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On Fibration Categories and Finitely Complete $(\infty, 1)$-categories

1 This is [not] an outline

The goal of this thesis is to redefine the functor: $F\colon N(\mathsf{FibCat}) \to \mathsf{Lex}_\infty$, from the nerve of the homotopical category of fibration categories to ∞-category of finitely complete ∞-categories and left exact functors, and prove that it induces an equivalence after localizing the domain at its weak equivalences.

The functor F is simply defined as the composite of the full embedding $\iota\colon N(\mathsf{FibCat}) \to \mathsf{FibCat}_\infty$, induced by the nerve functor, (should we instead work with a stronger notion of fibration ∞-category? also we need to define this ∞-category) and the functor $\mathbf{Ho}_\infty\colon \mathsf{FibCat}_\infty \to \mathsf{Lex}_\infty$ given by localizing fibration ∞-categories. We know by [Cis19, Thm. 7.5.6] that localizing a fibration ∞-category gives a finitely complete ∞-category and the mapping on the maps is specified by the universal property of localizations. A map between fibration ∞-categories is a left exact functor as defined in [Cis19, Def. 7.5.2] and [Cis19, Thm. 7.5.28] guarantees that it is indeed sent to a left exact functor, as needed.

To prove that F induces the desired equivalence of ∞-categories we define a (weak) fibrational structure on FibCat and FibCat_∞ and verify that the right derived functors associated to ι and $\mathbf{Ho}_\infty$ are equivalences.

We already have notions of weak equivalence and fibration internal to FibCat as provided in [KS19], so we only need to specify ones internal to FibCat_∞ satisfying the definition of (weak) fibration category given in [Cis19, Def. 7.4.12] (PLEASE LIST THE PROPERTIES). To do so, we try to extend the definition given in the 1-categorical context, so that the functor ι will preserve the desired structure. Hopefully we can just take the same definitions, without adaptations.

To show that the right derived functor of $\mathbf{Ho}_\infty$ induces an equivalence we remember that a finitely complete ∞-category has a canonical fibrational structure, where weak equivalences are the isomorphisms and fibrations are all the maps. Also, under this construction a left exact functor between the induced fibration ∞-categories is just a left exact functor in the usual sense, so we can fully embed Lex_∞ into FibCat_∞. This should define a right adjoint to $\mathbf{Ho}_\infty$, where the counit is the identity and the unit is given by the localization maps $\mathcal{P} \to L(\mathcal{P})$. We observe that such maps are weak equivalences in FibCat_∞, so localizing this ∞-category gives us an equivalence $L(\mathsf{FibCat}_\infty) \cong \mathsf{Lex}_\infty$ involving the right derived functor of $\mathbf{Ho}_\infty$, as we desired.

Once we have done this, we need to check that ι is left exact, i.e. it satisfies [Cis19, Def. 7.5.2] (PLEASE LIST THE PROPERTIES), for which we only need condition (iii). It should be easy to check because it is induced by the nerve functor.

To finally apply [Cis19, Thm. 7.6.15] to ι we still need to check two more conditions (PLEASE LIST THEM). Notice that (a) is trivial, so we actually only need to check (b).

2 This is [not] a fibration

One piece of the issue is to define a fibration ∞-category structure on FibCat_∞ and to do so we need to first provide the definitions of the elements at play.

Definition 2.1. A *fibration category* is a category $\mathcal{P}$ with a subcategory whose morphisms are called *weak equivalences* (denoted by $\xrightarrow{\sim}$) and one of *fibrations* (denoted by $\twoheadrightarrow$) subject to the following axioms:

1. $\mathcal{P}$ has a terminal object 1 and all objects are fibrant, that is every map to the terminal object is a fibration;
2. pullbacks along fibrations exist in $\mathcal{P}$ and (acyclic) fibrations are stable under pullback;
3. every morphism in $\mathcal{P}$ factors as a weak equivalence followed by a fibration;
4. weak equivalences satisfy the 2-out-of-6 property.

Definition 2.2. A functor between fibration categories is *left exact* if it preserves weak equivalences, fibrations, terminal objects and pullbacks along fibrations.

Notice that this corresponds to the definition of *exact functor* given in [KS19, Def. 2.7], however it also coincides with the one given in the more general context of fibration ∞-categories in [Cis19, Def. 7.5.2] (whose results we rely on), therefore to avoid conclusion I adopted the naming given in the latter.

The category FibCat is then constructed by taking small fibration categories as objects and left exact functors as arrows.

Definition 2.3. A morphism in FibCat is a *weak equivalence* if it induces an equivalence of categories on the associated homotopy categories.

Definition 2.4. A morphism $P\colon \mathcal{E} \to \mathcal{D}$ in FibCat is a *fibration* if:

1. it is an isofibration;
2. it lifts weak factorizations;
3. it lifts pseudo-factorizations.

The three conditions and the map being an isofibration are necessary to ensure that a fibration of fibration categories is sent to a fibration in Lex_∞.

Notice that this is just taken from [KS19, Def. 4.3] and, under our definition, every map to the terminal fibration category is a fibration.

Proposition 2.5. These classes of maps specify a fibration category structure on FibCat.

Proof. Check [KS19]. □

We want to switch to the ∞-categorical context, so let's introduce the corresponding notions.

Definition 2.6. A fibration ∞-category is a triple $(\mathcal{C}, W, Fib)$ where $W \subset \mathcal{C}$ is a sub-∞-category with the 2-out-of-3 property and Fib is a class of fibrations such that:

1. for any pullback square
$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "p'"'] & x \arrow[d, "p"] \\ y' \arrow[r, "v"'] & y \end{tikzcd}$$
in $\mathcal{C}$ in which p is a fibration between fibrant objects, if $p \in W$ then $p' \in W$;

2. given a map $f\colon x \to y$ in $\mathcal{C}$ with y fibrant, there exists a map $w\colon x \to x'$ in W and a fibration $p\colon x' \to y$ that $f = p \cdot w$.

The maps in W are generally called *weak equivalences*, while fibrations which are also in W are *trivial fibrations*.

Definition 2.7. Let $\mathcal{C}, \mathcal{D}$ be fibration ∞-categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *left exact* if it has the following properties:

1. it preserves terminal objects;

2. it preserves fibrations and trivial fibrations;

3. given a pullback square
$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "p'"'] & x \arrow[d, "p"] \\ y' \arrow[r, "v"'] & y \end{tikzcd}$$
in $\mathcal{C}$ where p is a fibration and y and y' are fibrant, the square
$$\begin{tikzcd} Fx' \arrow[r, "Fu"] \arrow[d, "Fp'"'] & Fx \arrow[d, "Fp"] \\ Fy' \arrow[r, "Fv"'] & Fy \end{tikzcd}$$
is a pullback in $\mathcal{D}$.

Let's then consider FibCat_∞, the ∞-category of small fibration ∞-categories and left exact functors. We can take the same definition of fibration for FibCat_∞ and use the following one for weak equivalences, which then should (CHECK!) make it a fibration ∞-category.

Definition 2.8. A left exact functor between fibration ∞-categories $F\colon \mathcal{P} \to \mathcal{Q}$ is a weak equivalence if it induces an equivalence $ho(L(\mathcal{P})) \to ho(L(\mathcal{Q}))$. Equivalently, if it satisfies the conditions of [Cis19, Thm. 7.6.15].

Proposition 2.9. (DESIDERATA) The ∞-category FibCat_∞ with the specified classes of maps is a fibration ∞-category.

Proof. The problem is showing the factorization condition (SHOW IT). Fibrations should be stable under pullback (which are computed as in **sSet** if we manage to describe FibCat_∞ as a full subcategory of functors with codomain in $\mathbf{Cat}_\infty$) and the same applies to trivial fibrations, with the proof being the same as the one given in the 1-categorical context. Every map to the terminal fibration ∞-category is naturally a fibration. $\square$

3 This is [not] an adjunction

In this section we provide a right adjoint to the localization functor $\mathbf{Ho}_\infty$.

