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Localizations of Models of Dependent Type Theory

1 This is [not] an outline

We can also rewrite the paper by Kapulkin about LCCC arising from TT using the language of localizations of quasi-categories. There they develop the relevant theory showing that under some conditions the frame associated to a fibration category is locally cartesian closed, but using Cisinski's results we can prove the same theorem directly using a more mainstream theory.

What should be included in such an overview?

1- Cisinski's theory of localizations (of fibration ∞-categories)

2- an introduction to contextual categories: where do they come from? Why are they useful? Check out Voevodsky's papers about C-systems

We explain what dependent type theory is (Martin-Lof's notes from 1984) and why it's an interesting foundation of mathematics. We mention Homotopy Type Theory as an effort to provide homotopical foundations which better model how we think about identities, which explains why intensional identity types are more interesting to us than extensional ones.

We move on to defining contextual categories (1211.2851, 1406.7413, 1507.02648) and what the Pi, Sigma and Id structures are (1406.7413, 1211.2851 Appendix B). To understand what the link between such structures and syntactically presented type theories we refer to 1507.02648, Sec. 1.1, while the statement of the conjectured correspondence is in 1304.0680, Sec. 2.1.

Where does the link between dependent type theories and ∞-categories come from? We see that ∞-categories intuitively model the behavior of type theories and their type constructions, especially when considering Homotopy Type Theory, however this relation is known only partially (references in the intro of 1507.02648). The idea is that the type theory we are interested in should be the internal language of some class of ∞-categories and a precise statement would require us to provide homotopical functors in both directions which induce an equivalence on the associated ∞-categories. The idea is to construct the functor from contextual categories as a localization functor, that is we need to provide a homotopical structure on contextual categories, as they do in 1507.02648 (there should be an older reference) which then provides an associated ∞-category. This is the object of the Initiality Conjecture, stated in 1610.00037, in the hope that such a correspondence will extend to Homotopy Type Theory and some notion of Elementary Higher Toposes, perhaps the one specified in 1805.03805. At the moment we know that HoTT can be interpreted in Higher Toposes with some structure. Current progress: 1709.09519, an upcoming paper by Nguyen-Uemura (HoTTest talk).

Our aim is to show that when taking contextual categories with the structure we specified earlier we obtain a locally cartesian closed ∞-category. To do so we provide a fibrational structure on contextual categories (1304.0680, 1507.02648), which as we anticipate will imply that their simplicial localizations are finitely complete. We also prove that the hypothesis of [Cis19, Thm. 7.6.16] are satisfied, informing that this will be sufficient to prove Kapulkin's main result from 1507.02648.

We then develop the theory of localizations of ∞-categories by Cisinski and specifically develop the results concerning ∞-categories with fibrations and weak equivalences. Localizations of such ∞-categories are finitely complete. The objective is to show [Cis19, Thm. 7.6.16]. How in depth should we go?

Why all of this is interesting: we are proving Kapulkin's result internalizing all of the discussion within the language of ∞-category theory and relying only on its simplicial model.

2 Contextual categories

There are many models of dependent type theory from category theory in the literature, like *category with attributes*, *categories with families* and *comprehension categories*, which are fairly similar among them, as shown by the adjunctions between their categories. Here we shall work with *contextual category*, which were first explored by Cartmell and Streicher in (INSERT REFS) and later by Voevodsky, under the name *C-systems*, in (MORE REFS). Lumsdaine in his phd thesis claims that contextual categories are the right framework PROP 1.2.5.

Definition 2.1. A *contextual category* $\mathcal{C}$ is a category with the following data:

1. a small category, which we also call $\mathcal{C}$, with a grading on objects $\operatorname{Ob}\mathcal{C} = \coprod_{n\in\mathbb{N}} \operatorname{Ob}_n \mathcal{C}$;
2. an object $* \in \operatorname{Ob}_0 \mathcal{C}$;
3. for each $n \in \mathbb{N}$, a map $ft_n \colon \operatorname{Ob}_{n+1}\mathcal{C} \to \operatorname{Ob}_n \mathcal{C}$, often simply denoted ft;
4. for each $n \in \mathbb{N}$ and $X \in \operatorname{Ob}_{n+1}\mathcal{C}$, a map $p_X \colon X \to ftX$;
5. for each $n \in \mathbb{N}$, $X \in \operatorname{Ob}_{n+1}\mathcal{C}$ and $f \colon Y \to ftX$, an object f^*X and a map $q(f,X)\colon f^*X \to X$;

such that:

1. $*$ is the unique element of $\operatorname{Ob}_0\mathcal{C}$;
2. $*$ is terminal;
3. for each n, $X \in \operatorname{Ob}_{n+1}\mathcal{C}$ and $f\colon Y \to ftX$, we have $ftf^*X = Y$ and the square

$$\begin{tikzcd} f^*X \arrow[r, "q(f,X)"] \arrow[d, "p_{f^*X}"'] & X \arrow[d, "p_X"] \\ Y \arrow[r, "f"'] & ftX \end{tikzcd}$$

   is a pullback;
4. for each $n \in \mathbb{N}$, $X \in \operatorname{Ob}_{n+1}\mathcal{C}$ and pair of maps $f\colon Y \to ftX$, $g\colon Z \to Y$, we have $(fg)^*X = g^*f^*X$, $1^*_{ftX}X = X$, $q(fg,X) = q(f,X)\cdot q(g,f^*X)$ and $q(1_{ftX},X) = 1_X$.

Remark 2.2. The last condition in the definition means that our choice of pullbacks is functorial, which allows us to see contextual categories as a strict model of dependent type theory, not requiring keeping track of coherency maps. Other models, like comprehension categories, do not have such requirements, which makes them more general but also interpretation harder, unless they are first strictified, that is replaced by an equivalent strict model of the same kind, like split comprehension categories (HOW DO YOU STRICTIFY THEM?). Famously, we also have homotopical models of dependent type theories, like tribes and fibration categories (REFERENCE), whose internal languages are precisely dependent type theories with intensional Id-types and Σ-types.

A motivating example for contextual categories is the *syntactic category of a dependent type theory*, which constructs from a dependent type theory T a contextual category $Syn(T)$ explicitly modeling it as we will show.

Construction 2.3. REF 1211
Given a dependent type theory T with the structural rules specified in REFS, its syntactic category $Syn(T)$ has:

1. $\operatorname{Ob}_n Syn(T)$ given by contexts $[x_1 : A_1, \ldots, x_n : A_n]$ of length n, modulo definitional equality and renaming of free variables;

2. maps are *context morphisms*, or *substitutions*, modulo definitional equalities and renaming of free variables. This means that a map

$$f\colon [x_1 : A_1, \ldots, x_n : A_n] \to [y_1 : B_1, \ldots, y_m : B_m(y_1, \ldots, y_{m-1})]$$

is an equivalence class of sequences of terms $f_1, \ldots, f_m$ such that

$$\begin{gathered} x_1 : A_1, \ldots, x_n : A_n \vdash f_1 : B_1, \\ \vdots \\ x_1 : A_1, \ldots, x_n : A_n \vdash f_m : B_m(f_1, \ldots, f_{m-1}), \end{gathered}$$

are all derivable judgements and two such sequences f, g are equivalent if we have

$$x_1 : A_1, \ldots, x_n : A_n \vdash f_i \equiv g_i \; B_i(f_1, \ldots, f_{i-1})$$

for every i, where $\equiv$ represents judgemental equality; we shall henceforth write $[f_i]$ for such sequences of terms up to equivalence specifying maps between contexts;

3. composition is given by substitution, that is given $[f]\colon \Gamma \to \Delta$, $[g]\colon \Delta \to \Theta$ we have a map $[g]\colon \Gamma \to \Theta$ induced by the same sequence of terms;

4. the identity $\Gamma \to \Gamma$ is given by the variables of Γ, considered as terms, that is we take the sequence $[x_i]$ given by

$$\begin{gathered} x_1 : A_1, \ldots, x_n : A_n \vdash x_1 : A_1, \\ \vdots \\ x_1 : A_1, \ldots, x_n : A_n \vdash x_n : A_n(x_1, \ldots, x_{n-1}); \end{gathered}$$

5. the terminal object is the empty context $[]$;

6. $ft([x_1 : A_1, \ldots, x_{n+1} : A_{n+1}]) = [x_1 : A_1, \ldots, x_n : A_n]$;

7. for $\Gamma = [x_1 : A_1, \ldots, x_n : A_{n+1}]$, $p_\Gamma\colon \Gamma \to ft\Gamma$ is the *dependent projection context morphism* $[x_1, \ldots, x_n]$, defined by

$$\begin{gathered} x_1 : A_1, \ldots, x_{n+1} : A_{n+1} \vdash x_1 : A_1, \\ \vdots \\ x_1 : A_1, \ldots, x_{n+1} : A_{n+1} \vdash x_n : A_n(x_1, \ldots, x_n) \end{gathered}$$

and thereby simply forgetting the last variable of Γ;

8. given contexts

$$\begin{aligned} \Gamma &= [x_1 : A_1, \ldots, x_{n+1} : A_{n+1}(x_1, \ldots, x_n)], \\ \Delta &= [y_1 : B_1, \ldots, y_m : B_m(y_1, \ldots, y_{m-1})] \end{aligned}$$

and a map $f = [f_i(y)]\colon \Delta \to ft\Gamma$ (where y is a vector of variables of length m), the pullback $f^*\Gamma$ is the context

$$[y_1 : B_1, \ldots, y_m : B_m(y_1, \ldots, y_{m-1}), y_{m+1} : A_{n+1}(f_1(y), \ldots, f_n(y))]$$

for some new variable y_{m+1}, while $q(\Gamma, f)\colon f^*\Gamma \to \Gamma'$ is specified by $[f_1, \ldots, f_n, y_{m+1}]$.

Remark 2.4. Given a dependent type theory T, the terms $t : A$ of a type over a context Γ can be recovered (up to definitional equality) from the syntactic category $Syn(T)$ by looking at sections of the basic dependent projection $p_{[\Gamma,x:A]} \colon [\Gamma, x : A] \to [\Gamma]$, which indeed act as identities over Γ and furthermore specify a term $\Gamma \vdash t : A$. Given the importance of such maps, we shall often simply write "sections" to refer to sections of basic dependent projections, without specifying which ones unless it creates ambiguity.

The above construction also tells us how to think about the other elements in the definition of contextual categories: basic dependent projections $p_B \colon \Gamma.A.B \to \Gamma.A$ represent dependent types $B(x)$ over $x : A$ in the context Γ and pulling back along a dependent projection corresponds to substituting variables, while a choice of a term corresponds to a choice of a section of a basic dependent projection and so on for the other objects in the definition.

Remark 2.5. We shall also make use of some conventions inspired by this construction. Namely, given a contextual category $\mathcal{C}$ and an object $\Gamma \in \mathrm{Ob}_n\,\mathcal{C}$, we shall write $(\Gamma, A_1, \ldots, A_k)$, $\Gamma.A_1.\ldots.A_k$ and $\Gamma.\Delta$ interchangeably for an object X in $\mathrm{Ob}_{n+k}\,\mathcal{C}$ with $ft^k X = \Gamma$ and call it a *context extension of* Γ *of length* k. We shall also write $p_{(A_1,\ldots,A_k)}$, $p_{A_1.\ldots.A_k}$ and p_Δ for the composition of the basic dependent projections $p_{\Gamma.A_1.\ldots.A_i}$, with i ranging from 1 to k, and the resulting map will be called a *dependent projection*. In the case where $k = 1$, p_{A_1} corresponds to a basic dependent projection and the context extension of Γ will be *simple*, while if $k = 0$ we have $p = \mathrm{Id}_\Gamma$ and then the context extension will be *trivial*. Greek letters shall be used to indicate context extensions of arbitrary length, while Latin ones will be reserved to simple extensions.

Continuing, given a dependent projection $p_{A_1.\ldots.A_k} = p_\Theta$ as above and a context morphism $f \colon \Delta \to \Gamma$, we define inductively $\Delta.f^*A_1.\ldots.f^*A_k = \Delta.f^*\Theta$ and $q(f, A_1.\ldots.A_k) = q(f,\Theta)$ by looking at the pasting of pullback squares

$$\begin{tikzcd}
\Delta.f^*A_1.\ldots.f^*A_k \arrow[r, "p_{f^*A_k}"'] \arrow[d, "q(f,A_1.\ldots.A_k)"'] \arrow[rrrr, bend left, "p_{f^*A_1.\ldots.f^*A_k}"] & \Delta.f^*A_1.\ldots.f^*A_{k-1} \arrow[r] \arrow[d, "q(f,A_1.\ldots.A_{k-1})"] & \cdots \arrow[r] & \Delta.f^*A_1 \arrow[r, "p_{f^*A_1}"'] \arrow[d, "q(f,A_1)"'] & \Delta \arrow[d, "f"] \\
\Gamma.A_1.\ldots.A_k \arrow[r, "p_{A_k}"] \arrow[rrrr, bend right, "p_{A_1.\ldots.A_k}"'] & \Gamma.A_1.\ldots.A_{k-1} \arrow[r] & \cdots \arrow[r] & \Gamma.A_1 \arrow[r, "p_{A_1}"] & \Gamma
\end{tikzcd},$$

where $q(f, A_1.\ldots.A_k) = q(q(f, A_1), A_2.\ldots.A_k) = q(q(f, A_1.\ldots.A_{k-1}), A_k)$. As usual, if $k = 0$ we have $q(f,\Theta) = f$, $f^*(\Gamma.\Theta) = \Gamma$, while for $k = 1$ we have $q(f, A_1) = q(f, \Gamma.A_1)$, $\Delta.f^*A_1 = f^*(\Gamma.A_1)$.

Finally, given a section $a \colon \Gamma \to \Gamma.A$ and a context morphism $f \colon \Delta \to \Gamma$, we also want to specify f^*a, that is the term of A which we get by switching context. This is given by the map $(1_\Delta, a \cdot f)$ specified by the pullback square

$$\begin{tikzcd}
\Delta \arrow[dr, dashed, "(1_\Delta,a\cdot f)"] \arrow[drr, bend left, "a\cdot f"] \arrow[ddr, bend right, equal] & & \\
& \Delta.f^*A \arrow[r, "q(f,A)"'] \arrow[d, "p_{f^*A}"] & \Gamma.A \arrow[d, "p_A"] \\
& \Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd},$$

and, as shown by the commutative diagram

$$\begin{tikzcd}
\Delta \arrow[r, "f"] \arrow[d, "f^*a"] \arrow[dd, bend right, equal] & \Gamma \arrow[d, "a"'] \arrow[dd, bend left, equal] \\
\Delta.f^*A \arrow[r, "q(f,A)"'] \arrow[d, "p_{f^*A}"] & \Gamma.A \arrow[d, "p_A"'] \\
\Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd},$$

it corresponds to the pullback of a along $q(f, A)$. By the techniques we provided earlier, we extend this construction to contexts of arbitrary length.

Definition 2.6. A *contextual functor* between contextual categories $F\colon \mathcal{C} \to \mathcal{D}$ is a functor on the underlying categories which preserves the grading, basic dependent projections and such that $q(Ff, FX) = F(q(f, X))$.

Remark 2.7. Our definition allows us to see contextual categories as models for an essentially algebraic theory with sorts indexed by $\mathbb{N} + \mathbb{N} \times \mathbb{N}$. In that context, we get a notion of morphisms between models of this theory, which coincides with the one we have just provided. The category of models for this theory will be the category of contextual categories, denoted by Cxl, which will be complete and cocomplete as the category of models of an essentially algebraic theory REFS.

