An infinite-times renewal equation: finite-dimensional approximation and long-time behaviour of the infinite-dimensional PDE

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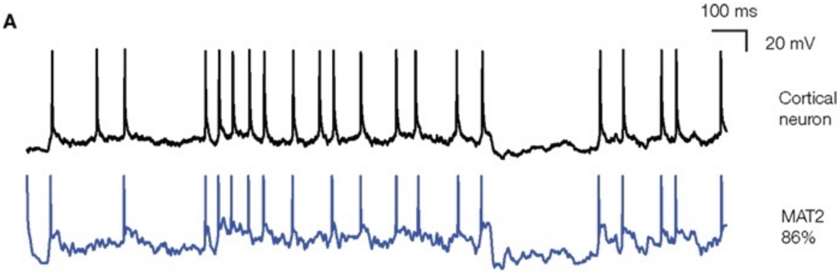
Motivations from Neuroscience

In neuroscience, membrane potential is widely used to describe the dynamics of a neuron. A **spike** refers to a rapid rising of the membrane potential for a neuron, which is illustrated in the figure below. Especially in a connected neural network, a spike of a neuron will greatly affect the dynamics of its adjacent neurons. Thus, the analysis of the **spike train** of a neuron is of great importance in mathematical neuroscience.



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In the language of stochastic process:

- State variable: $s_1(t) \geq 0$ the time since last spike.
- $p_1(s_1)$ the **rate** of having a new spike.
- If there is a new spike, reset $s_1(t) = 0$.
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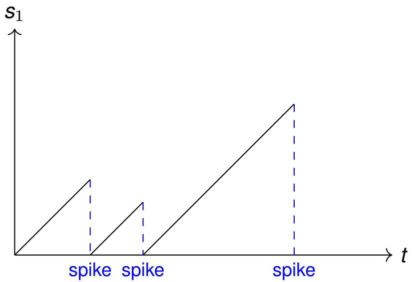

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- $p_2(s_1, s_2)$ the rate of having a new spike.
- If there is a new spike, $\big(s_1(t), s_2(t)\big) = \big(0, s_1(t^-)\big)$.
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Distribution described by a **2-times renewal equation** [Torres, Perthame, Salort 2022]:

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\begin{aligned}
&\partial_t n_2(t, s_1, s_2) + \sum_{i=1}^{2} \underbrace{\partial_{s_i} n_2(t, s_1, s_2)}_{\text{transport}} + \underbrace{p_2 n_2(t, s_1, s_2)}_{\text{renewal}} = 0 \\
&n_2(t, 0, s_1) = \int_{s_1}^{+\infty} n_2(t, s_1, s_2) p_2(s_1, s_2) ds_2
\end{aligned}
\tag{1}
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Naturally generalized to **finite-times renewal equation**

$$
\begin{aligned}
&\partial_t n_N(t, s_1, ..., s_N) + \sum_{i=1}^{N} \partial_{s_i} n_N(t, s_1, ..., s_N) + p_N n_N(t, s_1, ..., s_N) = 0 \\
&n_N(t, 0, s_1, ..., s_{N-1}) = \int n_2(t, s_1, ..., s_N) p_N(s_1, ..., s_N) ds_N
\end{aligned} \tag{2}
$$

Formal Construction

What if the spike rate depends on **all the spikes, possibly infinite**, in the past?

- $(s_1(t), ..., s_N(t), ...)$ the time since (N-th) last spike.
- $p_\infty(s_1, ..., s_N, ...)$ the rate of having a new spike.
- If there is a new spike, $s_1(t) = 0$ and $s_N(t) = s_{N-1}(t^-)$ for $N \neq 1$.
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$$\begin{tikzcd}
0 & s_1 & s_2 & s_3 & ... & s_N & ... \\
s_1 \arrow[ur] & s_2 \arrow[ur] & s_3 \arrow[ur] & \cdots & s_N \arrow[ur] & \cdots & \cdots
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\end{tikzcd}
$$