Remark 3.1. Given a finitely complete ∞-category, we can provide a fibrational structure on it by defining weak equivalences to be the isomorphisms and fibrations to be all of the maps. Under this construction we see that, given two finitely complete ∞-categories, a functor between the induced fibration ∞-categories is left exact if and only if it is left exact when seen as a functor between the underlying ∞-categories.

Proposition 3.2. There is an adjunction $\mathbf{Ho}_\infty\colon \mathsf{FibCat}_\infty \rightleftarrows \mathsf{Lex}_\infty\colon i$, where i is a full embedding, the unit is given by the localization maps $\gamma_{\mathcal{P}}\colon \mathcal{P} \to L(\mathcal{P})$ and the counit is the identity. This exhibits Lex_∞ as a reflexive sub-∞-category of FibCat_∞.

Proof. Let's consider for any finitely complete ∞-category $\mathcal{C}$ the pullback diagram

$$\begin{tikzcd} \mathbf{Ho}_\infty/\mathcal{C} \arrow[r] \arrow[d] & \mathsf{Lex}_\infty/\mathcal{C} \arrow[d] \\ \mathsf{FibCat}_\infty \arrow[r, "\mathbf{Ho}_\infty"'] & \mathsf{Lex}_\infty \end{tikzcd}.$$

We see that $\mathbf{Ho}_\infty/\mathcal{C}$ has a terminal object, namely the pair $(\mathcal{C}, \mathrm{id}_{\mathcal{C}})$, where $\mathcal{C}$ has the fibrational structure specified above: indeed, given any other object $(\mathcal{P}, F)$, the space of maps $(\mathbf{Ho}_\infty/\mathcal{C})((\mathcal{P}, F),(\mathcal{C}, \mathrm{id}_{\mathcal{C}}))$ is contractible as it corresponds to the fiber at F of the equivalence $\mathsf{FibCat}_\infty(\mathcal{P}, \mathcal{C}) \cong \mathsf{Lex}_\infty(L(\mathcal{P}), \mathcal{C})$ given by [Cis19, Prop. 7.5.11].

Unravelling everything, the constructed right adjoint is then the full embedding mapping everything in Lex_∞ to its copy in FibCat_∞. $\square$

Given this adjunction we present the following result, which reduces the problem of proving that the functor we constructed initially between $L(\mathsf{FibCat})$ and Lex_∞ is an equivalence to showing it for the right derived functor of ι.

Proposition 3.3. The right derived functor of $\mathbf{Ho}_\infty$ is an equivalence of ∞-categories $L(\mathsf{FibCat}_\infty) \cong \mathsf{Lex}_\infty$.

Proof. By 3.2 we know that $\mathbf{Ho}_\infty$ has a fully faithful right adjoint, so by [Cis19, Prop. 7.1.18] it is enough to show that a map in FibCat_∞ is mapped by $\mathbf{Ho}_\infty$ to an isomorphism if and only if it is a weak equivalence. By [Cis19, Prop. 7.6.11] a left exact functor between fibration ∞-categories induces an equivalences on the localizations if and only if it does so on the homotopy categories, which is also the condition under which it is a weak equivalence. □

4 This is [not] a ∞-category

Our approach relies on defining a ∞-category FibCat_∞. How to do this?

1- Construct a ∞-category where objects are pairs of functors (specifying weak equivalences and fibrations) with the same codomain between ∞-categories, that is $\mathrm{Fun}(\{0 \to 01 \leftarrow 1\}, \mathbf{Cat}_\infty)$. Then the morphisms are triples of functors making the proper diagram commute. We take the full subcategory of objects satisfying the axioms of a ∞-category with weak equivalences and fibrations, but how can we ask for pullbacks along fibrations to be preserved by the maps in this ∞-category? We also have to show that the hom-spaces are what we want. How to show then that a pullback along a fibration is a fibration? It may not be immediate because we are considering only triples of functors which make the diagrams commute as morphisms!

What if we take pairs of functors $\coprod_W \Delta^1 \to \mathcal{D}$, $Fib \to \mathcal{C}^{[1]} \to \mathcal{C}$, where the composite is a Cartesian fibration and $Fib \to \mathcal{C}^{[1]}$ preserves Cartesian morphisms? Essentially this is the definition of comprehension category, plus some extra data given by the first functor. We call fibrations the maps in the image of $Fib \to \mathcal{C}$ and their compositions, while weak equivalences are the maps identified by the first functor; we then ask for the axioms of fibration ∞-category to be satisfied. This gives us more freedom when specifying weak equivalences then just providing the definition we give when constructing a tribe from a comprehension category. Also, in this way weak equivalences are preserved.

Problem: maps between such objects may not be what we want! Namely, I want the functor $\mathcal{C}^{[1]} \to \mathcal{D}^{[1]}$ to be induced by post-composing with $\mathcal{C} \to \mathcal{D}$, but I don't know how to specify this. This (with the condition of $Fib \to \mathcal{C}$ being a cartesian fibration and the factorization $Fib \to \mathcal{C}^{[1]} \to \mathcal{C}$ preserving cartesian morphisms) would guarantee that pullbacks along fibrations are preserved by maps between fibration categories.

2- Take the 1-category of ∞-categories with fibrations and weak equivalences, then localize at categorical equivalences. What do the hom-spaces look like? If instead we localized directly at weak equivalences we would get directly $L(\mathsf{FibCat}_\infty)$, which supposedly is the "true" ∞-category of ∞-categories with fibrations and weak equivalences.

It's important to relate the hom-spaces $\mathsf{FibCat}_\infty(\mathcal{P}, \mathcal{Q})$ to the ones we want, that is $k(\mathrm{Fun}_{lex}(\mathcal{P}, \mathcal{Q}))$. Also, what are the ones of Lex_∞ like? Supposedly $k(\mathrm{Fun}_{lex}(\mathcal{C}, \mathcal{D}))$ and we may show both claims using [Cis19, Prop. 7.10.6], but I do not think it would be easy.

5 Other ideas

1- We could also try to get an indirect comparison by looking at the inclusion of the category of fibration categories and of the category of finitely complete ∞-categories into the one of ∞-categories with weak equivalences and fibrations. We then give to the latter the structure of a fibrations category by taking the definition above. The axioms should be easy to show because a pullback along a fibration will simply be a pullback in **sSet** and then we can give it the desired structure as in the context of fibration categories. Problem: showing the factorization condition.

We have then to show that the inclusion functors are left exact as functors between fibration categories. This means that the pullback of left exact functors between finitely complete ∞-categories is again a finitely complete ∞-category, which is proven in [Szu17, Lem. 2.11].

After this we have to show that the hypothesis of [Cis19, Thm. 7.6.15] hold. This is easy for (i), but for (ii) it's hard when it comes to the inclusion of fibration categories into ∞-categories with fibrations and weak equivalences. Namely, what if the domain of the morphism is a ∞-category? We have to find a span linking it to a weakly equivalent fibration category!

2- We can work with the 1-category of fibration ∞-categories and left exact functors and give it a fibrational structure. Problems: how do we show that after localizing a fibration of fibration categories we get an inner fibration? That is needed for the localization functor to be left exact and maybe we can use the ideas from [Szu17, Prop. 3.5]. Also, how to show condition (ii) of [Cis19, Thm. 7.6.15] for the inclusion of fibration categories into fibration ∞-categories? We need to approximate a map from a fibration ∞-category to a fibration category by one between fibration categories, the same problem as before. Everything else should be easy.

