Averages in the Plane over Convex Curves and Maximal Operators

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Contents

1 What's new in this note?

The writer added the proof omitted in Bourgain's paper, which is really common, and made the notations in the original paper clearer. Especially, the statement in the beginning of section 4 in Bourgain's paper is too strong to hold. The writer modified it so that the whole paper's proof still holds. Also, the writer gave a new proof of this statement based on (of course finite dimensional) compactness and an angle-preserving isometry (see lemma 4.1 in this note (7)). The writer completed the proof of lemma 3 in the original paper, which lacks some crucial conditions (such as the small boundedness of $\mu(t)$) not mentioned in the original paper (see lemma 4.2 in this note(10)).
For the contents clear enough in Bourgain's paper, the writer choose to omit them.

The proof of Bourgain is amazing, while there are something Bourgain may not have written down. This note is especially to make these clearer.

2 Summary of the Proof Line

To estimate $\|\sup_{t>0}|A_t f(x)|\|$, this paper breaks down the estimates in the following order:

$$\begin{aligned}\sup_{t>0}|A_t f(x)| &\Rightarrow \sup_{t>0}\{|A_t(E_{v(t)}[f])(x)| + \Sigma_{s\geq 0}|A_t(\Delta_{v(t)+s}[f])(x)|\} \\ &\Rightarrow \sup_{2^{-v}\leq t\leq 2^{-v+1}}|A_t(g)| \Rightarrow \sup_{1\leq t\leq 2}|\int g(x+ty)\sigma_k(y)| \\ &\Rightarrow |\int g(x+t(x)y)\sigma_k(y)| \\ &\Rightarrow \|\tilde{V}_x(y)dx\|_i, \quad i=1,2 \\ &\Rightarrow |\langle\tilde{V}_x,\tilde{V}_y\rangle| \Rightarrow \langle V_x,V_y\rangle \Rightarrow diam\ comp(V_x,V_y) \Rightarrow Geometric\ Estimates\end{aligned} \tag{1}$$

and we briefly explain this line in this section. And for the sake of simplicity, the writer won't explain what the notations are, so this section is intended for those who have read the Bourgain's paper or have read the following contents in this note. For readers who are not familiar with the notations, you are recommended to go to the next section or the original paper and get back here whenever you feel lost.

We first use a kind of dyadic decomposition to decompose the function $f(x)$, and the decomposition is determined by the t in $A_t(f)$. Next, we want to construct a universal bound for those different dyadic decompositions, where we use the Hardy-Littlewood maximal function to control the 'dyadic expectation' term. Next we need to estimate the 'dyadic difference' term, where we further decompose the singular arc-length measure σ into the sum of a group of less singular measures. The final task for us is to estimate the 'dyadic difference function with the less singular measure', where we transform it into a geometric problem compensating the lack of 'fast decay' (i.e. the term $n^{-\alpha}$) in pure analytical methods. The lucky thing is that we can use finite dimensional compactness argument to derive a series of geometric properties, which finally lead us to the end of the proof.

In addition, the spirit of the geometric estimates is that, we notice that the diameter of components of $V_x \cap V_y$ can be controlled by the edges, while those edges is a norm-preserving curve for one point, and a norm-changing curve for the other point. Thus we encode the information of edges into the equation and calculate its differential, while use curvature conditions to estimate the length of edges.

It should be noticed that although there are many different scales in the proof, we can always use the norm-preserving rescaling to prove at a specific and better scale.

3 Reduction of the Proof

Let Γ be a given convex, symmetric and smooth enough curve with non-vanishing curvature in $\mathbb{R}^2$. We first expalin some notations: We use the following notations:

$$\begin{cases} a \ll b := a < C(\Gamma)b \\ a \gg b := a > C(\Gamma)b \\ a \sim b := c(\Gamma)b < a < C(\Gamma)b \end{cases} \tag{2}$$

note that in certain contexts we will use $t \sim 2^{-v}$ to mean $2^{-v} < t < 2^{-v+1}$ and the readers will be able to tell it from the context. Please notice that the constants depend on the specific curve Γ.

