Sets & Set Operations.

Discrete math is study of the discrete structures used to represent discrete objects.

✓ Many discrete structures are built using sets.

- Examples of discrete structures built using sets are;
  - Combinations.
  - Relations.
  - Graphs.

Set Definition: A set is a (unordered) collection of objects. These objects are sometimes called elements or members of the set.

Examples:

- Vowels in the English alphabet.

  $V = \{a, e, i, o, u\}$.

- First seven prime numbers.

  $X = \{2, 3, 5, 7, 11, 13, 17\}$.

Representing Sets:

1). Listing (enumerating) the members of the set.

2). Using set builder notation i.e. definition by property.

$\{x \,/\, x \text{ has property } p\}$.

Example.

$E = \{50, 52, 54, 56, 58, 60, 62\}$.

$E = \{x \,/\, 50 \leq x \leq 63,\ x \text{ is an even integer}\}$.

If enumeration of the members is hard we often use ellipses.

Example: A set of integers b/w 1 and 100.

$A = \{1, 2, 3 \ldots\ldots 100\}$.

Important sets in discrete math.

- Natural Numbers

  $N = \{0, 1, 2, 3 \ldots\ldots \}$.
- Integers

  $Z = \{\ldots -2, -1, 0, 1, 2 \ldots\}$.
- Positive integers

  $Z^+ = \{1, 2, 3 \ldots\}$.
- Rational Numbers.

  $Q = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$.
- Real Numbers.

  $\mathbb{R}$

Equality:

Two sets are equal if and only if they have same elements.

Example: $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 2, 1, 3, 2\}$.

Note:
- Duplicates don't contribute anything new to a set, so remove them.
- The order of the elements in a set does not contribute anything new.

Example: Are $\{1, 2, 3, 4\}$ and $\{1, 2, 2, 4\}$ equal?

Ans: No.

Special sets:

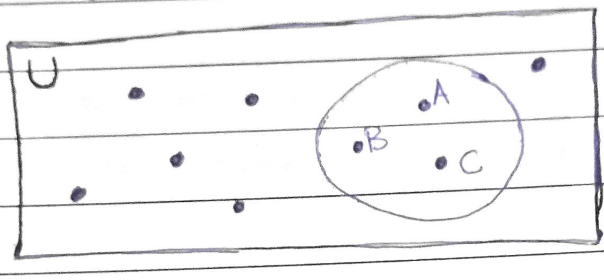
Universal Set: Denoted by U is set of all objects under consideration.

Empty set: Denoted by $\emptyset$ or $\{\ \}$, does not contain any element.

Venn Diagrams:

A set can be visualized using Venn Diagrams.

$V = \{A, B, c\}$.

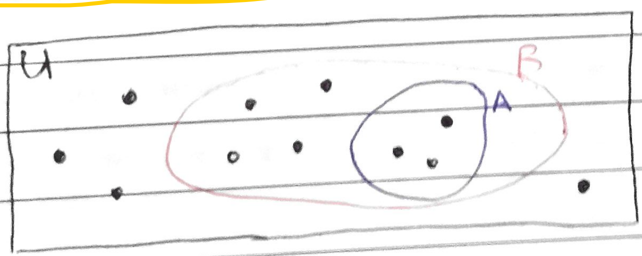


A Subset:

Definition: A set A is said to be subset of B if and only if every element of A is also an element of B.

"A is subset of B" is indicated as: $A \subseteq B$

$\forall x\,(x \in A) \rightarrow (x \in B)$



Subset Properties: [Empty set]

Theorem: An emty set is subset of any set. $\emptyset \subseteq S$

Proof: subset defintion, all elements of set A must be elements of set B: $\forall x\,(x \in A \rightarrow x \in B)$

- We must show that following implication holds for any S
  $\forall x\,(x \in \emptyset \rightarrow x \in B)$.
- Since the empty set does not contain any element, therefore $x \in \emptyset$ is always $\boxed{\text{false}}$.
- Then the implication is always True.

End of proof.

Subset Properties:

Theorem: Any set S is a subset of itself $S \subseteq S$.

Proof: Definition of subset says; all the elements of set A. must be also elements of set B: $\forall x(x \in A \rightarrow x \in B)$.

* Applying this we get:
  $\forall x\ (x \in S \rightarrow x \in S)$. which is trivially True.

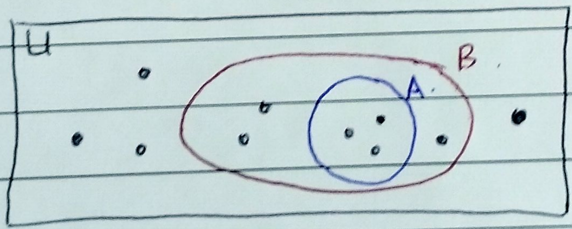
End of proof.

