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ALGANT Master Thesis

Localizations of Models of Dependent Type Theory

Author: Matteo Durante

Supervisor: Prof. dr. Denis-Charles Cisinski Cosupervisor: dr. Hoang Kim Nguyen

Universiteit Leiden

Universität Regensburg

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Introduction

We can also rewrite the paper by Kapulkin about LCCC arising from TT using the language of localizations of quasi-categories. There they develop the relevant theory showing that under some conditions the frame associated to a fibration category is locally cartesian closed, but using Cisinski's results we can prove the same theorem directly using a more mainstream theory.

What should be included in such an overview?

1- Cisinski's theory of localizations (of fibration ∞-categories)

2- an introduction to contextual categories: where do they come from? Why are they useful? Check out Voevodsky's papers about C-systems

We explain what dependent type theory is (Martin-Lof's notes from 1984) and why it's an interesting foundation of mathematics. We mention Homotopy Type Theory as an effort to provide homotopical foundations which better model how we think about identities, which explains why intensional identity types are more interesting to us than extensional ones.

We move on to defining contextual categories (1211.2851, 1406.7413, 1507.02648) and what the Pi, Sigma and Id structures are (1406.7413, 1211.2851 Appendix B). To understand what the link between such structures and syntactically presented type theories we refer to 1507.02648, Sec. 1.1, while the statement of the conjectured correspondence is in 1304.0680, Sec. 2.1.

Where does the link between dependent type theories and ∞-categories come from? We see that ∞-categories intuitively model the behavior of type theories and their type constructions, especially when considering Homotopy Type Theory, however this relation is known only partially (references in the intro of 1507.02648). The idea is that the type theory we are interested in should be the internal language of some class of ∞-categories and a precise statement would require us to provide homotopical functors in both directions which induce an equivalence on the associated ∞-categories. The idea is to construct the functor from contextual categories as a localization functor, that is we need to provide a homotopical structure on contextual categories, as they do in 1507.02648 (there should be an older reference) which then provides an associated ∞-category. This is the object of the Initiality Conjecture, stated in 1610.00037, in the hope that such a correspondence will extend to Homotopy Type Theory and some notion of Elementary Higher Toposes, perhaps the one specified in 1805.03805. At the moment we know that HoTT can be interpreted in Higher Toposes with some structure. Current progress: 1709.09519, an upcoming paper by Nguyen-Uemura (HoTTest talk).

Our aim is to show that when taking contextual categories with the structure we specified earlier we obtain a locally cartesian closed ∞-category. To do so we provide a fibrational structure on contextual categories (1304.0680, 1507.02648), which as we anticipate will imply that their simplicial localizations are finitely complete. We also prove that the hypothesis of [Cis19, Thm. 7.6.16] are satisfied, informing that this will be sufficient to prove Kapulkin's main result from 1507.02648.

We then develop the theory of localizations of ∞-categories by Cisinski and specifically develop the results concerning ∞-categories with fibrations and weak equivalences. Localizations of such ∞-categories are finitely complete. The objective is to show [Cis19, Thm. 7.6.16]. How in depth should we go?

Why all of this is interesting: we are proving Kapulkin's result internalizing all of the discussion within the language of ∞-category theory and relying only on its simplicial model.

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Contents

1 Background

We begin by presenting our dependent type theory and then introduce a class of algebraic models.

1.1 Logical Setting

We assume some familiarity with some kind of dependent type theory, preferably Martin-Löf Type Theory [ML84] or an extension, which should be enough to understand our setting. This has become a popular formal theory of logic to start from when creating an alternative foundation of mathematics since it provides a more nuanced concept of equality, the possibility of creating powerful proof assistants and, thanks to its relation to homotopy theory, cleaner and more general proofs of some theorems, like the Black-Massey theorem, presented in the language of Homotopy Type Theory [Uni13].

Notation 1.1.1. A context Γ is a sequence $a_1 : A_1, \ldots, a_n : A_n$ of types A_i with choices of terms $a_i : A_i$, each of them derivable from the previous ones and possibly dependent on them.

$$a_1 : A_1, \ldots, a_{i-1} : A_{i-1} \vdash A_i \text{ type}$$
$$x_1 : A_1, \ldots, x_{i-1} : A_{i-1} \vdash x_i : A_i.$$

We shall hide the dependency on previous terms in our notation (as we have just done) unless it is needed for clarity, in which case we may write $B(a_1, \ldots, a_n)$ for a type B dependent on $a_1 : A_1, \ldots, a_n : A_n$ and similarly $b(a_1, \ldots, a_n) : B(a_1, \ldots, a_n)$ for a dependent term or suppress individual dependencies.

We will be considering the structural rules from [KL12, App. A.1] and some logical ones as presented in [KL12, App. A.2]. Informally, we deal with:

1. *dependent sums* $\Sigma(A, B)$, also called Σ-*types*, and the associated introduction, elimination and computation rules;

2. *dependent products* $\Pi(A, B)$, also called Π-*types*, and the associated introduction, elimination and computation rules;

3. *(intensional) identity types* Id_A, also called Id-*types*, and the associated introduction, elimination and computation rules.

Furthermore, we consider the following two rules:

4. *η-conversion for functions*: for any type A and any type B dependent on $x : A$, given a function $f : \Pi(A, B)$, we have $\lambda(x : A).\, \mathsf{app}(f, x) \equiv f$;

5. *function extensionality*: for any type A and any type B dependent on $x : A$, given functions $f, g : \Pi(A, B)$, if $\mathsf{app}(f, x) = \mathsf{app}(g, x)$ for every $x : A$, then $f = g$.

Remark 1.1.2. Observe that when specifying η-conversion we used a *judgemental equality*, while for function extensionality we used a simple equality. The latter means that, given $h : \Pi(x : A, \mathrm{Id}_B(\mathsf{app}(f, x), \mathsf{app}(g, x)))$, we can present a term $\mathsf{ext}(f, g, h)$ of type $\mathsf{Id}_{\Pi(A,B)}(f, g)$.

Remark 1.1.3. It should be noted that function extensionality is a common assumption, however it is sometimes derived instead from other principles, like in Homotopy Type Theory, where it is a consequence of univalence [Uni13, Ch. 4.9]. Also, the other principles we are considering are fairly general and very desirable for any dependent type theory meant as a candidate foundation of mathematics, however they are still not enough for such a role: indeed, we lack introduction rules to construct any type or term starting from the empty context. This is fine, since it means that our results will apply to virtually any type-theoretic foundation of mathematics.

1.2 Contextual Categories

A problem of working with dependent type theory is the handling of the structural rules, namely context substitution, variable binding, variable substitution and so on. To avoid this issue, it is convenient to work within a *semantic model*, which we now present.

There are many algebraic models of dependent type theory in the literature meant to only model these rules, like *category with attributes*, *categories with families* and *comprehension categories*, which are fairly similar among them. Here we shall work with *contextual categories*, which were first explored by Cartmell and Streicher [Car78; Car86; Str91] and later by Voevodsky, under the name *C-systems* [Voe14b]. For informations on the others we refer to [nLab22].

Definition 1.2.1. A *contextual category* C is a small category, also denoted C, with the following data:

1. a grading on objects $\mathrm{Ob}\,\mathsf{C} = \coprod_{n\in\mathbb{N}} \mathrm{Ob}_n\,\mathsf{C}$, also called *contexts*;
2. an object $* \in \mathrm{Ob}_0\,\mathsf{C}$, the *empty context*;
3. for each $n \in \mathbb{N}$, a map $\mathsf{ft}_n\colon \mathrm{Ob}_{n+1}\,\mathsf{C} \to \mathrm{Ob}_n\,\mathsf{C}$, often simply denoted ft;
4. for each $n \in \mathbb{N}$ and $X \in \mathrm{Ob}_{n+1}\,\mathsf{C}$, a map $p_X\colon X \to \mathsf{ft}\,X$;
5. for each $n \in \mathbb{N}$, $X \in \mathrm{Ob}_{n+1}\,\mathsf{C}$ and $f\colon Y \to \mathsf{ft}\,X$, an object f^*X and a map $q(f, X)\colon f^*X \to X$;

such that:

1. $*$ is the unique element of $\mathrm{Ob}_0\,\mathsf{C}$;
2. $*$ is terminal;

3. for each n, $X \in \mathrm{Ob}_{n+1}\,\mathsf{C}$ and $f\colon Y \to \mathsf{ft}\,X$, we have $\mathsf{ft}(f^*X) = Y$ and the square

$$\begin{tikzcd} f^*X \arrow[r, "{q(f,X)}"] \arrow[d, "p_{f^*X}"'] & X \arrow[d, "p_X"] \\ Y \arrow[r, "f"'] & \mathsf{ft}\,X \end{tikzcd}$$

is a pullback;

4. for each $n \in \mathbb{N}$, $X \in \mathrm{Ob}_{n+1}\,\mathsf{C}$ and pair of maps $f\colon Y \to \mathsf{ft}\,X$, $g\colon Z \to Y$, we have $(fg)^*X = g^*f^*X$, $1^*_{\mathsf{ft}\,X}X = X$, $q(fg,X) = q(f,X)\cdot q(g,f^*X)$ and $q(1_{\mathsf{ft}\,X},X) = 1_X$.

Remark 1.2.2. The last condition in the definition means that our choice of pullbacks is functorial, which allows us to see contextual categories as a strict model of dependent type theory, not requiring to keep track of coherency maps. Other models, like comprehension categories, do not have such requirements, which makes them more general and interpretation harder, unless they are first strictified, that is replaced by an equivalent model of the same kind but with functorial pullbacks.

A motivating example for contextual categories is the *syntactic category of a dependent type theory*, which is constructed as follows.

Construction 1.2.3. [Car78, p. 2.6] Given a dependent type theory $\mathbf{T}$, its syntactic category $\mathsf{Syn}(\mathbf{T})$ has:

1. $\mathrm{Ob}_n\,\mathsf{Syn}(\mathbf{T})$ given by contexts $[x_1 : A_1,\dots,x_n : A_n]$ of length n, modulo judgemental equality and renaming of free variables;

2. maps are *context morphisms*, or *substitutions*, modulo judgemental equalities and renaming of free variables. This means that a map

$$f\colon [x_1 : A_1,\dots,x_n : A_n] \to [y_1 : B_1,\dots,y_m : B_m(y_1,\dots,y_{m-1})]$$

is an equivalence class of tuples of terms $(f_1,\dots,f_m)$ such that

$$\begin{gathered} x_1 : A_1,\dots,x_n : A_n \vdash f_1 : B_1, \\ \vdots \\ x_1 : A_1,\dots,x_n : A_n \vdash f_m : B_m(f_1,\dots,f_{m-1}), \end{gathered}$$

are all derivable judgements and two such tuples $(f_1,\dots,f_n),(g_1,\dots g_n)$ are equivalent if we have

$$x_1 : A_1,\dots,x_n : A_n \vdash f_i \equiv g_i : B_i(f_1,\dots,f_{i-1})$$

for every i; we shall henceforth also write $[f_i]$ for f;

3. composition is given by substitution;

4. the identity $\Gamma \to \Gamma$ is given by the variables of Γ, considered as terms, that is we take the sequence $[x_i]$ given by

$$\begin{gathered} x_1 : A_1, \dots, x_n : A_n \vdash x_1 : A_1, \\ \vdots \\ x_1 : A_1, \dots, x_n : A_n \vdash x_n : A_n(x_1, \dots, x_{n-1}); \end{gathered}$$

5. the terminal object is the empty context $[]$;

6. $\mathsf{ft}([x_1 : A_1, \dots, x_{n+1} : A_{n+1}]) = [x_1 : A_1, \dots, x_n : A_n]$;

7. for $\Gamma = [x_1 : A_1, \dots, x_n : A_{n+1}]$, $p_\Gamma : \Gamma \to \mathsf{ft}\,\Gamma$ is the *dependent projection context morphism* $[x_1, \dots, x_n]$, defined by

$$\begin{gathered} x_1 : A_1, \dots, x_{n+1} : A_{n+1} \vdash x_1 : A_1, \\ \vdots \\ x_1 : A_1, \dots, x_{n+1} : A_{n+1} \vdash x_n : A_n(x_1, \dots, x_n) \end{gathered}$$

   and thereby simply forgetting the last variable of Γ;

8. given contexts

$$\begin{aligned} \Gamma &= [x_1 : A_1, \dots, x_{n+1} : A_{n+1}(x_1, \dots, x_n)], \\ \Delta &= [y_1 : B_1, \dots, y_m : B_m(y_1, \dots, y_{m-1})] \end{aligned}$$

   and a map $f = [f_i(y)] : \Delta \to \mathsf{ft}\,\Gamma$ (where y is a vector of variables of length m), the pullback $f^*\Gamma$ is the context

$$[y_1 : B_1, \dots, y_m : B_m(y_1, \dots, y_{m-1}), y_{m+1} : A_{n+1}(f_1(y), \dots, f_n(y))]$$

   for some new variable y_{m+1}, while $q(\Gamma, f) : f^*\Gamma \to \Gamma'$ is specified by $[f_1, \dots, f_n, y_{m+1}]$.

Remark 1.2.4. Given a dependent type theory $\mathbf{T}$, the terms $a : A$ of a type over a context Γ can be recovered (up to definitional equality) from the syntactic category $\mathsf{Syn}(\mathbf{T})$ by looking at sections of the basic dependent projection $p_{[\Gamma, x:A]} : [\Gamma, x : A] \to [\Gamma]$, which indeed act as identities over Γ and furthermore specify a term $\Gamma \vdash a : A$. Given the importance of such maps, we shall often simply write “sections” to refer to sections of basic dependent projections, without specifying which projections unless it creates ambiguity.

The above construction also tells us how to think about the other elements in the definition of contextual categories: basic dependent projections $p_{\Gamma.A.B} : \Gamma.A.B \to \Gamma.A$ represent dependent types B over A in the context Γ, while pulling back along a dependent projection corresponds to switching context and so on for the other objects in the definition.

Notation 1.2.5. We shall also make use of some conventions inspired by this construction. Namely, given a contextual category C and an object $\Gamma \in \mathrm{Ob}_n\,\mathsf{C}$, we shall write $\Gamma.A_1.\dots.A_k$ and $\Gamma.\Delta$ interchangeably for an object X in $\mathrm{Ob}_{n+k}\,\mathsf{C}$ with $\mathsf{ft}^k X = \Gamma$ and call it a *context extension of* Γ *of length* k. We shall also write $p_{A_1.\dots.A_k}$ and p_Δ for the composition of the basic dependent projections $p_{\Gamma.A_1.\dots.A_i}$, with i ranging from 1 to k, and the resulting map will be called a *dependent projection*. In the case where $k = 1$, p_{A_1} corresponds to a basic dependent projection and the context extension of Γ will be *simple*, while if $k = 0$ we have $p = 1_\Gamma$ and then the context extension will be *trivial*. We shall also write $1_\Delta, 1_{A_1.\dots.A_k}, \dots, 1_{A_k}$ for $1_{\Gamma.\Delta}$, depending on what we want to emphasize. Greek letters shall be used to indicate context extensions of arbitrary length, while Latin ones will be reserved to simple extensions.

Continuing, given a dependent projection $p_{A_1.\dots.A_k} = p_\Theta$ as above and a context morphism $f\colon \Delta \to \Gamma$, we define inductively $f^*(\Gamma.\Theta) = \Delta.f^*\Theta = \Delta.f^*A_1.\dots.f^*A_k$ and $q(f, \Gamma.\Theta) = q(f, \Theta) = q(f, A_1.\dots.A_k)$ by looking at the pasting of pullback squares

$$
\begin{tikzcd}
\Delta.f^*A_1.\dots.f^*A_k \arrow[r, "p_{f^*A_k}"'] \arrow[d, "q(f,A_1.\dots.A_k)"'] \arrow[rrrr, bend left, "p_{f^*A_1.\dots.f^*A_k}"] & \Delta.f^*A_1.\dots.f^*A_{k-1} \arrow[r] \arrow[d, "q(f,A_1.\dots.A_{k-1})"] & \cdots \arrow[r] & \Delta.f^*A_1 \arrow[r, "p_{f^*A_1}"'] \arrow[d, "q(f,A_1)"'] & \Delta \arrow[d, "f"] \\
\Gamma.A_1.\dots.A_k \arrow[r, "p_{A_k}"] \arrow[rrrr, bend right, "p_{A_1.\dots.A_k}"'] & \Gamma.A_1.\dots.A_{k-1} \arrow[r] & \cdots \arrow[r] & \Gamma.A_1 \arrow[r, "p_{A_1}"] & \Gamma
\end{tikzcd},
$$

which also shows that

$$q(f, A_1.\dots.A_k) = q(q(f, A_1), A_2.\dots.A_k) = q(q(f, A_1.\dots.A_{k-1}), A_k).$$

As usual, if $k = 0$ we have $q(f, \Theta) = f$, $f^*(\Gamma.\Theta) = \Gamma$, while for $k = 1$ we have $q(f, A_1) = q(f, \Gamma.A_1)$, $\Delta.f^*A_1 = f^*(\Gamma.A_1)$ agreeing with the base structure of C.

