
On Fibration Categories and Finitely Complete $(\infty, 1)$ -categories

1 This is [not] an outline

The goal of this thesis is to redefine the functor: $F: N(\mathbf{FibCat}) \rightarrow \mathbf{Lex}_\infty$, from the nerve of the homotopical category of fibration categories to ∞ -category of finitely complete ∞ -categories and left exact functors, and prove that it induces an equivalence after localizing the domain at its weak equivalences.

The functor F is simply defined as the composite of the full embedding $\iota: N(\mathbf{FibCat}) \rightarrow \mathbf{FibCat}_\infty$, induced by the nerve functor, (should we instead work with a stronger notion of fibration ∞ -category? also we need to define this ∞ -category) and the functor $\mathbf{Ho}_\infty: \mathbf{FibCat}_\infty \rightarrow \mathbf{Lex}_\infty$ given by localizing fibration ∞ -categories. We know by [Cis19, Thm. 7.5.6] that localizing a fibration ∞ -category gives a finitely complete ∞ -category and the mapping on the maps is specified by the universal property of localizations. A map between fibration ∞ -categories is a left exact functor as defined in [Cis19, Def. 7.5.2] and [Cis19, Thm. 7.5.28] guarantees that it is indeed sent to a left exact functor, as needed.

To prove that F induces the desired equivalence of ∞ -categories we define a (weak) fibrational structure on \mathbf{FibCat} and \mathbf{FibCat}_∞ and verify that the right derived functors associated to ι and \mathbf{Ho}_∞ are equivalences.

We already have notions of weak equivalence and fibration internal to \mathbf{FibCat} as provided in [KS19], so we only need to specify ones internal to \mathbf{FibCat}_∞ satisfying the definition of (weak) fibration category given in [Cis19, Def. 7.4.12] (PLEASE LIST THE PROPERTIES). To do so, we try to extend the definition given in the 1-categorical context, so that the functor ι will preserve the desired structure. Hopefully we can just take the same definitions, without adaptations.

To show that the right derived functor of \mathbf{Ho}_∞ induces an equivalence we remember that a finitely complete ∞ -category has a canonical fibrational structure, where weak equivalences are the isomorphisms and fibrations are all the maps. Also, under this construction a left exact functor between the induced fibration ∞ -categories is just a left exact functor in the usual sense, so we can fully embed \mathbf{Lex}_∞ into \mathbf{FibCat}_∞ . This should define a right adjoint to \mathbf{Ho}_∞ , where the counit is the identity and the unit is given by the localization maps $\mathcal{P} \rightarrow L(\mathcal{P})$. We observe that such maps are weak equivalences

in \mathbf{FibCat}_∞ , so localizing this ∞ -category gives us an equivalence $L(\mathbf{FibCat}_\infty) \cong \mathbf{Lex}_\infty$ involving the right derived functor of \mathbf{Ho}_∞ , as we desired.

Once we have done this, we need to check that ι is left exact, i.e. it satisfies [Cis19, Def. 7.5.2] (PLEASE LIST THE PROPERTIES), for which we only need condition (iii). It should be easy to check because it is induced by the nerve functor.

To finally apply [Cis19, Thm. 7.6.15] to ι we still need to check two more conditions (PLEASE LIST THEM). Notice that (a) is trivial, so we actually only need to check (b).

2 This is [not] a fibration

One piece of the issue is to define a fibration ∞ -category structure on \mathbf{FibCat}_∞ and to do so we need to first provide the definitions of the elements at play.

Definition 2.1. A *fibration category* is a category \mathcal{P} with a subcategory whose morphisms are called *weak equivalences* (denoted by $\xrightarrow{\sim}$) and one of *fibrations* (denoted by \rightarrow) subject to the following axioms:

1. \mathcal{P} has a terminal object 1 and all objects are fibrant, that is every map to the terminal object is a fibration;
2. pullbacks along fibrations exist in \mathcal{P} and (acyclic) fibrations are stable under pull-back;
3. every morphism in \mathcal{P} factors as a weak equivalence followed by a fibration;
4. weak equivalences satisfy the 2-out-of-6 property.

Definition 2.2. A functor between fibration categories is *left exact* if it preserves weak equivalences, fibrations, terminal objects and pullbacks along fibrations.

Notice that this corresponds to the definition of *exact functor* given in [KS19, Def. 2.7], however it also coincides with the one given in the more general context of fibration ∞ -categories in [Cis19, Def. 7.5.2] (whose results we rely on), therefore to avoid confusion I adopted the naming given in the latter.

The category \mathbf{FibCat} is then constructed by taking small fibration categories as objects and left exact functors as arrows.

Definition 2.3. A morphism in \mathbf{FibCat} is a *weak equivalence* if it induces an equivalence of categories on the associated homotopy categories.

Definition 2.4. A morphism $P: \mathcal{E} \rightarrow \mathcal{D}$ in \mathbf{FibCat} is a *fibration* if:

1. it is an isofibration;
2. it lifts weak factorizations;
3. it lifts pseudo-factorizations.

The three conditions and the map being an isofibration are necessary to ensure that a fibration of fibration categories is sent to a fibration in \mathbf{Lex}_∞ .

Notice that this is just taken from [KS19, Def. 4.3] and, under our definition, every map to the terminal fibration category is a fibration.

Proposition 2.5. These classes of maps specify a fibration category structure on \mathbf{FibCat} .

