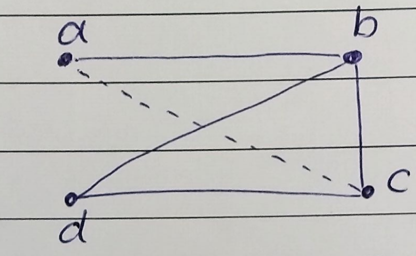
GRAPHS

Definition:

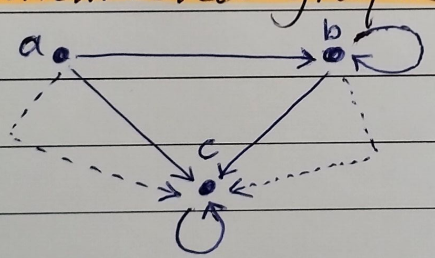
A graph $G = (V, E)$ consists of a nonempty set V of vertices (or nodes) and a set E of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

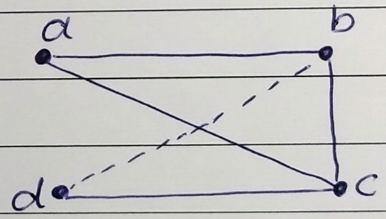
Example:



Basic types of graphs:

* Directed graphs.
* Undirected graphs.





Terminology:

* In a simple graph each edge connects two different vertices and no two edges connect same pair of vertices.
* Multigraphs may have multiple edges connecting the same two vertices. When m different edges connect the vertices u and v, we say that $\{u, v\}$ is an edge from multiplicity m.
* An edge that connects a vertex to itself is called a loop.
* A pseudograph may include loops, as well as multiple edges connecting the same pair of vertices.

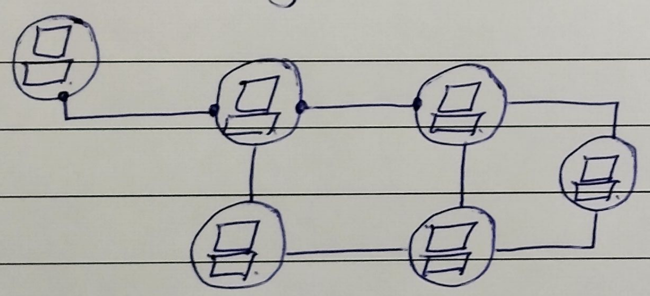
Graphs' applications.

Graphs and graph theory can be used to model:

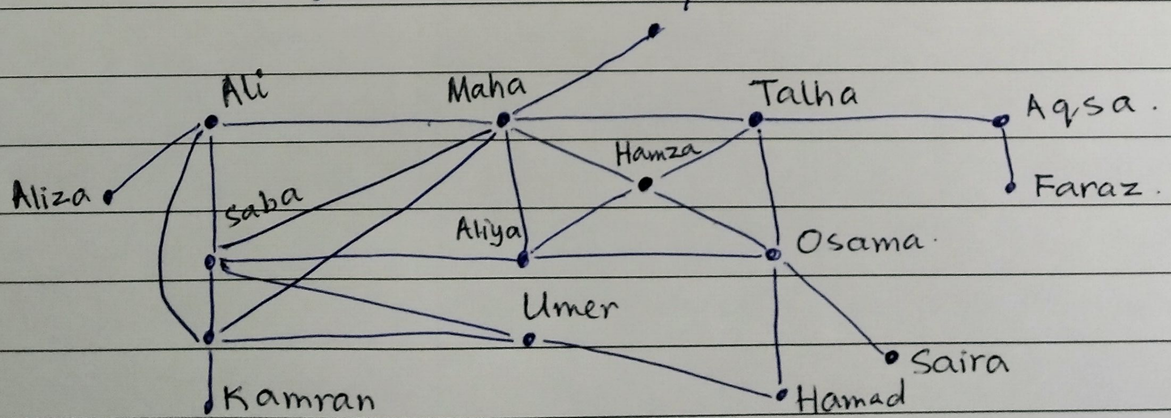
* Computer Networks.
* Social Networks
* Communications Networks.
* Information Networks.
* Software design.
* Transportation Networks.
* Biological Networks.

Computer Networks:.

- Nodes → Computers.
- Edges → Connections.

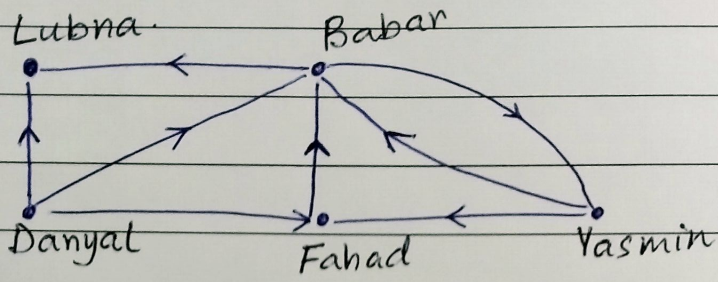


Social Networks: vertices represent individuals or organizations edges represent relationship b/w them.
e.g. – friendship graphs: undirected graphs where two people are connected if they are friends (in real world, on facebook, or in perticular virtual world, and so on).



