Project Report

Computational Methods and Tools

SIE BA3 2024

Group members:

- Marino Marco 372566
- Waig Fang Leo 373552

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Derivation from initial idea:

Initially, our project's goal was to create a functional simulation of the evolution of a given predator and prey population. We wanted to recreate the interactions between both species in order to use the model to predict comportments in the future as well as to study past behaviours. However, we found a data set with the two species populations according to the year. Therefore, we decided to, instead of approximating the populations with constants we would approximate the constants with our data set. After finding the parameter's values, we would use them in our model to create a graph that should resemble the reality. Nevertheless, it did not work, our graph did not follow the sinusoidal curve the Lotka-Volterra equation propose.

Instead, we decided to work on a sensibility test to analyse our constants and compare the results of the Lotka-Volterra equations to the graph of the actual evolution of population between lynx and hares as well as to create a population density graphic. Moreover, we decided to make a sensibility graph of our constants overtime to see what impact they had on the population's evolution.

Introduction to the problem:

Lotka-Volterra Equations:

In this project we are looking to study the evolution of two different populations interacting with each other, a prey and a predator population. In this case, we chose to study the lynx (predator) and hare (prey) populations. For this we used the Lotka-Volterra (L-V) equations:

$$\frac{dH}{dt} = \alpha H - \beta HL \qquad\qquad \frac{dL}{dt} = \delta HL - \gamma L$$

With:

- H = Hare population
- L = Lynx population
- α = Natural growth rate of hares
- β = Predation rate
- δ = Lynx reproduction rate
- γ = Lynx death rate

We found data of population per year of hares and lynx from 1847 to 1903. Having access to the data from Hudson's Bay Company we can easily compute $\frac{dH}{dt}$ and $\frac{dL}{dt}$. We just need to find $\alpha, \beta, \delta\ and\ \gamma$.

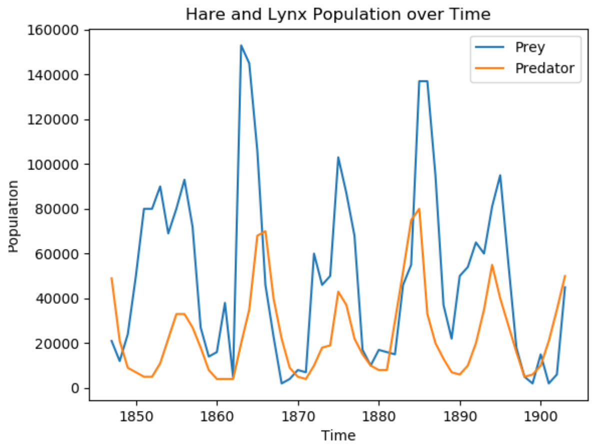

Figure 1: L-V simulation of Predator and Prey population over time

However, our first idea was to estimate ourselves the value of the parameters.

Problems we faced:

One of the first problems we faced was the lack of information on the parameters of the equation. We also had to search multiple times our data set as different sources had different values for the same year, but ultimately, we decided to stick with the first data we found. Another problem we had was the complexity of our problem, at first it seemed like only using the L-V equations was going to be too simple but when we added the sensitivity analysis, it became tougher to implement. One of the issues we had when programming was to decide when to stop. A lot of ideas have not been able to be coded due to time constraints. For instance, adding a random element to the equations, change the way of computing $\frac{dH}{dt}$ and $\frac{dL}{dt}$ adding a logistic growth and using Holling's type II functional response.

Results:

First, we used our equations using the parameters from *Matt Strimas-Mackey*. Which gave:

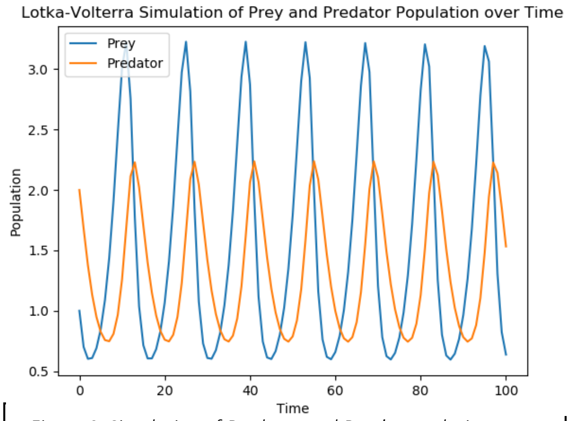

Figure 2: Simulation of Predator and Prey's populations over time (in years) with L-V

