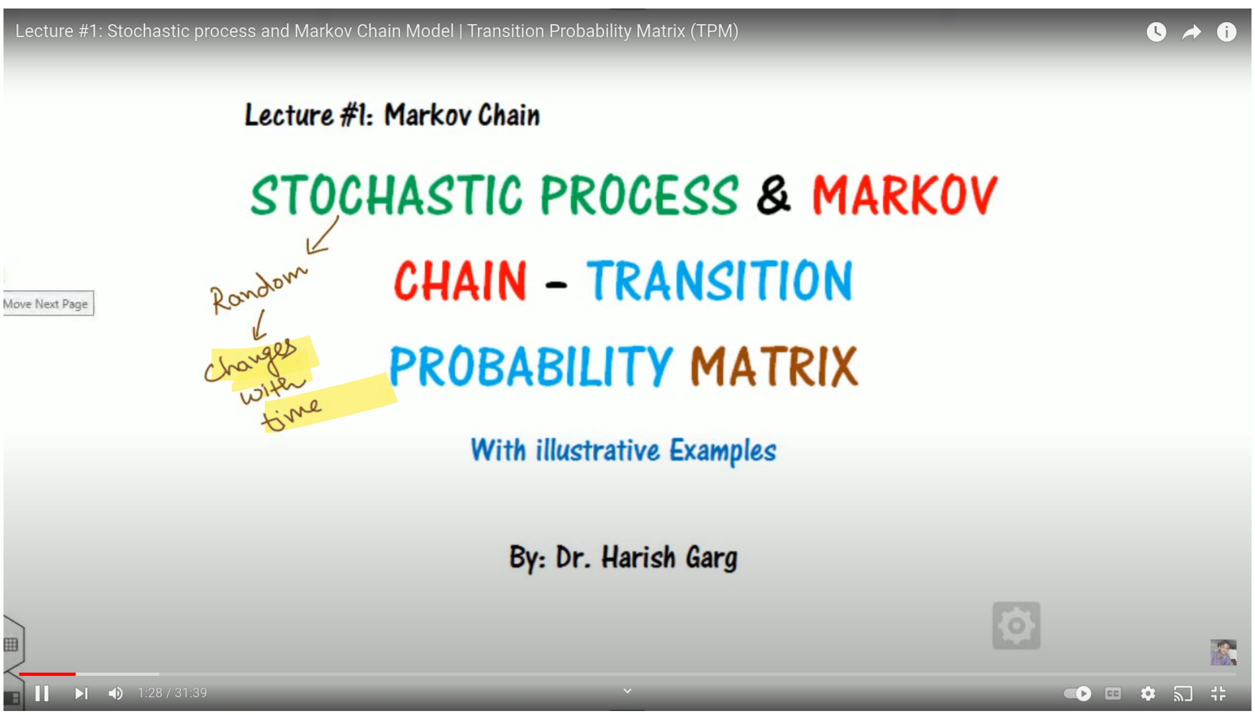
Defn: A stochastic process is any sequence of outcomes where the outcomes at any stage depend on chance.

A Markov process is a stochastic process where:

1) the set of possible outcomes is finite
2) the prob. of next outcome only depends on the outcome before
3) The probabilities are constant over time



Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)

Late on time

Late
On time

D

Case 2:

A professor tried not to be late for class too often. If he is late one day, he is 90% sure to be on time next time. If he is on time, then the next time there is a 30% chance of his being late.

In the long run, how often is he late for class?

4:35 / 31:39

Q

Case 3: Suppose that new razor blades were introduced in the market by three companies at the same time. When they were introduced each company has an equal share of the market, but during the year the following changes took place:

- Company A retained 90% of its customers and lost 3% to company B and 7% to company C.
- Company B retained 70% of its customers and lost 10% to Company A and 20% to company C.
- Company C retained 80% of its customers and lost 10% to Company A and 10% to company B.

Assuming that no change take place in the buying habits of the customers. What are the market shares of the three companies at the end of the first year and the second year?

Description of the STOCHASTIC PROCESS

In all these discussed cases (Case 1, Case 2, Case 3), the most important thing is the change the behaviour of the factor/states over the time.

Hence, we are interested in how a random variable changes over time.

The study which evolves over time includes Stochastic Processes.

Let

X_0 be number of students entered in a class at $\mathrm{t} = \mathrm{t}_0$

X_1 be number of students entered in a class at $\mathrm{t} = \mathrm{t}_0 + 1$

...

