**Definitions:**

2D Definitions:

3D Definitions:

**Trig Facts used in the proofs:**

We will use the following facts below:

**Convex Hull Facts For Finite Point Sets**

Define the vector be a unit vector that parameterizes the unit sphere in d dimensions.

Examples: In and in

Point in the interior of a convex hull satisfies the following equation:

Or equivalently

Point on the edge of the convex hull, but not a vertex of the convex hull satisfies the following equation:

Or

**NOTE:** The four lines read as follows, for each orientation of the point set, the point is either on the interior or on the edges of the convex hull. There is an orientation where it is on the edge. For orientations where it is on the edge, then there is another point also on the edge as well.

Point that are vertices of the convex hull satisfies the following equation:

Pairs of Points and that are non-adjacent vertices of the convex hull satisfies the following equation: (This should have significance in the extent formulas (at least for 2D))

Pairs of Points and that are adjacent vertices of the convex hull satisfies the following equation:

**Minimum Oriented Bounding Box**

The minimum oriented bounding box (MOBB) is the smallest area (volume) box around a set of points. We name the lengths of the sides of the MOBB Extent 1 and Extent 2 (and Extent 3 in 3D) which are denoted by and (and ) respectively

**2D Minimum Oriented Bounding Box**

**Critical Angles:**

Critical Angles for :

The critical angles for are:

The above is where is not differentiable.

NOTE: For pairs where and , then the is no longer well-defined and no longer a critical point (this can be seen by setting the derivatives equal at , which would make it no longer a non-differentiable point).

NOTE: For pairs where one or both points lies within the convex hull (not the boundary), then the above critical point is no longer a critical point as the max and min functions completely swallow the internal point(s).

Proof: A point on the interior of the convex satisfies: . Then it follows that and . This implies that Therefore, it follows that .

NOTE: For pairs where one or both points lie on the boundary of the convex hull (but are not the convex hull vertices), then the above critical point is no longer a critical point as these points become redundant for orientations when they are on the edge, or are completely swallowed by the max and min functions for other orientations.

Proof: A non-vertex boundary point on the convex hull satisfies:  Then it follows that and . This implies that Therefore, it follows that .

NOTE: For pairs where both points are convex hull vertices, but the line between the points passes through the interior of the convex hull (i.e. non-adjacent convex hull points), then the above critical point is no longer a critical point as the max and min functions make these pairing impossible. (TODO: prove this.)

The points where are:

NOTE: The above is only a critical point if there exists a such that or

TODO: Show that there is always if and only if and are both on the convex hull.

NOTE: For pairs where one or both points lie in the convex hull or on the boundary of the convex hull but are not vertices of the convex hull, then the above critical point is no longer a critical point. (TODO: prove this.)

NOTE: It is possible for both adjacent convex hull vertices and non-adjacent convex hull vertices to emit the above critical point. All that matters is that for the given orientation, both are on the ends of the range at the same time. (TODO: Prove this. Triangle inequality?)

Evaluating at the critical points (without removing the non-convex hull vertex points) gives the following two formulas:

NOTE: Here is a set of 5 points in 2D such that the minimum of and occurs at but is not critical for either at all: . This implies that the condition does not seem to play a role in the MOBB calculation.

Critical Angles for :

The critical angles for are:

The above is where is not differentiable.

NOTE: For pairs where and , then the is no longer well-defined and no longer a critical point (this can be seen by setting the derivatives equal at , which would make it no longer a non-differentiable point).

NOTE: For pairs where one or both points lies within the convex hull (not the boundary), then the above critical point is no longer a critical point as the max and min functions completely swallow the internal point(s). (TODO: prove this)

NOTE: For pairs where one or both points lie on the boundary of the convex hull (but are not the convex hull vertices), then the above critical point is no longer a critical point as the max and min functions completely swallow the internal point(s). (TODO: prove this)

NOTE: For pairs where both points are convex hull vertices, but the line between the points passes through the interior of the convex hull, then the above critical point is no longer a critical point as the max and min functions completely swallow the internal point(s). (TODO: prove this)

The points where are:

NOTE: The above is only a critical point if there exists a such that or

TODO: Show that there is always if and only if and are on the convex hull.

