

# Localizations of Models of Dependent Type Theory

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Dependent Type Theory is a candidate foundation for mathematics. Why is it interesting?

- 1 Closely linked to *computations* and *computer science*, makes proof assistants possible
- 2 Enough by itself, unlike Set Theory and Propositional Calculus
- 3 *Proofs* are internal objects
- 4 Better concept of *equality*

# Dependent Type Theory

## About

*Dependent types  $A$  over contexts  $\Gamma$  and their terms  $x : A$ .*

## Structural Rules

How to work with variables.

## Logical Rules

Construct new types and their terms from old, carry out computations, provide  $\Sigma$ -types  $\Sigma(A, B)$ ,  $\Pi$ -types  $\Pi(A, B)$  and  $\text{Id}$ -types  $\text{Id}_A$ , *natural-numbers-type*  $\text{Nat}$ ...

## Problem

Providing a model of Dependent Type Theory is hard.

## Solution

Defining a class of algebraic models.

# Modeling Structural Rules

## Definition (Contextual Categories)

A category  $\mathcal{C}$  with:

- ① a grading on objects  $\text{Ob } \mathcal{C} = \coprod_{n \in \mathbb{N}} \text{Ob}_n \mathcal{C}$ ;
- ② a map  $\text{ft}_n: \text{Ob}_{n+1} \mathcal{C} \rightarrow \text{Ob}_n \mathcal{C}$  for each  $n \in \mathbb{N}$ ;
- ③ *basic dependent projections*  $p_A: \Gamma.A \rightarrow \text{ft}_n(\Gamma.A) = \Gamma$ ;
- ④ a functorial choice of a pullback squares

$$\begin{array}{ccc} \Delta.f^*A & \xrightarrow{q(f,A)} & \Gamma.A \\ p_{f^*A} \downarrow & & \downarrow p_A \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

⑤ ...

# Modeling Logical Rules

## Extra Structure

Id-types require from  $\Gamma.A$  an object  $\Gamma.A.A$ .  $\text{Id}_A \dots$

$\Pi$ -types require from  $\Gamma.A.B$  an object  $\Gamma.\Pi(A, B)$ , a map  $\text{app}_{A,B} : \Gamma.\Pi(A, B).A \rightarrow \Gamma.A.B \dots$

## Definition

*A categorical model of type theory is a contextual category  $\mathcal{C}$  with  $\Sigma$ ,  $\text{Id}$  and  $\Pi$  structures.*

## Definition (Bi-Invertible Map)

A map  $f: \Gamma \rightarrow \Delta$  in a contextual category with  $\text{Id}$ -structure  $\mathcal{C}$  for which we can provide:

- ① maps  $g_1: \Delta \rightarrow \Gamma$ ,  $\eta: \Gamma \rightarrow \Gamma.(1_\Gamma, g_1 \cdot f)^* \text{Id}_\Gamma$ ;
- ② maps  $g_2: \Delta \rightarrow \Gamma$ ,  $\epsilon: \Delta \rightarrow \Delta.(1_\Delta, f \cdot g_2)^* \text{Id}_\Delta$ .

## Question

What if we localize at bi-invertible maps?

# Internal Languages Conjecture

## Conjecture (Kapulkin-Lumsdaine 2016)

*The horizontal maps, given by simplicial localization, induce equivalences of  $\infty$ -categories.*

$$\begin{array}{ccc} \mathrm{CxlCat}_{\Sigma,1,\mathrm{Id},\Pi} & \longrightarrow & \mathrm{LCCC}_{\infty} \\ \downarrow & & \downarrow \\ \mathrm{CxlCat}_{\Sigma,1,\mathrm{Id}} & \longrightarrow & \mathrm{Lex}_{\infty} \end{array}$$

Proof by Nguyen-Uemura has recently become available on arxiv. One hopes to extend this to an equivalence between  $\mathrm{CxlCat}_{\mathrm{HoTT}}$  and  $\mathrm{ElTopos}_{\infty}$ .



## Definition ( $\infty$ -Categories with Weak Equivalences and Fibrations)

*A triple  $(\mathcal{C}, W, \text{Fib})$  where:*

*...some conditions similar to fibration categories but weaker, with  $\mathcal{C}$  an  $\infty$ -category.*

## Theorem (Avigad-Kapulkin-Lumsdaine 2013)

*A contextual category with  $\Sigma$  and  $\text{Id}$  structures defines a fibration category, where weak equivalences are bi-invertible maps and fibrations are maps isomorphic to dependent projections.*

# Localizing Fibrational Categories

## Proposition (Cisinski)

*The localization at weak equivalences of an  $\infty$ -category with weak equivalences and fibrations  $\mathcal{C}$  is a finitely complete  $\infty$ -category.*

## Proposition (Cisinski)

*Given an  $\infty$ -category with weak equivalences and fibrations  $\mathcal{C}$ , if for every fibration  $f: x \rightarrow y$  between fibrant objects the pullback functor between fibrant slices  $f^*: \mathcal{C}(y) \rightarrow \mathcal{C}(x)$  has a right adjoint preserving trivial fibrations, then  $L(\mathcal{C})$  is locally cartesian closed.*

# Localizations of Models are Cartesian Closed

## Theorem (Kapulkin 2015)

*Given a categorical model of type theory  $C$ , its localization  $L(C)$  is a locally cartesian closed  $\infty$ -category.*

## Proof.

For any dependent projection  $p_\Delta: \Gamma.\Delta \rightarrow \Gamma$ , there exists a right adjoint to  $p_\Delta^*: C(\Gamma) \rightarrow C(\Gamma.\Delta)$  given by

$$(p_\Delta)_*(\Gamma.\Delta.\Theta) = \Gamma.\Pi(\Delta, \Theta)$$

with counit induced by  $\text{app}_{\Delta, \Theta}$ . It preserves the fibrational structure. □

Thank you for your attention!