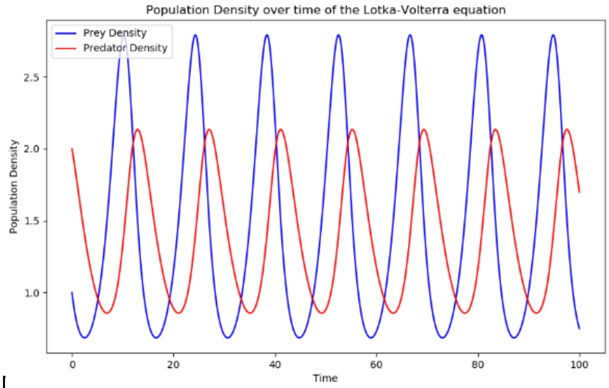

Figure 3: Population's density with L-V over time (in years)

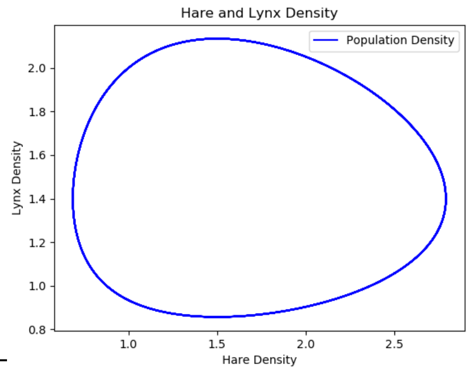

Figure 4 : Hare and lynx density in regards to each other's

The figure 2 shows the temporal dynamics of two populations, prey in blue and predator in orange. The populations oscillate over time, with prey peaks systematically preceding predator peaks. These oscillations illustrate the typical interactions between prey and predators: when prey increases, predators have more resources and also increase. Once the predators become numerous, they reduce the prey, which in turn reduces their own population. This qualitatively validates the predictions of the L-V equations, which predict periodic cycles. By superimposing figure 1 and figure 2, we notice that, although the reality is more complex than our model, the curves follow the same tendencies, thus, one could try to approximate reality using sinusoidal functions.

The figure 3 is similar to the previous one, but here the curves are distinguished by different colours, blue for prey density and red for predator density. The periodic oscillations remain clearly visible, reinforcing the predictable and rhythmic nature of prey-predator interactions.

The figure 4 represents a phase graph where the density of hares is on the x-axis and that of lynx on the y-axis. The plot forms a closed loop. The loop illustrates that the populations of prey and predators do not return to their initial state in a linear manner but follow a periodic trajectory in a phase plane. This shows the dependence of the two populations: each density state leads to a new state following defined interactions.

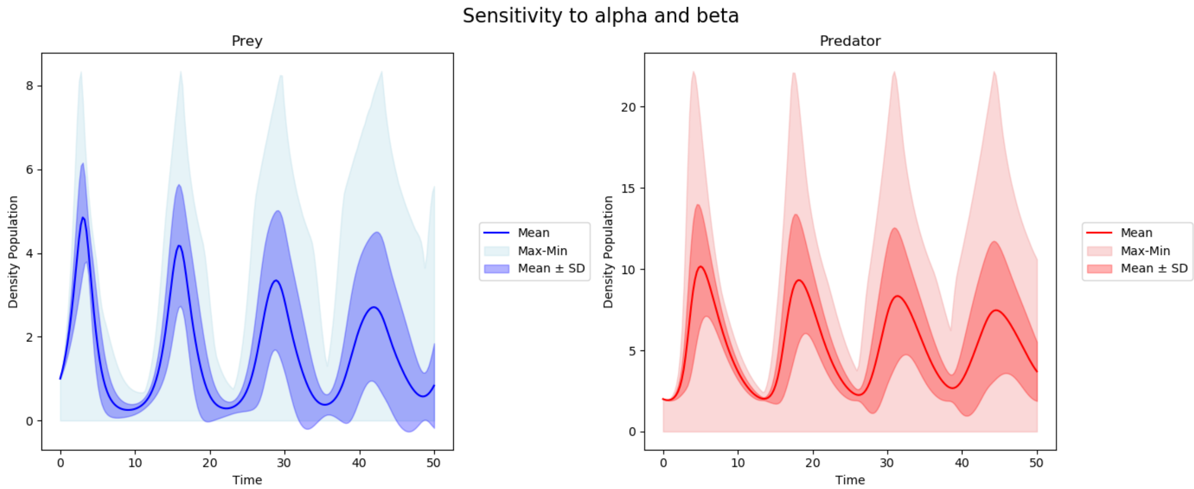

Figure 5: Changes in density population regarding α and β

Figure 5 shows two graphs side by side: in blue, the population density of prey over time. In red, the population density of predators over time. Each graph displays three main properties. The mean curve: Mean, the maximum and minimum curve: Max-Min, representing the full range of possible variations and, finally, a darker band: Mean ± SD, corresponding to the standard deviation around the mean. Prey oscillate with decreasing amplitude over time. Predators follow a similar cycle but with a time lag, predators peak after prey as we observed in figure 2. The uncertainty bands (light and dark) show that some periods are more sensitive than others. Prey have an increased variability at the beginning and after 15 years. Predators have a maximum variability in the early cycles and more stable towards the end.

