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# On Fibration Categories and Finitely Complete $(\infty, 1)$ -categories

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## 1 This is [not] an outline

The goal of this thesis is to redefine the functor:  $F: N(\mathbf{FibCat}) \rightarrow \mathbf{Lex}_\infty$ , from the nerve of the homotopical category of fibration categories to  $\infty$ -category of finitely complete  $\infty$ -categories and left exact functors, and prove that it induces an equivalence after localizing the domain at its weak equivalences.

The functor  $F$  is simply defined as the composite of the full embedding  $\iota: N(\mathbf{FibCat}) \rightarrow \mathbf{FibCat}_\infty$ , induced by the nerve functor, (should we instead work with a stronger notion of fibration  $\infty$ -category? also we need to define this  $\infty$ -category) and the functor  $\mathbf{Ho}_\infty: \mathbf{FibCat}_\infty \rightarrow \mathbf{Lex}_\infty$  given by localizing fibration  $\infty$ -categories. We know by [Cis19, Thm. 7.5.6] that localizing a fibration  $\infty$ -category gives a finitely complete  $\infty$ -category and the mapping on the maps is specified by the universal property of localizations. A map between fibration  $\infty$ -categories is a left exact functor as defined in [Cis19, Def. 7.5.2] and [Cis19, Thm. 7.5.28] guarantees that it is indeed sent to a left exact functor, as needed.

To prove that  $F$  induces the desired equivalence of  $\infty$ -categories we define a (weak) fibrational structure on  $\mathbf{FibCat}$  and  $\mathbf{FibCat}_\infty$  and verify that the right derived functors associated to  $\iota$  and  $\mathbf{Ho}_\infty$  are equivalences.

We already have notions of weak equivalence and fibration internal to  $\mathbf{FibCat}$  as provided in [KS19], so we only need to specify ones internal to  $\mathbf{FibCat}_\infty$  satisfying the definition of (weak) fibration category given in [Cis19, Def. 7.4.12] (PLEASE LIST THE PROPERTIES). To do so, we try to extend the definition given in the 1-categorical context, so that the functor  $\iota$  will preserve the desired structure. Hopefully we can just take the same definitions, without adaptations.

To show that the right derived functor of  $\mathbf{Ho}_\infty$  induces an equivalence we remember that a finitely complete  $\infty$ -category has a canonical fibrational structure, where weak equivalences are the isomorphisms and fibrations are all the maps. Also, under this construction a left exact functor between the induced fibration  $\infty$ -categories is just a left exact functor in the usual sense, so we can fully embed  $\mathbf{Lex}_\infty$  into  $\mathbf{FibCat}_\infty$ . This should define a right adjoint to  $\mathbf{Ho}_\infty$ , where the counit is the identity and the unit is given by the localization maps  $\mathcal{P} \rightarrow L(\mathcal{P})$ . We observe that such maps are weak equivalences

in  $\mathbf{FibCat}_\infty$ , so localizing this  $\infty$ -category gives us an equivalence  $L(\mathbf{FibCat}_\infty) \cong \mathbf{Lex}_\infty$  involving the right derived functor of  $\mathbf{Ho}_\infty$ , as we desired.

Once we have done this, we need to check that  $\iota$  is left exact, i.e. it satisfies [Cis19, Def. 7.5.2] (PLEASE LIST THE PROPERTIES), for which we only need condition (iii). It should be easy to check because it is induced by the nerve functor.

To finally apply [Cis19, Thm. 7.6.15] to  $\iota$  we still need to check two more conditions (PLEASE LIST THEM). Notice that (a) is trivial, so we actually only need to check (b).

## 2 This is [not] a fibration

One piece of the issue is to define a fibration  $\infty$ -category structure on  $\mathbf{FibCat}_\infty$  and to do so we need to first provide the definitions of the elements at play.

**Definition 2.1.** A *fibration category* is a category  $\mathcal{P}$  with a subcategory whose morphisms are called *weak equivalences* (denoted by  $\xrightarrow{\sim}$ ) and one of *fibrations* (denoted by  $\rightarrow$ ) subject to the following axioms:

1.  $\mathcal{P}$  has a terminal object 1 and all objects are fibrant, that is every map to the terminal object is a fibration;
2. pullbacks along fibrations exist in  $\mathcal{P}$  and (acyclic) fibrations are stable under pull-back;
3. every morphism in  $\mathcal{P}$  factors as a weak equivalence followed by a fibration;
4. weak equivalences satisfy the 2-out-of-6 property.