Finally, given a section $a\colon \Gamma \to \Gamma.A$ and a context morphism $f\colon \Delta \to \Gamma$, we also want to specify f^*a, that is the section which we get by switching context. This is given by the map $(1_\Delta, a \cdot f)$ specified by the pullback square

$$
\begin{tikzcd}
\Delta \arrow[dr, dotted, "(1_\Delta, a\cdot f)"] \arrow[drr, bend left, "a\cdot f"] \arrow[ddr, equal, bend right] & & \\
& \Delta.f^*A \arrow[r, "q(f,A)"'] \arrow[d, "p_{f^*A}"] & \Gamma.A \arrow[d, "p_A"] \\
& \Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd},
$$

and, as shown by the commutative diagram

$$
\begin{tikzcd}
\Delta \arrow[r, "f"] \arrow[d, "f^*a"] \arrow[dd, equal, bend right=60] & \Gamma \arrow[d, "a"'] \arrow[dd, equal, bend left=60] \\
\Delta.f^*A \arrow[r, "q(f,A)"'] \arrow[d, "p_{f^*A}"] & \Gamma.A \arrow[d, "p_A"'] \\
\Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd},
$$

it corresponds to the pullback of a along $q(f, A)$. By the techniques we provided earlier, we extend this construction to contexts extensions of arbitrary length.

Finally, when reasoning in a syntactic category, to specify a map $[f]: [\Gamma, a_1 : A_1, \ldots, a_n : A_n] \to [\Gamma, b_1 : B_1, \ldots, b_m : B_m]$ stable over Γ (that is acting like the identity on the corresponding variables) we can suppress the corresponding terms in the tuple. To produce such a map, we shall write then $(a_1, \ldots, a_n) \mapsto (f_1, \ldots, f_m)$, in order to clarify how to construct the f_i from the variables in the domain context.

Remark 1.2.6. When we pull back a dependent projection $p_\Delta : \Gamma.\Delta \to \Gamma$ along another one p_Ψ, we may also choose to do it the other way around, which induces an isomorphism $\Gamma.\Delta.p_\Delta^*\Psi \to \Gamma.\Psi.p_\Psi^*\Delta$ denoted by $\mathsf{exch}_{\Delta,\Psi}$. Syntactically, this amounts to swapping two context extensions not dependent on one another.

Definition 1.2.7. A *contextual functor* between contextual categories $F : \mathsf{C} \to \mathsf{D}$ is a functor on the underlying categories which preserves the grading, basic dependent projections and such that $q(Ff, FX) = F(q(f, X))$.

Remark 1.2.8. Our definition allows us to see contextual categories as models for a (small) essentially algebraic theory [AR94] with sorts indexed by $\mathbb{N} + \mathbb{N} \times \mathbb{N}$. In that context, we get a notion of morphism between models of this theory, which coincides with the one we have just provided. Its category of models will be the category of contextual categories, denoted by CxlCat, which is complete and cocomplete as the category of models of an essentially algebraic theory. The same can be done for the other algebraic models we mentioned.

As we stated, the defining properties of contextual categories are meant to model structural rules, so that we can reason about dependent type theories in terms of pullbacks, sections and so on while eliminating a lot of the bureaucracy needed to provide a model, which can now be done by constructing such a category.

Example 1.2.9. A constructive and formalization-ready approach to producing contextual categories was developed by Voevodsky in [Voe14a; Voe14a; Voe15b; Voe15a], where he starts from a *universe category.*

On the other hand, as noted in [KL12, par. 1.2], interpreting complicated structures in contextual categories quickly becomes unreadable, meaning that ideally we would like to be able to switch from a syntactic to a semantic presentation and viceversa freely, so that we may work with the most convenient one for a given situation. This would amount to some *soundness* and *correctness* theorems reminiscent of the ones of *first order predicate logic.* In categorical semantics, this corresponds to *initiality.*

Initiality Conjecture. *Given a dependent type theory* $\mathbf{T}$*, its syntactic category* $\mathsf{Syn}(\mathbf{T})$ *is initial in the category of contextual categories with the appropriate structure.*

Remark 1.2.10. Here by "appropriate structure" we mean extra algebraic structures meant to model the logical rules of the type theory, like Σ-types, Π-types and Id-types. Indeed, the definition of contextual category is not meant to deal with them. We will introduce the such structures in the next section.

Remark 1.2.11. This conjecture has been partially proven for a simple variant of dependent type theory in [Str91] and a proof formalized in Agda for Martin-Löf Type Theory was provided by de Boer and Brunerie [Boe20], however we still do not have a fully rigorous general statement. Such a statement would first require a general notion of dependent type theory, which has been worked on in [Isa16; Uem19; Bru20; BHL20; NU22]. (ARE YOU SURE ABOUT THESE CITATIONS?)

If we could prove it, then, under the proper conditions, for any contextual category C we would have a contextual functor $\mathsf{Syn}(\mathbf{T}) \to \mathsf{C}$ explaining how to interpret $\mathbf{T}$ in C. This is essentially an algorithmic problem: it reduces to explaining inductively to a computer how to construct the aforementioned functor.

Remark 1.2.12. From the Initiality Conjecture researchers have also derived some homotopical models, which are the great drivers of research in the field. Specifically, it was shown that ∞-*toposes* can model Homotopy Type Theory [Shu19] and, as a concrete example, in [KL12] one can find a model of Univalent Type Theory produced using Voevodsky's previously mentioned methods. More recent results concern the *Internal Languages Conjecture* [KS19], which was proven in [NU22] and is a direct descendant and a strengthening of Theorem 4.2.4. (UNIVALENT = HOMOTOPY???)

Remark 1.2.13. Another reason researchers often assume this conjecture to be true is that, thanks to what is known in folklore as the *syntax-semantic adjunction*, it provides *internal languages*, an intuitive but not yet formalized notation to reason about contextual categories. We shall not use such instruments and instead restrict ourselves to working with syntactic categories, with the idea that a generalization of our results to arbitrary contextual categories should be straightforward once we have a proof of the conjecture and all of the tools mentioned above. Unfortunately, this will also mean that we will not be able to rely on some results and definitions as originally stated.

Remark 1.2.14. It should be noted that, since CxlCat and its variants are complete, they do have initial objects.

1.3 Logical Structures

We now define the extra structures on contextual categories we mentioned earlier. Our definitions shall be taken from [KL12; KL18].

Definition 1.3.1. A Σ-*structure* on a contextual category C consists of:

1. for each $\Gamma.A.B \in \mathrm{Ob}_{n+2}\,\mathsf{C}$, an object $\Gamma.\,\Sigma(A,B) \in \mathrm{Ob}_{n+1}\,\mathsf{C}$;

2. for each $\Gamma.A.B \in \mathrm{Ob}_{n+2}\,\mathsf{C}$, a morphism $\mathsf{pair}_{A,B}\colon \Gamma.A.B \to \Gamma.\,\Sigma(A,B)$ over Γ;

3. for each $\Gamma.A.B, \Gamma.\,\Sigma(A,B).C \in \mathrm{Ob}_{n+2}\,\mathsf{C}$, and $d\colon \Gamma.A.B \to \Gamma.\,\Sigma(A,B).C$ with $p_C \cdot d = \mathsf{pair}_{A,B}$, a section $\mathsf{split}_d\colon \Gamma.\,\Sigma(A,B) \to \Gamma.\Sigma(A,B).C$ such that $\mathsf{split}_d \cdot \mathsf{pair}_{A,B} = d$;

4. where all of the above is compatible with context substitution, that is given a map $f \colon \Delta \to \Gamma$ we have

$$\begin{aligned} f^*(\Gamma.\,\Sigma(A,B)) &= \Delta.\Sigma(f^*A, f^*B), \\ f^*\,\mathsf{pair}_{A,B} &= \mathsf{pair}_{f^*A,f^*B}, \\ f^*\,\mathsf{split}_d &= \mathsf{split}_{f^*d}\,. \end{aligned}$$

Definition 1.3.2. A Id-*structure* on a contextual category C consists of:

1. for each $\Gamma.A \in \mathrm{Ob}_{n+1}\,\mathsf{C}$, an object $\Gamma.A.p_A^*A.\,\mathsf{Id}_A \in \mathrm{Ob}_{n+3}\,\mathsf{C}$;

2. for each $\Gamma.A \in \mathrm{Ob}_{n+1}\,\mathsf{C}$, a morphism $\mathsf{refl}_A \colon \Gamma.A \to \Gamma.A.p_A^*A.\,\mathsf{Id}_A$ such that $p_{\mathsf{Id}_A} \cdot \mathsf{refl}_A = (1_A, 1_A) \colon \Gamma.A \to \Gamma.A.p_A^*A$;

3. for each $\Gamma.A.p_A^*A.\,\mathsf{Id}_A\,.C$ and $d \colon \Gamma.A \to \Gamma.A.p_A^*A.\,\mathsf{Id}_A\,.C$ with $p_C \cdot d = \mathsf{refl}_A$, a section $\mathsf{J}_{C,d} \colon \Gamma.A.p_A^*A.\,\mathsf{Id}_A \to \Gamma.A.p_A^*A.\,\mathsf{Id}_A\,.C$ such that $\mathsf{J}_{C,d} \cdot \mathsf{refl}_A = d$;

4. where all of the above is compatible with context substitution, that is given a map $f \colon \Delta \to \Gamma$ we have

$$\begin{aligned} f^*(\Gamma.A.p_A^*A.\,\mathsf{Id}_A) &= \Delta.f^*A.p_{f^*A}^*(f^*A).\,\mathsf{Id}_{f^*A} \\ f^*\,\mathsf{refl}_A &= \mathsf{refl}_{f^*A} \\ f^*\,\mathsf{J}_{C,d} &= \mathsf{J}_{f^*C,f^*d}. \end{aligned}$$

Definition 1.3.3. A Π-*structure* on a contextual category C consists of:

1. for each $\Gamma.A.B \in \mathrm{Ob}_{n+2}\,\mathsf{C}$, an object $\Gamma.\,\Pi(A,B) \in \mathrm{Ob}_{n+1}\,\mathsf{C}$;

2. for each $\Gamma.A.B \in \mathrm{Ob}_{n+2}\,\mathsf{C}$, a map $\mathsf{app}_{A,B} \colon \Gamma.\,\Pi(A,B).p_{\Pi(A,B)}^*A \to \Gamma.A.B$ over Γ, that is such that $p_B \cdot \mathsf{app}_{A,B} = q(\Pi(A,B), A)$;

3. for each $\Gamma.A.B \in \mathrm{Ob}_{n+2}\,\mathsf{C}$ and section $b \colon \Gamma.A \to \Gamma.A.B$, a section $\lambda_{A,B}(b) \colon \Gamma \to \Gamma.\,\Pi(A,B)$;

4. such that for any sections $k \colon \Gamma \to \Gamma.\,\Pi(A,B)$, $a \colon \Gamma \to \Gamma.A$ the map $\mathsf{app}_{A,B}(k,a)$ defined as the composition of $\mathsf{app}_{A,B}$ with (k,a) specified by the factorization through the pullback

$$\begin{tikzcd} \Gamma \arrow[dr, dotted, "{(k,a)}"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "k"'] & & \\ & \Gamma.\,\Pi(A,B).p_{\Pi(A,B)}^*A \arrow[r, "{q(p_{\Pi(A,B)},A)}"'] \arrow[d, "p_{p_{\Pi(A,B)}^*A}"] & \Gamma.A \arrow[d, "p_A"] \\ & \Gamma.\,\Pi(A,B) \arrow[r, "p_{\Pi(A,B)}"'] & \Gamma \end{tikzcd},$$

we have $p_B \cdot \mathsf{app}_{A,B}(k,a) = a$;

5. such that for any $\Gamma.A.B$, $a{:}\,\Gamma \to \Gamma.A$ and $b{:}\,\Gamma.A \to \Gamma.A.B$ we have
$$\mathsf{app}(\lambda_{A,B}(b), a) = b \cdot a;$$
6. all of the above is compatible with context substitution, that is for any $f{:}\,\Delta \to \Gamma$ we have
$$\begin{aligned} f^*(\Gamma, \Pi(A,B)) &= (\Delta, \Pi(f^*A, f^*B)),\\ f^*\lambda_{A,B}(b) &= \lambda_{f^*A,f^*B}(f^*b),\\ f^*(\mathsf{app}_{A,B}(k,a)) &= \mathsf{app}_{f^*A,f^*B}(f^*k, f^*a). \end{aligned}$$

We shall say that the Π-structure satisfies the Π_η*-rule* if the equation
$$q(p_{\Pi(A,B)}, \Pi(A,B)) \cdot \lambda(1_{p^*_{\Pi(A,B)}A}, \mathsf{app}_{A,B}) = 1_{\Gamma.\,\Pi(A,B)}$$
is satisfied, in which case the structure will be called a Π_η*-structure*.

A Π-ext-*structure* on a contextual category with a Π-structure C is an operation giving for each $\Gamma.A.B \in \mathrm{Ob}_{n+2}\,\mathsf{C}$ a map
$$\begin{aligned} \mathsf{ext}_{A,B}{:}\,\Gamma.\,\Pi(A,B).p^*_{\Pi(A,B)}\,\Pi(A,B).\,\mathsf{Htp}_{A,B} \to&\\ \Gamma.\,\Pi(A,B).p^*_{\Pi(A,B)}\,\Pi(A,B).\,\mathsf{Id}_{\Pi(A,B)}& \end{aligned}$$
over $\Gamma.\,\Pi(A,B).p^*_{\Pi(A,B)}\,\Pi(A,B)$, stably in Γ, where $\mathsf{Htp}_{A,B}$ is a shorthand for the extension
$$\begin{aligned} \Pi\Big(&(p_{\Pi(A,B)}\cdot p_{p^*_{\Pi(A,B)}\,\Pi(A,B)})^*A,\\ &(\mathsf{app}_{A,B}\cdot q(p_{\Pi(A,B)}, A), \mathsf{app}_{A,B}\cdot q(q(p_{\Pi(A,B)}, \Pi(A,B)), A))^*\,\mathsf{Id}_B\Big), \end{aligned}$$
representing the type of homotopies from between two terms of $\Pi(A,B)$. The map then models function extensionality.

Remark 1.3.4. The definition we provided matches [KL18, Def. 2.5], while the one in [KL12, App. B.1.1] is mildly different as it starts from $\mathsf{app}(f,a)$ instead of $\mathsf{app}_{A,B}$ and constructs the latter from the former. We prefer this because it will allow us to more easily extend the Π-structure to arbitrary context extensions and check the necessary properties in the next chapter.

Notation 1.3.5. Given $\Gamma.A, \Gamma.B \in \mathrm{Ob}_{n+1}\,\mathsf{C}$, we write $\Gamma.[A,B]$ for $\Gamma.\,\Pi(A, p^*_A B)$.

Example 1.3.6. Given a dependent type theory $\mathbf{T}$, its syntactic category $\mathsf{Syn}(\mathbf{T})$ has all of the logical structures corresponding to its logical rules. We now proceed to show what some of the above maps correspond to in this category.

Remark 1.3.7. In $\mathsf{Syn}(\mathbf{T})$, the map $\mathsf{app}_{A,B}$ corresponds to
$$(f,a) \mapsto (a, \mathsf{app}(f,a)).$$

Also, the association $b \mapsto \lambda_{A,B}(b)$ for some section $\Gamma.A \to \Gamma.A.B$ corresponds to *lambda abstraction*, hence $\lambda_{A,B}(b)$ specifies the term $\lambda(a : A).b(a) : \Pi(A,B)$. The map on the left of the definition of the Π_η-rule is the *η-expansion map*, that is it corresponds syntactically to the map

$$f \mapsto \lambda(a : A).\,\mathsf{app}(f, a)$$

over Γ, thus the property means that we have a judgement

$$\Gamma \vdash f \equiv \lambda(a : A).\,\mathsf{app}(f, a) : \Pi(A,B)$$

which corresponds to η-conversion.

