

Localizations of Models of Dependent Type Theory

Author: Matteo Durante

Advisor: Hoang-Kim Nguyen

Regensburg University

July 5, 2022

A modern proof of the following theorem.

Theorem (Kapulkin 2015)

Given a categorical model of type theory C , its ∞ -categorical localization $L(C)$ is a locally cartesian closed ∞ -category.

Dependent Type Theory

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A theory of computations and a foundation of mathematics.

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Dependent types A over contexts Γ and their terms $x : A$.

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Structural rules

How to work with *variables*.

Logical rules

Construct new types and their terms from old, carry out computations. They provide Σ -types $\Sigma(A, B)$, Π -types $\Pi(A, B)$, Id-types Id_A , *natural-numbers-type* Nat . . .

Idea

To reason about a theory we can look at its interpretations.

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Problem

Providing a model of Dependent Type Theory is hard.

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Solution

Defining a class of algebraic models.

Modeling structural rules

Definition (contextual categories)

A category \mathcal{C} with:

- 1 a grading on objects (or *contexts*) $\text{Ob } \mathcal{C} = \coprod_{n \in \mathbb{N}} \text{Ob}_n \mathcal{C}$;
- 2 a unique and terminal object in $\text{Ob}_0 \mathcal{C}$, the *empty context*;
- 3 a map $\text{ft}_n: \text{Ob}_{n+1} \mathcal{C} \rightarrow \text{Ob}_n \mathcal{C}$ for each $n \in \mathbb{N}$;
- 4 *basic dependent projections* $p_A: \Gamma.A \rightarrow \text{ft}_n(\Gamma.A) = \Gamma$;
- 5 a functorial choice of pullback squares

$$\begin{array}{ccc} \Delta.f^*A & \xrightarrow{q(f,A)} & \Gamma.A \\ p_{f^*A} \downarrow & & \downarrow p_A \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

Modeling logical rules

Extra structure

Id-types require from $\Gamma.A$ an Id-object $\Gamma.A.A. \text{Id}_A \dots$

Π -types require from $\Gamma.A.B$ a Π -object $\Gamma.\Pi(A, B)$, an evaluation map $\text{app}_{A,B} : \Gamma.\Pi(A, B).A \rightarrow \Gamma.A.B$, $(f, a) \mapsto (a, \text{app}(f, a)) \dots$

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Definition

A categorical model of type theory is a contextual category \mathcal{C} with Σ , Id and Π structures.

Definition (bi-invertible map)

A map $f: \Gamma \rightarrow \Delta$ in a contextual category with Id -structure \mathcal{C} for which we can provide:

- ① maps $g_1: \Delta \rightarrow \Gamma$, $\eta: \Gamma \rightarrow \Gamma.(1_\Gamma, g_1 \cdot f)^* \text{Id}_\Gamma$;
- ② maps $g_2: \Delta \rightarrow \Gamma$, $\epsilon: \Delta \rightarrow \Delta.(1_\Delta, f \cdot g_2)^* \text{Id}_\Delta$.

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Question

What if we localize at bi-invertible maps?

Definition (∞ -categories with weak equivalences and fibrations)

A triple $(\mathcal{C}, W, \text{Fib})$ where:

...a weakening of the definition of fibration categories, with \mathcal{C} an ∞ -category.

Fibrational structure

Definition (∞ -categories with weak equivalences and fibrations)

A triple $(\mathcal{C}, W, \text{Fib})$ where:

...a weakening of the definition of fibration categories, with \mathcal{C} an ∞ -category.

Theorem (Avigad-Kapulkin-Lumsdaine 2013)

A contextual category with Σ and Id structures defines a fibration category, where weak equivalences are bi-invertible maps and fibrations are maps isomorphic to compositions of basic dependent projections $p_A: \Gamma.A \rightarrow \Gamma$.

In particular, it applies to categorical models of type theory.

Localizing fibrational categories

Proposition (Cisinski)

The localization at weak equivalences of an ∞ -category with weak equivalences and fibrations \mathcal{C} is a finitely complete ∞ -category.

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Given an ∞ -category with weak equivalences and fibrations \mathcal{C} , if for every fibration $f: x \rightarrow y$ between fibrant objects the pullback functor between fibrant slices $f^: \mathcal{C}(y) \rightarrow \mathcal{C}(x)$ has a right adjoint preserving trivial fibrations, then $L(\mathcal{C})$ is locally cartesian closed.*

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Proof.

For any basic dependent projection $p_A: \Gamma.A \rightarrow \Gamma$, there exists a right adjoint to $p_A^*: C(\Gamma) \rightarrow C(\Gamma.A)$ given by

$$(p_A)_*(\Gamma.A.\Theta) = \Gamma.\Pi(A, \Theta)$$

with counit induced by $\text{app}_{A,\Theta}$. It preserves the fibrational structure. □

Thank you for your attention!

Why is Dependent Type Theory cool?

- 1 Closely linked to *computations* and *computer science*, makes proof assistants possible
- 2 Enough by itself as a foundation, unlike Set Theory or Propositional Calculus
- 3 *Proofs* are internal objects
- 4 Better treatment of *equality*
- 5 Makes “fully faithful + essentially surjective = equivalence” independent from the Axiom of Choice
- 6 Homotopical interpretation in ∞ -groupoids

Internal Languages Conjecture

Conjecture (Kapulkin-Lumsdaine 2016)

The horizontal maps, given by simplicial localization, induce equivalences of ∞ -categories.

$$\begin{array}{ccc} \mathrm{CxlCat}_{\Sigma,1,\mathrm{Id},\Pi} & \longrightarrow & \mathrm{LCCC}_{\infty} \\ \downarrow & & \downarrow \\ \mathrm{CxlCat}_{\Sigma,1,\mathrm{Id}} & \longrightarrow & \mathrm{Lex}_{\infty} \end{array}$$

A proof by Nguyen-Uemura has recently become available on arxiv. One hopes to extend this to an equivalence between $\mathrm{CxlCat}_{\mathrm{HoTT}}$ and $\mathrm{ElTopos}_{\infty}$.