A problem of providing a model of dependent type theory is exhibiting the basic structural rules, namely context substitution, variable binding, variable substitution and so on for us. The defining properties of contextual categories are meant to model them and therefore take care of all of that for us, meaning that as long as we can show that something models a contextual category we automatically get an interpretation of dependent type theory, thereby eliminating a lot of bureaucracy. For this to be true, however, we need the following statement, which we shall assume because the results we need rely on it.

Conjecture 2.8 (Initiality)**.** Given a dependent type theory T, its syntactic category $Syn(T)$ is initial in the category of contextual categories with the appropriate structure.

Here by "appropriate structure" we mean extra structures meant to model the logical rules of the type theory, like Σ-types, Π-types and Id-types. Indeed, the definition of contextual category as we said deals with the structural rules, but nothing more. Similar results have been proven for some simple dependent type theories in STREICHER91 and HOFMANN95, however we still do not have a general statement. Such a statement would first require a general notion of dependent type theory, which has been worked on in 1904.04097, 2009.05539, 2205.00798, and there is an ongoing effort to provide a formalization and a flexible proof of some variants of the conjecture via proof assistants (HOTTEST TALK SLIDES).

If we could do that, then for any contextual category $\mathcal{C}$ we would have a contextual functor $Syn(T) \to \mathcal{C}$ explaining how to interpret T in $\mathcal{C}$. This is essentually an algorithmic problem: it reduces to explaining inductively to a computer how to construct the aforementioned functor.

We now define the extra structures on contextual categories we mentioned earlier. Our definitions shall be taken from 1211, 1808.

Definition 2.9. A Σ-*type structure* on a contextual category $\mathcal{C}$ consists of:

1. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$, an object $(\Gamma, \Sigma(A, B)) \in \mathrm{Ob}_{n+1}\,\mathcal{C}$;
2. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$, a morphism $pair_{A,B}\colon (\Gamma, A, B) \to (\Gamma, \Sigma(A, B))$ over Γ;
3. for each $(\Gamma, A, B), (\Gamma, \Sigma(A, B), C) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$, and $d\colon (\Gamma, A, B) \to (\Gamma, \Sigma(A, B), C)$ with $p_C \cdot d = pair_{A,B}$, a section $split_d\colon (\Gamma, \Sigma(A, B)) \to (\Gamma, \Sigma(A, B), C)$ such that $split_d \cdot pair_{A,B} = d$;
4. where all of the above is compatible with context substitution, that is given a map $f\colon \Delta \to \Gamma$ we have

$$\begin{aligned} f^*(\Gamma, \Sigma(A, B)) &= (\Delta, \Sigma(f^*A, f^*B)),\\ f^*pair_{A,B} &= pair_{f^*A, f^*B},\\ f^*split_d &= split_{f^*d}. \end{aligned}$$

Definition 2.10. A Id-*type structure* on a contextual category $\mathcal{C}$ consists of: COMPLETE

Definition 2.11. A Π-*type structure* on a contextual category $\mathcal{C}$ consists of:

1. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$, an object $(\Gamma, \Pi(A, B)) \in \mathrm{Ob}_{n+1}\,\mathcal{C}$;
2. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$, a map $\mathrm{app}_{A,B}\colon (\Gamma, \Pi(A, B), p^*_{\Pi(A,B)}A) \to (\Gamma, A, B)$ over Γ, that is such that $p_B \cdot app_{A,B} = q(\Pi(A, B), A)$;
3. for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$ and section $b\colon (\Gamma, A) \to (\Gamma, A, B)$, a section $\lambda_{A,B}(b)\colon \Gamma \to (\Gamma, \Pi(A, B))$;

4. such that for any sections $k\colon \Gamma \to (\Gamma, \Pi(A,B))$, $a\colon \Gamma \to A$ the map $\mathrm{app}_{A,B}(k,a)$ defined as the composition of $\mathrm{app}_{A,B}$ with (k,a) specified by the factorization through the pullback

$$\begin{tikzcd}
\Gamma \arrow[dr, dotted, "(k,a)"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "k"'] & & \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[r, "q(p_{\Pi(A,B)},A)"'] \arrow[d, "p_{p^*_{\Pi(A,B)}A}"] & \Gamma.A \arrow[d, "p_A"] \\
& \Gamma.\Pi(A,B) \arrow[r, "p_{\Pi(A,B)}"'] & \Gamma
\end{tikzcd},$$

we have $p_B \cdot \mathrm{app}_{A,B}(k,a) = a$;

5. such that for any (Γ, A, B), $a\colon \Gamma \to (\Gamma, A)$ and $b\colon (\Gamma, A) \to (\Gamma, A, B)$ we have

$$\mathrm{app}(\lambda_{A,B}(b), a) = b \cdot a;$$

6. all of the above is compatible with context substitution, that is for any $f\colon \Delta \to \Gamma$ we have

$$f^*(\Gamma, \Pi(A,B)) = (\Delta, \Pi(f^*A, f^*B)),$$
$$f^*\lambda_{A,B}(b) = \lambda_{f^*A, f^*B}(f^*b),$$
$$f^*(\mathrm{app}_{A,B}(k,a)) = \mathrm{app}_{f^*A,f^*B}(f^*k, f^*a).$$

We shall say that the the Π-type structure satisfies the Π_η*-rule* if the equation

$$q(p_{\Pi(A,B)}, \Pi(A,B)) \cdot \lambda(1_{p^*_{\Pi(A,B)}A}, \mathrm{app}_{A,B}) = 1_{(\Gamma, \Pi(A,B))}$$

is satisfied, in which case the structure will be called a Π_η*-type structure*. The map on the left is the η*-expansion map*, that is it models the map associating to the term $f : \Pi(A,B)$ the term $(\lambda(x : A).fx) : \Pi(A,B)$ over Γ. Also, $(1_{p^*_{\Pi(A,B)}A}, \mathrm{app}_{A,B})$ is specified by the following factorization through the pullback.

$$\begin{tikzcd}
\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[dr, dotted, "(1_{p^*_{\Pi(A,B)}A}, \mathrm{app}_{A,B})"] \arrow[drr, bend left, "\mathrm{app}_{A,B}"] \arrow[ddr, equal, bend right] & & \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A.p^*_{\Pi(A,B)}B \arrow[r, "q(p_{\Pi(A,B)},A.B)"'] \arrow[d, "p_{p^*_{\Pi(A,B)}A}"] & \Gamma.A.B \arrow[d, "p_A"] \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[r, "q(p_{\Pi(A,B)},A)"'] & \Gamma.A
\end{tikzcd}$$

(WE STILL NEED Π-EXT, WHICH REQUIRES Id-TYPE STRUCTS.)

Remark 2.12. The definition we provided matches the one of 1808, while in 1211 they give a mildly different (but equivalent) one. Namely, they require for each pair of sections $f\colon \Gamma \to \Gamma.\Pi(A,B)$, $a\colon \Gamma \to \Gamma.A$ a section $\mathrm{app}_{A,B}(f,a)\colon \Gamma \to \Gamma.A.B$ with the properties we expressed, from which they then construct a map $\mathrm{app}_{A,B}$ as the composite

$$\begin{aligned}
&q(q(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}A}, A), B) \cdot \\
&\qquad app((1_{\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A}, p_{p^*_{\Pi(A,B)}A})), (1_{\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A}, q(p_{\Pi(A,B)}, A)) \\
&\qquad\quad : \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \to \Gamma.A.B,
\end{aligned}$$

of which we shall more explicitly describe each component. We do not actually need this description, but for completeness we wanted to report it.

First we focus on $(1_{\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A}, p_{p^*_{\Pi(A,B)}A})$. This is given as the factorization through the pullback of the cone given by the two maps, as we will show in a moment.

$$\begin{tikzcd}
\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[drrr, bend left, "p_{p^*_{\Pi(A,B)}A}"] \arrow[dr, dotted] \arrow[ddr, equal] & & & \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A.(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}A})^*\Pi(A,B) \arrow[rr, "q(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}A}, \Pi(A,B))"'] \arrow[d, "p_{(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}A})^*\Pi(A,B)}"] & & \Gamma.\Pi(A,B) \arrow[d, "p_{\Pi(A,B)}"] \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[r, "p_{p^*_{\Pi(A,B)}A}"'] & \Gamma.\Pi(A,B) \arrow[r, "p_{\Pi(A,B)}"'] & \Gamma
\end{tikzcd}$$

Similarly, we obtain the other section in the argument.

$$\begin{tikzcd}
\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[drrr, bend left, "q(p_{\Pi(A,B)}, A)"] \arrow[dr, dotted] \arrow[ddr, equal] & & & \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A.(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B)}A})^*A \arrow[r] \arrow[d, "p_{p^*_{\Pi(A,B)}A}"'] & \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[r] \arrow[d, "p_{p^*_{\Pi(A,B)}A}"] & \Gamma.A \arrow[d, "p_A"] \\
& \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[r, "p_{p^*_{\Pi(A,B)}A}"'] & \Gamma.\Pi(A,B) \arrow[r, "p_{\Pi(A,B)}"'] & \Gamma
\end{tikzcd}.$$

The map app with the arguments specified above then provides a map $\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \to \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A.p^*_{p^*_{\Pi}}$
What follows is the other map appearing in the definition of $\mathrm{app}_{A,B}$.

$$\begin{tikzcd}
\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A.(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B}A})^*A.stuffB \arrow[rr, "q(p_{\Pi(A,B)} \cdot p_{p^*(A,B)A}, A.B)"] \arrow[d, "p_{stuffB}"'] & & \Gamma.A.B \arrow[d, "p_B"] \\
\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A.(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B}A})^*A \arrow[rr, "q(p_{\Pi(A,B)} \cdot p_{p^*(A,B)A}, A)"'] \arrow[d, "p_{(p_{\Pi(A,B)} \cdot p_{p^*_{\Pi(A,B}A})^*A}"'] & & \Gamma.A \arrow[d, "p_A"] \\
\Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \arrow[r, "p_{p^*_{\Pi(A,B)}A}"'] & \Gamma.\Pi(A,B) \arrow[r, "p_{\Pi(A,B)}"'] & \Gamma
\end{tikzcd}$$

3 Structures on Iterated Context Extensions

The aim of this section is to construct from a given contextual category $\mathcal{C}$ with extra structure another contextual category $\mathcal{C}^{cxt}$ with the same structure, but where objects are iterated context extensions, compatibly with a canonical contextual functor $\mathcal{C} \to \mathcal{C}^{cxt}$ defining an equivalence on the underlying categories, thereby generalizing our structures from simple context extensions to arbitrary ones. Indeed, we may then take a context extension, look at it in $\mathcal{C}^{cxt}$, apply the construction and then carry it back through the equivalence. This extension shall be heavily exploited in the final part of the thesis.

On the type theoretical side, extensions of logical rules for identity types to contexts were first explored by Str93 and Gambino REFS under the name of *identity contexts* and Lumsdaine refers to them in his PhD thesis, mentioning that it is possible to model this extension and similar ones for Σ and Π types in $\mathcal{C}^{cxt}$, which he constructs. At the time Kapulkin wrote 1507, nothing further was available in the literature and only in 1808 him and Lumsdaine gave more details on these matters, however they still did not flesh out the constructions in their entirety. In this section we aim to partially fix that for Π-structures specifically, which will be the main contribution of this work.

Construction 3.1. (LUMSDAINE PHD THESIS) Given a contextual category $\mathcal{C}$, we define a contextual category $\mathcal{C}^{cxt}$ in the following way:

1. the set $\mathrm{Ob}_n\,\mathcal{C}^{cxt}$ is given by n-iterated context extensions
$$\Gamma_1.\Gamma_2.\ldots.\Gamma_n$$
in $\mathcal{C}$;
2. morphisms $\Gamma_1.\Gamma_2.\ldots.\Gamma_n \to \Delta_1.\Delta_2.\ldots.\Delta_m$ are morphisms between them seen as objects of $\mathcal{C}$;
3. $*$ is the only element of $\mathrm{Ob}_0\,\mathcal{C}^{cxt}$;
4. $ft(\Gamma_1.\Gamma_2.\ldots.\Gamma_n.\Gamma_{n+1}) = \Gamma_1.\Gamma_2.\ldots.\Gamma_n$;
5. the map $p_{\Gamma_1.\Gamma_2.\ldots.\Gamma_n.\Gamma_{n+1}} : \Gamma_1.\Gamma_2.\ldots.\Gamma_{n+1} \to \Gamma_1.\Gamma_2.\ldots.\Gamma_n$ is the dependent projection exhibiting $\Gamma_1.\Gamma_2.\ldots.\Gamma_{n+1}$ as a context extension of $\Gamma_1.\Gamma_2.\ldots.\Gamma_n$ and will be denoted by $p_{\Gamma_{n+1}}$ unless there is ambiguity;
6. the chosen pullbacks are given by iterating the pullbacks along the basic dependent projections.

As we can see, any object of $\mathcal{C}^{cxt}$ is isomorphic to one in $\mathrm{Ob}_1\,\mathcal{C}^{cxt}$, that is the one which we get by looking at the associated object in $\mathcal{C}$ and then taking the dependent projection from it to the terminal object, which exhibits it as a 1-iterated context extension. The isomorphism is then given by the map in $\mathcal{C}^{cxt}$ corresponding to the identity of the object in $\mathcal{C}$. We now specify a monad $\mathcal{C} \mapsto \mathcal{C}^{cxt}$ on Cxl.

The unit $\mathcal{C} \to \mathcal{C}^{cxt}$ sends every n-object in $\mathcal{C}$ to the corresponding n-iterated (simple) context extension and every morphism to the one it represents.

Before we construct the multiplication, let's study this contextual functor. Every n-iterated context in $\mathcal{C}^{cxt}$ is isomorphic to one in the image of the unit, namely the one which we get by reducing it to an iterated simple context extension, meaning that the functor is essentially surjective. Also, it is fully faithful by construction and therefore it defines an equivalence on the underlying categories.

Let's construct the multiplication. An n-object of $(\mathcal{C}^{cxt})^{cxt}$ is an n-iterated context extension where each extension is itself an iterated context extension in $\mathcal{C}$, that is
$$(\Gamma_1.\ldots.\Gamma_{i_1}).(\Gamma_{i_1+1}.\ldots.\Gamma_{i_2}).\ldots.(\Gamma_{i_{n-1}+1}.\ldots.\Gamma_{i_n}).$$
Since composing dependent projections still gives dependent projections, seeing $\Gamma_{i_{j-1}+1}.\ldots.\Gamma_{i_j}$ as a single context extension Δ_j in $\mathcal{C}$, we can naturally map the object of $(\mathcal{C}^{cxt})^{cxt}$ to $\Delta_1.\ldots.\Delta_n$ in $\mathcal{C}^{cxt}$ and, again, every morphism in $(\mathcal{C}^{cxt})^{cxt}$ corresponds to a unique one in $\mathcal{C}^{cxt}$ once we specify domain and codomain. By construction, this functor is again contextual and an equivalence of categories.

The monad axioms follow from the fact that, essentially, both unit and counit are "identities" on objects and morphisms, which concludes our construction.

We are now ready to extend the Π-structure.

Construction 3.2. Let $\mathcal{C}$ be a contextual category with a Π-structure. Our objective for this section is, as anticipated, to construct one on $\mathcal{C}^{cxt}$. We will do so by induction on the length of the context extensions involved, taking the one from $\mathcal{C}$ in case we are working with objects corresponding to simple extensions. Remember however that in $\mathcal{C}^{cxt}$ we also have objects representing trivial extensions, which will require some minor attention from us.