These variations indicate that the behaviour of the system is strongly influenced by the parameters α, prey growth, and β, predation, in the early cycles. In the long term, the system seems to converge towards a more predictable and stable behaviour.

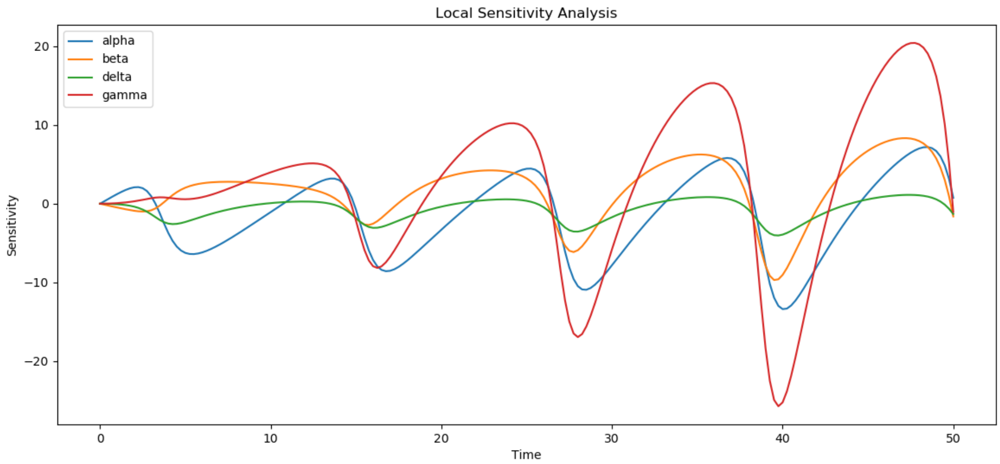

Figure 6: Each parameter's sensitivity in our model

Figure 6 shows the evolution of the impact that every parameter has over a time scale. Each parameter is represented by a separate curve. α, prey growth, in blue. β, predation rate, in orange. δ, predator efficiency, in green. γ, predator mortality, in red.

The curves oscillate between positive and negative values, indicating that the impact of each parameter varies over time. γ is the parameter that influence populations in a more dynamic and punctual way, with significant oscillations that reflect increased sensitivity at certain times. δ, being more stable, plays a constant and less variable role. α and β have an intermediate impact, influencing prey cyclically but less intensely than γ.

The cycles are synchronised with the oscillations of the populations. Moments of maximum or minimum sensitivity coincide with peaks or troughs in population densities.

Conclusion:

In conclusion, Lotka-Volterra equations can be used to determine the evolution of two different species with different hunting habits. However, this method is not perfect as in fact, the population of species does not vary following a pure sinusoidal function. We have seen that the reality is more complex that our model, but that does not mean that we cannot try to use it to approximate the evolution of multiple interacting species. Our code can currently be used to find each parameter's value per year, compare the distinct evolution of each species populations with data acquired with empirical experiences and then modify the parameters to try to match the real curves with hypothetical curves to approximate the constant values. It can be used to create a sensitivity graph as well. This can be employed to determine to which parameter the model is most sensitive to. Moreover, if our code was to be enhanced, one could analyse Predator-Prey dynamics to help foresee possible future behaviours that might have an enormous impact on one of the two studied species. In this cases, other simulations could take place to change the probable outcome. The parameter values such as α, β, δ and γ can be of great assistance when studying a particular ecosystem for they influence a dual-relation balance. Once the four parameters are estimated, the most determining factors would be at our disposal to be altered.

Appendix:

Estimation of Parameters:

To estimate the parameter α, the natural growth rate of hares, we assumed the growth of hares followed an exponential model in the absence of lynx. Thanks to our data, we know $H(t), H_0$ *and* t which means that we can isolate α and then compute the mean.

• $H(t)$ = Population of hares according to the year • H_0 = Initial hare population • t = Number of years since the start year	$H(t) = H_0 e^{\alpha t}$
• $\bar{\alpha}$ = mean α • α = estimated parameter • n = number of years	$\bar{\alpha} = \frac{1}{n}\sum_{t=1}^{n}\frac{ln(H(t)) - ln(H_0)}{t}$

Similarly, to estimate the parameter γ, the natural death rate of lynx. The decline of lynx population assuming no hares would follow an exponential decay. Thanks to our data, we know $L(t), L_0$ *and* t which means that we can isolate γ and then compute the mean value.

