Localizations of Models of Dependent Type Theory

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Introduction

Dependent Type Theory is a candidate foundation for mathematics. Why is it interesting?

1. Closely linked to *computations* and *computer science*, makes proof assistants possible
2. Enough by itself, unlike Set Theory and Propositional Calculus
3. *Proofs* are internal objects
4. Better treatment of *equality*

Dependent Type Theory

About
Dependent types A over contexts Γ and their *terms $x : A$*.

Structural Rules
How to work with variables.

Logical Rules
Construct new types and their terms from old, carry out computations, provide *Σ-types* $\Sigma(A, B)$, *Π-types* $\Pi(A, B)$ and *Id-types* Id_A, *natural-numbers-type* Nat...

Problem

Providing a model of Dependent Type Theory is hard.

Solution

Defining a class of algebraic models.

Modeling Structural Rules

Definition (Contextual Categories)

A category C with:

1. a grading on objects $\operatorname{Ob} \mathsf{C} = \coprod_{n \in \mathbb{N}} \operatorname{Ob}_n \mathsf{C}$;
2. a map $\operatorname{ft}_n \colon \operatorname{Ob}_{n+1} \mathsf{C} \to \operatorname{Ob}_n \mathsf{C}$ for each $n \in \mathbb{N}$;
3. *basic dependent projections* $p_A \colon \Gamma.A \to \operatorname{ft}_n(\Gamma.A) = \Gamma$;
4. a functorial choice of a pullback squares

$$
\begin{array}{ccc}
\Delta.f^*A & \xrightarrow{q(f,A)} & \Gamma.A \\
{\scriptstyle p_{f^*A}} \downarrow & & \downarrow {\scriptstyle p_A} \\
\Delta & \xrightarrow{f} & \Gamma
\end{array} ;
$$

5. ...

Modeling Logical Rules

Extra Structure

Id-types require from $\Gamma.A$ an object $\Gamma.A.A.\mathsf{Id}_A$...
Π-types require from $\Gamma.A.B$ an object $\Gamma.\Pi(A, B)$, a map
$\mathsf{app}_{A,B} \colon \Gamma.\Pi(A, B).A \to \Gamma.A.B$...

Definition

A categorical model of type theory is a contextual category C *with* Σ, Id *and* Π *structures.*

Bi-Invertibility

Definition (Bi-Invertible Map)

A map $f : \Gamma \to \Delta$ in a contextual category with Id-structure C for which we can provide:

1. *maps $g_1 : \Delta \to \Gamma$, $\eta : \Gamma \to \Gamma.(1_\Gamma, g_1 \cdot f)^* \, \mathrm{Id}_\Gamma$;*
2. *maps $g_2 : \Delta \to \Gamma$, $\epsilon : \Delta \to \Delta.(1_\Delta, f \cdot g_2)^* \, \mathrm{Id}_\Delta$.*

Question

What if we localize at bi-invertible maps?

Conjecture (Kapulkin-Lumsdaine 2016)

The horizontal maps, given by simplicial localization, induce equivalences of ∞-categories.

$$
\begin{array}{ccc}
\mathrm{CxlCat}_{\Sigma,1,\mathrm{Id},\Pi} & \longrightarrow & \mathrm{LCCC}_\infty \\
\downarrow & & \downarrow \\
\mathrm{CxlCat}_{\Sigma,1,\mathrm{Id}} & \longrightarrow & \mathrm{Lex}_\infty
\end{array}
$$

Proof by Nguyen-Uemura has recently become available on arxiv. One hopes to extend this to an equivalence between $\mathrm{CxlCat}_{\mathrm{HoTT}}$ and $\mathrm{ElTopos}_\infty$.

Fibrational Structure

Definition (∞-Categories with Weak Equivalences and Fibrations)

A triple $(\mathcal{C}, W, Fib)$ where:
...some conditions similar to fibration categories but weaker, with $\mathcal{C}$ an ∞-category.

Theorem (Avigad-Kapulkin-Lumsdaine 2013)

A contextual category with Σ and Id *structures defines a fibration category, where weak equivalences are bi-invertible maps and fibrations are maps isomorphic to dependent projections.*

Localizing Fibrational Categories

Proposition (Cisinski)

The localization at weak equivalences of an ∞-category with weak equivalences and fibrations $\mathcal{C}$ is a finitely complete ∞-category.

Proposition (Cisinski)

Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, if for every fibration $f: x \to y$ between fibrant objects the pullback functor between fibrant slices $f^*: \mathcal{C}(y) \to \mathcal{C}(x)$ has a right adjoint preserving trivial fibrations, then $L(\mathcal{C})$ is locally cartesian closed.

Localizations of Models are Cartesian Closed

Theorem (Kapulkin 2015)

Given a categorical model of type theory C, its localization L(C) is a locally cartesian closed ∞-category.

Proof.

For any dependent projection $p_\Delta \colon \Gamma.\Delta \to \Gamma$, there exists a right adjoint to $p_\Delta^* \colon C(\Gamma) \to C(\Gamma.\Delta)$ given by

$$(p_\Delta)_*(\Gamma.\Delta.\Theta) = \Gamma.\Pi(\Delta, \Theta)$$

with counit induced by $\mathrm{app}_{\Delta,\Theta}$. It preserves the fibrational structure. □

Thank you for your attention!