The map $(1_{p^*_{\Pi(A,B)}A}, \mathsf{app}_{A,B})$ is specified by the following factorization through the pullback.

$$\begin{tikzcd}
\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[drr, bend left, "\mathsf{app}_{A,B}"] \arrow[dr, dotted, "(1_{p^*_{\Pi(A,B)}A}, \mathsf{app}_{A,B})"] \arrow[ddr, equal] & & \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A.p^*_{\Pi(A,B)}B \arrow[r, "q(p_{\Pi(A,B)}, A.B)"'] \arrow[d, "p_{p^*_{\Pi(A,B)}A}"] & \Gamma.A.B \arrow[d, "p_A"] \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[r, "q(p_{\Pi(A,B)}, A)"'] & \Gamma.A
\end{tikzcd}.$$

Construction 1.3.8. Let C be a contextual category with a Π-structure. We define for any object $\Gamma.A$ a map $\mathsf{id}_A : \Gamma \to \Gamma.[A, A]$ as $\lambda_{A,p^*_A A}(1_A, 1_A)$.

Construction 1.3.9. Let C be a contextual category with a Π-structure, $f : \Gamma.A \to \Gamma.B$ a map over Γ. We want to provide a section $\tilde{f} : \Gamma \to \Gamma.\Pi(A,B)$ corresponding to f. We do so by looking at the commutative diagram

$$\begin{tikzcd}
\Gamma.A \arrow[drr, bend left, "f"] \arrow[dr, dotted, "(1_A, f)"] \arrow[ddr, equal] & & \\
& \Gamma.A.p^*_A B \arrow[r, "q(p_A, B)"'] \arrow[d, "p_{p^*_A B}"] & \Gamma.B \arrow[d, "p_B"] \\
& \Gamma.A \arrow[r, "p_A"'] & \Gamma
\end{tikzcd},$$

and then taking $\lambda_{A,p^*_A B}(1_A, f)$.

In $\mathsf{Syn}(\mathbf{T})$, using the fact that

$$\begin{aligned}
q(p_A, B) \cdot \mathsf{app}_{A,B} \cdot q(p_{p^*_A B}, \lambda_{A,p^*_A B}(1_A, f)) \cdot a &= q(p_A, B) \cdot \mathsf{app}_{A,B}(\lambda_{A,p^*_A B}(1_A, f), a) \\
&= q(p_A, B) \cdot (1_A, f) \cdot a \\
&= f \cdot a
\end{aligned}$$

we see that f acts as

$$
\begin{aligned}
&a\\
{\scriptstyle q(p_{p^*_A B},\lambda_{A,p^*_A B}(1_A,f))} &\mapsto (a,\tilde{f})\\
{\scriptstyle \mathsf{app}_{A,B}} &\mapsto (a,\mathsf{app}(\tilde{f},a))\\
{\scriptstyle q(p_A,B)} &\mapsto \mathsf{app}(\tilde{f},a).
\end{aligned}
$$

This justifies writing f for $\tilde{f}$ and $(1_A, f)$ then acts as $a \mapsto (a, \mathsf{app}(f,a))$, meaning that $\lambda_{A,p^*_A B}(1_A, f)$ syntactically corresponds to $\lambda(a : A).\, \mathsf{app}(f,a) : \Pi(A,B)$.

Construction 1.3.10. Let C be a contextual category with a Π-structure, $f\colon \Gamma.A.B \to \Gamma.A.B'$ a morphism over $\Gamma.A$, meaning that in $\mathsf{Syn}(\mathbf{T})$ it is a map $b \mapsto \mathsf{app}(f,b)$ for the term $f : \Pi(B,B')$ it induces. We want to construct a morphism

$$\Gamma.\,\Pi(A,f)\colon \Gamma.\,\Pi(A,B) \to \Gamma.\,\Pi(A,B')$$

modeling the postcomposition by f, that is $g \mapsto \lambda(a : A).\, \mathsf{app}(f, \mathsf{app}(g,a))$.

We do so by looking at the commutative diagram

$$
\begin{tikzcd}
\Gamma.\,\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[rr, "\mathsf{app}_{A,B}"] \arrow[dr, dotted, "(1_{p^*_{\Pi(A,B)}A},\, f\cdot \mathsf{app}_{A,B})"] \arrow[ddr, equal] & & \Gamma.A.B \arrow[d, "f"'] \arrow[dd, bend left, "p_B"] \\
& \Gamma.\,\Pi(A,B).p^*_{\Pi(A,B)}A.p^*_{\Pi(A,B)}B' \arrow[r, "q(p_{\Pi(A,B)},A.B')"'] \arrow[d, "p_{p^*_{\Pi(A,B)}B'}"] & \Gamma.A.B' \arrow[d, "p_{B'}"'] \\
& \Gamma.\,\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[r, "q(p_{\Pi(A,B)},A)"'] & \Gamma.A
\end{tikzcd}
$$

and then applying $\lambda_{p^*_{\Pi(A,B)}A,\, p^*_{\Pi(A,B)}B'}$ to the map given by the universal property of the pullback, which provides us with the section

$$\lambda_{p^*_{\Pi(A,B)}A,\, p^*_{\Pi(A,B)}B'}(1_{p^*_{\Pi(A,B)}A}, f \cdot \mathsf{app}_{A,B})\colon \Gamma.\,\Pi(A,B) \to \Gamma.\,\Pi(A,B).\,\Pi(A,B').$$

All we have to do now is postcompose with $q(p_{\Pi(A,B)}, \Pi(A,B'))$. Verifying that in $\mathsf{Syn}(\mathbf{T})$ the action on sections is the one we want is straightforward as we have done so far.

Remark 1.3.11. The previous construction is such that $\Gamma.\,\Pi(A, 1_B) = 1_{\Pi(A,B)}$ for every context $\Gamma.A.B$ in C if and only if the Π-structure satisfies the Π_η-rule.

Notation 1.3.12. As noted before, contextual categories with some extra logical structures still define small essentially algebraic theory with sorts indexed by $\mathbb{N} + \mathbb{N} \times \mathbb{N}$. Their categories of models are then specified by writing as subscript of CxlCat the corresponding structures, that is we write $\mathsf{CxlCat}_{\mathsf{Id}}$, $\mathsf{CxlCat}_{\Sigma,\mathsf{Id}}$ and so on.

2 Extended Logical Structures

The aim of this section is to construct from a given contextual category C with extra structure another contextual category C^{cxt} with the same structure, but where objects are iterated context extensions, compatibly with a canonical contextual functor $\mathsf{C} \to \mathsf{C}^{cxt}$ defining an equivalence on the underlying categories, thereby generalizing our structures from simple context extensions to arbitrary ones. Indeed, we may then take a context extension, look at it in C^{cxt}, apply the construction and then carry it back through the equivalence. This extension shall be heavily exploited in the final part of the thesis. This technique has been mentioned multiple times in the literature to justify various results at a certain level of generality, however nobody actually carried out the necessary constructions.

2.1 Constructions

On the type theoretical side, extensions of logical rules for identity types to contexts were first explored by Streicher [Str93], then by Gambino and Garner [GG08] under the name of *identity contexts*. We shall make use of their notation, meaning that a "term" of a context $x : \Delta$ is a tuple of choices $a_1 : A_1, \ldots, a_n : A_n$ and therefore we can split it as $a_1 : A_1, x' : \Delta'$.

Lumsdaine later claimed that it is possible to model this extension and similar ones for Σ- and Π-structures in C^{cxt} [Lum10, p. 26]. At the time Kapulkin proved the theorem this thesis focuses on, nothing further was available in the literature and only later him and Lumsdaine gave more details on these matters [KL18]. In this section we aim to partially fix that for Π-structures specifically, which will be the main contribution of this work. It should be noted that we may derive the corresponding logical rules for what would then be called Π-contexts, however we will not do so and focus on the semantics, maintaining that the translation should be straightforward and a simple duplication of work.

We begin by constructing the category of iterated contexts, where we will extend the structures.

Construction 2.1.1. [Lum10, p. 21] Given a contextual category C, we construct a contextual category C^{cxt} in the following way:

1. the set $\mathrm{Ob}_n \, \mathsf{C}^{cxt}$ is given by n-iterated non-trivial context extensions

$$\Gamma_1.\Gamma_2.\ldots.\Gamma_n$$

in C;

2. morphisms $\Gamma_1.\Gamma_2.\dots.\Gamma_n \to \Delta_1.\Delta_2.\dots.\Delta_m$ are morphisms between them seen as objects of C;

3. $*$ is the only element of $\mathrm{Ob}_0\,\mathsf{C}^{cxt}$;

4. $\mathsf{ft}(\Gamma_1.\Gamma_2.\dots.\Gamma_n.\Gamma_{n+1}) = \Gamma_1.\Gamma_2.\dots.\Gamma_n$;

5. the map $p_{\Gamma_1.\Gamma_2.\dots.\Gamma_n.\Gamma_{n+1}} \colon \Gamma_1.\Gamma_2.\dots.\Gamma_{n+1} \to \Gamma_1.\Gamma_2.\dots.\Gamma_n$ is the dependent projection exhibiting $\Gamma_1.\Gamma_2.\dots.\Gamma_{n+1}$ as a context extension of $\Gamma_1.\Gamma_2.\dots.\Gamma_n$;

6. the chosen pullbacks are given by iterating the pullbacks along the basic dependent projections, as in the original contextual category.

As we can see, any object of C^{cxt} either is the empty context or is isomorphic to one in $\mathrm{Ob}_1\,\mathsf{C}^{cxt}$, that is the one which we get by looking at the associated object in C and then taking the dependent projection from it to the terminal object, which exhibits it as a 1-iterated context extension. The isomorphism is then given by the map in C^{cxt} corresponding to the identity of the object in C. We now specify a monad $\mathsf{C} \mapsto \mathsf{C}^{cxt}$ on CxlCat.

The unit $\mathsf{C} \to \mathsf{C}^{cxt}$ sends every n-object in C to the corresponding n-iterated (simple) context extension and every morphism to the one it represents.

Before we construct the multiplication, let's study this contextual functor. Every n-iterated context in C^{cxt} is isomorphic to one in the image of the unit, namely the one which we get by reducing it to an iterated simple context extension, meaning that the functor is essentially surjective. Also, it is fully faithful by construction and therefore it defines an equivalence on the underlying categories.

Let's construct the multiplication. An n-object of $(\mathsf{C}^{cxt})^{cxt}$ is an n-iterated context extension where each extension is itself an iterated context extension in C, that is

$$(\Gamma_1.\dots.\Gamma_{i_1}).(\Gamma_{i_1+1}.\dots.\Gamma_{i_2}).\dots.(\Gamma_{i_{n-1}+1}.\dots.\Gamma_{i_n}).$$

Since composing dependent projections gives dependent projections, seeing $\Gamma_{i_{j-1}+1}.\dots.\Gamma_{i_j}$ as a single context extension Δ_j in C, we can naturally map the object of $(\mathsf{C}^{cxt})^{cxt}$ to $\Delta_1.\dots.\Delta_n$ in C^{cxt} and, again, every morphism in $(\mathsf{C}^{cxt})^{cxt}$ corresponds to a unique one in C^{cxt} once we specify domain and codomain. By construction, this functor is again contextual and an equivalence of categories.

The monad axioms follow from the fact that, essentially, both unit and multiplication are "identities" on objects and morphisms, which concludes our construction.

We are now ready, given a contextual category C, to extend the Π-structure to C^{cxt}. We shall do so in two parts, first in full generality and then under extra assumptions.

Given $\Gamma.\Delta.\Theta$ in C^{cxt}, where $l(\Gamma.\Delta.\Theta) = l(\Gamma.\Delta) + n = l(\Gamma) + m + n$ in C, we build $\Gamma.\Pi(\Delta,\Theta)$ by induction on the length of the context extensions involved, taking the one from C in case we are working with objects corresponding to simple extensions.

Construction 2.1.2 (Part 1)**.** If $n = 0$, then

$$\begin{aligned} \Gamma.\Pi(\Delta,\Theta) &= \Gamma, \\ \mathsf{app}_{\Delta,\Theta} &= 1_{\Gamma.\Delta}, \\ \lambda_{\Delta,\Theta}(b) &= 1_{\Gamma}. \end{aligned}$$

Notice that the only possible b in the latter case is given by $1_{\Gamma.\Theta}$. This is not really necessary to specify the $\mathsf{\Pi}$-structure, however we shall need it later in (WHERE?).

We now work with the case where $m > 0$, $n = 1$, thus we shall write $\Gamma.\Delta.\Theta = \Gamma.\Delta.B$. In the base case, $m = 1$, we have $\Gamma.\Delta = \Gamma.A$ and therefore we simply set our structure to be the one in C.

If $n-1>0$, we have $\Gamma.\Delta = \Gamma.\Delta'.A$ and then set

$$\Gamma.\mathsf{\Pi}(\Delta, B) = \Gamma.\mathsf{\Pi}(\Delta'.A, B) = \Gamma.\Pi(\Delta', \Pi(A, B))$$

$$\mathsf{app}_{\Delta,B}\colon \Gamma.\mathsf{\Pi}(\Delta', \Pi(A,B)).\Delta'.A \xrightarrow{q(\mathsf{app}_{\Delta',\mathsf{\Pi}(A,B)}, p^*_{\Pi(A,B)}A)} \Gamma.\Delta'.\Pi(A,B).A \xrightarrow{\mathsf{app}_{A,B}} \Gamma.\Delta'.A.B$$

$$\lambda_{\Delta,B}(b)\colon \Gamma \xrightarrow{\lambda_{\Delta',\mathsf{\Pi}(A,B)}(\lambda_{A,B}(b))} \Gamma.\mathsf{\Pi}(\Delta', \Pi(A,B)).$$

The idea here is to replicate the adjunction $\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, \mathbf{Set}(B, C))$. The map $\mathsf{app}_{\Delta,B}$ is then naturally interpreted as a sequence of partial evaluations and the phenomenon is commonly known as *currying-uncurrying*.

We have fully specified the construction for $l(\Gamma.\Delta.\Theta) = l(\Gamma.\Delta) + 1$ in C, hence we shall move on to the case where Δ has arbitrary length and construct the necessary structure by inducting on the length of Θ.

We will do so for syntactic categories and in contextual categories where a section $c\colon \Gamma \to \Gamma.A.B$ can be split as two sections $a\colon \Gamma \to \Gamma.A$, $b\colon \Gamma.A \to \Gamma.A.B$. We see that $p_B \cdot c = p_B \cdot b \cdot a = a$. Syntactically, this means that if we pick two terms $x : A$, $y : B(x)$ over Γ at once, then we can also specify for each choice of a term $x' : A$ over Γ a term $y' : B(x')$ such that $y \equiv y'$ when $x \equiv x'$. We shall later point out the difficulties encountered without our assumption.

Construction 2.1.3 (Part 2)**.** Suppose that $l(\Gamma.\Delta.\Theta) = l(\Gamma.\Delta) + n$, $n > 1$, and we have already provided the relevant constructions up to $n-1$. We again decompose the context as $\Gamma.\Delta.\Theta'.B$.

If we are working with a contextual category C where sections split, we make use of our assumption on $b\colon \Gamma.\Delta \to \Gamma.\Delta.\Theta'.B$ to get $b'' \cdot b'$.

$$\Gamma.\mathsf{\Pi}(\Delta, \Theta) = \Gamma.\mathsf{\Pi}(\Delta, \Theta'.B) = \Gamma.\Pi(\Delta, \mathsf{app}^*_{\Delta,\Theta'} B)$$

$$\mathsf{app}_{\Delta,\Theta}\colon \Gamma.\mathsf{\Pi}(\Delta, \Theta').\Pi(\Delta, \mathsf{app}^*_{\Delta,\Theta'} B).\Delta \xrightarrow{\mathsf{app}_{\Delta, \mathsf{app}^*_{\Delta,\Theta'} B}} \Gamma.\mathsf{\Pi}(\Delta, \Theta').\Delta.\mathsf{app}^*_{\Delta,\Theta'} B \xrightarrow{q(\mathsf{app}_{\Delta,\Theta'}, B)} \Gamma.\Delta.\Theta'.B$$

$$\lambda_{\Delta,\Theta}(b)\colon \Gamma \xrightarrow{\lambda_{\Delta,\Theta'}(b')} \Gamma.\mathsf{\Pi}(\Delta, \Theta') \xrightarrow{\lambda_{p^*_{\mathsf{\Pi}(\Delta,\Theta')}\Delta, \mathsf{app}^*_{\Delta,\Theta'} B}(\mathsf{app}^*_{\Delta,\Theta'} b'')} \Gamma.\mathsf{\Pi}(\Delta, \Theta').\Pi(\Delta, \mathsf{app}^*_{\Delta,\Theta'} B)$$

If we are instead in $\mathsf{Syn}(\mathbf{T})$, the only thing that changes is how we define $\lambda_{\Delta,\Theta}(b)$: we can indeed simply take the section specified by $\lambda(y : \Delta).b(y) : \mathsf{\Pi}(\Delta, \Theta)$, or equivalently

$$\lambda(y : \Delta).b'(y) : \mathsf{\Pi}(\Delta, \Theta'), \lambda(y : \Delta).b''(y, b'(y)) : \mathsf{\Pi}(y : \Delta, B(\mathsf{app}(\lambda(y : \Delta).b'(y), y))).$$

Given $a{:}\Gamma \to \Gamma.\Delta$, $f{:}\Gamma.\Delta \to \Gamma.\Delta.\Theta$, we construct $\mathsf{app}_{\Delta,\Theta}(f,a)$ as in the definition of Π-structures.

