The objective of this thesis is to provide a more modern proof of a result by Kapulkin stating that the quasi-categorical localization of a categorical model of type theory is a locally cartesian closed ∞-category. Henceforth, I will say "localization" to refer to the ∞-categorical one.

Dependent type theory talks about dependent types A over contexts Γ and their terms. It provides structural rules telling us that we can substitute definitionally equal variables and types. There are then logical rules specifying how to construct the objects we talk about, in particular Σ-types $\Sigma(A, B)$, that is the type of pairs (a, b) of elements, Π-types $\Pi(A, B)$, the type of maps from A to B, and Id-types Id_A, the type of proof of equalities between terms of A.

To study a theory, logicians work with its models, but here it's hard: structural rules require lots of checks. A solution is defining a class of algebraic models with simple axioms.

Such a class is given by contextual categories, categories specifying for each object a length and a father function to reduce them. Each object $\Gamma.A$ has a map to its reduction p_A which we call a basic dependent projection. The notation specifies that $\Gamma.A$ extends Γ by 1 and p_A wants to express that A is a dependent type in context Γ. Also, we want functorial pullbacks of basic dependent projections because they describe substitutions, which are strictly associative. This structure models the structural rules.

To model the logical rules we need some extra structure. For Id-types, for $\Gamma.A$ we need an identity object, for Π-types we want a hom object, a map describing function evaluation and so on. What we are interested in are categorical models of type theory, that is contextual categories C with Σ, Id and Π structures.

We want to talk about localizations. The right notion of invertibility in dependent type theory is bi-invertibility and for a map $f\colon \Gamma \to \Delta$ in a contextual category to model it it needs maps g_1, g_2 in the opposite direction and maps η, ϵ which tell us that they are pointwise left and right inverse to f. Thinking about types as spaces, as suggested by the groupoidal model, identity types are path spaces, hence η and ϵ are more like homotopies. What if we localize at bi-invertible maps? To answer that question we need some tools.

An ∞-category C with weak equivalences W and fibrations Fib is a triple generalizing the 1-categorical notion of fibration category. Essentially, we have some stability under pullbacks of fibrations, trivial fibrations, the factorization condition and other things. A contextual category with Σ and Id structures is one if we take bi-invertible maps as weak equivalences and maps isomorphic to dependent projections as fibrations, so this also applies to categorical models of type theory.

Cisinski's theory of localizations tells us that localizing such an ∞-category at weak equivalences always gives us a finitely complete ∞-category, but also if there is a right adjoint to the pullback functor induced by a fibration f from x to y, both fibrant, on the fibrant slices they define and the right adjoint preserves trivial fibrations, then the localization is also locally cartesian.

We now state the theorem we mentioned at the beginning. Given a categorical model of type theory C, its localization is a locally cartesian closed ∞-category. It is enough to construct a right adjoint to the pullback functor induced by p_A from $\Gamma.A$ to Γ, which we do by sending $\Gamma.A.\Theta$ to $\Gamma.\Pi(A, \Theta)$ and using the function evaluation map as a counit. Here the Π-object on the right is defined by extending the basic Π-structure.

Why is dependent type theory interesting? It's a foundation for mathematics which has become of interest because it is closely linked to computer science and it allows the creation of proof assistants like Agda and Lean, that is software which allows us to formalize mathematical reasoning, easily check proofs and even write some tedious bits automatically. There are already ongoing projects to formalize all of mathematics, like the math library. Secondly, this theory merges the two strata of classical foundations into one, generalizing both sets and propositions through types. Thirdly, we can reason about proofs within the theory, without a metatheory. Finally, equality is more structured: only objects of the same type can be compared and a proof of equality specifies in what sense two objects are equal.

Why is this interesting? Because it goes towards using ∞-categories to reason about dependent types and, viceversa, using dependent types to reason about ∞-categories, something which folkloristically we know to be doable. In 2016, Kapulkin and Lumsdaine conjectured that the functor induced by localizing contextual categories modeling some dependent type theories would induce equivalences between their ∞-categories of contextual categories and ∞-categories of finitely complete or locally cartesian closed ∞-categories, an equivalence which hopefully will extend to Homotopy Type Theory and Elementary ∞-toposes.