Note: Two sets are equal if each is subset of the other set.

A proper subset:

Definition: A set A is said to be proper subset of B. if and only if $A \subseteq B$ and $A \neq B$.

* "A is proper subset of B" is deneded as: $A \subset B$.



Example: $A = \{1, 2, 3\}$.
$B = \{1, 2, 3, 4, 5\}$.

Is $A \subset B$?.

Ans: Yes!

"CARDINALITY"

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a non-negative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

Examples:

$V = \{1 \quad 2 \quad 3 \quad 4 \quad 5\}$ $\qquad |V| = 5.$

$A = \{1, 2, 3 \ldots\ldots 20\}$ $\qquad |A| = 20.$

$|\emptyset| = 0.$

INFINITE SET.

Definition: A set is infinite if it is not finite. i.e. members are not countable.

Examples:
- Set of natural numbers. $N = \{1, 2, 3 \ldots\ldots\}$.
- Set of real numbers.

POWER SET.

Definition: Given a set S, the power set of S is the set of all subsets of S. It is denoted by $P(S)$.

Examples:

$S = \{1, 2, 3\}$.

$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

$|S| = 3$. $\quad |P(S)| = 8$.

$A = \{1\}$.

$P(A) = \{\phi, \{1\}\}$.

$|A| = 1$ $\quad |P(A)| = 2$

$B = \phi$

$P(B) = \{\phi\}$.

$|B| = 0$ $\quad |P(B)| = 1$.

If S is a set with $|S| = n$. then

$|P(S)| = 2^n$

N-TUPLE.

Sets are used to represent unordered collections.

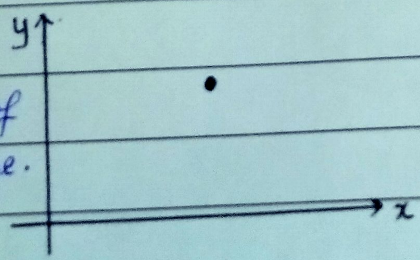
n-tuples are used to represent ordered collections.

Definition: An ordered n-tuple $(x1, x_2, \dots x_N)$ is the ordered collection that has x_1 as its first element, x_2 as its second element ...., and x_N its n^{th} element,

$N \geq 2$.

Example: coordinates of a point in 2D plane.

$(12, 18)$.



CARTECIAN PRODUCT

Definition: Let S and T be sets. The cartecian product of S and T, denoted by S x T, is the set ~~all~~ of all ordered pairs (s, t), wher s ∈ S and t ∈ T.

$$S \times T = \{(s,t) \mid s \in S \wedge t \in T\}$$

Examples:

$S = \{1, 2\}$ $T = \{a, b, c\}$.

$S \times T = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

$T \times S = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

NOTE: $S \times T \neq T \times S$.

Cardinality of the Cartecian Product.

$$\boxed{|S \times T| = |S| * |T|}$$

Example:

A = { Blue, Red, Green }.

B = { Circle, Triangle, Square }

A x B = { (Blue, Circle), (Blue, Triangle), (Blue, Square), (Red, Circle), (Red, Triangle), (Red, Square), (Green, Circle), (Green, Triangle), (Green, Square) }

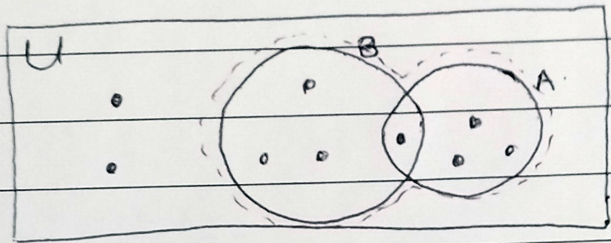
$|A| = 3$ $|B| = 3$. $\boxed{|A \times B| = 9 = |A| * |B|}$.

NOTE: A subset of cartecian product A x B is called a relation from the set A to the set B.

SET OPERATIONS.

Union: Let A and B bet sets. The union of A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B; or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

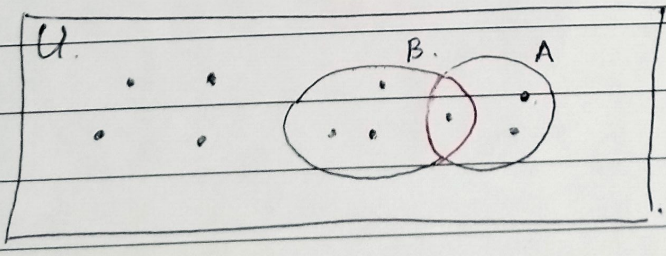


Example: $A = \{1, 2, 3, 6\}$ $B = \{2, 4, 6, 9\}$.

$A \cup B = \{1, 2, 3, 4, 6, 9\}$.