Formally, we can write the **infinite-times renewal equation**, with $[s]_\infty := (s_1, s_2, ...)$

$$
\begin{aligned}
&\partial_t n_\infty(t, [s]_\infty) + \sum_{i=1}^{+\infty} \partial_{s_i} n_\infty(t, [s]_\infty) + p_\infty n_\infty(t, [s]_\infty) = 0 \\
&n_\infty(t, 0, [s]_\infty) = n_\infty(t, [s]_\infty) p_\infty([s]_\infty)
\end{aligned}
\tag{3}
$$

Broader Connections

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- In these applications, the variable s_i refers to age but in many other areas multiple structures occur with different velocities, leading to more general equations under the form

$$\partial_t n + \sum_{i=1}^{N} \partial_{s_i} \left[g_i([s]_N) n \right] + p_N([s]_N, n_N) n_N = 0. \tag{4}$$

See [An, Jäger, Neuss-Radu 2015] for instance. Our analysis could cover such general settings.

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See [An, Jäger, Neuss-Radu 2015] for instance. Our analysis could cover such general settings.
- In **Hawkes process**, the renewal rate is of the form,
$$p_\infty([s]_\infty) = \sum_{i=1}^{+\infty} h(s_i). \tag{5}$$
This process has wide applications in neuroscience, genome analysis, and financial analysis etc.

Difficulties

- No Lebesgue measure on $\mathbb{R}^{\mathbb{N}}_{\geq 0} \Longrightarrow$ Weak solution.

$$\int \psi([s]_\infty) \sum_{i=1}^{+\infty} \partial_{s_i} n_\infty(d[s]_\infty) = -\int n_\infty(d[s]_\infty) \sum_{i=1}^{+\infty} \partial_{s_i} \psi([s]_\infty) \qquad (6)$$

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- **No convergence in total-variation norm.**
  Consider two Dirac masses δ_s and $\delta_{s'}$ respectively concentrated at $[s]_\infty$ and $[s']_\infty$ ($k_1 \neq k_2$),

$$\begin{aligned} [s]_\infty &= (k_1, 2k_1, ..., Nk_1, ...) \\ [s']_\infty &= (k_2, 2k_2, ..., Nk_2, ...) \end{aligned} \tag{7}$$

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Then,

$$\begin{aligned} &\text{supp } n^1_\infty(t) \subseteq \{[s]_\infty : \exists N \text{ s.t. } s_{n+1} - s_n = k_1, \quad \forall n \geq N\}, \\ &\text{supp } n^2_\infty(t) \subseteq \{[s]_\infty : \exists N \text{ s.t. } s_{n+1} - s_n = k_2, \quad \forall n \geq N\}. \end{aligned}$$

Thus,

$$\|n^1_\infty - n^2_\infty\|_{\mathrm{TV}} \equiv 2 \tag{8}$$

Notations

We use the following notations for elements and spaces,

- Order structure: $s_1 \leq s_2 \leq \ldots$; $[s]_N = (s_1, \ldots, s_N)$ and $[s]_\infty = (s_1, \ldots, s_N, \ldots)$; $[s]_{K,N} := (s_K, \ldots, s_N)$ and $[s]_{K,\infty} := (s_K, \ldots)$.
- Admissible region: $\mathcal{C}_N := \{[s]_N | s_1 \leq \ldots \leq s_N\}$, $\mathcal{C}_\infty := \{[s]_\infty | s_1 \leq s_2 \leq \ldots\}$.
- **Shift operator:** $\tau[s]_N = (0, [s]_{N-1})$, $\tau[s]_\infty = (0, [s]_\infty)$. **(No loss of memory)**
- Spaces with $1 \leq N \leq +\infty$: $\mathcal{P}(\mathcal{C}_N)$, $\mathcal{M}(\mathcal{C}_N)$, $C_b^i(\mathcal{C}_N)$, $C_w\big([0, +\infty); \mathcal{P}(\mathcal{C}_N)\big)$

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We use the following notations for functions and measures,