This fully specifies the data needed for a Π-structure on C^{cxt}, however we still have to check that it is indeed one, which we will do in the next section. First, however, we sketch the extension for a Σ- and a Id-structure on a contextual category.

Construction 2.1.4. Let C be a contextual category with an Id-structure. Given a dependent context $p_\Delta{:}\Gamma.\Delta \to \Gamma$ with $l(\Gamma.\Delta) = l(\Gamma) + n$ in C we proceed by induction on n.

If $n = 0$, then

$$\begin{aligned} \Gamma.\,\mathsf{Id}_\Delta &= \Gamma, \\ r_\Delta &= 1_\Gamma, \\ J_{\Theta,C,d} &= d. \end{aligned}$$

If $n = 1$,

$$\begin{aligned} \Gamma.\,\mathsf{Id}_A &= \Gamma.\,\mathsf{Id}_A, \\ r_A &= r_A, \\ J_{\Theta,C,d} &= J(\Theta, C, d), \end{aligned}$$

where the objects on the right are the ones given by the Id-structure on C.

If $n > 1$, assuming to have extended the Id-structure to shorter context extensions in C, we write instead

$$\begin{aligned} \Gamma.\,\mathsf{Id}_\Delta = \Gamma.\,\mathsf{Id}_{\Delta'.A} &= \Gamma.\,\mathsf{Id}_{\Delta'}.\,\mathsf{Id}_A \\ r_\Delta &= q(,) \cdot r_A \end{aligned}$$

2.2 Properties

In this section we verify that the desired properties hold.

Proposition 2.2.1. Given a contextual category with a Π-structure C, the above data defines a Π-structure on C^{cxt} which is compatible with the natural contextual functor $\mathsf{C} \to \mathsf{C}^{cxt}$.

Proof. We have to show that it is a Π-structure, which we will do inductively by verifying that at every step our proposed construction maintains the desired properties. The compatibility with the contextual functor will then follow directly from the way we defined the base case.

Let's consider an object $\Gamma.\Delta.\Theta$ in C^{cxt} such that $l(\Gamma.\Delta.\Theta) = l(\Gamma.\Delta) + n = l(\Gamma) + m + n$ in C. The only interesting case is the one where $m, n > 0$ and at least one of them is > 1: indeed, the desired properties in the other cases are either trivial or follow directly from the fact that they hold in C.

We start as before by working on $m > 1$, $n = 1$, so we write $\Gamma.\Delta.\Theta = \Gamma.\Delta.B = \Gamma.\Delta'.A.B$ and assume that the desired properties hold for shorter context extensions. We can then write

$$\begin{aligned}
p_B \cdot \mathsf{app}_{\Delta,B} &= p_B \cdot \mathsf{app}_{A,B} \cdot q(\mathsf{app}_{\Delta',\Pi(A,B)}, p^*_{\Pi(A,B)}A) \\
&= q(p_{\Pi(A,B)}, A) \cdot q(\mathsf{app}_{\Delta',\Pi(A,B)}, p^*_{\Pi(A,B)}A) \\
&= q(p_{\Pi(A,B)} \cdot \mathsf{app}_{\Delta',\Pi(A,B)}, A) \\
&= q(q(p_{\Pi(\Delta',\Pi(A,B))}, \Delta'), A) \\
&= q(p_{\Pi(\Delta',\Pi(A,B))}, \Delta'.A) \\
&= q(p_{\Pi(\Delta,B)}, \Delta) \\
p_B \cdot \mathsf{app}_{\Delta,B}(f,a) &= p_B \cdot \mathsf{app}_{\Pi(\Delta,B)} \cdot (f,a) \\
&= q(p_{\Pi(\Delta,B)}, \Delta) \cdot (f,a) \\
&= a \\
p_{\Pi(\Delta,B)} \cdot \lambda_{\Delta,B}(b) &= p_{\Pi(\Delta',\Pi(A,B))} \cdot \lambda_{\Delta',\Pi(A,B)}(\lambda_{A,B}(b)) \\
&= 1_\Gamma.
\end{aligned}$$

PLEASE VERIFY, LIKELY WRONG To justify $\mathsf{app}_{\Delta,B}(\lambda_{\Delta,B}(b), y) = b \cdot y$ we shall use type-theoretic reasoning. The section $b \colon \Gamma.\Delta \to \Gamma.\Delta.B$ is such that

$$y \equiv (y', a) \mapsto (y', a, b(y', a)) \equiv (y, b(y)),$$

thus applying $\lambda_{A,B}$ and then $\lambda_{\Delta',\Pi(A,B)}$ we get $\lambda(y' : \Delta').(\lambda(a : A).b(y', a)) \equiv \lambda(y : \Delta).b(y)$. Also, $\mathsf{app}_{\Delta,B}$ acts type-theoretically as

$$\begin{aligned}
&(f, y) \\
\overset{\text{split } y}{}&\equiv (f, y', a) \\
\overset{q(\mathsf{app}_{\Delta',\Pi(A,B)}, p^*_{\Pi(A,B)}A)}{}&\mapsto (y', \mathsf{app}(f, y'), a) \\
\overset{\mathsf{app}_{A,B}}{}&\mapsto (y', a, \mathsf{app}(\mathsf{app}(f, y'), a)) \\
&\equiv (y, \mathsf{app}(f, y)),
\end{aligned}$$

thus in this case we get

$$\begin{aligned}
&(\lambda(y : \Delta).b(y), y) \\
&\qquad \equiv (\lambda(y' : \Delta').(\lambda(a : A).b(y', a)), y', a) \\
&\qquad \mapsto (y', \lambda(a : A).b(y', a), a) \\
&\qquad \mapsto (y', a, b(y', a)) \\
&\qquad \equiv (y, b(y)),
\end{aligned}$$

where the inductive hypothesis has been used in the second step. Notice that we used the hypothesis about splitting sections, which was unnecessary in the corresponding part of the construction.

We now check inductively on the length of Θ.

$$\begin{aligned}
p_\Theta \cdot \mathsf{app}_{\Delta,\Theta} &= p_{\Theta'} \cdot p_B \cdot q(\mathsf{app}_{\Delta,\Theta'}, B) \cdot \mathsf{app}_{\Delta,\mathsf{app}^*_{\Delta,\Theta'}B} \\
&= p_{\Theta'} \cdot \mathsf{app}_{\Delta,\Theta'} \cdot p_{\mathsf{app}^*_{\Delta,\Theta'}B} \cdot \mathsf{app}_{\Delta,\mathsf{app}^*_{\Delta,\Theta'}B} \\
&= q(p_{\Pi(\Delta,\Theta)}, \Delta) \cdot q(p_{\Pi(\Delta,\mathsf{app}^*_{\Delta,\Theta'}B)}, \Delta) \\
&= q(p_{\Pi(\Delta,\Theta')} \cdot p_{\Pi(\Delta,\mathsf{app}^*_{\Delta,\Theta'}B)}, \Delta) \\
&= q(p_{\Pi(\Delta,\Theta').\Pi(\Delta,\mathsf{app}^*_{\Delta,\Theta'}B)}, \Delta) \\
&= q(p_{\Pi(\Delta,\Theta)}, \Delta) \\
p_\Theta \cdot \mathsf{app}_{\Delta,\Theta}(f,a) &= p_\Theta \cdot \mathsf{app}_{\Delta,\Theta} \cdot (a,f) \\
&= q(p_{\Pi(\Delta,\Theta)}, \Delta) \cdot (a,f) \\
&= a \\
p_{\Pi(\Delta,\Theta)} \cdot \lambda_{\Delta,\Theta}(b) &= p_{\Pi(\Delta,\Theta')} \cdot p_{\Pi(p^*_{\Pi(\Delta,\Theta')}\Delta,\mathsf{app}^*_{\Delta,\Theta'}B')} \cdot \lambda_{p^*_{\Pi(\Delta,\Theta')}\Delta,\mathsf{app}^*_{\Delta,\Theta'}B}(\mathsf{app}^*_{\Delta,\Theta'}b'') \cdot \lambda_{\Delta,\Theta'}(b') \\
&= p_{\Pi(\Delta,\Theta')} \cdot \lambda_{\Delta,\Theta'}(b') \\
&= 1_\Gamma.
\end{aligned}$$

Again, to justify $\mathsf{app}_{\Delta,\Theta}(\lambda(b), y) = b \cdot y$ we use internal reasoning. We know that, given a factorization $b = b'' \cdot b'$, b acts as

$$y \mapsto (y, b(y)) \equiv (y, b'(y), b''(y, b'(y))),$$

thus our construction provides us with

$$\lambda(y:\Delta).b(y) \equiv (\lambda(y:\Delta).b'(y), \lambda(y:\Delta).b''(y, b'(y))).$$

Also, $\mathsf{app}_{\Delta,\Theta}$ acts internally as

$$\begin{aligned}
&(f,y) \\
\text{split } f \quad &\equiv (f', f'', y) \\
\mathsf{app}_{\Delta,\mathsf{app}^*_{\Delta,\Theta'}B} \quad &\mapsto (f', y, \mathsf{app}(f'', y)) \\
q(\mathsf{app}_{\Delta,\Theta'},B) \quad &\mapsto (y, \mathsf{app}(f', y), \mathsf{app}(f'', y)) \\
&\equiv (y, \mathsf{app}(f, y)),
\end{aligned}$$

which here translates to

$$\begin{aligned}
&(\lambda(y:\Delta).b(y), y) \\
&\qquad \equiv (\lambda(y:\Delta).b'(y), \lambda(y:\Delta).b''(y, b'(y)), y) \\
&\qquad \mapsto (\lambda(y:\Delta).b'(y), y, b''(y, b'(y))) \\
&\qquad \mapsto (y, b'(y), b''(y, b'(y))) \\
&\qquad \equiv (y, b(y)),
\end{aligned}$$

where the inductive hypothesis has been used in the second step.

To conclude the proof one would still need to verify that the construction is compatible with context substitution. □

We shall also need the following lemma.

Lemma 2.2.2 ([Kap17])**.** Given an iterated context extension $\Gamma.\Delta.\Theta.\Psi$ in a contextual category with Π-types C, the contexts

$$\Gamma.\,\Pi(\Delta,\Theta.\Psi),\quad \Gamma.\,\Pi(\Delta,\Theta).\,\Pi(p^*_{\Pi(\Delta,\Theta)}\Delta,\mathsf{app}^*_{\Delta,\Theta}\,\Psi)$$

are equal in C. Also, $\Gamma.\,\Pi(\Delta,p_\Psi)=p_{\Pi(\Delta,\mathsf{app}^*_{\Delta,\Theta}\,\Psi)}$.

Proof. For the first claim, it is enough to notice that the two contexts reduce to the same one in C after applying the inductive construction we defined on C^{cxt} to reduce Ψ.

For the second claim instead we consider the chain of equalities

$$\begin{aligned}\Gamma.\,\Pi(\Delta,p_\Psi) &= q(p_{\Pi(\Delta,\Theta.\Psi)},\Pi(\Delta,\Theta))\cdot\lambda_{\Delta,\Theta}(1_{p^*_{\Pi(\Delta,\Theta)}\Delta},p_\Psi\cdot\mathsf{app}_{\Delta,\Theta.\Psi})\\ &= p_{\Pi(\Delta,\mathsf{app}^*_{\Delta,\Theta}\,\Psi)}\cdot p_{\Pi(\Delta,\Theta)}\cdot\lambda_{\Delta,\Theta}(1_{p^*_{\Pi(\Delta,\Theta)}\Delta},p_\Psi\cdot\mathsf{app}_{\Delta,\Theta.\Psi})\\ &= p_{\Pi(\Delta,\mathsf{app}^*_{\Delta,\Theta}\,\Psi)}.\end{aligned}$$

(PLEASE CHECK) □

Proposition 2.2.3. [Gar09, Prop. 3.3.1] Given a contextual category with an Id-structure C, C^{cxt} also carries a natural Id-structure compatible with the contextual functor $\mathsf{C}\to\mathsf{C}^{cxt}$ as described in REFS 0808.

Lemma 2.2.4. [**1808**] Given a contextual category with Id, Π_η structures and function extensionality C, the latter can also be extended to C^{cxt} compatibly with the extended Id and Π structures.

Remark 2.2.5. Lumsdaine noted that the lift of Id and Π structures is not compatible with the monad we provided earlier because they are not compatible with the multiplication. On the other hand, with a strategy along the line of the one we presented for Π-structures we can also lift Σ-structures compatibly with the monad, meaning that $(-)^{cxt}$ restricts to one on CxlCat_Σ.

3 Localizations of ∞-Categories

To prove that by localizing a syntactic category of a dependent type theory with some logical rules we get a locally cartesian closed ∞-category we need a theory of localizations of ∞-categories, which in our case will be quasi-categories. We shall provide then such a theory as developed by Cisinski [Cis19] with the aim of proving Theorem 3.2.31, which will do the heavy lifting in showing the desired result. Those familiar with the theory may skip the entire chapter while keeping in mind the aforementioned theorem, while those who do not know it may read it for a quick tour. We recommend however to read the original source, from which come all of the materials here presented.

Our exposition is divided in two parts: first we define localizations and show some results derived directly from it, while later we present some results which allow us to give a better condition of the localization when the starting ∞-category has a fibrational structure. The latter generalizes previous work by Szumilo and Kapulkin [Szu14; KS15; Kap17], who proved similar results for the simplicial localizations of *fibration categories* by studying the associated ∞-*categories of frames*, which provide concrete models, thereby avoiding fibrant replacements.

3.1 Universal Property

Localizations are defined in any context by a universal property of a certain form. Here we present ours and see what we can derive from it without extra assumptions.

Definition 3.1.1. Let C be a simplicial set and $W \subset C$ a simplicial subset. Given an ∞-category $\mathcal{D}$, we define $\underline{\mathrm{Hom}}_W(C, \mathcal{D})$ to be the full simplicial subset of $\underline{\mathrm{Hom}}(C, \mathcal{D})$ whose objects are the morphisms $f: C \to \mathcal{D}$ sending the 1-simplices in W to isomorphisms.

Remark 3.1.2. The above definition induces a canonical pullback square

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_W(C, \mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(C, \mathcal{D}) \arrow[d] \\
\underline{\mathrm{Hom}}_W(W, \mathcal{D}) \arrow[r] & \underline{\mathrm{Hom}}(W, \mathcal{D})
\end{tikzcd}$$

given by the inclusion $W \to C$.

Definition 3.1.3. Given a simplicial set C and a simplicial subset W, a *localization of C by W* is a morphism $\gamma: C \to L(C)$ in **sSet** such that:

1. $L(C)$ is an ∞-category;

2. γ sends the 1-simplices of W to isomorphisms in $L(C)$;

3. for any ∞-category $\mathcal{D}$ there is an equivalence of ∞-categories

$$\underline{\mathrm{Hom}}(L(C),\mathcal{D}) \to \underline{\mathrm{Hom}}_W(C,\mathcal{D})$$

given by precomposing with γ.

Proposition 3.1.4. The localization of C by W always exists and is essentially unique.

Proof. We begin by proving that a localization exists in the case where $W = C$.
In this context, $\underline{\mathrm{Hom}}_W(C,\mathcal{D}) \cong \underline{\mathrm{Hom}}(C,\mathcal{D}^{\cong})$ canonically, where $\mathcal{D}^{\cong}$ is the maximal subgroupoid of $\mathcal{D}$. Factoring $C \to \Delta^0$ in the Kan model structure, we find an anodyne map $C \to C'$. We then choose this map for γ and set $L(C) = C'$.
Remember that for any anodyne map $A \to B$ we get a trivial fibration $\underline{\mathrm{Hom}}(B,\mathcal{D}^{\cong}) \to \underline{\mathrm{Hom}}(A,\mathcal{D}^{\cong})$. Looking then at the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(L(C),\mathcal{D}^{\cong}) \arrow[r, two heads, "\gamma^*", "\sim"'] \arrow[d, "\cong"'] & \mathrm{Hom}_W(C,\mathcal{D}^{\cong}) \arrow[d, "\cong"] \\
\underline{\mathrm{Hom}}(L(C),\mathcal{D}) \arrow[r, "\gamma^*"'] & \underline{\mathrm{Hom}}_W(C,\mathcal{D})
\end{tikzcd},$$

we see that the lower γ^* is a trivial fibration, thus we have constructed a valid localization of C by W.