Theorem 1. *Let f be a bounded measurable function on the plane and define for $0 < t < \infty$ the average*

$$A_t f(x) = \int f(x + ty)\sigma(dy) \tag{3}$$

Denote Mf the corresponding maximal operator

$$Mf = \sup_{t>0} |A_t f| \tag{4}$$

Then for $2 < p < \infty$, there is an inequality

$$\|Mf\|_p \leq C(\Gamma, p)\|p\|_p \tag{5}$$

Here $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^2)$-norm.

To prove this theorem, we want to construct a bound $E(x)$ for $\sup_{t>0} A_t f(x)$. To achieve this, we find different bounds for different t in different dyadic intervals and then find a uniform bound for these separate bounds, and prove the L^p boundedness for this uniform bound $E(x)$.
Given a t, we find its dyadic interval $2^{-v} \leq t \leq 2^{-v+1}$, and consider a dyadic decomposition:

Definition 2. *We define dyadic expectation operator E_k as*

$$E_k(f) := E[f|\mathcal{D}_k] \tag{6}$$

and define dyadic difference operator $\Delta_k f$ as

$$\Delta_k f := E_{k+1} f - E_k f \tag{7}$$

then, for $2^{-v} \leq t \leq 2^{-v+1}$, we consider the decomposition

$$\begin{aligned} f &= E_v[f] + \Sigma_{k \geq v} \Delta_k f \\ A_t f &= A_t(E_v[f]) + \Sigma_{k \geq v} A_t(\Delta_k f) \end{aligned} \tag{8}$$

Notice that for a $2^{-v} \leq t \leq 2^{-v+1}$, the $\partial B(x,t)$ intersects at most $n(\Gamma) < \infty$ squares determined by the shape of Γ in $\mathcal{D}_v$, so

$$\begin{aligned} |A_t(E_v[f])| &\leq \Sigma_{1 \leq i \leq n(\Gamma) \partial B(x,t) \cap D_v^i} |f| dx \\ |A_t(E_v[f])| &\leq n(\Gamma)_{\cup_{1 \leq \Gamma} D_v^i} |f| dx \\ |A_t(E_v[f])| &\leq n(\Gamma) f^* \end{aligned} \tag{9}$$

Hence by the L^p-boundedness of the Hardy-Littlewood maximal function f^*, there is

$$\|\sup_t A_t(E_{v(t)}[f])\|_p \leq C\|f\|_p \tag{10}$$

where $v(t)$ is determined by t. Hence we now only need to prove the L^p-boundedness of

$$\sup_{v \geq 0} \sup_{2^{-v} \leq 2^{-v+1}} |\Sigma_{k \geq v} A_t(\Delta_k f)| \tag{11}$$

since

$$
\begin{aligned}
|\sup_t A_t f| &\le \sup |A_t(E_{v(t)}f)| + \sup_t \sup_{2^{-v(t)} \le 2^{-v(t)+1}} |\Sigma_{k\ge v(t)} A_t(\Delta_k f)| \\
|\sup_t A_t f| &\le C f^* + \sup_{v\ge 0} \sup_{2^{-v} \le 2^{-v+1}} |\Sigma_{k\ge v} A_t(\Delta_k f)|
\end{aligned} \tag{12}
$$

By the general Minkowski inequality, we have

$$
\begin{aligned}
&\{\Sigma_{v\ge 0}[\Sigma_{0\ge s\ge\infty} \sup_{2^{-v}\le t\le 2^{-v+1}} |A_t(\Delta_{v+s}f)|]^p\}^{\frac{1}{p}} \\
&= \|\|\cdot\|_1\|_p =: \|\cdot\|_{1,p} \\
&\le \|\|\cdot\|_p\|_1 := \|\cdot\|_{p,1} \\
&\le \Sigma_{0\le s\le\infty}\{\Sigma_{v\ge 0}[\sup_{t\sim 2^{-v}} |A_t(\Delta_{v+s}f)|]^p\}^{\frac{1}{p}}
\end{aligned} \tag{13}
$$