Let us consider then $\Gamma.\Delta.\Theta$ in $\mathcal{C}^{cxt}$, where $l(\Gamma.\Delta.\Theta) = l(\Gamma.\Delta) + n = l(\Gamma) + m + n$ in $\mathcal{C}$. If $m = 0$, then we set
$$\begin{aligned}\Gamma.\Pi(\Delta,\Theta) &= \Gamma.\Theta,\\ \mathrm{app}_{\Delta,\Theta} &= \mathrm{id}_{\Gamma.\Theta},\\ \lambda_{\Delta,\Theta}(b) &= b.\end{aligned}$$

Similarly, if $n = 0$, then

$$
\begin{aligned}
\Gamma.\Pi(\Delta, \Theta) &= \Gamma, \\
\mathrm{app}_{\Delta,\Theta} &= \mathrm{id}_{\Gamma.\Delta}, \\
\lambda_{\Delta,\Theta}(b) &= \mathrm{id}_{\Gamma}\,.
\end{aligned}
$$

Notice that the only possible b in the latter case is given by $\mathrm{id}_{\Gamma.\Theta}$. Also, the two constructions are compatible in the case where $m = n = 0$.

We now work with the case where $n > 0$, $m = 1$, thus we shall write $\Gamma.\Delta.\Theta = \Gamma.\Delta.B$. In the base case, $n = 1$, we have $\Gamma.\Delta = \Gamma.A$ and therefore we simply set our structure to be the one in $\mathcal{C}$.

If $n - 1 > 0$, we can write $\Gamma.\Delta$ as $\Gamma.\Delta'.A$ and then set

$$
\begin{aligned}
&\Gamma.\Pi(\Delta, B) = \Gamma.\Pi(\Delta'.A, B) = \Gamma.\Pi(\Delta', \Pi(A, B)) \\
&\mathrm{app}_{\Delta,B} \colon \Gamma.\Pi(\Delta', \Pi(A, B)).\Delta'.A \xrightarrow{q(\mathrm{app}_{\Delta',\Pi(A,B)}, p^*_{\Pi(A,B)}A)} \Gamma.\Delta'.\Pi(A, B).A \xrightarrow{\mathrm{app}_{A,B}} \Gamma.\Delta'.A.B \\
&\lambda_{\Delta,B}(b) \colon \Gamma \xrightarrow{\lambda_{\Delta',\Pi(A,B)}(\lambda_{A,B}(b))} \Gamma.\Pi(\Delta', \Pi(A, B)).
\end{aligned}
$$

The idea here is to replicate the adjunction $\mathbf{Set}(A\times B, C) \cong \mathbf{Set}(A, \mathbf{Set}(B, C))$. The map $\mathrm{app}_{\Delta,B}$ is then naturally interpreted as a sequence of partial evaluations and the phenomenon is commonly known as *currying-uncurrying*.

This fully specifies the construction when $l(\Gamma.\Delta.\Theta) = l(\Gamma.\Delta) + 1$ in $\mathcal{C}$, hence we shall move on to the case where Δ has arbitrary length and construct the necessary structure by inducting on the length of Θ. Suppose then that $l(\Gamma.\Delta.\Theta) = l(\Gamma.\Delta) + n$, $n > 1$, and we have already provided the relevant constructions up to $n - 1$. We again decompose the context as $\Gamma.\Delta.\Theta'.B$.

$$
\begin{aligned}
&\Gamma.\Pi(\Delta, \Theta) = \Gamma.\Pi(\Delta, \Theta'.B) = \Gamma.\Pi(\Delta, \mathrm{app}^*_{\Delta,\Theta'} B) \\
&\mathrm{app}_{\Delta,\Theta} \colon \Gamma.\Pi(\Delta, \Theta').\Pi(\Delta, \mathrm{app}^*_{\Delta,\Theta'} B).\Delta \xrightarrow{\mathrm{app}_{\Delta,\mathrm{app}^*_{\Delta,\Theta'} B}} \Gamma.\Pi(\Delta, \Theta').\Delta.\,\mathrm{app}^*_{\Delta,\Theta'} B \xrightarrow{q(\mathrm{app}_{\Delta,\Theta'}, B)} \Gamma.\Delta.\Theta'.B \\
&\lambda_{\Delta,\Theta}(b) \colon \Gamma \xrightarrow{\lambda_{\Delta,\Theta'}(p_B \cdot b)} \Gamma.\Pi(\Delta, \Theta') \xrightarrow{\lambda_{p^*_{\Pi(\Delta,\Theta')}\Delta,\mathrm{app}^*_{\Delta,\Theta'} B}(p^*_{\Pi(\Delta,\Theta')}a)} \Gamma.\Pi(\Delta, \Theta').\Pi(\Delta, \mathrm{app}^*_{\Delta,\Theta'} B)
\end{aligned}
$$

Our definition of $\lambda_{\Delta,\Theta}$ makes use of the map $p^*_{\Pi(\Delta,\Theta')}a$, which comes from the diagram

$$
\begin{tikzcd}
\Gamma.\Pi(\Delta,\Theta').p^*_{\Pi(\Delta,\Theta')}\Delta.\,\mathrm{app}^*_{\Delta,\Theta'} B \arrow[r, "q(\mathrm{app}_{\Delta,\Theta'},B)"] \arrow[dd, "p_{\mathrm{app}^*_{\Delta,\Theta'} B}"] & \Gamma.\Delta.\Theta'.B \arrow[d, "p_B"'] \\
 & \Gamma.\Delta.\Theta' \arrow[d, "p_{\Theta'}"'] \\
\Gamma.\Pi(\Delta,\Theta').p^*_{\Pi(\Delta,\Theta')}\Delta \arrow[uu, bend left, "q(p_{\Pi(\Delta,\Theta')},\Delta)^*a=p^*_{\Pi(\Delta,\Theta')}a"] \arrow[ur, "\mathrm{app}_{\Delta,\Theta'}"] \arrow[r, "q(p_{\Pi(\Delta,\Theta')},\Delta)"'] \arrow[d, "p_{p^*_{\Pi(\Delta,\Theta')}\Delta}"'] & \Gamma.\Delta \arrow[d, "p_\Delta"] \arrow[uu, bend right, "a"'] \\
\Gamma.\Pi(\Delta,\Theta') \arrow[r, "p_{\Pi(\Delta,\Theta')}"'] & \Gamma
\end{tikzcd}
,
$$

where the construction works because (as we shall verify in the next proposition) the triangle commutes.

Here we construct the actual $\lambda_{\Delta,\Theta}$. First, we consider a section $p_B \cdot b \colon \Gamma.\Delta \to \Gamma.\Delta.\Theta'$, to which we apply $\lambda_{\Delta,\Theta'}$ to get $\lambda_{\Delta,\Theta'} \colon \Gamma \to \Gamma.\Pi(\Delta, \Theta')$. After this, we look at the factorization

$$
\begin{tikzcd}
\Gamma.\Delta \arrow[drr, bend left, "b"] \arrow[dr, dotted, "(\lambda_{\Delta,\Theta'}(p_B\cdot b),q(\epsilon_{\Delta,\Theta'},B))"] \arrow[ddr, "\lambda_{\Delta,\Theta'}(p_B\cdot b)"'] \arrow[dddr, equal] & & \\
 & \Gamma.\Delta.\Pi(\Delta,\Theta').\epsilon^*_{\Gamma.\Delta.\Theta'}B \arrow[r, "q(\epsilon_{\Delta,\Theta'},B)"'] \arrow[d] & \Gamma.\Delta.\Theta'.B \arrow[d, "p_B"] \\
 & \Gamma.\Delta.\Pi(\Delta,\Theta') \arrow[r, "\epsilon_{\Delta,\Theta'}"'] \arrow[d, "p_{\Pi(\Delta,\Theta')}"] & \Gamma.\Delta.\Theta' \\
 & \Gamma.\Delta &
\end{tikzcd}
,
$$

getting then $\lambda_{\Delta,\Pi(\Delta,\Theta').\epsilon^*_{\Delta,\Theta'}B}(\lambda_{\Delta,\Theta'}(p_B\cdot b), q(\epsilon_{\Delta,\Theta'},B))\colon \Gamma\to\Gamma.\Pi(\Delta,\Pi(\Delta,\Theta').\epsilon^*_{\Gamma.\Delta.\Theta'}B)$.

Rewriting $\Gamma.\Pi(\Delta,\Pi(\Delta,\Theta').\epsilon_{\Gamma.\Delta.\Theta'}B)$ as $\Gamma.\Pi(\Delta,\Pi(\Delta,\Theta')).\Pi(\Delta,\mathrm{app}^*_{\Delta,\Pi(\Delta,\Theta')}\,\epsilon^*_{\Gamma.\Delta.\Theta'}B)$, we see that we can post-compose the section with the map $q(\Gamma.\Pi(\Delta,\epsilon_{\Gamma.\Delta.\Theta'}),\Pi(\Delta,\mathrm{app}^*_{\Delta,\Theta'}B))\colon \Gamma.\Pi(\Delta,\Pi(\Delta,\Theta')).\Pi(\Delta,\mathrm{app}^*_{\Delta,\Pi(\Delta,\Theta')}\,\epsilon^*_{\Gamma.\Delta.\Theta'}B)$ $\Gamma.\Pi(\Delta,\Theta').\Pi(\Delta,\mathrm{app}^*_{\Delta,\Theta'}B)$, which gives us our $\lambda_{\Delta,\Theta}(b)$.

Given $a\colon\Gamma\to\Gamma.\Delta$, $f\colon\Gamma.\Delta\to\Gamma.\Delta.\Theta$, we construct $\mathrm{app}(f,a)$ as in the definition of Π-structures.

This fully specifies the data needed for a Π-structure on $\mathcal{C}^{cxt}$, however we still have to check that it is indeed one.

Proposition 3.3. Given a contextual category with a Π-structure $\mathcal{C}$, the above data defines a Π-structure on $\mathcal{C}^{cxt}$ which is compatible with the natural contextual functor $\mathcal{C}\to\mathcal{C}^{cxt}$. Also, if the original Π-structure has the Π_η property, the same goes for the one on $\mathcal{C}^{cxt}$.

Proof. We have to show that it is a Π-structure, which we will do inductively by verifying that at every step our proposed construction maintains the desired properties. The compatibility with the contextual functor will then follow directly from the way we defined the base case.

Let's consider an object $\Gamma.\Delta.\Theta$ in $\mathcal{C}^{cxt}$. The only interesting case is the one where both Δ and Θ specify non-trivial context extensions in $\mathcal{C}$ and at least one of them is not basic: indeed, the desired properties in the other cases are either trivial or follow directly from the fact that they hold in $\mathcal{C}$.

We start as before by working on Δ, supposing that $\Gamma.\Delta.\Theta=\Gamma.\Delta.B=\Gamma.\Delta'.A.B$ and that the desired properties for context extensions over Γ of length up to $l(\Gamma.\Delta')-l(\Gamma)$. We can then write

$$\begin{aligned}
p_B\cdot\mathrm{app}_{\Delta,B} &= p_B\cdot\mathrm{app}_{A,B}\cdot q(\mathrm{app}_{\Delta',\Pi(A,B)},p^*_{\Pi(A,B)}A)\\
&= q(p_{\Pi(A,B)},A)\cdot q(\mathrm{app}_{\Delta',\Pi(A,B)},p^*_{\Pi(A,B)}A)\\
&= q(p_{\Pi(A,B)}\cdot\mathrm{app}_{\Delta',\Pi(A,B)},A)\\
&= q(q(p_{\Pi(\Delta',\Pi(A,B))},\Delta'),A)\\
&= q(p_{\Pi(\Delta',\Pi(A,B))},\Delta'.A)\\
&= q(p_{\Pi(\Delta,B)},\Delta)\\
p_B\cdot\mathrm{app}(f,a) &= p_B\cdot\mathrm{app}_{\Pi(\Delta,B)}\cdot p^*_{\Pi(\Delta,B))}a\cdot f\\
&= q(p_{\Pi(\Delta,B)},\Delta)\cdot p^*_{\Pi(\Delta,B)}a\cdot f\\
&= a\cdot p_{\Pi(\Delta,B))}\cdot f\\
&= a\\
\mathrm{app}(\lambda_{\Delta,B}(b),a) &= \mathrm{app}_{A,B}\cdot q(\mathrm{app}_{\Delta',\Pi(A,B)},A)\cdot p^*_{\Pi(\Delta',\Pi(A,B))}a\cdot\lambda_{\Delta,B}(b)\\
&= \mathrm{app}_{A,B}\cdot q(\mathrm{app}_{\Delta',\Pi(A,B))},A)\cdot p^*_{\Pi(\Delta',\Pi(A,B))}a\cdot\lambda_{\Delta',\Pi(A,B)}(\lambda_{A,B}(b))\\
&= \mathrm{app}_{A,B}\cdot q(\mathrm{app}_{\Delta',\Pi(A,B))},A)\cdot\mathrm{app}()\\
otherattempt... &= b\cdot a\\
&= b\cdot a\cdot p_{\Pi(\Delta,B)}\cdot\lambda_{\Delta,B}(b)\\
&= b\cdot q(p_{\Pi(\Delta,B)},\Delta)\cdot p^*_{\Pi(\Delta,B)}a\cdot\lambda_{\Delta,B}(b)\\
&=\\
\mathrm{app}(\lambda_{\Delta,B}(b),p_A\cdot a) &= \mathrm{app}(\lambda_{\Delta',\Pi(A,B)}(\lambda_{A,B}(b)),p_A\cdot a)\\
&= \lambda_{A,B}(b)\cdot p_A\cdot a\\
&= ???\\
p_{\Pi(\Delta,B)}\cdot\lambda_{\Delta,B}(b) &= p_{\Pi(\Delta',\Pi(A,B))}\cdot\lambda_{\Delta',\Pi(A,B)}(\lambda_{A,B}(b))\\
&= \mathrm{id}_\Gamma\,.
\end{aligned}$$

PLEASE VERIFY, LIKELY WRONG

We now check inductively on the length of Θ. (WE STILL HAVE TO FIND THE RIGHT λ!!!)

$$
\begin{aligned}
p_\Theta \cdot \mathrm{app}_{\Delta,\Theta} &= p_{\Theta'} \cdot p_B \cdot q(\mathrm{app}_{\Delta,\Theta'}, B) \cdot \mathrm{app}_{\Delta,\mathrm{app}^*_{\Delta,\Theta'} B} \\
&= p_{\Theta'} \cdot \mathrm{app}_{\Delta,\Theta'} \cdot p_{\mathrm{app}^*_{\Delta,\Theta'} B} \cdot \mathrm{app}_{\Delta,\mathrm{app}^*_{\Delta,\Theta'} B} \\
&= q(p_{\Pi(\Delta,\Theta)}, \Delta) \cdot q(p_{\Pi(\Delta,\mathrm{app}^*_{\Delta,\Theta'} B)}, \Delta) \\
&= q(p_{\Pi(\Delta,\Theta')} \cdot p_{\Pi(\Delta,\mathrm{app}^*_{\Delta,\Theta'} B)}, \Delta) \\
&= q(p_{\Pi(\Delta,\Theta').\Pi(\Delta,\mathrm{app}^*_{\Delta,\Theta'} B)}, \Delta) \\
&= q(p_{\Pi(\Delta,\Theta)}, \Delta) \\
p_\Theta \cdot \mathrm{app}(f,a) &= p_\Theta \cdot \mathrm{app}_{\Pi(\Delta,\Theta)} \cdot p^*_{\Pi(\Delta,\Theta)} a \cdot f \\
&= q(p_{\Pi(\Delta,\Theta)}, \Delta) \cdot p^*_{\Pi(\Delta,\Theta)} a \cdot f \\
&= a \cdot p_{\Pi(\Delta,\Theta)} \cdot f \\
&= a \\
need\lambda first...
\end{aligned}
$$

To conclude the proof one would still need to verify that the construction is compatible with context substitution. □

Construction 3.4. Given a map $f \colon \Gamma.\Delta.\Psi \to \Gamma.\Delta.\Theta$ over $\Gamma.\Delta$, we shall construct a map $\Gamma.\Pi(\Delta, f) \colon \Gamma.\Pi(\Delta,\Psi) \to \Gamma.\Pi(\Delta,\Theta)$ over Γ in the second argument of Π corresponding to postcomposition by f.