Proof. Check [KS19]. \square

We want to switch to the ∞ -categorical context, so let's introduce the corresponding notions.

Definition 2.6. A fibration ∞ -category is a triple $(\mathcal{C}, W, \mathit{Fib})$ where $W \subset \mathcal{C}$ is a sub- ∞ -category with the 2-out-of-3 property and Fib is a class of fibrations such that:

1. for any pullback square

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ p' \downarrow & & \downarrow p \\ y' & \xrightarrow{v} & y \end{array}$$

in \mathcal{C} in which p is a fibration between fibrant objects, if $p \in W$ then $p' \in W$;

2. given a map $f: x \rightarrow y$ in \mathcal{C} with y fibrant, there exists a map $w: x \rightarrow x'$ in W and a fibration $p: x' \rightarrow y$ that $f = p \cdot w$.

The maps in W are generally called *weak equivalences*, while fibrations which are also in W are *trivial fibrations*.

Definition 2.7. Let \mathcal{C}, \mathcal{D} be fibration ∞ -categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *left exact* if it has the following properties:

1. it preserves terminal objects;
2. it preserves fibrations and trivial fibrations;
3. given a pullback square

$$\begin{array}{ccc} x' & \xrightarrow{u} & x \\ p' \downarrow & & \downarrow p \\ y' & \xrightarrow{v} & y \end{array}$$

in \mathcal{C} where p is a fibration and y and y' are fibrant, the square

$$\begin{array}{ccc} Fx' & \xrightarrow{Fu} & Fx \\ Fp' \downarrow & & \downarrow Fp \\ Fy' & \xrightarrow{Fv} & Fy \end{array}$$

is a pullback in \mathcal{D} .

Let's then consider \mathbf{FibCat}_∞ , the ∞ -category of small fibration ∞ -categories and left exact functors. We can take the same definition of fibration for \mathbf{FibCat}_∞ and use the following one for weak equivalences, which then should (CHECK!) make it a fibration ∞ -category.

Definition 2.8. A left exact functor between fibration ∞ -categories $F: \mathcal{P} \rightarrow \mathcal{Q}$ is a weak equivalence if it induces an equivalence $ho(L(\mathcal{P})) \rightarrow ho(L(\mathcal{Q}))$. Equivalently, if it satisfies the conditions of [Cis19, Thm. 7.6.15].

Proposition 2.9. (DESIDERATA) The ∞ -category \mathbf{FibCat}_∞ with the specified classes of maps is a fibration ∞ -category.

Proof. The problem is showing the factorization condition (SHOW IT). Fibrations should be stable under pullback (which are computed as in \mathbf{sSet} if we manage to describe \mathbf{FibCat}_∞ as a full subcategory of functors with codomain in \mathbf{Cat}_∞) and the same applies to trivial fibrations, with the proof being the same as the one given in the 1-categorical context. Every map to the terminal fibration ∞ -category is naturally a fibration. \square

3 This is [not] an adjunction

In this section we provide a right adjoint to the localization functor \mathbf{Ho}_∞ .

Remark 3.1. Given a finitely complete ∞ -category, we can provide a fibrational structure on it by defining weak equivalences to be the isomorphisms and fibrations to be all of the maps. Under this construction we see that, given two finitely complete ∞ -categories, a functor between the induced fibration ∞ -categories is left exact if and only if it is left exact when seen as a functor between the underlying ∞ -categories.

Proposition 3.2. There is an adjunction $\mathbf{Ho}_\infty: \mathbf{FibCat}_\infty \rightleftarrows \mathbf{Lex}_\infty: i$, where i is a full embedding, the unit is given by the localization maps $\gamma_{\mathcal{P}}: \mathcal{P} \rightarrow L(\mathcal{P})$ and the counit is the identity. This exhibits \mathbf{Lex}_∞ as a reflexive sub- ∞ -category of \mathbf{FibCat}_∞ .

Proof. Let's consider for any finitely complete ∞ -category \mathcal{C} the pullback diagram

$$\begin{array}{ccc} \mathbf{Ho}_\infty / \mathcal{C} & \longrightarrow & \mathbf{Lex}_\infty / \mathcal{C} \\ \downarrow & & \downarrow \\ \mathbf{FibCat}_\infty & \xrightarrow{\mathbf{Ho}_\infty} & \mathbf{Lex}_\infty \end{array} .$$

We see that $\mathbf{Ho}_\infty / \mathcal{C}$ has a terminal object, namely the pair $(\mathcal{C}, \text{id}_{\mathcal{C}})$, where \mathcal{C} has the fibrational structure specified above: indeed, given any other object (\mathcal{P}, F) , the space of maps $(\mathbf{Ho}_\infty / \mathcal{C})((\mathcal{P}, F), (\mathcal{C}, \text{id}_{\mathcal{C}}))$ is contractible as it corresponds to the fiber at F of the equivalence $\mathbf{FibCat}_\infty(\mathcal{P}, \mathcal{C}) \cong \mathbf{Lex}_\infty(L(\mathcal{P}), \mathcal{C})$ given by [Cis19, Prop. 7.5.11].

Unravelling everything, the constructed right adjoint is then the full embedding mapping everything in \mathbf{Lex}_∞ to its copy in \mathbf{FibCat}_∞ . \square

Given this adjunction we present the following result, which reduces the problem of proving that the functor we constructed initially between $L(\mathbf{FibCat})$ and \mathbf{Lex}_∞ is an equivalence to showing it for the right derived functor of ι .

