FUNCTIONS ON REAL NUMBERS

Definition: Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then f_1+f_2 and $f_1 * f_2$ are also functions from A to $\mathbb{R}$ defined by;

$$(f_1+f_2)(x) = f_1(x) + f_2(x).$$

$$(f_1 * f_2)(x) = f_1(x) * f_2(x).$$

Examples:

Assume $f_1(x) = x-1$.

$f_2(x) = x^3+1$.

then

$$(f_1+f_2)(x) = x^3+x.$$

$$(f_1 * f_2)(x) = x^4 - x^3 + x - 1.$$

INCREASING & DECREASING FUNCTIONS

Definition:

A function f whose domain and codomain are subsets of real numbers is strictly increasing if $f(x) > f(y)$ whenever $x>y$.

Similarly f is called strictly decreasing if $f(x) < f(y)$. whenever $x>y$.

Example:

Let $g: \mathbb{R} \rightarrow \mathbb{R}$, where $g(x) = 2x-1$

Is it increasing function?

Ans. Yes. Proof:

For $x>y$ holds $2x > 2y$ and subsequently $2x-1 > 2y-1$.

Thus it is strictly increasing.

Note: Strictly increasing & strictly decreasing functions are one-to-one. $\longrightarrow$ Why?

By definition of one-to-one function: A function is one-to-one if $f(x) \neq f(y)$, whenever $x \neq y$.

IDENTITY FUNCTION.

Definition: Let A be a set. The identity function on A is the function $i_A: A \to A$ where $i_A(x) = x$.

Example:

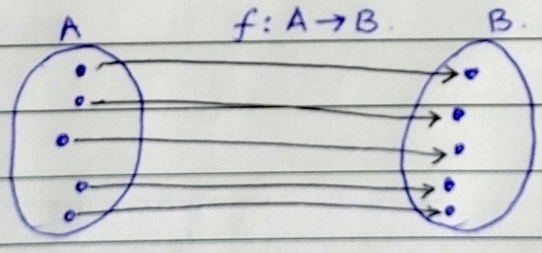
Let $A = \{1, 2, 3\}$.
Then $i_A(1) = ?$

Ans:

$i_A(1) = 1$
$i_A(2) = 2.$
$i_A(3) = 3.$

BIJECTIVE FUNCTION (Revision)

Definition: A function f is called bijection if it is both one-to-one and onto.



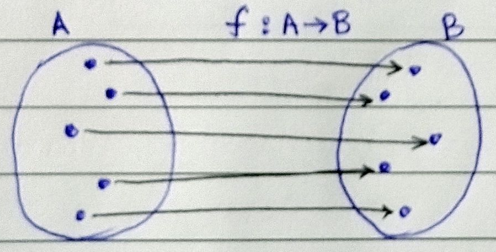
INVERSE FUNCTIONS.

Definition:

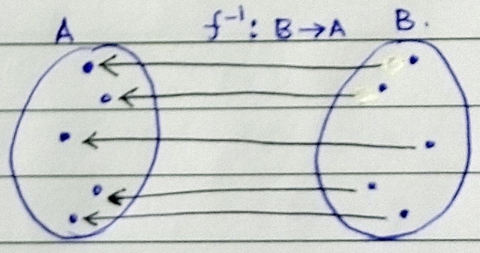
Let f be a bijection from set A to set B.
The inverse function of f is the function that assigns to an element b from B the unique element a ~~from~~ in A such that;
$f(a) = b$

The inverse function of f is denoted by f^{-1}.
Hence, $f^{-1}(b) = a$, when $f(a) = b$.

If the inverse function of f exists, f is called invertible.



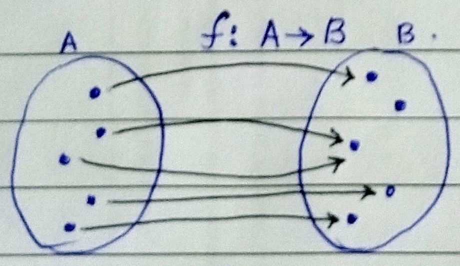
f is bijective



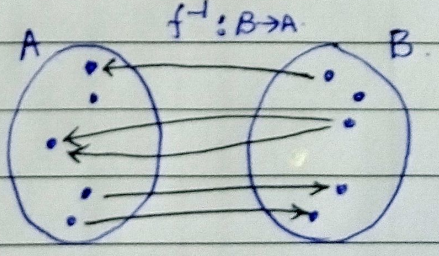
inverse of f.

Note: If f is not a bijection then it is not possible to define the inverse function of f.
Why?

i) Assume f is not one-to-one.



