\ v \in \Omega.$$

(b) F is said to be strictly monotone on Ω, if

$$(u-v)^T(F(u)-F(v)) > 0, \quad \forall u,\ v \in \Omega,\ u \neq v.$$

(c) F is said to be strongly monotone on Ω with modulus $\mu > 0$, if

$$(u-v)^T(F(u)-F(v)) \geqslant \mu\|u-v\|^2, \quad \forall u,\ v \in \Omega.$$

(d) F is said to be co-coercive on Ω with modulus $c > 0$, if

$$(u-v)^T(F(u)-F(v)) \geqslant c\|F(u)-F(v)\|^2, \quad \forall u,\ v \in \Omega.$$

(e) F is said to be Lipschitz continuous on Ω with modulus $L > 0$, if

$$\|F(u)-F(v)\| \leqslant L\|u-v\|, \quad \forall u,\ v \in \Omega.$$

From Definition 2.3, it is clear that co-coercive mappings are monotone but not necessarily strictly or strongly monotone. Conversely, strongly monotone and Lipschitz continuous mappings are co-coercive. Thus, co-coercive is an intermediate concept between simple and strong monotonicity.

Lemma 2.4. *Suppose that $F(\cdot)$ is co-coercive with constant c on Ω. Then,*

$$(u-u^*)^T e(u,\beta) \geqslant \left(1-\frac{\beta}{4c}\right)\|e(u,\beta)\|^2, \quad \forall u \in \Omega,\ u^* \in \Omega^*. \tag{21}$$

Proof. It follows from the definition of $\mathrm{VI}(F,\Omega)$, that

$$(u-e(u,\beta)-u^*)^T F(u^*) \geqslant 0. \tag{22}$$

From (15), we have

$$(u-u^*-e(u,\beta))^T(e(u,\beta)-\beta F(u)) \geqslant 0. \tag{23}$$

Inequalities (22) and (23) imply that

$$\{e(u,\beta)-\beta[F(u)-F(u^*)]\}^T\{(u-u^*)-e(u,\beta)\} \geqslant 0. \tag{24}$$

From (24), we get

$$(u-u^*)^T e(u,\beta) \geqslant \|e(u,\beta)\|^2 + \beta[F(u)-F(u^*)]^T(u-u^*) - \beta[F(u)-F(u^*)]^T e(u,\beta).$$

It follows from Definition 2.3,

$$\begin{aligned}(u-u^*)^T e(u,\beta) &\geqslant \|e(u,\beta)\|^2 - \beta[F(u)-F(u^*)]^T e(u,\beta) + \beta c\|F(u)-F(u^*)\|^2 \\ &= \|e(u,\beta)\|^2 - \beta[F(u)-F(u^*)]^T e(u,\beta) + \beta c\|F(u)-F(u^*)\|^2 + \frac{\beta}{4c}\|e(u,\beta)\|^2 - \frac{\beta}{4c}\|e(u,\beta)\|^2 \\ &\geqslant \left(1-\frac{\beta}{4c}\right)\|e(u,\beta)\|^2.\end{aligned}$$

This completes the proof. □

Now, we present a convergence theorem which is useful for the method studied in this paper.

Theorem 2.1. *Let $C_0 > 0$ be a constant, let $l \in \{0,1\}$ be a given integer, let $\{\beta_k\}$ be a positive sequence, and let $\inf\{\beta_k\} = \beta_{\min} > 0$. If the sequence $\{u^k\}$ generated by a method satisfies*

$$\|u^{k+1}-u^*\|^2 \leqslant \|u^k-u^*\|^2 - C_0\|e(u^{k+l},\beta_k)\|^2, \quad \forall u^* \in \Omega^*, \tag{25}$$

then $\{u^k\}$ converges to a solution set point of $\mathrm{VI}(F,\Omega)$.

Proof. Let u^* be a solution of $\mathrm{VI}(F,\Omega)$. First, from (25), we get

$$\sum_{k=0}^{\infty} C_0\|e(u^{k+l},\beta_k)\|^2 \leqslant \|u^0-u^*\|^2,$$

which means that

$$\lim_{k\to\infty} e(u^k,\beta_k) = 0.$$

Since $\beta_k \geqslant \beta_{\min}$, it follows from Lemma 2.2 that

$$\lim_{k\to\infty} e(u^k, \beta_{\min}) = 0.$$

Again, it follows from (25) that the sequence $\{u^k\}$ is bounded and therefore has at least one cluster point. Let u^* be a cluster point of $\{u^k\}$, and let $\{u^{k_j}\}$ the subsequence converging to u^*. Because $e(u, \beta_{\min})$ is continuous, taking limit along the subsequence,

$$e(u^*, \beta_{\min}) = \lim_{j\to\infty} e(u^{k_j}, \beta_{\min}) = 0.$$

Thus, it follows from Lemma 2.1 that u^* is a solution of VI(F, Ω).