We now move on to the general case. First of all, notice that as a particular case of the previous one we get that localizing Δ^1 at its non-trivial morphism our construction provides $L(\Delta^1) = J \sim \Delta^0$, while γ is the inclusion $\Delta^1 \to J$. Taking then $W \subset C$, we consider the commutative diagram

$$\begin{tikzcd}
\coprod_{f\in W_1}\Delta^1 \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr, bend left, "\gamma"] & \\
\coprod_{f\in W_1} J \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C' \arrow[dr, "\sim"'] & \\
& & L(C)
\end{tikzcd},$$

where $C' \xrightarrow{\sim} L(C)$ is an inner anodyne map obtained by taking the fibrant replacement of C' in the Joyal model structure. This can be done functorially via the small object argument.
For any ∞-category $\mathcal{D}$, we get a trivial fibration $\underline{\mathrm{Hom}}(L(C),\mathcal{D}) \overset{\sim}{\twoheadrightarrow} \underline{\mathrm{Hom}}(C',\mathcal{D})$ and a pullback square

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(C',\mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(C,\mathcal{D}) \arrow[d] \\
\prod_{f\in W_1}\underline{\mathrm{Hom}}(J,\mathcal{D}) \arrow[r] & \prod_{f\in W_1}\underline{\mathrm{Hom}}(\Delta^1,\mathcal{D})
\end{tikzcd},$$

which together with the pullback

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_W(C,\mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(C,\mathcal{D}) \arrow[d] \\
\Pi_{f\in W_1}\underline{\mathrm{Hom}}(\Delta^1,\mathcal{D}^{\cong}) \arrow[r] & \Pi_{f\in W_1}\underline{\mathrm{Hom}}(\Delta^1,\mathcal{D})
\end{tikzcd}$$

implies by pasting that

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(C',\mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_W(C,\mathcal{D}) \arrow[d] \\
\Pi_{f\in W_1}\underline{\mathrm{Hom}}(J,\mathcal{D}) \arrow[r, two heads, "\sim"] & \Pi_{f\in W_1}\underline{\mathrm{Hom}}(\Delta^1,\mathcal{D}^{\cong})
\end{tikzcd}$$

is also a pullback and therefore the upper arrow is a trivial fibration. Composing it with the other one we get $\gamma^*\colon \underline{\mathrm{Hom}}(L(C),\mathcal{D}) \to \underline{\mathrm{Hom}}_W(C,\mathcal{D})$, which is then a trivial fibration and therefore an equivalence of ∞-categories.

We now move on to proving that the localization is essentially unique. For this, we notice that γ^* establishes then an isomorphism between $\pi_0(k(\underline{\mathrm{Hom}}_W(C,-)))$ and $\pi_0(\underline{\mathrm{Hom}}(L(C),-)) = ho(\mathbf{sSet})(L(C),-)$ with respect to the Joyal model structure, thus by Yoneda the pair $(L(\mathcal{C}),\gamma)$ satisfying the universal property is unique up to unique isomorphism in $ho(\mathbf{sSet})$ and up to a contractible space of equivalences in $\mathbf{sSet}$. □

Remark 3.1.5. Through our construction above, one can always choose $L(C)$ so that γ is a bijection on objects because $C' \to L(C)$ is an inner anodyne extension and therefore a retract of a countable composition of sums of pushouts of maps which are the identity on objects, that is the inner horn inclusions.

Remark 3.1.6. 7.1.5

In this context, we may define $\overline{W}$, the saturation of W in C, as the maximal simplicial subset of C whose morphisms are the ones which become invertible in $L(C)$, thus we have $\overline{W} \cong k(L(C)) \times_{L(C)} C$ canonically, meaning that we have the following pullback square.

$$\begin{tikzcd}
\overline{W} \arrow[r, hook] \arrow[d] & C \arrow[d, "\gamma"] \\
k(L(C)) \arrow[r, hook] & L(C)
\end{tikzcd}$$

We have then inclusions $Sk_1(W) \subset W \subset \overline{W}$ and, for any ∞-category $\mathcal{D}$, this induces equalities

$$\underline{\mathrm{Hom}}_{Sk_1(W)}(C,\mathcal{D}) = \underline{\mathrm{Hom}}_W(C,\mathcal{D}) = \underline{\mathrm{Hom}}_{\overline{W}}(C,\mathcal{D}),$$

implying that $(L(C),\gamma)$ is also the localization of C by $Sk_1(W)$ and the one by $\overline{W}$. It can be noted that the inclusion $\overline{W} \to C$ is a fibration with respect to the Joyal model structure as it is the pullback of one, implying that if C is an is an ∞-category then so is $\overline{W}$.

We shall say that W is *saturated* if $W = \overline{W}$.

Remark 3.1.7. Given an ∞-category $\mathcal{C}$ and a simplicial subset W, the functor $ho(\mathcal{C}) \to ho(L(\mathcal{C}))$ exhibits $ho(L(\mathcal{C}))$ as the 1-categorical localization of $\mathcal{C}$ at $\mathrm{Arr}(\tau(W))$, as can be seen by using the universal property.
On the other hand, given a 1-category $\mathcal{C}$ and considered a set of morphisms W, not necessarily the induced map $L(N(\mathcal{C})) \to N(L(\mathcal{C}))$ is an isomorphism. Indeed, $L(N(\mathcal{C}))$ can have much better properties, as can be seen for example from 3.2.12, and in fact localizing 1-categories after taking their nerves gives every ∞-category as shown in [Cis19, Prop. 7.3.15].

Remark 3.1.8. Given a universe $\mathbf{U}$ and a simplicial subset W of a $\mathbf{U}$-small simplicial set C, let $\gamma: C \to L(C)$ be the associated localization. Then the functor

$$\gamma^*: \underline{\mathrm{Hom}}(L(C)^{\mathrm{op}}, \mathcal{S}) \to \underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathcal{S})$$

is fully faithful because it is obtained as

$$\underline{\mathrm{Hom}}(L(C)^{\mathrm{op}}, \mathcal{S}) \simeq \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(C^{\mathrm{op}}, \mathcal{S}) \to \underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathcal{S}).$$

It also has a left adjoint $\gamma_!$ and a right adjoint γ_* as shown in [Cis19, Ch. 6]. Full faithfulness then implies that for any presheaf $F: L(C)^{\mathrm{op}} \to \mathcal{S}$ the unit map $F \to \gamma_*\gamma^*(F)$ is invertible and, by adjunction, the same goes for the counit map $\gamma_!\gamma^*(F) \to F$.
Furthermore, its essential image consists exactly of those presheafs $F: C^{\mathrm{op}} \to \mathcal{S}$ such that, for any morphism $u: x \to y$ in W, the map $Fu: Fy \to Fx$ is invertible in $\mathcal{S}$. Indeed, restricting $\gamma_!$ and γ_* to $\underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(C^{\mathrm{op}}, \mathcal{S})$, we see that they are left and right adjoint to the equivalence of ∞-categories induced by γ^*, meaning that the counit map $\gamma^*\gamma_*(F) \to F$ and the unit map $F \to \gamma^*\gamma_!(F)$ are invertible.

Proposition 3.1.9. Given a simplicial set C and a simplicial subset W, the localization functor $\gamma: C \to L(C)$ is final and cofinal. In particular, if $e: \Delta^0 \to C$ encodes a final or a cofinal object, so does $\gamma(e)$.

Proof. First of all, the functor γ^{op} is also a localization, so it suffices to prove that γ is final. To do this, first we fix a universe $\mathbf{U}$ such that C is $\mathbf{U}$-small and then we remember that there is an adjunction $\gamma^*: \underline{\mathrm{Hom}}(L(C), \mathcal{S}) \rightleftarrows \underline{\mathrm{Hom}}(C, \mathcal{S}) : \gamma_*$ and, by the above remark, the unit of the adjunction is invertible, hence $1 \cong \gamma_*\gamma^*$. This gives us that

$$\lim_C F \cong \lim_C \gamma_*\gamma^*(F) \cong \lim_{L(C)} \gamma^*(F),$$

for any presheaf $F: C \to \mathcal{S}$, which is enough to prove that γ is final by [Cis19, Thm. 6.4.5]. $\square$

Proposition 3.1.10. Let's fix a universe $\mathbf{U}$, a $\mathbf{U}$-small simplicial set C and a simplicial subset W. Consider a morphism $f: C \to \mathcal{D}$, where $\mathcal{D}$ is a small ∞-category. Then f exhibits $\mathcal{D}$ as the localization of C by W if and only if the following conditions hold:

1. the morphism f sends the 1-simplices of W to invertible maps of $\mathcal{D}$;

2. the morphism f is essentially surjective;

3. the morphism f^* induces an equivalence of ∞-categories

$$f^*\colon \underline{\mathrm{Hom}}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \to \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(C^{\mathrm{op}}, \mathcal{S}).$$

Proof. If f is a localization then the conditions are satisfied (for (2) look at the construction in Prop. 3.1.4). For the converse, let's pick a localization $\gamma\colon C \to L(C)$ and, through condition (1), we get a factorization $g\colon L(C) \to \mathcal{D}$ such that $g \cdot \gamma \cong f$, giving us a triangle

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathcal{D}^{\mathrm{op}}, \mathcal{E}) \arrow[rr, "g^*"] \arrow[dr, "f^*"'] & & \underline{\mathrm{Hom}}(L(C)^{\mathrm{op}}, \mathcal{E}) \arrow[dl, "\gamma^*"] \\
& \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(C^{\mathrm{op}}, \mathcal{E}) &
\end{tikzcd}$$

commuting up to J-homotopy for any ∞-category $\mathcal{E}$. Picking $\mathcal{E} = \mathcal{S}$, γ^* and f^* are equivalences of ∞-categories, the latter by (3). It follows by 2-out-of-3 that g^* is one too, therefore the same applies to its left adjoint $g_!$, which is then fully faithful. This is equivalent to g being fully faithful by [Cis19, Prop. 6.1.15] and, since f is essentially surjective by (2), the same goes for g. It follows that g is an equivalence of ∞-categories. In the above triangle g^* is then an equivalence for any choice of $\mathcal{E}$ and the same applies to f^* by 2-out-of-3. We conclude by using $(-)^{\mathrm{op}}$. □

Proposition 3.1.11. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories with a right adjoint $g\colon \mathcal{D} \to \mathcal{C}$ and suppose that we are given simplicial subsets $V \subset \mathcal{C}$, $W \subset \mathcal{D}$ such that $f(V) \subset W$, $g(W) \subset V$. We can lift them to an adjunction $\overline{f}\colon L(\mathcal{C}) \rightleftarrows L(\mathcal{D}) :\overline{g}$ such that the diagrams

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "\gamma_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\gamma_{\mathcal{D}}"] \\
L(\mathcal{C}) \arrow[r, "\overline{f}"'] & L(\mathcal{D})
\end{tikzcd}, \quad
\begin{tikzcd}
\mathcal{D} \arrow[r, "g"] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{C} \arrow[d, "\gamma_{\mathcal{C}}"] \\
L(\mathcal{D}) \arrow[r, "\overline{g}"'] & L(\mathcal{C})
\end{tikzcd}$$

commute up to J-homotopy.

Proof. Let's write $\underline{\mathrm{Hom}}_V^W(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ whose objects are functors ϕ such that $\phi(V) \subset W$. The equivalence $\gamma_{\mathcal{C}}^*\colon \mathrm{Hom}(L(\mathcal{C}), L(\mathcal{D})) \to \underline{\mathrm{Hom}}_V(\mathcal{C}, L(\mathcal{D}))$ allows us to construct a functor $\underline{\mathrm{Hom}}_V^W(\mathcal{C}, \mathcal{D}) \to \underline{\mathrm{Hom}}_V(\mathcal{C}, L(\mathcal{D})) \to \underline{\mathrm{Hom}}(L(\mathcal{C}), L(\mathcal{D}))$ which associates to any ϕ as above a functor $\overline{\phi}$ making the square

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\phi"] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{C} \arrow[d, "\gamma_{\mathcal{C}}"] \\
L(\mathcal{D}) \arrow[r, "\overline{\phi}"'] & L(\mathcal{C})
\end{tikzcd}$$

commute up to J-homotopy.
The proof works by observing that our map also lifts natural transformations functorially, which allows us to show the triangle identities for the lifted unit and counit. □

Proposition 3.1.12. Let $u: \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories with a fully faithful right adjoint v and consider $W = k(\mathcal{D}) \times_{\mathcal{D}} \mathcal{C}$, the subcategory of maps of $\mathcal{C}$ which become invertible in $\mathcal{D}$. Then u exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ by W.

Proof. Given a localization $\gamma: \mathcal{C} \to L(\mathcal{C})$ by W, we get a functor $\gamma \cdot v: \mathcal{D} \to L(\mathcal{C})$ which, paired with the $\overline{u}$ obtained from the construction in the previous proof, lifts the adjunction $u \dashv v$ to the localizations (where $L(\mathcal{D}) = \mathcal{D}$ as we localize at the identities). Lifting maintains the counit invertible, which allows us to conclude that $\gamma \cdot v$ is fully faithful.

Essential surjectivity follows from the fact that, for any object c in $\mathcal{C}$, the unit η_c is such that $\epsilon_{u(c)} \cdot u(\eta_c) = 1_{u(c)}$ and, since ϵ is invertible, so is $u(\eta_c)$, thus η_c becomes invertible in $L(\mathcal{C})$ and shows that $(\gamma_{\mathcal{C}} \cdot v)(u(c)) = \gamma_{\mathcal{C}}(vu(c)) \cong c$. Notice that here we used that $L(\mathcal{C})_0 = \mathcal{C}_0$, which is permissible up to equivalence as noted in Remark 3.1.5.

(PLEASE CHECK PROOF) □

3.2 Localizations of Fibrational Structures

Definition 3.2.1. An ∞-*category with weak equivalences and fibrations* is a triple $(\mathcal{C}, W, \mathrm{Fib})$ where $\mathcal{C}$ is an ∞-category with a final object, $W \subset \mathcal{C}$ is a subcategory with the 2-out-of-3 property and $\mathrm{Fib} \subset \mathcal{C}$ a subsimplicial set such that:

1. for any morphism $p: x \to y$ in Fib (and W) with y fibrant, there is in $\mathcal{C}$ a pullback square

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "p'"'] & x \arrow[d, "p"] \\ y' \arrow[r, "v"'] & y \end{tikzcd}$$

   where p' also lies in Fib (and W);

2. for any map $f: x \to y$ with fibrant codomain can be factored as a map in W followed by one in Fib.

By *fibrant object* we mean an object whose map to the terminal one is in Fib.

We shall call *weak equivalences* the maps in W and *fibrations* the ones in Fib. Maps which are both shall be referred to as *trivial fibrations*.

Construction 3.2.2. Any finitely complete ∞-category $\mathcal{C}$ can be given the structure of an ∞-category with weak equivalences and fibrations by setting $W = k(\mathcal{C})$, $\mathrm{Fib} = \mathcal{C}$, which we will be doing henceforth.

Construction 3.2.3. For any ∞-category with weak equivalences and fibrations $\mathcal{C}$ and a fibrant object c, we can give to the slice category $\mathcal{C}/c$ the structure of an ∞-category with weak equivalences and fibrations by specifying as weak equivalences the morphisms which are mapped to weak equivalences of $\mathcal{C}$ by the projection $\mathcal{C}/c \to \mathcal{C}$ and similarly for the fibrations.

Definition 3.2.4. An ∞-category of fibrant objects is an ∞-category with weak equivalences and fibrations $\mathcal{C}$ in which all objects are fibrant.

Construction 3.2.5. For any ∞-category with weak equivalences and fibrations $\mathcal{C}$, its full subcategory given by fibrant objects is canonically an ∞-category of fibrant objects. We shall denote it by $\mathcal{C}_f$ and its weak equivalences are given by $W_f = W \cap \mathcal{C}_f$, its fibrations by $\mathrm{Fib}_f = \mathrm{Fib} \cap \mathcal{C}_f$.

