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Decision Support

Optimal advertising and pricing in a class of general new-product adoption models

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ABSTRACT

In [21], Sethi et al. introduced a particular new-product adoption model. They determine optimal advertising and pricing policies of an associated deterministic infinite horizon discounted control problem. Their analysis is based on the fact that the corresponding Hamilton–Jacobi–Bellman (HJB) equation is an ordinary non-linear differential equation which has an analytical solution. In this paper, generalizations of their model are considered. We take arbitrary adoption and saturation effects into account, and solve finite and infinite horizon discounted variations of associated control problems. If the horizon is finite, the HJB-equation is a 1st order non-linear partial differential equation with specific boundary conditions. For a fairly general class of models we show that these partial differential equations have analytical solutions. Explicit formulas of the value function and the optimal policies are derived. The controlled Bass model with isoelastic demand is a special example of the class of controlled adoption models to be examined and will be analyzed in some detail.

1. Introduction

In [21], Sethi et al. propose a new-product adoption model of durable goods and solve an associated deterministic infinite horizon control problem which involves two marketing instruments: price charged and advertising effort made. The authors assume a particular state dynamic as well as a special objective function. The state of the system represents (normalized) cumulative sales since a fixed market potential of customers is assumed to be given. Thus, the market share captured by time t can be described by the fraction of the market size, i.e. the state of the system is characterized by a number x within the unit interval. The dynamic, besides depending on the control variables p (price) and w (advertising rate), involves the state in terms of the factor $\sqrt{1-x}$. As explained in [21] Sethi et al. choose $\sqrt{1-x}$ as an approximation to the Bass functional term $(1-x)+x(1-x)=1-x^2$. They assume the discounted profit function to be of the special form

$$\int_0^\infty e^{-rt}(p(t)\dot{x}(t)-w(t)^2)dt,$$

where $p(t)$ and $w(t)$ denote admissible controls, $\dot{x}(t)$ is the rate of change of x, and r is a fixed discount rate. The associated Bellman equation is a non-linear ordinary differential equation (ODE). For the special cases that the state dynamic is either of the form $\dot{x}(t)=\mu w(t)/(1-\hat{\varepsilon}p(t))\sqrt{1-x(t)}$, $0\leqslant p(t)\leqslant 1/\hat{\varepsilon}$, μ and $\hat{\varepsilon}$ positive parameters, or a particular constant elasticity model $\dot{x}(t)=\mu w(t)p(t)^{-\varepsilon}\sqrt{1-x(t)}$, $w(t)\geqslant 0$, $p(t)>0$, $\varepsilon>1$, the ODE has – surprisingly – a very simple explicit solution of the form $const\cdot x$.

In this article we are going to consider generalizations of the model if demand is isoelastic and we shall solve the corresponding control problems. The extensions include general adoption and saturation effects, a (slight) generalization of the objective function, time dependent discount rates and, in particular, the corresponding finite horizon problems. If the horizon is finite the associated Bellman equation is a 1st order non-linear partial differential equation with specific boundary conditions. The structure which will be assumed, see below, implies that this class of partial differential equations has separable solutions in time and state.

Our analysis has been strongly influenced and has been motivated by revenue management applications of dynamic pricing. At first glance, adoption models and traditional pricing models seem to be two different worlds. In [9], it has been pointed out that such models, to some extent, are but opposite sides of the same coin: one simply interprets the fraction of the untapped market as the fraction of yet unsold items of an initial inventory.

When selling perishable products it is imperative to look at finite horizon problems. In the context of durable goods, infinite horizon problems are common. However, the marketing of movies or (potentially bestselling) books – entertainment products in general – is closely related to selling perishable products. In [9], finite horizon and infinite horizon problems have been considered simultaneously and we shall do the same in this article. To simplify the transfer of some ideas and results from [9] we shall adopt the following notation throughout: $x(t)$ denotes the (fractional) market share aquired by time t while $y(t):=y_0-x(t)$ is the fraction of

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the market still to be captured. An initial value y_0, e.g. $y_0 = 1$, should be thought of as selling a given inventory. This is equivalent to the situation when nobody has yet purchased the new product and the fraction of potential buyers is one. Motivated by the special power function $\sqrt{1-x}$, see above, we shall consider dynamics of the general form, $y(0) = y_0$,

$$\dot{y}(t) = -\lambda_t := -\lambda(t, p(t), w(t), y(t)) := -\mu(t)p(t)^{-\varepsilon}w(t)^{\delta}\psi(y(t)),$$

and profit functions of the form

$$\int_0^T e^{-R(t)}(p(t)\lambda_t - w(t)^a)dt.$$

The horizon T is allowed to be finite or infinite. The different factors of the sales intensity $\lambda(t, p, w, y)$ capture specific aspects of the adoption process. The first term $\mu(t)$, the arrival intensity at time t, is supposed to be a positive function on $[0,T]$. The second factor and the third one are the control activators. The fourth factor $\psi(y)$ is supposed to be a positive function on the open interval $(0, y_0)$ which satisfies some technical conditions, see Section 2.

The factor μ – seasonality is a prominent example of time influence – and the system function ψ determine the dynamic of the system if prices and advertising rates are given. We distinguish between a time factor and a state dependent behavior/communication factor. The latter depends on the present market share and this dependence is described by the function ψ.

The innocent looking function μ is most important. It is proportional to the number of people interested in buying the product in a short time interval around t. In applications, μ itself is typically a product or a more complicated expression depending on several quantities. It includes, for instance, a proportionality factor which commensurates dollars spent on advertising and generated revenue. Typically, the function $\mu(t)$ is monotone increasing, at least for a while. In addition, factors like market presence, consumer sentiment and the income evolution of potential buyers are all important. Such factors are assumed to be incorporated in μ. See [8] for a specific model which emphasizes when potential buyers can afford to buy the product. In [2], the generalized Bass model has been defined. The authors of [2] introduce a time factor which they call the "current marketing effort". While the two concepts are related they are not the same. An alternative model which can help specifying μ can be developed based on ideas explained in [4]. Bemmaor proposes mixing specific densities (defined on the non-negative real line) which capture an individual buyer's propensity to buy. He ends up with a density of first-purchase times across heterogeneous consumers. The function μ could be such a density. But other choices are possible too, see [12]. In some of the illustrative examples to be discussed below we shall – to simplify the presentation or to highlight certain aspects – assume μ to be a constant. But the fact that μ is typically a function which depends on time needs to be kept in mind, see the management recommendations in Section 3. The [constant] price elasticity ε will need to exceed 1, while the [constant] advertising elasticity δ has to satisfy the inequalities $0 \leqslant \delta < a$. It will turn out that the effectiveness-cost ratio $\Delta := \delta/a$ is a critical parameter. There is a non-negative function r which represents a variable discount rate; it determines $R(t) := \int_0^t r(s)ds$. This general form of R makes it possible to also analyze, for instance, problems with hyperbolic discounting. Note, the special parameter setting $\delta = 1$, $a = 2$, $\psi(y) = y^{1/2}$, $r(t) \equiv r > 0$ and $T = \infty$ specifies the model considered in [21].

In Section 4 we analyze the class of general power functions $\psi(y) = y^b$, $b > -(\varepsilon - 1)$, $b \in \mathbb{R}$, in detail. We obtain the results of [21] as particular examples, if b is set equal to 0.5. Power functions are the best known models to describe the influence of "external" factors or "innovation" on the diffusion of a product. If b is positive, then $\psi(y)$ is increasing in y, i.e. the impact of external factors on the adoption process decreases when more of the market has been captured; note, $x = 1 - y$. If b is negative one experiences just the opposite effect. We shall prove that the behavior of the optimal pricing strategy of such models critically depends on the exponent. For *time homogeneous* finite horizon models without discounting, a positive exponent induces a market skimming pricing strategy, i.e. prices are monotone decreasing. If b is negative, then optimal prices will increase monotonically over time. If $T = \infty$ and $r(t) \equiv r > 0$, the critical exponent b^* is no longer zero but equals $(a - \delta)/a$. Optimal prices are constant if $b = b^*$; they are decreasing if $b > b^*$, and they are increasing if $b < b^*$. Power functions and a few other classes of functions are special. The characteristic quantities of the optimally controlled system, i.e. the optimal system path $y(t)$, the optimal sales rate $\bar{\lambda}(t)$ along the optimal trajectory y and optimal (open loop) controls $\bar{p}(t)$ and $\bar{w}(t)$ together with the associated value function, all belong to the same function class as ψ. But even very simple polynomial functions ψ pose a challenge. The characteristics of the optimally controlled systems can no longer be expressed in terms of elementary functions. However, we shall derive explicit expressions of the characteristics in terms of solutions of particular Bernoulli differential equations. We also propose a robust and efficient method to numerically evaluate these expressions. This way, we are able to study the time evolution of very general systems. Based on the analytical expressions to be derived, advertising and pricing recommendations for management will be given.

