CSCI-UA 472 Artificial Intelligence

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Homework 04

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Problem 1

You have a box with 4 coins, all weighted.
1 coin comes up heads with probability 0.1 (Category 1).
1 coin comes up heads with probability 0.5 (Category 2).
2 coins come up heads with probability 0.8 (Category 3).

(A) You pick a coin out of the box at random and flip it. What is the probability that it will come up heads?

(B) You pick a coin out of the box and flip it twice. What is the probability of two heads?

(C) You pick two coins out of the box together (at the same time) and flip each of them once. What is the probability of two heads?

(D) You pick a coin of the box at random, flip it, put it back, again pick a coin at random, and flip it. What is the probability of two heads?

(E) You pick a coin out of the box at random and flip it. It comes up heads. What is the probability that it is Category 1? Category 2?

(F) You pick a coin out of the box at random and flip it. It comes up tails. What is the probability that it is Category 1? Category 2?

(G) You pick a coin out of the box at random and flip it. It comes up heads. You flip it again. What is the probability that the second flip will be heads?

(H) You pick a coin out of the box at random and flip it twice. It comes up heads both times. What is the probability that it is Category 1? Category 2?

(I) You pick a coin out of the box at random and flip it 10 times. What is the expected number of heads? (Hint: This is easy, once you've worked out part A. If you start calculating the probability distribution over the number of heads, you're on the wrong track.)

Solution to Problem 1

To answer this question, first, let us find probability for each category.

- Category 1 $\rightarrow \begin{cases} P(H) = 0.1 \\ P(T) = 1 - P(H) = 0.9 \end{cases}$

- Category 2 $\rightarrow \begin{cases} P(H) = 0.5 \\ P(T) = 1 - P(H) = 0.5 \end{cases}$

- Category 3 $\rightarrow \begin{cases} P(H) = 0.8 \\ P(T) = 1 - P(H) = 0.2 \end{cases}$

- With 4 coins, sample space for categories is (1233) i.e. a coin from category 1, 2, 3.

- Probability of picking a coin from box for each category $\rightarrow \begin{cases} C_1 = \frac{1}{4} \\ C_2 = \frac{1}{4} \\ C_3 = \frac{1}{2} \end{cases}$

Solutions:

(A) You pick a coin out of the box at random and flip it. What is the probability that it will come up heads?

- Sum the P(H) for each category $\rightarrow$ P(coin category) $\times$ P(head in that category)
- For k = 3, $\sum_{i=1}^{k} P(C_i) \times P(H)$
- P(H) = $\frac{1}{4} \times 0.1 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.8$
- P(H) = 0.025 + 0.125 + 0.4
- P(H) = 0.55 $\rightarrow$ answer

(B) You pick a coin out of the box and flip it twice. What is the probability of two heads?

- For k = 3, $\sum_{i=1}^{k} P(C_i) \times P(H) \times P(H)$
- P(HH) = $\frac{1}{4} \times 0.1 \times 0.1 + \frac{1}{4} \times 0.5 \times 0.5 + \frac{1}{2} \times 0.8 \times 0.8$
- P(HH) = 0.0025 + 0.0625 + 0.32
- P(HH) = 0.385 $\rightarrow$ answer

(C) You pick two coins out of the box together (at the same time) and flip each of them once. What is the probability of two heads?

- From category sample space, (1233) $\rightarrow \begin{cases} \text{Category possibilities are:} \\ P(12), P(13), P(23), P(33) \end{cases}$
- For $i = (1, 2, 3)$ and $j = (2, 3)$, $\sum_{i=1, j=2}^{i,j} P(i, j) \times P(H_i) \times P(H_j)$