Intersection: Let A and B be sets. The intersection of A and B, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



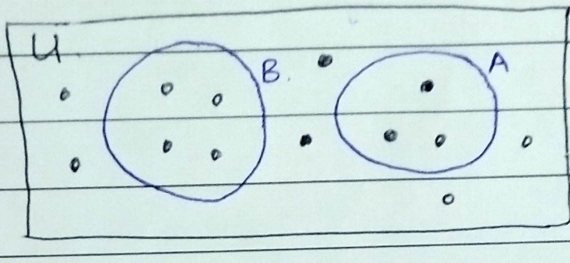
Example: $A = \{1, 2, 3, 6\}$ $B = \{2, 4, 6, 9\}$.

$A \cap B = \{2, 6\}$.

DISJOINT SETS.

Definition: Two sets are called disjoint if their intersection is empty.

A and B are disjoint if and only if $A \cap B = \emptyset$.



Example: $A = \{1, 2, 3, 6\}$ $B = \{4, 7, 8\}$

$A \cap B = \{\ \} = \emptyset$

Hence, A and B are disjoint sets.

Cardinality of the set Union.

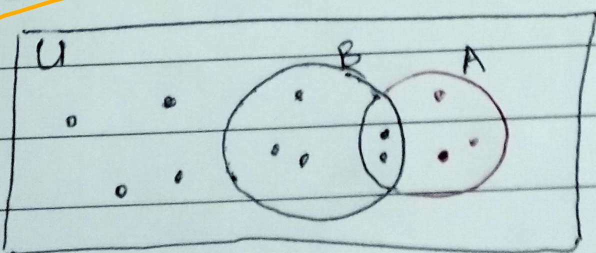
$$|A \cup B| = |A| + |B| - |A \cap B|$$

SET DIFFERENCE

Definition: Let A and B be sets. The difference of A and B, denoted by $A-B$, is the set containing those elements that are in A but not in B.

- The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$



Example: $A = \{1, 2, 3, 5, 7\}$ $B = \{1, 5, 6, 8\}$.

$A - B = \{2, 3, 7\}$.

COMPLEMENT OF A SET

Defintion: Let U be the universal set: the set of all objects under consideration.

Definition: The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U.

$$\bar{A} = \{x \mid x \notin A\}.$$

Example: $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ $\quad A = \{1, 3, 5, 7\}$

$\bar{A} = \{2, 4, 6, 8\}$.

SET IDENTITIES

Set identities (analogous to logical equivalences).

Identity

$A \cup \emptyset = A$.

$A \cap U = A$.

Domination

$A \cup U = U$

$A \cap \emptyset = \emptyset$

Idempotent.

$A \cup A = A$.

$A \cap A = A$.

Absorption Law.

$A \cup (A \cap B) = A$.

$A \cap (A \cup B) = A$.

Complement Law.

$A \cup \bar{A} = U$

$A \cap \bar{A} = \emptyset$.

Double complement.

$\bar{\bar{A}} = A$.

Commutative

$A \cup B = B \cup A$.

$A \cap B = B \cap A$.

Associative

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

Distributive.

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

DeMorgan's Laws.

$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.

$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.

Set identities can be proved using membership tables.

- List each combination of sets that an element can belong to then show that for each such a combination the element either belongs or does not belong to both sets in the identity. → Its. Like truthtables.

Prove: $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.

A	B	$\bar{A}$	$\bar{B}$	$\overline{A \cap B}$	$\bar{A} \cup \bar{B}$
1	1	0	0	0	0
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	1	1

GENERALIZED UNIONS & INTERSECTIONS.

Definition: The union of collection of sets is the set that contains those elements that are members of atleast one set in the collection.

$$\bigcup_{i=1}^{n} A_i = \{A_1 \cup A_2 \cup \cdots\cdots \cup A_n\}.$$

Definition: The intersection of collection of sets is the set that contains those elements that are members of all sets in the collection.

$$\bigcap_{i=1}^{n} A_i = \{A_1 \cap A_2 \cap \cdots \cap A_n\}.$$

Computer representation of sets.

How to represent sets in the computer?

- One solution: Data structures like a list
- Better solution:

Assign a bit in a bit string to each element in the universal set. and set the bit to 1 if the element is present otherwise 0.

Example:

All possible elements: $U = \{1\ 2\ 3\ 4\ 5\}$

* Assume $A = \{2, 5\}$

\- Computer representation $A = 01001$

* Assume $B = \{1, 5\}$.

\- Computer representation: $B = 10001$.

- The union is modeled with bitwise OR.

  $A \vee B = 11001$.

- The intersection is modeled with bitwise AND.

  $A \wedge B = 00001$.

- The complement is modeled with a bitwise negation

  $\bar{A} = 10110$.