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Near-field and far-field approximations by the Adomian and asymptotic decomposition methods

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ABSTRACT

The Adomian decomposition method and the asymptotic decomposition method give the near-field approximate solution and far-field approximate solution, respectively, for linear and nonlinear differential equations. The Padé approximants give solution continuation of series solutions, but the continuation is usually effective only on some finite domain, and it can not always give the asymptotic behavior as the independent variables approach infinity. We investigate the global approximate solution by matching the near-field approximation derived from the Adomian decomposition method with the far-field approximation derived from the asymptotic decomposition method for linear and nonlinear differential equations. For several examples we find that there exists an overlap between the near-field approximation and the far-field approximation, so we can match them to obtain a global approximate solution. For other nonlinear examples where the series solution from the Adomian decomposition method has a finite convergent domain, we can match the Padé approximant of the near-field approximation with the far-field approximation to obtain a global approximate solution representing the true, entire solution over an infinite domain.

1. Introduction

The Adomian decomposition method (DM) [1–5] provides a practical technique for the resolution of a large class of linear or nonlinear, ordinary or partial, differential equations and it has been applied to many fields of science and engineering, see e.g. [4–13].

The Adomian DM gives a series solution, or a convergent sequence of analytic approximants, generated by recursion. In some cases, the series solution converges slowly for the values of independent variables far from the initial values, see Example 1. In other cases, for some nonlinear differential equations the convergent domain of the series solution is not sufficiently large as compared to the domain of the true solution. In order to obtain a more extended solution in such situations, Adomian et al. [14,15] proposed one-step numerical algorithms derived from the Adomian DM, Wazwaz [16,17], Jiao et al. [18] and Chrysos et al. [19] used the Padé approximants of the decomposition solution, and Duan [20] developed a different initial iterative component. All of these methods begin with the series solution of the Adomian DM, and indeed obtain the approximate solution in a larger field. But if the true, entire solution exists on an infinite field then these methods will not be adequate for engineering models.

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In this article we introduce Adomian's lesser-known asymptotic DM [21–23], and use the matching of the near-field approximation derived from the Adomian DM with the far-field approximation derived from the asymptotic DM, or the matching of the Padé approximant of the near-field approximation with the far-field approximation, to overcome this difficulty of finite domains of convergence.

The asymptotic DM [21–23], also was introduced on pp. 72–87 and 139–140 in [3] and pp. 18–20 and 241–243 in [5], was developed for finding the asymptotic solution, including the asymptotes, of linear and nonlinear, ordinary and partial differential equations. Just as the Adomian DM, the asymptotic DM is a straightforward, powerful technique, which provides an efficient means for the analytic and numerical approximation even for a wide class of analytic nonlinearities, without having to appeal to the questionable practice of perturbation or linearization. In [24] Haldar and Datta applied the asymptotic DM to calculate integrals not expressible in terms of elementary functions nor adequately tabulated; for example, they elegantly derived Ramanujan's integral formula using the asymptotic DM. Evans and Hossen [25] obtained a favorable comparison between the approximate solution of the asymptotic DM and the finite element method for a nonlinear heat equation over a general domain. Also Hadizadeh and Maleknejad [26] solved the nonlinear model for a biosensor deriving the asymptotic analysis of the solution using decomposition.

The asymptotic DM is an interesting variation of the Adomian DM. Rather than the repeated integrations as required in the Adomian DM to compute the solution components, we instead determine the solution components by repeated differentiations in the asymptotic DM. In both of these methodologies, the solution is viewed as a decomposition of the pre-existent, unique, analytic function, which identically satisfies the mathematical model under consideration to be determined by recursion. We also have observed that the asymptotic DM can lead to a rapidly terminating series if the input function is a polynomial.

The text is divided into four sections. In next section we introduce the asymptotic DM for linear and nonlinear differential equations. In Section 3 we illustrate the technique for assembling global approximate solutions by matching near-field and far-field approximants. Section 4 summarizes our findings.

2. The asymptotic DM

We propose to illustrate the essence of the asymptotic DM by first developing solutions for the linear case and then for the nonlinear case.

Consider the linear ordinary differential equation

$$Lu + Ru = g(t). \tag{1}$$

For clarity of exposition we choose $L = \frac{d}{dt}$ and $R = \alpha(t)$ with the initial condition taken as $u(0) = 0$.

We remark that in the Adomian DM, we would invert the linear differential operator L as $L^{-1} = \int_0^t (\cdot) dt$, however in the asymptotic DM we instead choose to invert the remainder operator R as $R^{-1} = \frac{1}{\alpha(t)}$, where we necessarily restrict $\alpha(t) \neq 0$. First we solve for Ru to obtain

$$Ru = g(t) - Lu,$$

then we apply the inverse operator R^{-1}

$$R^{-1}Ru = R^{-1}g(t) - R^{-1}Lu.$$

Since $R^{-1}Ru = u$ for $R \neq 0$, we have derived

$$u = R^{-1}g(t) - R^{-1}Lu. \tag{2}$$

We assume that the solution u is decomposed into components u_n such that $u = \sum_{n=0}^{\infty} u_n$ and upon substitution we have

$$\sum_{n=0}^{\infty} u_n = R^{-1}g(t) - R^{-1}L\sum_{n=0}^{\infty} u_n.$$

Using Adomian's choice for the initial solution component u_0, we thence develop the recursion scheme for the solution components

$$u_0 = R^{-1}g(t), \quad u_{n+1} = -R^{-1}Lu_n, \quad n \geqslant 0. \tag{3}$$

We can thus obtain the asymptotic solution for the linear case in the form

$$u \simeq \sum_{n=0}^{\infty} (-1)^n (R^{-1}L)^n R^{-1} g(t), \tag{4}$$

or upon substitution for R and $g(t)$ we have

$$u \simeq \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\alpha(t)} \frac{d}{dt}\right)^n \frac{1}{\alpha(t)} g(t).$$

We use the symbol $\simeq$ to emphasize the series on the right hand side can be divergent. Practically we use the partial sum

$$\varphi_m(t) = \sum_{k=0}^{m-1} u_k(t). \tag{5}$$

as the approximation to the solution.