3- Starting from the fibration categories FibCat and Lex_∞ and the functor $\mathbf{Ho}_\infty$ between them we can show the approximation properties of [Cis19, Thm. 7.6.15]. The problems are the same as in (2), that is how can we prove that localizing a fibration in FibCat gives a fibration in Lex_∞? Also, the approximation property (i) is easy, while for (ii) given a morphism $\mathcal{C} \to L(\mathcal{P})$, we may try to consider a finitely complete «««< HEAD category $(\mathbf{sSet}_0 /\mathcal{C})^{\mathrm{op}}$, whose fibrational structure is the dual of the cofibrational one specified in [Szu17, Def. 4.1], specifically the fibrations are given by monomorphisms and weak equivalences by maps sent to isomorphisms under the functor $\lim\colon (\mathbf{sSet}_0 /\mathcal{C})^{\mathrm{op}} \to \mathcal{C}$; the proof that this is indeed a fibration category is the same as the one concerning $\mathbf{sSet}_\kappa /\mathcal{C}$ provided in [Szu17, Prop. 4.2]. If the map $\widetilde{\lim}\colon L(\mathbf{sSet}_0 /\mathcal{C})^{\mathrm{op}} \to \mathcal{C}$ is a weak equivalence (check the following proposition!), then we get the desired commutative square by looking at the category $\mathbf{sSet}_0 /\mathcal{C}$, whose fibrational structure is essentially the dual of the one specified in [Szu17, Def. 4.1], specifically the fibrations are given by monomorphisms and weak equivalences by maps sent to isomorphisms under the functor $\lim\colon (\mathbf{sSet}_0\,/\mathcal{C})^{\mathrm{op}} \to \mathcal{C}$; the proof that this is indeed a fibration category is the same as the one concerning $\mathbf{sSet}_\kappa\,/\mathcal{C}$ provided in [Szu17, Prop. 4.2]. If the map $\widetilde{\lim}\colon L(\mathbf{sSet}_0\,/\mathcal{C})^{\mathrm{op}} \to \mathcal{C}$ is a weak equivalence, then we get the desired commutative square by looking at

$$\begin{tikzcd} \mathcal{C} \arrow[r] & L(\mathcal{P}) \\ L(\mathbf{sSet}_0\,/\mathcal{C})^{\mathrm{op}} \arrow[u, "\widetilde{\lim}"] \arrow[r, equal] & L(\mathbf{sSet}_0\,/\mathcal{C})^{\mathrm{op}} \arrow[u] \end{tikzcd}$$

and somehow lift the map $L(\mathbf{sSet}_0\,/\mathcal{C})^{\mathrm{op}} \to L(\mathcal{P})$ through L,

(I do not believe this lift to always exist. Indeed, take the ∞-groupoid $\mathcal{C} = N(\{0 \leftrightarrows 1\})$, $\mathcal{P} = [1]$ with the minimal fibrational structure making $0 \to 1$ a weak equivalence and the identity as the upper map. Given in $(\mathbf{sSet}\,/\mathcal{C})^{\mathrm{op}}$ the objects $0\colon \Delta^0 \to \mathcal{C}$, $(1 \to 0)\colon \Delta^1 \to \mathcal{C}$ and the obvious morphism f between them, we see that the lifted map $(\mathbf{sSet}\,/\mathcal{C})^{\mathrm{op}} \to \mathcal{P}$ would send f to a morphism with domain 1 and codomain 0, which does not exist.)

Proposition 5.1. The map $\widetilde{\lim}$ is a categorical equivalence.

Proof. We shall verify that the hypothesis of [Cis19, Prop. 7.6.15] are satisfied.

First of all, the functor lim preserves finite limits as they commute with finite limits, therefore it is left exact. Also, a morphism in $(\mathbf{sSet}_0\,/\mathcal{C})^{\mathrm{op}}$ is a weak equivalence if and only if it is mapped to an isomorphism, thus condition (i) is trivially satisfied.

We are only missing condition (ii), so let's consider a morphism $\phi\colon c \to \lim_K D$ in $\mathcal{C}$. This corresponds to an object $(c, \hat{\phi})$ in $\mathcal{C}/D$, that is a map $\hat{\phi}\colon \Delta^0 * K \to \mathcal{C}$ such that $\hat{\phi}|_K = D$, $\hat{\phi}|_{\Delta^0} = c$. We have then a map $D \to \hat{\phi}$ in $\mathbf{sSet}_0\,/\mathcal{C}$ given by the inclusion $K \to \Delta^0 * K$ and inducing the morphism $\phi\colon c = \lim_{\Delta^0 * K} \hat{\phi} \to \lim_K D$, which allows us to construct the desired commutative square. ☐

We may also specify a weak equivalence $\mathcal{C} \to L(\mathbf{sSet}_0\,/\mathcal{C})$ as in [Szu17, Def. 4.6] (I don't really see how to adapt this). We extend $\mathcal{C} \to L(\mathcal{P})$ first to a functor $\mathbf{sSet}_0\,/\mathcal{C} \to L(\mathcal{P})$, $D\colon K \to \mathcal{C} \mapsto \lim_K D$ (doubtful about doing this) and then localize to get $L(\mathbf{sSet}_0\,/\mathcal{C}) \to L(\mathcal{P})$, granting us a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, equal] & L(P) \\ \mathcal{C} \arrow[r] & L(\mathbf{sSet}_0\,/\mathcal{C}) \arrow[u] \end{tikzcd} .$$

6 Plan B: LCCC

We can also rewrite the paper by Kapulkin about LCCC arising from TT using the language of localizations of quasi-categories. There they develop the relevant theory showing that under some conditions the frame associated to a fibration category is locally cartesian closed, but using Cisinski's results we can prove the same theorem directly using a more mainstream theory.

What should be included in such an overview?

1- Cisinski's theory of localizations (of fibration ∞-categories)

2- an introduction to contextual categories: where do they come from? Why are they useful? Check out Voevodsky's papers about C-systems

We explain what dependent type theory is (Martin-Lof's notes from 1984) and why it's an interesting foundation of mathematics. We mention Homotopy Type Theory as an effort to provide homotopical foundations which better model how we think about identities, which explains why intensional identity types are more interesting to us than extensional ones.

We move on to defining contextual categories (1211.2851, 1406.7413, 1507.02648) and what the Pi, Sigma and Id structures are (1406.7413, 1211.2851 Appendix B). To understand what the link between such structures and syntactically presented type theories we refer to 1507.02648, Sec. 1.1, while the statement of the conjectured correspondence is in 1304.0680, Sec. 2.1.

Where does the link between dependent type theories and ∞-categories come from? We see that ∞-categories intuitively model the behavior of type theories and their type constructions, especially when considering Homotopy Type Theory, however this relation is known only partially (references in the intro of 1507.02648). The idea is that the type theory we are interested in should be the internal language of some class of ∞-categories and a precise statement would require us to provide homotopical functors in both directions which induce an equivalence on the associated ∞-categories. The idea is to construct the functor from contextual categories as a localization functor, that is we need to provide a homotopical structure on contextual categories, as they do in 1507.02648 (there should be an older reference) which then provides an associated ∞-category. This is the object of the Initiality Conjecture, stated in 1610.00037, in the hope that such a correspondence will extend to Homotopy Type Theory and some notion of Elementary Higher Toposes, perhaps the one specified in 1805.03805. At the moment we know that HoTT can be interpreted in Higher Toposes with some structure. Current progress: 1709.09519, an upcoming paper by Nguyen-Uemura (HoTTest talk).

Our aim is to show that when taking contextual categories with the structure we specified earlier we obtain a locally cartesian closed ∞-category. To do so we provide a fibrational structure on contextual categories (1304.0680, 1507.02648), which as we anticipate will imply that their simplicial localizations are finitely complete. We also prove that the hypothesis of [Cis19, Thm. 7.6.16] are satisfied, informing that this will be sufficient to prove Kapulkin's main result from 1507.02648.

We then develop the theory of localizations of ∞-categories by Cisinski and specifically develop the results concerning ∞-categories with fibrations and weak equivalences. Localizations of such ∞-categories are finitely complete. The objective is to show [Cis19, Thm. 7.6.16]. How in depth should we go?

Why all of this is interesting: we are proving Kapulkin's result internalizing all of the discussion within the language of ∞-category theory and relying only on its simplicial model.

