Localizations of Models of dependent type theory

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Objective

A modern proof of the following theorem.

Theorem (Kapulkin 2015)

Given a dependent type theory $\mathbf{T}$ *with* Σ-, Id- *and* Π*-types, the* ∞*-localization of its syntactic category* $\mathrm{Syn}(\mathbf{T})$ *is a locally cartesian closed* ∞*-category.*

Dependent type theory

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A theory of computations and a foundation of mathematics.

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Dependent types A *over contexts* $\Gamma = (x_0 : A_0, \ldots, x_n : A_n)$ and their *terms* $x : A$.

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Structural rules

How to work with *variables*.

Logical rules

Construct new types and their terms from old, carry out computations. They provide Σ-*types* $\Sigma(A, B)$, Π-*types* $\Pi(A, B)$, Id-*types* Id_A, *natural-numbers-type* Nat...

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Solution

Defining a class of algebraic models.

Modeling structural rules

Definition (contextual categories)

A category C with:

1. a grading on objects (or *contexts*) $\operatorname{Ob} C = \coprod_{n \in \mathbb{N}} \operatorname{Ob}_n C$;
2. a unique and terminal object in $\operatorname{Ob}_0 C$, the *empty context*;
3. a map $\mathrm{ft}_n \colon \operatorname{Ob}_{n+1} C \to \operatorname{Ob}_n C$ for each $n \in \mathbb{N}$;
4. *basic dependent projections* $p_A \colon \Gamma.A \to \mathrm{ft}_n(\Gamma.A) = \Gamma$;
5. a functorial choice of pullback squares

$$\begin{tikzcd} \Delta.f^*A \arrow[r, "q(f,A)"] \arrow[d, "p_{f^*A}"'] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd} .$$

Example

Construction (syntactic category of a type theory Syn($\mathbf{T}$))

A category Syn($\mathbf{T}$) *where:*

1. *n-objects are contexts* $[x_0 : A_0, \dots, x_n : A_n]$;
2. *the empty context is* $[]$;
3. *morphisms* $[x_0 : A_0, \dots, x_n : A_n] \to [y_0 : B_0, \dots, y_m : B_m]$ *are tuples of terms* $(f_0 : B_0, \dots, f_m : B_m)$ *derivable from* $x_0 : A_0, \dots, x_n : A_n$;
4. *composition is given by subsitution;*
5. $\mathrm{ft}([x_0 : A_0, \dots, x_{n+1} : A_{n+1}]) = [x_0 : A_0, \dots, x_n : A_n]$;
6. *a basic projection is a tuple* $(x_0 : A_0, \dots, x_n : A_n)$;
7. *pullback squares are given by context substitution.*

Modeling logical rules

Extra structure

Id-types require from $\Gamma.A$ an Id-object $\Gamma.A.A.\,\mathrm{Id}_A$...

Π-types require from $\Gamma.A.B$ a Π-object $\Gamma.\Pi(A,B)$, an evaluation map $\mathrm{app}_{A,B}\colon \Gamma.\Pi(A,B).A \to \Gamma.A.B$, $(f,a) \mapsto (a, \mathrm{app}(f,a))$...

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Example

If $\mathbf{T}$ has some logical rules, then $\mathrm{Syn}(\mathbf{T})$ has the corresponding logical structures.

Bi-invertibility

Definition (bi-invertible map)

A map $f : \Gamma \to \Delta$ *in a contextual category with* Id*-structure* C *for which we can provide:*

1. *maps* $g_1 : \Delta \to \Gamma$, $\eta : \Gamma \to \Gamma.(1_\Gamma, g_1 \cdot f)^* \,\mathsf{Id}_\Gamma$*;*
2. *maps* $g_2 : \Delta \to \Gamma$, $\epsilon : \Delta \to \Delta.(1_\Delta, f \cdot g_2)^* \,\mathsf{Id}_\Delta$*.*

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Question

What if we localize at bi-invertible maps?

Fibrational structure

Definition (∞-categories with weak equivalences and fibrations)

A triple $(\mathcal{C}, W, Fib)$ where:
...a weakening of the definition of fibration categories, with $\mathcal{C}$ an ∞-category.

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Theorem (Avigad-Kapulkin-Lumsdaine 2013)

A contextual category with Σ- and Id-structures defines a fibration category, where weak equivalences are bi-invertible maps and fibrations are maps isomorphic to compositions of basic dependent projections $p_A \colon \Gamma.A \to \Gamma$.

Localizing fibrational categories

Proposition (Cisinski)

The localization at weak equivalences of an ∞-category with weak equivalences and fibrations $\mathcal{C}$ is a finitely complete ∞-category.

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Construction (fibrant slice $\mathcal{C}(x)$)

Given a fibrant object x in $\mathcal{C}$, lift the fibrational structure through $\mathcal{C}/x \to \mathcal{C}$ and then take the subcategory of fibrant objects of $\mathcal{C}/x$.

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Proposition (Cisinski)

Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, if for every fibration $f : x \to y$ between fibrant objects the pullback functor between fibrant slices $f^* : \mathcal{C}(y) \to \mathcal{C}(x)$ has a right adjoint preserving trivial fibrations, then $L(\mathcal{C})$ is locally cartesian closed.

Localizations of models are cartesian closed

Theorem (Kapulkin 2015)

Given a dependent type theory $\mathbf{T}$ *with* Σ-, Id- *and* Π-*types, the localization of its syntactic category* $\mathsf{Syn}(\mathbf{T})$ *is a locally cartesian closed* ∞-*category.*

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Proof.

For any basic dependent projection $p_A \colon \Gamma.A \to \Gamma$, there exists a right adjoint to $p_A^* \colon \mathrm{Syn}(\mathbf{T})(\Gamma) \to \mathrm{Syn}(\mathbf{T})(\Gamma.A)$ given by

$$(p_A)_*(\Gamma.A.\Theta) = \Gamma.\Pi(A, \Theta)$$

with counit induced by $\mathrm{app}_{A,\Theta}$. It preserves the fibrational structure. $\square$

Thank you for your attention!

Why is dependent type theory cool?

1. Closely linked to *computations* and *computer science*, makes proof assistants possible.
2. Enough by itself as a foundation, unlike set theory or propositional calculus.
3. *Proofs* are internal objects.
4. Better treatment of *equality*.
5. Makes “fully faithful + essentially surjective = equivalence” independent from the axiom of choice.
6. Homotopical interpretation in ∞-groupoids.

Internal languages conjecture

Conjecture (Kapulkin-Lumsdaine 2016)

The horizontal maps, given by simplicial localization, induce equivalences of ∞-categories.

$$\begin{tikzcd}
\mathsf{CxlCat}_{\Sigma,1,\mathsf{Id},\Pi} \arrow[r] \arrow[d] & \mathsf{LCCC}_{\infty} \arrow[d] \\
\mathsf{CxlCat}_{\Sigma,1,\mathsf{Id}} \arrow[r] & \mathsf{Lex}_{\infty}
\end{tikzcd}$$

A proof by Nguyen-Uemura has recently become available on arxiv. One hopes to extend this to an equivalence between $\mathsf{CxlCat}_{\mathsf{HoTT}}$ and $\mathsf{ElTopos}_{\infty}$.