Proposition 3.3. The right derived functor of \mathbf{Ho}_∞ is an equivalence of ∞ -categories $L(\mathbf{FibCat}_\infty) \cong \mathbf{Lex}_\infty$.

Proof. By 3.2 we know that \mathbf{Ho}_∞ has a fully faithful right adjoint, so by [Cis19, Prop. 7.1.18] it is enough to show that a map in \mathbf{FibCat}_∞ is mapped by \mathbf{Ho}_∞ to an isomorphism if and only if it is a weak equivalence. By [Cis19, Prop. 7.6.11] a left exact functor between fibration ∞ -categories induces an equivalences on the localizations if and only if it does so on the homotopy categories, which is also the condition under which it is a weak equivalence. \square

4 This is [not] a ∞ -category

Our approach relies on defining a ∞ -category \mathbf{FibCat}_∞ . How to do this?

1- Construct a ∞ -category where objects are pairs of functors (specifying weak equivalences and fibrations) with the same codomain between ∞ -categories, that is $\mathbf{Fun}(\{0 \rightarrow 01 \leftarrow 1\}, \mathbf{Cat}_\infty)$. Then the morphisms are triples of functors making the proper diagram commute. We take the full subcategory of objects satisfying the axioms of a ∞ -category with weak equivalences and fibrations, but how can we ask for pullbacks along fibrations to be preserved by the maps in this ∞ -category? We also have to show that the hom-spaces are what we want. How to show then that a pullback along a fibration is a fibration? It may not be immediate because we are considering only triples of functors which make the diagrams commute as morphisms!

What if we take pairs of functors $\coprod_W \Delta^1 \rightarrow \mathcal{D}$, $Fib \rightarrow \mathcal{C}^{[1]} \rightarrow \mathcal{C}$, where the composite is a Cartesian fibration and $Fib \rightarrow \mathcal{C}^{[1]}$ preserves Cartesian morphisms? Essentially this is the definition of comprehension category, plus some extra data given by the first functor. We call fibrations the maps in the image of $Fib \rightarrow \mathcal{C}$ and their compositions, while weak equivalences are the maps identified by the first functor; we then ask for the axioms of fibration ∞ -category to be satisfied. This gives us more freedom when specifying weak equivalences then just providing the definition we give when constructing a tribe from a comprehension category. Also, in this way weak equivalences are preserved.

Problem: maps between such objects may not be what we want! Namely, I want the functor $\mathcal{C}^{[1]} \rightarrow \mathcal{D}^{[1]}$ to be induced by post-composing with $\mathcal{C} \rightarrow \mathcal{D}$, but I don't know how to specify this. This (with the condition of $Fib \rightarrow \mathcal{C}$ being a cartesian fibration and the factorization $Fib \rightarrow \mathcal{C}^{[1]} \rightarrow \mathcal{C}$ preserving cartesian morphisms) would guarantee that pullbacks along fibrations are preserved by maps between fibration categories.

2- Take the 1-category of ∞ -categories with fibrations and weak equivalences, then localize at categorical equivalences. What do the hom-spaces look like? If instead we localized directly at weak equivalences we would get directly $L(\mathbf{FibCat}_\infty)$, which supposedly is the “true” ∞ -category of ∞ -categories with fibrations and weak equivalences.

It's important to relate the hom-spaces $\mathbf{FibCat}_\infty(\mathcal{P}, \mathcal{Q})$ to the ones we want, that is $k(\mathbf{Fun}_{lex}(\mathcal{P}, \mathcal{Q}))$. Also, what are the ones of \mathbf{Lex}_∞ like? Supposedly $k(\mathbf{Fun}_{lex}(\mathcal{C}, \mathcal{D}))$ and we may show both claims using [Cis19, Prop. 7.10.6], but I do not think it would be easy.

5 Other ideas

1- We could also try to get an indirect comparison by looking at the inclusion of the category of fibration categories and of the category of finitely complete ∞ -categories into the one of ∞ -categories with weak equivalences and fibrations. We then give to the latter the structure of a fibrations category by taking the definition above. The axioms should be easy to show because a pullback along a fibration will simply be a pullback in \mathbf{sSet} and then we can give it the desired structure as in the context of fibration categories. Problem: showing the factorization condition.

We have then to show that the inclusion functors are left exact as functors between fibration categories. This means that the pullback of left exact functors between finitely complete ∞ -categories is again a finitely complete ∞ -category, which is proven in [Szu17, Lem. 2.11].

After this we have to show that the hypothesis of [Cis19, Thm. 7.6.15] hold. This is easy for (i), but for (ii) it's hard when it comes to the inclusion of fibration categories into ∞ -categories with fibrations and weak equivalences. Namely, what if the domain of the morphism is a ∞ -category? We have to find a span linking it to a weakly equivalent fibration category!

2- We can work with the 1-category of fibration ∞ -categories and left exact functors and give it a fibrational structure. Problems: how do we show that after localizing a fibration of fibration categories we get an inner fibration? That is needed for the localization functor to be left exact and maybe we can use the ideas from [Szu17, Prop. 3.5]. Also, how to show condition (ii) of [Cis19, Thm. 7.6.15] for the inclusion of fibration categories into fibration ∞ -categories? We need to approximate a map from a fibration ∞ -category to a fibration category by one between fibration categories, the same problem as before. Everything else should be easy.