7 Localizations of ∞-Categories

To prove the that localizing a categorical model of type theory we get a locally cartesian closed ∞-category we need a theory of localizations. We shall provide one in the general context of ∞-categories as developed by Cisinski in *Higher Categories and Homotopical Algebra* with the aim of proving [Cis19, Thm. 7.6.16], which will do the heavy lifting in showing the desired result. Those familiar with the theory may skip the entire chapter while keeping in mind yadda yadda (LIST THE MAJOR RESULTS).

Definition 7.1. Let X be a simplicial set and $W \subset X$ a simplicial subset. Given an ∞-category $\mathcal{C}$, we define $\underline{\mathrm{Hom}}_W(X, \mathcal{C})$ to be the full simplicial subset of $\underline{\mathrm{Hom}}(X, \mathcal{C})$ whose objects are the morphisms $f\colon X \to \mathcal{C}$ sending the 1-simplices in W to isomorphisms.

Remark 7.2. The above definition induces a canonical pullback square

$$\begin{tikzcd} \underline{\mathrm{Hom}}_W(X, \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(X, \mathcal{C}) \arrow[d] \\ \underline{\mathrm{Hom}}_W(W, \mathcal{C}) \arrow[r] & \underline{\mathrm{Hom}}(W, \mathcal{C}) \end{tikzcd}$$

given by the inclusion $W \to X$.

Definition 7.3. Given an ∞-category $\mathcal{C}$ and $W \subset \mathcal{C}$, a *localization of* $\mathcal{C}$ *by* W is a functor $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ such that:

1. $L(\mathcal{C})$ is an ∞-category;
2. γ sends the 1-simplices of W to isomorphisms in $L(\mathcal{C})$;
3. for any ∞-category $\mathcal{D}$ there is an equivalence of ∞-categories
$$\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$$
given by precomposing with γ.

(CISINSKI DOES NOT ASK FOR $\mathcal{C}$ TO BE AN ∞-CATEGORY. SHOULD WE BE LESS GENERAL AS WE HAVE DONE?)

Proposition 7.4. Given an ∞-category $\mathcal{C}$ and a subsimplicial set W, the localization of $\mathcal{C}$ by W always exists and it is essentially unique.

Proof. We begin by proving that a localization exists in the case where $W = \mathcal{C}$.

In this context, $\underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \cong \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}^{\cong})$ canonically, where $\mathcal{D}^{\cong}$ is the maximal sub-∞-subgroupoid of $\mathcal{D}$. Factoring $\mathcal{C} \to \Delta^0$ in the Kan model structure, we find an anodyne map $\gamma\colon \mathcal{C} \to L(\mathcal{C})$.

Remember that for any anodyne map $A \to B$ we get a trivial fibration $\underline{\mathrm{Hom}}(B, \mathcal{D}^{\cong}) \to \underline{\mathrm{Hom}}(A, \mathcal{D}^{\cong})$. Looking then at the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(L(\mathcal{C}), \mathcal{D}^{\cong}) \arrow[r, two heads, "\gamma^*", "\sim"'] \arrow[d, "\cong"'] & \mathrm{Hom}_W(\mathcal{C}, \mathcal{D}^{\cong}) \arrow[d, "\cong"] \\ \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \arrow[r, "\gamma^*"'] & \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \end{tikzcd},$$

by the 2-out-of-3 property we see that the lower γ^* is an equivalence.

We now move on to the general case. First of all, notice that as a particular case of the previous one we get that localizing Δ^1 at its non-trivial morphism we obtain $\Delta^1 \to J = L(\Delta^1) \sim \Delta^0$. Taking then $W \subset \mathcal{C}$, we consider the commutative diagram

$$\begin{tikzcd} \coprod_{f \in W_1} \Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \arrow[ddr, bend left, "\gamma"] & \\ \coprod_{f \in W_1} J \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{C}' \arrow[dr, hook, "\sim"'] & \\ & & L(\mathcal{C}) \end{tikzcd},$$

where $\mathcal{C}' \to L(\mathcal{C})$ is an inner anodyne map obtained by taking the fibrant replacement of $\mathcal{C}'$ in the Joyal model structure. This can be done functorially via the small object argument.

For any ∞-category $\mathcal{D}$, we get a trivial fibration $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D})$ and a pullback square

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \Pi_{f \in W_1} \underline{\mathrm{Hom}}(J, \mathcal{D}) \arrow[r] & \Pi_{f \in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \end{tikzcd},$$

which together with the pullback

$$\begin{tikzcd} \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \Pi_{f \in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}^{\cong}) \arrow[r] & \Pi_{f \in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \end{tikzcd}$$

implies by pasting that

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \Pi_{f \in W_1} \underline{\mathrm{Hom}}(J, \mathcal{D}) \arrow[r, two heads, "\sim"] & \Pi_{f \in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}^{\cong}) \end{tikzcd}$$

is also a pullback and therefore the upper arrow is a trivial fibration. Composing it with the other one we get $\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$, which is then a trivial fibration and therefore an equivalence of ∞-categories.

7.1.4 Observe that, through this construction, one can always construct $L(\mathcal{C})$ so that γ is a bijection on objects because $\mathcal{C}' \to L(\mathcal{C})$ is an inner anodyne extension and therefore a retract of a countable composition of sums of pushouts of maps which are the identity on objects, that is the inner horn inclusions.

We now move on to proving that the localization is essentially unique. For this, we notice that γ establishes then an isomorphism between $\pi_0(k(\underline{\mathrm{Hom}}_W(\mathcal{C}, -)))$ and $\pi_0(\underline{\mathrm{Hom}}(L(\mathcal{C}), -)) = ho(\mathbf{sSet})(L(\mathcal{C}), -)$ with respect to the Joyal model structure, thus by Yoneda $(L(\mathcal{C}), \gamma)$ is unique up to unique isomorphism in $ho(\mathbf{sSet})$ and up to a contractible space of equivalences in $\mathbf{sSet}$. □

Remark 7.5. 7.1.5

In this context, we may define $\overline{W}$, the saturation of W in $\mathcal{C}$, as the cartesian square such that $\overline{W}$ is precisely the maximal simplicial subset of $\mathcal{C}$ whose morphisms are the ones which become invertible in $L(\mathcal{C})$.

We have then inclusions $Sk_1(W) \subset W \subset \overline{W}$ and, for any ∞-category $\mathcal{D}$, this induces equalities

$$\underline{\mathrm{Hom}}_{Sk_1(W)}(\mathcal{C}, \mathcal{D}) = \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})\, \underline{\mathrm{Hom}}_{\overline{W}}(\mathcal{C}, \mathcal{D}),$$

implying that $(L(\mathcal{C}), \gamma)$ is also the localization of $\mathcal{C}$ by $Sk_1(W)$ and the one by $\overline{W}$, however the inclusion $W \to \mathcal{C}$ is a fibration in the Joyal model category as it is the pullback of one, implying that $\overline{W}$ is itself an ∞-category.

We shall say that W is saturated if $W = \overline{W}$.

Remark 7.6. 7.1.6

The functor $ho(\mathcal{C}) \to ho(L(\mathcal{C}))$ exibits $ho(L(\mathcal{C}))$ as the 1-categorical localization of $\mathcal{C}$ at $\mathrm{Arr}(\tau(W))$, as can be seen by using the universal property.

On the other hand, given a 1-category $\mathcal{C}$ and localizing at a set of morphisms W, not necessarily the induced map $L(N(\mathcal{C})) \to N(L(\mathcal{C}))$ is an isomorphism. Indeed, $L(N(\mathcal{C}))$ can have much better properties, as can be seen for example from yadda yadda (PROPOSITION ABOUT FINITE COMPLETENESS) and in fact localizing 1-categories after taking their nerves gives every ∞-category (MORE PRECISE STATEMENT).

Proposition 7.7. 7.1.9

Given a universe $\mathbf{U}$ and W a simplicial subset of a $\mathbf{U}$-small ∞-category $\mathcal{C}$, let $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ be the associated localization. Then the functor $\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful and its essential image consists of all presheaves $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ sending maps $u\colon x \to y$ in W to invertible maps $u^*\colon Fy \to Fx$ in $\mathcal{S}$.