**Definition 2.2.** A functor between fibration categories is *left exact* if it preserves weak equivalences, fibrations, terminal objects and pullbacks along fibrations.

Notice that this corresponds to the definition of *exact functor* given in [KS19, Def. 2.7], however it also coincides with the one given in the more general context of fibration  $\infty$ -categories in [Cis19, Def. 7.5.2] (whose results we rely on), therefore to avoid confusion I adopted the naming given in the latter.

The category  $\mathbf{FibCat}$  is then constructed by taking small fibration categories as objects and left exact functors as arrows.

**Definition 2.3.** A morphism in  $\mathbf{FibCat}$  is a *weak equivalence* if it induces an equivalence of categories on the associated homotopy categories.

**Definition 2.4.** A morphism  $P: \mathcal{E} \rightarrow \mathcal{D}$  in  $\mathbf{FibCat}$  is a *fibration* if:

1. it is an isofibration;
2. it lifts weak factorizations;
3. it lifts pseudo-factorizations.

The three conditions and the map being an isofibration are necessary to ensure that a fibration of fibration categories is sent to a fibration in  $\mathbf{Lex}_\infty$ .

Notice that this is just taken from [KS19, Def. 4.3] and, under our definition, every map to the terminal fibration category is a fibration.

**Proposition 2.5.** These classes of maps specify a fibration category structure on  $\mathbf{FibCat}$ .

*Proof.* Check [KS19].  $\square$

We want to switch to the  $\infty$ -categorical context, so let's introduce the corresponding notions.

**Definition 2.6.** A fibration  $\infty$ -category is a triple  $(\mathcal{C}, W, \mathit{Fib})$  where  $W \subset \mathcal{C}$  is a sub- $\infty$ -category with the 2-out-of-3 property and  $\mathit{Fib}$  is a class of fibrations such that:

1. for any pullback square

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ p' \downarrow & & \downarrow p \\ y' & \xrightarrow{v} & y \end{array}$$

in  $\mathcal{C}$  in which  $p$  is a fibration between fibrant objects, if  $p \in W$  then  $p' \in W$ ;

2. given a map  $f: x \rightarrow y$  in  $\mathcal{C}$  with  $y$  fibrant, there exists a map  $w: x \rightarrow x'$  in  $W$  and a fibration  $p: x' \rightarrow y$  that  $f = p \cdot w$ .

The maps in  $W$  are generally called *weak equivalences*, while fibrations which are also in  $W$  are *trivial fibrations*.

**Definition 2.7.** Let  $\mathcal{C}, \mathcal{D}$  be fibration  $\infty$ -categories. A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is *left exact* if it has the following properties:

1. it preserves terminal objects;
2. it preserves fibrations and trivial fibrations;
3. given a pullback square

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ p' \downarrow & & \downarrow p \\ y' & \xrightarrow{v} & y \end{array}$$

in  $\mathcal{C}$  where  $p$  is a fibration and  $y$  and  $y'$  are fibrant, the square

$$\begin{array}{ccc} Fx' & \xrightarrow{Fu} & Fx \\ Fp' \downarrow & & \downarrow Fp \\ Fy' & \xrightarrow{Fv} & Fy \end{array}$$

is a pullback in  $\mathcal{D}$ .

Let's then consider  $\mathbf{FibCat}_\infty$ , the  $\infty$ -category of small fibration  $\infty$ -categories and left exact functors. We can take the same definition of fibration for  $\mathbf{FibCat}_\infty$  and use the following one for weak equivalences, which then should (CHECK!) make it a fibration  $\infty$ -category.

**Definition 2.8.** A left exact functor between fibration  $\infty$ -categories  $F: \mathcal{P} \rightarrow \mathcal{Q}$  is a weak equivalence if it induces an equivalence  $ho(L(\mathcal{P})) \rightarrow ho(L(\mathcal{Q}))$ . Equivalently, if it satisfies the conditions of [Cis19, Thm. 7.6.15].

**Proposition 2.9.** (DESIDERATA) The  $\infty$ -category  $\mathbf{FibCat}_\infty$  with the specified classes of maps is a fibration  $\infty$ -category.

*Proof.* The problem is showing the factorization condition (SHOW IT). Fibrations should be stable under pullback (which are computed as in  $\mathbf{sSet}$  if we manage to describe  $\mathbf{FibCat}_\infty$  as a full subcategory of functors with codomain in  $\mathbf{Cat}_\infty$ ) and the same applies to trivial fibrations, with the proof being the same as the one given in the 1-categorical context. Every map to the terminal fibration  $\infty$ -category is naturally a fibration.  $\square$

### 3 This is [not] an adjunction

In this section we provide a right adjoint to the localization functor  $\mathbf{Ho}_\infty$ .