Lemma 3.2.6 (Brown's Lemma)**.** 7.4.13

For any map $f\colon x \to y$ between fibrant objects in an ∞-category with weak equivalences and fibrations $\mathcal{C}$, there exists a commutative diagram of the form

$$\begin{tikzcd} & x \arrow[dl, equal] \arrow[d, "s"] \arrow[dr, "f"] & \\ x & z \arrow[l, two heads, "\sim", "p"'] \arrow[r, two heads, "q"'] & y \end{tikzcd},$$

where s is a weak equivalence, p a trivial fibration and q a fibration.

Proof. Since x and y are fibrant, the pullback of $x \to e$ and $y \to e$ exists and it corresponds to $x \times y$. The maps id_x, f define a cone over our cospan which induces a map $g\colon x \to x \times y$ and we then factor the latter as a weak equivalence $s\colon x \to z$ followed by a fibration $\pi\colon z \to x \times y$. We get then the desired maps $p = p_x \cdot \pi$, $q = p_y \cdot \pi$, where p_x, p_y denote the projections $x \times y \to x$, $x \times y \to y$ respectively. □

Corollary 3.2.7. Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations, $\mathcal{D}$ an ∞-category and $V \subset \mathcal{D}$ a subcategory with the 2-out-of-3 property. If F sends trivial fibrations between fibrant objects into V, then it also sends weak equivalences between fibrant objects into V.

Proof. Looking at the commutative diagram

$$\begin{tikzcd} x \arrow[dd, equal] \arrow[dr, "s"] \arrow[drr, bend left, "f"] & & \\ & z \arrow[r, "q"] \arrow[dl, two heads, "p"] & y \\ x \arrow[urr, bend right, "f"'] & & \end{tikzcd},$$

given by Brown's Lemma 3.2.6 we see that Fp lies in V and therefore the same goes for Fs. Also, since f and s are weak equivalences we know that q is too, hence it is a trivial fibration. It follows that Fq is in V and the same goes for $Ff = Fq \cdot Fs$. □

Construction 3.2.8. Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$ and a fibrant object z in it, we write $\mathcal{C}(z)$ for $(\mathcal{C}/z)_f$, that is the full subcategory $\mathcal{C}/z$ given by the fibrations $x \to z$ of $\mathcal{C}$; we shall refer to $\mathcal{C}(z)$ as the *fibrant slice of* $\mathcal{C}$ *over* z. For any morphism $f\colon x \to y$ between fibrant objects, we have a left exact functor $f^*\colon \mathcal{C}(y) \to \mathcal{C}(x)$ induced pulling back along f (CAREFUL WITH THIS! YOU ARE USING IT LATER ON; CHECK [Cis19, Prop. 7.4.15]). The existence follows from the fact that pullbacks along fibrations with fibrant codomain exist, while left exactness comes from limits commuting and weak equivalences being preserved as a consequence of 3.2.7. (PERHAPS MORE DETAIL?)

Definition 3.2.9. Let $\mathcal{C}$, $\mathcal{D}$ be ∞-categories with weak equivalences and fibrations. A functor $F: \mathcal{C} \to \mathcal{D}$ is *left exact* if it has the following properties:

1. the functor F preserves final objects;
2. the functor F sends (trivial) fibrations between fibrant objects to (trivial) fibrations;
3. the functor F preserves any pullback square in $\mathcal{C}$

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "p'"'] & x \arrow[d, "p"] \\ y' \arrow[r, "v"'] & y \end{tikzcd}$$

where p is a fibration and y, y' are fibrant objects.

Remark 3.2.10. By Brown's Lemma, a left exact functor preserves weak equivalences between fibrant objects.

Remark 3.2.11. When considering a functor F between finitely complete ∞-categories, left exactness is equivalent to preserving finite limits.

Proposition 3.2.12. 7.5.6
Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, the localization $L(\mathcal{C}_f)$ has finite limits and the localization functor $\mathcal{C}_f \to L(\mathcal{C}_f)$ is left exact. Moreover, for any ∞-category $\mathcal{D}$ with finite limits and any left exact functor $f: \mathcal{C}_f \to \mathcal{D}$, the induced functor $\overline{F}: L(\mathcal{C}_f) \to \mathcal{D}$ is left exact.

Proof. Maybe do not prove it? It relies on a bunch of results from ch. 7.2, 7.3, 7.4 which we do not really want to prove.

7.1.10, 7.2.18, 7.2.25, 7.3.27, 7.4.13, 7.4.16

We know by 3.1.9 that $L(\mathcal{C}_f)$ has a final object, hence to show completeness it is enough to prove that it also has pullbacks ([Cis19, Thm. 7.3.27]). This can be done using the fact that any morphism in $L(\mathcal{C}_f)$ can be seen as a composition $\gamma(p) \cdot \gamma(s)^{-1}$, where s is a trivial fibration, for which Cisinski uses the theory of the *right calculus of fractions*, and the fact that γ_f preserves pullbacks along fibrations. The proof also shows us that all pullback squares in $L(\mathcal{C}_f)$ are isomorphic to images of pullback squares in $\mathcal{C}_f$ in which all maps are fibrations. □

Proposition 3.2.13. 7.5.16
Let x be a fibrant object in an ∞-category with weak equivalences and fibrations $\mathcal{C}$. The induced functor $\mathcal{C}_f/\gamma_f(x) \to \mathcal{C}/\gamma(x)$ is final.

Proof. We have that $\mathcal{C}_f/\gamma_f(x) = L(\mathcal{C}_f)/\gamma_f(x) \times_{L(\mathcal{C}_f)} \mathcal{C}_f$ and $\mathcal{C}/\gamma(x) = L(\mathcal{C})/\gamma(x) \times_{L(\mathcal{C})} \mathcal{C}$ and the functor we are considering is induced by $\overline{\iota}: L(\mathcal{C}_f) \to L(\mathcal{C})$.

To prove that it is final, it is sufficient to show that for any object (c, u) of $L(\mathcal{C})/\gamma(x)$ the coslice $(c,u)\backslash(\mathcal{C}_f/\gamma_f(x))$ is weakly contractible and, to do this, by [Cis19, Lem. 4.3.15] we can show that any functor $F: E \to (c,u)\backslash(\mathcal{C}_f/\gamma_f(x))$, where E is the nerve of a finite partially ordered set, is Δ^1-homotopic to a constant functor. This can be done through the theory of Reedy fibrant diagrams developed in [Cis19, Ch. 7.4]. □

Proposition 3.2.14. 7.5.17

Let $\mathbf{U}$ be a universe and $\mathcal{C}$ a $\mathbf{U}$-small ∞-category with weak equivalences and fibrations. For any ∞-category $\mathcal{D}$ with $\mathbf{U}$-small colimits and any functor $F\colon \mathcal{C} \to \mathcal{D}$, we have an isomorphism

$$(\gamma_f)_! \iota^*(F) \cong \bar{\iota}^* \gamma_!(F)$$

induced by the square

$$\begin{tikzcd} \mathcal{C}_f \arrow[r, "\iota"] \arrow[d, "\gamma_f"'] & \mathcal{C} \arrow[d, "\gamma"] \\ L(\mathcal{C}_f) \arrow[r, "\bar{\iota}"'] & L(\mathcal{C}) \end{tikzcd} ,$$

which commutes up to J-homotopy.

Proof. We only need to prove that the evaluation of the canonical map $(\gamma_f)_! \iota^*(F) \cong \bar{\iota}^* \gamma_!(F)$ at any object x of $\mathcal{C}_f$ is invertible. This evaluation is equivalent by [Cis19, Prop. 6.4.9] to the map

$$\operatorname{colim}_{\mathcal{C}_f/\gamma_f(x)} i^*(F)/\gamma_f(x) \to \operatorname{colim}_{\mathcal{C}/\gamma(x)} F/\gamma(x),$$

where $F/\gamma(x)$ is define by composing F with the canonical projection $\mathcal{C}/\gamma(x) \to \mathcal{C}$ and similarly for $i^*(F)/\gamma_f(x)$. Using 3.2.13 and the commutativity of the square above, we get that the desired map is indeed invertible for all x. □

Proposition 3.2.15. 7.5.18

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. The canonical functor $\bar{\iota}\colon L(\mathcal{C}_f) \to L(\mathcal{C})$ is an equivalence of ∞-categories, hence the ∞-category $L(\mathcal{C})$ is finitely complete and the localization functor $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ is left exact.

Proof. 7.5.6, 7.5.17

We already know that $\bar{\iota}$ is essentially surjective as every object in $\mathcal{C}$ is weakly equivalent to one in $\mathcal{C}_f$ and the localization functors are essentially surjective themselves, thus it is enough to prove that it is fully faithful. To do this, we may fix a universe $\mathbf{U}$ such that $\mathcal{C}$ is $\mathbf{U}$-small and prove that the functor

$$\bar{\iota}_!\colon \underline{\operatorname{Hom}}(L(\mathcal{C}_f), \mathcal{S}) \to \underline{\operatorname{Hom}}(L(\mathcal{C}), \mathcal{S})$$

is fully faithful and use [Cis19, Prop. 6.1.15]. Remember that this full faithfulness condition is equivalent to the unit map $1 \to \bar{\iota}^* \bar{\iota}_!$ of the adjunction $\bar{\iota}_! \dashv \bar{\iota}^*$ being invertible.

We know that $\bar{\iota}_*$ and $\bar{\iota}^*$ both have right adjoints, thus they preserve colimits. Also, every $\mathcal{S}$-valued functor indexed by a $\mathbf{U}$-small ∞-category can be obtained as a colimit of representable ones, hence it is enough to check that the condition holds for any representable functor F. Also, γ_f is essentially surjective, which means that it is sufficient to check that map $(\gamma_f)_! \to \bar{\iota}^* \bar{\iota}_! (\gamma_f)_!$ which we get by precomposing the unit with $(\gamma_f)_!$ is invertible.

We have then the chain of isomorphisms

$$\begin{aligned} (\gamma_f)_! &\cong (\gamma_f)_! \bar{\iota}^* \bar{\iota}_! \\ &\cong \bar{\iota}^* \gamma_f \iota_! \\ &\cong \bar{\iota}^* \bar{\iota}_! (\gamma_f)_!, \end{aligned}$$

where the first isomorphism comes from the full faithfulness of ι, the second one from 3.2.14 and the last one the fact that $\bar{\iota} \cdot \gamma_f \cong \gamma \cdot \iota$, as noted in 3.2.14.

The second claim follows directly from the first one and 3.2.12. □

Remark 3.2.16. Here we see that the theory of localizations of ∞-categories with weak equivalences and fibrations provides much better results the 1-categorical equivalent, embodied by the homotopy theory of model categories and fibration categories (which we will define in the next chapter): indeed, these are particular cases of the ∞-analogue, however their homotopy categories, i.e. their 1-categorical localizations by weak equivalences, are almost never finitely complete.

Corollary 3.2.17. 7.5.19

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. For a morphism between fibrant objects $p: x \to y$, the following conditions are equivalent:

1. the morphism p has a section in $ho(L(\mathcal{C}))$;
2. there exists a morphism $p': x' \to x$ s.t. the composition of p' and p is a weak equivalence;
3. there exists a fibration $p': x' \to x$ s.t. the composition of p' and p is a weak equivalence.

Proof. 7.5.18

We see that (iii) trivially implies (ii), therefore we shall focus on the other implication. Given then such a morphism p', we factor it as $qi = p'$, a weak equivalence followed by a fibration. Since $p \cdot p' = p \cdot (q \cdot i) = (p \cdot q) \cdot i$, by 2-out-of-3 $p \cdot q$ is a weak equivalence, giving us what we wanted.

Should we prove (i)? Uses right calculus of fractions, but it's rather simple. □

Construction 3.2.18. 7.5.22

Given an ∞-category $\mathcal{C}$ with weak equivalences and fibrations, we can get another one $\overline{\mathcal{C}}$ with the same underlying ∞-category and class of fibrations, but where the weak equivalences are given by the saturation $\overline{W}$ as described in 3.1.6. We have that $L(\mathcal{C}) \cong L(\overline{\mathcal{C}})$, hence in general we can substitute $\mathcal{C}$ by $\overline{\mathcal{C}}$ with no issues. Also, the substitution commutes with the formation of slices over fibrant objects, that is, for any fibrant object x of $\mathcal{C}$, a map in $\mathcal{C}/x$ induces an invertible map in $L(\mathcal{C}/x)$ if and only if its image becomes invertible in $L(\mathcal{C})$, which can be seen as a consequence of 3.2.17.

Remark 3.2.19. Let $\mathcal{C}$ be an ∞-category with weak equivalences W and $F: \mathcal{C} \to \mathcal{D}$ be a functor. The precomposition functor $\gamma^*: \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ does not have a left adjoint in general, but we may ask whether $\mathrm{Hom}(F, \gamma^*(-))$ is representable in $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D})$. If it is, a representative is denoted by $\mathbf{R}F: L(\mathcal{C}) \to \mathcal{D}$ and is called the *right derived functor of F*. Beware that to be precise one would have to specify the natural transformation $F \to \mathbf{R}F \cdot \gamma$ exhibiting it as such. Dually, a representative of $\mathrm{Hom}(\gamma^*(-), F)$ is the *left derived functor of F*.

Proposition 3.2.20. 7.5.24

If $F: \mathcal{C} \to \mathcal{D}$ sends weak equivalences to isomorphisms, then the functor $\overline{F}: L(\mathcal{C}) \to \mathcal{D}$, associated to F by the universal property of $L(\mathcal{C})$, is the right derived functor of F.

Proof. Let's fix a universe $\mathbf{U}$ such that $\mathcal{C}$ and $\mathcal{D}$ are $\mathbf{U}$-small and let $G: L(\mathcal{C}) \to \mathcal{D}$ be any functor. Then the invertible map $\overline{F} \cdot \gamma \cong F$ and the equivalence of ∞-categories $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \simeq \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$ induce invertible maps $\mathrm{Hom}(\overline{F}, G) \simeq \mathrm{Hom}(\overline{F} \cdot \gamma, G \cdot \gamma) \simeq \mathrm{Hom}(F, G \cdot \gamma)$ in $\mathcal{S}$, functorially in G. □

Construction 3.2.21. (NOT COMPLETE, ONE MAY SHOW THAT OUR CONSTRUCTION DOES GIVE THE RIGHT DERIVED FUNCTOR)

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. Any functor $F: \mathcal{C} \to \mathcal{D}$ sending weak equivalences between fibrant objects to invertible maps then has a right derived functor $\mathbf{R}F$, which may be constructed as follows.

First we choose a quasi-inverse $R: L(\mathcal{C}) \to L(\mathcal{C}_f)$ of the equivalence of ∞-categories specified in 3.2.15, then we pick a functor $\overline{F}: L(\mathcal{C}_f) \to \mathcal{D}$ and a natural isomorphism $j: \overline{F} \cdot \gamma_f \to F \cdot \iota$. We set then $\mathbf{R}F = \overline{F} \cdot R$.

What we are doing in this construction is selecting for every object in $\mathcal{C}$ a fibrant replacement, exactly like when we talk about right derived functors in the context of model categories. This is necessary because, a priori, we are not sending all weak equivalences to invertible maps in $\mathcal{D}$, hence we would have to show that before applying the universal property of localizations. Also, for any other functor $G: \mathcal{D} \to \mathcal{E}$, we have that $G \cdot \mathbf{R}F = \mathbf{R}(G \cdot F)$.

Definition 3.2.22. Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$ and an ∞-category with weak equivalences $\mathcal{D}$, let's consider a functor $F: \mathcal{C} \to \mathcal{D}$ preserving weak equivalences between fibrant objects of $\mathcal{C}$. We call the *right derived functor of F* the right derived functor of the composition

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\gamma_{\mathcal{D}}} L(\mathcal{D}),$$

where $\gamma_{\mathcal{D}}$ is the localization functor of of $\mathcal{D}$ at its weak equivalences. This right derived functor of F is denoted by $\mathbf{R}F$, that is $\mathbf{R}F = \mathbf{R}(\gamma_{\mathcal{D}} \cdot F): L(\mathcal{C}) \to L(\mathcal{D})$, which makes sense since we can apply the construction 3.2.21.

There are some interesting remarks which may be included!!!!

Proposition 3.2.23. 7.5.28

For any left exact functor $F: \mathcal{C} \to \mathcal{D}$ between ∞-categories with weak equivalences and fibrations, the right derived functor $\mathbf{R}F: L(\mathcal{C}) \to L(\mathcal{D})$ is left exact.