The behavior function $\psi(y) = \Omega y + \Gamma y(1 - y)$, Ω and Γ real valued constants (typically non-negative), is a special example of a very simple polynomial function. If Ω and Γ are positive, it characterizes the controlled [generalized] Bass model, see [3,7,13] and references therein. If $\Omega = \Gamma = 1$ and $\delta = 0$ the general control model becomes a version of the Robinson and Lakhani model [18]. The choice $\Omega = 1$, $\Gamma = 0$, $\delta = 0$ is a generalization of the Sethi and Bass model [20]. The case $\Omega = 0$, $\Gamma > 0$ is the controlled version of Mansfield's model [17]. The general case, Ω, $\Gamma \geqslant 0$, $0 \leqslant \delta < a$, $\varepsilon > 1$, $r(t) \equiv r \geqslant 0$, and T finite or infinite, will be analyzed in detail in Section 5.

This paper is organized as follows. A detailed description of the class of control problems and the notation that will be adopted throughout will be given in Section 2. As a first result we shall state a particular Dorfman-Steiner identity to be employed later on. In Section 3 we will derive formulas of the value function and the optimal policies of the general class of finite as well as infinite horizon problems in feedback form. Closed-form solutions of all characteristics of an optimally controlled system in the time domain will be given as well. We shall also sketch the basic ideas of a simple but powerful numerical scheme to compute the evolution of the optimally controlled system. In Section 4 we shall consider the special class of power functions. We shall exploit the general expressions derived in Section 3 and describe the value function and the optimal policies explicitly in terms of elementary functions. A detailed sensitivity study will also be undertaken in this section. In Section 5 we shall analyze the dynamic which motivated the authors of [21] to consider a functional term like $\sqrt{1-x}$, and we shall analyze versions of the controlled Bass model in some detail. In particular, we shall characterize the time when optimal prices of the controlled Bass model might peak. In Section 6 we offer some concluding remarks. For general references and literature reviews we refer to [5,7,10,13,14,16] and [21]. For (especially) stimulating reading and fundamental ideas pertaining to the present work we refer to [1,2,11,19] and [21].

2. Specification of the model

We consider a monopolist who introduces a new durable good in a deterministic environment. We choose the ratio of [potential] buyers that currently have not yet purchased the product and all potential buyers as state variable of the controlled system. The evolution of the system from its initial position y_0, e.g. $y_0 = 1$, is supposed to be governed by the rate of sales $\lambda(t, p, w, y)$. We assume λ to be of the separable multiplicative form $\mu(t)p^{-\varepsilon}w^{\delta}\psi(y)$ with constant price and advertising elasticities ε and δ. The firm is supposed to choose a positive admissible feedback pricing strategy $p(t, y)$, $0 \leqslant y \leqslant y_0$, $0 \leqslant t \leqslant T$, and a non-negative advertising strategy $w(t, y)$ such that the ordinary differential equation

$$\dot{y}(t) = \begin{cases} -\lambda(t, p(t, y(t)), w(t, y(t)), y(t)), & t < \tau, \\ 0, & t \geqslant \tau, \end{cases} \tag{2.1}$$

where $\tau := \inf\{t | t \in [0, T], y(t) = 0\}$, has a unique solution. Admissible controls are all control functions p and w such that (2.1), and all integrals to be defined below, are well defined. The objective of the decision maker is to choose an admissible pair of control functions (p, w) such that the *positive* pricing policy $p_t := p(t, y(t))$ and the *non-negative* advertising policy $w_t := w(t, y(t))$ maximize the pay-off

$$\int_0^{T\wedge\tau} e^{-R(t)} p_t \lambda_t dt - \int_0^{T\wedge\tau} e^{-R(t)} w_t^a dt. \tag{2.2}$$

The following conditions are supposed to be satisfied throughout. The function $\mu(t)$ is positive and measurable on $[0, T]$; the horizon T is allowed to be finite or infinite. The parameters are such that $\varepsilon > 1$ and $a > \delta \geqslant 0$. The function ψ is positive and $\psi^{\frac{1}{\varepsilon-1}}$ is integrable on the open unit interval (0, 1). There is a non-negative integrable function $r(t)$ which represents a varying discount rate; $R(t) = \int_0^t r(s)ds,\ t \geqslant 0$.

We shall use dynamic programming, see Fleming and Soner [6], Chapter I, to solve the deterministic optimal control problem given by (2.1) and (2.2). To this end, let $V(t, y)$ denote the value function of the advertising and pricing problem when y is the fraction of the untapped market at time t. Obviously, cf. (2.1), the value function V satisfies the boundary conditions $V(t, 0) = 0$, $0 \leqslant t \leqslant T$, and $V(T, y) = 0$, $0 \leqslant y \leqslant y_0$. It also satisfies the Bellman equation, see [6],

$$r(t)V(t, y) = \dot{V}(t, y) + \sup_{p>0, w\geqslant 0} \{\lambda(t, p, w, y)(p - V'(t, y)) - w^a\}, \tag{2.3}$$

where $\dot{V}$, V' resp., denotes the time derivative, state derivative resp., of V. For each t and y, assuming $V'(t, y) > 0$, the necessary optimality conditions of the maximization problem (2.3) imply

$$p^*(t, y) := \frac{\varepsilon}{\varepsilon - 1} V'(t, y) \quad \text{and}$$

$$w^*(t, y) := \left(\frac{\delta}{a\varepsilon}\mu(t)\psi(y)p^*(t, y)^{-(\varepsilon-1)}\right)^{\frac{1}{a-\delta}}.$$

The condition $\delta < a$ guarantees that the second order optimality conditions hold. Intuitively, should $\delta \geqslant a$ then the cost of advertising spending w^a would grow at the same rate or more slowly (in w) than the factor w^{δ} of λ, λ the rate of sales. In such a situation revenue would tend to infinity. Elementary analysis shows that for every (t, y) the control values $p^*(t, y)$ and $w^*(t, y)$ are unique maximizers of the optimization problem which appears on the right hand side of (2.3). The explicit form of the non-linear PDE (2.3) will be analyzed in Section 3. Exploiting the optimality conditions we obtain a dynamic version of the Dorfman-Steiner identity.

Proposition 2.1. *Let $\lambda^*(t, y) := \mu(t)p^*(t, y)^{-\varepsilon}\, w^*(t, y)^{\delta}\psi(y)$, then*

$$\frac{w^*(t, y)^a}{p^*(t, y)\lambda^*(t, y)} = \frac{\delta}{a\varepsilon}. \tag{2.4}$$

The Dorfman-Steiner identity is a powerful result. For example, in the isoelastic case it is an immediate implication of (2.4) that the optimal revenue and the optimal advertising cost are fixed fractions of the optimal profit over any interval $[t, T]$, $0 \leqslant t \leqslant T$, and any initial value y_t, $0 < y_t$. More precisely,

$$e^{R(t)} \int_t^T e^{-R(s)} p_s^* \cdot \lambda_s^* ds = \frac{a\varepsilon}{a\varepsilon - \delta} V(t, y_t),$$

and

$$e^{R(t)} \int_t^T e^{-R(s)} (w_s^*)^a ds = \frac{\delta}{a\varepsilon - \delta} V(t, y_t),$$

see Theorem 3.3 for explicit expressions of the open loop controls. Further applications of (2.4) will be discussed in Sections 3 and 4.

3. Explicit solution expressions

From now on we shall use the following abbreviations:

$$\Delta := \frac{\delta}{a}, \qquad \gamma := \frac{\varepsilon - \Delta}{1 - \Delta} \qquad \text{and}$$

$$\eta(t) := \gamma \frac{1 - \Delta}{\Delta}\left(\frac{\varepsilon - 1}{\varepsilon}\right)^{\gamma - 1}\left(\frac{\Delta}{\varepsilon}\mu(t)\right)^{\frac{1}{1-\Delta}}. \tag{3.1}$$

Note, $0 \leqslant \Delta < 1$, $\gamma - 1 = \frac{\varepsilon-1}{1-\Delta}$ and $\frac{\gamma-1}{\gamma} = \frac{\varepsilon-1}{\varepsilon-\Delta}$. The value function $V(t, y)$ satisfies the HJB equation, cf. (2.3),

$$r(t)V = \dot{V} + \eta(t)\frac{1}{\gamma} V'^{-(\gamma-1)}\psi(y)^{\frac{a}{a-\delta}} \tag{3.2}$$

with the boundary conditions $V(T, y) = 0$, $0 \leqslant y \leqslant y_0$, and $V(t, 0) = 0$, $0 \leqslant t \leqslant T$. Since the non-linear partial differential Eq. (3.2) has a similar structure as the difference-differential equations analyzed in [9] and [15], we try a separable solution of the form

$$V(t, y) = \alpha(t)\beta(y).$$

We require β to satisfy the Bernoulli equation, $y \geqslant 0$,

$$\beta(y)\beta'(y)^{\gamma-1} = \psi(y)^{\frac{a}{a-\delta}}, \quad \beta(0) = 0, \tag{3.3}$$

and α to satisfy the Bernoulli equation with terminal condition $\alpha(T) = 0$, $0 \leqslant t \leqslant T$,

$$r(t)\alpha(t) = \dot{\alpha}(t) + \frac{\eta(t)}{\gamma}\alpha(t)^{-(\gamma-1)}. \tag{3.4}$$

The following result is an immediate consequence of the equations (3.3) and (3.4).