X_n be number of students entered in a class at $t = t_n$

Let $\{X_n, n = 0,1,2, \ldots\}$ be a stochastic process that takes on a finite or countable number of possible values.

In probability theory, a Markov model is a stochastic model used to model randomly changing systems over time.

For example,

- The amount of money the gambler possesses after toss "t"

- No of students enter in a class before time "t"

11:00 am 11:05 am → ??
11:07 am → ??

Notation:

$$\boxed{X_n = i}$$

It means process is said to be in state i at time n.

$X_0 = 3 \quad X_1 = 4$

Notation:

$$X_n = i$$

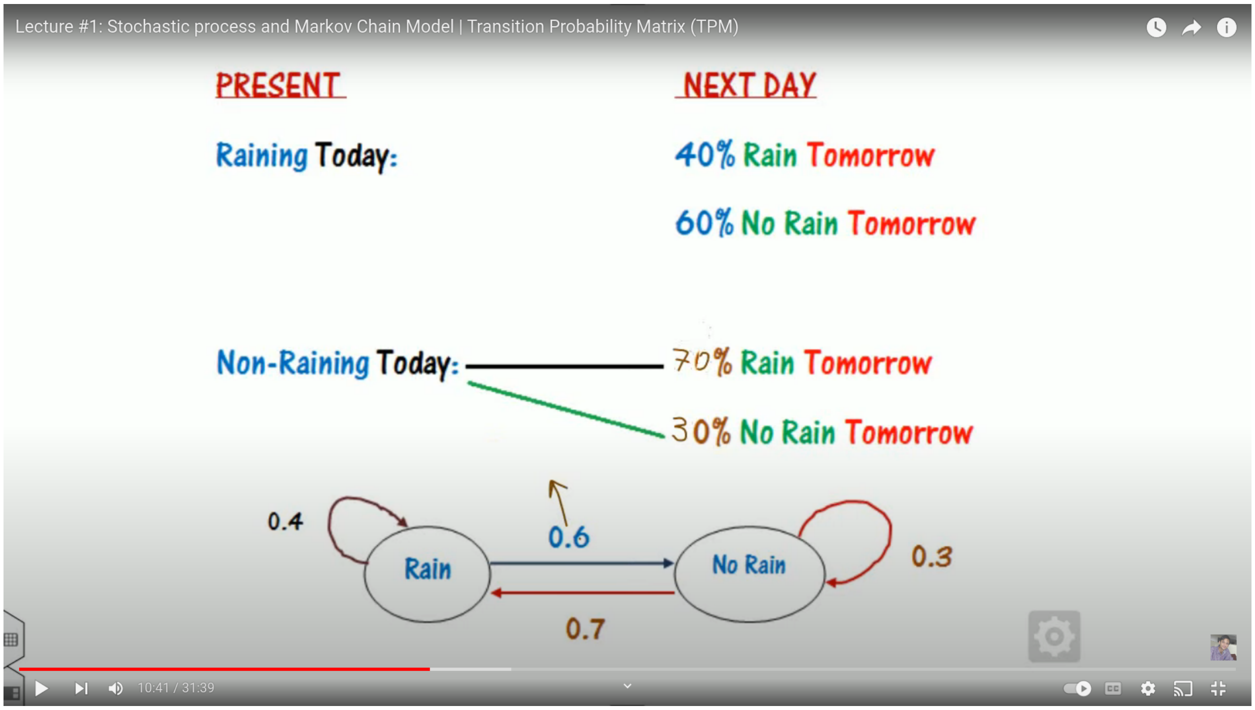
It means process is said to be in state i at time n.