Proof. 7.5.6

We have a square

$$\begin{tikzcd} L(\mathcal{C}_f) \arrow[r, "\overline{F}"] \arrow[d] & L(\mathcal{D}_f) \arrow[d] \\ L(\mathcal{C}) \arrow[r, "\mathbf{R}F"'] & L(\mathcal{D}) \end{tikzcd}$$

commuting up to J-homotopy, where the vertical maps are equivalences of ∞-categories and $\overline{F}$ is the functor obtained by restricting F to the subcategories of fibrant objects $\mathcal{C}_f$ and $\mathcal{D}_f$. It therefore suffices to show that $\overline{F}$ is left exact, but this follows from 3.2.12. $\square$

Remark 3.2.24. (WHY DO WE NEED TO SPECIFY THIS?)

For the remainder of this chapter, given an ∞-category $\mathcal{C}$, subcategories of weak equivalences $W \subset \mathcal{C}$ are such that the inclusion $W \to \mathcal{C}$ is an inner fibration. This means that a simplex $x: \Delta^n \to \mathcal{C}$ lies in W if and only if its edges $x|_{\Delta^{\{i,i+1\}}}: \{i, i+1\} \to \mathcal{C}$ lie in W for $0 \leq i < n$.

W then contains all invertible maps of $\mathcal{C}$ if and only if the aforementioned inclusion is an isofibration.

Definition 3.2.25. Let $\mathcal{C}$, $\mathcal{D}$ be ∞-categories with subcategories of weak equivalences $W \subset \mathcal{C}$, $W' \subset \mathcal{D}$. A functor $f: \mathcal{C} \to \mathcal{D}$ has the *right approximation property* if the following conditions hold:

1. a morphism in $\mathcal{C}$ is in W if and only if its image under f is in W';

2. given objects c, d in $\mathcal{C}$, $\mathcal{D}$ respectively and a map $\psi: d \to f(c)$ in $\mathcal{D}$, there is a map $\phi: c' \to c$ in $\mathcal{C}$ and a weak equivalence $u: d \to f(c')$ in $\mathcal{D}$ such that the triangle

$$\begin{tikzcd} d \arrow[r, "\psi"] \arrow[d, "u"'] & f(c) \\ f(c') \arrow[ur, "f(\phi)"'] & \end{tikzcd}$$

   commutes.

Proposition 3.2.26. 7.6.2

A functor $f: \mathcal{C} \to \mathcal{D}$ between ∞-categories such that the induced functor on the homotopy categories $ho(f): ho(\mathcal{C}) \to ho(\mathcal{D})$ is an equivalence of categories has the right approximation property.

Proof. Consider a map $\psi: d \to f(c)$. Since $ho(f)$ is essentially surjective, there exists an invertible map $d \to ho(f)(c') = f(c')$ in $ho(\mathcal{D})$, which comes from an invertible map $d \to f(c')$ of $\mathcal{D}$. In $ho(\mathcal{D})$ we can then complete this to a triangle

$$\begin{tikzcd} d \arrow[r, "{[\psi]}"] \arrow[d, "\sim"'] & f(c) \\ f(c') \arrow[ur, "{[\phi]}"'] & \end{tikzcd}$$

and, since $ho(f)$ is fully faithful, ϕ can be lifted to $\tilde{\phi}: c' \to c$ in $ho(\mathcal{C})$. This gives us a commutative triangle

$$\begin{tikzcd} d \arrow[r, "\psi"] \arrow[d, "\sim"'] & f(c) \\ f(c') \arrow[ur, "f(\tilde{\phi})"'] & \end{tikzcd}$$

in $\mathcal{D}$. $\square$

Example 3.2.27. Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, the inclusion $\mathcal{C}_f \to \mathcal{C}$ has the right approximation property.

Example 3.2.28. Given a saturated ∞-category with weak equivalences and fibrations $\mathcal{C}$, the localization functor $\mathcal{C} \to L(\mathcal{C})$ has the right approximation property ([Cis19, Ex. 7.6.4]).

Theorem 3.2.29. 7.6.10

Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories with finite limits. If f commutes with them, then the following conditions are equivalent:

1. the functor f is an equivalence of ∞-categories;

2. the functor $ho(f)\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories;

3. the functor f has the right approximation property.

Proof. 7.3.29, 7.6.2, 7.6.5, 7.6.7

We trivially have that (i) implies (ii) and, by 3.2.26, (iii) follows from (ii), hence we only have to show that (iii) gives (i). Let's assume then that f has the right approximation property.

Given a final object e of $\mathcal{C}$, $f(e)$ is still final in $\mathcal{D}$ by 3.1.9, thus for any object d of $\mathcal{D}$ we have a map $d \to f(e)$ and, by the right approximation property, we get a commutative triangle with an isomorphism (SPECIFY WHICH STRUCTURE YOU ARE CONSIDERING ON THE ∞-CATEGORIES) $d \to f(c)$ for some c in $\mathcal{C}$, which gives us essential surjectivity.

We are still missing full faithfulness. To do this, we use that the right approximation property implies that we have an equivalence of ∞-groupoids $k(f)\colon k(\mathcal{C}) \to k(\mathcal{D})$ ([Cis19, Lem. 7.6.7]) and that, for any object c of $\mathcal{C}$, the map $\mathcal{C}/c \to \mathcal{D}/f(c)$ induced on the slices still has the right approximation property ([Cis19, Prop. 7.6.7]), therefore again we get an equivalence of ∞-groupoids $k(\mathcal{C}/c) \to k(\mathcal{D}/f(c))$.

Keeping these facts in mind, let's look at the projection $\mathcal{C}/c \to \mathcal{C}$. This functor is conservative, thus the square

$$\begin{tikzcd} k(\mathcal{C}/c) \arrow[r] \arrow[d] & \mathcal{C}/c \arrow[d] \\ k(\mathcal{C}) \arrow[r] & \mathcal{C} \end{tikzcd}$$

is a pullback. We observe by pasting that the pullback of $k(\mathcal{C}/c) \to k(\mathcal{C})$ along $c'\colon \Delta^0 \to k(\mathcal{C})$ is $\mathcal{C}(c', c)$, as it is clear from the diagram

$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r] \arrow[d] & k(\mathcal{C}/c) \arrow[r] \arrow[d] & \mathcal{C}/c \arrow[d] \\ \Delta^0 \arrow[r, "c'"'] & k(\mathcal{C}) \arrow[r] & \mathcal{C} \end{tikzcd} .$$

In the same way, we get that the pullback of $k(\mathcal{D}/f(c)) \to k(\mathcal{D})$ along $f(c')\colon \Delta^0 \to k(\mathcal{D})$ is $\mathcal{D}(f(c'), f(c))$. Since we have a commutative square

$$\begin{tikzcd} k(\mathcal{C}/c) \arrow[r] \arrow[d] & k(\mathcal{D}/f(c)) \arrow[d] \\ k(\mathcal{C}) \arrow[r] & k(\mathcal{D}) \end{tikzcd}$$

where the horizontal maps are equivalences of ∞-groupoids, the induced map $\mathcal{C}(c', c) \to \mathcal{D}(f(c'), f(c))$ is again an equivalence of ∞-groupoids, which is what we wanted.

Since f is essentially surjective and fully faithful, it is an equivalence of ∞-categories. □

Corollary 3.2.30. 7.6.13

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations and consider a localization functor $\gamma\colon \mathcal{C} \to L(\mathcal{C})$. For any fibrant object x of $\mathcal{C}$, the canonical functor $\mathcal{C}/x \to L(\mathcal{C})/\gamma(x)$, $t \mapsto \gamma(t)$, induces an equivalence of ∞-categories $L(\mathcal{C}/x) \simeq L(\mathcal{C})/\gamma(x)$.

Proof. 7.5.18, 7.5.22, 7.5.24, 7.5.28, 7.6.4, 7.6.10

By 3.2.18, we can assume that $\mathcal{C}$ is saturated. Our objective is to show that the induced functor $\phi\colon L(\mathcal{C}/x) \to L(\mathcal{C})/\gamma(x)$ has the right approximation property and it preserves finite limits, which will allow us to apply 3.2.29 and conclude.

To show condition (1) we only need to prove that ϕ is conservative, which can be reduced to showing that a map in $\mathcal{C}/x$ becomes invertible in $L(\mathcal{C}/x)$ if and only if it becomes an insomorphism in $L(\mathcal{C})$. This however is true by saturation of $\mathcal{C}$. We still need to check condition (2), which can be done on $\mathcal{C}/x \to L(\mathcal{C})/\gamma(x)$, but this follows from the fact that γ has it, as mentioned in 3.2.28.

To apply 3.2.29 we still need to show that ϕ preserves limits. To do this, we use the fact that $\mathcal{C}/x$ has the structure of an ∞-category with weak equivalences and fibrations. Given that γ is left exact, the functor $\mathcal{C}/x \to L(\mathcal{C})/\gamma(x)$ maps weak equivalences to isomorphisms and we can apply 3.2.20 to prove that ϕ is its right derived functor. Finally, through 3.2.23 we get that ϕ is also left exact. □

Theorem 3.2.31. 7.6.16

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. Given a fibrant object x, let $\mathcal{C}(x)$ be the full subcategory of fibrant objects of $\mathcal{C}/x$ (INCLUDE 7.6.12), which will be an ∞-category of fibrant objects. Assume that, for any fibration between fibrant objects $p\colon x \to y$, the pullback functor $p^*\colon \mathcal{C}(y) \to \mathcal{C}(x)$, $(y' \to y) \mapsto (y' \times_y x \to x)$ has a right adjoint $p_*\colon \mathcal{C}(x) \to \mathcal{C}(y)$ preserving trivial fibrations. Then, for any map $p\colon x \to y$ in $L(\mathcal{C})$, the pullback functor $p^*\colon \mathcal{C}(y) \to \mathcal{C}(x)$ has a right adjoint.

Proof. 6.1.6, 6.1.7, 6.1.8, 7.1.14, 7.4.14, 7.5.18, 7.6.13

Given a localization functor $\gamma\colon \mathcal{C} \to L(\mathcal{C})$, one reduces the problem to proving that, for any fibration between fibrant objects $p\colon x \to y$, the pullback functor

$$\gamma(p)^*\colon L(\mathcal{C})/\gamma(y) \to L(\mathcal{C})/\gamma(x)$$

has a right adjoint.

A consequence of Brown's Lemma 3.2.6 is that any functor preserving trivial fibrations between fibrant objects also preserves weak equivalences, and, since $p^*\colon \mathcal{C}(y) \to \mathcal{C}(x)$ has both a right and a left adjoint named p_* and $p_!$ (the latter given by post-composing with p) which do preserve them, by 3.1.11 we have a pair of adjunctions on the localizations, namely

$$\overline{p}^*\colon L(\mathcal{C}(y)) \rightleftarrows L(\mathcal{C}(x)) :\overline{p}_*,$$

$$\overline{p}_!\colon L(\mathcal{C}(x)) \rightleftarrows L(\mathcal{C}(y)) :\overline{p}^*.$$

Given that $(\mathcal{C}/z)_f = \mathcal{C}(z)$ for all fibrant objects z of $\mathcal{C}$, by 3.2.15 we have that $L(\mathcal{C}(z)) \simeq L(\mathcal{C}/z)$ and, by 3.2.30, we also know that $L(\mathcal{C}/z) \simeq L(\mathcal{C})/\gamma(z)$, hence $L(\mathcal{C}(z)) \simeq L(\mathcal{C})/\gamma(z)$. Notice that $\overline{p}_!$ is equivalent to $\gamma(p)_!\colon L(\mathcal{C})/\gamma(x) \to L(\mathcal{C})/\gamma(y)$ and, by essential uniqueness of the adjoints, this extends to $\gamma(p)^*$ and $\overline{p}^*$, therefore $\gamma(p)^*$ has a right adjoint induced by $\overline{p}_*$. □

4 Localizations of Syntactic Categories

In this chapter we finally talk about localizing syntactic categories. To do this, we first specify a class of maps we localize at, explaining the rationale behind our choice, and then from this we present a fibrational structure which will allow us to apply the results from the previous section to prove Theorem 4.2.4. We also briefly mention the *Internal Languages Conjecture*.

4.1 Bi-Invertibility

As anticipated, we introduce a notion of weak equivalence in our context.

Definition 4.1.1. Given a contextual category with Id-structure C, a morphism $f\colon \Gamma.A \to \Gamma.B$ over Γ is *simply bi-invertible over* Γ if there exist:

1. a morphism $g_1\colon \Gamma.B \to \Gamma.A$;
2. a section $\eta\colon \Gamma.A \to \Gamma.A.(1_A, g_1 f)^* \mathsf{Id}_A$;
3. a morphism $g_2\colon \Gamma.B \to \Gamma.A$;
4. a section $\epsilon\colon \Gamma.B \to \Gamma.B.(1_B, f g_2)^* \mathsf{Id}_B$.

We now generalize the above definition to arbitrary context extensions in $\mathsf{Syn}(\mathbf{T})$ by working as usual with $\mathsf{Syn}(\mathbf{T})^{cxt}$ to provide the notion given in [Kap17, Def. 1.4].

Definition 4.1.2. Given a dependent type theory with Id-types $\mathbf{T}$, a morphism $f\colon \Gamma.\Delta \to \Gamma.\Theta$ over Γ in $\mathsf{Syn}(\mathbf{T})$ is *bi-invertible over* Γ if it is simply bi-invertible over Γ as a morphism between the two corresponding simple context extensions of Γ in $\mathsf{Syn}(\mathbf{T})^{cxt}$. It is then called *bi-invertible* if Γ is the empty context.

Remark 4.1.3. A morphism $f\colon \Gamma.\Delta \to \Gamma.\Theta$ bi-invertible over Γ is also bi-invertible since we can extend a section $\Gamma.\Delta \to \Gamma.\Delta.(1_\Delta, gf)^* \mathsf{Id}_\Delta$ to a section $\Gamma.\Delta \to \Gamma.\Delta.(1_{\Gamma.\Delta}, gf)^* \mathsf{Id}_{\Gamma.\Delta}$ thanks to the map $\mathsf{refl}_\Gamma\colon \Gamma \to \Gamma.p_\Gamma^*\Gamma.\,\mathrm{Id}_\Gamma$.

Remark 4.1.4. Our definition of bi-invertible map is less general than the original one (which concerns arbitrary contextual categories with an Id-structure) because that one relies on strong Σ-types, which we do not assume. They suggest using the Id-structure on iterated context extensions as an alternative tool to get the same result in the general case, however we did not provide the construction for arbitrary contextual categories.

Remark 4.1.5. Our interest in simply bi-invertible morphisms stems from the fact that they model the right notion of invertible map in dependent type theory: indeed, from the required data for a simply bi-invertible map f over Γ we can provide a section $\Gamma \to \Gamma.\mathsf{isHIso}(f)$ (MAYBE ACTUALLY DO IT?) as defined in [KL12, Def. B.3.3], which itself is the translation into the language of contextual categories of the notion of bi-invertible map in type theory [Uni13, Def 4.3.1].

It is important to note that, while type theory has no way to encode internally the concept of isomorphism of the contextual model, it does have its own internal notion of isomorphism. However, given a map f, the type $\mathsf{isIso}(f)$ is not, in general, a *mere proposition* [Uni13, Def. 3.3.1], unlike $\mathsf{isHIso}(f)$ [Uni13, Thm. 4.3.2], which makes the latter preferable. Also, every bi-invertible map can be given the structure of an isomorphism and viceversa, hence they are closely related.

Remark 4.1.6. Assuming the Initiality Conjecture, localizing a contextual category with an Id-structure at bi-invertible maps gives us an ∞-category modeling a dependent type theory with Id-types. It has been conjectured that such a type theory should provide the internal language of ∞-categories, but a precise statement of this correspondence has not yet been produced and only a few results in this direction have been proven so far. A conjecture of this kind would amount to an equivalence between an ∞-category of contextual categories with some extra structures and an ∞-category of structured ∞-categories induced by the localization at bi-invertible maps, which means that first have to determine what properties the localization has. Notable in this sense is the Internal Languages Conjecture mentioned in Remark 1.2.12.

To study the localizations of syntactic categories we want to apply the results from the previous section, which require us to specify a fibrational structure by also providing a class of fibrations.

Definition 4.1.7 ([Bro73])**.** A *fibration category* is a triple $(\mathcal{P}, W, \mathrm{Fib})$ where $\mathcal{P}$ is a category and W, Fib are wide subcategories such that:

1. $\mathcal{P}$ has a terminal object;
2. maps to the terminal object lie in Fib;
3. Fib and $W \cap \mathrm{Fib}$ are closed under pullback along any map in $\mathcal{P}$;
4. every map in $\mathcal{P}$ can be factored as a map in W followed by one in Fib;
5. W has the 2-out-of-6 property.

Remark 4.1.8. Seeing $\mathcal{P}$, W and Fib in the above definition as ∞-categories, we notice that the triple has canonically the structure of an ∞-category of fibrant objects, hence we shall adopt the conventions we used in that context.