- Cases $\rightarrow \begin{cases} Coin_{1,2} = 2 \times \dfrac{1}{4} \times \dfrac{1}{3} = \dfrac{1}{6} \\ Coin_{1,3} = 2 \times \dfrac{1}{4} \times \dfrac{2}{3} = \dfrac{1}{3} \\ Coin_{2,3} = 2 \times \dfrac{1}{4} \times \dfrac{2}{3} = \dfrac{1}{3} \\ Coin_{3,3} = \dfrac{1}{2} \times \dfrac{1}{3} = \dfrac{1}{6} \end{cases}$
- Probability for Heads in each coin tuple: $\begin{cases} coin_1 = 0.1 \times 0.5 & 0.05 \\ coin_2 = 0.1 \times 0.8 & 0.08 \\ coin_3 = 0.5 \times 0.8 & 0.4 \\ coin_4 = 0.8 \times 0.8 & 0.64 \end{cases}$
- P(H)P(H) $= \dfrac{1}{6} \times 0.05 + \dfrac{1}{3} \times 0.08 + \dfrac{1}{3} \times 0.4 + \dfrac{1}{6} \times 0.64$
- P(H)P(H) = 0.275 $\rightarrow$ answer

(D) You pick a coin of the box at random, flip it, put it back, again pick a coin at random, and flip it. What is the probability of two heads?

- Take the solution from (A) and multiply it with itself.
- For k = 3, $(\sum_{i=1}^{k} P(C_i) \times P(H)) \times (\sum_{i=1}^{k} P(C_i) \times P(H))$
- Answer from (A) i.e. P(H) = 0.55. Thus,
- P(Head, put back coin, then another head) $= P(H) \times P(H) = 0.55 \times 0.55$
- P(H)P(H) = 0.3025 $\rightarrow$ answer

(E) You pick a coin out of the box at random and flip it. It comes up heads. What is the probability that it is Category 1? Category 2?

- For Category 1:
  - Probability of heads in category 1 divided by probability of heads in all categories.
  - P(heads of C_1) $= \dfrac{P(H_{C_1})}{P(H)}$, thus:
  - $P(H_{C_1}) = \dfrac{1}{4} \times 0.1$
  - P(H) = answer from (A) i.e. $\frac{1}{4} \times 0.1 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.8$
  - P(heads of C_1) $= \dfrac{\frac{1}{4} \times 0.1}{\frac{1}{4} \times 0.1 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.8}$
  - P(heads of C_1) $= \dfrac{0.025}{0.55}$
  - P(heads of C_1) = 0.0455 (3 dp) $\rightarrow$ answer

- Category 2:
  - Probability of heads in category 2 divided by probability of heads in all categories.
  - P(heads of C_2) $= \dfrac{P(H_{C_2})}{P(H)}$, thus:
  - $P(H_{C_2}) = \dfrac{1}{4} \times 0.5$
  - P(H) = answer from (A) i.e. $\frac{1}{4} \times 0.1 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.8$
  - P(heads of C_2) $= \dfrac{\frac{1}{4} \times 0.5}{\frac{1}{4} \times 0.1 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.8}$
  - P(heads of C_2) $= \dfrac{0.125}{0.55}$
  - P(heads of C_2) = 0.227 (3 dp) $\rightarrow$ answer

(F) You pick a coin out of the box at random and flip it. It comes up tails. What is the probability that it is Category 1? Category 2?

- For Category 1:
  - Probability of tails in category 1 divided by probability of tails in all categories.
  - P(tails of C_1) $= \dfrac{P(T_{C_1})}{P(T)}$, thus:
  - $P(T_{C_1}) = \dfrac{1}{4} \times 0.9$
  - P(T) $= \frac{1}{4} \times 0.9 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.2$
  - P(tails of C_1) $= \dfrac{\frac{1}{4} \times 0.9}{\frac{1}{4} \times 0.9 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.2}$
  - P(tails of C_1) $= \dfrac{0.225}{0.45}$
  - P(tails of C_1) = 0.5 $\rightarrow$ answer
- Category 2:
  - Probability of tails in category 2 divided by probability of tails in all categories.
  - P(tails of C_2) $= \dfrac{P(T_{C_2})}{P(T)}$, thus:
  - $P(T_{C_2}) = \dfrac{1}{4} \times 0.5$
  - P(T) $= \frac{1}{4} \times 0.9 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.2$

– P(tails of C_2) = $\dfrac{\frac{1}{4} \times 0.5}{\frac{1}{4} \times 0.9 + \frac{1}{4} \times 0.5 + \frac{1}{2} \times 0.2}$

– P(tails of C_2) = $\dfrac{0.125}{0.45}$

– P(tails of C_2) = 0.278 (3 dp) $\rightarrow$ answer

(G) You pick a coin out of the box at random and flip it. It comes up heads. You flip it again. What is the probability that the second flip will be heads?