Next we develop the asymptotic DM for the nonlinear case. Consider the nonlinear ordinary differential equation

$$Lu + Nu = g(t). \tag{6}$$

For clarity of exposition we choose $L = \frac{d}{dt}$ and the nonlinearity $Nu = f(u)$, where f is assumed to be analytic in u.

Again our goal is to solve for the solution by not inverting the linear differential operator L, but by inverting the nonlinear operator Nu and hence determining the asymptotic solution u. First we solve for Nu to obtain

$$Nu = g(t) - Lu,$$

or

$$f(u) = g(t) - Lu.$$

Next we assume the decomposition series $u = \sum_{n=0}^{\infty} u_n$ for the solution and that the analytic function $f(u)$ is decomposed into the series of Adomian polynomials such that

$$f(u) = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the nth Adomian polynomial is a function $A_n = A_n(u_0, \ldots, u_n)$. Upon substitution we have

$$\sum_{n=0}^{\infty} A_n = g(t) - L \sum_{n=0}^{\infty} u_n,$$

from where we develop the recursion scheme

$$A_0 = g(t), \quad A_n = -Lu_{n-1}, \quad n \geqslant 1,$$

or when emphasizing the dependency of the Adomian polynomials

$$A_0(u_0) = g(t), \quad A_n(u_0, \ldots, u_n) = -Lu_{n-1}, \quad n \geqslant 1. \tag{8}$$

In order to explicitly solve for the solution components u_n, we require the formulas of the Adomian polynomials for the specific nonlinearity $f(u)$. The definition of the Adomian polynomials A_n was first provided in [2] by the formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[f\left(\sum_{m=0}^{\infty} u_m \lambda^m \right) \right] \Bigg|_{\lambda=0},$$

and where λ is simply a grouping parameter of convenience. New, efficient algorithms in MATHEMATICA for rapid computer-generation of the Adomian polynomials are provided in [20,27] by Duan. The first few Adomian polynomials are

$$\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= f'(u_0)u_1, \\
A_2 &= f'(u_0)u_2 + f''(u_0)\frac{u_1^2}{2!}, \\
A_3 &= f'(u_0)u_3 + f''(u_0)u_1u_2 + f^{(3)}(u_0)\frac{u_1^3}{3!}, \\
A_4 &= f'(u_0)u_4 + f''(u_0)\left[\frac{1}{2}u_2^2 + u_1u_3\right] + f^{(3)}(u_0)\frac{1}{2}u_1^2u_2 + f^{(4)}(u_0)\frac{1}{4!}u_1^4, \\
&\ldots,
\end{aligned}$$

where $f^{(n)}(u_0) = \frac{d^n}{du_0^n} f(u_0)$. Note that if we define the convenient shorthand

$$A_n = f'(u_0)u_n + B_n, n \geqslant 1, \tag{9}$$

where the B_n are designated the complementary polynomials

$$\begin{aligned}
B_1 &= 0, \\
B_2 &= f''(u_0)\frac{u_1^2}{2!}, \\
B_3 &= f''(u_0)u_1u_2 + f^{(3)}(u_0)\frac{u_1^3}{3!}, \\
B_4 &= f''(u_0)\left[\frac{1}{2}u_2^2 + u_1u_3\right] + f^{(3)}(u_0)\frac{1}{2}u_1^2u_2 + f^{(4)}(u_0)\frac{1}{4!}u_1^4, \\
&\ldots
\end{aligned}$$

Then we rewrite the recursion scheme (8) as

$$A_0(u_0) = g(t), f'(u_0)u_n + B_n(u_0, \ldots, u_{n-1}) = -Lu_{n-1}, n \geqslant 1,$$

or

$$f(u_0) = g(t) \Rightarrow u_0 = f^{-1}[g(t)], \tag{10}$$

$$f'(u_0)u_n + B_n = -Lu_{n-1} \Rightarrow u_n = \frac{-B_n - Lu_{n-1}}{f'(u_0)}, \quad n \geqslant 1, \tag{11}$$

for $f(u_0) \neq 0$, and where f^{-1} is the inverse function of f. Thus we obtain the m-term asymptotic approximation $\varphi_m(t) = \sum_{n=0}^{m-1} u_n(t)$ for the nonlinear equation.

For the nonlinear differential equation in the form

$$Lu + Ru + Nu = g(t), \tag{12}$$

where $L = \frac{d}{dt}$, $R = \alpha(t)$, and $Nu = f(u)$ is an analytic nonlinearity. We can consider to invert the operator R as in the case of linear equations

$$u = R^{-1}g(t) - R^{-1}Lu - R^{-1}f(u),$$

where $R \neq 0$. Suppose the decomposition $u = \sum_{n=0}^{\infty} u_n$ for the solution and decompose $f(u)$ as above $f(u) = \sum_{n=0}^{\infty} A_n$. The recursion scheme

$$u_0 = R^{-1}g(t), \quad u_n = -R^{-1}Lu_{n-1} - R^{-1}A_{n-1}, \quad n \geqslant 1, \tag{13}$$

determines the analytic components of the asymptotic solution.