In the following, we prove the sequence $\{u^k\}$ has exactly one cluster point. Assume that $\tilde{u}$ is another cluster point, and denote $\delta := \|\tilde{u} - u^*\| > 0$. Because u^* is a cluster point of the sequence $\{u^k\}$, there is a $k_0 > 0$ such that

$$\|u^{k_0} - u^*\| \leqslant \frac{\delta}{2}.$$

On the other hand, since $u^* \in \Omega^*$ and thus

$$\|u^k - u^*\| \leqslant \|u^{k_0} - u^*\|, \quad \forall k \geqslant k_0.$$

It follows that

$$\|u^k - \tilde{u}\| \geqslant \|\tilde{u} - u^*\| - \|u^{k_0} - u^*\| \geqslant \frac{\delta}{2}, \quad \forall k \geqslant k_0.$$

This contradicts the assumption, thus the sequence $\{u^k\}$ converges to $u^* \in \Omega^*$. □

In the rest of this paper, for convergence analysis, it is important to make the iteration sequence generated by the algorithm satisfy (25) in Theorem 2.1.

3. The algorithm

We are now in the position to describe our algorithm formally.

Algorithm 3.1

S0. Choose $u^0 \in \Omega$, $c > 0$, $[\beta_l, \beta_u] \subset (0, 4c)$, $\beta_0 \in [\beta_l, \beta_u]$, $\gamma \in (0, 2)$, and $\theta \in (0, 2)$.
Set $k := 0$.

S1. Check the stopping criterion. If it is satisfied, stop; otherwise, go to the next step.

S2. (generate the temporary point)

$$\bar{u}^k = P_\Omega[u^k - \theta\alpha_k^* e(u^k, \beta_k)], \tag{26}$$

where $\alpha_k^* = 1 - \frac{\beta_k}{4c}$.

S3. (generate the new point)

$$u^{k+1} = P_\Omega[u^k - \gamma(\eta_k^* e(u^k, \beta_k) + \tau_k^*(u^k - \bar{u}^k))], \tag{27}$$

where

$$\eta_k^* = \frac{2\left(1 - \frac{\beta_k}{4c}\right)\|e(u^k, \beta_k)\|^2\|u^k - \bar{u}^k\|^2 - (\Upsilon(\theta\alpha_k^*) + \|u^k - \bar{u}^k\|^2)e(u^k, \beta_k)^T(u^k - \bar{u}^k)}{2(\|e(u^k, \beta_k)\|^2\|u^k - \bar{u}^k\|^2 - (e(u^k, \beta_k)^T(u^k - \bar{u}^k))^2)}, \tag{28}$$

$$\tau_k^* = \frac{\Upsilon(\theta\alpha_k^*) + \|u^k - \bar{u}^k\|^2}{2\|u^k - \bar{u}^k\|^2} - \eta_k^* \frac{e(u^k, \beta_k)^T(u^k - \bar{u}^k)}{\|u^k - \bar{u}^k\|^2}, \tag{29}$$

$$\Upsilon(\theta\alpha_k^*) = \|u^k - \bar{u}^k\|^2 + 2\theta\alpha_k^*\left(1 - \frac{\beta_k}{4c}\right)\|e(u^k, \beta_k)\|^2 - 2\theta\alpha_k^* e(u^k, \beta_k)^T(u^k - \bar{u}^k). \tag{30}$$

S4. Compute

$$\omega_k = \frac{\beta_k\|F(u^{k+1}) - F(u^k)\|}{\|u^{k+1} - u^k\|},$$

and set

$$\beta_{k+1} = \begin{cases} \min\{\beta_u, 2.5\beta_k\} & \text{if } \omega_k < 0.4, \\ \max\{\beta_l, \frac{2}{3}\beta_k\} & \text{if } \omega_k > 1.4, \\ \beta_k & \text{otherwise}. \end{cases} \tag{31}$$

Set $k := k + 1$; go to Step 1.

Remark 3.1. In step 1, based on different problem, stopping criterion is different, for example, when considering complementarity problems, we usually choose

$$\|\min\{u^k, F(u^k)\}\| \leqslant \varepsilon$$

as stopping criterion. In step 3, when computing η_k^*, we first check $\|e(u^k,\beta_k)\|^2\|u^k - \bar{u}^k\|^2 - (e(u^k,\beta_k)^T(u^k - \bar{u}^k))^2$. If it is too small, we set $\eta_k^* = 0$. In step 4, our strategy of adjusting β_k is similar to the technique presented in [1], which is in the spirit of balancing the quantities $\|u^{k+1} - u^k\|$ and $\beta_k\|F(u^{k+1}) - F(u^k)\|$ during iteration.

4. Convergence analysis

In order to construct our new method, we first construct the profitable direction, beginning with a series of lemmas. The following lemma give the property of $\Upsilon(\theta\alpha^*)$ which given by (30).