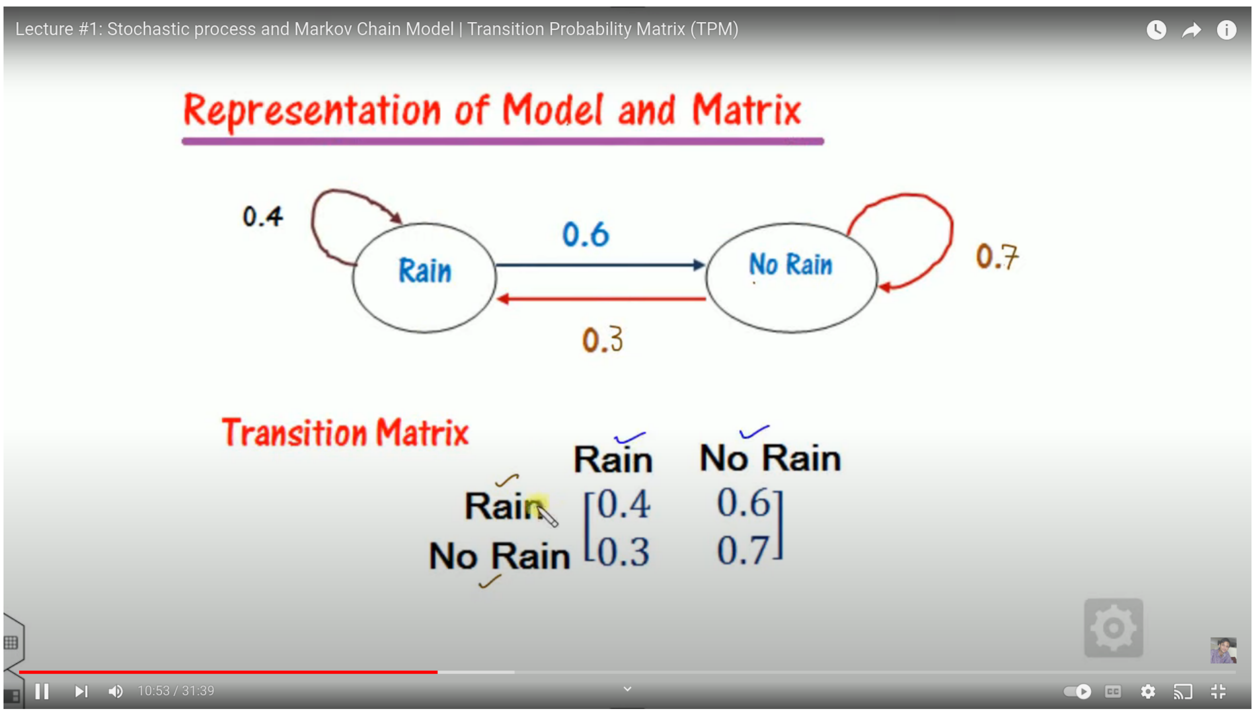
Here, we will see the new term STATE Mentioned. WHAT is STATE ???

In Markov Model, two terms play an important role:

- States
- Transition Probability

To understand the MARKOV MODEL, we consider a very SIMPLE EXAMPLE.


Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)
PRESENT
NEXT DAY
Raining Today:
40% Rain Tomorrow
60% No Rain Tomorrow
Non-Raining Today:
70% Rain Tomorrow
30% No Rain Tomorrow
0.4
Rain
0.6
No Rain
0.3
0.7
10:41 / 31:39


Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)
Representation of Model and Matrix
0.4
Rain
0.6
No Rain
0.7
0.3
Transition Matrix
Rain No Rain
Rain [0.4 0.6]
No Rain [0.3 0.7]
10:53 / 31:39

Markov Chain

$X_n = i$ → State

time period

It is the process $X_0, X_1, X_2, \ldots$

- Here system states are observable and fully autonomous
- Simplest of all Markov models.

The state of the Markov chain at time t is the value of X_t

For example, if

$$X_t = 5$$

we say the the process is in state 5 at time t.

