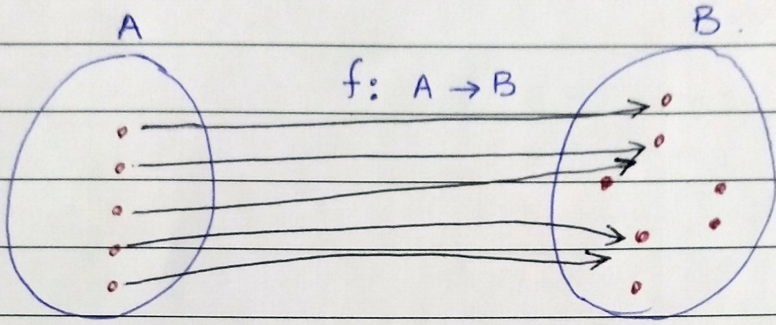
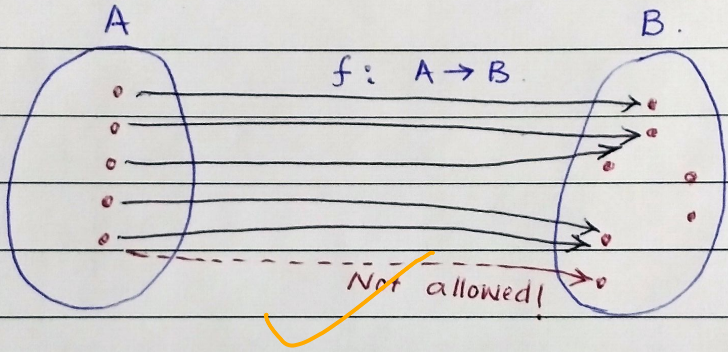
FUNCTIONS

Definition: Let A and B be two sets.
A function from A to B, denoted as $f: A \to B$, is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ to denote the assignment of b to an element of A by the function.





Representations of functions:

① * Explicitly state the assignments in between elements of the two ~~functions~~ sets.

② * Compactly by formula. (using 'standard' functions)

Example 1:

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$

- Assume f is defined as:
  - $1 \to c$.
  - $2 \to a$.
  - $3 \to c$.

Is f a function?

Ans: Yes, since $f(1) = c$, $f(2) = a$, $f(3) = c$, each element of A is assigned an element from B.

Example #2

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

Assume g is defined as

$1 \to c$

$1 \to b$.

$2 \to a$.

$3 \to c$.

Is g a function?

Ans: No. $g(1)$ is assigned both c and b.

Example #3.

$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $B = \{0, 1, 2\}$.

Define $h: A \to B$ as:

$h(x) = x \bmod 3$

Assignments:

$0 \to 0$.	$3 \to 0$	$6 \to 0$	.....
$1 \to 1$	$4 \to 1$	.....	
$2 \to 2$	$5 \to 2$.	.....	

Important sets:

Definition: Let f be a function from A to B.

- We say that A is the domain of f and B is codomain
- If $f(a) = b$, b is the image of a and a is a pre-image of b.
- The range of f is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

Assume f is defined as: $1 \to c$, $2 \to a$, $3 \to c$

What is the image of 1? Ans: c is the image of 1.

What is the pre-image of a? Ans: 2 is pre-image of a.

Domain of f? Ans: $\{1, 2, 3\}$.

Codomain of f? Ans: $\{a, b, c\}$.

Range of f? Ans: $\{a, c\}$.