Useful graph models of social networks include:

- Influence graphs: → directed graphs where there is an edge from one person to another person if the first person can influence the second person.



Collaboration graphs. undirected graphs where two people are connected if they collaborate in a specific way.

Examples:

The Hollywood graph: models the collaboration of actors in flims.

- We represent two actors by vertices and we connect two vertices if the actors they represent have appeared in the same movie.
- Kevin Bacon numbers (six degrees of Kevin Bacon).

An academic collaboration graph: models the collaboration of researchers who have jointly written a paper in a perticular subject.

- We represent researchers in a perticular academic discipline using vertices.
- We connect the vertices representing two researchers in this discipline if they are coauthors of a paper.

- Erdos number.

INFORMATION GRAPHS:

- Graphs can be used to model different types of network that link different types of information

- **Web graph:** In a web graph web-pages are represented by vertices and links are represented by directed edges.

- **Citation Network:** Research paper in perticular discipline are represented by vertices.
  - Wen a paper cites a second paper as reference, there is an edge from the vertex representing this paper to vertex representing second paper.

TRANSPORTATION GRAPHS:

Graphs models are extensively used in the study of transportation networks.

- Airline networks: are modeled using directed multigraphs
  - airports are represented using vertices.
  - each flight is represented by a directed edge from the vertex representing the departure airport to the vertex representing the destination airport.
- Road Networks: can be modeled using graphs where;
  - vertices represent intersections and edges represent roads.
  - undirected edge represent two-way roads and directed edges represent one-way roads.

BIOLOGICAL NETWORKS:

In molecular & population biology.

- determining a protein's or gene's function.
- protein-protein interaction networks.
- biochemical networks
- signal transduction networks.

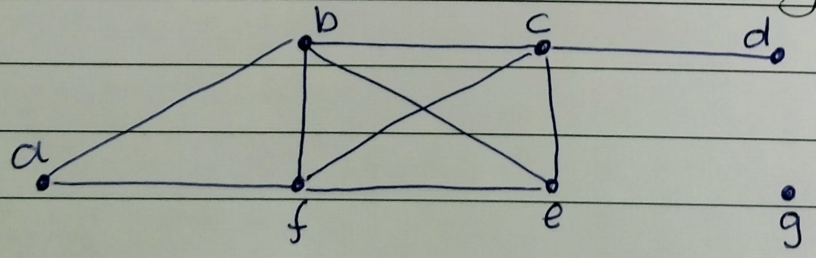
Graph Characteristics:
(Undirected Graphs).

Definition #1: Two vertices u and v in an undirected graph G are called adjacent (neighbors) in G if there is an edge e between u and v.
- Such an edge e is called incident with vertices u and v and e is said to connect u and v.

Definition #2: The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$ is called neighborhood of v.

Definition #3: The degree of a vertex in a undirected graph is the number of edges incident with it, except that a loop at vertex contributes two to the degree of that vertex. The of the vertex v is denoted by $\deg(v)$.

Example: What are the degrees and neighborhoods of the vertices in the graph G?



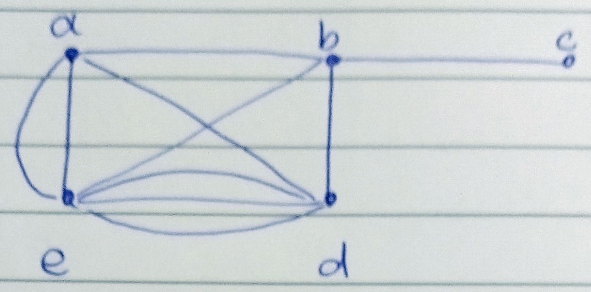
Graph – G.

Solution:

G: $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$
$\deg(e) = 3$, $\deg(g) = 0$.

$N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$,
$N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$
$N(g) = \emptyset$.

Example#2. What are the degrees and neighborhoods of the vertices in the graph H.



Graph- H.

Solution:

H: deg (a) = 4, deg (b) = deg (e) = 6, deg (c) = 1, deg (d) = 5

N(a) = {b, d, e}, N(b) = {a, b, c, d, e}
N(c) = {b}, N(d) = {a, b, e}, N(e) = {a, b, d}.

Graph Characterstics (Undirected Graphs).

Theorem #1 (Handshaking Theorem)

If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof: Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and righ-hand sides of this equation equal twice the number of edges.