Lemma 3.1. *Let ψ be a positive measurable function on $(0, y_0)$ such that $\psi^{\frac{1}{\varepsilon-1}}$ is integrable on any subinterval $(0, y)$, $0 < y \leqslant y_0$. Define the diffusion potential,*

$$B(y) := \frac{\gamma}{\gamma - 1}\int_0^y \psi(z)^{\frac{1}{\varepsilon-1}} dz.$$

Let r be a non-negative measurable function on $(0, \infty)$ such that $R(t) = \int_0^t r(s)ds$ is finite for any $t > 0$. Assume that for any $T > 0$ and $t \in [0, T]$ the time-to-go potential $A^{(0)}(t)$, the future potential $A(t)$ resp.,

$$A^{(0)}(t; T) := A^{(0)}(t) := \int_t^T e^{-\gamma R(s)}\eta(s)ds,$$

$$A(t; T) := A(t) := e^{\gamma R(t)}A^{(0)}(t) \text{ resp.}, \tag{3.5}$$

are well defined. Then,

(i) $\beta(y) := B(y)^{\frac{\gamma-1}{\gamma}}$ solves the non-linear ODE (3.3);
(ii) $\alpha(t) := A(t; T)^{\frac{1}{\gamma}}$ solves (3.4) and satisfies $\alpha(T) = 0$.

Remark 3.1. If $r(t) \equiv r > 0$ and $\eta(t) \equiv \eta$, then

$$A(t;T) = \eta\frac{1-e^{-\gamma r(T-t)}}{\gamma r} \triangleq \begin{cases} \stackrel{r\to 0}{\to} & \eta(T-t), \\ \stackrel{T\to\infty}{\to} & \frac{\eta}{\gamma r}. \end{cases} \tag{3.6}$$

Simple algebra together with Lemma 3.1 yields explicit formulas of the value function and the optimal feedback policies $p^*(t, y)$ and $w^*(t, y)$ if the horizon is finite. These formulas, when combined with (3.6), also yield formulas of the optimal policies of a time-homogeneous infinite horizon problem. It is instructive to consider the infinite horizon problem separately, see Corollary 3.1 below, and to display the formulas of the future value $V_{(\infty)}$ and the optimal policies $p^*_{(\infty)}$ and $w^*_{(\infty)}$.

Theorem 3.1. *Assume all conditions of Lemma 3.1 to hold. Then, $0 < y \leqslant y_0$, $0 \leqslant t < T$,*

$$p^*(t,y) = \frac{\varepsilon}{\varepsilon-1}A(t;T)^{\frac{1}{\gamma}}B(y)^{-\frac{1}{\gamma}}\psi(y)^{\frac{1}{\varepsilon-1}},$$

and

$$w^*(t,y) = \left(\frac{\Delta}{\varepsilon-\Delta}\eta(t)A(t;T)^{-\frac{\gamma-1}{\gamma}}B(y)^{\frac{\gamma-1}{\gamma}}\right)^{\frac{1}{a}}$$

are optimal feedback policies of the control problem specified by (2.1) and (2.2). The value function V is also a product of factors which involve both potentials,

$$V(t,y) = A(t;T)^{\frac{1}{\gamma}}B(y)^{\frac{\gamma-1}{\gamma}} =: e^{rt}V^{(0)}(t,y).$$

Corollary 3.1. *Let $T = \infty$, $r(t) \equiv r > 0$ and $\eta(t) \equiv \eta$. Then,*

$$p^*_{(\infty)}(y) = \left(\frac{\eta}{\gamma r}\right)^{\frac{1}{\gamma}}B(y)^{-\frac{1}{\gamma}}, \quad \text{and} \quad w^*_{(\infty)}(y) = \left(\frac{\eta}{\gamma r}\right)^{-\left(\frac{\gamma-1}{\gamma}\right)}B(y)^{\frac{\gamma-1}{\gamma}};$$

$$V_{(\infty)}(y) := V(y) = \left(\frac{\eta}{\gamma r}\right)^{\frac{1}{\gamma}}B(y)^{\frac{\gamma-1}{\gamma}}.$$

Since we have explicit formulas of the optimal policies of the finite horizon model and the infinite horizon one we can compute the optimal rates of sales in both cases. These rates can be concisely described in terms of the derivatives of the log-transforms of the functions $\beta(y)$ and $A^{(0)}(t;T)$. The following formulas are easily deduced from Theorem 3.1.

Proposition 3.1. *The optimal rates of sales λ^*, if $T < \infty$, and $\lambda^*_{(\infty)}$, if $T = \infty$, are given by:*

$$\lambda^*(t,y) = \frac{\gamma-1}{\gamma}\frac{\beta(y)}{\beta'(y)}\left(\frac{-\dot{A}^{(0)}(t;T)}{A^{(0)}(t;T)}\right), \quad \text{and} \quad \lambda^*_{(\infty)}(y) = r(\gamma-1)\frac{\beta(y)}{\beta'(y)}.$$

Proposition 3.1 enables us to solve the differential equation which characterizes an optimal trajectory $y(t)$, i.e.,

$$\dot{y}(t) = -\lambda^*(t, y(t)), \quad y(0) = y_0. \tag{3.7}$$

The optimally controlled system will always capture the full market potential over any given finite time interval $[0, T]$, i.e. $y(T) = 0$, independent of T being small or large. This is an important property of the isoelastic model. It is a consequence of the constant price elasticity of demand and the possibility to set prices arbitrarily low.

Theorem 3.2. *Assume all conditions of Lemma 3.1 to hold. Then the optimal trajectory $y(t)$ of the finite horizon problem with initial value y_0 is given by, $0 \leqslant t \leqslant T$,*

$$y(t) = B^{-1}\left(B(y_0)\frac{A^{(0)}(t;T)}{A^{(0)}(0;T)}\right), \quad \text{i.e.} \quad x(t) = y_0 - y(t). \tag{3.8}$$

Proof. Using Proposition 3.1, the differential Eq. (3.7) is equivalent to, in the proof we shall be using the notation $A^{(0)}(t)$ instead of $A^{(0)}(t;T)$,

$$\beta'(y(t))\dot{y}(t) = \frac{\gamma-1}{\gamma}\overbrace{\ln(A^{(0)}(t))}^{\bullet}\beta(y(t)).$$

Thus, $Z(t) := \beta(y(t))$ is the solution of the linear initial value problem

$$\dot{Z}(t) = \frac{\gamma-1}{\gamma}\overbrace{\ln\left(A^{(0)}(t)\right)}^{\bullet}Z(t), \quad Z(0) = \beta(y_0).$$

It is given by

$$Z(t) = \beta(y_0)\left(\frac{A^{(0)}(t)}{A^{(0)}(0)}\right)^{\frac{\gamma-1}{\gamma}}. \tag{3.9}$$

By definition of β, Eq. (3.9) is equivalent to

$$B(y(t)) = B(y_0)\frac{A^{(0)}(t)}{A^{(0)}(0)}. \tag{3.10}$$

Since B has an inverse function B^{-1}, formula (3.8) follows. □

Remark 3.2. Expression (3.8) is a closed form solution formula of the trajectory $y(t)$. Identity (3.10) suggests a robust algorithm which computes (approximating) decreasing step functions of $y(t)$. These step functions are accurate approximation of y for basically any function $\psi^{\frac{1}{\varepsilon-1}}$. For any t and initial value y_0 the right hand side of (3.10) can be computed off-line. Numerical integration methods provide functional values of $B(y)$. Discretizing the interval $[0, 1]$ and choosing, for any given t, a value $y(t)$ belonging to the grid such that $B(y(t))$ is the best approximation to $B(y_0) \cdot A^{(0)}(t)/A^{(0)}(0)$ yields a step function. It approximates the solution of (3.7) as accurately as the grid allows it to do.

Theorems 3.1 and 3.2 together with Proposition 3.1 enable us to compute the optimal open loop controls $\bar{p}(t) := p^*(t, y(t))$ and $\bar{w}(t) := w^*(t, y(t))$, when $y(t)$ solves (3.7), for an arbitrary initial value y_0. Moreover, we can evaluate V along the optimal trajectory. $\overline{V}(t) := V(t, y(t))$ is the continuation value which captures the accumulated (optimal) profits from t onwards. We derive several important relationships between all these quantities. These results will be further exploited in the next section. The upcoming formulas are derived using Theorem 3.1 and identity (3.10). From a practitioner's point of view these results are most helpful. They provide the analytical underpinning of the management recommendations to be given below. In particular, the results reveal that for time homogeneous finite horizon models without discounting and for any feasible function ψ the optimal advertising rate is constant over time. Moreover, for this particular case $\overline{V}$ is a straight line with negative slope. If the arrival intensity factor μ is time independent then these formulas can be further simplified, see Section 4.3.