Image of a subset

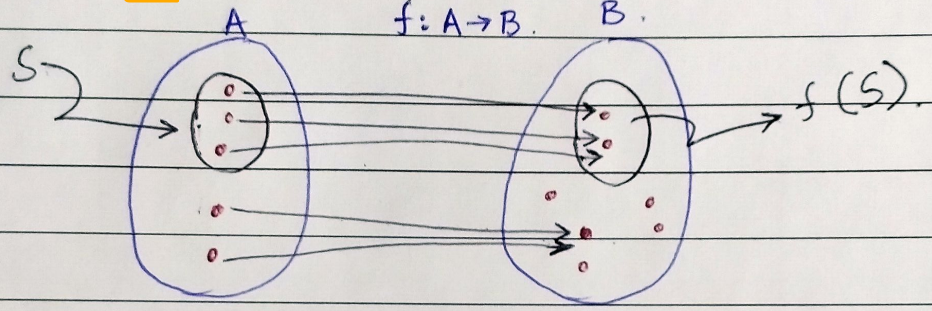
Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is the subset of B that ~~contains~~ consists of the images of the elements of S.

We denote the image of S by $f(S)$, so that

$$f(S) = \{ f(s) \mid s \in S \}.$$

or

$$f(S) = \{ t \mid \exists s \in S \ (t = f(s)) \}$$



Example:

Let $A = \{1, 2, 3\}$ $B = \{a, b, c\}$

f: $1 \to c$, $2 \to a$, $3 \to c$

Let $S = \{1, 3\}$ then image $f(S) = ?$

Ans: $f(S) = \{c\}$.

Addition & Multiplication of functions.

Two real-valued functions with same domains can be added & multiplied.

Definition:

Let f_1 and f_2 be two functions from A to $\mathbb{R}$.

Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x).$$

$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Example: $f_1(x) = x^2$ $\quad f_2(x) = x - x^2$.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x.$$

$$(f_1 f_2)(x) = f_1(x) \cdot f_2(x) = x^2(x - x^2) = x^3 - x^4.$$

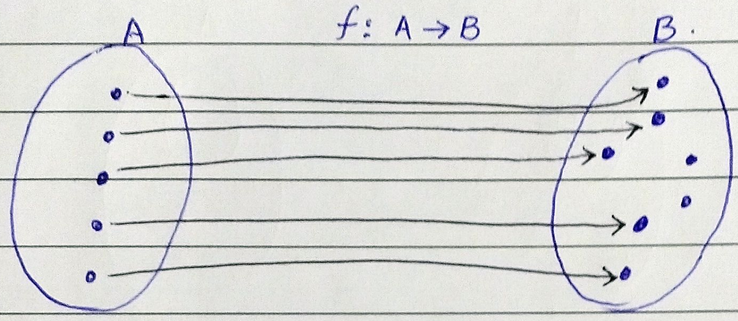
INJECTIVE FUNCTION

Definition: A function f is said to be one-to-one or injective, if and only if $f(x)=f(y)$ implies $x=y$ for all x, y in the domain of f.

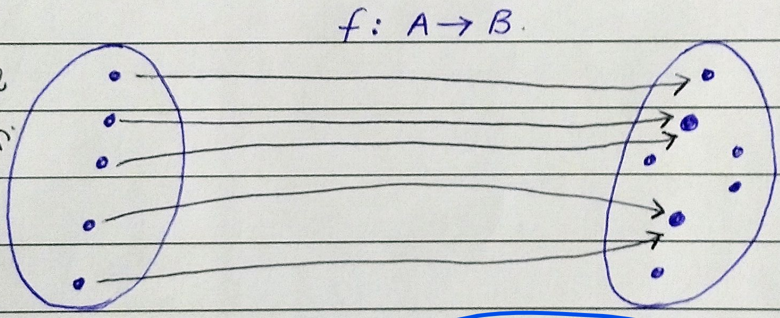
A function is said to be injective if it is one-to-one.

A function is one-to-one if and only if $f(x) \neq f(y)$ whenever $x \neq y$. This is the contrapositive of definition.

Injective Function.



Not an Injective Function.



Example#1

Let $A = \{1, 2, 3\}$ $\quad B = \{a, b, c\}$.

Function f is defined as; $1 \to c$, $2 \to a$, $3 \to c$

Is f one-to-one?