Proof. The map γ gives us a morphism

$$\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \simeq \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}) \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}),$$

which has a left adjoint $\gamma_!$ and a right adjoint γ_*. Now, for any presheaf $F\colon L(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}$, the unit map $F \to \gamma_*\gamma^*F$ is invertible and, by adjunction, the same goes for the counit map $\gamma_! \gamma^* F \to F$, which means that u^* is fully faithful. On the other hand, given a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ sending 1-simplices in W to invertible maps, the counit $\gamma^*\gamma_* F \to F$ and the unit $F \to \gamma^*\gamma_! F$ are both invertible since the restrictions of these adjunctions to $\underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ form adjoint equivalences of ∞-categories as γ^* is an equivalence. □

Proposition 7.8. 7.1.10

Given an ∞-category $\mathcal{C}$ and a simplicial subset W, the localization functor $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ is final and cofinal. In particular, if $e\colon \Delta^0 \to \mathcal{C}$ encodes a final or a cofinal object, so does $\gamma(e)$.

Proof. 1702.02681, prop. 5.13. □

Proposition 7.9. 7.1.11 Let's fix a universe $\mathbf{U}$, a $\mathbf{U}$-small ∞-category $\mathcal{C}$ and a simplicial subset W. Consider then a functor $f\colon \mathcal{C} \to \mathcal{D}$, where X is a small ∞-category. Then f exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ by W if and only if the following conditions hold:

1. the functor f sends the 1-simplices of W to invertible maps of $\mathcal{D}$;
2. the functor f is essentially surjective;
3. the functor f^* induces an equivalence of ∞-categories
$$f^*\colon \underline{\mathrm{Hom}}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \to \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}).$$

Proof. One implication is trivial (for (2) look at the construction in 7.4). For the converse, let's pick a localization $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ and, through condition (1), we get a factorization $g\colon L(\mathcal{C}) \to \mathcal{D}$ such that $g \cdot \gamma \cong f$, giving us a triangle

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathcal{D}^{\mathrm{op}}, \mathcal{E}) \arrow[rr, "g^*"] \arrow[dr, "f^*"'] & & \underline{\mathrm{Hom}}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{E}) \arrow[dl, "\gamma^*"] \\
& \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}) &
\end{tikzcd}$$

commuting up to J-homotopy for any ∞-category $\mathcal{E}$. Picking $\mathcal{E} = \mathcal{S}$, γ^* and f^* are equivalences of ∞-categories, the latter by (3). It follows by 2-out-of-3 that g^* is one too, and therefore the same applies to its left adjoint $g_!$, which is then fully faithful. This is equivalent to g being fully faithful (FUN THEOREM, MAYBE STATE IT AT LEAST 6.1.5) and, since f is essentially surjective by (2), the same goes for g. It follows that g is an equivalence of ∞-categories. □

Proposition 7.10. 7.1.14

Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functors between ∞-categories with a right adjoint $g\colon \mathcal{D} \to \mathcal{C}$ and suppose that we are given simplicial subsets $V \subset \mathcal{C}$, $W \subset \mathcal{D}$ such that $f(V) \subset W$, $g(W) \subset V$. Then we can lift them to an adjunction $\overline{f}\colon L(\mathcal{C}) \rightleftarrows L(\mathcal{D}) : \overline{g}$ such that the diagrams

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "\gamma_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\gamma_{\mathcal{D}}"] \\
L(\mathcal{C}) \arrow[r, "\overline{f}"'] & L(\mathcal{D})
\end{tikzcd}, \quad
\begin{tikzcd}
\mathcal{D} \arrow[r, "g"] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{C} \arrow[d, "\gamma_{\mathcal{C}}"] \\
L(\mathcal{D}) \arrow[r, "\overline{g}"'] & L(\mathcal{C})
\end{tikzcd}$$

Proof. Let's write $\underline{\mathrm{Hom}}_V^W(\mathcal{C},\mathcal{D})$ for the full subcategory of $\underline{\mathrm{Hom}}(\mathcal{C},\mathcal{D})$ whose objects are functors ϕ such that $\phi(V)\subset W$. The equivalence $\gamma_{\mathcal{C}}^*\colon \mathrm{Hom}(L(\mathcal{C}),L(\mathcal{D}))\to\underline{\mathrm{Hom}}_V(\mathcal{C},L(\mathcal{D}))$ allows us to construct a functor $\underline{\mathrm{Hom}}_V^W(\mathcal{C},\mathcal{D})\to\underline{\mathrm{Hom}}_V(\mathcal{C},L(\mathcal{D}))\to\underline{\mathrm{Hom}}(L(\mathcal{C},\mathcal{D}))$ which associates to any ϕ as above a functor $\overline{\phi}$ making the square

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\phi"] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{C} \arrow[d, "\gamma_{\mathcal{C}}"] \\
L(\mathcal{D}) \arrow[r, "\overline{\phi}"'] & L(\mathcal{C})
\end{tikzcd}$$

commute up to J-homotopy.

The proof works by observing that our map also lifts natural transformations functorially, which allows us to show the triangle identities for the lifted unit and counit. □

Proposition 7.11. 7.1.18

Let $u\colon\mathcal{C}\to\mathcal{D}$ be a functor between ∞-categories with a fully faithful right adjoint v and consider $W=k(\mathcal{D})\times_{\mathcal{D}}\mathcal{C}$, the subcategory of maps of $\mathcal{C}$ which become invertible in $\mathcal{D}$. Then u exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ by $W>$

Proof. Given a localization $\gamma\colon\mathcal{C}\to L(\mathcal{C})$ by W, we get a functor $\gamma\cdot v\colon\mathcal{D}\to L(\mathcal{C})$ which, paired with the $\overline{u}$ obtained from the construction in the previous proof, lifts the adjunction $u\dashv v$ to the localizations (where $L(\mathcal{D})\cong\mathcal{D}$ as we localize at the identities). Lifting maintains the counit invertible, which allows us to conclude that $\gamma\cdot v$ is fully faithful.

Essential surjectivity follows from the fact that, for any object c in $\mathcal{C}$, the unit η_c is such that $\epsilon_{u(c)}\cdot u(\eta_c)=\mathrm{id}_{u(c)}$ and, since ϵ is invertible, so is $u(\eta_c)$, thus η_c becomes invertible in $L(\mathcal{C})$ and shows that $(\gamma_{\mathcal{C}}\cdot v)(u(c))=\gamma_{\mathcal{C}}(vu(c))\cong c$. Notice that here we used that $L(\mathcal{C})_0=\mathcal{C}_0$, which is permissible up to equivalence as previously noted.

(PLEASE CHECK PROOF) □

Corollary 7.12. 7.2.5

Proof. 7.1.5 □

Corollary 7.13. 7.2.8

Proof. 7.1.5, 7.1.18, 7.2.5 □

Corollary 7.14. 7.2.10

Corollary 7.15. 7.2.15

Corollary 7.16. 7.2.16

Corollary 7.17. 7.2.18

Proof. 7.2.16 □

Corollary 7.18. 7.2.25

Proof. 7.2.8, 7.2.10, 7.2.15 □

Corollary 7.19. 7.3.2

Corollary 7.20. 7.3.4

Corollary 7.21. 7.3.5

Proof. 7.1.10, 7.3.2, 7.3.4 □

Corollary 7.22. 7.3.8

Proof. 7.1.10, 7.1.12 □

Corollary 7.23. 7.3.9

Proof. 7.3.8 □

Corollary 7.24. 7.3.10

Proof. 7.3.9 □

Proposition 7.25. 7.3.11

Corollary 7.26. 7.3.15

Proof. 7.1.12, 7.3.9, 7.3.10 □

Corollary 7.27. 7.3.16

Proof. 7.3.5, 7.3.15 □

Corollary 7.28. 7.3.26

Corollary 7.29. 7.3.27

Proof. 7.3.16, 7.3.26 □

Corollary 7.30. 7.3.28

Corollary 7.31. 7.3.29

Proof. 7.3.27, 7.3.28 □

Proposition 7.32. 7.4.2

Proposition 7.33. 7.4.11

Proof. 7.3.11, 7.4.2 □

Definition 7.34. An ∞-*category with weak equivalences and fibrations* is a triple $(\mathcal{C}, W, Fib)$ where $\mathcal{C}$ is an ∞-category with a final object, $W \subset \mathcal{C}$ is a subcategory with the 2-out-of-3 property whose maps are called weak equivalences and $Fib \subset \mathcal{C}$ is a class of fibrations such that:

1. for any pullback square of $\mathcal{C}$

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "p'"'] & x \arrow[d, "p"] \\ y' \arrow[r, "v"'] & y \end{tikzcd}$$

   in which y is fibrant and p lies in Fib (and W), the same goes for p';

2. for any map $f\colon x \to y$ with fibrant codomain can be factored as a map in W followed by one in Fib.