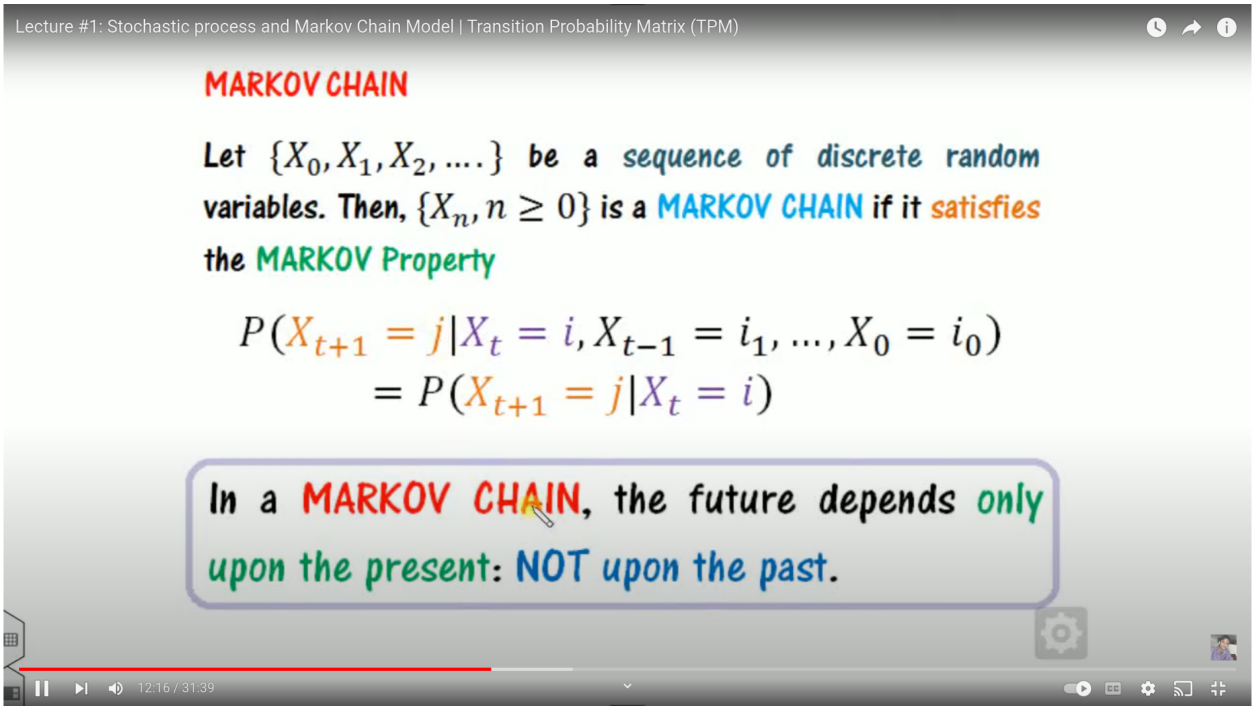
Markov Property or memoryless

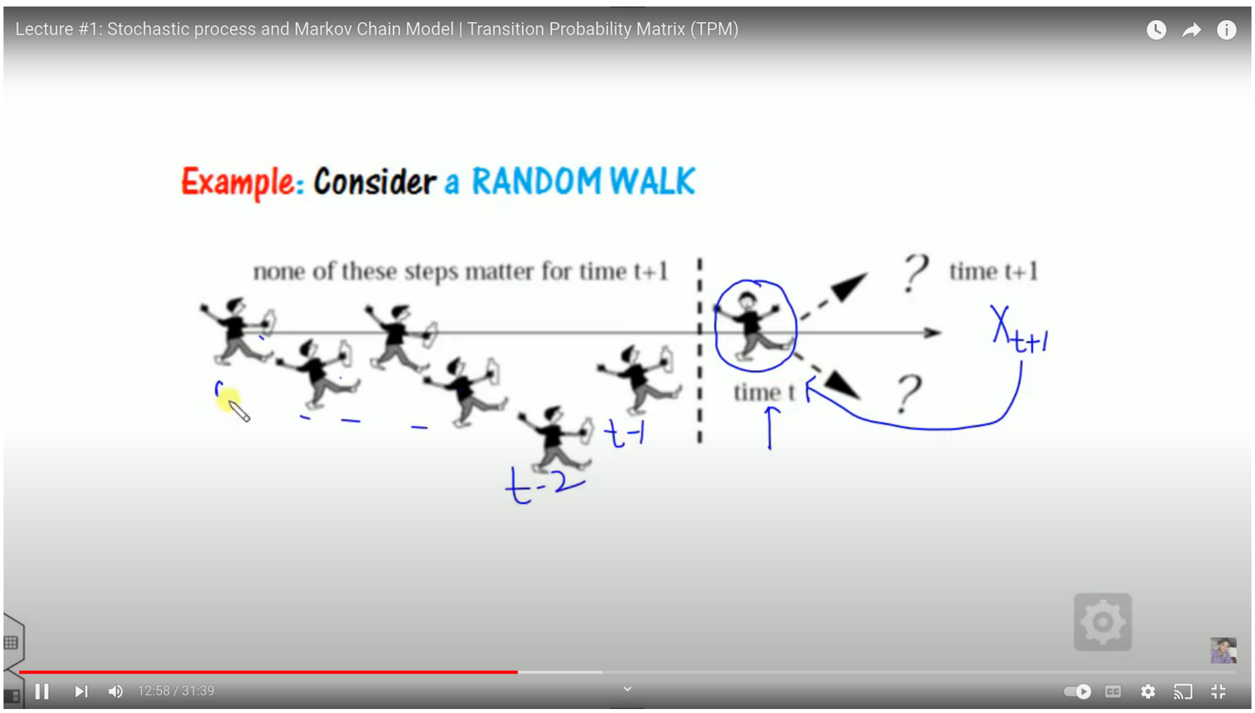
The basic property of the Markov chain is that X_{t+1} depends upon X_t, but it does not depend upon $X_{t-1}, X_{t-2}, \ldots, X_1, X_0$.