Theorem 3.3. *Let the assumptions of Lemma 3.1 hold. Then*

$$\overline{V}(t) = A^{(0)}(t;T)e^{R(t)}\left(\frac{B(y_0)}{A^{(0)}(0;T)}\right)^{\frac{\gamma-1}{\gamma}},$$

$$\bar{p}(t) = \frac{\varepsilon}{\varepsilon-1}e^{R(t)}\psi(y(t))^{\frac{1}{\varepsilon-1}}\left(\frac{B(y_0)}{A^{(0)}(0;T)}\right)^{-\frac{1}{\gamma}},$$

$$\bar{w}^a(t) = \frac{\Delta}{\varepsilon - \Delta}\eta(t)e^{-(\gamma-1)R(t)}\left(\frac{B(y_0)}{A^{(0)}(0;T)}\right)^{\frac{\gamma-1}{\gamma}},$$

$$\bar{\lambda}(t) = \frac{\gamma-1}{\gamma}\eta(t)e^{-\gamma R(t)}\psi(y(t))^{-\frac{1}{\varepsilon-1}}\frac{B(y_0)}{A^{(0)}(0;T)}.$$

Proof. We shall prove the formula for $\overline{V}$. The other three formulas can be derived by the same kind of reasoning. Due to the Dorfman-Steiner identity, cf. Proposition 2.1, it is actually sufficient to verify just one of these three formulas; any of the other two will then follow. Employing Theorem 3.1 and expression (3.10) we get – recall, $A(t) = A(t;\,T)$, $A^{(0)}(t) = A^{(0)}(t;\,T)$ –

$$\begin{aligned}\overline{V}(t) := V(t, y(t)) &= A(t)^{\frac{1}{\gamma}}B(y(t))^{\frac{\gamma-1}{\gamma}}\\ &= e^{R(t)}A^{(0)}(t)^{\frac{1}{\gamma}}B(y_0)^{\frac{\gamma-1}{\gamma}}\left(\frac{A^{(0)}(t)}{A^{(0)}(0)}\right)^{\frac{\gamma-1}{\gamma}}\\ &= e^{R(t)}A^{(0)}(t)\left(\frac{B(y_0)}{A^{(0)}(0)}\right)^{\frac{\gamma-1}{\gamma}}. \qquad \square\end{aligned}$$

Corollary 3.2. *Let $\mu(t) \equiv \mu > 0$ and $r(t) \geqslant 0$.*

(i) If $T < \infty$ and $r(t) \equiv 0$, then $\bar{w}(t)$ is constant and $\overline{V}(t)$ is a linear function with negative slope.
(ii) If $T = \infty$ and $r(t) > 0$, then $e^{(\gamma-1)R(t)}\bar{w}(t)^a$ and $e^{-R(t)}\overline{V}(t)$ are constant functions; $\bar{w}(t)$ is monotone decreasing.
(iii) Except for the special case when $\psi(y) \equiv 1$ optimal prices are dynamic, i.e. they change with time.

Remark 3.3. The discounted optimal prices $\bar{p}(t)$ depend explicitly on the current evaluations of the market share by the system function ψ but – surprisingly – do not directly depend on the fluctuations of $\mu(t)$ [for $\bar{p}$, only the time-to-go potential matters]. Prices, of course, depend on the magnitude of the arrival intensity. For instance, should the constant value of μ double then prices will increase by a certain percentage depending on the parameters of the model.

The optimal advertising rate $\bar{w}(t)$, the continuation value $\overline{V}(t)$ resp., is isochronous with $\mu(t)$, with $A^{(0)}(t;\,T)$ resp. Both characteristics do not directly depend on the current diffusion value $\psi(y(t))$ [for $\bar{w}$, it is only the diffusion potential $B(y_0)$ that matters], but – through B – depend on the magnitude of ψ.

The optimal trajectory is characterized by the property that at any time t, $0 \leqslant t \leqslant T$, the ratio of the remaining diffusion potential and the initial one equals the ratio of the time-to-go potential at t and its initial value. The characterization of $\bar{\lambda}$ in the time domain reveals the following extraordinary feature of the controlled problem when no other constraint but positivity is imposed on p: While the adoption rate of the uncontrolled system is proportional to $\psi(y(t))$, the rate of the optimally controlled system is proportional to a power of its reciprocal! This property has to be recalled whenever one is looking at graphs of $\bar{\lambda}$. It is useful to recall the Dorfman-Steiner identity, see Proposition 2.1, since the graphs of all three functions, $\bar{\lambda}(t)$, $\bar{p}(t)$ and $\bar{w}(t)$, are interconnected.

For the stylized model considered in this article Theorem 3.3 justifies the following management recommendations. Additional recommendations can be given for special subclasses of the general model, see Section 4.3.

MR I (advertising): Optimal advertising spending should be synchronized with arrival intensity. *Without discounting*, if μ is constant then the optimal advertising rates should not change over time (the optimal rates are constant). In general, *with discounting*, advertising spending should be reduced if the discounting term dominates the arrival intensity. The exact values of $\bar{w}$ depend on the non-trivial interplay of both terms.

MR II (pricing): Whenever the controlled system explicitly depends on the state of the system then optimal pricing strategies are dynamic. Optimal price paths will be monotone increasing if the behavior function ψ is monotone decreasing (in y); discounting intensifies the growth of prices, see Sections 4 and 5 for simple examples. A *skimming strategy*, i.e. prices are monotone decreasing, is optimal if ψ is monotone increasing and the discount rate is zero. *With discounting*, the interplay of both factors determines the price trajectory. The system function ψ together with the discount rate r determine whether or not a market *penetration pricing strategy* will be optimal. See Section 5 for examples and details when increasing prices during the first phase of the adoption process followed by a phase of declining prices are optimal. A large price elasticity value ε flattens the optimal price path. To reduce price variations large elasticity values should be chosen in early stages of prelaunch advertising studies.

4. The class of power functions $\psi(y) = y^b$

Motivated by the case $\psi(y) = y^{1/2}$, recall $y = 1 - x$, cf. [21], we shall derive explicit formulas of $\overline{V}$, $\bar{p}$, $\bar{w}$, etc., if $\psi(y) = y^b$, $b > -(\varepsilon - 1)$, and $a > \delta \geqslant 0$. The lower bound on b guarantees that $y^{\frac{b}{\varepsilon-1}}$ is integrable in a neighborhood of zero, cf. Lemma 3.1. A negative power b of the system function ψ is a stylized model of an "imitation force" influencing the adoption process. A positive value of b fits market situations when "innovation" is the driving force of the product diffusion. For the class of general power functions ψ, explicit expressions of the results derived in Section 3, see Theorems 3.1 and 3.2, are obtained by elementary calculations. All these expressions turn out to be power functions as well. The class of exponential functions is a second example of system functions ψ when all these expressions have the same functional form (in y). Due to the elementary form of all these functions they can be evaluated to any precision by using any of the standard math packages. The accuracy of general numerical methods can be checked against these numbers. Hence, the class of power functions is an ideal test-bed for the numerical procedure sketched in Remark 3.2. We like to point out that all plots displayed in this section and the next one are based on the numbers computed by this method. The results are excellent when compared with the analytical ones and provide credibility to all other numerical computations.

4.1. Power functions combined with a general time factor $\mu(t)$

The following two preliminary results together with Proposition 4.1 and Theorem 4.1 summarize the results of the elementary calculations referred to above. These results display all power expressions as functions of time and of the initial value y_0 in detail. The first lemma follows by definition and simple integration. The second one shows that for power functions ψ the optimal rate of sales expressed in feedback form is a linear function of y, a fact which is an immediate consequence of Lemma 4.1 and Proposition 3.1. Since the rate of sales function is linear in y, see Lemma 4.2, one can easily compute the optimal state trajectory $y(t)$ for any initial value. Once the state trajectory is known the results of Section 3 give formulas of all characteristics, e.g., of the optimal price path $\bar{p}(t) := p^*(t, y(t))$, the optimal advertising trajectory $\bar{w}(t) := w^*(t, y(t))$, etc.

Lemma 4.1. *Let $a > \delta \geqslant 0$, $b > -(\varepsilon - 1)$ and $\psi(y) = y^b$, $0 \leqslant y \leqslant y_0$. Then,*

$$\beta(y)=\left(\frac{\varepsilon-\Delta}{b+\varepsilon-1}\right)^{\frac{\varepsilon-1}{\varepsilon-\Delta}}\cdot y^{\frac{b+\varepsilon-1}{\varepsilon-\Delta}}\quad \text{and}\quad B(y)=\left(\frac{\varepsilon-\Delta}{b+\varepsilon-1}\right)y^{1+\frac{b}{\varepsilon-1}}.$$

Lemma 4.2. *The optimal rates of sales $\lambda^*(t, y)$ and $\lambda^*_{(\infty)}(y)$ are given by*

$$\lambda^*(t,y)=\frac{\varepsilon-1}{b+\varepsilon-1}\frac{\eta(t)}{A(t;T)}\cdot y,\quad \text{and}\quad \lambda^*_{(\infty)}(y)=\frac{\varepsilon-1}{b+\varepsilon-1}\gamma r\cdot y.$$

Proposition 4.1. *For any $y_0 > 0$, $0\leqslant t\leqslant T$,*

$$y(t)=y_0\left(\frac{A^{(0)}(t;T)}{A^{(0)}(0;T)}\right)^{\frac{\varepsilon-1}{b+\varepsilon-1}},$$

and if $\mu(t)$ and $r(t)$ are constants, we obtain the explicit formulas

$$y(t)=\begin{cases} y_0\left(\frac{e^{-\gamma rt}-e^{-\gamma rT}}{1-e^{-\gamma rT}}\right)^{\frac{\varepsilon-1}{b+\varepsilon-1}}, & \text{if } r>0 \text{ and } T<\infty,\\ y_0\left(1-\frac{t}{T}\right)^{\frac{\varepsilon-1}{b+\varepsilon-1}}, & \text{if } r=0 \text{ and } T<\infty,\\ y_0 e^{-\frac{(\varepsilon-1)\gamma r}{b+\varepsilon-1}t}, & \text{if } r>0 \text{ and } T=\infty.\end{cases}$$