Lemma 4.1. *For given u^k and $\beta_k > 0$, $0 < \theta < 2$, let $\bar{u}^k$ be defined in (26). Then we have*

$$\Upsilon_k(\theta\alpha_k^*) \geqslant \theta(2-\theta)\left(1 - \frac{\beta_u}{4c}\right)^2 \|e(u^k,\beta_l)\|^2 \geqslant 0. \tag{32}$$

Proof. From the definition of $\Upsilon_k(\theta\alpha_k^*)$, we have

$$\begin{aligned}
\Upsilon_k(\theta\alpha_k^*) &= \|u^k - \bar{u}^k\|^2 + 2\theta\alpha_k^*\left(1 - \frac{\beta_k}{4c}\right)\|e(u^k,\beta_k)\|^2 - 2\theta\alpha_k^*(u^k - \bar{u}^k)^T e(u^k,\beta_k) \\
&= \|u^k - \bar{u}^k - \theta\alpha_k^* e(u^k,\beta_k)\|^2 - (\theta\alpha_k^*)^2\|e(u^k,\beta_k)\|^2 + 2\theta\alpha_k^*\left(1 - \frac{\beta_k}{4c}\right)\|e(u^k,\beta_k)\|^2 \\
&\geqslant 2\theta\alpha_k^*\left(1 - \frac{\beta_k}{4c}\right)\|e(u^k,\beta_k)\|^2 - (\theta\alpha_k^*)^2\|e(u^k,\beta_k)\|^2 = \theta(2-\theta)(\alpha_k^*)^2\|e(u^k,\beta_k)\|^2 \\
&\geqslant \theta(2-\theta)\left(1 - \frac{\beta_u}{4c}\right)^2\|e(u^k,\beta_k)\|^2 \geqslant \theta(2-\theta)\left(1 - \frac{\beta_u}{4c}\right)^2\|e(u^k,\beta_l)\|^2,
\end{aligned} \tag{33}$$

where the last inequality follows from Lemma 2.2. This completes the proof. □

The following lemma can be derived from Lemma 4.3 of [1]; for completeness, a proof is provided.

Lemma 4.2. *Let u^* be any finite point in Ω^*, $\bar{u}^k$ be generated by (26), and $F(\cdot)$ is co-coercive on Ω, then we have*

$$2(u^k - u^*)^T(u^k - \bar{u}^k) \geqslant \Upsilon_k(\theta\alpha_k^*) + \|u^k - \bar{u}^k\|^2 \geqslant 0, \tag{34}$$

where $\Upsilon_k(\theta\alpha_k^*)$ is given by (30).

Proof. Because

$$\bar{u}^k = P_\Omega[u^k - \theta\alpha_k^* e(u^k,\beta_k)],$$

from (18), it follows that

$$\begin{aligned}
\|\bar{u}^k - u^*\|^2 &= \|P_\Omega[u^k - \theta\alpha_k^* e(u^k,\beta_k)] - u^*\|^2 \leqslant \|u^k - \theta\alpha_k^* e(u^k,\beta_k) - u^*\|^2 - \|u^k - \theta\alpha_k^* e(u^k,\beta_k) - \bar{u}^k\|^2 \\
&= \|u^k - u^*\|^2 - 2\theta\alpha_k^* e(u^k,\beta_k)^T(u^k - u^*) + 2\theta\alpha_k^* e(u^k,\beta_k)^T(u^k - \bar{u}^k) - \|u^k - \bar{u}^k\|^2.
\end{aligned}$$

Thus, from Lemma 2.4, we get

$$\|\bar{u}^k - u^*\|^2 \leqslant \|u^k - u^*\|^2 - 2\theta\alpha_k^*\left(1 - \frac{\beta_k}{4c}\right)\|e(u^k,\beta_k)\|^2 + 2\theta\alpha_k^* e(u^k,\beta_k)^T(u^k - \bar{u}^k) - \|u^k - \bar{u}^k\|^2, \tag{35}$$

and after rearranging terms, we get

$$\|u^k - u^*\|^2 - \|\bar{u}^k - u^*\|^2 \geqslant 2\theta\alpha_k^*\left(1 - \frac{\beta_k}{4c}\right)\|e(u^k,\beta_k)\|^2 - 2\theta\alpha_k^* e(u^k,\beta_k)^T(u^k - \bar{u}^k) + \|u^k - \bar{u}^k\|^2.$$

Thus, from (30)

$$\|u^k - u^*\|^2 - \|\bar{u}^k - u^*\|^2 \geqslant \Upsilon_k(\theta\alpha_k^*). \tag{36}$$

Substituting the equality

$$\bar{u}^k - u^* = (u^k - u^*) - (u^k - \bar{u}^k),$$

into (36), we have

$$2(u^k-u^*)^T(u^k-\bar{u}^k)-\|u^k-\bar{u}^k\|^2 \geqslant \Upsilon_k(\theta\alpha_k^*),$$

which establishes the first inequality in (34) and the second inequality is obvious according to Lemma 4.1. □

From Lemmas 4.2, 4.1 and Definition 2.2, we can see $u^k-\bar{u}^k$ is a profitable direction, e.g., $-(u^k-\bar{u}^k)$ is a descent direction for $\frac{1}{2}\|u-u^*\|^2$ at the point $u=u^k$. From Lemma 2.4, we see $e(u^k,\beta_k)$ is also a profitable direction, so we get the following lemma which present a new profitable direction.