Evaluating at the critical points gives the following two formulas:

Critical Angles for :

The critical differentiable angles (where the maximum can be attained) for are of the form:

Where i,j,k, and l refer to the points , , , and and . This can be shown by selecting points for the max and min functions and differentiating and setting the derivative equal to 0 (i is the max point for x, j is the min point for x, k is the max point for y, and l is the min point for y).

NOTE:

From the above, it is easy to prove that:

And

And thus:

The non-differentiable critical angles (where a minimum can be attained for the area) for are of the form:

Where i and j refer to the points and and . This can be shown by looking at the non-differentiable points of the max and min functions from the function definition (Either combining and into one formula or directly computing it)

NOTE: For pairs of points with or (but not both), then the critical angle degenerates to .

NOTE: For pairs where and , then the is no longer well-defined and no longer a critical point (this can be seen by setting the derivatives equal at , which would make it no longer a non-differentiable point).

NOTE: For pairs where one or both points lies within the convex hull (not the boundary), then the above critical point is no longer a critical point as the max and min functions completely swallow the internal point(s). (TODO: prove this)

NOTE: For pairs where one or both points lie on the boundary of the convex hull (but are not the convex hull vertices), then the above critical point is no longer a critical point as the max and min functions completely swallow the internal point(s). (TODO: prove this)

NOTE: For pairs where both points are convex hull vertices, but the line between the points passes through the interior of the convex hull, then the above critical point is no longer a critical point as the max and min functions completely swallow the internal point(s). (TODO: prove this)

From the above, it is easy to prove that:

This means that instead of there being unique angles (formed from all pairs of points), there are pairs of unique angles to consider (getting rid of pairs with points with itself and pairs that are reversed).

NOTE: A geometric interpretation of the angle is the angle such that an edge of the MOBB is parallel to the line between the pair of points defining the critical angle (TODO: Need proof)

Min-Max Representation of Area at minimum points:

Plugging a critical differentiable point into equation (1) gives the following formula for the Area at that critical point:

Define:

Then:

This means that the maximum angle (from the two indices) can be found with the following formula:

This leads to a simple brute force algorithm:

|  |
| --- |
| **Algorithm 1** Maximum Oriented Bounding Box Brute Force (Ignore convex hull facts) |
| **Input:** Set S of n points  **Output:** Pair of points that corresponds to the maximum angle |
| 1:  2:  3: **For** **do**  4: **For** j **do**  **5:**  =  **6: if** :  **7: For**  **do**  **8: For**  **do**  **9:**  =  10**: if** :  11**:**  12**:**  13**:**  14**:**  15**: For**  **do**  16**: if**  17**:**  18**: if**  19**:**  20**: if**  21**:**  22**: if**  23**:**  24**:**  25**: if**  >  26**:**  27**:**  28**:**  29**:**  30**:**  31: **if**  32: **return** (,,)  33: **else**  34: **return** undefined |

Analyzing the complexity, the first two loops take iterations, the next two loops take iterations, and the inner most loop takes n iterations. This gives a total of .

NOTE: The above formula and algorithm does not make use of the fact that i is the max point for x, j is the min point for x, k is the max point for y, and l is the min point for y and essentially recalculates them using m. There might be ways to decrease the complexity by using this fact. The loop bounds come from the symmetry of the angles.

NOTE: The above does not consider that points on the interior of the convex hull do not need to be considered.