Theorem 4.1. *Assume a, δ, ε and b to satisfy the conditions stated in Lemma 4.1. Let $c_{\bar p}:=\frac{\varepsilon}{\varepsilon-1}\left(\frac{b+\varepsilon-1}{\varepsilon-\Delta}\right)^{\frac{1}{\gamma}}A^{(0)}(0;T)^{-\frac{\gamma-1}{\gamma}\frac{b-1+\Delta}{b+\varepsilon-1}}$, $c_{\bar w^a}:=\left(\frac{\Delta}{\varepsilon}\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{1}{1-\Delta}}\cdot\left(\frac{\varepsilon-\Delta}{b+\varepsilon-1}\frac{1}{A^{(0)}(0;T)}\right)^{\frac{\gamma-1}{\gamma}}$, and $c_{\overline V}:=\left(\frac{\varepsilon-\Delta}{b+\varepsilon-1}/A^{(0)}(0;T)\right)^{\frac{\gamma-1}{\gamma}}$. Then*

$$\begin{aligned}
\bar p(t) &= c_{\bar p}\cdot e^{R(t)}A^{(0)}(t;T)^{\frac{b}{b+\varepsilon-1}}\cdot y_0^{\frac{b-1+\Delta}{\varepsilon-\Delta}} &&\overset{r=0}{=} c_{\bar p}A(t;T)^{\frac{b}{b+\varepsilon-1}}y_0^{\frac{b-1+\Delta}{\varepsilon-\Delta}},\\
\bar w(t)^a &= c_{\bar w^a}\cdot\mu(t)^{\frac{1}{1-\Delta}}e^{-\frac{\gamma-1}{\gamma}R(t)}\cdot y_0^{\frac{b+\varepsilon-1}{\varepsilon-\Delta}} &&\overset{r=0}{=} c_{\bar w^a}\mu(t)^{\frac{1}{1-\Delta}}y_0^{\frac{b+\varepsilon-1}{\varepsilon-\Delta}},\\
\overline V(t) &= c_{\overline V}\cdot e^{R(t)}A^{(0)}(t;T)\cdot y_0^{\frac{b+\varepsilon-1}{\varepsilon-\Delta}} &&\overset{r=0}{=} c_{\overline V}A(t;T)y_0^{\frac{b+\varepsilon-1}{\varepsilon-\Delta}}.
\end{aligned}$$

4.2. Illustration of the case when μ is constant

We illustrate the dependence of the characteristics of the model, i.e. the optimal state trajectory y, the rate $\bar\lambda_T$ and the optimal control paths, on the *size* and the *sign* of the parameter b. In this subsection, in order not to clutter the issue, we assume that the factor μ is a positive constant, see Section 4.4 for the general case. Time independence of the arrival intensity has far reaching consequences, see the discussion below and Section 4.3. For the purpose of illustration, the values of b are chosen to vary within the set $\{-0.8, -0.5, -0.25, 0, 0.25, 0.5, 2, 10\}$; the other model parameters are: $\mu = 1$, $r = 0$, $T = 10$, $y_0 = 1$, $a = 2$, $\delta = 1$, and $\varepsilon = 2$, i.e. we consider a finite horizon time homogeneous model without discounting. At the end of this section we shall briefly comment on the cases $r > 0$ and T finite or infinite. Some of the b values are carefully chosen, e.g. $b=\pm\frac{a-\delta}{a}=\pm 0.5$ and $b = 0$, to illustrate special properties of the system, see below. A power function is a good behavior model when either "innovation" or "external" factors are the driving forces of the adoption process. The choice $b = 1$ is one of the two extreme versions of the Bass model, see Introduction and Section 5 (choose $\Omega = 1$ and $\Gamma = 0$). The other special case of a Bass model is Mansfield's model, i.e. $\Omega = 0$ and $\Gamma = 1$. All properties of the characteristics y, $\bar\lambda$, $\bar p$, $\bar w$ and $\overline V$ which will be illustrated by the graphs can be easily verified using the formulas stated in Theorem 4.1, see also the next subsection.

If $b = 0$, the untapped market share $y(t)$ decreases linearly in time, see Window 1 of Fig. 1. This particular choice of b corresponds to the situation when the dynamic does not explicitly depend on the state y, and neither "innovation" behavior nor "word-of-mouth" communication are playing any role in the diffusion of the product. The affine structure of $y(t)$ in t, if $b = 0$, is reflected by the fact that $\bar\lambda_T$ is constant, viz. $\bar\lambda_T(t)\equiv 0.1$, see Window 2. For negative powers b, the functions $y(t)$ are always convex, while they are concave if b is positive. If $b = -0.5$, this is one of the special values of the parameter set, then $\bar\lambda_T$ depends linearly on t. Optimal rates of sales are monotone increasing and convex in time whenever the exponent b is positive; negative exponents force the rates to be monotone decreasing functions. But the rates can be convex or concave, depending on whether or not $b>-\frac{a-\delta}{a}=-0.5$.

For the case of a constant function μ, Fig. 2 shows the remarkable dependence of the optimal control paths on b. If $b = 0$, then $\bar p(t)\equiv$ *const*. For negative b values, i.e. latecomers to the market are most eager to buy, prices are convex and are increasing functions over time. If the ψ function has a positive exponent – a typical innovation model – then prices are concave and monotone decreasing; they drop rapidly at the end of a finite planning horizon, see Window 2.1. The dependence of the advertising rate on b is most special, see Window 2.2. If μ is constant, then optimal advertising rates $\bar w(t)$ are independent of time! The easy explanation of this fact is offered by Theorem 4.1; see also MR I. A consequence of this fact, when combined with Proposition 2.1, is that the graphs of the reciprocal of $\bar p$ and $\bar\lambda$, see Figs. 1 and 2, are actually proportional to each other.

In a similar manner we can exploit Theorem 4.1 when analyzing the cases of time homogeneous models with a constant discount rate and a finite or infinite planning horizon. We briefly describe two results of our analysis which highlight the differences between a case *without* discounting and a case *with* discounting. A crucial difference between these two cases is that the optimal advertising rate $\bar w$ is no longer a constant but a decreasing function, if r is positive. More precisely, for any b, the function $\bar w$ is exponentially decreasing with rate $\frac{\gamma-1}{\gamma}\frac{r}{a}$. Furthermore, optimal price paths are exponential functions too; their rate equals

$$r\left(1-\frac{b}{b+\varepsilon-1}\cdot\frac{\varepsilon-\Delta}{1-\Delta}\right),$$

and this rate can be positive or negative. Whenever $b = 1 - \Delta$, the rate is zero, and the optimal price trajectory is constant. A special case of such a parameter constellation is $a = 2$, $\delta = 1$, $b = 1/2$ and $T = \infty$, the model analyzed in [21]. Theorem 4.1 together with the identity $b = 1 - \Delta$ offer an explanation of this exceptional phenomena, see also Section 5.1. Another remarkable case is specified by $b = 0$, i.e. there is no adoption effect present. In this special case optimal prices will be increasing but the discounted optimal prices are constant, i.e. the price of the durable good must rise at the interest rate!

4.3. Comparative statics

The explicit expressions of the characteristics of a general optimally controlled system described by (2.1), see Theorem 3.3, enable us to perform a detailed sensitivity analysis of all characteristics with respect to the various model parameters. In order not to overburden the presentation we shall restrict our sensitivity analysis to the case of power functions $\psi(y) = y^b$ and time independent functions μ. To further simplify the presentation we shall assume $y_0 = 1$, and $r(t)\equiv 0$. Comparative statics for more general systems will be published elsewhere. Whenever $y_0\equiv 1$, $r(t)\equiv 0$ and $\mu(t)\equiv\mu > 0$ the formulas given in Theorem 4.1 simplify. Let

$$C:=(\varepsilon-\Delta)\left(\frac{\mu}{\varepsilon}\Delta^{\Delta}\left(\frac{\varepsilon-1}{\varepsilon}\frac{1}{b+\varepsilon-1}\right)^{\varepsilon-1}\right)^{\frac{1}{\varepsilon-\Delta}}. \tag{4.1}$$

Then, recall $\gamma=\frac{\varepsilon-\Delta}{1-\Delta}$,

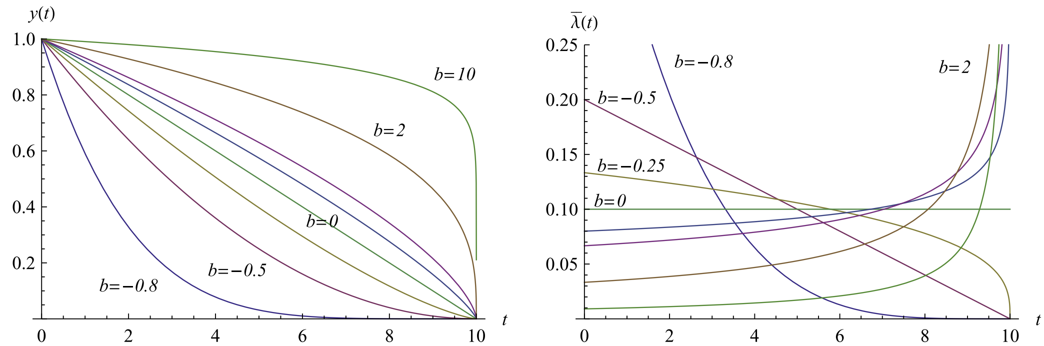

Fig. 1. Optimal trajectories $y(t)$ of the market-share-to-be-captured, and the corresponding rates of sales $\bar{\lambda}(t)$ for different parameter values b; $T < \infty$ and $r \equiv 0$.