Example: Think about the graph where vertices represent the people at a party and an edge connects two people who have shaken hands.

Theorem # 2.

An undirected graph has an even number of vertices of odd degree.

Proof:

Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in a undirected graph $G=(V,E)$ with m edges.

Then;

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

must be even since $\deg(v)$ is even for each $v \in V_1$.

This sum must be even because $2m$ is even and the sum of the degrees of the vertices of even degrees is also even. Because this is the sum of the degrees of all vertices of odd degree in the graph, there must be even number of such vertices.

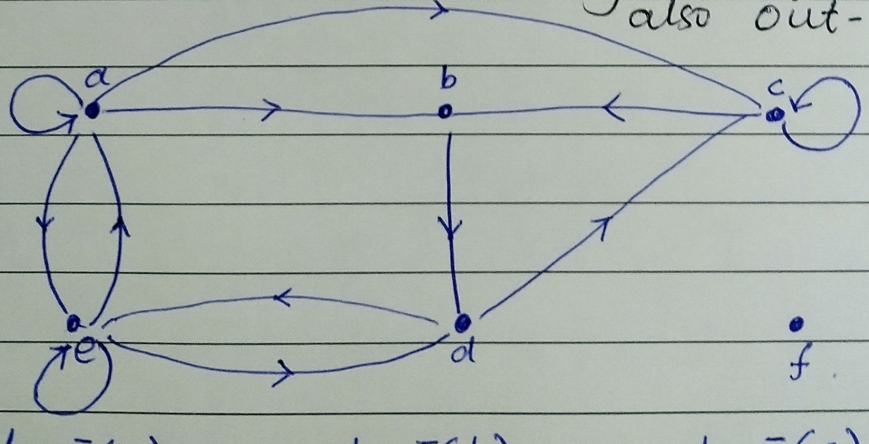
DIRECTED GRAPHS.

Definition: A directed graph $G = (V, E)$ consists of V, a non-empty set of vertices (or nodes) and E, a set of directed edges or arcs. Each edge (u, v) is an ordered pair of vertices. The directed edge (u, v) is said to start at u and end at v.

Definition: Let (u, v) be an edge in G. Then u is initial vertex of this edge and is adjacent to v and v is the terminal vertex of this edge and is adjacent from u. The initial and terminal vertices of a loop are same.

Definition: The in-degree of vertex v, denoted by $deg^-(v)$ is the number of edges which terminate at v. The out-degree of vertex v, denoted by $deg^+(v)$ is the number of edges with v as their initial vertex. Note that loop at a vertex contributes 1 to both in-degree and out-degree of the vertex.

Example: What are in-degree of vertices? also out-degrees?



Solution: $deg^-(a) = 2$, $deg^-(b) = 2$, $deg^-(c) = 3$

$deg^-(d) = 2$, $deg^-(e) = 3$, $deg^-(f) = 0$.

$deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$

$deg^+(d) = 2$, $deg^+(e) = 3$, $deg^+(f) = 0$.

THEOREM:

Let $G = (V, E)$ be a graph with directed edeges. Then:

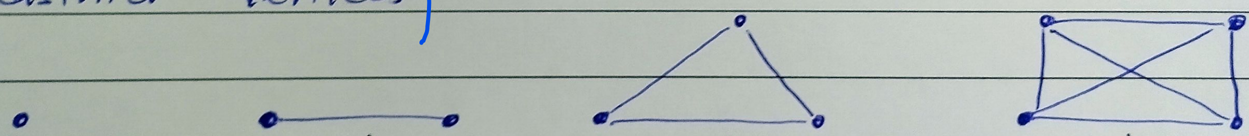
$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

Proof:

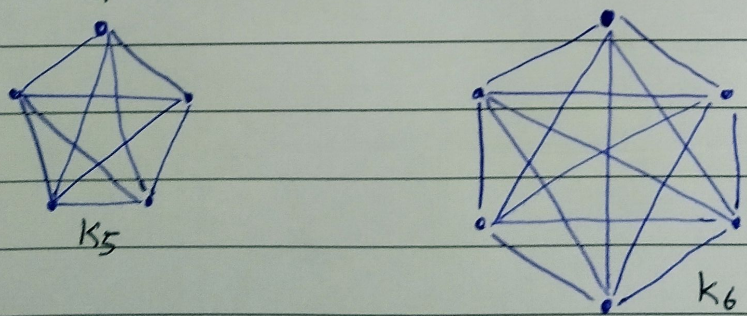
The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. It follows that both sums equal number of edges in the graphs.

COMPLETE GRAPHS:

A complete graph on n vertices, denoted by K_n is the simple graph that contains (exactly one edge between each pair of distinct vertices.)