- In this question, we use answers from part (A) and part (B).
- P(HH) = $\dfrac{P(H_1 \bigcap H_2)}{P(H_1)}$
- $P(H_1 \bigcap H_2)$ = answer (B) = 0.385
- $P(H_1)$ = answer (A) = 0.55
- Hence, $\dfrac{0.385}{0.55}$
- P(HH) = 0.7 $\rightarrow$ answer

(H) You pick a coin out of the box at random and flip it twice. It comes up heads both times. What is the probability that it is Category 1? Category 2?

- Category 1:
  - From (A), P(H) = 0.55.
  - From (E), $P(H_{C_1}) = 0.0455$.
  - Thus, after flipping the coin twice, the probability of getting heads both times is:
  - P(Category 1, 2 flips, 2 heads) = $\dfrac{P(H_{C_1}) \times P(H_{C_1})}{P(H) \times P(H)}$
  - P(Category 1, 2 flips, 2 heads) = $\dfrac{0.0455 \times 0.0455}{0.55 \times 0.55}$
  - P(Category 1, 2 flips, 2 heads) = $\dfrac{0.00207}{0.3025}$
  - P(Category 1, 2 flips, 2 heads) = 0.006843 $\rightarrow$ answer
- Category 2:
  - From (A), P(H) = 0.55.
  - From (E), $P(H_{C_2}) = 0.227$.
  - Thus, after flipping the coin twice, the probability of getting heads both times is:
  - P(Category 2, 2 flips, 2 heads) = $\dfrac{P(H_{C_2}) \times P(H_{C_2})}{P(H) \times P(H)}$
  - P(Category 2, 2 flips, 2 heads) = $\dfrac{0.227 \times 0.227}{0.55 \times 0.55}$

– P(Category 2, 2 flips, 2 heads) $= \dfrac{0.05153}{0.3025}$
– P(Category 2, 2 flips, 2 heads) $= 0.17035 \rightarrow$ answer

(I) You pick a coin out of the box at random and flip it 10 times. What is the expected number of heads? (Hint: This is easy, once you've worked out part A. If you start calculating the probability distribution over the number of heads, you're on the wrong track.)

- Here, let us use information from (A).
- For 10 tosses, Exp(H_1) $= 10 \times 0.1$.
- For 10 tosses, Exp(H_2) $= 10 \times 0.5$.
- For 10 tosses, Exp(H_3) $= 10 \times 0.8$.
- For each category: $\begin{cases} \mathrm{H}_1 \times \frac{1}{4} & category 1 \\ \mathrm{H}_2 \times \frac{1}{4} & category 2 \\ \mathrm{H}_3 \times \frac{1}{2} & category 3 \end{cases}$
- Total expected number of heads $= \dfrac{1}{4} \times$ Exp(H_1) $+ \dfrac{1}{4} \times$ Exp(H_2) $+ \dfrac{1}{2} \times$ Exp(H_3)
- Exp(H) $= \dfrac{1}{4} \times 1 + \dfrac{1}{4} \times 5 + \dfrac{1}{2} \times 8$
- Exp(H) $= 5.5 \rightarrow$ answer

Problem 2

Suppose that X and Y are random variables with the following joint distribution

	Y=2	**Y=4**	**Y=5**
X=2	0.20	0.1	0.05
X=3	0.15	0.2	0.1
X=4	0.1	0.05	0.05

(A) Compute the following quantities: (a) P(X=2). (b) P(Y=4). (c) P(X=Y) (d) P(Y=4|X=2). (e) P(X=2|Y=4). (f) Exp(X). (g) Exp(Y).