**Remark 3.1.** Given a finitely complete  $\infty$ -category, we can provide a fibrational structure on it by defining weak equivalences to be the isomorphisms and fibrations to be all of the maps. Under this construction we see that, given two finitely complete  $\infty$ -categories, a functor between the induced fibration  $\infty$ -categories is left exact if and only if it is left exact when seen as a functor between the underlying  $\infty$ -categories.

**Proposition 3.2.** There is an adjunction  $\mathbf{Ho}_\infty: \mathbf{FibCat}_\infty \rightleftarrows \mathbf{Lex}_\infty: i$ , where  $i$  is a full embedding, the unit is given by the localization maps  $\gamma_{\mathcal{P}}: \mathcal{P} \rightarrow L(\mathcal{P})$  and the counit is the identity. This exhibits  $\mathbf{Lex}_\infty$  as a reflexive sub- $\infty$ -category of  $\mathbf{FibCat}_\infty$ .

*Proof.* Let's consider for any finitely complete  $\infty$ -category  $\mathcal{C}$  the pullback diagram

$$\begin{array}{ccc} \mathbf{Ho}_\infty / \mathcal{C} & \longrightarrow & \mathbf{Lex}_\infty / \mathcal{C} \\ \downarrow & & \downarrow \\ \mathbf{FibCat}_\infty & \xrightarrow{\mathbf{Ho}_\infty} & \mathbf{Lex}_\infty \end{array} .$$

We see that  $\mathbf{Ho}_\infty / \mathcal{C}$  has a terminal object, namely the pair  $(\mathcal{C}, \text{id}_{\mathcal{C}})$ , where  $\mathcal{C}$  has the fibrational structure specified above: indeed, given any other object  $(\mathcal{P}, F)$ , the space of maps  $(\mathbf{Ho}_\infty / \mathcal{C})((\mathcal{P}, F), (\mathcal{C}, \text{id}_{\mathcal{C}}))$  is contractible as it corresponds to the fiber at  $F$  of the equivalence  $\mathbf{FibCat}_\infty(\mathcal{P}, \mathcal{C}) \cong \mathbf{Lex}_\infty(L(\mathcal{P}), \mathcal{C})$  given by [Cis19, Prop. 7.5.11].

Unravelling everything, the constructed right adjoint is then the full embedding mapping everything in  $\mathbf{Lex}_\infty$  to its copy in  $\mathbf{FibCat}_\infty$ .  $\square$

Given this adjunction we present the following result, which reduces the problem of proving that the functor we constructed initially between  $L(\mathbf{FibCat})$  and  $\mathbf{Lex}_\infty$  is an equivalence to showing it for the right derived functor of  $\iota$ .

**Proposition 3.3.** The right derived functor of  $\mathbf{Ho}_\infty$  is an equivalence of  $\infty$ -categories  $L(\mathbf{FibCat}_\infty) \cong \mathbf{Lex}_\infty$ .

*Proof.* By 3.2 we know that  $\mathbf{Ho}_\infty$  has a fully faithful right adjoint, so by [Cis19, Prop. 7.1.18] it is enough to show that a map in  $\mathbf{FibCat}_\infty$  is mapped by  $\mathbf{Ho}_\infty$  to an isomorphism if and only if it is a weak equivalence. By [Cis19, Prop. 7.6.11] a left exact functor between fibration  $\infty$ -categories induces an equivalences on the localizations if and only if it does so on the homotopy categories, which is also the condition under which it is a weak equivalence.  $\square$

## 4 This is [not] a $\infty$ -category

Our approach relies on defining a  $\infty$ -category  $\mathbf{FibCat}_\infty$ . How to do this?

1- Construct a  $\infty$ -category where objects are pairs of functors (specifying weak equivalences and fibrations) with the same codomain between  $\infty$ -categories, that is  $\mathbf{Fun}(\{0 \rightarrow 01 \leftarrow 1\}, \mathbf{Cat}_\infty)$ . Then the morphisms are triples of functors making the proper diagram commute. We take the full subcategory of objects satisfying the axioms of a  $\infty$ -category with weak equivalences and fibrations, but how can we ask for pullbacks along fibrations to be preserved by the maps in this  $\infty$ -category? We also have to show that the hom-spaces are what we want. How to show then that a pullback along a fibration is a fibration? It may not be immediate because we are considering only triples of functors which make the diagrams commute as morphisms!

