# Localizations of Models of Dependent Type Theory

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A modern proof of the following theorem.

### Theorem (Kapulkin 2015)

*Given a categorical model of type theory C, its $\infty$-categorical localization $L(C)$ is a locally cartesian closed $\infty$-category.*

# Dependent Type Theory

## What

A theory of computations and a foundation of mathematics.

## Objects

*Dependent types A over contexts* Γ *and their terms* $x : A$.

## Structural rules

How to work with variables.

## Logical rules

Construct new types and their terms from old, carry out computations, provide *Σ-types* $\Sigma(A, B)$, *Π-types* $\Pi(A, B)$ and *Id-types* $\mathrm{Id}_A$, *natural-numbers-type* Nat...

# Models

**Problem**

Providing a model of Dependent Type Theory is hard.

**Solution**

Defining a class of algebraic models.

# Modeling structural rules

## Definition (contextual categories)

A category C with:

1. a grading on objects $\mathrm{Ob}\,C = \coprod_{n \in \mathbb{N}} \mathrm{Ob}_n\,C$;
2. a map $\mathrm{ft}_n \colon \mathrm{Ob}_{n+1}\,C \to \mathrm{Ob}_n\,C$ for each $n \in \mathbb{N}$;
3. *basic dependent projections* $p_A \colon \Gamma.A \to \mathrm{ft}_n(\Gamma.A) = \Gamma$;
4. a functorial choice of pullback squares

$$
\begin{array}{ccc}
\Delta.f^*A & \xrightarrow{q(f,A)} & \Gamma.A \\
{\scriptstyle p_{f^*A}}\downarrow & & \downarrow{\scriptstyle p_A} \\
\Delta & \xrightarrow{\quad f \quad} & \Gamma
\end{array} \; ;
$$

5. ...

# Modeling logical rules

## Extra structure

Id-types require from $\Gamma.A$ an object $\Gamma.A.A.\,\mathsf{Id}_A$...
$\Pi$-types require from $\Gamma.A.B$ an object $\Gamma.\Pi(A,B)$, a map
$\mathsf{app}_{A,B}\colon \Gamma.\Pi(A,B).A \to \Gamma.A.B$...

## Definition

*A categorical model of type theory is a contextual category $\mathsf{C}$ with $\Sigma$, Id and $\Pi$ structures.*

# Bi-invertibility

## Definition (Bi-invertible map)

A map $f : \Gamma \to \Delta$ in a contextual category with Id-structure C for which we can provide:

1. maps $g_1 : \Delta \to \Gamma$, $\eta : \Gamma \to \Gamma.(1_\Gamma, g_1 \cdot f)^* \operatorname{Id}_\Gamma$;

2. maps $g_2 : \Delta \to \Gamma$, $\epsilon : \Delta \to \Delta.(1_\Delta, f \cdot g_2)^* \operatorname{Id}_\Delta$.

## Question

What if we localize at bi-invertible maps?

# Fibrational structure

## Definition ($\infty$-categories with weak equivalences and fibrations)

*A triple $(\mathcal{C}, W, Fib)$ where:*
*...a weakening of the definition of fibration categories, with $\mathcal{C}$ an $\infty$-category.*

## Theorem (Avigad-Kapulkin-Lumsdaine 2013)

*A contextual category with $\Sigma$ and Id structures defines a fibration category, where weak equivalences are bi-invertible maps and fibrations are maps isomorphic to dependent projections.*

# Localizing fibrational categories

## Proposition (Cisinski)

*The localization at weak equivalences of an $\infty$-category with weak equivalences and fibrations $\mathcal{C}$ is a finitely complete $\infty$-category.*

## Proposition (Cisinski)

*Given an $\infty$-category with weak equivalences and fibrations $\mathcal{C}$, if for every fibration $f : x \to y$ between fibrant objects the pullback functor between fibrant slices $f^* \colon \mathcal{C}(y) \to \mathcal{C}(x)$ has a right adjoint preserving trivial fibrations, then $L(\mathcal{C})$ is locally cartesian closed.*

# Localizations of models are Cartesian closed

## Theorem (Kapulkin 2015)

*Given a categorical model of type theory* C, *its localization* $L(C)$ *is a locally cartesian closed $\infty$-category.*

## Proof.

For any basic dependent projection $p_A \colon \Gamma.A \to \Gamma$, there exists a right adjoint to $p_A^* \colon C(\Gamma) \to C(\Gamma.A)$ given by

$$(p_A)_*(\Gamma.A.\Theta) = \Gamma.\Pi(A, \Theta)$$

with counit induced by $\mathrm{app}_{A,\Theta}$. It preserves the fibrational structure. □

Thank you for your attention!

# Why is Dependent Type Theory cool?

1. Closely linked to *computations* and *computer science*, makes proof assistants possible
2. Enough by itself as a foundation, unlike Set Theory or Propositional Calculus
3. *Proofs* are internal objects
4. Better treatment of *equality*

# Internal Languages Conjecture

## Conjecture (Kapulkin-Lumsdaine 2016)

*The horizontal maps, given by simplicial localization, induce equivalences of $\infty$-categories.*

$$
\begin{array}{ccc}
\mathrm{CxlCat}_{\Sigma,1,\mathrm{Id},\Pi} & \longrightarrow & \mathrm{LCCC}_\infty \\
\downarrow & & \downarrow \\
\mathrm{CxlCat}_{\Sigma,1,\mathrm{Id}} & \longrightarrow & \mathrm{Lex}_\infty
\end{array}
$$

A proof by Nguyen-Uemura has recently become available on arxiv. One hopes to extend this to an equivalence between $\mathrm{CxlCat}_{\mathrm{HoTT}}$ and $\mathrm{ElTopos}_\infty$.