- For a fixed t: $n_N(t) \in \mathcal{P}(\mathcal{C}_N)$, $n_\infty(t) \in \mathcal{P}(\mathcal{C}_\infty)$.
- **Marginal distribution** for $1 \leq N \leq +\infty$,

$$n_N^{(K)}(t, [s]_K) = \int n_N(t, [s]_K, [s]_{K+1,N}) d[s]_{K+1,N} \tag{9}$$

Assumptions

Generally speaking, we are working under the following assumptions:

- **Boundedness**: for $1 \leq N \leq +\infty$, we have

$$0 < a_- \leq p_N([s]_N) \leq a_+ < +\infty. \tag{10}$$

- **Finite-dimensional Approximation of renewal rate:**

$$\begin{aligned} p_\infty([s]_\infty) &= \sum_{i=1}^{+\infty} \varphi_i([s]_i), \\ \sum_{i=1}^{+\infty} \|\varphi_i\|_{L^\infty} &< +\infty. \end{aligned} \tag{11}$$

- Usually, given a p_∞, we will use p_N as $p_N([s]_N) = \sum_{i=1}^{N} \varphi_i([s]_i)$.

Formulation of Weak Solution: BBGKY Hierarchy

- For a system with many of infinite particles, BBGKY hierarchy can be used to describe the dynamics using only **finite-dimensional marginals**,

$$\begin{aligned} & n_N^{(K)}(t, [s]_K) = \int n_N(t, [s]_K, [s]_{K+1,N}) d[s]_{K+1,N} \\ & \partial_t n_N^{(K)} = \mathcal{L} n_N^{(K)} + \textbf{High-Dimensional Error} \end{aligned} \tag{12}$$

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- Although in our problem, $[s]_\infty$ refers to infinite state variables of a particle (neuron etc.), the method of BBGKY hierarchy can still be used, giving Equation (12).
- Form of solution: $\{n_\infty^{(K)}(t)\}_K$, with each $n_\infty^{(K)} \in C_w\big([0,+\infty);\mathcal{P}(\mathcal{C}_K)\big)$.
- **Consistency condition:**

$$\int n_\infty^{(K)}(t,[s]_K) = \int n_\infty^{(K+1)}(t,[s]_K,s_{K+1})ds_{K+1} \tag{13}$$

- Advantage: Don't need to deal with $\mathcal{C}_\infty \subset \mathbb{R}^{\mathbb{N}}$.

Formulation of Weak Solution: BBGKY Hierarchy

In the spirit of BBGKY hierarchy, we define the **hierarchy solution**,

$$\begin{cases} \partial_t n_\infty^{(K)} + \sum_{i=1}^{K} \partial_{s_i} n_\infty^{(K)} + p_K([s]_K) n_\infty^{(K)} + E_\infty^{(K)}(t,[s]_K) = 0, \\ n_\infty^{(K)}(t, s_1 = 0, s_2, ... s_K) = \int_{u=0}^{\infty} \left[p_K n_\infty^{(K)} + E_\infty^{(K)} \right](t, s_2, ..., s_K, u) du, \\ n_\infty^{(K)}(t=0,[s_K]) = n_\infty^{0,(K)}([s_K]), \end{cases} \tag{14}$$

with the **high-dimensional error** defined as,

$$E_\infty^{(K)}(t,[s]_K) = \sum_{i=K+1}^{\infty} \int_{s_{K+1}=0}^{\infty} ... \int_{s_i=0}^{\infty} \varphi_i([s]_i) n_\infty^{(i)}(t,[s]_i)\, d[s]_{K+1,i}, \tag{15}$$

Formulation of Weak Solution: Kolmogorov's Extension

- **Kolmogorov extension theorem:** For a consistent family of probability measures $\{n_\infty^{(K)} \in \mathcal{P}(\mathcal{C}_K)\}_K$, there exists a unique $n_\infty \in \mathcal{P}(\mathcal{C}_\infty)$ such that,

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- Since the hierarchy solution $\{n_\infty^{(K)}(t)\}_K$ is consistent, we can use the extension theorem to give the extension,