What if we take pairs of functors  $\coprod_W \Delta^1 \rightarrow \mathcal{D}$ ,  $Fib \rightarrow \mathcal{C}^{[1]} \rightarrow \mathcal{C}$ , where the composite is a Cartesian fibration and  $Fib \rightarrow \mathcal{C}^{[1]}$  preserves Cartesian morphisms? Essentially this is the definition of comprehension category, plus some extra data given by the first functor. We call fibrations the maps in the image of  $Fib \rightarrow \mathcal{C}$  and their compositions, while weak equivalences are the maps identified by the first functor; we then ask for the axioms of fibration  $\infty$ -category to be satisfied. This gives us more freedom when specifying weak equivalences then just providing the definition we give when constructing a tribe from a comprehension category. Also, in this way weak equivalences are preserved.

Problem: maps between such objects may not be what we want! Namely, I want the functor  $\mathcal{C}^{[1]} \rightarrow \mathcal{D}^{[1]}$  to be induced by post-composing with  $\mathcal{C} \rightarrow \mathcal{D}$ , but I don't know how to specify this. This (with the condition of  $Fib \rightarrow \mathcal{C}$  being a cartesian fibration and the factorization  $Fib \rightarrow \mathcal{C}^{[1]} \rightarrow \mathcal{C}$  preserving cartesian morphisms) would guarantee that pullbacks along fibrations are preserved by maps between fibration categories.

2- Take the 1-category of  $\infty$ -categories with fibrations and weak equivalences, then localize at categorical equivalences. What do the hom-spaces look like? If instead we localized directly at weak equivalences we would get directly  $L(\mathbf{FibCat}_\infty)$ , which supposedly is the “true”  $\infty$ -category of  $\infty$ -categories with fibrations and weak equivalences.

It's important to relate the hom-spaces  $\mathbf{FibCat}_\infty(\mathcal{P}, \mathcal{Q})$  to the ones we want, that is  $k(\mathbf{Fun}_{lex}(\mathcal{P}, \mathcal{Q}))$ . Also, what are the ones of  $\mathbf{Lex}_\infty$  like? Supposedly  $k(\mathbf{Fun}_{lex}(\mathcal{C}, \mathcal{D}))$  and we may show both claims using [Cis19, Prop. 7.10.6], but I do not think it would be easy.

## 5 Other ideas

1- We could also try to get an indirect comparison by looking at the inclusion of the category of fibration categories and of the category of finitely complete  $\infty$ -categories into the one of  $\infty$ -categories with weak equivalences and fibrations. We then give to the latter the structure of a fibrations category by taking the definition above. The axioms should be easy to show because a pullback along a fibration will simply be a pullback in  $\mathbf{sSet}$  and then we can give it the desired structure as in the context of fibration categories. Problem: showing the factorization condition.

We have then to show that the inclusion functors are left exact as functors between fibration categories. This means that the pullback of left exact functors between finitely complete  $\infty$ -categories is again a finitely complete  $\infty$ -category, which is proven in [Szu17, Lem. 2.11].

After this we have to show that the hypothesis of [Cis19, Thm. 7.6.15] hold. This is easy for (i), but for (ii) it's hard when it comes to the inclusion of fibration categories into  $\infty$ -categories with fibrations and weak equivalences. Namely, what if the domain of the morphism is a  $\infty$ -category? We have to find a span linking it to a weakly equivalent fibration category!

2- We can work with the 1-category of fibration  $\infty$ -categories and left exact functors and give it a fibrational structure. Problems: how do we show that after localizing a fibration of fibration categories we get an inner fibration? That is needed for the localization functor to be left exact and maybe we can use the ideas from [Szu17, Prop. 3.5]. Also, how to show condition (ii) of [Cis19, Thm. 7.6.15] for the inclusion of fibration categories into fibration  $\infty$ -categories? We need to approximate a map from a fibration  $\infty$ -category to a fibration category by one between fibration categories, the same problem as before. Everything else should be easy.

3- Starting from the fibration categories  $\mathbf{FibCat}$  and  $\mathbf{Lex}_\infty$  and the functor  $\mathbf{Ho}_\infty$  between them we can show the approximation properties of [Cis19, Thm. 7.6.15]. The problems are the same as in (2), that is how can we prove that localizing a fibration in  $\mathbf{FibCat}$  gives a fibration in  $\mathbf{Lex}_\infty$ ? Also, the approximation property (i) is easy, while for (ii) given a morphism  $\mathcal{C} \rightarrow L(\mathcal{P})$ , we may try to consider a finitely complete « $\lll$ » < HEAD category  $(\mathbf{sSet}_0/\mathcal{C})^{\text{op}}$ , whose fibrational structure is the dual of the cofibrational one specified in [Szu17, Def. 4.1], specifically the fibrations are given by monomorphisms and weak equivalences by maps sent to isomorphisms under the functor  $\lim: (\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow \mathcal{C}$ ; the proof that this is indeed a fibration category is the same as the one concerning  $\mathbf{sSet}_\kappa/\mathcal{C}$  provided in [Szu17, Prop. 4.2]. If the map  $\widetilde{\lim}: L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow \mathcal{C}$  is a weak equivalence (check the following proposition!), then we get the desired commutative square by looking at ===== category  $\mathbf{sSet}_0/\mathcal{C}$ , whose