3- Starting from the fibration categories \mathbf{FibCat} and \mathbf{Lex}_∞ and the functor \mathbf{Ho}_∞ between them we can show the approximation properties of [Cis19, Thm. 7.6.15]. The problems are the same as in (2), that is how can we prove that localizing a fibration in \mathbf{FibCat} gives a fibration in \mathbf{Lex}_∞ ? Also, the approximation property (i) is easy, while for (ii) given a morphism $\mathcal{C} \rightarrow L(\mathcal{P})$, we may try to consider a finitely complete « \lll » < HEAD category $(\mathbf{sSet}_0/\mathcal{C})^{\text{op}}$, whose fibrational structure is the dual of the cofibrational one specified in [Szu17, Def. 4.1], specifically the fibrations are given by monomorphisms and weak equivalences by maps sent to isomorphisms under the functor $\lim: (\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow \mathcal{C}$; the proof that this is indeed a fibration category is the same as the one concerning $\mathbf{sSet}_\kappa/\mathcal{C}$ provided in [Szu17, Prop. 4.2]. If the map $\widetilde{\lim}: L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow \mathcal{C}$ is a weak equivalence (check the following proposition!), then we get the desired commutative square by looking at the category $\mathbf{sSet}_0/\mathcal{C}$, whose fibra-

tional structure is essentially the dual of the one specified in [Szu17, Def. 4.1], specifically the fibrations are given by monomorphisms and weak equivalences by maps sent to isomorphisms under the functor $\lim: (\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow \mathcal{C}$; the proof that this is indeed a fibration category is the same as the one concerning $\mathbf{sSet}_\kappa/\mathcal{C}$ provided in [Szu17, Prop. 4.2]. If the map $\widetilde{\lim}: L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow \mathcal{C}$ is a weak equivalence, then we get the desired commutative square by looking at

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & L(\mathcal{P}) \\ \widetilde{\lim} \uparrow & & \uparrow \\ L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} & \xlongequal{\quad} & L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \end{array}$$

and somehow lift the map $L(\mathbf{sSet}_0/\mathcal{C})^{\text{op}} \rightarrow L(\mathcal{P})$ through L ,

(I do not believe this lift to always exist. Indeed, take the ∞ -groupoid $\mathcal{C} = N(\{0 \rightrightarrows 1\})$, $\mathcal{P} = [1]$ with the minimal fibrational structure making $0 \rightarrow 1$ a weak equivalence and the identity as the upper map. Given in $(\mathbf{sSet}/\mathcal{C})^{\text{op}}$ the objects $0: \Delta^0 \rightarrow \mathcal{C}$, $(1 \rightarrow 0): \Delta^1 \rightarrow \mathcal{C}$ and the obvious morphism f between them, we see that the lifted map $(\mathbf{sSet}/\mathcal{C})^{\text{op}} \rightarrow \mathcal{P}$ would send f to a morphism with domain 1 and codomain 0, which does not exist.)

Proposition 5.1. The map $\widetilde{\lim}$ is a categorical equivalence.

Proof. We shall verify that the hypothesis of [Cis19, Prop. 7.6.15] are satisfied.

First of all, the functor \lim preserves finite limits as they commute with finite limits, therefore it is left exact. Also, a morphism in $(\mathbf{sSet}_0/\mathcal{C})^{\text{op}}$ is a weak equivalence if and only if it is mapped to an isomorphism, thus condition (i) is trivially satisfied.

We are only missing condition (ii), so let's consider a morphism $\phi: c \rightarrow \lim_K D$ in \mathcal{C} . This corresponds to an object $(c, \hat{\phi})$ in \mathcal{C}/D , that is a map $\hat{\phi}: \Delta^0 * K \rightarrow \mathcal{C}$ such that $\hat{\phi}|_K = D$, $\hat{\phi}|_{\Delta^0} = c$. We have then a map $D \rightarrow \hat{\phi}$ in $\mathbf{sSet}_0/\mathcal{C}$ given by the inclusion $K \rightarrow \Delta^0 * K$ and inducing the morphism $\phi: c = \lim_{\Delta^0 * K} \hat{\phi} \rightarrow \lim_K D$, which allows us to construct the desired commutative square. \square

We may also specify a weak equivalence $\mathcal{C} \rightarrow L(\mathbf{sSet}_0/\mathcal{C})$ as in [Szu17, Def. 4.6] (I don't really see how to adapt this). We extend $\mathcal{C} \rightarrow L(\mathcal{P})$ first to a functor $\mathbf{sSet}_0/\mathcal{C} \rightarrow L(\mathcal{P})$, $D: K \rightarrow \mathcal{C} \mapsto \lim_K D$ (doubtful about doing this) and then localize to get $L(\mathbf{sSet}_0/\mathcal{C}) \rightarrow L(\mathcal{P})$, granting us a diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & L(\mathcal{P}) \\ \parallel & & \uparrow \\ \mathcal{C} & \longrightarrow & L(\mathbf{sSet}_0/\mathcal{C}) \end{array} .$$

6 Plan B: LCCC

We can also rewrite the paper by Kapulkin about LCCC arising from TT using the language of localizations of quasi-categories. There they develop the relevant theory

showing that under some conditions the frame associated to a fibration category is locally cartesian closed, but using Cisinski's results we can prove the same theorem directly using a more mainstream theory.

What should be included in such an overview?