Remark 4.1.9 ([Shu14])**.** Our definition differs slightly from the original one by Brown because it asks for W to be closed under the 2-out-of-6 property instead of the more classical 2-out-of-3, but, as shown by Cisinski, if all of the other axioms are satisfied then the following are equivalent:

1. W has the 2-out-of-6 property;

2. W has the 2-out-of-3 and is saturated, that is a morphism in $\mathcal{P}$ becomes invertible in $\mathbf{Ho}(\mathcal{P})$ if and only if it lies in W.

See [Rad06, Thm. 7.2.7]. Another difference is that Brown only requires factorizations of the diagonal maps $X \to X \times X$, but then he derives our factorization condition from the other properties.

Now we have enough to specify the fibrational structure.

Proposition 4.1.10 ([AKL15])**.** Any contextual category with Σ-, Id-, Nat- and 1-structures C carries the structure of a fibration category, where maps isomorphic to dependent projections are the fibrations and bi-invertible ones are the weak equivalences.

The above result can be however generalized.

Proposition 4.1.11. A contextual category with Σ- and Id- structures C carries the structure of a fibration category given by the same classes of maps as above.

Proof. It suffices to note that at no point the proof by Lumsdaine uses the other structures. □

Remark 4.1.12. We shall refer to the above classes of maps as weak equivalences and fibrations even in absence of a Σ-structure.

Remark 4.1.13. The results as stated rely on the Initiality Conjecture (since in their reasoning the authors used internal languages) and their proof makes use of strong Σ-types, which they adopted to say that every dependent projection is isomorphic to a basic one. We can avoid relying on strong Σ-types by constructing the fibrational structure on $\mathsf{Syn}(\mathbf{T})^{cxt}$ (where the condition on dependent projections holds by construction) to later carry it back to $\mathsf{Syn}(\mathbf{T})$ through the equivalence, while we do not need the Initiality Conjecture because we can argue in $\mathsf{Syn}(\mathbf{T})^{cxt}$ using the dependent type theory $\mathbf{T}$.

Corollary 4.1.14. Given a dependent type theory with Σ- and Id-types $\mathbf{T}$, the ∞-category $L(\mathsf{Syn}(\mathbf{T}))$ is finitely complete.

Proof. It follows directly from Proposition 4.1.11 and Proposition 3.2.15. □

4.2 Local Cartesian Closure

We are now ready to provide a few results needed to show that, given a dependent type theory with Σ-, Id-, Π_η-types and function extensionality $\mathbf{T}$, the hypothesis of Theorem 3.2.31 are satisfied by $\mathsf{Syn}(\mathbf{T})$ with respect to the above fibrational structure. Henceforth, we shall write C for $\mathsf{Syn}(\mathbf{T})$.

Lemma 4.2.1 ([Kap17])**.** For any dependent projection $p_\Delta : \Gamma.\Delta \to \Gamma$ in C, the pullback functor $p_\Delta^* : \mathsf{C}(\Gamma) \to \mathsf{C}(\Gamma.\Delta)$ between the fibrant slices admits a right adjoint.

Proof. Let's set $(p_\Delta)_*(\Gamma.\Delta.\Theta) = \Gamma.\Pi(\Delta,\Theta)$. Our counit shall be given by

$$\epsilon_{\Gamma.\Delta.\Theta}\colon \Gamma.\Delta.p_\Delta^* \Pi(\Delta,\Theta) \xrightarrow{\mathsf{exch}_{\Delta,\Pi(\Delta,\Theta)}} \Gamma.\Pi(\Delta,\Theta).p^*_{\Pi(\Delta,\Theta)}\Delta \xrightarrow{\mathsf{app}_{\Delta,\Theta}} \Gamma.\Delta.\Theta$$

and it is then sufficient to prove that, for any context morphism $f\colon \Gamma.\Delta.p_\Delta^*\Psi \to \Gamma.\Delta.\Theta$ over $\Gamma.\Delta$, there is a unique $\tilde f\colon \Gamma.\Psi \to \Gamma.\Pi(\Delta,\Theta)$ making the diagram

$$\begin{tikzcd}
\Gamma.\Delta.p_\Delta^*\Psi \arrow[d, dotted, "p_\Delta^*(\tilde f)"'] \arrow[dr, "f"] & \\
\Gamma.\Delta.p_\Delta^*\Pi(\Delta,\Theta) \arrow[r, "\epsilon_{\Gamma.\Delta.\Theta}"'] & \Gamma.\Delta.\Theta
\end{tikzcd}$$

commute. This will then uniquely define how the right adjoint acts on the morphisms.

We start by specifying the unit $\eta_{\Gamma.\Psi}\colon \Gamma.\Psi \to \Gamma.\Pi(\Delta, p_\Delta^*\Psi)$.

Let's consider the commutative square

$$\begin{tikzcd}
\Gamma.\Psi.p_\Psi^*\Pi(\Delta,p_\Delta^*\Psi) \arrow[r, "q(p_\Psi,\Pi(\Delta,p_\Delta^*\Psi))"] \arrow[d, "p_{\Pi(\Delta,p_\Delta^*\Psi)}"'] & \Gamma.\Pi(\Delta,p_\Delta^*\Psi) \arrow[d, "p_{\Pi(\Delta,p_\Delta^*\Psi)}"] \\
\Gamma.\Psi \arrow[r, "p_\Psi"'] & \Gamma
\end{tikzcd}$$

where the map $q(p_\Psi, \Pi(\Delta, p_\Delta^*\Psi))$ acts as $(y, f) \mapsto f$. If we can provide a section of the left map corresponding to $y \mapsto (y, \lambda(x : \Delta).y)$ we are done as we can then compose it with $q(p_\Psi, \Pi(\Delta, p_\Delta^*\Psi))$ to get our unit $y \mapsto \lambda(x : \Delta).y$.

To construct it first we look at the commutative square

$$\begin{tikzcd}
\Gamma.\Psi \arrow[dr, dotted, "(1_\Psi,1_\Psi)"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& \Gamma.\Psi.p_\Psi^*\Psi \arrow[r, "q(p_\Psi,\Psi)"'] \arrow[d, "p_{p_\Psi^*\Psi}"] & \Gamma.\Psi \arrow[d, "p_\Psi"] \\
& \Gamma.\Psi \arrow[r, "p_\Psi"'] & \Gamma
\end{tikzcd},$$

where the factorization corresponds to $x \mapsto (x, x)$. Then, we pull back along $p_{p_\Psi^*\Delta}$, getting a section $p^*_{p_\Psi^*\Delta}(1_\Psi, 1_\Psi)\colon \Gamma.\Psi.p_\Psi^*\Delta \to \Gamma.\Psi.p_\Psi^*\Delta.p^*_{p_\Psi^*\Delta}\Psi$.

We then apply $\lambda_{\Delta,\Psi}$, which gives us a section

$$\lambda_{\Delta,\Psi}(p^*_{p_\Psi^*\Delta}(1_\Psi,1_\Psi)) = \lambda(1_{p_\Psi^*\Delta}, p_{p_\Psi^*\Delta})\colon \Gamma.\Psi \to \Gamma.\Psi.p_\Psi^*\Pi(\Delta, p_\Delta^*\Psi),$$

which is exactly what we were looking for.

We then define our lift $\tilde f$ as the composite $\Gamma.\Pi(\Delta, f)\cdot\eta_{\Gamma.\Psi}$. The commutativity of the above triangle follows by internal reasoning from

$$
\begin{aligned}
& (x, y) \\
\eta_{\Gamma.\Psi} & \mapsto (x, \lambda(x:\Delta).y) \\
\Gamma.\,\Pi(\Delta,f) & \mapsto (x, \lambda(x:\Delta).\,\mathsf{app}(f, \mathsf{app}(\lambda(x:\Delta).y, x))) \\
\beta\text{-reduction} & \equiv (x, \lambda(x:\Delta).\,\mathsf{app}(f,y)) \\
\mathsf{exch}_{\Pi(\Delta,\Psi),\Delta} & \mapsto (\lambda(x:\Delta).\,\mathsf{app}(f,y), x) \\
\mathsf{app}_{\Delta,\Psi} & \mapsto (x, \mathsf{app}(\lambda(x:\Delta).\,\mathsf{app}(f,y), x)) \\
\beta\text{-reduction} & \equiv (x, \mathsf{app}(f,y))
\end{aligned}
$$

and our description of $f\colon \Gamma.\Delta.p_\Delta^*\Psi \to \Gamma.\Delta.\Theta$ in Remark 1.3.9. (USED INTERNAL REASONING AGAIN) while the uniqueness of the $\tilde{f}$ follows from the fact that we can not construct, from every term $y:\Psi$, another term $g(y):\Pi(\Delta,\Theta)$ such that $\mathsf{app}(g(y),x) = \mathsf{app}(f,x)$. Indeed, the only way would be through η-expansion on $\lambda(x:\Delta).\,\mathsf{app}(f,y)$, but as we noted the Π_η-rule implies that this process returns the same term.

This also shows that $(p_\Delta)_*(f) = \Gamma.\,\Pi(\Delta,f)$, meaning that the construction of $\Gamma.\,\Pi(\Delta,f)$ is functorial. □

We know that every fibration in C is isomorphic to a dependent projection, so this tells us that every fibration induces an adjunction between fibrant slices.

To apply 3.2.31, we need to show that $(p_\Delta)_*$ is left exact (actually, we need to prove that it preserves trivial fibrations, but it essentially requires the same amount of work), which we shall do in two steps.

Lemma 4.2.2 ([KL18])**.** Consider a bi-invertible map $f\colon \Gamma.\Delta.\Theta \to \Gamma.\Delta.\Psi$ over $\Gamma.\Delta$ in C. The map $\Gamma.\,\Pi(\Delta,f)$ is then bi-invertible over Γ.

Proof. We shall construct an homotopical right inverse to $\Gamma.\,\Pi(\Delta,f)$ by working with g_1 and η since the other part of the construction involving g_2 and ϵ is essentially identical. To do so, we consider the commutative diagram

$$
\begin{tikzcd}
\Gamma.\Delta.\Psi.(1_\Psi, fg_1)^*\,\mathsf{Id}_\Psi \arrow[r, "q((1_\Psi,fg_1),\mathsf{Id}_\Psi)"] \arrow[d, "p_{(1_\Psi,fg_1)^*\,\mathsf{Id}_\Psi}"] & \Gamma.\Delta.\Psi.p_\Psi^*\Psi.\,\mathsf{Id}_\Psi \arrow[d, "p_{\mathsf{Id}_\Psi}"] \\
\Gamma.\Delta.\Psi \arrow[r, "(1_\Psi,fg_1)"'] \arrow[u, bend left, "\eta"] & \Gamma.\Delta.\Psi.p_\Psi^*\Psi
\end{tikzcd}
$$

in $\mathsf{C}(\Gamma.\Delta)$ and apply the right adjoint $(p_\Delta)_*$, which gives us

$$
\begin{tikzcd}
\Gamma.\,\Pi(\Delta.\Psi.(1_\Psi, fg_1)^*\,\mathsf{Id}_\Psi) \arrow[r, "\Gamma.\,\Pi(\Delta,q((1_\Psi,fg_1),\mathsf{Id}_\Psi))"] \arrow[d, "\Gamma.\,\Pi(\Delta,p_{(1_\Psi,fg_1)^*\,\mathsf{Id}_\Psi})"] & \Gamma.\,\Pi(\Delta.\Psi.p_\Psi^*\Psi.\,\mathsf{Id}_\Psi) \arrow[d, "\Gamma.\,\Pi(\Delta,p_{\mathsf{Id}_\Psi})"] \\
\Gamma.\,\Pi(\Delta.\Psi) \arrow[r, "\Gamma.\,\Pi(\Delta,(1_\Psi,fg_1))"'] \arrow[u, bend left, "\Gamma.\,\Pi(\Delta,\eta)"] & \Gamma.\,\Pi(\Delta,\Psi.p_\Psi^*\Psi)
\end{tikzcd}
$$

in $\mathsf{C}(\Gamma)$.

We have that

$$
\begin{aligned}
\Gamma.\,\Pi(\Delta, p_{(1_\Psi, fg_1)^* \mathsf{Id}_\Psi}) &= \Gamma.\,\Pi(\Delta,\Psi).\,\Pi(\Delta,\Psi).\,\Pi(\Delta,\mathsf{Id}_\Psi) \\
\Gamma.\,\Pi(\Delta,(1_\Psi, fg_1)) &= (\Gamma.\,\Pi(\Delta,1_\Psi), \Gamma.\,\Pi(\Delta, fg_1)) = (1_{\Pi(\Delta,\Psi)}, \Gamma.\,\Pi(\Delta,f)\cdot\Gamma.\,\Pi(\Delta,g_1)), \\
\Gamma.\,\Pi(\Delta, p_{(1_\Psi, fg_1)^* \mathsf{Id}_\Psi}) &= p_{\Pi(\Delta,\mathsf{Id}_\Psi)}, \\
\Gamma.\,\Pi(\Delta, q((1_\Psi, fg_1),\mathsf{Id}_\Psi)) &= q((1_{\Pi(\Delta,\Psi)}, \Gamma.\,\Pi(\Delta, fg_1)), \Pi(\Delta,\mathsf{Id}_\Psi)), \\
\Gamma.\,\Pi(\Delta.\Psi.(1_\Psi, fg_1)^* \mathsf{Id}_\Psi) &= \Gamma.\,\Pi(\Delta,\Psi).(1_{\Pi(\Delta,\Psi)}, \Gamma.\,\Pi(\Delta, fg_1))^* \mathsf{Id}_{\Pi(\Delta,\Psi)}\,.
\end{aligned}
$$

Notice that on the right we have suppressed quite a bit of notation in order to avoid making everything unreadable.

Remember that function extensionality provides us with a map

$$
\Gamma.\,\Pi(\Delta,\Psi).\,\Pi(\Delta,\Psi).\,\Pi(\Delta,\mathsf{Id}_\Psi) \to \Gamma.\,\Pi(\Delta,\Psi).\,\Pi(\Delta,\Psi).\,\mathsf{Id}_{\Pi(\Delta,\Psi)}
$$

over $\Gamma\,\Pi(\Delta,\Psi).\,\Pi(\Delta,\Psi)$, which composed with the top map and $\Gamma.\,\Pi(\Delta,\eta)$ induces a factorization

$$
\begin{tikzcd}
\Gamma.\,\Pi(\Delta,\Psi) \arrow[drr, bend left] \arrow[dr, dotted] \arrow[ddr, equal, bend right] & & \\
& \Gamma.\,\Pi(\Delta,\Psi).(1_{\Psi(\Delta,\Psi)}, \Gamma.\,\Pi(\Delta,(1_\Psi, fg_1))^* \mathsf{Id}_{\Pi(\Delta,\Psi)} \arrow[r] \arrow[d, "p_{(1_{\Psi(\Delta,\Psi)}, \Gamma.\,\Pi(\Delta,(1_\Psi, fg_1))^* \mathsf{Id}_{\Pi(\Delta,\Psi)})}"] & \Gamma.\,\Pi(\Delta,\Psi).\,\Pi(\Delta,\Psi).\,\mathsf{Id}_{\Pi(\Delta,\Psi)} \arrow[d, "p_{\mathsf{Id}(\Delta,\Psi)}"] \\
& \Gamma.\,\Pi(\Delta,\Psi) \arrow[r, "(1_{\Psi(\Delta,\Psi)}, \Gamma.\,\Pi(\Delta,(1_\Psi, fg_1)))"'] & \Gamma.\,\Pi(\Delta,\Psi).\,\Pi(\Delta,\Psi)
\end{tikzcd}
,
$$

which is what we needed. □

Lemma 4.2.3 ([Kap17])**.** In the above conditions, the functor $(p_\Delta)_*\colon \mathsf{C}(\Gamma.\Delta) \to \mathsf{C}(\Gamma)$ is left exact.

Proof. As a right adjoint, $(p_\Delta)_*$ preserves limits and in particular pullbacks and the terminal object. Also, by Lemma 2.2.2, it preserves dependent projections and Lemma 4.2.2 tells us that the same goes for weak equivalences, which concludes the proof. □

Again, the above extends to all fibrations in C and it allows to prove our desired result.

Theorem 4.2.4 ([Kap17])**.** The ∞-category $L(\mathsf{C})$ is locally cartesian closed.

Proof. We already know that it is finitely complete by 4.1.14. The hypothesis of Theorem 3.2.31 are satisfied by Lemma 4.2.1 and Lemma 4.2.3. □

4.3 Conclusions

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