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- Weak solution: for arbitrary $K < +\infty$ and $\psi \in C_b^1(\mathcal{C}_K)$,

$$\int_{\mathcal{C}_\infty} \psi(T,[s]_K) n_\infty(T, d[s]_\infty) - \int_{\mathcal{C}_\infty} \psi(0,[s]_K) n_\infty(0, d[s]_\infty)$$
$$= \int_0^T \int_{\mathcal{C}_\infty} n_\infty(t, d[s]_\infty) \Big[(\partial_t + \sum_{i=1}^K \partial_{s_i}) \psi(t,[s]_K) + p_\infty([s]_\infty)\big(\psi(t, \tau[s]_K) - \psi(t,[s]_K)\big)\Big] dt$$

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- **Two formulation of solutions are equivalent.**

Finite-dimensional Approximation

Consider the finite-times renewal equation:

$$\begin{aligned}
&\partial_t n_N(t,[s]_N) + \sum_{i=1}^{N} \partial_{s_i} n_N(t,[s]_N) + p_N([s]_N) n_N(t,[s]_N) = 0 \\
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- Initial data: $n_N(0) \in \mathcal{P}(\mathcal{C}_N)$.
- Well-posedness: characteristic line + fixed-point method;
- Solution: $n_N(t) \in C_w\big([0,+\infty);\mathcal{P}(\mathcal{C}_N)\big)$.
- Exponential convergence in $\|\cdot\|_{\mathcal{M}^1}$: use Doeblin's method.

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We will use the solution of finite-times renewal equation **to approximate, and converge to**, the solution of infinite times renewal equation.

Finite-dimensional Approximation: Strong Topology

Assume that for each $1 \leq N \leq +\infty$, the renewal rate is,

$$0 < a_- \leq p_N([s]_N) = \sum_{i=1}^{+\infty} \varphi_i([s]_i) \leq a_+ < +\infty \tag{19}$$

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Under proper assumptions on initial data, for the sequence $\{n_N(t)\}_N$ of finite-dimensional solutions, we have,

(i) For all $T > 0$ and $K \in \mathbb{N}$, the sequence $\{n_N^{(K)}\}_N$ is a Cauchy sequence in $C\big([0,T]; L^1(\mathcal{C}_K)\big)$ and thus it has a consistent limit $n_\infty^{(K)} \in C\big((0,\infty); L^1(\mathcal{C}_K)\big)$.

(ii) $E_N^{(K)}(t,[s]_K) \to E_\infty^{(K)}(t,[s]_K)$ in $C\big((0,T); L^1(\mathcal{C}_K)\big)$ as $N \to \infty$.

(iii) $\{n_\infty^{(K)}\}_K$ is the unique consistent weak solution of the hierarchy system (14)–(15).

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(iii) $\{n_\infty^{(K)}\}_K$ is the unique consistent weak solution of the hierarchy system (14)–(15).

- If $\|\varphi_i\|_\infty$ has a 'rapid' decay relative to i, then the local-in-time strong approximation can be extended to unifrom-in-time,

$$n_N^{(K)} \to n_\infty^{(K)} \quad \text{in } C\big([0, +\infty); L^1(\mathcal{C}_K)\big). \tag{20}$$

Finite-times Renewal Equation: Exponential Convergence in $\|\cdot\|_{\mathcal{M}^1}$

Doeblin Theorem:

- Doeblin's condition: P_t a Markov semi-group, there exists $t_0 > 0$ and $\nu \in \mathcal{P}(\mathcal{X})$ s.t.

$$P_{t_0}\mu \geq \alpha\nu, \quad \forall \mu \in \mathcal{P}(\mathcal{X}) \tag{21}$$

- Assume P_t satisfies Doeblin's condition. Then the semigroup has a unique equilibrium $\mu^* \in \mathcal{P}(\mathcal{X})$ and,

$$\|P_t\mu - \mu^*\|_{\mathcal{M}^1} \leq \frac{1}{1-\alpha} e^{-\lambda t} \|\mu - \mu^*\|_{\mathcal{M}^1}, \quad \forall t \geq 0. \tag{22}$$

  Here $\lambda = -\frac{\ln(1-\alpha)}{t_0} > 0$.
- Can we choose a $t_N^* > 0$ such that $n_N(t_N^*)$ has a lower bound measure, uniform for arbitrary initial distribution $n_N(0)$?
- Given a particle at $[s']_N$, what is the **probability density** of it jumping to $[s]_N$, after time s_N?