(B) What is the probability distribution for the random variable X+Y? Compute Exp(X+Y) directly from that probability distribution.

Solution to Problem 2

(A) Computing the following:

(a) Technically, in this case, probability of a certain event is divided by total probability of all events. However, in this case, total probability is 1, thus, I ignore division by 1 in the following computations.

(b) P(X = 2)

- P(X = 2) = P(X=2 and Y=2) + P(X=2 and Y=4) + P(X=2 and Y=5)
- P(X = 2) = 0.20 + 0.1 + 0.05
- P(X = 2) = 0.35

(c) P(Y = 4)

- P(Y = 4) = P(Y=4 and X=2) + P(Y=4 and X=3) + P(Y=4 and X=4)
- P(Y = 4) = 0.1 + 0.2 + 0.05
- P(Y = 4) = 0.35

(d) P(X = Y)

- P(X = Y) = P(X=2 and Y=2) + P(X=4 and Y=4)
- P(X = Y) = 0.20 + 0.05
- P(X = Y) = 0.25

(e) P(Y=4|X=2)

- Conditional Probability distribution: P(A|B) = $\dfrac{P(A) \bigcap P(B)}{P(B)}$
- Thus, P(Y=4|X=2) = $\dfrac{P(Y = 4, X = 2)}{P(X = 2)}$
- P(Y=4|X=2) = $\dfrac{0.1}{0.20 + 0.1 + 0.05}$
- P(Y=4|X=2) = $\dfrac{0.1}{0.35}$
- P(Y=4|X=2) = 0.286 (3 dp) $\rightarrow$ answer

(f) P(X=2|Y=4)

- Conditional Probability distribution: P(A|B) = $\dfrac{P(A) \bigcap P(B)}{P(B)}$
- Thus, P(X=2|Y=4) = $\dfrac{P(X = 2, Y = 4)}{P(Y = 4)}$
- P(X=2|Y=4) = $\dfrac{0.1}{0.1 + 0.20 + 0.05}$
- P(X=2|Y=4) = $\dfrac{0.1}{0.35}$
- P(X=2|Y=4) = 0.286 (3 dp) $\rightarrow$ answer

(g) Exp(X)

- Exp(X) = $\sum_{\text{all x}} XP(X)$
- Exp(X) $= 2 \times (0.20 + 0.1 + 0.05) + 3 \times (0.15 + 0.2 + 0.1) + 4 \times (0.1 + 0.05 + 0.05)$
- Exp(x) $= 2 \times (0.35) + 3 \times (0.45) + 4 \times (0.2) = 0.7 + 1.35 + 0.8$
- Exp(X) $= 2.85$

(h) Exp(Y)

- Exp(Y) = $\sum_{\text{all y}} YP(Y)$
- Exp(Y) $= 2 \times (0.20 + 0.15 + 0.1) + 4 \times (0.1 + 0.2 + 0.05) + 5 \times (0.05 + 0.1 + 0.05)$
- Exp(Y) $= 2 \times (0.45) + 4 \times (0.35) + 5 \times (0.2) = 0.9 + 1.4 + 1.0$
- Exp(Y) $= 3.3$

(B) Probability Distribution for random variable X+Y:

- X+Y in the given joint distribution gives, (X+Y) = {4,5,6,7,8,9}
- For each X+Y value, calculate its probability.
  - P(X + Y = 4) = P(X = 2, Y = 2) = 0.2
  - P(X + Y = 5) = P(X = 3, Y = 2) = 0.15
  - P(X + Y = 6) = P(X = 2, Y = 4) + P(X = 4, Y = 2) = 0.1 + 0.1 = 0.2
  - P(X + Y = 7) = P(X = 2, Y = 5) + P(X = 3, Y = 4) = 0.05 + 0.2 = 0.25
  - P(X + Y = 8) = P(X = 3, Y = 5) + P(X = 4, Y = 4) = 0.1 + 0.05 = 0.15
  - P(X + Y = 9) = P(X = 4, Y = 5) = 0.05
- Hence, using the above calculations, we get the probability distribution of random variable X+Y i.e. {0.2, 0.15, 0.2, 0.25, 0.15, 0.05}
- Computing Exp(X+Y)
  - Exp(X+Y) = $\sum_{\text{all x+y}} (X + Y)P(X + Y)$
  - Exp(X+Y) $= 4 \times 0.2 + 5 \times 0.15 + 6 \times 0.2 + 7 \times 0.25 + 8 \times 0.15 + 9 \times 0.05$
  - Ep(X+Y) $= 6.15$