1- Cisinski's theory of localizations (of fibration ∞ -categories)

2- an introduction to contextual categories: where do they come from? Why are they useful? Check out Voevodsky's papers about C-systems

We explain what dependent type theory is (Martin-Lof's notes from 1984) and why it's an interesting foundation of mathematics. We mention Homotopy Type Theory as an effort to provide homotopical foundations which better model how we think about identities, which explains why intensional identity types are more interesting to us than extensional ones.

We move on to defining contextual categories (1211.2851, 1406.7413, 1507.02648) and what the Π , Σ and Id structures are (1406.7413, 1211.2851 Appendix B). To understand what the link between such structures and syntactically presented type theories we refer to 1507.02648, Sec. 1.1, while the statement of the conjectured correspondence is in 1304.0680, Sec. 2.1.

Where does the link between dependent type theories and ∞ -categories come from? We see that ∞ -categories intuitively model the behavior of type theories and their type constructions, especially when considering Homotopy Type Theory, however this relation is known only partially (references in the intro of 1507.02648). The idea is that the type theory we are interested in should be the internal language of some class of ∞ -categories and a precise statement would require us to provide homotopical functors in both directions which induce an equivalence on the associated ∞ -categories. The idea is to construct the functor from contextual categories as a localization functor, that is we need to provide a homotopical structure on contextual categories, as they do in 1507.02648 (there should be an older reference) which then provides an associated ∞ -category. This is the object of the Initiality Conjecture, stated in 1610.00037, in the hope that such a correspondence will extend to Homotopy Type Theory and some notion of Elementary Higher Toposes, perhaps the one specified in 1805.03805. At the moment we know that HoTT can be interpreted in Higher Toposes with some structure. Current progress: 1709.09519, an upcoming paper by Nguyen-Uemura (HoTTest talk).

Our aim is to show that when taking contextual categories with the structure we specified earlier we obtain a locally cartesian closed ∞ -category. To do so we provide a fibrational structure on contextual categories (1304.0680, 1507.02648), which as we anticipate will imply that their simplicial localizations are finitely complete. We also prove that the hypothesis of [Cis19, Thm. 7.6.16] are satisfied, informing that this will be sufficient to prove Kapulkin's main result from 1507.02648.

We then develop the theory of localizations of ∞ -categories by Cisinski and specifically develop the results concerning ∞ -categories with fibrations and weak equivalences. Localizations of such ∞ -categories are finitely complete. The objective is to show [Cis19, Thm. 7.6.16]. How in depth should we go?

Why all of this is interesting: we are proving Kapulkin's result internalizing all of the discussion within the language of ∞ -category theory and relying only on its simplicial

model.

7 Localizations of ∞ -Categories

To prove the that localizing a categorical model of type theory we get a locally cartesian closed ∞ -category we need a theory of localizations. We shall provide one in the general context of ∞ -categories as developed by Cisinski in *Higher Categories and Homotopical Algebra* with the aim of proving [Cis19, Thm. 7.6.16], which will do the heavy lifting in showing the desired result. Those familiar with the theory may skip the entire chapter while keeping in mind yadda yadda ([LIST THE MAJOR RESULTS](#)).

Definition 7.1. Let X be a simplicial set and $W \subset X$ a simplicial subset. Given an ∞ -category \mathcal{C} , we define $\underline{\mathrm{Hom}}_W(X, \mathcal{C})$ to be the full simplicial subset of $\underline{\mathrm{Hom}}(X, \mathcal{C})$ whose objects are the morphisms $f: X \rightarrow \mathcal{C}$ sending the 1-simplices in W to isomorphisms.

Remark 7.2. The above definition induces a canonical pullback square

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_W(X, \mathcal{C}) & \longrightarrow & \underline{\mathrm{Hom}}(X, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathrm{Hom}}_W(W, \mathcal{C}) & \longrightarrow & \underline{\mathrm{Hom}}(W, \mathcal{C}) \end{array}$$

given by the inclusion $W \rightarrow X$.

Definition 7.3. Given an ∞ -category \mathcal{C} and $W \subset \mathcal{C}$, a *localization of \mathcal{C} by W* is a functor $\gamma: \mathcal{C} \rightarrow L(\mathcal{C})$ such that:

1. $L(\mathcal{C})$ is an ∞ -category;
2. γ sends the 1-simplices of W to isomorphisms in $L(\mathcal{C})$;
3. for any ∞ -category \mathcal{D} there is an equivalence of ∞ -categories

$$\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \rightarrow \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$$

given by precomposing with γ .

(CISINSKI DOES NOT ASK FOR \mathcal{C} TO BE AN ∞ -CATEGORY. SHOULD WE BE LESS GENERAL AS WE HAVE DONE?)

Proposition 7.4. Given an ∞ -category \mathcal{C} and a subsimplicial set W , the localization of \mathcal{C} by W always exists and it is essentially unique.

Proof. We begin by proving that a localization exists in the case where $W = \mathcal{C}$.

In this context, $\underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \cong \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}^{\cong})$ canonically, where \mathcal{D}^{\cong} is the maximal sub- ∞ -subgroupoid of \mathcal{D} . Factoring $\mathcal{C} \rightarrow \Delta^0$ in the Kan model structure, we find an anodyne map $\gamma: \mathcal{C} \rightarrow L(\mathcal{C})$.

