Localizations of Models of Dependent Type Theory

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June 23, 2022

Introduction

Dependent Type Theory is a candidate foundation for mathematics. Why is it interesting?

1. Closely linked to *computations* and *computer science*, makes proof assistants possible
2. Sufficient by itself, unlike Set Theory and Propositional Calculus
3. *Proofs* are internal objects
4. Better concept of *equality*

Dependent Type Theory

About *dependent types* A *over contexts* Γ and their *terms* $x : A$.
Structural rules: how to work with variables.
Logical rules: construct new types and their terms from old, carry out computations; provide Σ-*types* $\Sigma(A, B)$, Π-*types* $\Pi(A, B)$ and Id-*types* Id_A, *natural-numbers-type* Nat...

Problem

Providing a model of Dependent Type Theory is hard.

Solution

Defining a class of algebraic models.

Contextual Categories

Definition

A category C with:

1. a grading on objects $\operatorname{Ob} \mathrm{C} = \coprod_{n\in\mathbb{N}} \operatorname{Ob}_n \mathrm{C}$;
2. a map $\mathrm{ft}_n \colon \operatorname{Ob}_{n+1} \mathrm{C} \to \operatorname{Ob}_n \mathrm{C}$ for each $n \in \mathbb{N}$;
3. *basic dependent projections* $p_A \colon \Gamma.A \to \mathrm{ft}_n(\Gamma.A) = \Gamma$;
4. a functorial choice of a pullback squares

$$\begin{tikzcd} \Delta.f^*A \arrow[r, "q(f,A)"] \arrow[d, "p_{f^*A}"'] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd};$$

5. ...

Extra Structure

Modeling logical rules requires some extra structure.
Id-types require from $\Gamma.A$ an object $\Gamma.A.A.\mathrm{Id}_A$...
Π-types require from $\Gamma.A.B$ an object $\Gamma.\Pi(A,B)$, a map $\mathrm{app}_{A,B} : \Gamma.\Pi(A,B).A \to \Gamma.A.B$...

Definition

A categorical model of type theory is a contextual category C *with* Σ, Id *and* Π *structures.*

Bi-Invertibility

Definition

In a contextual category with Id*-structure* C*, a map* $f : \Gamma \to \Delta$ *is bi-invertible if we can give:*

1. *maps* $g_1 : \Delta \to \Gamma$, $\eta : \Gamma \to \Gamma.(1_\Gamma, g_1 f)^* \,\mathsf{Id}_\Gamma$*;*
2. *maps* $g_2 : \Delta \to \Gamma$, $\epsilon : \Delta \to \Delta.(1_\Delta, f g_2)^* \,\mathsf{Id}_\Delta$*.*

Question

What happens if we localize a contextual category at bi-invertible maps?

Internal Languages Conjecture

Conjecture (Kapulkin-Lumsdaine 2016)

The horizontal maps, given by simplicial localization, induce equivalences of ∞-categories.

$$\begin{tikzcd} \mathrm{CxlCat}_{\Sigma,1,\mathrm{Id},\Pi} \arrow[r] \arrow[d] & \mathrm{LCCC}_\infty \arrow[d] \\ \mathrm{CxlCat}_{\Sigma,1,\mathrm{Id}} \arrow[r] & \mathrm{Lex}_\infty \end{tikzcd}$$

Proof by Nguyen-Uemura has recently become available on arxiv. One hopes to extend this to an equivalence between $\mathrm{CxlCat}_{\mathrm{HoTT}}$ and $\mathrm{ElTopos}_\infty$.

Fibrational Structure

Definition

An ∞-category with weak equivalences and fibrations is a triple $(\mathcal{C}, W, Fib)$ where:
...some conditions similar to fibration categories but weaker, with $\mathcal{C}$ an ∞-category.

Theorem (Avigad-Kapulkin-Lumsdaine 2013)

A contextual category with Σ and Id structures defines a fibration category, where weak equivalences are bi-invertible maps and fibrations are maps isomorphic to dependent projections.

Localizing Fibrational Categories

Proposition (Cisinski)

The localization at weak equivalences of an ∞-category with weak equivalences and fibrations $\mathcal{C}$ is a finitely complete ∞-category.

Proposition (Cisinski)

Given an ∞-category with weak equivalences and fibrations $\mathcal{C}$, if for every fibration $f : x \to y$ between fibrant objects the pullback functor between fibrant slices $f^* \colon \mathcal{C}(y) \to \mathcal{C}(x)$ has a right adjoint preserving trivial fibrations, then $L(\mathcal{C})$ is locally cartesian closed.

Localizations of Models are Cartesian Closed

Lemma (Kapulkin 2015)

Given a categorical model of type theory C *and a dependent projection* $p_\Delta \colon \Gamma.\Delta \to \Gamma$*, there exists a right adjoint to* p_Δ^* *given by*

$$(p_\Delta)^*(\Gamma.\Delta.\Theta) = \Gamma.\Pi(\Delta, \Theta)$$

preserving the fibrational structure. $\mathrm{app}_{\Delta,\Theta}$ *induces the counit.*

Theorem (Kapulkin 2015)

Given a categorical model of type theory C*, its localization* $L(\mathsf{C})$ *is a locally cartesian closed* ∞*-category.*

Thank you for your attention!