Problem 3

Suppose you have the same box as in problem 1.

(A) Someone offers you the following bet: You will take a coin at random out of the box and flip it. If it comes up heads, he pays you \$10; if it comes up tails, you pay him \$10.

Should you take the bet? What is your expected profit (positive) or loss (negative) if you take the bet? Draw the decision tree, including the decision whether to take or reject the bet.

(B) The other person now gives you a third option. You will pull a coin at random out of the box. You can flip the coin once, to test it; however, he will charge you $1 for that. When you see the outcome, you have a choice: You can quit, in which case the game is over, or you can flip the coin, in which case he will pay you $10 if it is heads and you will pay him $10 if it is tails.

What, now, is your best strategy? What is your expected profit or loss if you follow your best strategy? Draw the decision tree.

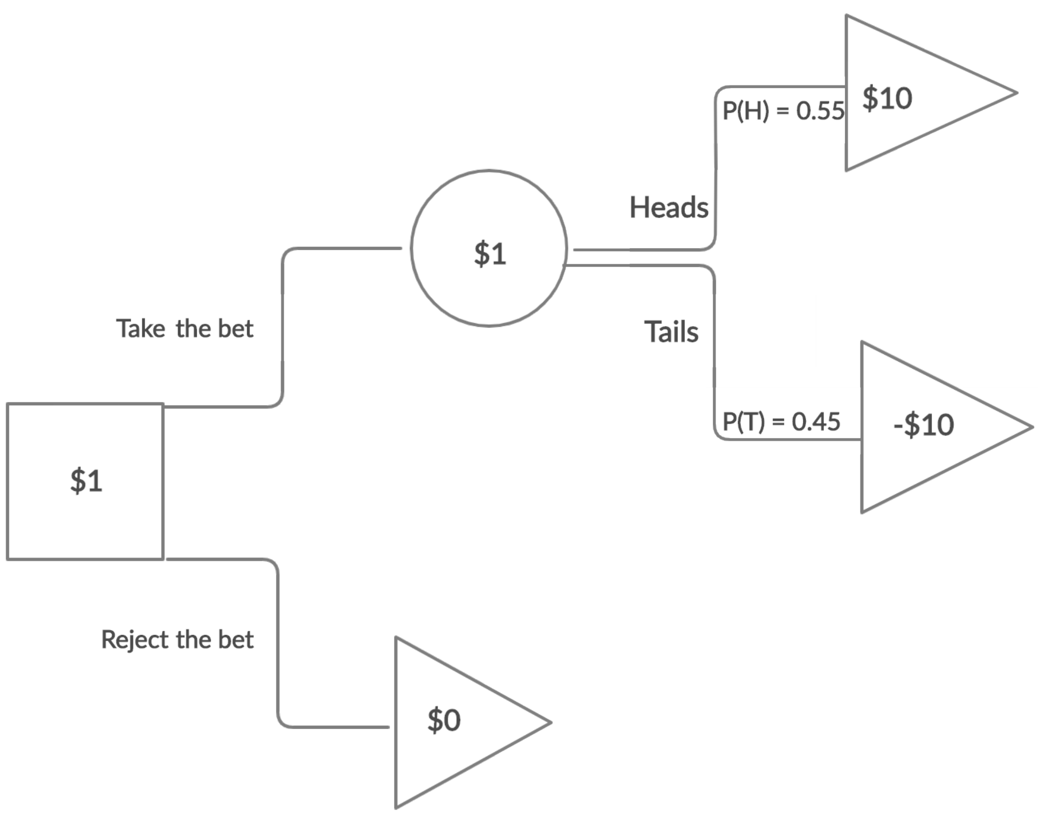
Solution to Problem 3

(A) For part A, refer to the following diagram.

