INTERPOLATION:

NEWTON DIVIDED DIFFERENCE (POLYNOMIAL METHOD)

Derivation:

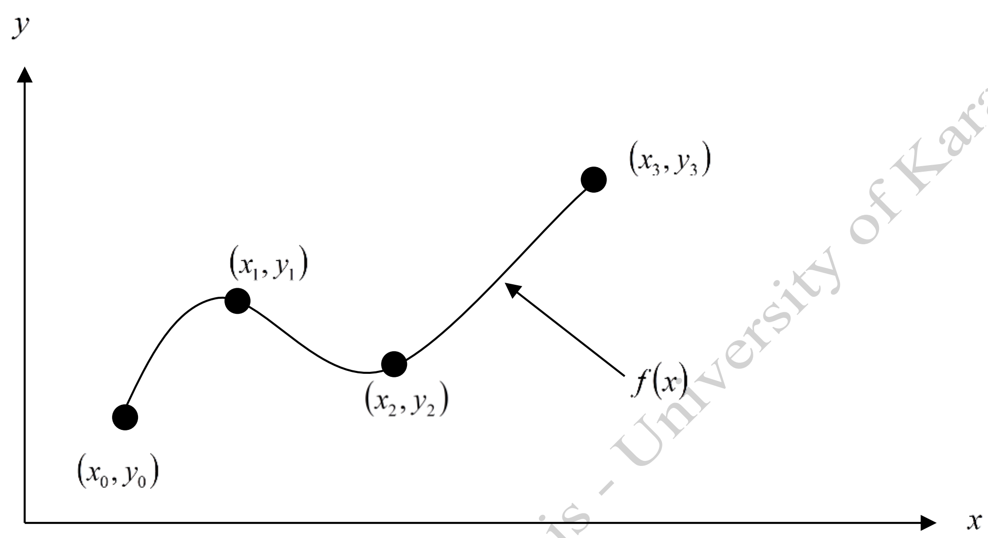

Figure 1 Interpolation of discrete data.

Linear Interpolation

Given (x_0, y_0) **and** (x_1, y_1), **fit a linear interpolant through the data. Noting** $y = f(x)$ **and** $y_1 = f(x_1)$, **assume the linear interpolant** $f_1(x)$ **is given by (Figure 2)**

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$
$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

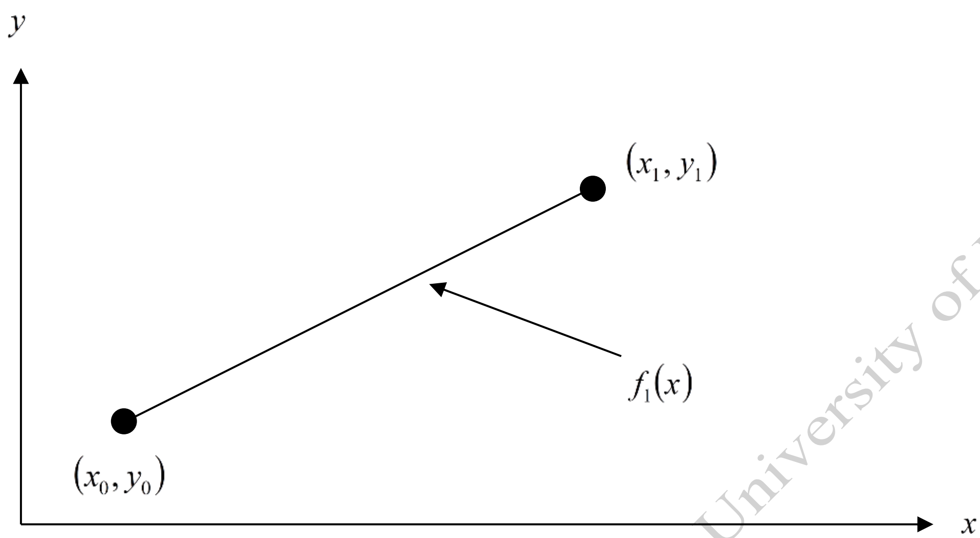

Figure 2 **Linear interpolation.**

Quadratic Interpolation

Given (x_0, y_0), (x_1, y_1), **and** (x_2, y_2), **fit a quadratic interpolant through the data. Noting** $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, **and** $y_2 = f(x_2)$, **assume the quadratic interpolant** $f_2(x)$ **is given by**

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$,

$$= b_0$$

$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$$

$$= f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) + \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}(x-x_0)(x-x_1)$$

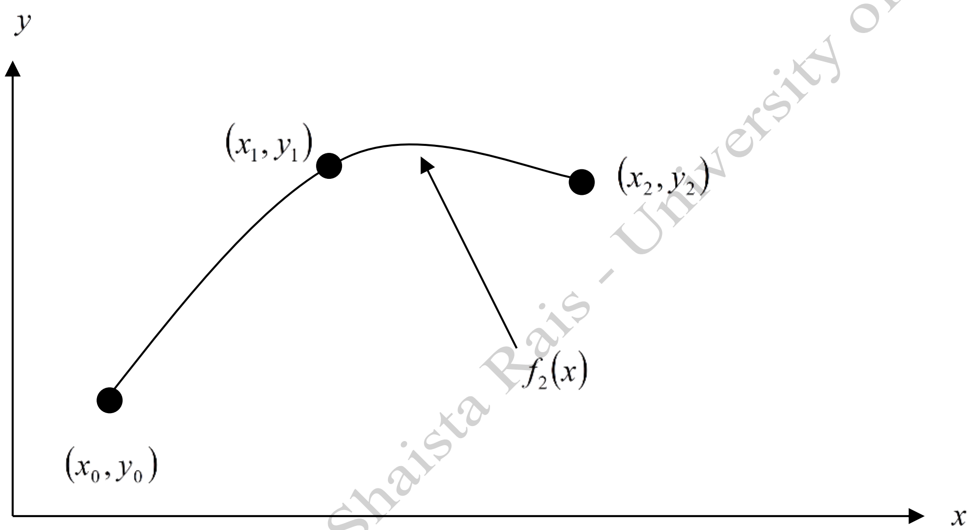

Figure 3: **Quadratic interpolation.**

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$b_2 = \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

Note that b_0, b_1, and b_2 are finite divided differences. b_0, b_1, and b_2 are the first, second, and third finite divided differences, respectively. We denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, **and** $f[x_2, x_1, x_0]$ **are called bracketed functions of their variables enclosed in square brackets.**

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ **data points,** $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, **as**

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$\begin{aligned} b_0 &= f[x_0] \\ b_1 &= f[x_1, x_0] \\ b_2 &= f[x_2, x_1, x_0] \\ &\vdots \\ b_{n-1} &= f[x_{n-1}, x_{n-2}, \ldots., x_0] \\ b_n &= f[x_n, x_{n-1}, \ldots., x_0] \end{aligned}$$

where the definition of the m^{th} **divided difference is**

$$\begin{aligned} b_m &= f[x_m, \ldots\ldots\ldots, x_0] \\ &= \frac{f[x_m, \ldots\ldots\ldots, x_1] - f[x_{m-1}, \ldots\ldots\ldots, x_0]}{x_m - x_0} \end{aligned}$$

From the above definition, it can be seen that the divided differences are calculated recursively.