We do so by looking at the commutative diagram

$$
\begin{tikzcd}
\Gamma.\Pi(\Delta,p^*_\Delta\Psi).p^*_{\Pi(\Delta,p^*_\Delta\Psi)}\Delta \arrow[rr, "\mathrm{app}_{\Delta,p^*_\Delta\Psi}"] \arrow[dr, dotted, "(1_{p^*_{\Pi(\Delta,p^*_\Delta\Psi)}\Delta}, f\cdot\mathrm{app}_{\Delta,p^*_\Delta\Psi})"] \arrow[ddr, equal] & & \Gamma.\Delta.p^*_\Delta\Psi \arrow[d, "f"] \arrow[dd, bend left, "p_{p^*_\Delta\Psi}"] \\
& \Gamma.\Pi(\Delta,p^*_\Delta\Psi).\Delta.\Theta \arrow[r, "q(q(p_{\Pi(\Delta,p^*_\Delta\Psi)},\Delta),\Theta)"] \arrow[d, "p_{p^*_{p^*_\Delta\Theta}\Psi}"] & \Gamma.\Delta.\Theta \arrow[d, "p_\Psi"'] \\
& \Gamma.\Pi(\Delta,p^*_\Delta\Psi).p^*_{\Pi(\Delta,p^*_\Delta\Psi)}\Delta \arrow[r, "q(p_{\Pi(\Delta,p^*_\Delta\Psi)},\Delta)"'] & \Gamma.\Delta
\end{tikzcd}
$$

and then applying $\lambda_{\Delta,\Theta}$ to the map given by the universal property of the pullback, which gives us a section $\lambda_{\Delta,\Theta}(1_{p^*_{\Pi(\Delta,p^*_\Delta\Psi)}\Delta}, f \cdot \mathrm{app}_{\Delta,p^*_\Delta\Psi}) \colon \Gamma.\Pi(\Delta,p^*_\Delta\Psi) \to \Gamma.\Pi(\Delta,p^*_\Delta\Psi).\Pi(\Delta,\Theta)$. All we have to do now is post-compose with $q(p_{\Pi(\Delta,\Psi)}, \Pi(\Delta,\Theta))$.

We shall also need the following lemma in the proof of FINAL THEOREM.

Lemma 3.5. Given an iterated context extension $\Gamma.\Delta.\Theta.\Psi$ in a contextual category with Π-types $\mathcal{C}$, the contexts

$$\Gamma.\Pi(\Delta,\Theta.\Psi), \quad \Gamma.\Pi(\Delta,\Theta).\Pi(p^*_{\Pi(\Delta,\Theta)}\Delta, \mathrm{app}^*_{\Delta,\Theta}\Psi)$$

are equal in $\mathcal{C}$. Also, $\Gamma.\Pi(\Delta, p_\Psi) = p_{\Pi(\Delta,\mathrm{app}^*_{\Delta,\Theta}\Psi)}$.

Proof. For the first claim, it is enough to notice that the two contexts reduce to the same one in $\mathcal{C}$ after applying the inductive construction we defined on $\mathcal{C}^{cxt}$ to reduce Ψ.

For the second claim instead we consider the chain of equalities

$$
\begin{aligned}
\Gamma.\Pi(\Delta,p_\Psi) &= q(p_{\Pi(\Delta,\Theta.\Psi)}, \Pi(\Delta,\Theta)) \cdot \lambda_{\Delta,\Theta}(1_{p^*_{\Pi(\Delta,\Theta)}\Delta}, p_\Psi \cdot \mathrm{app}_{\Delta,\Theta.\Psi}) \\
&= p_{\Pi(\Delta,\mathrm{app}^*_{\Delta,\Theta})} \cdot p_{\Pi(\Delta,\Theta)} \cdot \lambda_{\Delta,\Theta}(1_{p^*_{\Pi(\Delta,\Theta)}\Delta}, p_\Psi \cdot \mathrm{app}_{\Delta,\Theta.\Psi}) \\
&= p_{\Pi(\Delta,\mathrm{app}^*_{\Delta,\Theta})}.
\end{aligned}
$$

(PLEASE CHECK) □

We now do the same for an Id-structure on a contextual category.

Construction 3.6. Let $\mathcal{C}$ be a contextual category with an Id-structure. Given a dependent context $p_\Delta \colon \Gamma.\Delta \to \Gamma$ with $l(\Gamma.\Delta) = l(\Gamma) + n$ in $\mathcal{C}$ we proceed by induction on n.

If $n = 0$, then

$$\begin{aligned} \Gamma.\operatorname{Id}_\Delta &= \Gamma, \\ r_\Delta &= 1_\Gamma, \\ J_{\Theta,C,d} &= d. \end{aligned}$$

If $n = 1$,

$$\begin{aligned} \Gamma.\operatorname{Id}_A &= \Gamma.\operatorname{Id}_A, \\ r_A &= r_A, \\ J_{\Theta,C,d} &= J(\Theta, C, d), \end{aligned}$$

where the objects on the right are the ones given by the Id-structure on $\mathcal{C}$.

If $n > 1$, assuming to have extended the Id-structure to shorter context extensions in $\mathcal{C}$, we write instead

$$\begin{aligned} \Gamma.\operatorname{Id}_\Delta &= \Gamma.\operatorname{Id}_{\Delta'.A} = \Gamma.\operatorname{Id}_{\Delta'}.\operatorname{Id}_A \\ r_\Delta &= q(,) \cdot r_A \end{aligned}$$

Proposition 3.7. [Gar09b] Given a contextual category with an Id-structure $\mathcal{C}$, $\mathcal{C}^{cxt}$ also carries a natural Id-structure compatible with the contextual functor $\mathcal{C} \to \mathcal{C}^{cxt}$ as described in REFS 0808.

Lemma 3.8. [**1808**] Given a contextual category with Id, Π_η structures and function extensionality $\mathcal{C}$, the latter can also be extended to $\mathcal{C}^{cxt}$ compatibly with the Id and Π structures specified above. In particular, if we have two sections $b, b' \colon \Gamma.\Delta \to \Gamma.\Delta.\Theta$ and a homotopy $h : b \sim b'$, that is a section $h \colon \Gamma.\Delta \to \Gamma.\Delta.(b,b')^* \operatorname{Id}_\Theta$, then there is also a homotopy between $\lambda_{\Delta,\Theta}(b)$ and $\lambda_{\Delta,\Theta}(b')$ and these constructions are functorial in $\mathcal{C}$.

Corollary 3.9. [**1808**] Under the conditions of the previous theorem, if $f \colon \Gamma.\Delta.\Theta \to \Gamma.\Delta.\Psi$ is bi-invertible over $\Gamma.\Delta$, then the same goes for the induced map $\Gamma.\Pi(\Delta,\Theta) \to \Gamma.\Pi(\Delta,\Psi)$.

(WE COULD ALSO REFER TO 5.9 FROM 1203.3253, WHERE THEY GIVE A PROOF FOR TRIBES WHICH MAY BE ADAPTED; THM 4.8 FROM THERE TELLS US THAT THEY ARE CANONICALLY SPLIT TYPE-THEORETIC FIBRATION CATEGORIES, WHICH ALLOWS US TO APPLY THE RESULT)

(MAYBE INTRODUCE THE LAST TWO RESULTS LATER? THEY HOLD FOR SIMPLE EXTENSIONS IN ANY CONTEXTUAL CATEGORY, HENCE THEY ALSO HOLD FOR SIMPLE EXTENSIONS IN THE ITERATED CONTEXTUAL CAT AND FOR ITERATED EXTENSIONS IN ANY CONTEXTUAL CAT. ALSO, WE NEED TO SAY WHAT BI-INVERTIBLE MAPS ARE, WHICH I WOULD LIKE TO DO LATER; LET'S KEEP THIS SECTION ABOUT STUFF STRICTLY ABOUT ITERATED CONTEXTS)

Remark 3.10. In Lumsdaine's PhD thesis it is noted that the lift of Id and Π structures is not compatible with the monad we provided earlier because they are not compatible with the multiplication. On the other hand, with a strategy along the line of the one we presented for Π-structures we can also lift Σ-structures compatibly with the monad, meaning that $(-)^{cxt}$ restricts to one on Cxl_Σ.

4 Localizations of ∞-Categories

To prove that localizing a categorical model of type theory we get a locally cartesian closed ∞-category we need a theory of localizations. We shall provide one in the general context of ∞-categories as developed by Cisinski in *Higher Categories and Homotopical Algebra* with the aim of proving [Cis19, Thm. 7.6.16], which will do the heavy lifting in showing the desired result. Those familiar with the theory may skip the entire chapter while keeping in mind yadda yadda (LIST THE MAJOR RESULTS).

Definition 4.1. Let X be a simplicial set and $W \subset X$ a simplicial subset. Given an ∞-category $\mathcal{C}$, we define $\underline{\mathrm{Hom}}_W(X, \mathcal{C})$ to be the full simplicial subset of $\underline{\mathrm{Hom}}(X, \mathcal{C})$ whose objects are the morphisms $f \colon X \to \mathcal{C}$ sending the 1-simplices in W to isomorphisms.

Remark 4.2. The above definition induces a canonical pullback square

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_W(X, \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Hom}}(X, \mathcal{C}) \arrow[d] \\
\underline{\mathrm{Hom}}_W(W, \mathcal{C}) \arrow[r] & \underline{\mathrm{Hom}}(W, \mathcal{C})
\end{tikzcd}$$

given by the inclusion $W \to X$.

Definition 4.3. Given an ∞-category $\mathcal{C}$ and $W \subset \mathcal{C}$, a *localization of* $\mathcal{C}$ *by* W is a functor $\gamma \colon \mathcal{C} \to L(\mathcal{C})$ such that:

1. $L(\mathcal{C})$ is an ∞-category;
2. γ sends the 1-simplices of W to isomorphisms in $L(\mathcal{C})$;
3. for any ∞-category $\mathcal{D}$ there is an equivalence of ∞-categories
$$\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$$
given by precomposing with γ.

(CISINSKI DOES NOT ASK FOR $\mathcal{C}$ TO BE AN ∞-CATEGORY. SHOULD WE BE LESS GENERAL AS WE HAVE DONE?)

Proposition 4.4. Given an ∞-category $\mathcal{C}$ and a subsimplicial set W, the localization of $\mathcal{C}$ by W always exists and it is essentially unique.

Proof. We begin by proving that a localization exists in the case where $W = \mathcal{C}$.

In this context, $\underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \cong \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}^{\cong})$ canonically, where $\mathcal{D}^{\cong}$ is the maximal subgroupoid of $\mathcal{D}$. Factoring $\mathcal{C} \to \Delta^0$ in the Kan model structure, we find an anodyne map $\gamma \colon \mathcal{C} \to L(\mathcal{C})$.

Remember that for any anodyne map $A \to B$ we get a trivial fibration $\underline{\mathrm{Hom}}(B, \mathcal{D}^{\cong}) \to \underline{\mathrm{Hom}}(A, \mathcal{D}^{\cong})$. Looking then at the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(L(\mathcal{C}), \mathcal{D}^{\cong}) \arrow[r, two heads, "\gamma^*", "\sim"'] \arrow[d, "\cong"'] & \mathrm{Hom}_W(\mathcal{C}, \mathcal{D}^{\cong}) \arrow[d, "\cong"] \\
\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \arrow[r, "\gamma^*"'] & \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})
\end{tikzcd},$$

by the 2-out-of-3 property we see that the lower γ^* is an equivalence.

We now move on to the general case. First of all, notice that as a particular case of the previous one we get that localizing Δ^1 at its non-trivial morphism we obtain $\Delta^1 \to J = L(\Delta^1) \sim \Delta^0$. Taking then $W \subset \mathcal{C}$, we consider the commutative diagram

$$\begin{tikzcd}
\coprod_{f \in W_1} \Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \arrow[ddr, bend left, "\gamma"] & \\
\coprod_{f \in W_1} J \arrow[r] & \mathcal{C}' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\sim"'] & \\
& & L(\mathcal{C})
\end{tikzcd},$$

where $\mathcal{C}' \to L(\mathcal{C})$ is an inner anodyne map obtained by taking the fibrant replacement of $\mathcal{C}'$ in the Joyal model structure. This can be done functorially via the small object argument.

For any ∞-category $\mathcal{D}$, we get a trivial fibration $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D})$ and a pullback square

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \Pi_{f\in W_1} \underline{\mathrm{Hom}}(J, \mathcal{D}) \arrow[r] & \Pi_{f\in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \end{tikzcd},$$

which together with the pullback

$$\begin{tikzcd} \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \Pi_{f\in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}^{\cong}) \arrow[r] & \Pi_{f\in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \end{tikzcd}$$

implies by pasting that

$$\begin{tikzcd} \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D}) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \Pi_{f\in W_1} \underline{\mathrm{Hom}}(J, \mathcal{D}) \arrow[r, two heads, "\sim"] & \Pi_{f\in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}^{\cong}) \end{tikzcd}$$

is also a pullback and therefore the upper arrow is a trivial fibration. Composing it with the other one we get $\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$, which is then a trivial fibration and therefore an equivalence of ∞-categories.

7.1.4 Observe that, through this construction, one can always construct $L(\mathcal{C})$ so that γ is a bijection on objects because $\mathcal{C}' \to L(\mathcal{C})$ is an inner anodyne extension and therefore a retract of a countable composition of sums of pushouts of maps which are the identity on objects, that is the inner horn inclusions.

We now move on to proving that the localization is essentially unique. For this, we notice that γ establishes then an isomorphism between $\pi_0(k(\underline{\mathrm{Hom}}_W(\mathcal{C}, -)))$ and $\pi_0(\underline{\mathrm{Hom}}(L(\mathcal{C}), -)) = ho(\mathbf{sSet})(L(\mathcal{C}), -)$ with respect to the Joyal model structure, thus by Yoneda $(L(\mathcal{C}), \gamma)$ is unique up to unique isomorphism in $ho(\mathbf{sSet})$ and up to a contractible space of equivalences in **sSet**. □

Remark 4.5. 7.1.5

In this context, we may define $\overline{W}$, the saturation of W in $\mathcal{C}$, as the cartesian square

such that $\overline{W}$ is precisely the maximal simplicial subset of $\mathcal{C}$ whose morphisms are the ones which become invertible in $L(\mathcal{C})$, that is $\overline{W} \cong k(L(\mathcal{C})) \times_{L(\mathcal{C})} \mathcal{C}$ canonically.

We have then inclusions $Sk_1(W) \subset W \subset \overline{W}$ and, for any ∞-category $\mathcal{D}$, this induces equalities

$$\underline{\mathrm{Hom}}_{Sk_1(W)}(\mathcal{C}, \mathcal{D}) = \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) = \underline{\mathrm{Hom}}_{\overline{W}}(\mathcal{C}, \mathcal{D}),$$

implying that $(L(\mathcal{C}), \gamma)$ is also the localization of $\mathcal{C}$ by $Sk_1(W)$ and the one by $\overline{W}$, however the inclusion $\overline{W} \to \mathcal{C}$ is a fibration in the Joyal model category as it is the pullback of one, implying that $\overline{W}$ is itself an ∞-category.