Mathematically, the Markov property is stated as

$$P(X_{t+1} = j | X_t = i, \cancel{X_{t-1} = i_1, \ldots,} X_0 = i_0) = P(X_{t+1} = j | X_t = i)$$

for all $t = 1,2,3, \ldots$ and for all states.


Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)
MARKOV CHAIN
Let $\{X_0, X_1, X_2, \dots\}$ be a sequence of discrete random variables. Then, $\{X_n, n \geq 0\}$ is a MARKOV CHAIN if it satisfies the MARKOV Property
$P(X_{t+1} = j | X_t = i, X_{t-1} = i_1, \dots, X_0 = i_0) = P(X_{t+1} = j | X_t = i)$
In a MARKOV CHAIN, the future depends only upon the present: NOT upon the past.
12:16 / 31:39


Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)
Example: Consider a RANDOM WALK
none of these steps matter for time t+1
time t+1
?
?
time t
X_{t+1}
t-1
t-2
12:58 / 31:39

Example:

States

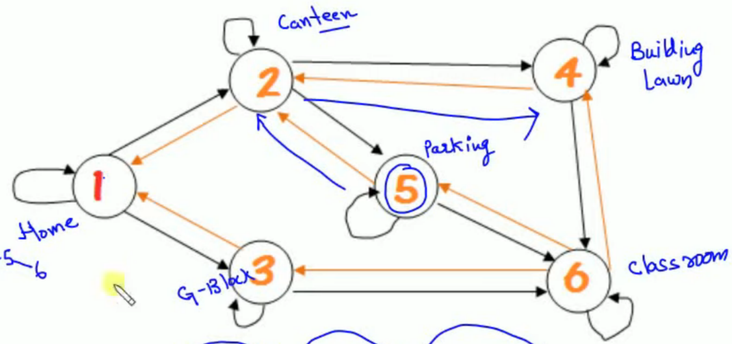
5 – 2 – 1 – 3 – 6

5 – 2 – 4 – 6

6

5 – 2 – 2 – 1 – 3 – 1 – 2 – 5 – 6

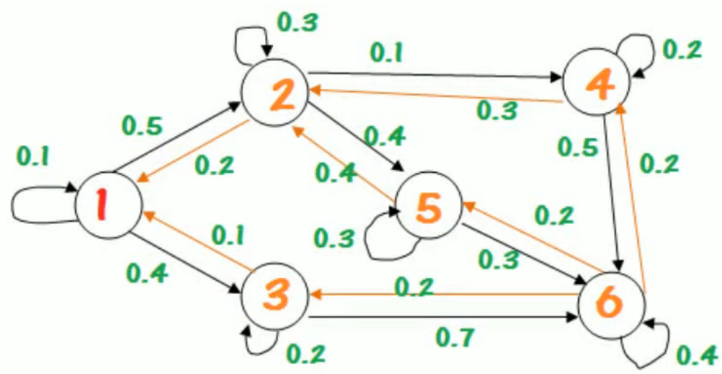
5 – 6



$$P(X_3 = 6 | X_2 = 4, X_1 = 2, X_0 = 5)$$

??