Corresponding to the n-term asymptotic approximation $\varphi_n(t)$, we use $\phi_m(t)$ to denote the m-term approximation of the solution derived from the Adomian DM, where m does not necessarily equal n. We find that the m-term approximation $\phi_m(t)$ from the Adomian DM can serve as the near-field approximation of the solution $u(t)$, i.e., t close to the initial value $t = t_0$, while the asymptotic approximation $\varphi_n(t)$ from the asymptotic DM can serve as the far-field approximation, i.e., t far from the initial value $t = t_0$.

3. The global approximate solution by means of matching the asymptotic DM approximation

We illustrate by some examples the matching of the near-field approximations with the far-field approximations, and in other cases the matching of the Padé approximants of the near-field approximations with the far-field approximations.

Example 1. Consider the first order linear differential equation

$$\frac{du}{dt} + (at^2 + bt + c)u = 1, \quad t > 0, \; u(0) = 0, \tag{14}$$

where a, b, c are constants satisfying $a^2 + b^2 \neq 0$ and $at^2 + bt + c > 0$ for all $t > 0$.

In operator notation, we have

$$Lu + Ru = g, \tag{15}$$

where $L = \frac{d}{dt}$, $R = at^2 + bt + c$ and $g = 1$.

Solving by the Adomian DM, we rewrite the Eq. (15) as

$$Lu = g - Ru.$$

Applying the inverse operator $L^{-1}(\cdot) = \int_0^t (\cdot) dt$ to both sides gives

$$u = u(0) + L^{-1}g - L^{-1}Ru,$$

The Adomian DM decomposes the solution as a series $u = \sum_{n=0}^{\infty} u_n$, and substitution gives

$$\sum_{n=0}^{\infty} u_n = u(0) + L^{-1}g - L^{-1}R \sum_{n=0}^{\infty} u_n.$$

The components of the series solution are given by the recursion scheme

$$u_0 = u(0) + L^{-1}g, \quad u_n = -L^{-1}Ru_{n-1}, \quad n = 1, 2, \ldots \tag{16}$$

In this example, we have $u(0) = 0$, $g(t) = 1$ and $R = at^2 + bt + c$. Therefore we compute the solution components

$$\begin{aligned}
u_0 &= t,\\
u_1 &= -\frac{ct^2}{2} - \frac{bt^3}{3} - \frac{at^4}{4},\\
u_2 &= \frac{c^2t^3}{6} + \frac{5}{24}bct^4 + \frac{b^2t^5}{15} + \frac{3}{20}act^5 + \frac{7}{72}abt^6 + \frac{a^2t^7}{28},\\
&\ldots,\\
u_n &= -\int_0^t (at^2 + bt + c)u_{n-1}dt, n = 1, 2, \ldots.
\end{aligned}$$

The partial sums $\phi_m = \sum_{n=0}^{m-1} u_n$ of the Adomian decomposition series can serve as a near-field approximate solution.

Solving (15) by the asymptotic DM, we rewrite it as

$$Ru = g - Lu.$$

Applying the inverse operator $R^{-1} = \frac{1}{at^2+bt+c}$ to both sides gives

$$u = R^{-1}g - R^{-1}Lu.$$

Also the asymptotic DM decomposes the solution as a series $u = \sum_{n=0}^{\infty} u_n$, and substitution gives

$$\sum_{n=0}^{\infty} u_n = R^{-1}g - R^{-1}L\sum_{n=0}^{\infty} u_n.$$

The components of the asymptotic series solution are given by the recursion scheme

$$u_0 = R^{-1}g, \quad u_n = -R^{-1}Lu_{n-1}, \quad n = 1, 2, \ldots \tag{17}$$

In this example, substituting the values of $g(t)$ and R^{-1} leads to the components of the asymptotic solution

$$\begin{aligned}
u_0 &= \frac{1}{at^2 + bt + c},\\
u_1 &= \frac{2at + b}{(at^2 + bt + c)^3},\\
u_2 &= \frac{3b^2 - 2ac + 10abt + 10a^2t^2}{(at^2 + bt + c)^5},\\
&\ldots,\\
u_n &= -\frac{u'_{n-1}}{at^2 + bt + c}, \quad n \geqslant 1.
\end{aligned}$$

For the special case of $a = c = 0$, $b = 2$, the differential equation in (14) becomes

$$\frac{du}{dt} + 2tu = 1, \quad u(0) = 0. \tag{18}$$

According to the above method we obtain the Adomian decomposition solution

$$u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} n! t^{2n+1}}{(2n+1)!}, \tag{19}$$

and the near-field approximation

$$\phi_m(t) = \sum_{n=0}^{m-1} u_n = \sum_{n=0}^{m-1} \frac{(-1)^n 2^{2n} n! t^{2n+1}}{(2n+1)!}. \tag{20}$$

Also we obtain the asymptotic decomposition series

$$u \simeq \sum_{n=0}^{\infty} \frac{(2n+1)!}{(2n+1)2^{2n+1} n! t^{2n+1}}, \tag{21}$$

and the asymptotic approximation

$$\varphi_m(t) = \sum_{n=0}^{m-1} u_n = \sum_{n=0}^{m-1} \frac{(2n+1)!}{(2n+1)2^{2n+1} n! t^{2n+1}}. \tag{22}$$

We recognize that, in this case, the asymptotic decomposition series in (21) is a divergent series.

For suitable m and n, computing $\phi_m(t)$ and $\varphi_n(t)$, we can discover that there is significant overlap in some region of t. For example, $\phi_{40}(t)$ and $\varphi_6(t)$ overlap in the approximate region $2.6 < t < 3.6$, see Table 1.