And it's obvious that for the term $\|\cdot\|_{p,1}$, by the general Minkowski inequality, its L^p norm can be dominated by

$$
\|\cdot\|_{p,1,p} \le \|\cdot\|_{p,p,1} := \Sigma_{0\le s\le\infty}\{\Sigma_{v\ge 0}\|\sup_{t\sim 2^{-v}} |A_t(\Delta_{v+s}f)|\|_p^p\}^{\frac{1}{p}} \tag{14}
$$

Hence we only need to prove

$$
\|\sup_{t\sim 2^{-v}} |A_t(g)|\|_p \le C 2^{-\alpha(p)s}\|g\|_p, \quad provided\ E_{v+s}[g] = 0 \tag{15}
$$

since if this equation holds, there will be

$$
\begin{aligned}
&\Sigma_{0\le s\le\infty}\{\Sigma_{v\ge 0}\|\sup_{t\sim 2^{-v}} |A_t(\Delta_{v+s}f)|\|_p^p\}^{\frac{1}{p}} \\
&\le \Sigma_{0\le s\le\infty} 2^{-\alpha(p)s}\{\Sigma_{v\ge 0}\|\Delta_{v+s}f\|_p^p\}^{\frac{1}{p}} \\
&\le \Sigma_{0\le s\le\infty} 2^{-\alpha(p)s}\{\|\Sigma_{v\ge 0}\Delta_{v+s}f\|_p^p\}^{\frac{1}{p}} \\
&\le \Sigma_{0\le s\le\infty} 2^{-\alpha(p)s}\{\|E_s[f]\|_p^p\}^{\frac{1}{p}}, \quad by\ Minkowski\ Inequality \\
&\le \Sigma_{0\le s\le\infty} 2^{-\alpha(p)s}\|f\|_p, \quad by\ Hölder\ Inequality \\
&\le C\|f\|_p
\end{aligned} \tag{16}
$$

By scaling method, it's enough to assume $v = 0$, since the rescaling transform $f_\lambda(x) := \lambda f(\lambda x)$ preserves L^p-norm.
Now we aim to prove the following lemma:

Lemma 3. *Provided $E_{v+s}[g] = 0$, we have the following estimate*

$$
\|\sup_{t\sim 2^{-v}} |A_t(g)|\|_p \le C 2^{-\alpha(p)s}\|g\|_p \tag{17}
$$

To get better estimate, we want to reduce the singularity of $\sigma(x)$ in the following way:

Definition 4. *We consider the following decomposition of arc length measure σ of Γ:*

$$
\sigma = \sigma_0 + \Sigma_{k=1}^{\infty} 2^{k-1}\sigma_k \tag{18}
$$

where

$$\sigma_0 := \chi_{\{1\le\|y\|\le 2\}} \tag{19}$$

and

$$\sigma_k := \chi_{\{1\le\|y\|1+2^{-k}\}} - \chi_{\{1+2^{-k}\le\|y\|\le 1+2^{-k+1}\}} \tag{20}$$

Obviously, we have the equation

$$|A_t f(x)| \le |\int f(x+ty)\sigma_0(y)dy| + \Sigma_{k=1}^{\infty} 2^{k-1}|\int f(x+ty)\sigma_k(y)dy| \tag{21}$$

So our goal now is to prove

$$\|\sup_{1\le t\le 2}|\int f(x+ty)\sigma_k(y)dy|\|_p \le C2^{-k(1+\alpha)}\|f\|_p \tag{22}$$

We omit the detailed reason since it's quite clear in Bourgain's original work.
Let k be fixed and denote $n = 2^k$, then we have the following definition:

Definition 5. *Consider a radius function $t(x)$ ranging in $[1, 2]$ and satisfying at the point x*

$$\sup_{1<t<2}|\int f(x+ty)\sigma_k(y)dy| = |\int f(x+t(x)y)\sigma_k(y)dy| \tag{23}$$

and we use the following notation:

$$\tilde{V}_x = \sigma_k(\frac{y-x}{t(x)}), \quad and\ V_x = |\tilde{V}_x| \tag{24}$$

Then we have

$$\sup_{1<t<2}|\int f(x+ty)\sigma_k(y)dy| = |\int f(x+t(x)y)\sigma_k(y)dy| = |\int f(y)\tilde{V}_x(y)dy| \tag{25}$$

Choose test functions g as

$$(g) \subset \{|x| \le c\},\ \|g\|_q = 1,\ q = \frac{p}{p-1} \tag{26}$$

since bounded L^p-functions are dense in L^p.
Then we can use dualization and interchanging integration order to show that the sufficient goal will be to prove the following inequality (for the detail please see the original paper):

$$\|\int g(x)\tilde{V}_x dx\|_{L^q([0,1]^2)} \le C2^{-(1+\alpha)k}\|g\|_{L^q([0,1]^2)} \tag{27}$$

Note that we can choose the $L^q([0, 1]^2)$-norm because we can also use the norm-preserving rescaling here. To prove this inequality, we have the following lemma:

Lemma 6. *To prove $\|\int g(x)\tilde{V}_x dx\|_q \le C2^{-(1+\alpha)k}\|g\|_q$, it suffices to consider a region $\Omega \subset \{|x| \le c\}$ and estimate ($n = 2^k$)*

$$\|\int_\Omega \tilde{V}_x(y)dx\|_q \le n^{-1-\alpha}|\Omega|^{\frac{1}{q}} \tag{28}$$

Proof. By $H\ddot{o}lder$ inequality, if the equation holds, we have

$$\begin{aligned}
&\|\int_\Omega g(x)\tilde{V}_x(y)dx\|_q \\
=&\|\|g(x)\tilde{V}_x(y)\|_{L^1(x)}\|_q \\
\leq&\|\|g(x)\|_q\|\tilde{V}_x(y)\|_p\|_q \\
\leq&\|g\|_q\|\int_\Omega \tilde{V}_x(y)dx\|_q \\
\leq&\|g\|_q C2^{-(1+\alpha)k}|\Omega|^{\frac{1}{q}}
\end{aligned} \tag{29}$$

□

To prove this inequality, we first use $H\ddot{o}lder$ to dominate the term with L^1-norm and L^2-norm:

$$\begin{aligned}
&\|\int_\Omega \tilde{V}_x(y)\|_q \leq \\
=&\|(\int_\Omega \tilde{V}_x(y))^{1-\theta}(\int_\Omega \tilde{V}_x(y))^{\theta}\|_q \\
\leq&\|(\int_\Omega \tilde{V}_x(y))^{1-\theta}\|_{\frac{1}{1-\theta}}\|(\int_\Omega \tilde{V}_x(y))^{\theta}\|_{\frac{2}{\theta}} \leq \\
=&\|\int_\Omega \tilde{V}_x(y)\|_1^{1-\theta}\|\int_\Omega \tilde{V}_x(y)\|_2^{\theta}
\end{aligned} \tag{30}$$

hence we have

$$\|\int_\Omega \tilde{V}_x(y)\|_q \leq \|\int_\Omega \tilde{V}_x(y)\|_1^{1-\theta}\|\int_\Omega \tilde{V}_x(y)\|_2^{\theta} \tag{31}$$