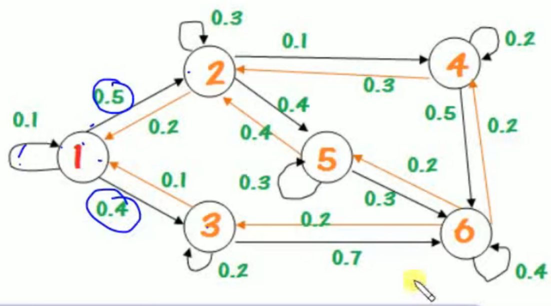
Example:



$$P(X_3 = 6 | X_2 = 4, X_1 = 2, X_0 = 5)$$
$$= P(X_3 = 6 | X_2 = 4)$$

Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)

Example:



Probability Matrix

state	1	2	3	4	5	6
1	0.1	0.5	0.4	0	0	0
2	0.2	0.3	0.0	0.1	0.4	0
3	0.1	0	0.2	0	0	0.7
4	0	0.3	0	0.2	0	$\boxed{0.5}$
5	0	0.4	0	0	0.3	0.3
6	0	0	0.2	0.2	0.2	0.4

$$P(X_3 = 6 | X_2 = 4, X_1 = 2, X_0 = 5)$$
$$= P(X_3 = 6 | X_2 = 4)$$
$$= p_{46}^{(1)}$$
$$= 0.5$$

17:41 / 31:39

Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)

Imp

Transition probabilities

Probabilities **FROM** state i **TO** state j after 1 step time period, denoted by p_{ij} is defined as,

$$\boxed{p_{ij} = P\{X_{n+1} = j | X_n = i\}}$$

Examples:

$$P(X_2 = 3 | X_1 = 2) = p_{23}^{(1)}$$
$$P(X_2 = 2 | X_1 = 0) = p_{02}$$

...

1-day

n-step probabilities

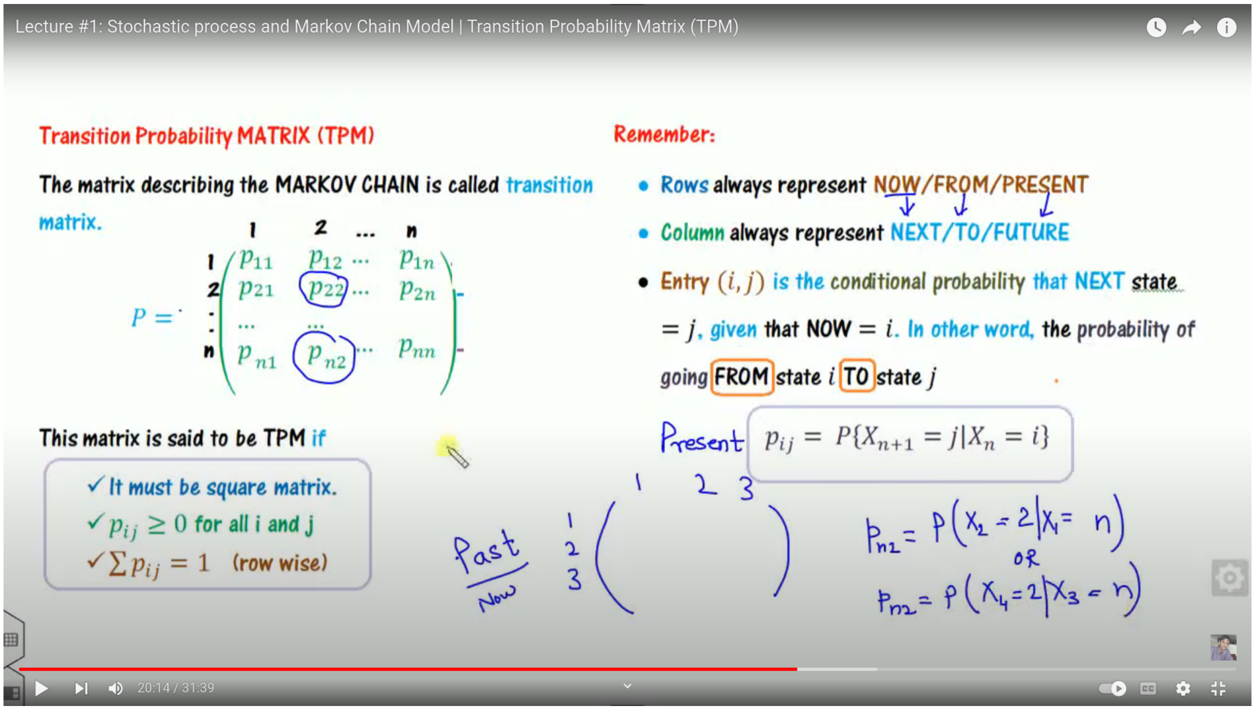
Probabilities **FROM** state i **TO** state j after n- step time period, denoted by $p_{ij}(n)$ or $p_{ij}^{(n)}$ is defined as,