Therefore we can assemble a global approximate solution by matching the two approximants simply as

$$\tilde{u}(t) = \phi_{40}(t)h(\xi - t) + \varphi_6(t)h(t - \xi), \tag{23}$$

where ξ belongs to the region of t where $\phi_{40}(t)$ and $\varphi_6(t)$ overlap, $h(t)$ is the unit step function

$$h(t) = \begin{cases} 0, & t < 0, \\ 1/2, & t = 0, \\ 1, & t > 0. \end{cases} \tag{24}$$

In this particular example the exact solution can be expressed in terms of the imaginary error function

$$u^*(t) = \frac{\sqrt{\pi}}{2} e^{-t^2} \operatorname{erfi}(t), \tag{25}$$

where $\operatorname{erfi}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{t^2} dt$. In Fig. 1 we plot the curves of the near-field approximation $\phi_{40}(t)$, the far-field approximation $\varphi_6(t)$ and the exact solution $u^*(t)$.

By increasing the orders m and n of the near-field approximation $\phi_m(t)$ and the far-field approximation $\varphi_n(t)$ we can match or blend the respective approximants with higher accuracy. For example, the matching of $\phi_{60}(t)$ with $\varphi_{10}(t)$ is more accurate as demonstrated in Table 2.

We remark that although the Adomian series solution (19) is convergent for all t, but the convergent speed slows dramatically as t increases. Here using the far-field approximation $\varphi_n(t)$ is very efficient since we do not require n to become excessively large for larger and larger t. Also we have verified the Padé approximants of the near-field approximations $\phi_{40}(t)$ and $\phi_{60}(t)$ by MATHEMATICA. The results show that the Padé approximants $\left[\frac{40}{40}\right]\{\phi_{40}(t)\}$ and $\left[\frac{60}{60}\right]\{\phi_{60}(t)\}$ deviate from the exact solution $u^*(t)$, respectively for $t > 12$ and $t > 20$.

Other cases are verified to have similar characteristics.

Example 2. Consider the second order linear differential equation

$$u''(t) + 2u'(t) + 10u(t) = \sqrt{t}, \quad t > 0, \tag{26}$$

subject to the initial conditions $u(0) = 0$, $u'(0) = 0$.

By the Adomian DM, integrating the equation and utilizing the initial conditions, we get

$$u = \frac{4}{15} t^{5/2} - 2L^{-1}u - 10L^{-1}L^{-1}u,$$

Table 1
The overlap region of $\phi_{40}(t)$ and $\varphi_6(t)$.

t	2.6	2.85	3.1	3.35	3.6
$\lvert\phi_{40} - \varphi_6\rvert$	0.0003034	0.0001998	0.00008541	0.00003108	0.0001377

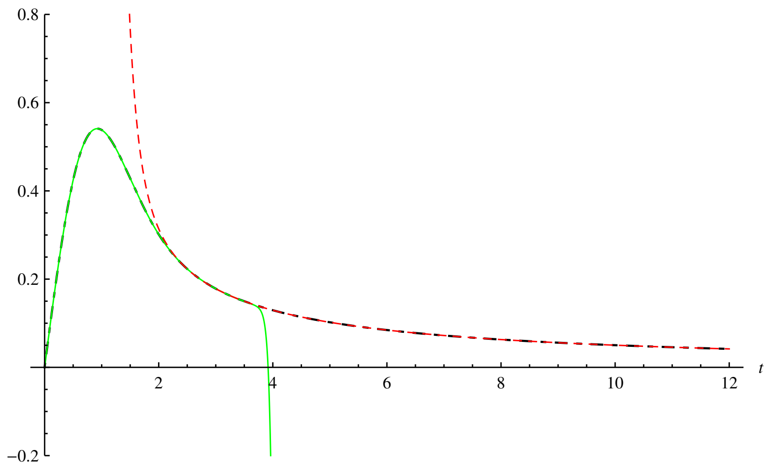

Fig. 1. The near-field approximation $\phi_{40}(t)$ (solid line), far-field approximation $\varphi_6(t)$ (dashed line) and exact solution $u^*(t)$ (dot-sideline).

Table 2
The overlap region of $\phi_{60}(t)$ and $\varphi_{10}(t)$.

t	3.3	3.55	3.8	4.05	4.3
$\lvert\phi_{60}-\varphi_{10}\rvert$	4.35×10^{-6}	2.16×10^{-6}	6.60×10^{-7}	1.72×10^{-7}	4.27×10^{-7}

where $L^{-1}(\cdot)=\int_0^t(\cdot)dt$. The components of the m-term near-field approximation $\phi_m=\sum_{n=0}^{m-1}u_n$ are given by

$$u_0=\frac{4t^{5/2}}{15},\quad u_n=-2L^{-1}u_{n-1}-10L^{-1}L^{-1}u_{n-1},\quad n\geqslant 1. \tag{27}$$

Computing gives

$$\begin{aligned}u_1&=-\frac{16t^{7/2}}{105}-\frac{32t^{9/2}}{189},\\ u_2&=\frac{64t^{9/2}}{945}+\frac{256t^{11/2}}{2079}+\frac{1280t^{13/2}}{27027},\\ &\cdots\end{aligned}$$

For the far-field approximation by the asymptotic DM, we write Eq. (26) as

$$u=\frac{\sqrt{t}}{10}-\frac{1}{10}u''-\frac{1}{5}u'.$$

Applying the iterative scheme

$$u_0=\frac{\sqrt{t}}{10},\quad u_n=-\frac{1}{10}u''_{n-1}-\frac{1}{5}u'_{n-1},\quad n\geqslant 1, \tag{28}$$

we obtain the components of the far-field approximation $\varphi_n(t)$

$$\begin{aligned}u_1&=\frac{1}{400t^{3/2}}-\frac{1}{100\sqrt{t}},\\ u_2&=-\frac{3}{3200t^{7/2}}+\frac{3}{2000t^{5/2}}-\frac{1}{1000t^{3/2}},\\ &\cdots\end{aligned}$$

We note that $e^{-t}\cos 3t$ and $e^{-t}\sin 3t$ are two linear independent solutions of the homogenous differential equation corresponding to Eq. (26). By the method of variation of constants we obtain the exact solution of the initial value problem in this example

$$u^*(t)=\frac{1}{3}\int_0^t\sqrt{s}e^{-(t-s)}\sin 3(t-s)ds. \tag{29}$$

In Fig. 2 we plot the curves of the near-field approximation $\phi_{50}(t)$, the far-field approximation $\varphi_{20}(t)$ and the exact solution $u^*(t)$. Obviously the overlap of $\phi_{50}(t)$ and $\varphi_{20}(t)$ appears in the approximate region $5<t<6$.