Finite-times Renewal Equation: Exponential Convergence in $\|\cdot\|_{\mathcal{M}^1}$

Using knowledge of exponential distribution,

- A particle has at least a_- probability to have a jump **right now**.
- A particle has at least e^{-a_+u} probability to do not have a jump **during** time u.

Thus, $[s']_N$ jump to $[s]_N$ after time s_N, **with probability at least** $a_-^N e^{-a_+ s_N}$.

$$\begin{tikzcd} {[s']_N} \arrow[r, "1 \text{ jump}"] & (0, [s']_{N-1}) \arrow[r, bend right, "\text{time } s_N", "N-1 \text{ jump}"'] & (s_1, \dots, s_N) \end{tikzcd}$$

Intuition:

- a_-^N refers to the N times of jump.
- $e^{-a_+ s_N}$ refers to the time without jump of length s_N.

Finite-times Renewal Equation: Exponential Convergence in $\|\cdot\|_{\mathcal{M}^1}$

- Thus, $[s']_N$ jump to $[s]_N$ after time s_N, **with probability at least** $a_-^N e^{-a_+ s_N}$.
- For a chosen t_N^* and $[s]_N$ with $s_N \leq t_N^*$, we have the lower bound,

$$n_N(t,[s]_N) \geq a_-^N e^{-a_+ s_N} \int n_N(t,[s']_N) d[s']_N = a_-^N e^{-a_+ s_N} \tag{23}$$

- By integration over the domain $s_N \leq t_N^*$, we have the total measure of the lower bound approximately as $\left(\frac{a_-}{a_+}\right)^N$.
- Now we have the rate of exponential convergence **approximately as** $\left(\frac{a_-}{a_+}\right)^N$.

As dimension $N \to +\infty$, the $\mathcal{M}^1$-convergence rate will goes to 0

- Recall the counterexample $\|n_{\infty,1}(t) - n_{\infty,2}(t)\|_{\mathcal{M}^1}$.

Need to find other metrics on $\mathcal{P}(\mathcal{C}_\infty)$.

Exponential Convergence in Monge-Kantorovich Distance

For $1 \leq N \leq +\infty$, given a tranport cost function $V([s]_N, [s']_N)$, we have the **Monge-Kantorovich (M.-K.) distance** on $\mathcal{P}(\mathcal{C}_N)$ defined as,

$$\begin{cases} \mathcal{T}_V(n_N, m_N) := \inf\limits_{\omega_N \in \mathcal{H}(n_N, m_N)} \iint V([s]_N, [s']_N)\omega_N(d[s]_N, d[s']_N), \\ \\ \mathcal{H}(n_N, m_N) = \{\omega_N \in \mathcal{P}\big(\mathcal{C}_N \times \mathcal{C}_N\big) \text{ with marginals } n_N \text{ and } m_N\}. \end{cases} \tag{24}$$

We will define and use the cost function $V_{\beta,a}([s]_\infty, [s']_\infty)$,

$$V_{\beta,a}([s]_\infty, [s']_\infty) := \sum_{i=1}^{\infty} \frac{|s_i - s'_i| \wedge a}{(1+\beta)^i} \leq \frac{a}{\beta}, \tag{25}$$

- β: decay of the weight.
- a: truncation, to make $V_{\beta,a}([s]_\infty, [s']_\infty)$ finite.