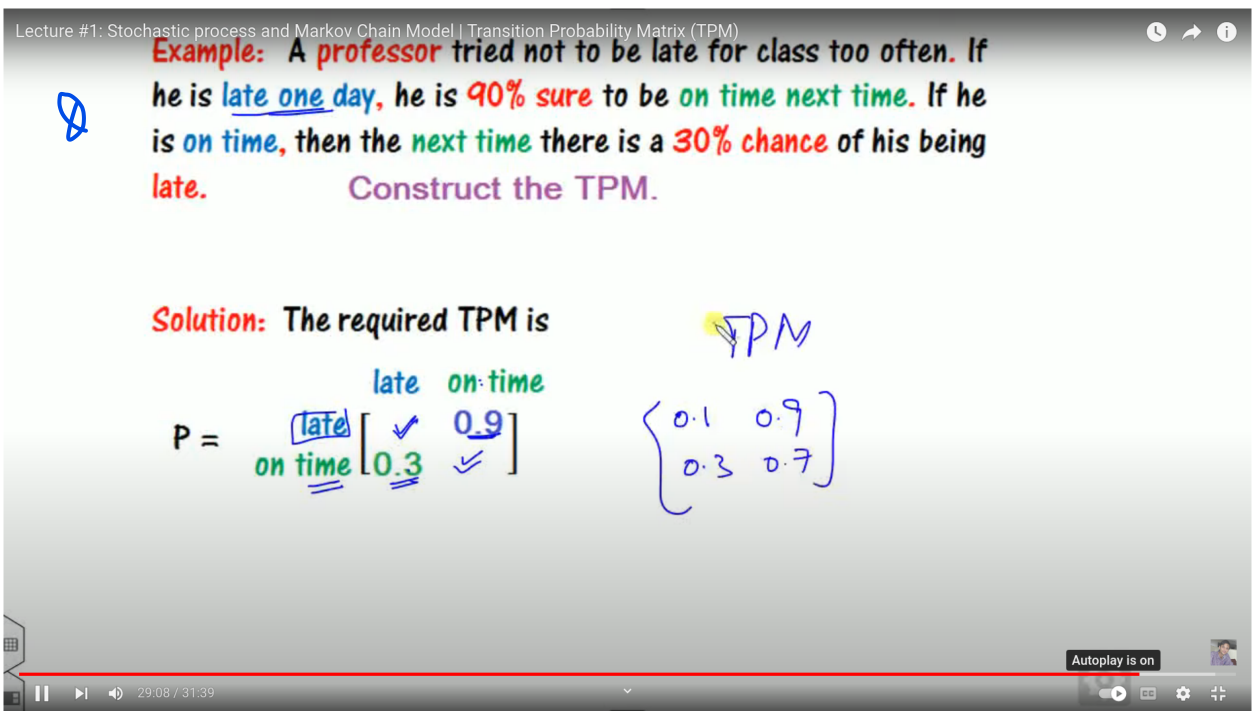
$$\boxed{p_{ij}^{(n)} = P\{X_{n+1} = j | X_1 = i\}}$$