Lemma 4.3. *Suppose that $F(\cdot)$ is co-coercive on Ω, let $\eta>0,\ \tau>0$, then*

$$\eta e(u^k,\beta_k)+\tau(u^k-\bar{u}^k),$$

is a profitable direction.

In the following, we investigate the technique of identifying the optimal step size along the profitable direction, then we prove the convergence of the new method. For convenience of the coming analysis, we replace u^k and β_k by u and β, we use the notation

$$\bar{u}=P_\Omega[u-\theta\alpha^*e(u,\beta)], \tag{37}$$

for the temporary point in Algorithm 3.1. The new iterate of the method can be written

$$u_+=P_\Omega[u-\eta e(u,\beta)-\tau(u-\bar{u})]. \tag{38}$$

Lemma 4.4. *Let $\bar{u}$ be given by (37), then*

$$\frac{(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2} \geqslant \frac{1}{4}\{C_0\|e(u,\beta_l)\|^2+\|u-\bar{u}\|^2\}, \tag{39}$$

where $C_0=\theta(2-\theta)\left(1-\frac{\beta_u}{4c}\right)^2$.

Proof. From the second inequality of (32), we obtain

$$\frac{\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2}{2\|u-\bar{u}\|^2} \geqslant \frac{1}{2}. \tag{40}$$

It follows from (40) that

$$\frac{(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2} \geqslant \frac{1}{4}(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2).$$

Using the first inequality of (32), we get

$$\frac{(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2} \geqslant \frac{1}{4}\left\{\theta(2-\theta)\left(1-\frac{\beta_u}{4c}\right)^2\|e(u,\beta_l)\|^2+\|u-\bar{u}\|^2\right\}.$$

Thus, (39) holds with $C_0=\theta(2-\theta)\left(1-\frac{\beta_u}{4c}\right)^2$. □

Now, let us observe the difference between $\|u-u^*\|^2$ and $\|u_+-u^*\|^2$.

Lemma 4.5. *We have the following inequality*

$$\Theta(\eta,\tau)=\|u-u^*\|^2-\|u_+-u^*\|^2 \geqslant \Phi(\eta,\tau), \tag{41}$$

where

$$\Phi(\eta,\tau)=2\eta\left(1-\frac{\beta}{4c}\right)\|e(u,\beta)\|^2+\tau(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)-\eta^2\|e(u,\beta)\|^2-2\eta\tau e(u,\beta)^T(u-\bar{u})-\tau^2\|u-\bar{u}\|^2. \tag{42}$$

Proof. Because $u_+=P_\Omega[u-\eta e(u,\beta)-\tau(u-\bar{u})]$, by setting $v=u-\eta e(u,\beta)-\tau(u-\bar{u})$ and $\omega=u^*$ in (17), we get

$$\|u_+-u^*\|^2 \leqslant \|u-\eta e(u,\beta)-\tau(u-\bar{u})-u^*\|^2=\|u-u^*\|^2-2(u-u^*)^T(\eta e(u,\beta)+\tau(u-\bar{u}))+\|\eta e(u,\beta)+\tau(u-\bar{u})\|^2.$$

Substituting it into the equality in (41), we obtain

$$\begin{aligned}\Theta(\eta,\tau) &\geqslant 2(u-u^*)^T(\eta e(u,\beta)+\tau(u-\bar{u}))-\|\eta e(u,\beta)+\tau(u-\bar{u})\|^2\\ &=2\eta(u-u^*)^Te(u,\beta)+2\tau(u-u^*)^T(u-\bar{u})-\eta^2\|e(u,\beta)\|^2-2\eta\tau e(u,\beta)^T(u-\bar{u})-\tau^2\|u-\bar{u}\|^2.\end{aligned}$$

From Lemmas 2.4 and 4.2, it follows that

$$\Theta(\eta,\tau) \geqslant 2\eta\left(1-\frac{\beta}{4c}\right)\|e(u,\beta)\|^2 + \tau(\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2) - \eta^2\|e(u,\beta)\|^2 - 2\eta\tau e(u,\beta)^T(u-\bar{u}) - \tau^2\|u-\bar{u}\|^2,$$

which proves the inequality in (41). □

Note that $\Phi(\eta,\tau)$ is a lower bound of $\Theta(\eta,\tau)$. Therefore, it is reasonable to choose the optimal value of $\eta,\ \tau$ that maximize $\Phi(\eta,\tau)$ in order to accelerate the convergence of our method.