By *fibrant object* we mean an object whose map to the terminal one is in Fib.

We shall call *weak equivalences* the maps in W and *fibrations* the ones in Fib. Maps which are both shall be referred to as *trivial fibrations*.

Proposition 7.35. 7.4.13

Corollary 7.36. 7.4.14

Proof. 7.4.13 □

Proposition 7.37. 7.4.16

Proof. 7.4.13 □

Proposition 7.38. 7.4.19

Proof. 7.4.11 □

Proposition 7.39. 7.5.5

Proposition 7.40. 7.5.6

Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, the localization $L(\mathcal{C}_f)$ has finite limites and the localization functor $\mathcal{C}_f \to L(\mathcal{C}_f)$ is left exact. Moreover, for any ∞-category $\mathcal{D}$ with finite limits and any left exact functor $f\colon \mathcal{C}_f \to \mathcal{D}$, the induced functor $\overline{F}\colon L(\mathcal{C}_f) \to \mathcal{D}$ is left exact.

Proof. Maybe do not prove it? It relies on a bunch of results from ch. 7.2, 7.3, 7.4 which we do not really want to prove.

7.1.10, 7.2.18, 7.2.25, 7.3.27, 7.4.13, 7.4.16 □

Definition 7.41. An ∞-category of fibrant objects is an ∞-category with weak equivalences and fibrations $\mathcal{C}$ in which all objects are fibrant.

Example 7.42. For any ∞-category with weak equivalences and fibrations $\mathcal{C}$, its full subcategory given by fibrant objects is an ∞-category of fibrant objects. We shall denote it by $\mathcal{C}_f$.

Proposition 7.43. 7.5.16

Let x be a fibrant object in an ∞-category with weak equivalences and fibrations $\mathcal{C}$. The induced functor $\mathcal{C}_f/\gamma_f(x) \to \mathcal{C}/\gamma(x)$ is final.

Proof. We have that $\mathcal{C}_f/\gamma_f(x) = L(\mathcal{C}_f)/\gamma_f(x) \times_{L(\mathcal{C}_f)} \mathcal{C}_f$ and $\mathcal{C}/\gamma(x) = L(\mathcal{C})/\gamma(x) \times_{L(\mathcal{C})} \mathcal{C}$ and the functor we are considering is induced by $\bar{\iota}\colon L(\mathcal{C}_f) \to L(\mathcal{C})$.

To prove that it is final, it is sufficient to show that for any object (c, u) of $L(\mathcal{C})/\gamma(x)$ the coslice (c, u)
$(\mathcal{C}_f/\gamma_f(x))$ is weakly contractible and, to do this, by [Cis19, Lem. 4.3.15] we can show that any functor $F\colon E \to (c, u)$
$(\mathcal{C}_f/\gamma_f(x))$, where E is the nerve of a finite partially ordered set, is Δ^1-homotopic to a constant functor. This can be done through the theory of Reedy fibrant diagrams developed in [Cis19, Ch. 7.4]. □

Proposition 7.44. 7.5.17

Let $\mathbf{U}$ be a universe and $\mathcal{C}$ a $\mathbf{U}$-small ∞-category with weak equivalences and fibrations. For any ∞-category $\mathcal{D}$ with $\mathbf{U}$-small colimits and any functor $F\colon \mathcal{C} \to \mathcal{D}$, we have an isomorphism

$$(\gamma_f)_! \iota^*(F) \cong \bar{\iota}^* \gamma_!(F)$$

induced by the square

$$\begin{tikzcd} \mathcal{C}_f \arrow[r, "\iota"] \arrow[d, "\gamma_f"'] & \mathcal{C} \arrow[d, "\gamma"] \\ L(\mathcal{C}_f) \arrow[r, "\bar{\iota}"'] & L(\mathcal{C}) \end{tikzcd},$$

which commutes up to J-homotopy.

Proof. We only need to prove that the evaluation of the canonical map $(\gamma_f)_! \iota^*(F) \cong \bar{\iota}^* \gamma_!(F)$ at any object x of $\mathcal{C}_f$ is invertible. This evaluation is equivalent by [Cis19, Prop. 6.4.9] to the map

$$\operatorname{colim}_{\mathcal{C}_f/\gamma_f(x)} i^*(F)/\gamma_f(x) \to \operatorname{colim}_{\mathcal{C}/\gamma(x)} F/\gamma(x),$$

where $F/\gamma(x)$ is define by composing F with the canonical projection $\mathcal{C}/\gamma(x) \to \mathcal{C}$ and similarly for $i^*(F)/\gamma_f(x)$. Using 7.43 and the commutativity of the square above, we get that the desired map is indeed invertible for all x. □

Proposition 7.45. 7.5.18

Proof. 7.5.6, 7.5.17

We already know that $\bar{\iota}$ is essentially surjective as every object in $\mathcal{C}$ is weakly equivalent to one in $\mathcal{C}_f$ and the localization functors are essentially surjective themselves, thus it is enough to prove that it is fully faithful. To do this, we may fix a universe $\mathbf{U}$ such that $\mathcal{C}$ is $\mathbf{U}$-small and prove that the functor

$$\bar{\iota}_!\colon \underline{\mathrm{Hom}}(L(\mathcal{C}_f), \mathcal{S}) \to \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{S})$$

is fully faithful by [Cis19, Prop. 6.1.15], which is equivalent to proving that the unit map $1 \to \bar{\iota}^*\bar{\iota}_!$ of the adjunction $\bar{\iota}^* \dashv \bar{\iota}_!$ is invertible.

We know that $\bar{\iota}_*$ and $\bar{\iota}^*$ both have right adjoints, thus they preserve colimits. Also, every $\mathcal{S}$-valued functor indexed by a $\mathbf{U}$-small ∞-category can be obtained as a colimit of representable ones, hence it is enough to check that the condition holds for any representable functor F. Also, γ_f is essentially surjective, which means that it is sufficient to check that map $(\gamma_f)_! \to \bar{\iota}^*\bar{\iota}_!(\gamma_f)_!$ which we get by precomposing the unit with $(\gamma_f)_!$ is invertible.

We have then the chain of isomorphisms

$$\begin{aligned}(\gamma_f)_! &\cong (\gamma_f)_!\bar{\iota}^*\bar{\iota}_! \\ &\cong \bar{\iota}^*\gamma_f\iota_! \\ \bar{\iota}^*\bar{\iota}_!(\gamma_f)_!,\end{aligned}$$

where the first isomorphism comes from the full faithfulness of ι, the second one from 7.44 and the last one the fact that $\bar{\iota} \cdot \gamma_f \cong \gamma \cdot \iota$, as noted in 7.44.

The second claim follows directly from the first one and 7.40. □

Corollary 7.46. 7.5.19

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. For a morphism between fibrant objects $p\colon x \to y$, the following conditions are equivalent:

1. the morphism p has a section in $ho(L(\mathcal{C}))$;
2. there exists a morphism $p'\colon x' \to x$ s.t. the composition of p' and p is a weak equivalence;
3. there exists a fibration $p'\colon x' \to x$ s.t. the composition of p' and p is a weak equivalence.