Exponential Convergence in Monge-Kantorovich Distance

A vital constant is the **Lipschitz constant** of $p_\infty([s]_\infty)$ relative to $V_{\beta,a}([s]_\infty, [s']_\infty)$,

$$\delta := \sup_{[s]_\infty, [s']_\infty} \frac{|p_\infty([s]_\infty) - p_\infty([s']_\infty)|}{V_{\beta,a}([s]_\infty, [s']_\infty)} \tag{26}$$

To prove the convergence of M.-K. distance, we need a **quantative condition** for δ,

$$\gamma := \frac{\beta a_-}{1+\beta} - \frac{a\delta}{\beta} > 0 \tag{27}$$

- Recall a_- is the lower bound of $p_\infty([s]_\infty)$.
- The two terms in γ represents **two competing effects**, which we will see later.

Theorem. Assume $p_\infty([s]_\infty)$ satisfies (10) and (11), and δ satisfies the quantitative condition. Given $n_\infty(0), m_\infty(0) \in \mathcal{P}(\mathcal{C}_\infty)$, the unique $n_\infty(t)$ and $m_\infty(t)$ satisfy,

$$\mathcal{T}_{V_{\beta,a}}(n_\infty(t), m_\infty(t)) \le \mathcal{T}_{V_{\beta,a}}(n_\infty(0), m_\infty(0)) e^{-\gamma t} \tag{28}$$

Exponential Convergence: Coupling Measure

- Control $\mathcal{T}_{V_{\beta,a}}(n_\infty(t), m_\infty(t))$: $\omega_\infty(t)$ a **coupling measure** of $n_\infty(t)$ and $m_\infty(t)$,

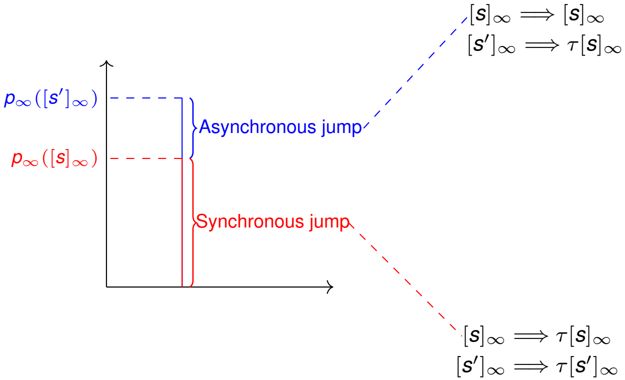
$$\int_{\mathcal{C}_\infty \times \mathcal{C}_\infty} \omega_\infty(t, d[s]_\infty, d[s']_\infty) V_{\beta,a}([s]_\infty, [s']_\infty) \geq \mathcal{T}_{V_{\beta,a}}(n_\infty(t), m_\infty(t))$$

- How to construct a proper coupling measure?

Couple the stochastic process $[s(t)]_\infty$ and $[s'(t)]_\infty$.

- $[s(t)]_\infty$ has jump rate $p_\infty([s]_\infty)$; $[s'(t)]_\infty$ has jump rate $p_\infty([s']_\infty)$.
  - **Synchronous jump:**
    $[s(t)]_\infty$ and $[s'(t)]_\infty$ has rate $p_\infty([s]_\infty) \wedge p_\infty([s']_\infty)$ to jump at the same time.
  - **Asynchronous jump:**
    $[s(t)]_\infty$ jump with $[s'(t)]_\infty$ fixed, of rate $\big(p_\infty([s]_\infty) - p_\infty([s']_\infty)\big)_+$
    $[s'(t)]_\infty$ jump with $[s(t)]_\infty$ fixed, of rate $\big(p_\infty([s']_\infty) - p_\infty([s]_\infty)\big)_+$.