• $L(t)$ = Population of lynx according to the year • L_0 = Initial lynx population • t = Number of years since the start year	$L(t) = L_0 e^{-\gamma t}$
• $\bar{\gamma}$ = mean γ • γ = estimated parameter • n = number of years	$\bar{\gamma} = \frac{1}{n}\sum_{t=1}^{n}\frac{\ln(L_0) - \ln(L(t))}{t}$

However, for the β and the δ we changed our approach. We know all our variables except for β and δ, so we isolate them, calculate their values for every year and then compute their mean value.

Predation and lynx reproduction equations	
• H = Hare population • L = Lynx population • α = Natural growth rate of hares • β = Predation rate • δ = Lynx reproduction rate • γ = Lynx death rate	$\frac{dH}{dt} = \alpha H - \beta HL$ $\frac{dL}{dt} = \delta HL - \gamma L$

- $\bar{\beta} =$ mean β
- $\frac{dH}{dt} = \Delta H = H(t) - H(t-1)$
- $\bar{\delta} =$ mean δ
- $\frac{dL}{dt} = \Delta L = L(t) - L(t-1)$
- $n =$ number of years

$$\bar{\beta} = \frac{1}{n}\sum_{t=1}^{n}\frac{\alpha H - \Delta H}{HL}$$

$$\bar{\delta} = \frac{1}{n}\sum_{t=1}^{n}\frac{\Delta L + \gamma L}{HL}$$

Once we got all our mean values, we decided to use them in our graph. However, we did not get the expected results as our graphics resembled more to straight lines than to a sinusoidal function.

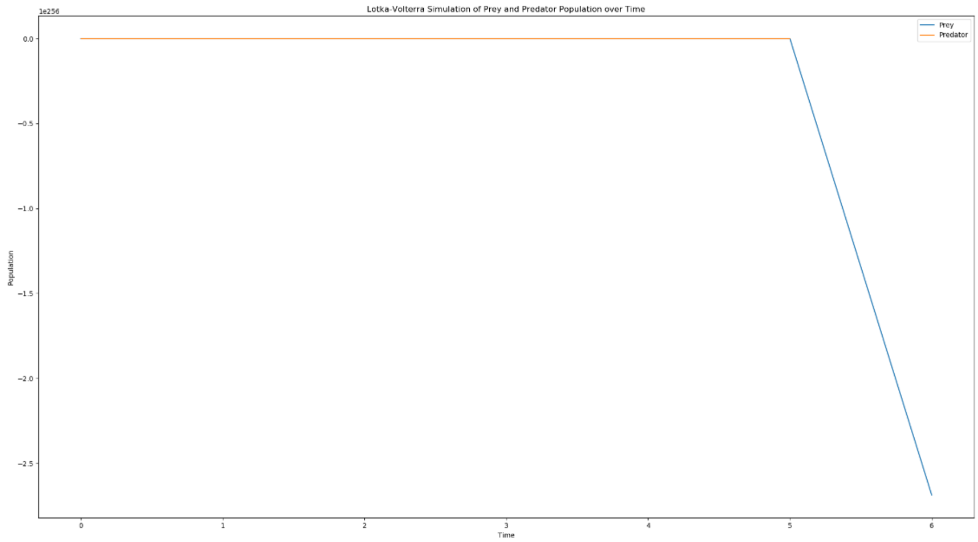

Figure 2: L-V simulation of Predator and Prey population overtime

Problems we faced:

The first problem we encountered was at the very beginning and it was the lack of information about what we first wanted: approximate the populations using already known constants to then compare them with the reality. Our solution was to approximate the constants with a known population over the years and then compare our computed populations with the real one to see if it worked.

Afterwards, we had another problem: the way to approximate the constants α and γ. To find β and δ we use the initial equations, and we need the α and γ as well. After thorough research, we decided to calculate them by using the natural growth and the natural death rates, respectively, and it worked. With them we calculated the β and the δ. So, we had values for the α, β, δ and γ per year.

However, as the L-V equations only work with unique general constants and not constants whose values change every year, we needed to figure out how to get a general value for each constant from our vast values. We opted for calculating the mean values of each constant. Our results were not what we expected which make sense as our α values are in the interval [-0.559616;0.334376], β values are in the interval [-0.000134; 0.001626], our δ values are in the interval [-0.000390;0.000270] and our γ values are in the interval [-0.018772;0.847298].

Taking this into account, the mean values are not an accurate way of estimating the general constants.

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