Proof. 7.5.18

Should we prove it? Uses right calculus of fractions, but it's rather simple. □

Construction 7.47. 7.5.22

Proof. 7.5.19, 7.5.20 □

Remark 7.48. Let $\mathcal{C}$ be an ∞-category with weak equivalences W and $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. The precomposition functor $\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ does not have a left adjoint in general, but we may ask whether $\mathrm{Hom}(F, \gamma^*(-))$ is representable in $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D})$. If it is, a representative is denoted by $\mathbf{R}F\colon L(\mathcal{C}) \to \mathcal{D}$ and is called the *right derived functor of F*. Beware that to be precise one would have to specify the natural transformation $F \to \mathbf{R}F \cdot \gamma$ exhibiting it as such. Dually, a representative of $\mathrm{Hom}(\gamma^*(-), F)$ is the *left derived functor of F*.

Proposition 7.49. 7.5.24

If $F\colon \mathcal{C} \to \mathcal{D}$ sends weak equivalences to isomorphisms, then the functor $\overline{F}\colon L(\mathcal{C}) \to \mathcal{D}$, associated to F by the universal property of $L(\mathcal{C})$, is the right derived functor of F.

Proof. Let's fix a universe $\mathbf{U}$ such that $\mathcal{C}$ and $\mathcal{D}$ are $\mathbf{U}$-small and let $G\colon L(\mathcal{C}) \to \mathcal{D}$ be any functor. Then the invertible map $\overline{F} \cdot \gamma \cong F$ and the equivalence of ∞-categories $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \simeq \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$ induce invertible maps $\mathrm{Hom}(\overline{F}, G) \simeq \mathrm{Hom}(\overline{F} \cdot \gamma, G \cdot \gamma) \simeq \mathrm{Hom}(F, G \cdot \gamma)$ in $\mathcal{S}$, functorially in G. □

Construction 7.50. (NOT COMPLETE)

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. Given any functor $F\colon \mathcal{C} \to \mathcal{D}$ sending weak equivalences between fibrant objects to invertible maps then has a right derived functor $\mathbf{R}F$, which may be constructed as follows.

First we choose a quasi-inverse $R\colon L(\mathcal{C}) \to L(\mathcal{C}_f)$ of the equivalence of ∞-categories specified in **??**, then we choose a functor $\overline{F}\colon L(\mathcal{C}_f) \to \mathcal{D}$ and a natural isomorphism $j\colon \overline{F} \cdot \gamma_f \to F \cdot \iota$. We set then $\mathbf{R}F = \overline{F} \cdot R$.

There are some interesting remarks which may be included!!!!

Proposition 7.51. 7.5.28

For any left exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories with weak equivalences and fibrations, the right derived functor $\mathbf{R}F\colon L(\mathcal{C}) \to L(\mathcal{D})$ is left exact.

Proof. 7.5.6

We have a square

$$\begin{tikzcd} L(\mathcal{C}_f) \arrow[r, "\overline{F}"] \arrow[d] & L(\mathcal{D}_f) \arrow[d] \\ L(\mathcal{C}) \arrow[r, "\mathbf{R}F"'] & L(\mathcal{D}) \end{tikzcd}$$

commuting up to J-homotopy, where the vertical maps are equivalences of ∞-categories and $\overline{F}$ is the functor obtained by restricting F to the subcategories of fibrant objects $\mathcal{C}_f$ and $\mathcal{D}_f$. It therefore suffices to show that $\overline{F}$ is left exact, but this follows from 7.40. □

Lemma 7.52. 7.6.2

Proof. 7.2.10.4, 7.2.18, 7.4.13 □

Lemma 7.53. 7.6.4

Proof. 7.2.10.4, 7.2.18, 7.4.13 □

Lemma 7.54. 7.6.5

Proof. 4.2.9 □

Lemma 7.55. 7.6.7

Proof. 4.3.15, 7.4.19, 7.5.5 □

Theorem 7.56. 7.6.10

Proof. 7.3.29, 7.6.2, 7.6.5, 7.6.7 ☐

Corollary 7.57. 7.6.13

Proof. 7.5.18, 7.5.22, 7.5.24, 7.5.28, 7.6.4, 7.6.10 ☐

Theorem 7.58. 7.6.16

Proof. 6.1.6, 6.1.7, 6.1.8, 7.1.14, 7.4.14, 7.5.18, 7.6.13 ☐

8 Categorical Models of TT as Locally Cartesian Closed Fibration Categories

Definition 8.1. A fibration category $\mathcal{P}$ is *locally cartesian closed* if, for any fibration $p\colon a \to b$, the pullback functor $p^*\colon \mathcal{P} \downarrow b \to \mathcal{P} \downarrow a$ admits a right adjoint p_* which is an exact functor.

How does Kapulkin prove that a categorical model of Type Theory is a locally cartesian closed fibration category?

First of all, he refers to AKL15 to show that $\mathcal{P}$ has a fibrational structure, then he goes on to show the following results, whose proofs are extremely terse and therefore should be expanded.

(I DON'T UNDERSTAND WHAT f^*b SHOULD BE IN 1211.2851, DEF. 1.2.4. WHAT FOLLOWS IS MY IDEA.)

Definition 8.2. Given $p_A\colon \Gamma.A \to \Gamma$, a section $a\colon \Gamma \to \Gamma.A$ and $f\colon \Delta \to \Gamma$, we look at the commutative diagram

$$\begin{tikzcd}
\Delta \arrow[dr, dotted, "f^*a"] \arrow[drr, bend left, "a\cdot f"] \arrow[ddr, bend right, equal] & & \\
& \Delta.f^*A \arrow[r, "q(f,A)"'] \arrow[d, "p_{f^*A}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\
& \Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd},$$

which gives us f^*a as the factorization through the pullback square of the pair $(\mathrm{id}_\Delta, a\cdot f)$.

Lemma 8.3. For any dependent projection $p_\Delta\colon \Gamma.\Delta \to \Gamma$ in a categorical model of type theory $\mathcal{C}$, the pullback functor $p_\Delta^*\colon \mathcal{C} \downarrow \Gamma \to \mathcal{C} \downarrow \Gamma.\Delta$ admits a right adjoint.

Proof. Kapulkin 1507.02648, Lemma 5.5. ☐

We know that every fibration in $\mathcal{C}$ is isomorphic to a composite of dependent projections , so this tells us that every fibration induces an adjunction between fibrational slices (YOU SHOULD DEFINE THEM).

Lemma 8.4. Consider an iterated context extension $\Gamma.\Delta.\Theta.\Psi$ in a categorical model of type theory $\mathcal{C}$. Then the contexts

$$\Gamma.\Pi(\Delta,\Theta).\Pi(p^*_{\Pi(\Delta,\Theta)}\Delta.app^*_{\Delta,\Theta}.\Psi) \text{ and } \Gamma.\Pi(\Delta,\Theta.\Psi)$$

are isomorphic (actually equal) in $\mathcal{C}$.

Proof. Kapulchin 1507.02648, Lemma 5.5. It simply refers to the construction of the Π-structure on $\mathcal{C}^{cxt}$. ☐

We are finally ready to prove the result leading to the final one we want.

Proposition 8.5. A categorical model of type theory $\mathcal{C}$ is a locally cartesian closed fibration category.

Proof. Kapulkin 1507.02648, Proposition 5.4. ☐

Theorem 8.6. Given a categorical model of type theory $\mathcal{C}$, the ∞-category $L(\mathcal{C})$ is locally cartesian closed.

Proof. Since a fibration category is more generally a ∞-category with fibrations and weak equivalences, we can apply [Cis19, Prop. 7.6.16] as the hypothesis are satisfied by 8.5. ☐

9 Pushforward

One may ask whether cocartesian fibrations in **sSet** model Pi types, which is a piece needed to understand a novel model of dependent type theory provided by **sSet**. To answer this question, an explicit description of the right adjoint p_* of the pullback functor $p^*\colon \mathbf{sSet}/Y \to \mathbf{sSet}/X$ induced by a morphism $p\colon X \to Y$ is needed.