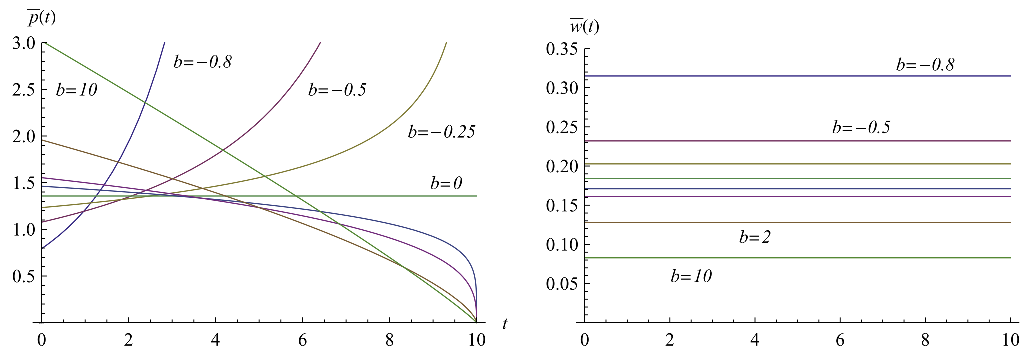

Fig. 2. Optimal price trajectories $\bar{p}(t)$ and optimal advertising rates $\bar{w}(t)$ parametrized by b; $T < \infty$ and $r \equiv 0$.

$$y(t) = \left(1 - \frac{t}{T}\right)^{\frac{\varepsilon-1}{b+\varepsilon-1}}, \tag{4.2}$$

$$\bar{\lambda}(t) = \frac{\varepsilon - 1}{b + \varepsilon - 1}\frac{1}{T}\left(1 - \frac{t}{T}\right)^{\frac{-b}{b+\varepsilon-1}}, \tag{4.3}$$

$$\bar{p}(t) = C\frac{\varepsilon}{\varepsilon - 1}\frac{b + \varepsilon - 1}{\varepsilon - \Delta}\left(1 - \frac{t}{T}\right)^{\frac{b}{b+\varepsilon-1}} \cdot T^{1/\gamma}, \tag{4.4}$$

$$\bar{w}(t)^a = C\frac{\Delta}{\varepsilon - \Delta}T^{-\frac{\gamma-1}{\gamma}}, \tag{4.5}$$

$$\overline{V}(t) = C\left(1 - \frac{t}{T}\right) \cdot T^{1/\gamma}. \tag{4.6}$$

Note, since μ does not depend on time we already know that $\bar{w}(t)$ is constant. To prove qualitative and quantitative results on the dependence of the characteristics on a, b, δ, ε and T we simply take derivatives of the functions (4.2)–(4.6) or of the log of these functions, with respect to the parameters and analyze the sign of the derivative functions. For example, since $\overline{V}$ involves μ as a power expression, the elasticity of the revenue expression along the optimal path with respect to μ is constant and equals $1/(\varepsilon - \Delta)$. Hence, $\partial\overline{V}/\partial\mu$ is positive and expected revenue increases if the arrival intensity becomes larger, a property which is intuitively clear. In addition, the formulae reveal that the μ-elasticities of $\bar{p}$ and $\bar{w}^a$ are not only positive but equal $1/(\varepsilon - \Delta)$ too. This fact can also be easily derived using the Dorfman-Steiner identity combined with the property that the optimal (fractional) rate of sales $\bar{\lambda}(t;\mu)$ is independent of μ. Table 1 summarizes the results of our sensitivity analysis.

The entries +, −, and 0 indicate that a characteristic (a row in Table 1) is either a monotone increasing function (+), is a monotone decreasing function (−) or is independent (0) of the parameter (a column of Table 1). The entry "±" indicates that no general statement is possible and that the behavior of the quantity of interest depends on the relative magnitude of two or more of the parameters. Besides the information provided in Table 1 various formulas of elasticities can be derived, or relationships between different elasticities can be proved. To give just one example, it can be easily seen that

Table 1
A summary of the sensitivity analysis of the time homogeneous case without discounting.

	b	Δ	ε	μ	t	T
y	+	0	±	0	−	+
$\bar{\lambda}$	±	0	±	0	+, if $b > 0$ 0, if $b = 0$ −, if $b < 0$	−, if $b \geq 0$ ±, if $b < 0$
$\bar{p}$	±	±	±	+	−, if $b > 0$ 0, if $b = 0$ +, if $b < 0$	+, if $b \geq 0$ −, if $b < 0$
$\bar{w}$	−	±	±	+	0	−
$\overline{V}$	−	±	±	+	−	+

$$El_{\Delta}\bar{p}(t;\Delta) = El_{\Delta}\overline{V}(t;\Delta) + \frac{\Delta}{\varepsilon - \Delta};$$

thus, the difference of both elasticities is independent of t, $0 \leqslant t < T$, and of T as well.

Fig. 3 explains the many ± entries in Table 1 and, at the same time, illustrates the dependence of optimal price trajectories $\bar{p}(t)$ on price elasticity ε and discount rate r. The first window of Fig. 3 shows the graphs of $\bar{p}(t)$ for several ε-values. Typically, increased price sensitivity of customers implies lower prices over time. In general, however, price trajectories are not isotone but may intersect. To clearly show this phenomena we have chosen $\psi(y) = y^b$, $b = -0.05$, and ε small, e.g. 1.1 and 1.2. The intersection of the paths is due to the non-trivial interaction of the various parameters of the model, cf. Theorems 3.3 and 4.1. The second

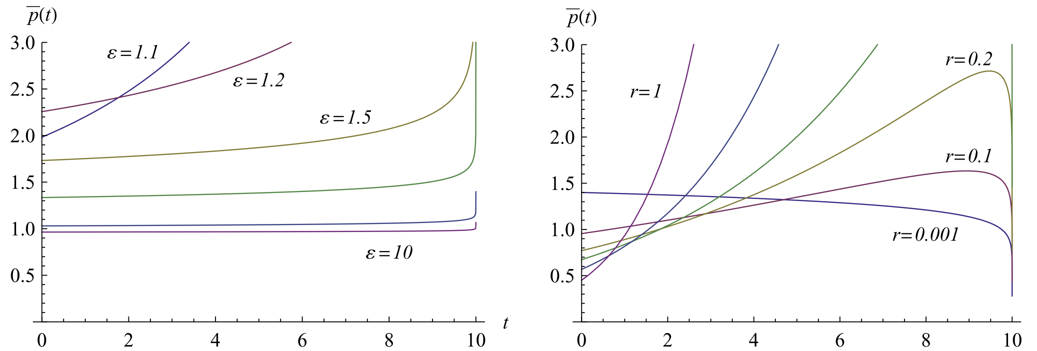

Fig. 3. Optimal price trajectories $\bar{p}(t)$ as functions of the price elasticity (window on the left, Fig. 3a) and as functions on the discount rate (on the right, Fig. 3b); the parameters are: $b = -0.05$, $r = 0$, $\varepsilon \in \{1.1, 1.2, 1.5, 2, 4, 10\}$ in Fig. 3a, and $b = 0.1$, $\varepsilon = 2$, $r \in \{0.001, 0.1, 0.2, 0.3, 0.5, 1\}$ in Fig. 3b.

window of Fig. 3 reveals the essential features of the dependence of $\bar{p}(t)$ on r. Higher discount rates imply lower prices during (short) initial phases of the product diffusion. But the larger r becomes, the more rapidly optimal prices will increase. Price paths are not isotone (in r) and may intersect. Although it is possible to deduce a formula of the point of intersection, we are not going to present these details. We also do not provide graphs of $\bar{w}(t;\varepsilon)$ and $\bar{w}(t;r)$. To a certain extent, the dependence of optimal advertising spending on the parameters ε and r can be deduced from the properties of $\bar{p}(t;\varepsilon)$ and $\bar{p}(t;r)$. To this end, one exploits the relationship between $\bar{w}(t)$ and $\bar{p}(t)$, see formulas following (2.3). This relationship reveals that smaller values of ε typically imply higher advertising spending. Larger discount rates imply that more money is spent on advertising at the beginning of the life cycle of a product, but the spending is reduced faster for larger values of r. Again, the paths $\bar{w}(t)$ depend on ε and r in a convoluted manner, and numerical studies can and should be performed for individual cases, cf. Remark 3.2 and the results of Section 3 and this section.