Lemma 4.6. *$\Phi(\eta,\tau)$ reach its maximum at*

$$\eta^* = \frac{2(1-\frac{\beta}{4c})\|e(u,\beta)\|^2\|u-\bar{u}\|^2 - (\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2)e(u,\beta)^T(u-\bar{u})}{2(\|e(u,\beta)\|^2\|u-\bar{u}\|^2 - (e(u,\beta)^T(u-\bar{u}))^2)}, \tag{43}$$

$$\tau^* = \frac{\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2}{2\|u-\bar{u}\|^2} - \eta^*\frac{e(u,\beta)^T(u-\bar{u})}{\|u-\bar{u}\|^2}. \tag{44}$$

Proof. Note that $\Phi(\eta,\tau)$ is differentiable and has its maximum. From (42) and $\frac{\partial\Phi(\eta,\tau)}{\partial\tau}=0$, we get

$$\tau = \frac{\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2}{2\|u-\bar{u}\|^2} - \eta\frac{e(u,\beta)^T(u-\bar{u})}{\|u-\bar{u}\|^2}. \tag{45}$$

Thus, it follows that

$$\Phi(\eta,\tau) = \frac{(\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2} + \eta\frac{2(1-\frac{\beta}{4c})\|e(u,\beta)\|^2\|u-\bar{u}\|^2 - (\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2)e(u,\beta)^T(u-\bar{u})}{\|u-\bar{u}\|^2} - \eta^2 \times \frac{\|e(u,\beta)\|^2\|u-\bar{u}\|^2 - (e(u,\beta)^T(u-\bar{u}))^2}{\|u-\bar{u}\|^2}. \tag{46}$$

The right-hand side of (46) is a quadratic of η, and it reaches its maximum at

$$\eta^* = \frac{2(1-\frac{\beta}{4c})\|e(u,\beta)\|^2\|u-\bar{u}\|^2 - (\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2)e(u,\beta)^T(u-\bar{u})}{2(\|e(u,\beta)\|^2\|u-\bar{u}\|^2 - (e(u,\beta)^T(u-\bar{u}))^2)}. \tag{47}$$

Combining (45) and (47), we have

$$\tau^* = \frac{\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2}{2\|u-\bar{u}\|^2} - \eta^*\frac{e(u,\beta)^T(u-\bar{u})}{\|u-\bar{u}\|^2},$$

and the assertion is established immediately. □

From computational experience (see also [14]), we prefer to attach a relax factor $\gamma\in(0,2)$ to η^* and τ^*, arriving the new step (27). Denote

$$u_{new} = P_\Omega[u - \gamma(\eta^* e(u,\beta) + \tau^*(u-\bar{u}))].$$

Now, we analyze the global convergence of our algorithm.

Lemma 4.7. *Let $\gamma\in(0,2)$ and $\bar{u}$ be defined by (37). It holds that*

$$\Phi(\gamma\eta^*,\gamma\tau^*) = \gamma(2-\gamma)\left(\frac{(\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2} + (\eta^*)^2\right). \tag{48}$$

Proof. From (42), we get

$$\begin{aligned}\Phi(\gamma\eta^*,\gamma\tau^*) &= 2\gamma\eta^*\left(1-\frac{\beta}{4c}\right)\|e(u,\beta)\|^2 + \gamma\tau^*(\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2) - \gamma^2(\eta^*)^2\|e(u,\beta)\|^2 - 2\gamma^2\eta^*\tau^* e(u,\beta)^T(u-\bar{u}) \\ &\quad - \gamma^2(\tau^*)^2\|u-\bar{u}\|^2 = \gamma\Bigg(2\eta^*\left(1-\frac{\beta}{4c}\right)\|e(u,\beta)\|^2 + \tau^*(\Upsilon(\theta\alpha^*) + \|u-\bar{u}\|^2) - \gamma(\eta^*)^2\|e(u,\beta)\|^2 \\ &\quad -2\gamma\eta^*\tau^* e(u,\beta)^T(u-\bar{u}) - \gamma(\tau^*)^2\|u-\bar{u}\|^2\Bigg).\end{aligned}$$

Then, it follows from (44) and (43) that

$$\begin{aligned}\Phi(\gamma\eta^*,\gamma\tau^*) &= \gamma\Bigg((2-\gamma)\frac{(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2}+\eta^*\frac{2\left(1-\frac{\beta}{4c}\right)\|e(u,\beta)\|^2\|u-\bar{u}\|^2-(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)e(u,\beta)^T(u-\bar{u})}{\|u-\bar{u}\|^2}\\ &\quad -\gamma(\eta^*)^2\frac{\|e(u,\beta)\|^2\|u-\bar{u}\|^2-(e(u,\beta)^T(u-\bar{u}))^2}{\|u-\bar{u}\|^2}\Bigg)\\ &= \gamma(2-\gamma)\left(\frac{(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2}+(\eta^*)^2\right).\end{aligned}$$

