MATRICES.

Definition:

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called $m \times n$ matrix.

Square matrix:

A matrix with same number of rows as columns is called a square matrix.

Equal Matrices:

Two matrices are equal if they have same number of rows and same number of columns and the corresponding entries in every position are equal.

Example: Equal matrices:

$$A = \begin{bmatrix} 1 & 0.5 \\ 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1/2 \\ 2^2 & 2 \end{bmatrix}$$

- Matrix A and matrix B are equal.

- Let A is $m \times n$ matrix i.e. A has m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The entry at i^{th} row and j^{th} column is represented as a_{ij} and is called the corresponding element of A.
- A short notation to represent matrix A is $A = [a_{ij}]$.

MATRICES OPERATIONS/ARITHMETIC.

MATRIX ADDITION:

Let A and B be two matrices of same order $m \times n$. The sum of A and B, denoted by $A+B$, is the matrix that has $a_{ij}+b_{ij}$ as its $(i,j)^{th}$ element.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}.$$

Note: Matrices of different dimensions/sizes cannot be added.

MATRIX MULTIPLICATION:

Let A be an $m \times k$ matrix and B be a $k \times n$ matrix. The product of A and B, denoted by AB, is the $m \times n$ matrix that has its $(i,j)^{th}$ element equal to the sum of products of the corresponding elements from the i^{th} row of A and j^{th} column of B.

$A = [a_{ik}]$, $B = [b_{kj}]$.

$AB = [C_{ij}]$ where $C_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots a_{jk} b_{kj}$.

Example:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

Note: Product of two matrices possible only when number of columns in first matrix and no. of rows in second matrix are equal.

$AB \neq BA$.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1\times2+2\times1 & 1\times1+2\times3 \\ 3\times2+4\times1 & 3\times1+4\times3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 10 & 15 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 2\times1+1\times3 & 2\times2+1\times4 \\ 1\times1+3\times3 & 1\times2+3\times4 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 10 & 14 \end{bmatrix}.$$

$AB \neq BA$ Proved.

IDENTITY MATRIX.

The identity matrix of order n is the $n\times n$ matrix $I_n = [\delta_{ij}]$, wher $\delta_{ij} = 1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Properties:

* Assume A is an $m\times n$ matrix, then $AI_n = A$ and $I_m A = A$.

Power of square matrices:

* Assume A is an $n\times n$ matrix Then $A^0 = I_n$. and

$$A^r = \underbrace{A\,A\,A\cdots A}_{r \text{ times}}.$$

MATRIX TRANSPOSE.

- Let $A = [a_{ij}]$ be an $m \times n$ matrix. The transpose of A, denoted by A^t, is the $n \times m$ matrix obtained by interchanging the rows and columns of A.

If $A^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2 \cdots n$ and $j = 1, 2 \cdots m$.

Example:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{then} \quad A^t = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Rotate an image by 90° = ?

MATRIX INVERSE

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The inverse of A, denoted by A^{-1}, is the $n \times n$ matrix such that $\boxed{A A^{-1} = A^{-1} A = I}$.

Note: Inverse of the matrix A may not exist.

SYMMETRIC MATRIX.

A square matrix A is called symmetric if $A^t = A$.

Example:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix} \qquad A^t = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}.$$

- A is symmetric matrix.

ZERO-ONE MATRICES.

A matrix in which elements are either 0 or 1 and no other numbers is called zero-one-matrix

- Zero-one matrices are often used to represent discrete structures and boolean arithmetic based on boolean operations is performed on such structures

Join of Two matrices: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zero-one matrices. The join of A and B is the zero-one matrix with $(i, j)^{th}$ entry $a_{ij} \vee b_{ij}$. The join of A and B is rep. by $A \vee B$.

Meet of two matrices: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zero-one matrices. The meet of A and B, rep. by $A \wedge B$ is a $m \times n$ non-zero matrix where $(i, j)^{th}$ element is $a_{ij} \wedge b_{ij}$.

Boolean product of two zero-one matrices: Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix. The boolean product of A and B, denoted by $A \odot B$ is an $m \times n$ matrix, where c_{ij} is each $(i, j)^{th}$ element as defined bellow;

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj})$$

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad B \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad A \odot B = ?$

$$A \odot B = \begin{bmatrix} \cancel{(1 \wedge 1) \vee (0 \wedge 0)} & \cancel{(1 \wedge 1) \vee (0 \wedge 1)} & \cancel{(1 \wedge 0) \vee (0 \wedge 1)} \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ Ans.}$$

The r^{th} Boolean power.

Definition: Let A be a square zero-one matrix and let r be positive integer.

The r^{th} boolean power of A is the Boolean product of r factors of A. It is denoted by $A^{[r]}$.

$$\boxed{A^{[r]} = \underbrace{A \odot A \odot A \cdots \odot A}_{r \text{ times}}}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Find $A^{[n]}$ for all positive integers n.

$$A^{[2]} = A \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{[3]} = A^{[2]} \odot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{[4]} = A^{[3]} \odot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$A^{[n]} = A^{[5]}$ for all positive integers n with $n \geq 5$.