If one wishes to calculate the corners of the MOBB instead, then the following calculation can be done instead of steps 26-30:

Plugging a non-differentiable critical point into equation (1) gives the following formula for the Area at that critical point:

This means that the minimum angle (from the two indices) can be found with the following formula:

This leads to a simple brute force algorithm:

|  |
| --- |
| **Algorithm 1** Minimum Oriented Bounding Box Brute Force (Ignore convex hull facts) |
| **Input:** Set S of n points  **Output:** Pair of points that corresponds to the minimum angle |
| 1:  2:  3: **For** **do**  4: **For** j **do**  5: =  6: **if** :  7:  8:  9:  10: **For**  **do**  11: **if**  12:  13: **if**  14:  11: **if**  12:  13: **if**  14:  15:  16: **if**  17:  18:  19:  20: **if**  21: **return** (,)  22: **else**  23: **return** undefined |

The outer loop and the second loop together take iterations and the inner loop (plus the initialization) takes n iterations. In the worst case, this gives a runtime of .

If one wishes to calculate the corners of the MOBB instead, then the following calculation can be done instead of steps 18-19:

Next, if we consider that the non-differentiable points only occur at angles that are in line with edges of the convex hull instead of any arbitrary pair of points, this leads to the following algorithm:

NOTE: This algorithm makes use of the following fact:

|  |
| --- |
| **Subroutine 1** Convex Hull 2D |
| **Input:** Set S of n points  **Output:** Sequence of Points such that each adjacent pair of points (and end points) form an edge of the convex hull (no duplicates) |

|  |
| --- |
| **Algorithm 2** Brute Force on the Convex Hull |
| **Input:** Set S of n points  **Output:** Pair of points that corresponds to the minimum angle |
| 1:  2:  3.  4: **For** **do**  4: j = i + 1 (mod )  5: =  7:  8:  9:  10: **For**  **do**  11: **if**  12:  13: **if**  14:  11: **if**  12:  13: **if**  14:  15:  16: **if**  17:  18:  19:  20: **if**  21: **return** (,)  22: **else**  23: **return** undefined |

The Convex Hull 2D is known to take at most . The outer loop takes iterations and in the worst case, the inner loop (plus the initialization) takes n iterations. In the worst case, this gives a runtime of .

Geometric/Combinatorial Representation of Extent 1 at minimum points:

NOTE: For , all angles are minimum and . This case is not included in the table below

Table 1:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | | | |
|  |  |  |  |  |
| 2 |  | --- | --- | --- |
| 3 |  |  |  | --- |
| 4 |  |  |  |  |
| 5 |  |  |  |  |
| 6 |  |  |  |  |
| 7 |  |  |  |  |
| 8 |  |  |  |  |

Geometric/Combinatorial Representation of Extent 2 at minimum points:

NOTE: For , all angles are minimum and . This case is not included in the table below

Table 2:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | | | |
|  |  |  |  |  |
| 2 |  | --- | --- | --- |
| 3 |  |  |  | --- |
| 4 |  |  |  |  |
| 5 |  |  |  |  |
| 6 |  |  |  |  |
| 7 |  |  |  |  |
| 8 |  |  |  |  |

Geometric/Combinatorial Representation of Area at minimum points:

NOTE: For , all angles are minimum and . This case is not included in the table below.

FACT: The minimum area corresponds to an angle that makes one of the MOBB’s edges in line with a pair of points from the set (need tighten it to a pair of points from the convex hull as shown by Toussaint)

Table 3:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | | | |
|  |  |  |  |  |
| 2 |  | --- | --- | --- |
| 3 |  |  |  | --- |
| 4 |  |  |  |  |
| 5 |  |  |  |  |
| 6 |  |  |  |  |
| 7 |  |  |  |  |
| 8 |  |  |  |  |

Geometric/Combinatorial Representation of Extent 1 at minimum points (Alternate formulas):

Table 1:

|  |  |
| --- | --- |
|  |  |
| 4 |  |
| 4 |  |
| 5 |  |
| 5 |  |

**Conjecture:** The angle that minimizes or and maximizes the other extent also minimizes A

**3D Minimum Oriented Bounding Box**

**Critical Angles:**

Critical Angles for :

The critical angles for are:

Euler Angles:

Vector-Angle:

?