We shall say that $\mathcal{C}$ is saturated if the canonical inclusion $W \to \overline{W}$ is an isomorphism of ∞-categories.

Remark 4.6. 7.1.6

The functor $ho(\mathcal{C}) \to ho(L(\mathcal{C}))$ exhibits $ho(L(\mathcal{C}))$ as the 1-categorical localization of $\mathcal{C}$ at $\mathrm{Arr}(\tau(W))$, as can be seen by using the universal property.

On the other hand, given a 1-category $\mathcal{C}$ and localizing at a set of morphisms W, not necessarily the induced map $L(N(\mathcal{C})) \to N(L(\mathcal{C}))$ is an isomorphism. Indeed, $L(N(\mathcal{C}))$ can have much better properties, as can be seen for example from 4.23, and in fact localizing 1-categories after taking their nerves gives every ∞-category as shown in [Cis19, Prop. 7.3.15].

Proposition 4.7. 7.1.9

Given a universe $\mathbf{U}$ and W a simplicial subset of a $\mathbf{U}$-small ∞-category $\mathcal{C}$, let $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ be the associated localization. Then the functor $\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful and its essential image consists of all presheaves $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ sending maps $u\colon x \to y$ in W to invertible maps $u^*\colon Fy \to Fx$ in $\mathcal{S}$.

Proof. The map γ gives us a morphism

$$\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C})^{\mathrm{op}},\mathcal{S}) \simeq \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}},\mathcal{S}) \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}},\mathcal{S}),$$

which has a left adjoint $\gamma_!$ and a right adjoint γ_*. Now, or any presheaf $F\colon L(\mathcal{C})^{\mathrm{op}} \to \mathcal{S}$, the unit map $F \to \gamma_*\gamma^* F$ is invertible and, by adjunction, the same goes for the counit map $\gamma_!\gamma^* F \to F$, which means that u^* is fully faithful. On the other hand, given a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ sending 1-simplices in W to invertible maps, the counit $\gamma^*\gamma_* F \to F$ and the unit $F \to \gamma^*\gamma_! F$ are both invertible since the restrictions of these adjunctions to $\underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}},\mathcal{S})$ form adjoint equivalences of ∞-categories as γ^* is an equivalence. □

Proposition 4.8. 7.1.10
Given an ∞-category $\mathcal{C}$ and a simplicial subset W, the localization functor $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ is final and cofinal. In particular, if $e\colon \Delta^0 \to \mathcal{C}$ encodes a final or a cofinal object, so does $\gamma(e)$.

Proof. First of all, the functor γ^{op} is also a localization, so it suffices to prove that γ is final. To do this, first we fix a universe $\mathbf{U}$ such that $\mathcal{C}$ is $\mathbf{U}$-small and then we remember that there is an adjunction $\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C}),\mathcal{S}) \rightleftarrows \underline{\mathrm{Hom}}(\mathcal{C},\mathcal{S}) :\gamma_*$. Since γ^* induces an equivalence when restricting the codomain to $\underline{\mathrm{Hom}}_W(\mathcal{C},\mathcal{S})$, we know that it is fully faithful, thus the unit of the adjunction is invertible, hence $1 \cong \gamma_*\gamma^*$. This gives us that

$$\lim_{\mathcal{C}} F \cong \lim_{\mathcal{C}} \gamma_*\gamma^* F \cong \lim_{L(\mathcal{C})} \gamma^* F,$$

for any presheaf $F\colon \mathcal{C} \to \mathcal{S}$, which is enough to prove that γ is final ([Cis19, Thm. 6.4.5]). □

Proposition 4.9. 7.1.11 Let's fix a universe $\mathbf{U}$, a $\mathbf{U}$-small ∞-category $\mathcal{C}$ and a simplicial subset W. Consider then a functor $f\colon \mathcal{C} \to \mathcal{D}$, where X is a small ∞-category. Then f exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ by W if and only if the following conditions hold:

1. the functor f sends the 1-simplices of W to invertible maps of $\mathcal{D}$;
2. the functor f is essentially surjective;
3. the functor f^* induces an equivalence of ∞-categories

$$f^*\colon \underline{\mathrm{Hom}}(\mathcal{D}^{\mathrm{op}},\mathcal{S}) \to \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}},\mathcal{S}).$$

Proof. One implication is trivial (for (2) look at the construction in 4.4). For the converse, let's pick a localization $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ and, through condition (1), we get a factorization $g\colon L(\mathcal{C}) \to \mathcal{D}$ such that $g \cdot \gamma \cong f$, giving us a triangle

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathcal{D}^{\mathrm{op}},\mathcal{E}) \arrow[rr, "g^*"] \arrow[dr, "f^*"'] & & \underline{\mathrm{Hom}}(L(\mathcal{C})^{\mathrm{op}},\mathcal{E}) \arrow[dl, "\gamma^*"] \\
& \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}},\mathcal{E}) &
\end{tikzcd}$$

commuting up to J-homotopy for any ∞-category $\mathcal{E}$. Picking $\mathcal{E} = \mathcal{S}$, γ^* and f^* are equivalences of ∞-categories, the latter by (3). It follows by 2-out-of-3 that g^* is one too, and therefore the same applies to its left adjoint $g_!$, which is then fully faithful. This is equivalent to g being fully faithful (FUN THEOREM, MAYBE STATE IT AT LEAST 6.1.5) and, since f is essentially surjective by (2), the same goes for g. It follows that g is an equivalence of ∞-categories. □

Proposition 4.10. 7.1.14
Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functors between ∞-categories with a right adjoint $g\colon \mathcal{D} \to \mathcal{C}$ and suppose that we are given simplicial subsets $V \subset \mathcal{C}$, $W \subset \mathcal{D}$ such that $f(V) \subset W$, $g(W) \subset V$. Then we can lift them to an adjunction $\overline{f}\colon L(\mathcal{C}) \rightleftarrows L(\mathcal{D}) :\overline{g}$ such that the diagrams

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "\gamma_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\gamma_{\mathcal{D}}"] \\
L(\mathcal{C}) \arrow[r, "\overline{f}"'] & L(\mathcal{D})
\end{tikzcd}, \quad
\begin{tikzcd}
\mathcal{D} \arrow[r, "g"] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{C} \arrow[d, "\gamma_{\mathcal{C}}"] \\
L(\mathcal{D}) \arrow[r, "\overline{g}"'] & L(\mathcal{C})
\end{tikzcd}$$

Proof. Let's write $\underline{\mathrm{Hom}}_V^W(\mathcal{C},\mathcal{D})$ for the full subcategory of $\underline{\mathrm{Hom}}(\mathcal{C},\mathcal{D})$ whose objects are functors ϕ such that $\phi(V) \subset W$. The equivalence $\gamma_{\mathcal{C}}^*\colon \mathrm{Hom}(L(\mathcal{C}), L(\mathcal{D})) \to \underline{\mathrm{Hom}}_V(\mathcal{C}, L(\mathcal{D}))$ allows us to construct a functor $\underline{\mathrm{Hom}}_V^W(\mathcal{C},\mathcal{D})$ $\underline{\mathrm{Hom}}_V(\mathcal{C}, L(\mathcal{D})) \to \underline{\mathrm{Hom}}(L(\mathcal{C}), L(\mathcal{D}))$ which associates to any ϕ as above a functor $\overline{\phi}$ making the square

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\phi"] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{C} \arrow[d, "\gamma_{\mathcal{C}}"] \\
L(\mathcal{D}) \arrow[r, "\overline{\phi}"'] & L(\mathcal{C})
\end{tikzcd}$$

commute up to J-homotopy.

The proof works by observing that our map also lifts natural transformations functorially, which allows us to show the triangle identities for the lifted unit and counit. □

Proposition 4.11. 7.1.18

Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories with a fully faithful right adjoint v and consider $W = k(\mathcal{D}) \times_{\mathcal{D}} \mathcal{C}$, the subcategory of maps of $\mathcal{C}$ which become invertible in $\mathcal{D}$. Then u exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ by W.

Proof. Given a localization $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ by W, we get a functor $\gamma \cdot v\colon \mathcal{D} \to L(\mathcal{C})$ which, paired with the $\overline{u}$ obtained from the construction in the previous proof, lifts the adjunction $u \dashv v$ to the localizations (where $L(\mathcal{D}) \cong \mathcal{D}$ as we localize at the identities). Lifting maintains the counit invertible, which allows us to conclude that $\gamma \cdot v$ is fully faithful.

Essential surjectivity follows from the fact that, for any object c in $\mathcal{C}$, the unit η_c is such that $\epsilon_{u(c)} \cdot u(\eta_c) = \mathrm{id}_{u(c)}$ and, since ϵ is invertible, so is $u(\eta_c)$, thus η_c becomes invertible in $L(\mathcal{C})$ and shows that $(\gamma_{\mathcal{C}} \cdot v)(u(c)) = \gamma_{\mathcal{C}}(vu(c)) \cong c$. Notice that here we used that $L(\mathcal{C})_0 = \mathcal{C}_0$, which is permissible up to equivalence as previously noted.

(PLEASE CHECK PROOF) □

Definition 4.12. An ∞-*category with weak equivalences and fibrations* is a triple $(\mathcal{C}, W, Fib)$ where $\mathcal{C}$ is an ∞-category with a final object, $W \subset \mathcal{C}$ is a subcategory with the 2-out-of-3 property and $Fib \subset \mathcal{C}$ a subsimplicial set such that:

1. for any morphism $p\colon x \to y$ in Fib (and W) with y fibrant, there is in $\mathcal{C}$ a pullback square

$$\begin{tikzcd}
x' \arrow[r, "u"] \arrow[d, "p'"'] & x \arrow[d, "p"] \\
y' \arrow[r, "v"'] & y
\end{tikzcd}$$

   where p' also lies in Fib (and W);

2. for any map $f\colon x \to y$ with fibrant codomain can be factored as a map in W followed by one in Fib.

By *fibrant object* we mean an object whose map to the terminal one is in Fib.

We shall call *weak equivalences* the maps in W and *fibrations* the ones in Fib. Maps which are both shall be referred to as *trivial fibrations*.

Construction 4.13. Any finitely complete ∞-category $\mathcal{C}$ can be given the structure of an ∞-category with weak equivalences and fibrations by setting $W = k(\mathcal{C})$, $Fib = \mathcal{C}$, which we will be doing henceforth.

Construction 4.14. For any ∞-category with weak equivalences and fibrations $\mathcal{C}$ and a fibrant object c, we can give to the slice category $\mathcal{C}/c$ the structure of an ∞-category with weak equivalences and fibrations by specifying as weak equivalences the morphisms which are mapped to weak equivalences of $\mathcal{C}$ by the projection $\mathcal{C}/c \to \mathcal{C}$ and similarly for the fibrations.

Definition 4.15. An ∞-category of fibrant objects is an ∞-category with weak equivalences and fibrations $\mathcal{C}$ in which all objects are fibrant.

Construction 4.16. For any ∞-category with weak equivalences and fibrations $\mathcal{C}$, its full subcategory given by fibrant objects is canonically an ∞-category of fibrant objects. We shall denote it by $\mathcal{C}_f$ and its weak equivalences are given by $W_f = W \cap \mathcal{C}_f$, its fibrations by $Fib_f = Fib \cap \mathcal{C}_f$.

Proposition 4.17 (Brown's Lemma)**.** 7.4.13
For any map $f\colon x \to y$ between fibrant objects in an ∞-category with weak equivalences and fibrations $\mathcal{C}$, there exists a commutative diagram of the form

$$\begin{tikzcd} & x \arrow[dl, equal] \arrow[d, "s"] \arrow[dr, "f"] & \\ x & z \arrow[l, two heads, "\sim", "p"'] \arrow[r, two heads, "q"'] & y \end{tikzcd},$$

where s is a weak equivalence, p a trivial fibration and q a fibration.

Proof. Since x and y are fibrant, the pullback of $x \to e$ and $y \to e$ exists and it corresponds to $x \times y$. The maps id_x, f define a cone over our cospan which induces a map $g\colon x \to x \times y$ and we then factor the latter as a weak equivalence $s\colon x \to z$ followed by a fibration $\pi\colon z \to x \times y$. We get then the desired maps $p = p_x \cdot \pi$, $q = p_y \cdot \pi$, where p_x, p_y denote the projections $x \times y \to x$, $x \times y \to y$ respectively. $\square$

Corollary 4.18. Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations, $\mathcal{D}$ an ∞-category and $V \subset \mathcal{D}$ a subcategory with the 2-out-of-3 property. If F sends trivial fibrations between fibrant objects into V, then it also sends weak equivalences between fibrant objects into V.

Proof. Looking at the commutative diagram

$$\begin{tikzcd} x \arrow[dd, equal] \arrow[dr, "s"] \arrow[drr, bend left, "f"] & & \\ & z \arrow[r, "q"] \arrow[dl, "p"] & y \\ x \arrow[urr, bend right, "f"'] & & \end{tikzcd},$$

given by Brown's Lemma 4.17 we see that Fp lies in V and therefore the same goes for Fs. Also, since f and s are weak equivalences we know that q is too, hence it is a trivial fibration. It follows that Fq is in V and the same goes for $Ff = Fq \cdot Fs$. $\square$

Construction 4.19. Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$ and a fibrant object z in it, we write $\mathcal{C}(z)$ for $(\mathcal{C}/z)_f$, that is the full subcategory $\mathcal{C}/z$ given by the fibrations $x \to z$ of $\mathcal{C}$. For any morphism $f\colon x \to y$ between fibrant objects, we have a left exact functor $f^*\colon \mathcal{C}(y) \to \mathcal{C}(x)$ induced pulling back along f (CAREFUL WITH THIS! YOU ARE USING IT LATER ON; CHECK [Cis19, Prop. 7.4.15]). The existence follows from the fact that pullbacks along fibrations with fibrant codomain exist, while left exactness comes from limits commuting and weak equivalences being preserved as a consequence of 4.18. (PERHAPS MORE DETAIL?)

Definition 4.20. Let $\mathcal{C}$, $\mathcal{D}$ be ∞-categories with weak equivalences and fibrations. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is *left exact* if it has the following properties:

1. the functor F preserves final objects;
2. the functor F sends (trivial) fibrations between fibrant objects to (trivial) fibrations;
3. the functor F preserves any pullback square in $\mathcal{C}$

$$\begin{tikzcd} x' \arrow[r, "u"] \arrow[d, "p'"'] & x \arrow[d, "p"] \\ y' \arrow[r, "v"'] & y \end{tikzcd}$$

where p is a fibration and y, y' are fibrant objects.

Remark 4.21. By Brown's Lemma, a left exact functor preserves weak equivalences between fibrant objects.

Remark 4.22. When considering a functor F between finitely complete ∞-categories, left exactness is equivalent to preserving finite limits.

Proposition 4.23. 7.5.6

Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, the localization $L(\mathcal{C}_f)$ has finite limits and the localization functor $\mathcal{C}_f \to L(\mathcal{C}_f)$ is left exact. Moreover, for any ∞-category $\mathcal{D}$ with finite limits and any left exact functor $f\colon \mathcal{C}_f \to \mathcal{D}$, the induced functor $\overline{F}\colon L(\mathcal{C}_f) \to \mathcal{D}$ is left exact.