This completes the proof. □

Denote

$$\Theta(\gamma\eta^*,\gamma\tau^*)=\|u-u^*\|^2-\|u_{new}-u^*\|^2.$$

Lemma 4.8. *Let u^* be any finite point in Ω^*, $\gamma\in(0,2)$, and $\bar{u}$ be given by (37). Then we have*

$$\Theta(\gamma\eta^*,\gamma\tau^*)\geqslant\frac{\gamma(2-\gamma)}{4}\{C_0\|e(u,\beta_l)\|^2+\|u-\bar{u}\|^2\}\geqslant\gamma(2-\gamma)\frac{C_0}{4}\|e(u,\beta_l)\|^2, \tag{49}$$

where $C_0=\theta(2-\theta)\left(1-\frac{\beta_u}{4c}\right)^2$.

Proof. From (41), we get

$$\Theta(\gamma\eta^*,\gamma\tau^*)\geqslant\Phi(\gamma\eta^*,\gamma\tau^*).$$

Furthermore, it follows from Lemma 4.7 that

$$\Theta(\gamma\eta^*,\gamma\tau^*)\geqslant\gamma(2-\gamma)\left(\frac{(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2}+(\eta^*)^2\right)\geqslant\gamma(2-\gamma)\frac{(\Upsilon(\theta\alpha^*)+\|u-\bar{u}\|^2)^2}{4\|u-\bar{u}\|^2}.$$

From Lemma 4.4, it follows that

$$\Theta(\gamma\eta^*,\gamma\tau^*)\geqslant\frac{\gamma(2-\gamma)}{4}\{C_0\|e(u,\beta_l)\|^2+\|u-\bar{u}\|^2\}\geqslant\gamma(2-\gamma)\frac{C_0}{4}\|e(u,\beta_l)\|^2.$$

This completes the proof. □

We summarize the analytical result of this section in the following theorem.

Theorem 4.1. *Let $\gamma\in(0,2)$, $\theta\in(0,2)$, $u^*\in\Omega^*$, for given $u^k\in\Omega$, β_k is chose like (31). Suppose that $F(\cdot)$ is co-coercive on Ω with modulus c. Then the method*

$$\begin{aligned}\bar{u}^k &= P_\Omega[u^k-\theta\alpha_k^*e(u^k,\beta_k)],\quad \alpha_k^*=1-\frac{\beta_k}{4c},\\ u^{k+1} &= P_\Omega[u^k-\gamma(\eta_k^*e(u^k,\beta_k)+\tau_k^*(u^k-\bar{u}^k))]\end{aligned}$$

with step size

$$\eta_k^*=\frac{2\left(1-\frac{\beta_k}{4c}\right)\|e(u^k,\beta_k)\|^2\|u^k-\bar{u}^k\|^2-(\Upsilon(\theta\alpha_k^*)+\|u^k-\bar{u}^k\|^2)e(u^k,\beta_k)^T(u^k-\bar{u}^k)}{2(\|e(u^k,\beta_k)\|^2\|u^k-\bar{u}^k\|^2-(e(u^k,\beta_k)^T(u^k-\bar{u}^k))^2)}, \tag{50}$$

$$\tau_k^*=\frac{\Upsilon(\theta\alpha_k^*)+\|u^k-\bar{u}^k\|^2}{2\|u^k-\bar{u}^k\|^2}-\eta_k^*\frac{e(u^k,\beta_k)^T(u^k-\bar{u}^k)}{\|u^k-\bar{u}^k\|^2}, \tag{51}$$

$$\Upsilon(\theta\alpha_k^*)=\|u^k-\bar{u}^k\|^2+2\theta\alpha_k^*\left(1-\frac{\beta_k}{4c}\right)\|e(u^k,\beta_k)\|^2-2\theta\alpha_k^*e(u^k,\beta_k)^T(u^k-\bar{u}^k), \tag{52}$$

generates a sequence of iterates $\{u^k\}$ satisfying

$$\|u^{k+1}-u^*\|^2\leqslant\|u^k-u^*\|^2-\frac{\gamma(2-\gamma)}{4}C_0\|e(u^k,\beta_l)\|^2,$$

where $C_0=\theta(2-\theta)\left(1-\frac{\beta_u}{4c}\right)^2>0$.

From Theorems 2.1 and 4.1, we obtain the following theorem that states the global convergence of Algorithm 3.1.

Theorem 4.2. *Suppose $F(\cdot)$ is co-coercive on Ω, and the solution set Ω^* is nonempty, then the sequence $\{u^k\}\subset R^n$ generated by Algorithm 3.1 converges to a solution of $VI(F,\Omega)$.*

5. Computational results

In this section, we give some preliminary computational results of the proposed algorithm. We implement our Algorithm 3.1 in MATLAB to solve some complementary problems. Our main purpose is to show the advantages of the proposed method over the old one. To this end, we also code algorithm of Li et al. [1].