- We used P(H) from question 1. P(H) = 0.55, P(T) = 1 - P(H) = 0.45.
- We calculate expected value through as:

$$E(val) = (0.55 \times 10) + (0.45 \times -10) = 1$$

- Since, the expected value is greator than zero, i.e. $1, bet should be taken.
- Respectively, the expected profit in this situation is equal to $1.



(B) For part B, please refer to the following figure.

- Using problem 1, we know that $P(H_2|H_1) = 0.7$
- We find $P(T_2|T_1)$ by the following equation:

$$P(T_2|T_1) = \frac{0.285}{0.45} = 0.633$$

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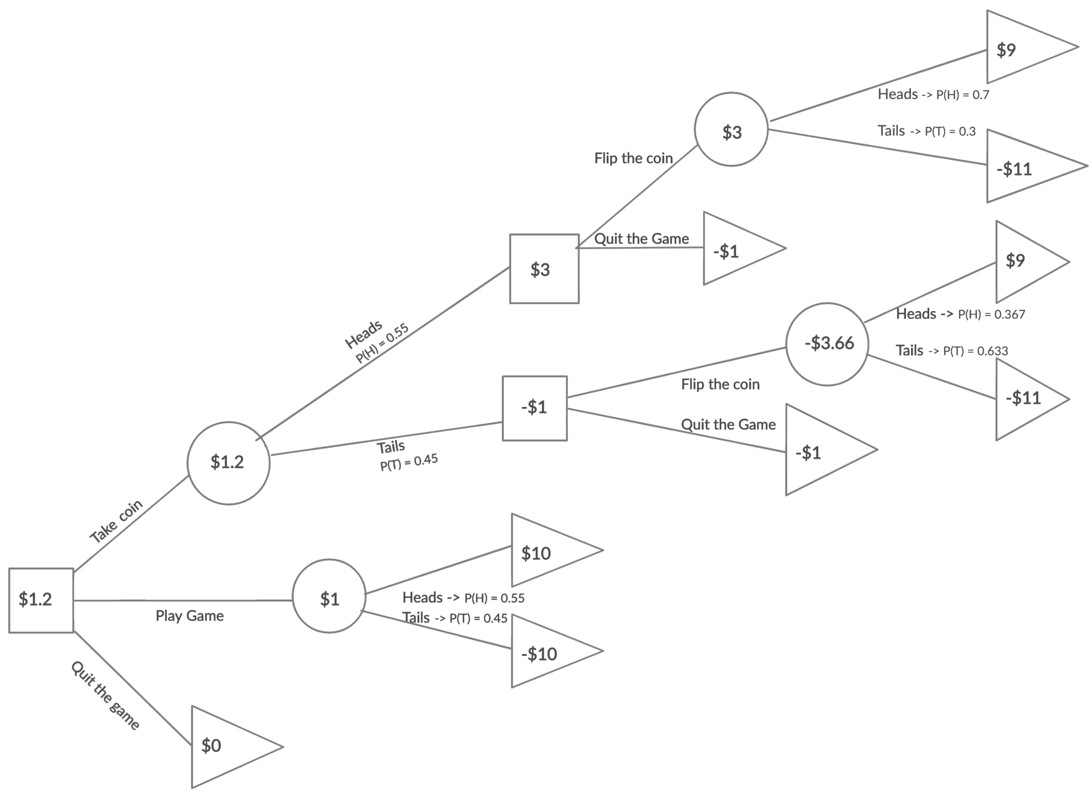
- Thus, using the information, we draw the following decision tree.
- After the test flip, if we get Heads, we SHOULD take the bet as expected value is \$3.
- If test flip gives tail, we SHOULD NOT take the bet as expected value is less than \$0.



End of Assignment. Thank you!