With details in Bourgain's paper, we notice that

$$\begin{cases} \|\int_\Omega \tilde{V}_x(y)\|_1 \ll \frac{1}{n}|\Omega| \\ \|\int_\Omega \tilde{V}_x(y)\|_2 \ll \frac{logn}{n}|\Omega|^{\frac{1}{2}} \end{cases} \tag{32}$$

so to achieve the estimate

$$\|\int_\Omega \tilde{V}_x(y)dx\|_q \leq n^{-1-\alpha}|\Omega|^{\frac{1}{q}} \tag{33}$$

we want to split $\Omega = \Omega_0 \cup \Omega_1$, where

$$\Omega_0 = \bigcup_{s=1}^{S} \Omega_{0,s} \tag{34}$$

with the following properties to compensate the $n^{-\alpha}$-term lacked in the previous estimations:

$$\begin{cases} \|\int_{\Omega_{0,s}} \tilde{V}_x(y)\|_1 \ll n^{-1-\alpha}|\Omega_{0,s}| \\ \|\int_{\Omega_1} \tilde{V}_x(y)\|_2 \ll n^{-1-\alpha}|\Omega|^{\frac{1}{2}} \end{cases} \tag{35}$$

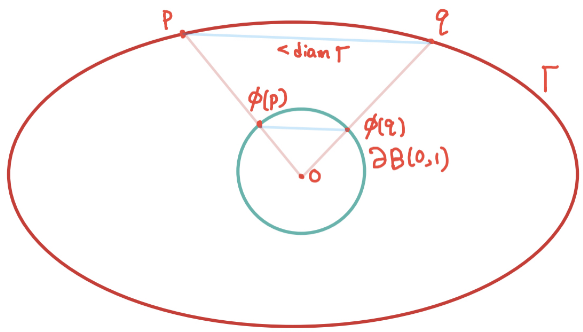

Figure 1: This is about how ϕ acts. When $|\phi(p) - \phi(q)|$ is too small, we can use the condition of bounded curvatures of Γ, and combined with the boundedness of $\|p-q\|$, we can get the desired equation.

4 Geometric Estimates

Lemma 7. *Let $\|\cdot\|$ be the norm induced by the convex smooth curve Γ on $\mathbb{R}^2$. Suppose $\|p\| \sim 1 \sim \|q\|$ and $\|p-q\| < c\|p\|$ and $Angle(p,q) \in [0, \pi - \varepsilon]$ with $\varepsilon > 0$. Then we have that*

$$dist(q, \mathbb{R}p) \sim (\|p-q\| - |\|p\| - \|q\||)^{\frac{1}{2}} \|p-q\|^{\frac{1}{2}} \tag{36}$$

Remark 8. *It should be noticed that the original formulation of this lemma in Bourgain's paper is wrong, since we can choose $p = -q$ then $dist(q, \mathbb{R}p) = 0$ with the right term larger than 0. However this does not affect the whole proof, since the assumption $0 \le Angle(p,q) \le \pi - \varepsilon$ is always satisfied in the later proof.*

Proof. We consider the following smooth enough map

$$\phi(x) = \frac{x}{\frac{|x|}{\|x\|}} = \frac{x\|x\|}{|x|} \tag{37}$$

Notice that this map is a angle-preserving and isometric map from $(\mathbb{R}^2, \|\cdot\|)$ to $(\mathbb{R}^2, |\cdot|)$, mapping Γ to $\partial B(0,1)$, so that

$$|sinAngle(p-q,q)| = |sinAngle(\phi(p-q), \phi(q))| \tag{38}$$

and

$$\|\frac{q}{\|q\|} - \frac{p-q}{\|p-q\|}\| \sim |\phi(\frac{q}{\|q\|}) - \phi(\frac{p-q}{\|p-q\|})| \tag{39}$$

for the reason please see (1).