Exponential Convergence: Coupling Measure



Exponential Convergence: Coupling Measure

Recall the shift operator τ (for a jump),

$$\tau[s]_\infty = (0, [s]_\infty) \tag{29}$$

Vital Observation: Synchronous jump $\Longrightarrow$ Exponential decay

$$V_{\beta,a}(\tau[s]_\infty, \tau[s']_\infty) = \frac{1}{1+\beta} V_{\beta,a}([s]_\infty, [s']_\infty) \tag{30}$$

This is because by $V_{\beta,a}([s]_\infty, [s']_\infty) = \sum_{i=1}^{+\infty} \frac{|s_i - s_i'| \wedge a}{(1+\beta)^i}$,

$$\begin{tikzcd}
\frac{|s_1-s_1'|\wedge a}{1+\beta} \arrow[dr] & \frac{|s_2-s_2'|\wedge a}{(1+\beta)^2} \arrow[dr] & \cdots\cdots & \cdots\cdots \\
\frac{|0-0|\wedge a}{1+\beta} = 0 & \frac{|s_1-s_1'|\wedge a}{(1+\beta)^2} & \frac{|s_2-s_2'|\wedge a}{(1+\beta)^3} & \cdots\cdots
\end{tikzcd}$$

Exponential Convergence: Competing Effects

A differential equation for M.-K. distance's upper bound,

$$\frac{d}{dt}\iint_{\mathcal{C}_\infty\times\mathcal{C}_\infty} V_{\beta,a}([s]_\infty,[s']_\infty)\omega_\infty(t,d[s]_\infty,d[s']_\infty)$$

$$= \iint_{\mathcal{C}_\infty\times\mathcal{C}_\infty}\Bigg(\Big[V_{\beta,a}(\tau[s]_\infty,\tau[s']_\infty) - V_{\beta,a}([s]_\infty,[s']_\infty)\Big]\big(\overbrace{p_\infty([s]_\infty)\wedge p_\infty([s']_\infty)}^{\text{Synchronous}}\big)$$

$$+\Big[V_{\beta,a}(\tau[s]_\infty,[s']_\infty) - V_{\beta,a}([s]_\infty,[s']_\infty)\Big]\big(p_\infty([s]_\infty) - p_\infty([s']_\infty)\big)_+$$

$$+\Big[V_{\beta,a}([s]_\infty,\tau[s']_\infty) - V_{\beta,a}([s]_\infty,[s']_\infty)\Big]\big(\underbrace{p_\infty([s']_\infty) - p_\infty([s]_\infty)}_{\text{Asynchronous}}\big)_+\Bigg)\omega_\infty dt$$

Exponential Convergence: Competing Effects

For the synchronous jump,

- recall a_- is the lower bound of p_∞.

$$\begin{aligned}&\Big[V_{\beta,a}(\tau[s]_\infty,\tau[s']_\infty)-V_{\beta,a}([s]_\infty,[s']_\infty)\Big]\big(p_\infty([s]_\infty)\wedge p_\infty([s']_\infty)\big)\\ \le&-\frac{\beta}{1+\beta}V_{\beta,a}([s]_\infty,[s']_\infty)a_-\end{aligned} \tag{31}$$

For the asynchronous jump,

- recall $\frac{a}{\beta}$ is the upper bound of $V_{\infty,\beta,a}$.
- recall δ is the Lipschitz constant of p_∞ relative to $V_{\infty,\beta,a}$.

$$\Big|V_{\beta,a}(\tau[s]_\infty,[s']_\infty)-V_{\beta,a}([s]_\infty,[s']_\infty)\Big|\le\frac{a}{\beta},$$

$$\Big|V_{\beta,a}([s]_\infty,\tau[s']_\infty)-V_{\beta,a}([s]_\infty,[s']_\infty)\Big|\le\frac{a}{\beta},$$

$$(p_\infty([s]_\infty)-p_\infty([s']_\infty))_+ + (p_\infty([s']_\infty)-p_\infty([s]_\infty))_+\le\delta V_{\beta,a}([s]_\infty,[s']_\infty).$$