Proof. Maybe do not prove it? It relies on a bunch of results from ch. 7.2, 7.3, 7.4 which we do not really want to prove.

7.1.10, 7.2.18, 7.2.25, 7.3.27, 7.4.13, 7.4.16

We know by 4.8 that $L(\mathcal{C}_f)$ has a final object, hence to show completeness it is enough to prove that it also has pullbacks ([Cis19, Thm. 7.3.27]). This can be done using the fact that any morphism in $L(\mathcal{C}_f)$ can be seen as a composition $\gamma(p) \cdot \gamma(s)^{-1}$, where s is a trivial fibration, for which Cisinski uses the theory of the *right calculus of fractions*, and the fact that γ_f preserves pullbacks along fibrations. The proof also shows us that all pullback squares in $L(\mathcal{C}_f)$ are isomorphic to images of pullback squares in $\mathcal{C}_f$ in which all maps are fibrations. □

Proposition 4.24. 7.5.16

Let x be a fibrant object in an ∞-category with weak equivalences and fibrations $\mathcal{C}$. The induced functor $\mathcal{C}_f/\gamma_f(x) \to \mathcal{C}/\gamma(x)$ is final.

Proof. We have that $\mathcal{C}_f/\gamma_f(x) = L(\mathcal{C}_f)/\gamma_f(x) \times_{L(\mathcal{C}_f)} \mathcal{C}_f$ and $\mathcal{C}/\gamma(x) = L(\mathcal{C})/\gamma(x) \times_{L(\mathcal{C})} \mathcal{C}$ and the functor we are considering is induced by $\bar{\iota}\colon L(\mathcal{C}_f) \to L(\mathcal{C})$.

To prove that it is final, it is sufficient to show that for any object (c, u) of $L(\mathcal{C})/\gamma(x)$ the coslice $(c, u)\backslash(\mathcal{C}_f/\gamma_f(x))$ is weakly contractible and, to do this, by [Cis19, Lem. 4.3.15] we can show that any functor $F\colon E \to (c, u)\backslash(\mathcal{C}_f/\gamma_f(x))$ where E is the nerve of a finite partially ordered set, is Δ^1-homotopic to a constant functor. This can be done through the theory of Reedy fibrant diagrams developed in [Cis19, Ch. 7.4]. □

Proposition 4.25. 7.5.17

Let $\mathbf{U}$ be a universe and $\mathcal{C}$ a $\mathbf{U}$-small ∞-category with weak equivalences and fibrations. For any ∞-category $\mathcal{D}$ with $\mathbf{U}$-small colimits and any functor $F\colon \mathcal{C} \to \mathcal{D}$, we have an isomorphism

$$(\gamma_f)_! \iota^*(F) \cong \bar{\iota}^* \gamma_!(F)$$

induced by the square

$$\begin{tikzcd} \mathcal{C}_f \arrow[r, "\iota"] \arrow[d, "\gamma_f"'] & \mathcal{C} \arrow[d, "\gamma"] \\ L(\mathcal{C}_f) \arrow[r, "\bar{\iota}"'] & L(\mathcal{C}) \end{tikzcd},$$

which commutes up to J-homotopy.

Proof. We only need to prove that the evaluation of the canonical map $(\gamma_f)_! \iota^*(F) \cong \bar{\iota}^* \gamma_!(F)$ at any object x of $\mathcal{C}_f$ is invertible. This evaluation is equivalent by [Cis19, Prop. 6.4.9] to the map

$$\operatorname{colim}_{\mathcal{C}_f/\gamma_f(x)} i^*(F)/\gamma_f(x) \to \operatorname{colim}_{\mathcal{C}/\gamma(x)} F/\gamma(x),$$

where $F/\gamma(x)$ is define by composing F with the canonical projection $\mathcal{C}/\gamma(x) \to \mathcal{C}$ and similarly for $i^*(F)/\gamma_f(x)$. Using 4.24 and the commutativity of the square above, we get that the desired map is indeed invertible for all x. □

Proposition 4.26. 7.5.18

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. The canonical functor $\bar{\iota}\colon L(\mathcal{C}_f) \to L(\mathcal{C})$ is an equivalence of ∞-categories, hence the ∞-category $L(\mathcal{C})$ is finitely complete and the localization functor $\gamma\colon \mathcal{C} \to L(\mathcal{C})$ is left exact.

Proof. 7.5.6, 7.5.17

We already know that $\overline{\iota}$ is essentially surjective as every object in $\mathcal{C}$ is weakly equivalent to one in $\mathcal{C}_f$ and the localization functors are essentially surjective themselves, thus it is enough to prove that it is fully faithful. To do this, we may fix a universe $\mathbf{U}$ such that $\mathcal{C}$ is $\mathbf{U}$-small and prove that the functor

$$\overline{\iota}_!\colon \underline{\mathrm{Hom}}(L(\mathcal{C}_f), \mathcal{S}) \to \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{S})$$

is fully faithful and use [Cis19, Prop. 6.1.15]. Remember that this full faithfulness condition is equivalent to the unit map $1 \to \overline{\iota}^*\overline{\iota}_!$ of the adjunction $\overline{\iota}_! \dashv \overline{\iota}^*$ being invertible.

We know that $\overline{\iota}_*$ and $\overline{\iota}^*$ both have right adjoints, thus they preserve colimits. Also, every $\mathcal{S}$-valued functor indexed by a $\mathbf{U}$-small ∞-category can be obtained as a colimit of representable ones, hence it is enough to check that the condition holds for any representable functor F. Also, γ_f is essentially surjective, which means that it is sufficient to check that map $(\gamma_f)_! \to \overline{\iota}^*\overline{\iota}_!(\gamma_f)_!$ which we get by precomposing the unit with $(\gamma_f)_!$ is invertible.

We have then the chain of isomorphisms

$$\begin{aligned}(\gamma_f)_! &\cong (\gamma_f)_!\overline{\iota}^*\overline{\iota}_! \\ &\cong \overline{\iota}^*\gamma_f\iota_! \\ &\cong \overline{\iota}^*\overline{\iota}_!(\gamma_f)_!,\end{aligned}$$

where the first isomorphism comes from the full faithfulness of ι, the second one from 4.25 and the last one the fact that $\overline{\iota}\cdot\gamma_f \cong \gamma\cdot\iota$, as noted in 4.25.

The second claim follows directly from the first one and 4.23. ☐

Remark 4.27. Here we see that the theory of localizations of ∞-categories with weak equivalences and fibrations provides much better results the 1-categorical equivalent, embodied by the homotopy theory of model categories and fibration categories (which we will define in the next chapter): indeed, these are particular cases of the ∞-analogue, however their homotopy categories, i.e. their 1-categorical localizations by weak equivalences, are almost never finitely complete.

Corollary 4.28. 7.5.19

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. For a morphism between fibrant objects $p\colon x \to y$, the following conditions are equivalent:

1. the morphism p has a section in $ho(L(\mathcal{C}))$;
2. there exists a morphism $p'\colon x' \to x$ s.t. the composition of p' and p is a weak equivalence;
3. there exists a fibration $p'\colon x' \to x$ s.t. the composition of p' and p is a weak equivalence.

Proof. 7.5.18

We see that (iii) trivially implies (ii), therefore we shall focus on the other implication. Given then such a morphism p', we factor it as $qi = p'$, a weak equivalence followed by a fibration. Since $p\cdot p' = p\cdot(q\cdot i) = (p\cdot q)\cdot i$, by 2-out-of-3 $p\cdot q$ is a weak equivalence, giving us what we wanted.

Should we prove (i)? Uses right calculus of fractions, but it's rather simple. ☐

Construction 4.29. 7.5.22

Given an ∞-category $\mathcal{C}$ with weak equivalences and fibrations, we can get another one $\overline{\mathcal{C}}$ with the same underlying ∞-category and class of fibrations, but where the weak equivalences are given by the saturation $\overline{W}$ as described in 4.5. We have that $L(\mathcal{C}) \cong L(\overline{\mathcal{C}})$, hence in general we can substitute $\mathcal{C}$ by $\overline{\mathcal{C}}$ with no issues. Also, the substitution commutes with the formation of slices over fibrant objects, that is, for any fibrant object x of $\mathcal{C}$, a map in $\mathcal{C}/x$ induces an invertible map in $L(\mathcal{C}/x)$ if and only if its image becomes invertible in $L(\mathcal{C})$, which can be seen as a consequence of 4.28.

Remark 4.30. Let $\mathcal{C}$ be an ∞-category with weak equivalences W and $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. The precomposition functor $\gamma^*\colon \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \to \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ does not have a left adjoint in general, but we may ask whether $\mathrm{Hom}(F, \gamma^*(-))$ is representable in $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D})$. If it is, a representative is denoted by $\mathbf{R}F\colon L(\mathcal{C}) \to \mathcal{D}$ and is called the *right derived functor of F*. Beware that to be precise one would have to specify the natural transformation $F \to \mathbf{R}F\cdot\gamma$ exhibiting it as such. Dually, a representative of $\mathrm{Hom}(\gamma^*(-), F)$ is the *left derived functor of F*.

Proposition 4.31. 7.5.24

If $F\colon \mathcal{C} \to \mathcal{D}$ sends weak equivalences to isomorphisms, then the functor $\overline{F}\colon L(\mathcal{C}) \to \mathcal{D}$, associated to F by the universal property of $L(\mathcal{C})$, is the right derived functor of F.

Proof. Let's fix a universe $\mathbf{U}$ such that $\mathcal{C}$ and $\mathcal{D}$ are $\mathbf{U}$-small and let $G\colon L(\mathcal{C}) \to \mathcal{D}$ be any functor. Then the invertible map $\overline{F} \cdot \gamma \cong F$ and the equivalence of ∞-categories $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \simeq \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$ induce invertible maps $\mathrm{Hom}(\overline{F}, G) \simeq \mathrm{Hom}(\overline{F} \cdot \gamma, G \cdot \gamma) \simeq \mathrm{Hom}(F, G \cdot \gamma)$ in $\mathcal{S}$, functorially in G. □

Construction 4.32. (NOT COMPLETE, ONE MAY SHOW THAT OUR CONSTRUCTION DOES GIVE THE RIGHT DERIVED FUNCTOR)

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. Any functor $F\colon \mathcal{C} \to \mathcal{D}$ sending weak equivalences between fibrant objects to invertible maps then has a right derived functor $\mathbf{R}F$, which may be constructed as follows.

First we choose a quasi-inverse $R\colon L(\mathcal{C}) \to L(\mathcal{C}_f)$ of the equivalence of ∞-categories specified in 4.26, then we pick a functor $\overline{F}\colon L(\mathcal{C}_f) \to \mathcal{D}$ and a natural isomorphism $j\colon \overline{F} \cdot \gamma_f \to F \cdot \iota$. We set then $\mathbf{R}F = \overline{F} \cdot R$.

What we are doing in this construction is selecting for every object in $\mathcal{C}$ a fibrant replacement, exactly like when we talk about right derived functors in the context of model categories. This is necessary because, a priori, we are not sending all weak equivalences to invertible maps in $\mathcal{D}$, hence we would have to show that before applying the universal property of localizations. Also, for any other functor $G\colon \mathcal{D} \to \mathcal{E}$, we have that $G \cdot \mathbf{R}F = \mathbf{R}(G \cdot F)$.

Definition 4.33. Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$ and an ∞-category with weak equivalences $\mathcal{D}$, let's consider a functor $F\colon \mathcal{C} \to \mathcal{D}$ preserving weak equivalences between fibrant objects of $\mathcal{C}$. We call the *right derived functor of* F the right derived functor of the composition

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\gamma_{\mathcal{D}}} L(\mathcal{D}),$$

where $\gamma_{\mathcal{D}}$ is the localization functor of of $\mathcal{D}$ at its weak equivalences. This right derived functor of F is denoted by $\mathbf{R}F$, that is $\mathbf{R}F = \mathbf{R}(\gamma_{\mathcal{D}} \cdot F)\colon L(\mathcal{C}) \to L(\mathcal{D})$, which makes sense since we can apply the construction 4.32.

There are some interesting remarks which may be included!!!!

Proposition 4.34. 7.5.28

For any left exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories with weak equivalences and fibrations, the right derived functor $\mathbf{R}F\colon L(\mathcal{C}) \to L(\mathcal{D})$ is left exact.

Proof. 7.5.6

We have a square

$$\begin{tikzcd} L(\mathcal{C}_f) \arrow[r, "\overline{F}"] \arrow[d] & L(\mathcal{D}_f) \arrow[d] \\ L(\mathcal{C}) \arrow[r, "\mathbf{R}F"'] & L(\mathcal{D}) \end{tikzcd}$$

commuting up to J-homotopy, where the vertical maps are equivalences of ∞-categories and $\overline{F}$ is the functor obtained by restricting F to the subcategories of fibrant objects $\mathcal{C}_f$ and $\mathcal{D}_f$. It therefore suffices to show that $\overline{F}$ is left exact, but this follows from 4.23. □

Remark 4.35. (WHY DO WE NEED TO SPECIFY THIS?)

For the remainder of this chapter, given an ∞-category $\mathcal{C}$, subcategories of weak equivalences $W \subset \mathcal{C}$ are such that the inclusion $W \to \mathcal{C}$ is an inner fibration. This means that a simplex $x\colon \Delta^n \to \mathcal{C}$ lies in W if and only if its edges $x|_{\Delta^{\{i,i+1\}}}\colon \{i, i+1\} \to \mathcal{C}$ lie in W for $0 \leq i < n$.

W then contains all invertible maps of $\mathcal{C}$ if and only if the aforementioned inclusion is an isofibration.

Definition 4.36. Let $\mathcal{C}$, $\mathcal{D}$ be ∞-categories with subcategories of weak equivalences $W \subset \mathcal{C}$, $W' \subset \mathcal{D}$. A functor $f\colon \mathcal{C} \to \mathcal{D}$ has the *right approximation property* if the following conditions hold:

1. a morphism in $\mathcal{C}$ is in W if and only if its image under f is in W';

2. given objects c, d in $\mathcal{C}$, $\mathcal{D}$ respectively and a map $\psi\colon d \to f(c)$ in $\mathcal{D}$, there is a map $\phi\colon c' \to c$ in $\mathcal{C}$ and a weak equivalence $u\colon d \to f(c')$ in $\mathcal{D}$ such that the triangle

$$\begin{tikzcd} d \arrow[r, "\psi"] \arrow[d, "u"'] & f(c) \\ f(c') \arrow[ur, "f(\phi)"'] & \end{tikzcd}$$

commutes.

Proposition 4.37. 7.6.2
A functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories such that the induced functor on the homotopy categories $ho(f)\colon ho(\mathcal{C}) \to ho(\mathcal{D})$ is an equivalence of categories has the right approximation property.

Proof. Consider a map $\psi\colon d \to f(c)$. Since $ho(f)$ is essentially surjective, there exists an invertible map $d \to ho(f)(c') = f(c')$ in $ho(\mathcal{D})$, which comes from an invertible map $d \to f(c')$ of $\mathcal{D}$. In $ho(\mathcal{D})$ we can then complete this to a triangle

$$\begin{tikzcd} d \arrow[r, "{[\psi]}"] \arrow[d, "\sim"'] & f(c) \\ f(c') \arrow[ur, "{[\phi]}"'] & \end{tikzcd}$$

and, since $ho(f)$ is fully faithful, ϕ can be lifted to $\tilde{\phi}\colon c' \to c$ in $ho(\mathcal{C})$. This gives us a commutative triangle

$$\begin{tikzcd} d \arrow[r, "\psi"] \arrow[d, "\sim"'] & f(c) \\ f(c') \arrow[ur, "f(\tilde{\phi})"'] & \end{tikzcd}$$

in $\mathcal{D}$. $\square$

Example 4.38. Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, the inclusion $\mathcal{C}_f \to \mathcal{C}$ has the right approximation property.