Remember that for any anodyne map $A \rightarrow B$ we get a trivial fibration $\underline{\mathrm{Hom}}(B, \mathcal{D}^\cong) \rightarrow \underline{\mathrm{Hom}}(A, \mathcal{D}^\cong)$. Looking then at the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(L(\mathcal{C}), \mathcal{D}^\cong) & \xrightarrow[\sim]{\gamma^*} & \mathrm{Hom}_W(\mathcal{C}, \mathcal{D}^\cong) \\ \cong \downarrow & & \downarrow \cong \\ \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) & \xrightarrow{\gamma^*} & \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \end{array},$$

by the 2-out-of-3 property we see that the lower γ^* is an equivalence.

We now move on to the general case. First of all, notice that as a particular case of the previous one we get that localizing Δ^1 at its non-trivial morphism we obtain $\Delta^1 \rightarrow J = L(\Delta^1) \sim \Delta^0$. Taking then $W \subset \mathcal{C}$, we consider the commutative diagram

$$\begin{array}{ccc} \coprod_{f \in W_1} \Delta^1 & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{f \in W_1} J & \longrightarrow & \mathcal{C}' \end{array} \quad \begin{array}{c} \searrow \gamma \\ \nearrow \sim \\ \downarrow \\ L(\mathcal{C}) \end{array},$$

where $\mathcal{C}' \rightarrow L(\mathcal{C})$ is an inner anodyne map obtained by taking the fibrant replacement of \mathcal{C}' in the Joyal model structure. This can be done functorially via the small object argument.

For any ∞ -category \mathcal{D} , we get a trivial fibration $\underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \rightarrow \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D})$ and a pullback square

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}) \\ \downarrow & & \downarrow \\ \prod_{f \in W_1} \underline{\mathrm{Hom}}(J, \mathcal{D}) & \longrightarrow & \prod_{f \in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \end{array},$$

which together with the pullback

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D}) \\ \downarrow & & \downarrow \\ \prod_{f \in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}^\cong) & \longrightarrow & \prod_{f \in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \end{array}$$

implies by pasting that

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{D}) & \longrightarrow & \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \\ \downarrow & & \downarrow \\ \prod_{f \in W_1} \underline{\mathrm{Hom}}(J, \mathcal{D}) & \xrightarrow{\sim} & \prod_{f \in W_1} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}^\cong) \end{array}$$

is also a pullback and therefore the upper arrow is a trivial fibration. Composing it with the other one we get $\gamma^*: \underline{\mathrm{Hom}}(L(\mathcal{C}), \mathcal{D}) \rightarrow \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D})$, which is then a trivial fibration and therefore an equivalence of ∞ -categories.

7.1.4 Observe that, through this construction, one can always construct $L(\mathcal{C})$ so that γ is a bijection on objects because $\mathcal{C}' \rightarrow L(\mathcal{C})$ is an inner anodyne extension and therefore a retract of a countable composition of sums of pushouts of maps which are the identity on objects, that is the inner horn inclusions.

We now move on to proving that the localization is essentially unique. For this, we notice that γ establishes then an isomorphism between $\pi_0(k(\underline{\mathrm{Hom}}_W(\mathcal{C}, -)))$ and $\pi_0(\underline{\mathrm{Hom}}(L(\mathcal{C}), -)) = ho(\mathbf{sSet})(L(\mathcal{C}), -)$ with respect to the Joyal model structure, thus by Yoneda $(L(\mathcal{C}), \gamma)$ is unique up to unique isomorphism in $ho(\mathbf{sSet})$ and up to a contractible space of equivalences in \mathbf{sSet} . \square

Remark 7.5. 7.1.5

In this context, we may define \overline{W} , the saturation of W in \mathcal{C} , as the cartesian square such that \overline{W} is precisely the maximal simplicial subset of \mathcal{C} whose morphisms are the ones which become invertible in $L(\mathcal{C})$.

We have then inclusions $Sk_1(W) \subset W \subset \overline{W}$ and, for any ∞ -category \mathcal{D} , this induces equalities

$$\underline{\mathrm{Hom}}_{Sk_1(W)}(\mathcal{C}, \mathcal{D}) = \underline{\mathrm{Hom}}_W(\mathcal{C}, \mathcal{D}) \underline{\mathrm{Hom}}_{\overline{W}}(\mathcal{C}, \mathcal{D}),$$

implying that $(L(\mathcal{C}), \gamma)$ is also the localization of \mathcal{C} by $Sk_1(W)$ and the one by \overline{W} , however the inclusion $W \rightarrow \mathcal{C}$ is a fibration in the Joyal model category as it is the pullback of one, implying that \overline{W} is itself an ∞ -category.

We shall say that W is saturated if $W = \overline{W}$.

Remark 7.6. 7.1.6

The functor $ho(\mathcal{C}) \rightarrow ho(L(\mathcal{C}))$ exhibits $ho(L(\mathcal{C}))$ as the 1-categorical localization of \mathcal{C} at $\mathrm{Arr}(\tau(W))$, as can be seen by using the universal property.

On the other hand, given a 1-category \mathcal{C} and localizing at a set of morphisms W , not necessarily the induced map $L(N(\mathcal{C})) \rightarrow N(L(\mathcal{C}))$ is an isomorphism. Indeed, $L(N(\mathcal{C}))$ can have much better properties, as can be seen for example from yadda yadda ([PROPOSITION ABOUT FINITE COMPLETENESS](#)) and in fact localizing 1-categories after taking their nerves gives every ∞ -category ([MORE PRECISE STATEMENT](#)).