Example 3. Consider the differential equation of Riccati type

$$\frac{du}{dt}+(at+b)u^2=ct+d,\quad t>0, \tag{30}$$

with the initial value $u(0)=0$, where a, b, c, d are constants satisfying $a,b,c,d\geqslant 0$, $a^2+b^2\neq 0$, $c^2+d^2\neq 0$ and $ad\neq bc$.
By the Adomian DM we have by applying the integral operator $L^{-1}=\int_0^t(\cdot)dt$

$$u=dt+\frac{ct^2}{2}-L^{-1}(at+b)u^2,$$

Decompose u and u^2 as

$$u=\sum_{n=0}^{\infty}u_n,\quad u^2=\sum_{n=0}^{\infty}A_n, \tag{31}$$

where the Adomian polynomials A_n for the nonlinearity $f(u)=u^2$ are [3,13,20,28,29]

$$A_0=u_0^2, A_1=2u_0u_1, A_2=2u_0u_2+u_1^2, A_3=2u_0u_3+2u_1u_2,\ldots \tag{32}$$

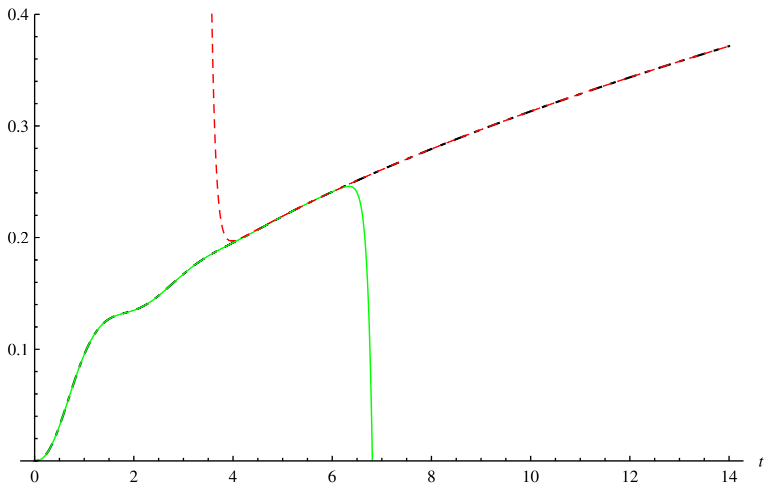

Fig. 2. The near-field approximation $\phi_{50}(t)$ (solid line), far-field approximation $\varphi_{20}(t)$ (dashed line) and the exact solution $u^*(t)$ (dot-sideline).

The components of the near-field approximation $\phi_m(t)$ are determined recursively

$$u_0 = dt + \frac{ct^2}{2},$$
$$u_1 = -\frac{1}{3}bd^2t^3 + \left[-\frac{1}{4}bcd - \frac{ad^2}{4}\right]t^4 + \left[-\frac{bc^2}{20} - \frac{acd}{5}\right]t^5 - \frac{1}{24}ac^2t^6,$$
$$\dots,$$
$$u_n = -\int_0^t (at+b)A_{n-1}dt, n \geqslant 1.$$

By the asymptotic DM we have from (30)

$$u^2 = \frac{ct+d}{at+b} - \frac{1}{at+b}u',$$

Using the same decompositions for u and u^2 as in the Adomian DM gives the iterative scheme for the asymptotic DM

$$A_0 = \frac{ct+d}{at+b}, \quad A_n = -\frac{1}{at+b}u'_{n-1}, \quad n \geqslant 1. \tag{33}$$

Utilizing the Adomian polynomials (32) we obtain the components of the far-field approximation $\varphi_n(t)$ according to the scheme (10) and (11)

$$u_0 = \sqrt{\frac{d+ct}{b+at}},$$
$$u_1 = \frac{-bc+ad}{4(b+at)^2(d+ct)},$$
$$u_2 = \frac{(-bc+ad)(5bc+7ad+12act)}{32(b+at)^{7/2}(d+ct)^{5/2}},$$
$$\dots$$

Next we consider the special case of $a = d = 1$, $b = c = 0$ in this example, i.e., the nonlinear differential equation

$$\frac{du}{dt} + tu^2 = 1, \quad t > 0, \ u(0) = 0, \tag{34}$$

By the Adomian DM the components u_k of near-field approximation $\phi_m = \sum_{k=0}^{m-1} u_k$ are evaluated

$$u_0 = t, u_1 = -\frac{t^4}{4}, u_2 = \frac{t^7}{14}, u_3 = -\frac{23t^{10}}{1120}, \dots$$

By the asymptotic DM the components of the far-field approximation $\varphi_n = \sum_{k=0}^{n-1} u_k$ are derived

$$u_0 = \frac{1}{\sqrt{t}}, u_1 = \frac{1}{4t^2}, u_2 = \frac{7}{32t^{7/2}}, u_3 = \frac{21}{64t^5}, \dots$$

Computation shows that this Adomian decomposition series has a finite radius of convergence. By plotting the curves of $\phi_m(t)$ and $\varphi_n(t)$ for several values of m and n we do not find any regions of overlap. In this case we connect the two approximations by the Padé approximant of $\phi_m(t)$, or simply replace $\phi_m(t)$ by its Padé approximant, then match the Padé approximant with $\varphi_n(t)$.