At first, we implement original algorithm in [1] and our Algorithm 3.1 to an economic equilibrium problem. The details of the problem are taken from [1] as follows: denote $u = \begin{pmatrix} y \\ z \end{pmatrix}$, and $F(u) = \begin{pmatrix} s^{max} - s(u) \\ d(u) - d^{min} \end{pmatrix}$, the problem is

$$u \geqslant 0, \quad F(u) \geqslant 0, \quad u^T F(u) = 0, \tag{53}$$

where $s(u) = Mx(u),\ d(u) = Nx(u)$,

$$M = \begin{pmatrix} e_n^T & & & \\ & e_n^T & & \\ & & \cdots & \\ & & & e_n^T \end{pmatrix} \in R^{m\times nm}, \quad N = (\, I_n \quad I_n \quad \cdots \quad I_n \,) \in R^{n\times nm},$$

$x(u)$ is the solution of

$$(\tilde{x} - x)^T \{t(x) + M^T[g(Mx) + y] - N^T[h(Nx) + z]\} \geqslant 0, \quad \forall \tilde{x} \geqslant 0. \tag{54}$$

In our test, we use the same functions $g,\ h,\ t$ as in [1]. The entries of s^{max} and d^{min} are 150 and 40, respectively. We take $\theta = \gamma = 1.8$, the co-coercive modulus of c, β_l and β_u is same as in [1], $\beta_0 = 0.7 * c$, starting point $u^0 = 0$. The stopping criterion for solving the problem (53) is

$$\| \min\{u^k, F(u^k)\}\|_\infty \leqslant 10^{-6}.$$

In the implementation of the algorithms, we apply the method in [5] to solve the involved linear complementarity problem (54), and thus obtain solution $x(u)$ of the concerned lower-level problem. Table 1 shows the computational results.

The computational results are given in Table 1 for some m and n, where 'Nit' denotes number of iterations and 'CPU' denotes Cputime in seconds. From this table, we can see that for all scale of the problem, the number of iterations is less (not larger than) that of [1] and the CPU time need is much less than that of [1]. The results confirm that the improvement is effective.

We now consider another complementarity problem

$$u \geqslant 0, \quad F(u) \geqslant 0, \quad \langle u, F(u)\rangle = 0.$$

In our test, we take

$$F(u) = D(u) + Mu + q,$$

where $D(u)$ and $Mu + q$ are the nonlinear part and the linear part of $F(u)$, respectively. We form $F(u)$ similarly as in [12]. The matrix $M = A^T A + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, +5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500, 500)$. In $D(u)$, the nonlinear part of $F(u)$, the components are $D_j(u) = a_j * \arctan(u_j)$ and a_j is a random variable in $(-1, 0)$. A similar type of the problem was tested in [15–17].

We solve this problem with our Algorithm 3.1 and the original algorithm of Li et al. [1] with different starting points. The parameters in the algorithms are $\gamma = \theta = 1.8,\ c = \frac{15}{n},\ \beta_l = \frac{0.015c}{n},\ \beta_u = \frac{0.09c}{n}$, $\beta_0 = \frac{0.07c}{n}$, where n is the scale of the problem. The stopping criterion for solving the problem is also

$$\| \min\{u^k, F(u^k)\}\|_\infty \leqslant 10^{-6}.$$

Tables 2–4 give the computational results.

Table 1
Computational results for different scale.

m	n	mn	Algorithm in [1]		Algorithm 3.1	
			Nit	CPU (s)	Nit	CPU (s)
10	20	200	19	0.2970	17	0.2500
10	30	300	31	3.0630	26	2.3280
15	30	450	17	3.9540	15	3.0780
20	40	800	17	11.8910	14	8.6870
25	50	1250	16	47.0310	15	37.8130
30	60	1800	16	84.0160	15	70.0790
30	80	2400	25	267.1250	25	266.5470

Table 2
Computational results for u^0 generated uniformly in (0,1).

Dim (n)	Algorithm in [1]		Our Algorithm 3.1	
	It. Num.	Cpu	It. Num.	Cpu
100	61	0.0150	51	0.0150
200	115	0.0160	109	0.0160
300	158	0.0940	145	0.1090
400	232	0.2190	225	0.2340
500	263	0.4070	239	0.3900
600	301	0.6410	294	0.6720
700	336	1.0160	324	1.0160
800	371	1.3120	361	1.3120
900	376	1.7970	357	1.7350
1000	445	2.4370	437	2.4220
2000	621	13.5630	594	13.0940
3000	667	31.3900	629	29.8430

Table 3
Computational results for $u^0 = (0, 0, \ldots, 0)$.