4.4. Illustration of the case when μ depends on time

To show the impact that time dependence of the arrival intensity has on the characteristics of a system which is driven by a power function $\psi(y) = y^b$, we shall consider three μ-scenarios which are depicted in Fig. 4. To have nice and meaningful analytical expressions of such functions we use the construction proposed by Bemmaor in [4], see Introduction. Thus, scenario *II* is a mixture of densities characterized by two parameters, 0.45 and 10, see [4], page 206 for details. Scenarios *I* and *III* are but shifted versions of this basic μ function. The first one is generated by shifting μ five (time) units to the left, while the third one is obtained by a shift of the same magnitude to the right. The three scenarios capture typical marketing situations. The first case illustrates fading customer interest right from the start. The third case describes just the opposite phenomena, viz. growing product awareness. The second one, in between the two extremes, portrays the "normal" situation of a steady, increasing interest in buying the product at the beginning of its life-cycle, followed by fading interest towards the end. In this subsection we concentrate on a finite horizon problem. Infinite horizon problems with time dependent μ functions will be discussed in Section 5.

Window 2 of Fig. 4 illustrates the management recommendation MR I, see Section 3, when ψ is a power function. According to Theorem 4.1, see also Theorem 3.3, the optimal advertising rate is isochronous with μ. The shifts in height of the various plots are due to the different $c_{\bar{w}}$ values for different b values, see Theorem 4.1. Note, $c_{\bar{w}}$ is monotone decreasing in b if the initial value y_0 equals one. The optimal price evolution does not explicitly depend on the fluctuations of μ. Nevertheless, prices are influenced by the time dependence of the arrival intensity of potential buyers via their implicit dependence on $y(t)$. This is due to the presence of the term $\psi(y(t))$ in the optimal pricing formula, see Theorem 3.3. This dependence is convoluted since the composition of the two functions ψ and y depends on both b and μ in a non-trivial manner. Separate plots of $\psi(1-y)$ and $y(t)$ and their compositions for different b values would give an impression of this convoluted dependence.

Window 1 of Fig. 5 shows the time evolution of the nine different plots of the untapped market share. The graphs *II*−, *II*0 and *II*+ have the typical S-shape form one expects to see, recall $y = 1 - x$. During the introductory phase of the adoption process of a product it takes time for the market share to take off before it starts converging during the final phase of the life cycle. The different y evolutions clearly reflect the three μ scenarios and display the

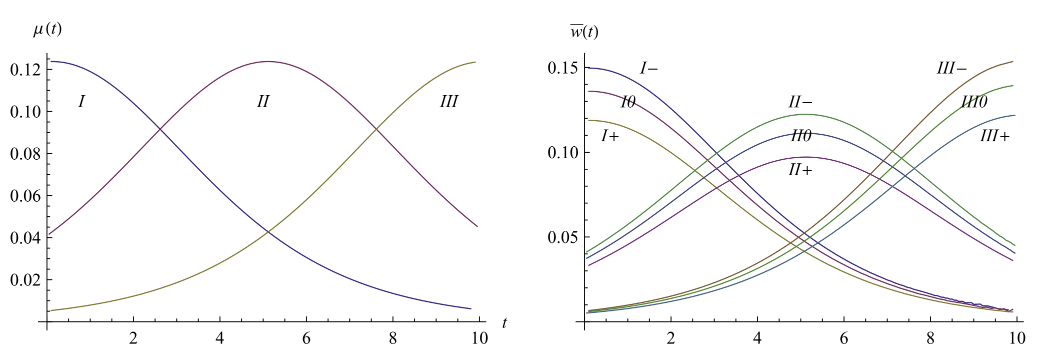

Fig. 4. Three μ-scenarios, labeled *I*, *II* and *III* (the graphs on the left). The corresponding optimal advertising rates for three different power functions (all together nine plots) are shown on the right; the label *I*−, for instance, refers to scenario *I* and the negative power value $b = -0.25$, see below; label *II*0 indicates the combination of scenario *II* and $b = 0$. The parameters are: $y_0 = 1$, $a = 2$, $\delta = 1$, $\varepsilon = 2$, $T = 10$, $r = 0$, and b is an element of the set $\{-0.25, 0, 0.25\}$.

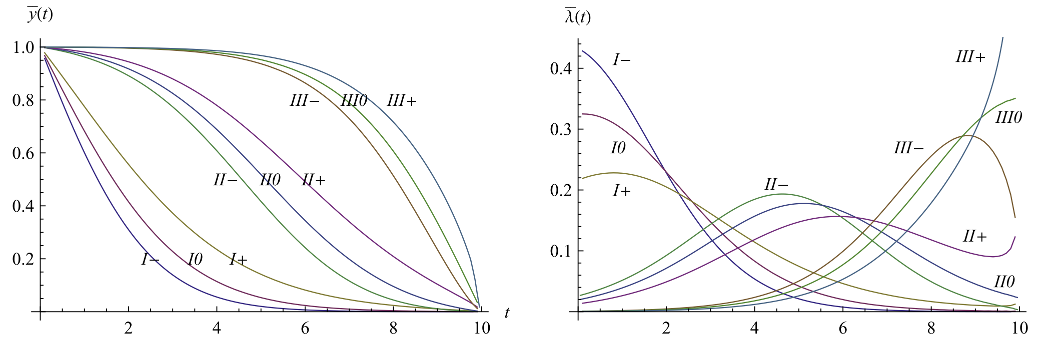

Fig. 5. The evolution of the untapped market share and the corresponding rates of sale of all nine cases; the parameters are the same as before.

spectrum of possible market share developments. The exact formula of each function y is given by Proposition 4.1. The individual collection of y-functions is monotone increasing in b. The graphs of y are complemented by the graphs of $\bar{\lambda}$, see the window on the right in Fig. 5.

These plots give an impression of the many possibilities how market shares and rates of sales can evolve, especially when the horizon is finite and the factor μ is still large close to the expiration date. Rates of sales can be decreasing all the time, see possibility $I-$, depending on the time dependence of the factor μ (scenario I) and the properties of the system function ($b = -0.25$); or rates can peak early, like in the case $I+$. The scenarios $II+$ and $III-$ show the complexity of the evolution of rates of sales due to the interplay of a finite horizon and system parameters. The two scenarios nicely illustrate the impact the system function has. The positive exponent in scenario $II+$ is responsible for the up-tick of the rates at the end, while the downturn of $\bar{\lambda}_{III-}$ is due to the negative b exponent. To summarize, simple and general statements about how rates of sales will develop under all possible circumstances are – at best – misleading and, most often, flat wrong. It usually requires detailed numerical studies to predict the exact time evolution of the rates of sales and the corresponding market share if models have complex arrival intensities and behavioral structures.

5. Controlled flexible Bass models

In this section we shall exploit the results of Section 3 and analyze controlled Bass models, i.e. the case $\psi(y) = \Omega y + \Gamma y(1-y)$, Ω, $\Gamma \geqslant 0$, in greater detail. Note, ψ is not a simple power function and the results of Section 4 do not apply, except for the very special case when $\Gamma = 0$. We start by studying the special case $\Omega = \Gamma = 1$, and we check how well the results based on the approximation of $\psi(y)$ by $y^{1/2}$, proposed in [21], fit the exact characteristics y_E, $\bar{\lambda}_E$, $\bar{p}_E$, $\bar{w}_E$ and $\overline{V}_E$; the subscript E indicates the exact values. We compute the true values of the characteristics using Theorem 3.3.

To begin with, we take a look at the very special situation when the price elasticity parameter $\varepsilon = 2$. Fig. 6 shows that the two functions $\sqrt{y}$ and $y + y(1-y)$ are not really close when one looks at both functions on the whole unit interval [0, 1]. However, the values of the diffusion potential of both functions at $y_0 = 1$, i.e. the integral $\int_0^1 \psi(y)^{\frac{1}{\varepsilon-1}} dy$, cf. Lemma 3.1, are the same if $\varepsilon = 2$. Hence, the optimal advertising rates and the continuation values $\overline{V}(t)$ are identical as well; for this special choice of ε this is actually the case no matter what the values a, δ and the function μ are. There are slight differences between the pricing policies and the optimal trajectories, see Table 2, row 3.

If $\varepsilon \neq 2$, for instance, $\varepsilon = 1.5$ and $a = 2$, $\delta = 1$, $\mu \equiv 1$, $y_0 = 1$, and T is finite or infinite, there are (small) differences between the optimal advertising rates. These rates are constant since μ is assumed to be a constant. There are also small differences between the continuation values $\overline{V}$ of both systems. These differences are small since the values of the two diffusion potentials at 1 are almost the same (the deviation of $\varepsilon = 1.5$ from $\varepsilon = 2$ is "small"). The differences between the paths $y(t)$ of both systems, and between the rates $\bar{\lambda}(t)$ of the two systems, are minor if one just looks at the graphs, although the relative percentage errors can be substantial. But the optimal price paths of both systems clearly deviate from each other and evolve quite differently over time, see Fig. 6. While $p_S(t)$ is constant, $p_E(t)$ converges to zero after an initial growth period; the subscript S refers to Sethi et al., see [21].

Table 2 provides a few representative error numbers which support the general observations. The relative errors are the quotients of the difference of the two functions relative to the value of the exact curve, and these ratios are averaged over the time range which comprises 95 % of the total market share (of both models).