Example 4.39. Given a saturated ∞-category with weak equivalences and fibrations $\mathcal{C}$, the localization functor $\mathcal{C} \to L(\mathcal{C})$ has the right approximation property ([Cis19, Ex. 7.6.4]).

Theorem 4.40. 7.6.10
Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories with finite limits. If f commutes with them, then the following conditions are equivalent:

1. the functor f is an equivalence of ∞-categories;
2. the functor $ho(f)\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories;
3. the functor f has the right approximation property.

Proof. 7.3.29, 7.6.2, 7.6.5, 7.6.7
We trivially have that (i) implies (ii) and, by 4.37, (iii) follows from (ii), hence we only have to show that (iii) gives (i). Let's assume then that f has the right approximation property.

Given a final object e of $\mathcal{C}$, $f(e)$ is still final in $\mathcal{D}$ by 4.8, thus for any object d of $\mathcal{D}$ we have a map $d \to f(e)$ and, by the right approximation property, we get a commutative triangle with an isomorphism (SPECIFY WHICH STRUCTURE YOU ARE CONSIDERING ON THE ∞-CATEGORIES) $d \to f(c)$ for some c in $\mathcal{C}$, which gives us essential surjectivity.

We are still missing full faithfulness. To do this, we use that the right approximation property implies that we have an equivalence of ∞-groupoids $k(f)\colon k(\mathcal{C}) \to k(\mathcal{D})$ ([Cis19, Lem. 7.6.7]) and that, for any object c of $\mathcal{C}$, the map $\mathcal{C}/c \to \mathcal{D}/f(c)$ induced on the slices still has the right approximation property ([Cis19, Prop. 7.6.7]), therefore again we get an equivalence of ∞-groupoids $k(\mathcal{C}/c) \to k(\mathcal{D}/f(c))$.

Keeping these facts in mind, let's look at the projection $\mathcal{C}/c \to \mathcal{C}$. This functor is conservative, thus the square

$$\begin{tikzcd} k(\mathcal{C}/c) \arrow[r] \arrow[d] & \mathcal{C}/c \arrow[d] \\ k(\mathcal{C}) \arrow[r] & \mathcal{C} \end{tikzcd}$$

is a pullback. We observe by pasting that the pullback of $k(\mathcal{C}/c) \to k(\mathcal{C})$ along $c' \colon \Delta^0 \to k(\mathcal{C})$ is $\mathcal{C}(c', c)$, as it is clear from the diagram

$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r] \arrow[d] & k(\mathcal{C}/c) \arrow[r] \arrow[d] & \mathcal{C}/c \arrow[d] \\ \Delta^0 \arrow[r, "c'"'] & k(\mathcal{C}) \arrow[r] & \mathcal{C} \end{tikzcd} .$$

In the same way, we get that the pullback of $k(\mathcal{D}/f(c)) \to k(\mathcal{D})$ along $f(c') \colon \Delta^0 \to k(\mathcal{D})$ is $\mathcal{D}(f(c'), f(c))$. Since we have a commutative square

$$\begin{tikzcd} k(\mathcal{C}/c) \arrow[r] \arrow[d] & k(\mathcal{D}/f(c)) \arrow[d] \\ k(\mathcal{C}) \arrow[r] & k(\mathcal{D}) \end{tikzcd}$$

where the horizontal maps are equivalences of ∞-groupoids, the induced map $\mathcal{C}(c', c) \to \mathcal{D}(f(c'), f(c))$ is again an equivalence of ∞-groupoids, which is what we wanted.

Since f is essentially surjective and fully faithful, it is an equivalence of ∞-categories. $\square$

Corollary 4.41. 7.6.13

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations and consider a localization functor $\gamma \colon \mathcal{C} \to L(\mathcal{C})$. For any fibrant object x of $\mathcal{C}$, the canonical functor $\mathcal{C}/x \to L(\mathcal{C})/\gamma(x)$, $t \mapsto \gamma(t)$, induces an equivalence of ∞-categories $L(\mathcal{C}/x) \simeq L(\mathcal{C})/\gamma(x)$.

Proof. 7.5.18, 7.5.22, 7.5.24, 7.5.28, 7.6.4, 7.6.10

By 4.29, we can assume that $\mathcal{C}$ is saturated. Our objective is to show that the induced functor $\phi \colon L(\mathcal{C}/x) \to L(\mathcal{C})/\gamma(x)$ has the right approximation property and it preserves finite limits, which will allow us to apply 4.40 and conclude.

To show condition (1) we only need to prove that ϕ is conservative, which can be reduced to showing that a map in $\mathcal{C}/x$ becomes invertible in $L(\mathcal{C}/x)$ if and only if it becomes an insomorphism in $L(\mathcal{C})$. This however is true by saturation of $\mathcal{C}$. We still need to check condition (2), which can be done on $\mathcal{C}/x \to L(\mathcal{C})/\gamma(x)$, but this follows from the fact that γ has it, as mentioned in 4.39.

To apply 4.40 we still need to show that ϕ preserves limits. To do this, we use the fact that $\mathcal{C}/x$ has the structure of an ∞-category with weak equivalences and fibrations. Given that γ is left exact, the functor $\mathcal{C}/x \to L(\mathcal{C})/\gamma(x)$ maps weak equivalences to isomorphisms and we can apply 4.31 to prove that ϕ is its right derived functor. Finally, through 4.34 we get that ϕ is also left exact. $\square$

Theorem 4.42. 7.6.16

Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. Given a fibrant object x, let $\mathcal{C}(x)$ be the full subcategory of fibrant objects of $\mathcal{C}/x$ (INCLUDE 7.6.12), which will be an ∞-category of fibrant objects. Assume that, for any fibration between fibrant objects $p \colon x \to y$, the pullback functor $p^* \colon \mathcal{C}(y) \to \mathcal{C}(x)$, $(y' \to y) \mapsto (y' \times_y x \to x)$ has a right adjoint $p_* \colon \mathcal{C}(x) \to \mathcal{C}(y)$ preserving trivial fibrations. Then, for any map $p \colon x \to y$ in $L(\mathcal{C})$, the pullback functor $p^* \colon \mathcal{C}(y) \to \mathcal{C}(x)$ has a right adjoint.

Proof. 6.1.6, 6.1.7, 6.1.8, 7.1.14, 7.4.14, 7.5.18, 7.6.13

Given a localization functor $\gamma\colon \mathcal{C} \to L(\mathcal{C})$, one reduces the problem to proving that, for any fibration between fibrant objects $p\colon x \to y$, the pullback functor

$$\gamma(p)^*\colon L(\mathcal{C})/\gamma(y) \to L(\mathcal{C})/\gamma(x)$$

has a right adjoint.

A consequence of Brown's Lemma 4.17 is that any functor preserving trivial fibrations between fibrant objects also preserves weak equivalences, and, since $p^*\colon \mathcal{C}(y) \to \mathcal{C}(x)$ has both a right and a left adjoint named p_* and $p_!$ (the latter given by post-composing with p) which do preserve them, by 4.10 we have a pair of adjunctions on the localizations, namely

$$\overline{p}^*\colon L(\mathcal{C}(y)) \rightleftarrows L(\mathcal{C}(x)) :\overline{p}_*,$$
$$\overline{p}_!\colon L(\mathcal{C}(x)) \rightleftarrows L(\mathcal{C}(y)) :\overline{p}^*.$$

Given that $(\mathcal{C}/z)_f = \mathcal{C}(z)$ for all fibrant objects z of $\mathcal{C}$, by 4.26 we have that $L(\mathcal{C}(z)) \simeq L(\mathcal{C}/z)$ and, by 4.41, we also know that $L(\mathcal{C}/z) \simeq L(\mathcal{C})/\gamma(z)$, hence $L(\mathcal{C}(z)) \simeq L(\mathcal{C})/\gamma(z)$. Notice that $\overline{p}_!$ is equivalent to $\gamma(p)_!\colon L(\mathcal{C})/\gamma(x) \to L(\mathcal{C})/\gamma(y)$ and, by essential uniqueness of the adjoints, this extends to $\gamma(p)^*$ and $\overline{p}^*$, therefore $\gamma(p)^*$ has a right adjoint induced by $\overline{p}_*$. □

5 Categorical Models of TT as Locally Cartesian Closed Fibration Categories

To apply the results from the previous section, we first need to specify a fibrational structure on categorical models of type theory.

Definition 5.1. A *fibration category* is a triple $(\mathcal{P}, W, Fib)$ where $\mathcal{P}$ is a category and W, Fib are wide subcategories such that:

1. $\mathcal{P}$ has a terminal object;
2. maps to the terminal object lie in Fib;
3. Fib and $W \cap Fib$ are closed under pullback along any map in $\mathcal{P}$;
4. every map in $\mathcal{P}$ can be factored as a map in W followed by one in Fib;
5. W has the 2-out-of-6 property.

Remark 5.2. Seeing $\mathcal{P}$, W and Fib in the above definition as ∞-categories, we notice that the triple has canonically the structure of an ∞-category with weak equivalences and fibrations with fibrant objects, hence we shall adopt the conventions we used in that context.

Now we have to specify the classes of maps which will provide the desired structure.

Definition 5.3. Given a categorical model of type theory $\mathcal{C}$, a morphism $f\Gamma.A \to \Gamma.B$ over Γ is *simply bi-invertible over* Γ if there exist:

1. a morphism$g_1\colon \Gamma.B \to \Gamma.A$;
2. a section $\eta\colon \Gamma.A \to \Gamma.(1_{\Gamma.A}, g_1 f)^* \operatorname{Id}_A$ of the canonical projection $p_{(1_{\Gamma.A}, g_1 f)^* \operatorname{Id}_A}\colon \Gamma.(1_{\Gamma.A}, g_1 f)^* \operatorname{Id}_A \to \Gamma.A$;
3. a morphism $g_2\colon \Gamma.B \to \Gamma.A$;
4. a section $\epsilon\colon \Gamma.A \to \Gamma.(1_{\Gamma.A}, fg_2)^* \operatorname{Id}_A$ of the canonical projection $p_{(1_{\Gamma.A}, fg_2)^* \operatorname{Id}_A}\colon \Gamma.(1_{\Gamma.A}, fg_2)^* \operatorname{Id}_A \to \Gamma.A$.

We now generalize the above definition to general context extensions by working as usual with $\mathcal{C}^{cxt}$.

Definition 5.4. Given a categorical model of type theory $\mathcal{C}$, a morphism $f\colon \Gamma.\Delta \to \Gamma.\Theta$ over Γ is *bi-invertible over* Γ if it is as a morphism in $\mathcal{C}^{cxt}$. It is simply called *bi-invertible* when Γ is the terminal context.

Definition 5.5. A fibration category $\mathcal{P}$ is *locally cartesian closed* if, for any fibration $p\colon a \to b$, the pullback functor $p^*\colon \mathcal{P} \downarrow b \to \mathcal{P} \downarrow a$ admits a right adjoint p_* which is an exact functor.

Remark 5.6. Our interest in bi-invertible morphisms stems from the fact that they model the right notion of invertible map in dependent type theory: indeed, from the required data for a simply bi-invertible map f over Γ we can provide a section $\Gamma \to \Gamma.isHIso(f)$ of the dependent projection $p_{\Gamma.isHIso(f)}\colon \Gamma.isHIso(f) \to \Gamma$ REFS, which is the translation into the language of contextual category of the notion of bi-invertible map in type theory MORE REFS.

It is important to note that, while type theory has no way to encode internally the concept of isomorphism of the contextual model, it does have its own internal notion of isomorphism. However, given a map f, the type $isIso(f)$ is not, in general, a *mere proposition*. On the other hand, the more lax notion of bi-invertibility is such that $isHIso(f)$ is a mere proposition (UNDER WHICH HP? LEMMA 5.12 from 1203.3253), which makes it preferable. Also, every bi-equivalent map can be given the structure of an isomorphism and viceversa, hence they are closely related.

Proposition 5.7. (REFS, 1507, AKL15, 1610.00037) A categorical model of type theory $\mathcal{C}$ carries the structure of a fibration category where maps isomorphic to dependent projections are the fibrations and bi-invertible ones are the weak equivalences.

(CAREFUL: AS STATED IN AKL15, WE ALSO NEED THE UNIT TYPE AND OTHER STUFF! BUT KAPULKIN DOESN'T TAKE IT IN 1507)

The above result can be however generalized.

Proposition 5.8. A contextual category with an Id-structure $\mathcal{C}$ carries the structure of a fibration category similar to the one above.

Proof. As noted in AKL15, 1507 and 1610.00037, the previous result does not depend on a Σ-structure because all we need is that every dependent projection is isomorphic to a basic one, which we have by working with context extensions in $\mathcal{C}^{cxt}$. Indeed, after constructing the fibrational structure there, we may lift it back through the equivalence of categories $\mathcal{C} \to \mathcal{C}^{cxt}$. $\square$

How does Kapulkin prove that a categorical model of Type Theory is a locally cartesian closed fibration category?

First of all, he refers to AKL15 to show that $\mathcal{P}$ has a fibrational structure, then he goes on to show the following results, whose proofs are extremely terse and therefore should be expanded.

Definition 5.9. Given $p_A\colon \Gamma.A \to \Gamma$, a section $a\colon \Gamma \to \Gamma.A$ and $f\colon \Delta \to \Gamma$, we look at the commutative diagram

$$\begin{tikzcd}
\Delta \arrow[dr, dotted, "f^*a"] \arrow[drr, bend left, "a\cdot f"] \arrow[ddr, bend right, equal] & & \\
& \Delta.f^*A \arrow[r, "q(f,A)"'] \arrow[d, "p_{f^*A}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\
& \Delta \arrow[r, "f"'] & \Gamma
\end{tikzcd},$$

which gives us f^*a as the factorization through the pullback square of the pair $(\mathrm{id}_\Delta, a \cdot f)$.

(THIS CAN BE JUSTIFIED BY LOOKING AT LEMMA 2.15 IN 1706.03605. THE FOLLOWING RESULT SHOWS THAT THIS ASSIGNMENT IS ALSO FUNCTORIAL.)

Lemma 5.10. For any dependent projection $p_\Delta\colon \Gamma.\Delta \to \Gamma$ in a categorical model of type theory $\mathcal{C}$, the pullback functor $p_\Delta^*\colon \mathcal{C} \downarrow \Gamma \to \mathcal{C} \downarrow \Gamma.\Delta$ admits a right adjoint.

Proof. Let's set $(p_\Delta)_*(\Gamma.\Delta.\Theta) = \Gamma.\Pi(\Delta.\Theta)$. Our counit shall be given by

$$\epsilon_{\Gamma.\Delta.\Theta} \colon \Gamma.\Delta.p_\Delta^*\Pi(\Delta,\Theta) \xrightarrow{exch_{\Delta,\Pi(\Delta,\Theta)}} \Gamma.\Pi(\Delta,\Theta).p_{\Pi(\Delta,\Theta)}^*\Delta \xrightarrow{\mathrm{app}_{\Delta,\Theta}} \Gamma.\Delta.\Theta$$

and it is then sufficient to prove that, for any context morphism $f \colon \Gamma.\Delta.p_\Delta^*\Psi \to \Gamma.\Delta.\Theta$ over $\Gamma.\Delta$, there is a unique $\tilde{f} \colon \Gamma.\Psi \to \Gamma.\Pi(\Delta,\Theta)$ making the diagram

$$\begin{tikzcd}
\Gamma.\Delta.p_\Delta^*\Psi \arrow[d, dotted, "p_\Delta^*\tilde{f}"'] \arrow[dr, "f"] & \\
\Gamma.\Delta.p_\Delta^*\Pi(\Delta,\Theta) \arrow[r, "\epsilon_{\Gamma.\Delta.\Theta}"'] & \Gamma.\Delta.\Theta
\end{tikzcd}$$

commute. This will then uniquely specify how the right adjoint acts on the morphisms.