Exponential Convergence: Competing Effects

Collecting different terms, we have,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{C}_\infty\times\mathcal{C}_\infty} V_{\beta,a}([s]_\infty,[s']_\infty)\omega_\infty(t,d[s]_\infty,d[s']_\infty)\\
\leq &-\iint_{\mathcal{C}_\infty\times\mathcal{C}_\infty}\frac{\beta a_-}{1+\beta}V_{\beta,a}([s]_\infty,[s']_\infty)\omega_\infty(t,d[s]_\infty,d[s']_\infty)\\
&+\iint_{\mathcal{C}_\infty\times\mathcal{C}_\infty}\frac{a\delta}{\beta}V_{\beta,a}([s]_\infty,[s']_\infty)\omega_\infty(t,d[s]_\infty,d[s']_\infty).
\end{aligned}
$$

This can be written in a more compact way,

$$
\frac{d}{dt}T_{V_{\beta,a}}(\omega_\infty(t)) \leq T_{V_{\beta,a}}(\omega_\infty(t))\Big(-\frac{\beta a_-}{1+\beta}+\frac{a\delta}{\beta}\Big) = -\gamma T_{V_{\beta,a}}(\omega_\infty(t)),
$$

$$
\gamma = \frac{\beta a_-}{1+\beta} - \frac{a\delta}{\beta}
$$

Thus we have the desired exponential convergence,

$$
\mathcal{T}_{V_{\beta,a}}(n_\infty(t),m_\infty(t)) \leq \mathcal{T}_{V_{\beta,a}}(n_\infty(0),m_\infty(0))e^{-\gamma t} \tag{32}
$$

Exponential Convergence: Competing Effects

synchronous rate

asynchronous rate

$$\gamma = \frac{\beta}{1+\beta} a_- - \frac{a}{\beta} \delta$$

synchronous decay

asynchronous growth

Exponential Convergence: Explicit Conditions

We define the **weighted Lipschtz constant and fluctuation bound**,

$$\begin{cases} L_\infty(\beta) := \max_{1\le i\le\infty} \sup_{[s]_i,[s']_i} (1+\beta)^i \frac{|\varphi_i([s]_i)-\varphi_i([s']_i)|}{|s_i-s_i'|}, \\ F_\infty(\beta) := \max_{1\le i\le\infty} \sup_{[s]_i,[s']_i} (1+\beta)^i \Big|\varphi_i([s]_i)-\varphi_i([s']_i)\Big|. \end{cases} \tag{33}$$

We can express the Lipshitz constant δ of p_∞ relative $V_{\beta,a}$,

$$\begin{aligned} \frac{|p_\infty([s]_\infty)-p_\infty([s']_\infty)|}{V_{\beta,a}([s]_\infty,[s']_\infty)} &\le \frac{\sum_{i=1}^\infty |\varphi_i([s]_i)-\varphi_i([s']_i)|}{\sum_{i=1}^\infty \frac{1}{(1+\beta)^i}(|s_i-s_i'|\wedge a)} \le \max_{1\le i\le\infty}\left\{\frac{|\varphi_i([s]_i)-\varphi_i([s']_i)|}{\frac{1}{(1+\beta)^i}(|s_i-s_i'|\wedge a)}\right\} \\ &\le \frac{F_\infty(\beta)}{a} \vee L_\infty(\beta) := \delta. \end{aligned}$$

To make δ small, we only need to make the fluctuation and Lipschitz constant of φ_i to exponentially decay.

Exponential decay in t $\Longleftrightarrow$ Exponential decay in φ_i along i

EXponential Convergence: Examples of Renewal Rate

- We can set the renewal rate as,

$$p_\infty([s]_\infty) = \sum_{i=1}^{+\infty} \frac{(a_- \vee s_i) \wedge Ca_-}{(1+\beta)^i}.$$

- Here $F_\infty(\beta) = (C-1)a_-$ and $L_\infty(\beta) \leq 1$.
- We generalize the renewal rate to,

$$p_\infty([s]_\infty) = \sum_{i=1}^{+\infty} \frac{(a_- \vee f(s_i)) \wedge Ca_-}{(1+\beta)^i},$$

where $f(\cdot)$ is a Lipschitz continuous function.

Thanks for your attention!

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