For example, we investigate ϕ_{20} and φ_8. The curves of the near-field approximation $\phi_{20}(t)$ and the far-field approximation $\varphi_8(t)$ are plotted in Fig. 3. Also the curve of the exact solution $u^*(t)$ is plotted in Fig. 3, where $u^*(t)$ can be given in terms of the Airy functions

$$u^*(t) = \frac{\sqrt{3}\mathrm{Ai}(t) - \mathrm{Bi}(t)}{\sqrt{3}\mathrm{Ai}'(t) - \mathrm{Bi}'(t)}. \tag{35}$$

Since $\phi_{20}(t)$ is a polynomial of degree 58, we compute its Padé approximant $\left[\frac{29}{29}\right]\{\phi_{20}(t)\}$ by MATHEMATICA and find the Padé approximant $\left[\frac{29}{29}\right]\{\phi_{20}(t)\}$ and the far-field approximation $\varphi_8(t)$ overlap in the approximate region $3 < t < 20$, see Fig. 4. Thus we can match them

$$\tilde{u}(t) = \left[\frac{29}{29}\right]\{\phi_{20}(t)\}h(\xi - t) + \varphi_8(t)h(t - \xi), \tag{36}$$

which is a global approximation, where ξ belongs to the region of overlap.

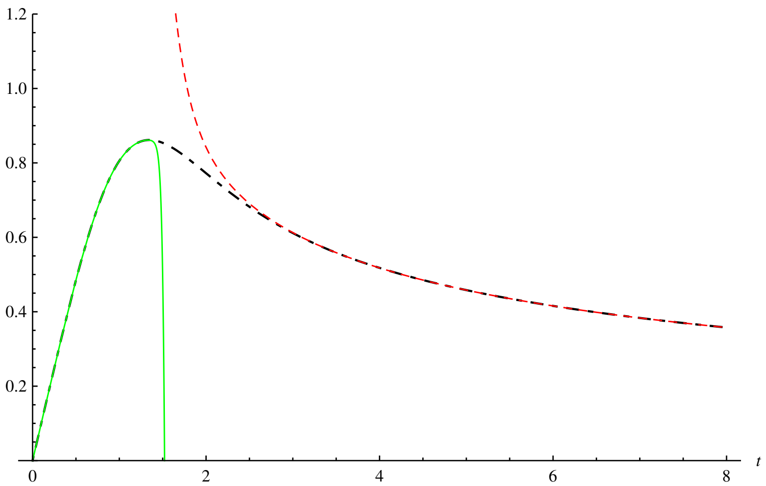

Fig. 3. The near-field approximation $\phi_{20}(t)$ (solid line), far-field approximation $\varphi_8(t)$ (dashed line) and exact solution $u^*(t)$ (dot-sideline).

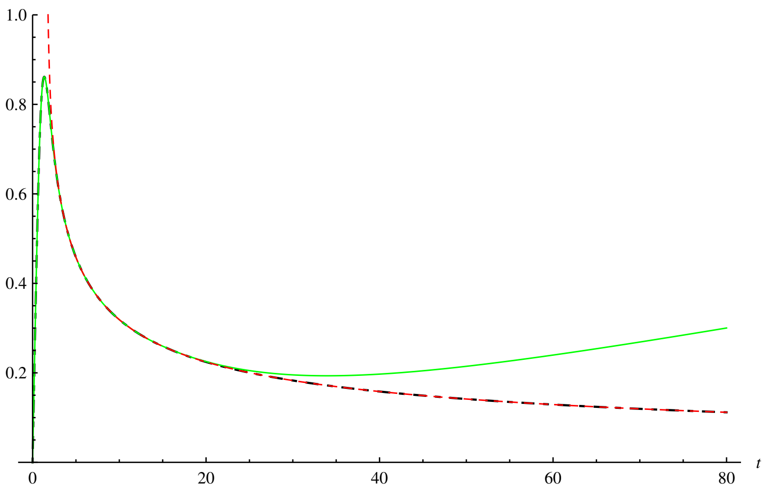

Fig. 4. The Padé approximant $\left[\frac{29}{29}\right]\{\phi_{20}(t)\}$ (solid line), far-field approximation $\varphi_8(t)$ (dashed line) and the exact solution $u^*(t)$ (dot-sideline).

We remark that the Padé approximant $\left[\frac{29}{29}\right]\{\phi_{20}(t)\}$ deviates from the exact solution $u^*(t)$ for $t > 20$. In fact the Padé approximant $\left[\frac{29}{29}\right]\{\phi_{20}(t)\}$ approaches infinity as $t \to +\infty$. So it can not give the asymptotic behavior of the exact solution as $t \to +\infty$.

Other cases have been verified to have similar properties. For example, for the case of $a = d = 0$, $b = c = 1$ in the example, the differential equation

$$\frac{du}{dt} + u^2 = t, \quad t > 0, \ u(0) = 0, \tag{37}$$

has an Adomian decomposition series with a finite domain of convergence. The curves of $\phi_{24}(t)$, $\varphi_9(t)$ and the exact solution $u^*(t)$ are plotted in Fig. 5, where the exact solution is

$$u^*(t) = \frac{\sqrt{3}\mathrm{Ai}'(t) + \mathrm{Bi}'(t)}{\sqrt{3}\mathrm{Ai}(t) + \mathrm{Bi}(t)}, \tag{38}$$

which can be obtained by MAPLE. We verified that the Padé approximant to the near-field approximation $\phi_{24}(t)$ matches the far-field approximation $\varphi_9(t)$ perfectly.