Consider an object $f\colon T \to X$ in $\mathbf{sSet}/X$. What is $p_*(f)\colon T' \to Y$? We know that a n-simplex t' of T' corresponds bijectively to a map $t'\colon \Delta^n \to T'$, which in turn corresponds bijectively to a commutative diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "t'"] \arrow[dr, "y"'] & & T' \arrow[dl, "p_*(f)"] \\ & Y & \end{tikzcd}$$

and, under the adjunction $p^* \dashv p_*$, we get bijectively another commutative diagram

$$\begin{tikzcd} U \arrow[rr, "t"] \arrow[dr, "p^*(y)"'] & & T \arrow[dl, "f"] \\ & X & \end{tikzcd},$$

from which follows that

$$T'_n \cong \{(y,t) \mid y \in Y_n,\ t \in \mathbf{sSet}/X(p^*(y), f)\}$$

and the map $p_*(f)$ then sends $(y,t) \in T'_n$ to $y \in Y_n$.

The same method can be extended to give us the pushforward along a map of marked simplicial sets $p\colon (X, E_X) \to (Y, E_Y)$ in **mSet**. Specifically, our previous construction can be adapted to give us the n-simplices by starting from maps $(\Delta^n)_\flat \to p_*(T, E_T) = (T', E_{T'})$, telling us again that

$$T'_n \cong \{(y,t) \mid y \in Y_n,\ t \in \mathbf{mSet}/X(p^*(y), f)\},$$

while to get the markings we notice that every marked edge in $p_*(T, E_T)$ corresponds to a unique map $(\Delta^1)_\natural \to p_*(T, E_T)$ and the same procedure allows us to write

$$E_{T'} = \{(y,t) \mid y \in E_Y,\ t \in \mathbf{mSet}/X(p^*(y), f)\},$$

which fully specifies the needed data.

Now, under which conditions on p does this specify a Quillen adjunction when the slices of **mSet** are equipped with the contravariant model structure? If it is a coCartesian fibration, it generally doesn't, but it does when it is a Cartesian fibration. How can we specify an approximation q of p_* such that, after localizing in the infinity-sense, we get an adjunction $p^* \dashv q$?

Idea: use the theory of bifibrations. From a coCartesian fibration $\phi\colon X \to Y$ we can construct a bifibration $E \to X \times Y$ by constructing the maps $p\colon E \to X$, $q\colon E \to Y$ by first taking the pullback of ϕ along $ev_0\colon Y^{\Delta^1} \to Y$ and then composing the map $E \to Y^{\Delta^1}$ with ev_1.

$$\begin{tikzcd}
E \arrow[r, "p"] \arrow[d] \arrow[dd, bend right=60, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\
Y^{\Delta^1} \arrow[r, "ev_0"] \arrow[d, "ev_1"] & Y \\
Y &
\end{tikzcd}$$

We want to show that, for any (coCartesian?) morphism $f\colon A \to X$, we have an isomorphism between $\phi_*(f)$ and $q_*p^*(f)$.

Canonically, we have

$$E_n = \{(x, \Delta^n \times \Delta^1 \xrightarrow{g} Y) \mid x \in X_n,\ \phi(x) = g|_{\Delta^n \times \{0\}}\}$$

and, by pasting the pullback squares, we also get

$$(\mathrm{dom}(p^*f))_n = \{(a, \Delta^n \times \Delta^1 \xrightarrow{g} Y) \mid a \in A_n,\ f(a) = g|_{\Delta^n \times \{0\}}\}$$

and therefore

$$\begin{aligned}(\mathrm{dom}(q_*p^*(f)))_n &= \{(y, q^*(y) \xrightarrow{g} p^*(f)) \mid y \in Y_n\} \\ &= \{(y, p_!q^*(y) \xrightarrow{g} f) \mid y \in Y_n\},\end{aligned}$$

which we want to relate to $\phi_*(f)$.

To do this, we want to understand the maps $p_! q^*(y) \xrightarrow{g} f$ and somehow relate them to $\phi^*(y) \to f$. By definition,

$$\begin{aligned}\operatorname{dom}(p_! q^*(y))_k &= \operatorname{dom}(q^*(y))_k \\ &= \{(x, \Delta^k \times \Delta^1 \xrightarrow{h} Y, t) \mid x \in X_k,\ \phi(x) = h|_{\Delta^k \times \{0\}},\ t \in (\Delta^n)_k,\ y(t) = h|_{\Delta^k \times \{1\}}\},\end{aligned}$$

with $q^*(y)(x, h, t) = (x, h)$, thus $p_! q^*(y)(x, h, t) = x$.

On the other hand, we have

$$\operatorname{dom}(\phi^*(y))_k = \{(x, t) \mid x \in X_k,\ t \in (\Delta^n)_k,\ \phi(x) = y(t)\}$$

and $\phi^*(y)(t, x) = x$.

If we can create a bijection between morphisms of the form $p_! q^*(y) \to f$ and $\phi^*(y) \to f$ we are done. Unfortunately, I do not see how we can do this: any morphism $p_! q^*(y) \to f$ induces a morphism $\phi^*(y) \to f$ by precomposing with the inclusion $\phi^*(y) \to p_! q^*(y)$, $(x, t) \mapsto (x, h_{\phi(x)}, t)$, where $h_{\phi(x)}$ is obtained by precomposing $\phi(x) \colon \Delta^k \to Y$ with $p_{\Delta^k} \colon \Delta^k \times \Delta^1 \to \Delta^k$, but this association is only injective, not surjective, and I have no good idea about how to construct others.

To construct the bijection I may start from a morphism $p_! q^*(y) - > f$ and construct another one with $\phi^*(y)$ as domain by lifting morphisms $\Delta^k \times \Delta^1 \to Y$ to decide where to map $(x, h, t) \in \operatorname{dom}(p_! q^*(y))_k$, however this involves solving a coherence problem and I would have to do so coherently to define a morphism of simplicial sets as desired. Perhaps these restrictions actually allow a solution, but I do not believe so.

It may also be possible that the injective morphism we mentioned earlier is a weak equivalence with respect to our model structure, which may be enough.

We provide a counterexample to the previous claim in the context of right fibrations. Consider $\phi \colon \partial\Delta^1 \to \Delta^1$, $i \mapsto 0$, which is a right fibration. We have that $\phi^*(1) = 0$, the empty simplicial set, thus $\phi_*(f)^{-1}(1) \cong \Delta^0$. On the other hand, $(p_! q^*)(1) = U \amalg V$, thus $q_* p^*(f)^{-1}(1)$ can be a disjoint union of non-zero simplicial sets, which would then not be an equivalent ∞-groupoid. It follows that our map is not, in general, a weak equivalence in the model structure of right fibrations on slices of **sSet**.

If instead from a right fibration ϕ we first take the opposites of ϕ and f, then do the pushforward and finally we take again the opposites we get $\phi_*(f)$, which is encoded in the following commutative diagram where the vertical maps are isomorphisms.

$$\begin{tikzcd} f \colon Z \to X \arrow[d, mapsto] & \mathbf{sSet}/X \arrow[r, "\phi_*"] \arrow[d, "\mathrm{op}"'] & \mathbf{sSet}/Y \arrow[d, "\mathrm{op}"] & f \colon Z \to Y \arrow[d, mapsto] \\ f^{\mathrm{op}} \colon Z^{\mathrm{op}} \to X^{\mathrm{op}} & \mathbf{sSet}/X^{\mathrm{op}} \arrow[r, "\phi_*^{\mathrm{op}}"'] & \mathbf{sSet}/Y^{\mathrm{op}} & f^{\mathrm{op}} \colon Z^{\mathrm{op}} \to Y^{\mathrm{op}} \end{tikzcd}$$

References

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