Derivation of Binet's Formula

1 Binet's Formula

Binet's formula is an explicit expression for the n-th Fibonacci number, which avoids the need for recursion or iteration.Binet's formula gives a way to directly compute Fibonacci numbers without having to compute all the previous numbers in the sequence. However, for large n, the formula may result in inaccuracies due to the limitations of floating-point arithmetic.

Binet's Formula for Fibonacci Numbers

The Fibonacci numbers are defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1$$

We seek an explicit formula for F_n in terms of n. To do this, we solve the recurrence relation using methods from solving linear recurrence relations.

Step 1: Set up the characteristic equation

We assume a solution of the form $F_n = r^n$, where r is a constant to be determined. Substituting this into the recurrence relation:

$$r^n = r^{n-1} + r^{n-2}$$

Dividing through by r^{n-2}, we get:

$$r^2 = r + 1$$

Rearranging the terms:

$$r^2 - r - 1 = 0$$

This is the characteristic equation for the Fibonacci sequence.

Step 2: Solve the characteristic equation

To solve the quadratic equation $r^2 - r - 1 = 0$, we apply the quadratic formula:

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus, the two roots are:

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

These roots are the golden ratio ϕ and its conjugate $\bar{\phi}$:

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \bar{\phi} = \frac{1-\sqrt{5}}{2}$$

Step 3: General solution to the recurrence

The general solution to the recurrence relation is a linear combination of the two independent solutions ϕ^n and $\bar{\phi}^n$. Thus, we have:

$$F_n = A\phi^n + B\bar{\phi}^n$$

where A and B are constants to be determined from the initial conditions.

Step 4: Use initial conditions to solve for constants

Using the initial conditions, we solve for A and B.
- When $n = 0$, $F_0 = 0$:

$$F_0 = A\phi^0 + B\bar{\phi}^0 = A + B = 0$$

This gives the equation:

$$A + B = 0 \quad \Rightarrow \quad B = -A$$

- When $n = 1$, $F_1 = 1$:

$$F_1 = A\phi^1 + B\bar{\phi}^1 = A\phi + B\bar{\phi} = A\phi - A\bar{\phi} = A(\phi - \bar{\phi})$$

We know that:

$$\phi - \bar{\phi} = \sqrt{5}$$

Thus:

$$1 = A\sqrt{5}$$

Solving for A, we get:

$$A = \frac{1}{\sqrt{5}}$$

Since $B = -A$, we have:

$$B = -\frac{1}{\sqrt{5}}$$

Step 5: Final solution

Now that we have determined the constants A and B, we can write the general solution for F_n:

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \bar{\phi}^n \right)$$

Thus, we have derived Binet's formula for the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

This formula provides a direct computation of the n-th Fibonacci number.