Example 4. Consider the nonlinear differential equation with a parameter power

$$\frac{du}{dt} + tu^a = 1, \quad t > 0, \ u(0) = 0, \tag{39}$$

where a is a constant such that $0 < a < 2$.

We remark that Adomian and Rach [30] previously considered the Adomian DM for the nonlinear differential equations with decimal power nonlinearities.

By the Adomian DM we have

$$u = t - L^{-1}tu^a,$$

where $L^{-1} = \int_0^t(\cdot)dt$. Set up the iterations

$$u_0 = t, \quad u_n = -L^{-1}tA_{n-1}, \quad n = 1, 2, \ldots \tag{40}$$

Computing the Adomian polynomials for u^a yields [3,13,20,27,28]

$$\begin{aligned}
A_0 &= u_0^a,\\
A_1 &= au_0^{a-1}u_1,\\
A_2 &= \frac{a(a-1)}{2}u_0^{a-2}u_1^2 + au_0^{a-1}u_2,\\
A_3 &= \frac{a(a-2)(a-1)}{6}u_0^{a-3}u_1^3 + a(a-1)u_0^{a-2}u_1u_2 + au_0^{a-1}u_3,\\
&\ \ \cdots
\end{aligned}$$

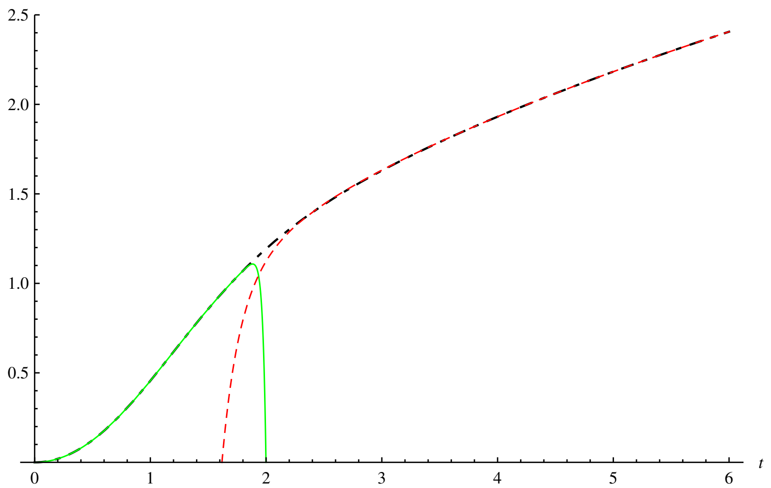

Fig. 5. The near-field approximation $\phi_{24}(t)$ (solid line), far-field approximation $\varphi_9(t)$ (dashed line) and exact solution $u^*(t)$ (dot-sideline).

By the recursion scheme (40) we obtain the components of the near-field approximation $\phi_m = \sum_{k=0}^{m-1} u_k$

$$u_1 = -\frac{t^{2+a}}{2+a}, u_2 = \frac{at^{3+2a}}{6+7a+2a^2}, \ldots$$

By the asymptotic DM we have

$$u^a = \frac{1}{t} - \frac{1}{t}\frac{d}{dt}u. \tag{41}$$

Using the recursion scheme

$$A_0 = \frac{1}{t}, \quad A_n = -\frac{1}{t}\frac{d}{dt}u_{n-1}, \quad n = 1, 2, \ldots, \tag{42}$$

we compute the asymptotic components from the Adomian polynomials in turn

$$\begin{aligned}
&A_0 = \frac{1}{t} \Rightarrow u_0 = t^{-1/a},\\
&A_1 = -\frac{1}{t}\frac{d}{dt}u_0 = \frac{t^{-2-\frac{1}{a}}}{a} \Rightarrow u_1 = \frac{1}{a^2}t^{-1-\frac{2}{a}},\\
&A_2 = -\frac{1}{t}\frac{d}{dt}u_1 = \frac{(2+a)t^{-3-\frac{2}{a}}}{a^3} \Rightarrow u_2 = \frac{5+a}{2a^4}t^{-2-\frac{3}{a}},\\
&\quad\cdots
\end{aligned}$$

We check that for the values of a approaching 1 the near-field approximation $\phi_m(t)$ and far-field approximation $\varphi_n(t)$ can be found to almost have a small region of overlap. Fig. 6 shows the case of $a = 0.8$, where $\phi_{41}(t)$ and $\varphi_7(t)$ overlap almost in the approximate region $3.1 < t < 3.7$, the numeric solution $u^*(t)$ is obtained by MATHEMATICA with 24 digits of precision.

We simulate different cases for $0 < a < 2$ and find that the more a is close to 0 or 2, the more difficult it is to match $\phi_m(t)$ with $\varphi_n(t)$. Whereas we can match the Padé approximant of $\phi_m(t)$ with the far-field approximation $\varphi_n(t)$.