Proposition 7.7. 7.1.9

Given a universe \mathbf{U} and W a simplicial subset of a \mathbf{U} -small ∞ -category \mathcal{C} , let $\gamma: \mathcal{C} \rightarrow L(\mathcal{C})$ be the associated localization. Then the functor $\gamma^*: \underline{\mathrm{Hom}}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \rightarrow \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful and its essential image consists of all presheaves $F: \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{S}$ sending maps $u: x \rightarrow y$ in W to invertible maps $u^*: Fy \rightarrow Fx$ in \mathcal{S} .

Proposition 7.8. The map γ gives us a morphism

$$\gamma^*: \underline{\mathrm{Hom}}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \simeq \underline{\mathrm{Hom}}_{W^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}) \rightarrow \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}),$$

which has a left adjoint $\gamma_!$ and a right adjoint γ_* . Now, for any presheaf $F: L(\mathcal{C})^{\mathrm{op}} \rightarrow \mathcal{S}$, the unit map $F \rightarrow \gamma_*\gamma^*F$ is invertible and, by adjunction, the same goes for the counit

map $\gamma_! \gamma^* F \rightarrow F$, which means that u^* is fully faithful. On the other hand, given a presheaf $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}$ sending 1-simplices in W to invertible maps, the counit $\gamma^* \gamma_* F \rightarrow F$ and the unit $F \rightarrow \gamma^* \gamma_! F$ are both invertible since the restrictions of these adjunctions to $\underline{\text{Hom}}_{W^{\text{op}}}(\mathcal{C}^{\text{op}}, \mathcal{S})$ form adjoint equivalences of ∞ -categories as γ^* is an equivalence.

Proposition 7.9. 7.1.10

Given an ∞ -category \mathcal{C} and a simplicial subset W , the localization functor $\gamma: \mathcal{C} \rightarrow L(\mathcal{C})$ is final and cofinal.

Corollary 7.10. 7.1.11 Let's fix a universe \mathbf{U} , a \mathbf{U} -small ∞ -category \mathcal{C} and a simplicial subset W . Consider then a functor $f: \mathcal{C} \rightarrow \mathcal{D}$, where \mathcal{D} is a small ∞ -category. Then f exhibits \mathcal{D} as the localization of \mathcal{C} by W if and only if the following conditions hold:

1. the functor f sends the 1-simplices of W to invertible maps of \mathcal{D} ;
2. the functor f is essentially surjective;
3. the functor f^* induces an equivalence of ∞ -categories

$$f^*: \underline{\text{Hom}}(\mathcal{D}^{\text{op}}, \mathcal{S}) \rightarrow \underline{\text{Hom}}_{W^{\text{op}}}(\mathcal{C}^{\text{op}}, \mathcal{S}).$$

Corollary 7.11. 7.1.14

Corollary 7.12. 7.1.18

Proof. 7.1.9 □

Corollary 7.13. 7.2.5

Proof. 7.1.5 □

Corollary 7.14. 7.2.8

Proof. 7.1.5, 7.1.18, 7.2.5 □

Corollary 7.15. 7.2.10

Corollary 7.16. 7.2.15

Corollary 7.17. 7.2.16

Corollary 7.18. 7.2.18

Proof. 7.2.16 □

Corollary 7.19. 7.2.25

Proof. 7.2.8, 7.2.10, 7.2.15 □

Corollary 7.20. 7.3.2

Corollary 7.21. 7.3.4

Corollary 7.22. 7.3.5

Proof. 7.1.10, 7.3.2, 7.3.4

□

Corollary 7.23. 7.3.8

Proof. 7.1.10, 7.1.12

□

Corollary 7.24. 7.3.9

Proof. 7.3.8

□

Corollary 7.25. 7.3.10

Proof. 7.3.9

□

Proposition 7.26. 7.3.11

Corollary 7.27. 7.3.15

Proof. 7.1.12, 7.3.9, 7.3.10

□

Corollary 7.28. 7.3.16

Proof. 7.3.5, 7.3.15

□

Corollary 7.29. 7.3.26

Corollary 7.30. 7.3.27

Proof. 7.3.16, 7.3.26

□

Corollary 7.31. 7.3.28

Corollary 7.32. 7.3.29

Proof. 7.3.27, 7.3.28

□

Proposition 7.33. 7.4.2

Proposition 7.34. 7.4.11

Proof. 7.3.11, 7.4.2

□

Proposition 7.35. 7.4.13

Corollary 7.36. 7.4.14

Proof. 7.4.13

□

Proposition 7.37. 7.4.16

<i>Proof.</i> 7.4.13	□
Proposition 7.38. 7.4.19	
<i>Proof.</i> 7.4.11	□
Proposition 7.39. 7.5.5	
Proposition 7.40. 7.5.6	
<i>Proof.</i> 7.1.10, 7.2.18, 7.2.25, 7.3.27, 7.4.13, 7.4.16	□
Proposition 7.41. 7.5.17	
Proposition 7.42. 7.5.18	
<i>Proof.</i> 7.5.6, 7.5.17	□
Proposition 7.43. 7.5.19	
<i>Proof.</i> 7.5.18	□
Proposition 7.44. 7.5.22	
<i>Proof.</i> 7.5.19	□
Proposition 7.45. 7.5.24	
Proposition 7.46. 7.5.28	
<i>Proof.</i> 7.5.6	□
Lemma 7.47. 7.6.2	
<i>Proof.</i> 7.2.10.4, 7.2.18, 7.4.13	□
Lemma 7.48. 7.6.4	
<i>Proof.</i> 7.2.10.4, 7.2.18, 7.4.13	□
Lemma 7.49. 7.6.5	
<i>Proof.</i> 4.2.9	□
Lemma 7.50. 7.6.7	
<i>Proof.</i> 4.3.15, 7.4.19, 7.5.5	□
Theorem 7.51. 7.6.10	
<i>Proof.</i> 7.3.29, 7.6.2, 7.6.5, 7.6.7	□
Corollary 7.52. 7.6.13	
<i>Proof.</i> 7.5.18, 7.5.22, 7.5.24, 7.5.28, 7.6.4, 7.6.10	□
Theorem 7.53. 7.6.16	
<i>Proof.</i> 6.1.6, 6.1.7, 6.1.8, 7.1.14, 7.4.14, 7.5.18, 7.6.13	□