Hence to prove

$$|sinAngle(p-q,q)| \sim \|\frac{q}{\|q\|} - \frac{p-q}{\|p-q\|}\| \tag{40}$$

we only need to prove, for $|x| = |y| = 1$ under the angle condition, there is

$$|sinAngle(x,y)| \ll |x-y| \tag{41}$$

which is clear since

$$
\begin{aligned}
|x-y| = |2sin(\frac{Angle(x,y)}{2})| \\
|2sin(\frac{Angle(x,y)}{2})cos(\frac{Angle(x,y)}{2})| < |2sin(\frac{sin(Angle(x,y))}{2})| \\
|sin(Angle(x,y))| \ll |2sin(\frac{Angle(x,y)}{2})| \\
|sinAngle(x,y)| \ll |x-y|
\end{aligned}
\tag{42}
$$

To prove $|sinAngle(x,y)| \gg |x-y|$, notice that:

$$
|x-y| \ll Angle(x,y) \ll sinAngle(x,y) \quad , \quad 0 \le Angle(x,y) < \pi - \varepsilon,\ \varepsilon > 0 \tag{43}
$$

Hence we have proved $|sin(Angle(p-q,q))| \sim \|\frac{q}{\|q\|} - \frac{p-q}{\|p-q\|}\|$, and now we only need to prove

$$
\|\frac{q}{\|q\|} - \frac{p-q}{\|p-q\|}\| \sim (\frac{\delta}{\|p-q\|})^{\frac{1}{2}} \tag{44}
$$

where $\delta := \|q\| + \|p-q\| - \|p\|$. To prove this, notice that we have

$$
\frac{\delta}{\|p-q\|} = \frac{\|p-q\| + \|q\| - \|p\|}{\|p-q\|} \tag{45}
$$

so we only need to prove

$$
(\|\frac{q}{\|q\|} - \frac{p-q}{\|p-q\|}\|)^2 \sim \frac{\|p-q\| + \|q\| - \|p\|}{\|p-q\|} \tag{46}
$$

We want to prove that $\|q'\| \Rightarrow \|\frac{p'+q'}{2}\| \ge 1 - c'\|p'-q'\|^2$ under assumption $0 \le Angle(p',q') \ge \pi - \varepsilon$. The proof is obvious since we can set c' large enough, say c^* to make the equation holds when $Angle(p',q') < \varepsilon$, then we set $c(\theta, q') := \frac{1-\|\frac{p'+q'}{2}\|}{\|p'-q'\|^2}$ with $Angle(p',q') = \theta$ and $\|p'\| = \|q'\| = 1$. Notice that the function is continuous and non zero with respect to θ on the two dimensional compact space $\theta \in [\varepsilon, \pi-\varepsilon] \times \Gamma$, so we make $c' = \max\{c^*, \sup_{\theta \in [\varepsilon,\pi-\varepsilon], q' \in \Gamma} c(\theta, q') \neq 0\}$. Now we have proved $\|q'\| \Rightarrow \|\frac{p'+q'}{2}\| \le 1 - c'\|p'-q'\|^2$.
Notice that we could use the similar compactness method t prove another statement:

$$
\|p'\| = 1 = \|q'\| \Rightarrow \|\frac{p'+q'}{2}\| \le 1 - c\|p'-q'\|^2 \tag{47}
$$

Thus we have

$$
(\|\frac{q}{\|q\|} - \frac{p-q}{\|p-q\|}\|)^2 \sim 1 - \|\frac{\frac{q}{\|q\|} + \frac{p-q}{\|p-q\|}}{2}\| \tag{48}
$$

which means we need to prove

$$
1 - \|\frac{\frac{q}{\|q\|} + \frac{p-q}{\|p-q\|}}{2}\| \sim \frac{\|p-q\| + \|q\| - \|p\|}{\|p-q\|} \tag{49}
$$

By assumption $0 \leq Angle(p,q) \leq \pi - \varepsilon$ and $\|p\| > \|q\|$, to prove the equation above we need to construct a somewhat subtle compact space. Notice that by assumption we have