Example 5. Consider the nonlinear differential equation with the fractional power nonlinearity

$$\frac{du}{dt} + u^{3/4} + tu = t, \quad t > 0, \ u(0) = 0, \tag{43}$$

By the Adomian DM we have the integral equation

$$u = \frac{t^2}{2} - \int_0^t (tu + u^{3/4})dt. \tag{44}$$

The components of the near-field approximation $\phi_m(t)$ are determined by

$$u_0 = \frac{t^2}{2}, \quad u_n = -\int_0^t (tu_{n-1} + A_{n-1})dt, \quad n \geqslant 1.$$

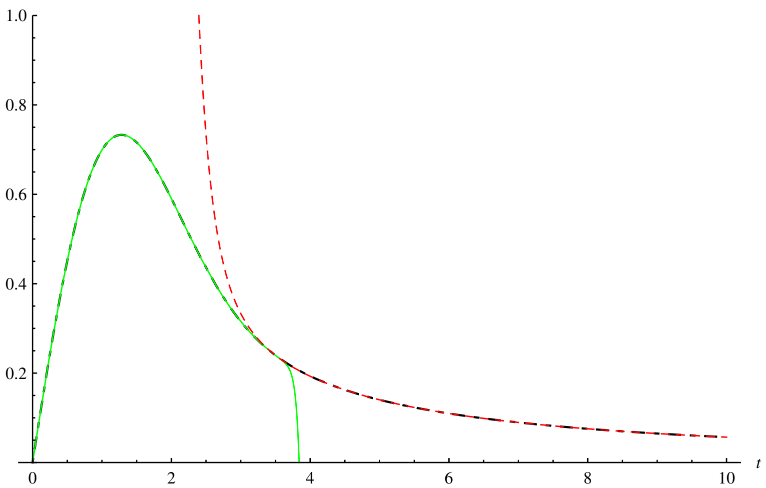

Fig. 6. The near-field approximation $\phi_{41}(t)$ (solid line), far-field approximation $\varphi_7(t)$ (dashed line) and the numeric solution $u^*(t)$ (dot-sideline) in the case of $a = 0.8$.

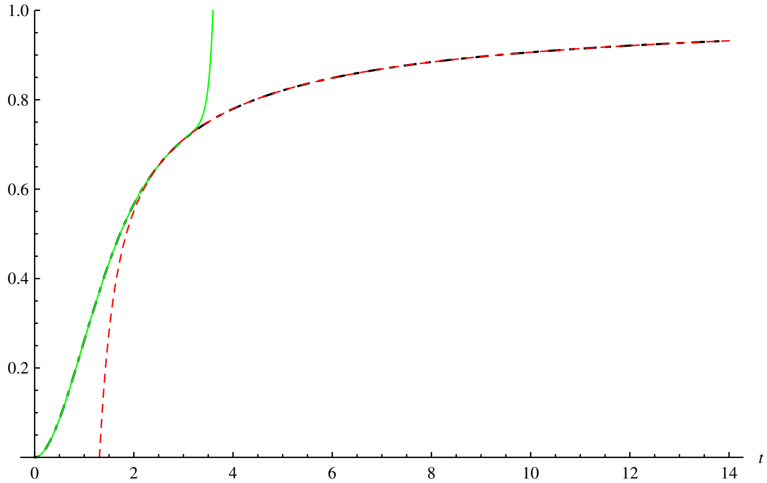

Fig. 7. The near-field approximations $\phi_{24}(t)$ (solid line), far-field approximations $\varphi_4(t)$ (dashed line) and numeric solution $u^*(t)$ (dot-sideline).

The Adomian polynomials A_n for the nonlinearity $f(u) = u^{3/4}$ are given by Example 4 for $a = \frac{3}{4}$. By computation we have

$$u_1 = -\frac{2^{1/4}t^{5/2}}{5} - \frac{t^4}{8},$$
$$u_2 = \frac{t^3}{10\sqrt{2}} + \frac{47t^{9/2}}{360 \times 2^{3/4}} + \frac{t^6}{48},$$
$$\ldots$$

By the asymptotic DM we rewrite Eq. (43) according to the method for Eq. (12)

$$u = 1 - \frac{1}{t}u^{3/4} - \frac{1}{t}u'. \tag{45}$$

This gives the iterative scheme

$$u_0 = 1, \quad u_n = -\frac{1}{t}A_{n-1} - \frac{1}{t}u'_{n-1}, \quad n \geqslant 1. \tag{46}$$

The components of the far-field approximation $\varphi_n(t)$ are computed

$$u_1 = -\frac{1}{t}, u_2 = -\frac{1}{t^3} + \frac{3}{4t^2}, u_3 = -\frac{3}{t^5} + \frac{9}{4t^4} - \frac{15}{32t^3}, \ldots$$

By computation we find that the near-field approximation $\phi_{24}(t)$ and the far-field approximation $\varphi_4(t)$ overlap almost in the approximate region $2 < t < 3$, see Fig. 7. So they can match to form the global approximate solution.

Finally, we present two degenerate examples. For the equation $u' + u^2 = 1$, the asymptotic DM leads to the asymptote lines $u = \pm 1$ for the solutions

$$u = \frac{(1 + u(0))e^{2t} - 1 + u(0)}{(1 + u(0))e^{2t} + 1 - u(0)}.$$

For the equation $u'' + 2u' + u = t^2$, the asymptotic DM leads to a terminating series $\varphi(t) = t^2 - 4t + 6$, which is readily verified to be a particular solution, and an asymptote curve for the solution subject to any initial conditions.

4. Conclusion

We consider the solution continuation of the series obtained by the Adomian DM by means of the asymptotic DM. Solution continuation by the Padé approximants is shown to be effective only within a finite region, while we illustrate a method providing a global approximate solution by means of matching the asymptotic DM.

For some differential equations, the near-field approximation from the Adomian DM and the far-field approximation from the asymptotic DM exhibit a region of overlap, hence we can match the two approximations to assemble a global approximate solution. For other nonlinear differential equations, although the near-field approximation and the far-field approximation do not overlap, the Padé approximant of the near-field approximation matches with the far-field approximation perfectly forming a global approximation.

A large number of computations have been carried out by MATHEMATICA 7.0 and in Examples 4 and 5 the calculation of the Adomian polynomials used a new, improved algorithm presented in [20,27] by Duan.

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