fibrational structure is essentially the dual of the one specified in [Szu17, Def. 4.1], specifically the fibrations are given by monomorphisms and weak equivalences by maps sent to isomorphisms under the functor  $\lim: (\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow \mathcal{C}$ ; the proof that this is indeed a fibration category is the same as the one concerning  $\mathbf{sSet}_\kappa/\mathcal{C}$  provided in [Szu17, Prop. 4.2]. If the map  $\widetilde{\lim}: L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow \mathcal{C}$  is a weak equivalence, then we get the desired commutative square by looking at  $\gggg> 1ec446aacab0c480b70982ec08fc5c331a7c94e5$

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & L(\mathcal{P}) \\ \widetilde{\lim} \uparrow & & \uparrow \\ L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} & \xlongequal{\quad} & L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \end{array}$$

and somehow lift the map  $L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow L(\mathcal{P})$  through  $L$ .  $\llll< \text{HEAD}$

**Proposition 5.1.** The map  $\widetilde{\lim}$  is a categorical equivalence.

*Proof.* We shall verify that the hypothesis of [Cis19, Prop. 7.6.15] are satisfied.

First of all, the functor  $\lim$  preserves finite limits as they commute with finite limits, therefore it is left exact. Also, a morphism in  $(\mathbf{sSet}_0/\mathcal{C})^{\text{op}}$  is a weak equivalence if and only if it is mapped to an isomorphism, thus condition (i) is trivially satisfied.

We are only missing condition (ii), so let's consider a morphism  $\phi: c \rightarrow \lim_K D$  in  $\mathcal{C}$ . This corresponds to an object  $(c, \hat{\phi})$  in  $\mathcal{C}/D$ , that is a map  $\hat{\phi}: \Delta^0 * K \rightarrow \mathcal{C}$  such that  $\hat{\phi}|_K = D$ ,  $\hat{\phi}|_{\Delta^0} = c$ . We have then a map  $D \rightarrow \hat{\phi}$  in  $\mathbf{sSet}_0/\mathcal{C}$  given by the inclusion  $K \rightarrow \Delta^0 * K$  and inducing the morphism  $\phi: c = \lim_{\Delta^0 * K} \hat{\phi} \rightarrow \lim_K D$ , which allows us to construct the desired commutative square.  $\square$

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We may also specify a weak equivalence  $\mathcal{C} \rightarrow L(\mathbf{sSet}_0/\mathcal{C})$  as in [Szu17, Def. 4.6] (I don't really see how to adapt this). We extend  $\mathcal{C} \rightarrow L(\mathcal{P})$  first to a functor  $\mathbf{sSet}_0/\mathcal{C} \rightarrow L(\mathcal{P})$ ,  $D: K \rightarrow \mathcal{C} \mapsto \lim_K D$  (doubtful about doing this) and then localize to get  $L(\mathbf{sSet}_0/\mathcal{C}) \rightarrow L(\mathcal{P})$ , granting us a diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & L(\mathcal{P}) \\ \parallel & & \uparrow \\ \mathcal{C} & \longrightarrow & L(\mathbf{sSet}_0/\mathcal{C}) \end{array} .$$

## 6 Plan B: LCCC

We can also rewrite the paper by Kapulkin about LCCC arising from TT using the language of localizations of quasi-categories. There they develop the relevant theory showing that under some conditions the frame associated to a fibration category is locally cartesian closed, but using Cisinski's results we can prove the same theorem directly using a more mainstream theory.

What should be included in such an overview?

- 1- Cisinski's theory of localizations (of fibration  $\infty$ -categories)
- 2- an introduction to contextual categories: where do they come from? Why are they useful? Check out Voevodsky's papers about C-systems

## References

- [Cis19] D. Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2019. ISBN: 9781108473200. URL: <https://books.google.it/books?id=RawqvQEACAAJ> (cit. on pp. 1, 2, 4–7).
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