The above is where is not differentiable.

NOTE: If , then the above degenerates to

NOTE: If (but ), then the above degenerates to

NOTE: If the , then is no longer well defined and not a critical point anymore.

NOTE: The above angles occur when

Evaluating at the critical points of (without removing the non-convex hull vertex points) gives the following formulas:

TODO: Graph the above and verify that the minimum is still at a non-differentiable point (it is true, however, the original assumption that the minimum of the 4d graph is at a non-differentiable point has not been vetted)

Solving again for its non-differentiable points gives:

Solving in the above for its non-differentiable points gives:

TODO: Get a good desmos justification that the minimum occurs at the non-differentiable points

Solving in the above (instead of ) for its non-differentiable points gives:

TODO: Get a good desmos justification that the minimum occurs at the non-differentiable points

Solving for its non-differentiable points in the first set of equations gives:

Solving for its non-differentiable points in the first set of equations gives:

Solving in the above for its non-differentiable points gives:

Note: when n == i or when n == j

Which implies:

Solving in the above (instead of ) for its non-differentiable points gives:

Where

Solutions only exist when

Solving in the above (instead of or ) for its non-differentiable points gives:

Critical Angles for :

The critical angles for are:

Euler Angles:

The above is where is not differentiable.

NOTE: If , then the above degenerates to

NOTE: If (but ), then the above degenerates to

NOTE: If the , then is no longer well defined and not a critical point anymore.

NOTE: The above angles occur when

Evaluating at the critical points of (without removing the non-convex hull vertex points) gives the following formulas:

Solving in the above gives:

Solving for in the first set gives:

Solving for in the first set gives:

Solving for in the above gives:

Solving for in the above gives:

Solving for in the above gives:

Critical Angles for :

The critical angles for are:

Euler Angles:

Vector -Angle:

?

The above is where is not differentiable.

NOTE: If , then the above degenerates to:

Euler Angles

Vector -Angle:

?

NOTE: If , then the above degenerates to

NOTE: If , then the above degenerates to

NOTE: If , then the above degenerates to

NOTE: If the , then is no longer well defined and not a critical point anymore.

NOTE: The above angles occur when

Evaluating at the critical points of (without removing the non-convex hull vertex points) gives the following formulas:

Euler Angles:

Vector -Angle:

Solving in the above gives:

Solving for in the first set gives:

Solving for in the first set gives:

Solving in the above gives:

Solving in the above gives:

Critical Angles for :

The non-differentiable critical angles (where a minimum can be attained for the area) for are of the form:

**Maximum Oriented Bounding Box**

The maximum oriented bounding box is an oriented bounding box that maximizes the area for the minimum oriented bounding box.

Extent 1 at maximum points:

Extent 2 at maximum points:

Area at maximum points:

NOTE: For , all angles are maximum and . This case is not included in the table below

NOTE: For 2 points, there is a nice description of the maximum angle: however, this equation doesn’t seem to describe the maximum angle for sets of more than 2 points. There must exist a more general formula…

Table 6:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | | | |
|  |  |  |  |  |
| 2 |  | --- | --- | --- |
| 3 |  |  |  | --- |
| 4 |  |  |  |  |
| 5 |  |  |  |  |
| 6 |  |  |  |  |
| 7 |  |  |  |  |
| 8 |  |  |  |  |

Sources:

[A minimum bounding box algorithm and its application to rapid prototyping](https://utw10945.utweb.utexas.edu/Manuscripts/1999/1999-019-Chan.pdf)

[Fast oriented bounding box optimization on the rotation group SO(3,R)](https://perso.uclouvain.be/chia-tche.chang/resources/CGM12_slides.pdf) ([Fast oriented bounding box optimization on the rotation group SO(3,ℝ) | ACM Transactions on Graphics](https://dl.acm.org/doi/10.1145/2019627.2019641))