Dim (n)	Algorithm in [1]		Our Algorithm 3.1	
	It. Num.	Cpu	It. Num.	Cpu
100	61	0.0150	55	0.0150
200	117	0.0150	115	0.0160
300	163	0.1090	147	0.1250
400	212	0.2190	198	0.2190
500	254	0.3750	235	0.3750
600	286	0.5940	271	0.5940
700	325	0.8750	312	0.8750
800	353	1.2350	346	1.2340
900	387	1.7030	376	1.7030
1000	416	2.0790	385	1.9690
2000	608	14.2030	586	13.7650
3000	726	34.0470	692	32.6250

Table 4
Computational results for $u^0 = (1, 1, \ldots, 1)$.

Dim (n)	Algorithm in [1]		Our Algorithm 3.1	
	It. Num.	Cpu	It. Num.	Cpu
100	60	0.0150	56	0.0160
200	109	0.0310	105	0.0310
300	164	0.1250	159	0.1250
400	215	0.2040	191	0.2030
500	243	0.3750	222	0.3590
600	300	0.6250	291	0.6250
700	341	0.9370	335	0.9370
800	339	1.1720	331	1.1720
900	364	1.2500	348	1.2340
1000	430	2.1560	419	2.1250
2000	589	13.7340	558	13.0620
3000	687	32.8440	652	30.9840

From Tables 2–4, we can also observe that the improvement strategy is effective. The number of iterations is less than that of [1] and the CPU time need in our algorithm is much less than that of [1] when the scale of the problem is large. In addition, for a set of similar problems, it seems that the number of iterations is not very sensitive to starting point.

6. Conclusion

Based on the method of Li et al. [1], we observe a new descent direction, and present a modified descent method for solving variational inequalities. The corresponding optimal step sizes along the descent direction are also identified to accelerate the convergence. Under the condition that $F(\cdot)$ is co-coercive, the convergence of the algorithm is proved and our preliminary computational results indicated the efficiency.

In our implementation, in both examples, we need to set co-coercivity constant c, we just give a guess about c. In fact, choosing a suitable parameter c is difficult in practice, the same as choosing the strong monotonicity modulus and the Lipschitz continuous constant. Thus, it is important to find a self-adaptive scheme to choose such parameters self-adaptively. This is one of our ongoing research topics.

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References

[1] M. Li, L.Z. Liao, X.M. Yuan, A modified projection method for co-coercive variational inequality, European Journal of Operational Research 189 (2008) 310–323.
[2] F. Facchinei, J.S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, vols. I and II, Springer-Verlag, Berlin, 2003.
[3] A.A. Goldstein, Convex programming in Hilbert space, Bulletin of the American Mathematical Society 70 (1964) 709–710.
[4] E.S. Levitin, B.T. Polyak, Constrained minimization problems, USSR Computational Mathematics and Mathematical Physics 6 (1966) 1–50.
[5] B.S. He, H. Yang, Q. Meng, D.R. Han, Modified Goldstein–Levitin–Polyak projection method for asymmetric strongly monotone variational inequalities, Journal of Optimization Theory and Applications 112 (2002) 129–143.
[6] D.R. Han, W.Y. Sun, A new modified Goldstein–Levitin–Polyak projection method for variational inequality problems, Computers and Mathematics with Applications 47 (2004) 1817–1825.
[7] B.S. He, A class of projection and contraction methods for monotone variational inequalities, Applied Mathematics and Optimization 35 (1997) 69–76.
[8] D. Sun, A class of iterative methods for solving nonlinear projection equations, Journal of Optimization Theory and Applications 91 (1) (1996) 123–140.
[9] M.V. Solodov, P. Tseng, Modified projection-type methods for monotone variational inequalities, SIAM Journal of Control and Optimization 34 (1996) 1814–1830.
[10] D.R. Han, Hong K. Lo, Two new self-adaptive projection methods for variational inequality problems, Computers and Mathematics with Application 43 (2002) 1529–1537.
[11] D.P. Bertsekas, J.N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods, Prentice-Hall, Englewood Cliffs, NJ, 1989.
[12] B.S. He, L.Z. Liao, Improvements of some projection methods for monotone nonlinear variational inequalities, Journal of Optimization Theory and Applications 112 (1) (2002) 111–128.
[13] E.H. Gafni, D.P. Bertsekas, Two-metric projection methods for constrained optimization, SIAM Journal of Control and Optimization 22 (1984) 936–964.
[14] D.R. Han, Hong K. Lo, Solving non-additive traffic assignment problems: a descent method for co-coercive variational inequalities, European Journal of Operational Research 159 (2004) 529–544.
[15] P.T. Harker, J.S. Pang, A damped newton method for the linear complementarity problem, Lecture in Applied Mathematics 26 (1990) 256–284.
[16] P. Marcotee, J.P. Dussault, A note on a globally convergent newton method for solving variational inequalities, Operation Research Letters 6 (1987) 35–42.
[17] K. Taji, M. Fukushima, T. Ibaraki, A globally convergent newton method for solving strongly monotone variational inequalities, Mathematical Programming 58 (1993) 369–383.