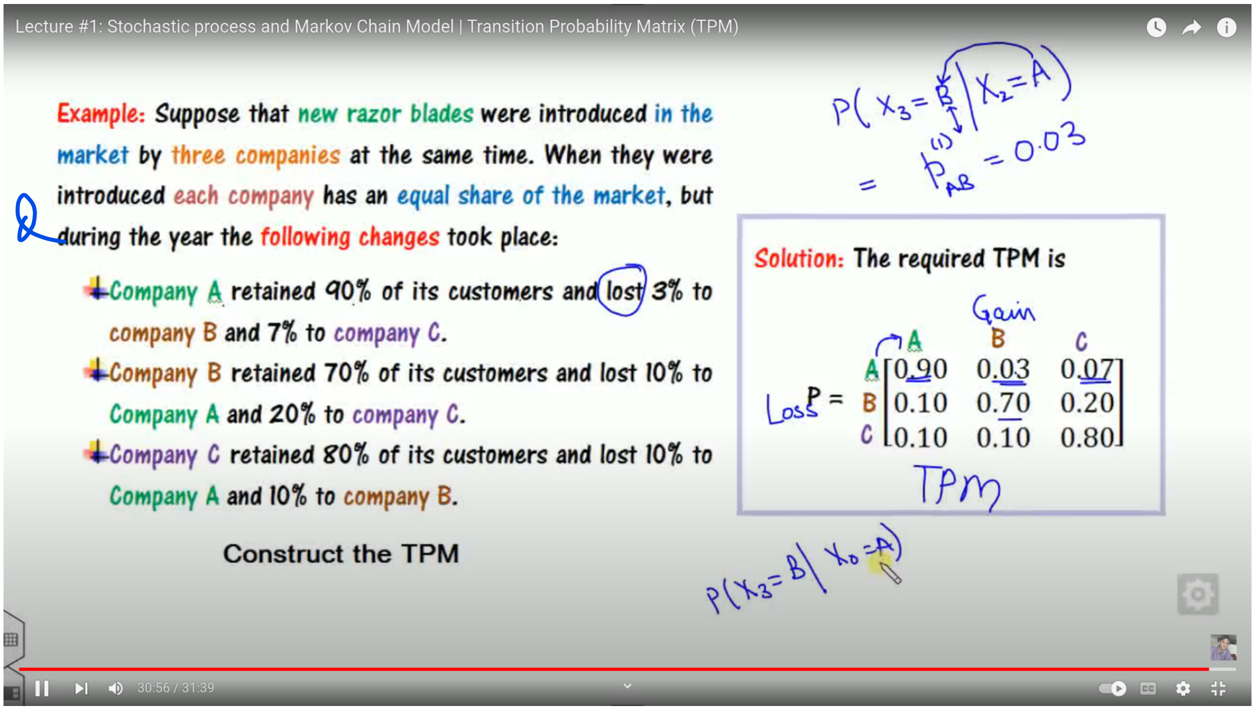
Example:

$$P(X_2 = 3 | X_0 = 2) = p_{23}^{(2)}$$
$$P(X_3 = 2 | X_1 = 1) = p_{12}^{(2)}$$

17:45 / 31:39


Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)
Transition Probability MATRIX (TPM)
The matrix describing the MARKOV CHAIN is called transition matrix.
$P = \begin{array}{c|cccc} & 1 & 2 & \dots & n \\ \hline 1 & p_{11} & p_{12} & \cdots & p_{1n} \\ 2 & p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \dots & \dots & & \\ n & p_{n1} & p_{n2} & \cdots & p_{nn} \end{array}$
This matrix is said to be TPM if
✓ It must be square matrix.
✓ $p_{ij} \geq 0$ for all i and j
✓ $\sum p_{ij} = 1$ (row wise)
Remember:
Rows always represent NOW/FROM/PRESENT
Column always represent NEXT/TO/FUTURE
Entry (i, j) is the conditional probability that NEXT state $= j$, given that NOW $= i$. In other word, the probability of going FROM state i TO state j
Present $p_{ij} = P\{X_{n+1} = j \mid X_n = i\}$
Past
Now
$P_{n2} = P(X_2 = 2 \mid X_1 = n)$
OR
$P_{n2} = P(X_4 = 2 \mid X_3 = n)$
20:14 / 31:39


Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)
Example: A professor tried not to be late for class too often. If he is late one day, he is 90% sure to be on time next time. If he is on time, then the next time there is a 30% chance of his being late.
Construct the TPM.
Solution: The required TPM is
TPM
$P = \begin{array}{c|cc} & \text{late} & \text{on time} \\ \hline \text{late} & ✓ & 0.9 \\ \text{on time} & 0.3 & ✓ \end{array}$
$\begin{pmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{pmatrix}$
Autoplay is on
29:08 / 31:39



Lecture #1: Stochastic process and Markov Chain Model | Transition Probability Matrix (TPM)
Example: Suppose that new razor blades were introduced in the market by three companies at the same time. When they were introduced each company has an equal share of the market, but during the year the following changes took place:
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Company C retained 80% of its customers and lost 10% to Company A and 10% to company B.
Construct the TPM
Solution: The required TPM is
$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.90 & 0.03 & 0.07 \\ B & 0.10 & 0.70 & 0.20 \\ C & 0.10 & 0.10 & 0.80 \end{array}$$
Gain
Loss
TPM
$P(X_3 = B \mid X_2 = A) = p_{AB}^{(1)} = 0.03$
$P(X_3 = B \mid X_0 = A)$
30:56 / 31:39