We start by specifying the unit $\eta_{\Gamma.\Psi} \colon \Gamma.\Psi \to \Gamma.\Pi(\Delta, p_\Delta^*\Psi)$.

Let's consider the commutative square

$$\begin{tikzcd}
\Gamma.\Psi.p_\Psi^*\Pi(\Delta,p_\Delta^*\Psi) \arrow[r, "q(p_\Psi,\Pi(\Delta,p_\Delta^*\Psi))"] \arrow[d, "p_{\Pi(\Delta,p_\Delta^*\Psi)}"'] & \Gamma.\Pi(\Delta,p_\Delta^*\Psi) \arrow[d, "p_{\Pi(\Delta,p_\Delta^*\Psi)}"] \\
\Gamma.\Psi \arrow[r, "p_\Psi"'] & \Gamma
\end{tikzcd}$$

where the map $q(p_\Psi,\Pi(\Delta,p_\Delta^*\Psi))$ acts by forgetting the term of Ψ. If we can provide a section of the vertical map on the left pointing to the term $\lambda a.const_a$, we are done as we can then compose it with $q(p_\Psi,\Pi(\Delta,p_\Delta^*\Psi))$ to get our unit.

We construct it by looking at the commutative square

$$\begin{tikzcd}
\Gamma.\Psi \arrow[dr, dotted, "{(1_\Psi,1_\Psi)}"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& \Gamma.\Psi.p_\Psi^*\Psi \arrow[r, "{q(p_\Psi,\Psi)}"'] \arrow[d, "p_{p_\Psi^*\Psi}"] & \Gamma.\Psi \arrow[d, "p_\Psi"] \\
& \Gamma.\Psi \arrow[r, "p_\Psi"'] & \Gamma
\end{tikzcd},$$

which gives us the term $[\Gamma, x : \Psi, x : \Psi]$. Then, we pull back along $p_{p_\Psi^*\Delta}$, getting a section $p_{p_\Psi^*\Delta}^*(1_\Psi, 1_\Psi) \colon \Gamma.\Psi.p_\Psi^*\Delta \to \Gamma.\Psi.p_\Psi^*\Delta.p_{p_\Psi^*\Delta}^*\Psi$. We then apply λ, which gives us a section

$$\lambda(p_{p_\Psi^*\Delta}^*(1_\Psi, 1_\Psi)) = \lambda(1_{p_\Psi^*\Delta}, p_{p_\Psi^*\Delta}) \colon \Gamma.\Psi \to \Gamma.\Psi.p_\Psi^*\Pi(\Delta, p_\Delta^*\Psi)$$

and we can then conclude by post-composing with $q(p_\Psi,\Pi(\Delta,p_\Delta^*\Psi)) \colon \Gamma.\Psi.p_\Psi^*\Pi(\Delta,p_\Delta^*\Psi) \to \Gamma.\Pi(\Delta,p_\Delta^*\Psi)$, which provides our unit.

We then define our lift $\tilde{f}$ as the composite $\Gamma.\Pi(\Delta,f) \cdot \eta_{\Gamma.\Psi}$. The commutativity of the above triangle follows by β-reduction, while the uniqueness of the $\tilde{f}$ giving the desired factorization by the Π_η-property. (PLEASE PROVE IT)

This also shows that $(p_\Delta)_*(f) = \Gamma.\Pi(\Delta,f)$, meaning that the construction of $\Gamma.\Pi(\Delta,f)$ is functorial. $\square$

We know that every fibration in $\mathcal{C}$ is isomorphic to a composite of dependent projections , so this tells us that every fibration induces an adjunction between fibrational slices (YOU SHOULD DEFINE THEM, MAYBE AS INFTY-CATS OF FIBRANT OBJECTS INDUCED FROM THE SLICES).

We are finally ready to prove the result leading to the final one we want.

Proposition 5.11. A categorical model of type theory $\mathcal{C}$ is a locally cartesian closed fibration category.

Proof. Kapulkin 1507.02648, Proposition 5.4.

We already know that it is a fibration category by PREVIOUS RESULT. Also, for any basic dependent projection $\Gamma.\Delta \to \Gamma$, the pullback functor between fibrant slices p^*_Δ has a right adjoint $(p_\Delta)_*$ by the previous lemma. We only need to check that this right adjoint is compatible with the fibrational structure.

As a right adjoint, $(p_\Delta)_*$ preserves limits and in particular pullbacks and the terminal object. Also, by LEMMA ABOUT PI OBJECTS BEING EQUAL, it preserves dependent projections and, by LEMMA ABOUT PRESERVING BI-INVERTIBLE MAPS, weak equivalences. ☐

Theorem 5.12. Given a categorical model of type theory $\mathcal{C}$, the ∞-category $L(\mathcal{C})$ is locally cartesian closed.

Proof. Since a fibration category is more generally a ∞-category with fibrations and weak equivalences, we can apply 4.42 as the hypothesis are satisfied by 5.11. ☐

6 What the hell are these things?

In the proofs various objects are mentioned. Here we explain the constructions.

First of all, in 1211, Section B.1.1, to specify what it means to have a Π structure they define the *app* map taking sections $a\colon \Gamma \to \Gamma.A$ and $f\colon \Gamma \to \Gamma.\Pi(A,B)$ of the obvious dependent projections and returning a section $app(f,a)\colon \Gamma \to \Gamma.A.B$ such that $p_b \cdot app(f,a) = a$. This tells us that $app(f,a) : B(a)$ is essentially what we get when we apply the map $f : \Pi(A,B)$ to $a : A$.

Also, given a section $b\colon \Gamma.A \to \Gamma.A.B$ we have a map $\lambda(b)\colon \Gamma \to \Gamma.\Pi(A,B)$. This turns a term $b : B(x)$ dependent on $x : A$ into a map $\lambda(b)\Pi(A,B)$ associating $x : A$ to b and indeed we have that $app(\lambda(b),a) = b \cdot a$.

The context substitution properties are as expected.

We want to describe explicitly what the map $app_{A,B}\colon \Gamma.\Pi(A,B).p^*_{\Pi(A,B)}A \to \Gamma.A.B$ is. Intuitively, it should model the map sending a term $f : \Pi(A,B)$ and a term $a : A$ to $f(a) : B$. To construct it we first need to specify a few things.

MOVE IT

7 Pushforward

One may ask whether cocartesian fibrations in **sSet** model Pi types, which is a piece needed to understand a novel model of dependent type theory provided by **sSet**. To answer this question, an explicit description of the right adjoint p_* of the pullback functor $p^*\colon \mathbf{sSet}/Y \to \mathbf{sSet}/X$ induced by a morphism $p\colon X \to Y$ is needed.

Consider an object $f\colon T \to X$ in $\mathbf{sSet}/X$. What is $p_*(f)\colon T' \to Y$? We know that a n-simplex t' of T' corresponds bijectively to a map $t'\colon \Delta^n \to T'$, which in turn corresponds bijectively to a commutative diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "t'"] \arrow[dr, "y"'] & & T' \arrow[dl, "p_*(f)"] \\ & Y & \end{tikzcd}$$

and, under the adjunction $p^* \dashv p_*$, we get bijectively another commutative diagram

$$\begin{tikzcd} U \arrow[rr, "t"] \arrow[dr, "p^*(y)"'] & & T \arrow[dl, "f"] \\ & X & \end{tikzcd},$$

from which follows that

$$T'_n \cong \{(y,t) \mid y \in Y_n,\ t \in \mathbf{sSet}/X(p^*(y), f)\}$$

and the map $p_*(f)$ then sends $(y,t) \in T'_n$ to $y \in Y_n$.

The same method can be extended to give us the pushforward along a map of marked simplicial sets $p\colon (X, E_X) \to (Y, E_Y)$ in **mSet**. Specifically, our previous construction can be adapted to give us the n-simplices by starting from maps $(\Delta^n)_\flat \to p_*(T, E_T) = (T', E_{T'})$, telling us again that

$$T'_n \cong \{(y, t) \mid y \in Y_n,\ t \in \mathbf{mSet}\,/X(p^*(y), f)\},$$

while to get the markings we notice that every marked edge in $p_*(T, E_T)$ corresponds to a unique map $(\Delta^1)_\natural \to p_*(T, E_T)$ and the same procedure allows us to write

$$E_{T'} = \{(y, t) \mid y \in E_Y,\ t \in \mathbf{mSet}\,/X(p^*(y), f)\},$$

which fully specifies the needed data.

Now, under which conditions on p does this specify a Quillen adjunction when the slices of **mSet** are equipped with the contravariant model structure? If it is a coCartesian fibration, it generally doesn't, but it does when it is a Cartesian fibration. How can we specify an approximation q of p_* such that, after localizing in the infinity-sense, we get an adjunction $p^* \dashv q$?

Idea: use the theory of bifibrations. From a coCartesian fibration $\phi\colon X \to Y$ we can construct a bifibration $E \to X \times Y$ by constructing the maps $p\colon E \to X$, $q\colon E \to Y$ by first taking the pullback of ϕ along $ev_0\colon Y^{\Delta^1} \to Y$ and then composing the map $E \to Y^{\Delta^1}$ with ev_1.

$$\begin{tikzcd}
E \arrow[r, "p"] \arrow[d] \arrow[dd, bend right, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\
Y^{\Delta^1} \arrow[r, "ev_0"] \arrow[d, "ev_1"] & Y \\
Y &
\end{tikzcd}$$

We want to show that, for any coCartesian morphisms $f\colon A \to Y$, $g\colon A \to X$, we have an equivalence between $\mathbf{Map}_(\phi^* f, g)$ and $\mathbf{Map}(f, q_* p^*(g))$.

Canonically, we have

$$E_n = \{(x, \Delta^n \times \Delta^1 \xrightarrow{g} Y) \mid x \in X_n,\ \phi(x) = g|_{\Delta^n \times \{0\}}\}$$

and, by pasting the pullback squares, we also get

$$(\mathrm{dom}(p^* f))_n = \{(a, \Delta^n \times \Delta^1 \xrightarrow{g} Y) \mid a \in A_n,\ f(a) = g|_{\Delta^n \times \{0\}}\}$$

and therefore

$$\begin{aligned}
(\mathrm{dom}(q_* p^*(f)))_n &= \{(y, q^*(y) \xrightarrow{g} p^*(f)) \mid y \in Y_n\} \\
&= \{(y, p_! q^*(y) \xrightarrow{g} f) \mid y \in Y_n\},
\end{aligned}$$

which we want to relate to $\phi_*(f)$.

To do this, we want to understand the maps $p_! q^*(y) \xrightarrow{g} f$ and somehow relate them to $\phi^*(y) \to f$. By definition,

$$\begin{aligned}
\mathrm{dom}(p_! q^*(y))_k &= \mathrm{dom}(q^*(y))_k \\
&= \{(x, \Delta^k \times \Delta^1 \xrightarrow{h} Y, t) \mid x \in X_k,\ \phi(x) = h|_{\Delta^k \times \{0\}},\ t \in (\Delta^n)_k,\ y(t) = h|_{\Delta^k \times \{1\}}\},
\end{aligned}$$

with $q^*(y)(x, h, t) = (x, h)$, thus $p_! q^*(y)(x, h, t) = x$.

On the other hand, we have

$$\mathrm{dom}(\phi^*(y))_k = \{(x, t) \mid x \in X_k,\ t \in (\Delta^n)_k,\ \phi(x) = y(t)\}$$

and $\phi^*(y)(t, x) = x$.

If we can create a bijection between morphisms of the form $p_! q^*(y) \to f$ and $\phi^*(y) \to f$ we are done. Unfortunately, I do not see how we can do this: any morphism $p_! q^*(y) \to f$ induces a morphism $\phi^*(y) \to f$ by precomposing with the inclusion $\phi^*(y) \to p_! q^*(y)$, $(x,t) \mapsto (x, h_{\phi(x)}, t)$, where $h_{\phi(x)}$ is obtained by precomposing $\phi(x)\colon \Delta^k \to Y$ with $p_{\Delta^k}\colon \Delta^k \times \Delta^1 \to \Delta^k$, but this association is only injective, not surjective, and I have no good idea about how to construct others.

To construct the bijection I may start from a morphism $p_! q^*(y) -> f$ and construct another one with $\phi^*(y)$ as domain by lifting morphisms $\Delta^k \times \Delta^1 \to Y$ to decide where to map $(x,h,t) \in \mathrm{dom}(p_! q^*(y))_k$, however this involves solving a coherence problem and I would have to do so coherently to define a morphism of simplicial sets as desired. Perhaps these restrictions actually allow a solution, but I do not believe so.

It may also be possible that the injective morphism we mentioned earlier is a weak equivalence with respect to our model structure, which may be enough.

We provide a counterexample to the previous claim in the context of right fibrations. Consider $\phi\colon \partial\Delta^1 \to \Delta^1$, $i \mapsto 0$, which is a right fibration. We have that $\phi^*(1) = 0$, the empty simplicial set, thus $\phi_*(f)^{-1}(1) \cong \Delta^0$. On the other hand, $(p_! q^*)(1) = U \amalg V$, thus $q_* p^*(f)^{-1}(1)$ can be a disjoint union of non-zero simplicial sets, which would then not be an equivalent ∞-groupoid. It follows that our map is not, in general, a weak equivalence in the model structure of right fibrations on slices of **sSet**. (MAYBE WRONG: YOU CAN'T CHECK THIS ON FIBERS BECAUSE THESE ARE NOT FIBRANT OBJECTS IN $\mathbf{sSet}/Y$! NEED TO USE THE DEFINITION CONCERNING HOMOTOPY CLASSES OF MAPS INTO FIBRANT OBJECTS IN THE SLICE)

If instead from a right fibration ϕ we first take the opposites of ϕ and f, then do the pushforward and finally we take again the opposites we get $\phi_*(f)$, which is encoded in the following commutative diagram where the vertical maps are isomorphisms.

$$\begin{tikzcd}
f\colon Z \to X \arrow[d, mapsto] & \mathbf{sSet}/X \arrow[r, "\phi_*"] \arrow[d, "\mathrm{op}"'] & \mathbf{sSet}/Y \arrow[d, "\mathrm{op}"] & f\colon Z \to Y \arrow[d, mapsto] \\
f^{\mathrm{op}}\colon Z^{\mathrm{op}} \to X^{\mathrm{op}} & \mathbf{sSet}/X^{\mathrm{op}} \arrow[r, "\phi_*^{\mathrm{op}}"'] & \mathbf{sSet}/Y^{\mathrm{op}} & f^{\mathrm{op}}\colon Z^{\mathrm{op}} \to Y^{\mathrm{op}}
\end{tikzcd}$$

The same argument extends to show that any map $\phi_*(f) \to q_* p^*(f)$ or in the other direction is not a weak equivalence in general. A concrete example can be given by taking $f = \phi$.

References

[Cis19] D. Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2019. ISBN: 9781108473200. URL: `https://books.google.it/books?id=RawqvQEACAAJ` (cit. on pp. 1, 12, 14, 15, 17–19, 21, 22).