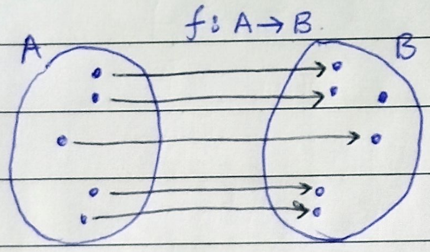
f



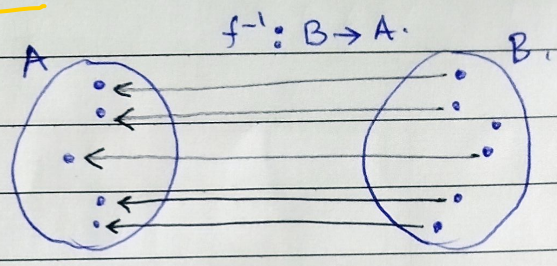
Inverse.

Inverse is not a function. One element of B is mapped to two different elements.

ii) Assume f is not onto. Then?



f



Inverse.

Here also, inverse is not a function. One element in B is not assigned any value in A.

Example # 1.

Let $A = \{1, 2, 3\}$. and i_A be the identity function

$i_A(1) = 1$. $\qquad i_A^{-1}(1) = 1$

$i_A(2) = 2$. $\qquad i_A^{-1}(2) = 2$.

$i_A(3) = 3$. $\qquad i_A^{-1}(3) = 3$.

Therefore, the inverse function of i_A is i_A.

Example #2.

Let $g: \mathbb{R} \to \mathbb{R}$, where $g(x) = 2x - 1$.

What is the inverse function g^{-1} ?

Solution:

$y = 2x - 1 \Rightarrow y + 1 = 2x$.

$\Rightarrow \boxed{(y+1)/2 = x}$.

Therefore $g^{-1}(y) = x = (y+1)/2$

Test the correctness of inverse:

$g(3) = 2*3 - 1 = 5$. $\qquad g(10) = 2*10 - 1 = 19$.

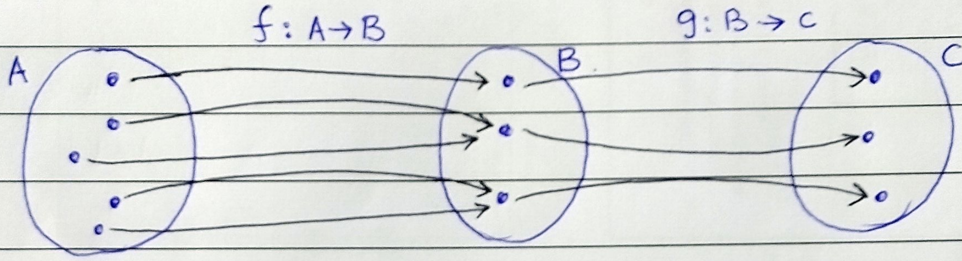
$g^{-1}(5) = (5+1)/2 = 3$. $\qquad g^{-1}(19) = (19+1)/2 = 10$.

Composition Of Functions.

Definition: Let f be a function from set A to set B and let g be a function from set B to set C.

The composition of functions g and f, denoted by $g \circ f$ is defined by;

$$(g \circ f)(a) = g(f(a)).$$



Example#1.

Let $A = \{1, 2, 3\}$ $B = \{a, b, c, d\}$.

$g: A \to A$;

$1 \to 3$

$2 \to 1$

$3 \to 2$

$f: A \to B$.

$1 \to b$

$2 \to a$.

$3 \to d$.

$f \circ g: A \to B$.

$1 \to d$

$2 \to b$.

$3 \to a$.

Example #2.

Let f and g be two functions from Z to Z, where

$f(x) = 2x$.

$g(x) = x^2$.

$(f \circ g) : Z \to Z$

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)). \\
&= f(x^2). \\
&= 2x^2.
\end{aligned}
$$

$g \circ f : Z \to Z$.

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)). \\
&= g(2x). \\
&= (2x)^2 \\
&= 4x^2.
\end{aligned}
$$

Example # 3.

$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ for all x.

Let $f : \mathbb{R} \to \mathbb{R}$, wher $f(x) = 2x - 1$. and.

$f^{-1}(x) = (x+1)/2$

$$
\begin{aligned}
(f \circ f^{-1})(x) &= f(f^{-1}(x)). \\
&= f((x+1)/2). \\
&= 2((x+1)/2) - 1. \\
&= x + 1 - 1 \\
&= x.
\end{aligned}
$$

$$
\begin{aligned}
(f^{-1} \circ f)(x) &= f^{-1}(f(x)). \\
&= f^{-1}(2x-1). \\
&= 2x/2. \\
&= x
\end{aligned}
$$

Floor Function:

The floor function assigns a real number x the largest integer that is less than or equal to x.
The floor function is denoted by $\lfloor x \rfloor$.

Ceiling Function:

The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x.
Ceiling function is denoted by $\lceil x \rceil$.