8 Categorical Models of TT as Locally Cartesian Closed Fibration Categories

Definition 8.1. A fibration category \mathcal{P} is *locally cartesian closed* if, for any fibration $p: a \rightarrow b$, the pullback functor $p^*: \mathcal{P} \downarrow b \rightarrow \mathcal{P} \downarrow a$ admits a right adjoint p_* which is an exact functor.

How does Kapulkin prove that a categorical model of Type Theory is a locally cartesian closed fibration category?

First of all, he refers to AKL15 to show that \mathcal{P} has a fibrational structure, then he goes on to show the following results, whose proofs are extremely terse and therefore should be expanded.

Lemma 8.2. For any dependent projection $p_\Delta: \Gamma.\Delta \rightarrow \Gamma$ in a categorical model of type theory \mathcal{C} , the pullback functor $p_\Delta^*: \mathcal{C} \downarrow \Gamma \rightarrow \mathcal{C} \downarrow \Gamma.\Delta$ admits a right adjoint.

Proof. Kapulkin 1507.02648, Lemma 5.5. □

We know that every fibration in \mathcal{C} is isomorphic to a composite of dependent projections, so this tells us that every fibration induces an adjunction between fibrational slices [\(YOU SHOULD DEFINE THEM\)](#).

Lemma 8.3. Consider an iterated context extension $\Gamma.\Delta.\Theta.\Psi$ in a categorical model of type theory \mathcal{C} . Then the contexts

$$\Gamma.\Pi(\Delta, \Theta).\Pi(p_{\Pi(\Delta, \Theta)}^* \Delta.app_{\Delta, \Theta}^* \Psi) \text{ and } \Gamma.\Pi(\Delta, \Theta.\Psi)$$

are isomorphic (actually equal) in \mathcal{C} .

Proof. Kapulkin 1507.02648, Lemma 5.5. It simply refers to the construction of the Π -structure on \mathcal{C}^{ext} . □

We are finally ready to prove the result leading to the final one we want.

Proposition 8.4. A categorical model of type theory \mathcal{C} is a locally cartesian closed fibration category.

Proof. Kapulkin 1507.02648, Proposition 5.4. □

Theorem 8.5. Given a categorical model of type theory \mathcal{C} , the ∞ -category $L(\mathcal{C})$ is locally cartesian closed.

Proof. Since a fibration category is more generally a ∞ -category with fibrations and weak equivalences, we can apply [Cis19, Prop. 7.6.16] as the hypothesis are satisfied by 8.4. □

9 Pushforward

One may ask whether cocartesian fibrations in **sSet** model Pi types, which is a piece needed to understand a novel model of dependent type theory provided by **sSet**. To answer this question, an explicit description of the right adjoint p_* of the pullback functor $p^*: \mathbf{sSet}/Y \rightarrow \mathbf{sSet}/X$ induced by a morphism $p: X \rightarrow Y$ is needed.

Consider an object $f: T \rightarrow X$ in \mathbf{sSet}/X . What is $p_*(f): T' \rightarrow Y$? We know that a n -simplex t' of T' corresponds bijectively to a map $t': \Delta^n \rightarrow T'$, which in turn corresponds bijectively to a commutative diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{t'} & T' \\ & \searrow y & \swarrow p_*(f) \\ & Y & \end{array}$$

and, under the adjunction $p^* \dashv p_*$, we get bijectively another commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{t} & T \\ & \searrow p^*(y) & \swarrow f \\ & X & \end{array},$$

from which follows that

$$T'_n \cong \{(y, t) \mid y \in Y_n, t \in \mathbf{sSet}/X(p^*(y), f)\}$$

and the map $p_*(f)$ then sends $(y, t) \in T'_n$ to $y \in Y_n$.

The same method can be extended to give us the pushforward along a map of marked simplicial sets $p: (X, E_X) \rightarrow (Y, E_Y)$ in **mSet**. Specifically, our previous construction can be adapted to give us the n -simplices by starting from maps $(\Delta^n)_b \rightarrow p_*(T, E_T) = (T', E_{T'})$, telling us again that

$$T'_n \cong \{(y, t) \mid y \in Y_n, t \in \mathbf{mSet}/X(p^*(y), f)\},$$

while to get the markings we notice that every marked edge in $p_*(T, E_T)$ corresponds to a unique map $(\Delta^1)_b \rightarrow p_*(T, E_T)$ and the same procedure allows us to write

$$E_{T'} = \{(y, t) \mid y \in E_Y, t \in \mathbf{mSet}/X(p^*(y), f)\},$$

which fully specifies the needed data.

Now, under which conditions on p does this specify a Quillen adjunction when the slices of **mSet** are equipped with the contravariant model structure?

References

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