Localizations of Models of Dependent Type Theory

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Objective

A modern proof of the following theorem.

Theorem (Kapulkin 2015)

Given a categorical model of type theory C*, its* ∞*-categorical localization* $L(\mathrm{C})$ *is a locally cartesian closed* ∞*-category.*

Dependent Type Theory

What

A theory of computations and a foundation of mathematics.

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Objects

Dependent types A *over contexts* Γ and their *terms* $x : A$.

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Logical rules

Construct new types and their terms from old, carry out computations, provide Σ-*types* $\Sigma(A, B)$, Π-*types* $\Pi(A, B)$ and Id-*types* Id_A, *natural-numbers-type* Nat...

Problem

Providing a model of Dependent Type Theory is hard.

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Solution

Defining a class of algebraic models.

Modeling structural rules

Definition (contextual categories)

A category C with:

1. a grading on objects $\operatorname{Ob} \mathsf{C} = \coprod_{n \in \mathbb{N}} \operatorname{Ob}_n \mathsf{C}$;
2. a map $\mathrm{ft}_n \colon \operatorname{Ob}_{n+1} \mathsf{C} \to \operatorname{Ob}_n \mathsf{C}$ for each $n \in \mathbb{N}$;
3. *basic dependent projections* $p_A \colon \Gamma.A \to \mathrm{ft}_n(\Gamma.A) = \Gamma$;
4. a functorial choice of pullback squares

$$\begin{tikzcd} \Delta.f^*A \arrow[r, "q(f{,}A)"] \arrow[d, "p_{f^*A}"'] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd};$$

5. ...

Modeling logical rules

Extra structure

Id-types require from $\Gamma.A$ an object $\Gamma.A.A.\mathrm{Id}_A$...
Π-types require from $\Gamma.A.B$ an object $\Gamma.\Pi(A, B)$, a map $\mathrm{app}_{A,B} : \Gamma.\Pi(A, B).A \to \Gamma.A.B$, $(f, a) \mapsto (a, \mathrm{app}(f, a))$...

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Definition

A categorical model of type theory is a contextual category C *with* Σ, Id *and* Π *structures.*

Bi-invertibility

Definition (bi-invertible map)

A map $f : \Gamma \to \Delta$ in a contextual category with Id*-structure* C *for which we can provide:*

1. *maps* $g_1 : \Delta \to \Gamma$, $\eta : \Gamma \to \Gamma.(1_\Gamma, g_1 \cdot f)^* \, \mathrm{Id}_\Gamma$;
2. *maps* $g_2 : \Delta \to \Gamma$, $\epsilon : \Delta \to \Delta.(1_\Delta, f \cdot g_2)^* \, \mathrm{Id}_\Delta$.

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Question

What if we localize at bi-invertible maps?

Fibrational structure

Definition (∞-categories with weak equivalences and fibrations)

A triple $(\mathcal{C}, W, Fib)$ where:
...a weakening of the definition of fibration categories, with $\mathcal{C}$ an ∞-category.

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Theorem (Avigad-Kapulkin-Lumsdaine 2013)

A contextual category with Σ and Id structures defines a fibration category, where weak equivalences are bi-invertible maps and fibrations are maps isomorphic to compositions of basic dependent projections $p_A \colon \Gamma.A \to \Gamma$.

Localizing fibrational categories

Proposition (Cisinski)

The localization at weak equivalences of an ∞-category with weak equivalences and fibrations $\mathcal{C}$ is a finitely complete ∞-category.

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Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, if for every fibration $f : x \to y$ between fibrant objects the pullback functor between fibrant slices $f^* \colon \mathcal{C}(y) \to \mathcal{C}(x)$ has a right adjoint preserving trivial fibrations, then $L(\mathcal{C})$ is locally cartesian closed.

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Proof.

For any basic dependent projection $p_A \colon \Gamma.A \to \Gamma$, there exists a right adjoint to $p_A^* \colon \mathsf{C}(\Gamma) \to \mathsf{C}(\Gamma.A)$ given by

$$(p_A)_*(\Gamma.A.\Theta) = \Gamma.\Pi(A, \Theta)$$

with counit induced by $\mathrm{app}_{A,\Theta}$. It preserves the fibrational structure. □

Thank you for your attention!

Why is Dependent Type Theory cool?

1. Closely linked to *computations* and *computer science*, makes proof assistants possible
2. Enough by itself as a foundation, unlike Set Theory or Propositional Calculus
3. *Proofs* are internal objects
4. Better treatment of *equality*
5. Makes "fully faithful + essentially surjective = equivalence" independent from the Axiom of Choice
6. Homotopical interpretation in ∞-groupoids

Internal Languages Conjecture

Conjecture (Kapulkin-Lumsdaine 2016)

The horizontal maps, given by simplicial localization, induce equivalences of ∞-categories.

$$\begin{tikzcd}
\mathsf{CxlCat}_{\Sigma,1,\mathsf{Id},\Pi} \arrow[r] \arrow[d] & \mathsf{LCCC}_{\infty} \arrow[d] \\
\mathsf{CxlCat}_{\Sigma,1,\mathsf{Id}} \arrow[r] & \mathsf{Lex}_{\infty}
\end{tikzcd}$$

A proof by Nguyen-Uemura has recently become available on arxiv. One hopes to extend this to an equivalence between $\mathsf{CxlCat}_{\mathsf{HoTT}}$ and $\mathsf{ElTopos}_{\infty}$.