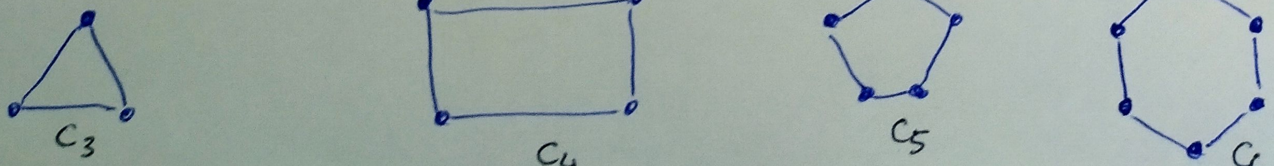
K_1 K_2 K_3 K_4



K_5 K_6.

A CYCLE:

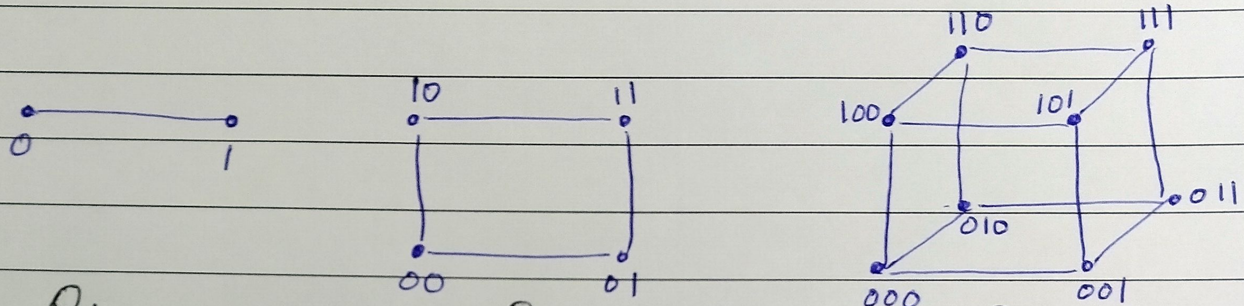
A cycle C_n for $n \geq 3$ consists of n vertices $v_1, v_2 \dots v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\} \dots \{v_{n-1}, v_n\}, \{v_n, v_1\}$



C_3 C_4 C_5 C_6

N-dimensional hypercube.

An n-dimensional hypercuber or n-cube, Q_n, is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position.



Q_1 Q_2 Q_3.

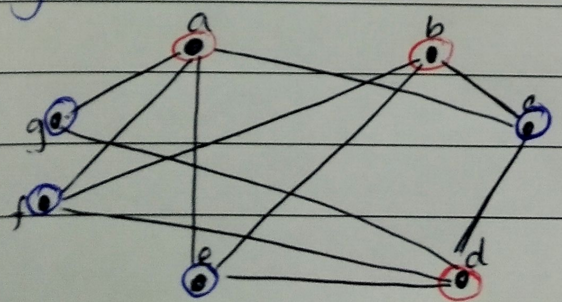
Bipartite Graphs.

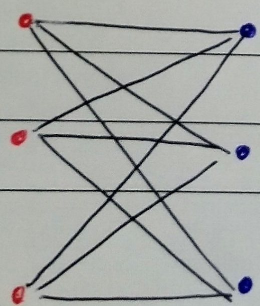
Definition:

A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, (there are no edges which connects two vertices in V_1 or V_2.)

Note:

An equivalent definition of bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color.



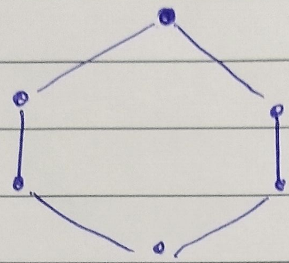


A Complete ← Bipartite Graph.

Coding Theory → Turbo codes.

Petrinet in computer science is a mathematical modeling tool used in analysis & simulation of concurrent syst

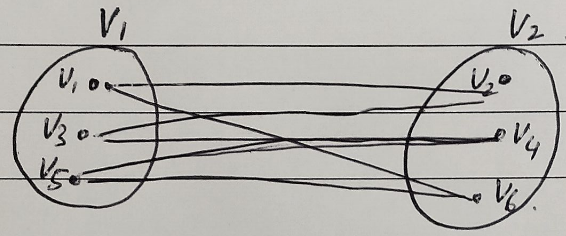
Example: Show that C_6 is bipartite.



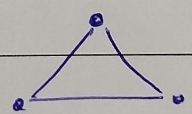
C_6.

Solution:

We can partition the vertex set into $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_2, v_4, v_6\}$. so that every edge of C_6 connects a vertex in V_1 and V_2.



Example: Show that C_3 is not bipartite.



C_3.

Solution:

If we devide the vertex set of C_3 into two nonempty sets, one of the two must contain two vertices. But in C_3 every vertex is connected to every other vertex. Therefore two vertices in the same partition are connected. Hence C_3 is not bipartite.