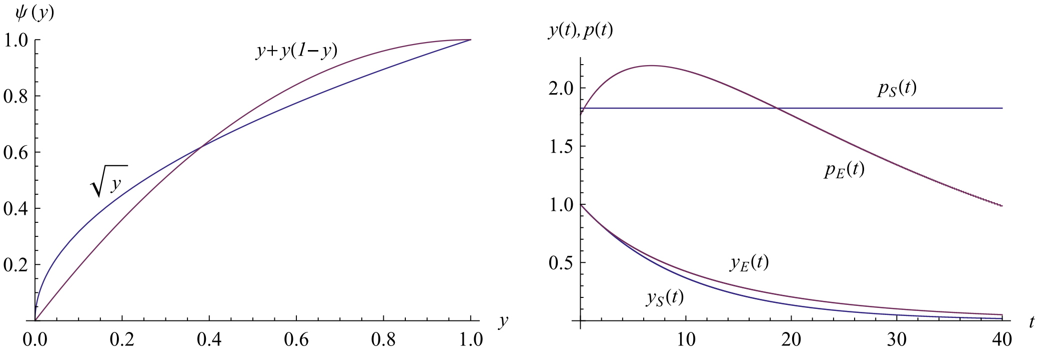

Fig. 6. A comparison of the Sethi et al. approximation, i.e. $a = 2$ and $\delta = 1$, and the true model; other parameters are $\mu = 1$ and $\varepsilon = 1.5$, see also Table 2. The plot on the left shows the graph of the functions $\psi(y) = y^{\frac{a-\delta}{a}} = \sqrt{y}$ and $\psi(y) = y + y(1-y)$ restricted to the unit interval. The window on the right shows the exact optimal price function $\bar{p}_E(t)$ together with the optimal pricing policy $\bar{p}_S(t)$ proposed by Sethi et al. The second set of graphs on the right depicts the optimal state evolutions $y_E(t)$ and $y_S(t)$, see [21].

Table 2
Relative average percentage errors (see this subsection for an explanation), as a function of ε, of the exact characteristics $\bar{w}$, $\bar{p}$, etc. and the characteristics of the approximating system proposed in [21]; $a = 2$, $\delta = 1$, $\mu \equiv 1$, $T = \infty$ and $r \equiv 0.1$.

ε	Percentage errors of:				
	y	$\bar{\lambda}$	$\bar{p}$	$\bar{w}$	$\overline{V}$
1.1	25.76	18.09	18.57	3.92	7.74
1.5	23.87	12.15	13.17	1.60	3.18
2	20.06	9.83	10.78	0	0
3	14.74	7.17	7.97	0.74	1.48
5	9.49	4.60	5.14	0.75	1.50
10	4.99	2.42	2.70	0.47	0.95

Note, if ε increases then the relative errors get smaller. If ε tends to one then the errors can be substantial. However, based on extensive numerical studies over a wide range of parameter values we claim that the approximation of $y + y(1 - y)$ by the power function y^b, $b = 0.5$, is – overall – acceptable, especially, if only values of $\overline{V}$ and $\bar{w}$ are important. Over a wide range of parameter values, the two models yield very similar results. But this fact does not necessarily imply that the optimal controls $\bar{p}_s$ and $\bar{w}_s$ are good controls if the system function is ψ_E.

Next, we take a brief look at time independent controlled Bass models and time dependent ones when Ω and Γ are positive parameters. For positive parameters any Bass model is a combination of the special cases $\psi(y) = y$, analyzed in Section 4, and Mansfield's model $\psi(y) = y(1 - y)$. The linear function $\psi(y) = y$ assumes that the adoption of the new product can be solely attributed to innovation (external factors), and that there is no word-of-mouth influence present. The Mansfield function stylizes the situation when the word-of-mouth communication channel not only is the dominating one, but the innovation channel is completely shut down. Mansfield originally proposed his model to help explain differences among innovations in the rate of imitation. He investigated factors which determine how rapidly the use of a new technique spreads from one firm to another.

The time independent controlled Mansfield model can be analyzed using Theorem 3.3. Due to the symmetry of the function $y(1 - y)$ on the unit interval the results can be easily predicted: if T is finite, $r = 0$ and μ is constant, the optimal price path will look very much like the upper part of an elongated ellipse. It is perfectly symmetric with respect to the midpoint of the planning horizon. The optimal rates of sales will have a bathtub shape. Rates rapidly fall from $+\infty$ until they reach a plateau level, stay there for most of the time, before shooting up towards the end, being a mirror image of the rates prior to half-time of the life cycle of the product. If μ is constant then optimal advertising rates are constant too. The characteristics of a finite horizon model with discounting will show a similar behavior as the characteristics in cases without discounting, except that symmetry of all functions should no longer be expected. For most of the time, optimal prices will be lower and will be skewed towards $t = 0$; they will peak earlier. Rates of sales will have the same kind of asymptotic behavior, with and without discounting, but if r is positive then rates reach their minimum value during the second half of the life time cycle. Due to the presence of discounting, optimal advertising rates will be monotone decreasing on $[0, T]$.

In general, it is worthwhile to study how the evolution of optimal price paths depends on the quotient Ω/Γ. In the case when μ is a constant, *market penetration* pricing policies or *skimming* policies can be optimal. Which kind of policy will be optimal depends on the relative strength of the two communication channels. Both possibilities can happen in the finite horizon case or the infinite horizon case with discounting. For instance, if $T < \infty$ and $r = 0$ a market penetration pricing strategy is optimal as long as Ω is "small"; more precisely, as long as $\Omega/\Gamma < 1$. If $T = \infty$ and $r > 0$ the evolution of optimal price paths is similar, but the decline is [naturally] much more gradual. With discounting, the critical value of the quotient Ω/Γ depends on r; for individual cases, it can be numerically determined.

For time homogeneous controlled Bass models, our final result precisely describes when optimal prices will peak. The proof of part (i) of the following proposition exploits the general characterizations given in Theorem 3.2; elementary calculus yields part (ii).

Proposition 5.1. *For a controlled Bass model, let $\Gamma = 1$, $y_0 = 1$, $0 \leqslant \delta < a$, $\varepsilon > 1$ and $\Omega \geqslant 0$. Define $\omega := 1 + \Omega$.*

(i) *Let T be finite, $r \equiv 0$ and μ a positive constant. If $\Omega < 1$, then the optimal price trajectory will peak at a unique point $\hat{t} := \hat{t}(\Omega, 1)$ in the open interval $(0, T)$. As a function of Ω, the mapping $\hat{t}$ is monotone decreasing. If $\Omega \geqslant 1$, i.e. the innovation parameter is greater or equal than the imitation coefficent, then optimal price paths are monotone decreasing; price skimming is optimal.*

(ii) *Let $\varepsilon = 2$ and $\Omega \leqslant 1$. Then, $\omega = 1 + \Omega \leqslant 2$,*

$$\hat{t}(\Omega, 1) = T\left(1 - \frac{\omega^3}{6(\omega - \frac{2}{3})}\right).$$

If μ is time dependent, for instance μ is one of the scenarios considered in Section 4.4, the evolution of optimal prices and the evolution of other characteristics depend on the interplay of the μ-function and the Bass functional ψ, as well as the discount rate r. Since the advertising rates $\bar{w}$ are synchronized with μ the evolution of $\bar{w}$ can be easily predicted. Furthermore, for scenario I all optimal advertising rates are monotone decreasing in t, and the rates are isotone in Ω. For the second scenario, the time dependence of $\bar{w}$ corresponds to the one of μ_{II}, and the rates are again higher when Ω becomes larger. As far as the evolution of the market shares is concerned, they stay fairly close over a range of Ω values while the sales rates can be quite different during the initial phase of the adoption period. For small Ω values the optimal rates $\bar{\lambda}$ are monotone decreasing. If $\Omega \gg \Gamma$ then the rates will have a concave-convex shape reaching a maximum value fairly quickly.

6. Conclusions

In contrast to previous studies we analyze *time* and *general state dependent* controlled adoption models characterized by isoelastic price and advertising dynamics. In addition, we allow for non-negative time dependent discount rates and consider the control problem over a finite horizon and an infinite horizon. Except for advertising costs no other cost expressions are taken into account.

We use our general theoretical results to analyze, in particular, time dependent controlled Bass models and models where the system function ψ is a power function of y. The theory provides the analytical underpinning of specific management recommendations. The computational method which we propose makes it possible to numerically study isoelastic models with arbitrary time and state factors. Specifically, should sales and/or optimal prices of a model with complex characteristics peak at some points in time then the corresponding numbers can be easily computed.

An important implication of the various formulas and results is the fact that the properties of the system function ψ, for instance $\psi(y)$ is monotone increasing, determine the time evolution of optimal prices. The prices are transformed values of the adoption process. Since the time factor μ only affects optimal prices in terms of an integral expression, μ influences the price level but not the evolution of optimal prices.

The optimal advertising rate is subordinated to the function μ and only depends on ψ via the diffusion potential.

These insights, combined with proper choices of the arrival intensity μ, facilitate the process of fitting models to data.

Finally, our analysis sheds new light on questions raised in [2,7], e.g., "Why does the Bass model fit without decision variables?", and "Why does the Bass model lead to odd optimal advertising policies?".

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