Ans: No. It is not one-to-one since $f(1) = f(3) = c$ and $1 \neq 3$.

Example#2. Let $g: Z \to Z$, where $g(x) = 2x-1$

Is g one-to-one?

Ans: Yes. Suppose $g(a) = g(b)$ i.e. $2a-1 = 2b-1$

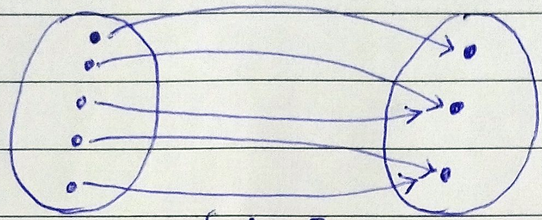
$\Rightarrow 2a = 2b$.

$\Rightarrow a = b$.

SURJECTIVE FUNCTIONS

Definition: A function f from A to B is called onto or surjective, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$

All codomain elements are covered.



$f: A \to B$.

Example#1 Let $A = \{1, 2, 3\}$ $B = \{a, b, c\}$
Function f is defined as; $1 \to c$, $2 \to a$, $3 \to c$
Is f an onto?

Ans : No. f is not onto, since $b \in B$ has no pre-image.

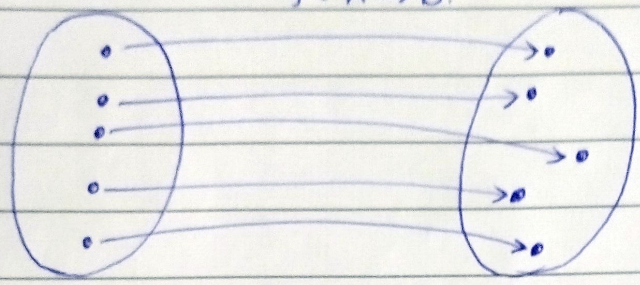
Example#2. $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $B = \{0, 1, 2\}$
Function h from $A \to B$ is defined as; $h(x) = x \bmod 3$
Is h an onto function?

Ans: Yes. h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

BIJECTIVE FUNCTIONS

Definition: A function f is called a bijection or it is both one-to-one and onto.

$f: A \to B$.



Example #1

Let $A = \{1, 2, 3\}$ $B = \{a, b, c\}$

Function f is defined as; $1 \to c$, $2 \to a$, $3 \to b$

Is f a bijection?

Ans: Yes. It is both one-to-one and onto.

Note: Let f be a function from a set A to itself, where A is finite.

f is one-to-one only iff it is onto.

* This is not true for A an infinite set.

Define $f: Z \to Z$, wher $f(z) = 2 * z$.

f is one-to-one but not onto (3 has no pre-image).

Example #2.

Define $g: \omega \to \omega$ (whole numbers), where $g(n) = [n/2]$ (floor function)

$0 \to [0/2] \Rightarrow [0]$

$1 \to [1/2] \Rightarrow [0]$

$2 \to [2/2] \Rightarrow [1]$.

$3 \to [3/2] \Rightarrow [1]$. Is g a bijection?

Ans: No. g is onto but not one-to-one.

$g(0) = g(1) = 0$ but $0 \neq 1$.

Theorem: Let f be a function $f: A \to A$, from a set A to itself, where A is finite.
Then f is one-to-one if and only if f is onto.

Proof: $\rightarrow$ A is finite and f is one-to-one (injective)
Is f an onto function (surjection)?

Yes. Every element points to exactly one element. Injection assures they are different. So we have $|A|$ different elements A points to. Since $f: A \to A$, the codomain is covered thus the function is also onto (surjection) .... and a bijection.

$\leftarrow$ A is finite and f is an onto function
Is the function one-to-one?

Yes. Every element maps to exactly one element and all elements in A are covered. Thus maping must be one-to-one.

Note: Above is not true when A is an infinite set