$$\|p\| \sim \|q\| \sim \|p-q\| \sim 1 \tag{50}$$

We assume for $p, q, p-q$, there is $\alpha \leq \|\cdot\| \leq \beta$, then we consider the compact space $(\|p\|, \|q\|, \|p-q\|) \in [\alpha, \beta]^3$, and given the norms of them, the possible angles between them and the assumption $0 \leq Angle(p,q) \leq \pi - \varepsilon$ also form a compact space. So given their norms, we consider the function $\frac{\frac{\|p-q\|+\|q\|-\|p\|}{\|p-q\|}}{1-\|\frac{\frac{q}{\|q\|}+\frac{p-q}{\|p-q\|}}{2}\|}$, since it's continuous everywhere and differentiable with nonzero derivatives (being differentiable comes from the enough smoothness of Γ), we can give its upper and lower bound, and by uniform continuity we have that the bounds change continuously on the compact space $[\alpha, \beta]^3$, we then have the total upper and lower bounds.
Now we have

$$1 - \|\frac{\frac{q}{\|q\|} + \frac{p-q}{\|p-q\|}}{2}\| \sim \frac{\|p-q\| + \|q\| - \|p\|}{\|p-q\|} \tag{51}$$

Finally there is

$$dist(q, \mathbb{R}p) \sim \|p-q\||sinAngle(p-q,q)| \sim \delta^{\frac{1}{2}}\|p-q\|^{\frac{1}{2}}. \tag{52}$$

Proof completed. □

Assumptions below are assumed in the proof of the following lemma.
(1)We assume that

$$|x-y| < c < \frac{1}{2}(r_x + r_y) \tag{53}$$

(2)We assume that

$$\|x-y\| \sim |x-y| \sim |a| + |1-b| < 1/2 \tag{54}$$

Please note that the only added assumption is $|a| + |1-b| < \frac{1}{2}$.
(3)Assume certain conditions satisfying (7), we then have that

$$|a| \sim dist(y, \mathbb{R}x) \sim (\|x-y\| - |r_x - r_y|)^{\frac{1}{2}} \|x-y\|^{\frac{1}{2}} \tag{55}$$

(4)Curvature hypothesis
(5)We assume that

$$|x-y| \sim |a| + |1-b| \gg n^{-1} \tag{56}$$

(6)(i)We assume that

$$\|x-y\| - |r_x - r_y| \gg n^{-1} \tag{57}$$

or equivalently by (3), we assume that

$$na^2 \gg |x-y| \tag{58}$$

(ii)If (i) does not hold, i.e we assume

$$na^2 < C|x-y| \tag{59}$$

(7)We may suppose

$$|a| < |1-b| \tag{60}$$

(8)We assume that

$$n(\|x-y\| - |r_x - r_y|) < C \tag{61}$$

Remark 9. *We can prove $\|x-y\| - |r_x - r_y| \gg n^{-1}$ iff $na^2 \gg |x-y|$ by the following calculation*

$$
\begin{aligned}
(\|x-y\| - |r_x - r_y|)^{\frac{1}{2}} \|x-y\|^{\frac{1}{2}} &\sim |a| \\
(\|x-y\| - |r_x - r_y|)^{\frac{1}{2}} &\sim \|x-y\|^{-\frac{1}{2}} |a| \\
\|x-y\|^{-\frac{1}{2}} |a| &\gg n^{-\frac{1}{2}}, \quad \textit{by } \|x-y\| - |r_x - r_y| \gg n^{-1} \\
na^2 &\gg \|x-y\|
\end{aligned}
\tag{62}
$$

Lemma 10. *Denote r_x the radius $t(x)$ introduced in previous sections when defining V_x. Then*
(1)The diameter of each of the component of $V_x \cap V_y$ (resp. of the unique component in case of coincidence) is

$$
diam(V_x \cap V_y) \le \frac{C}{n} \left(|x-y| + \frac{1}{n} \right)^{-\frac{1}{2}} \left(|\|x-y\| - |r_x - r_y|| + \frac{1}{n} \right)^{-\frac{1}{2}} \tag{63}
$$

(2)

$$
|\langle \tilde{V_x}, \tilde{V_y} \rangle| \le C \frac{\langle V_x, V_y \rangle}{1 + n|\|x-y\| - |r_x - r_y||} \tag{64}
$$

Before going to the proof, we first give the following crucial lemma:

Lemma 11. *For $|t| < \frac{1}{n}$, let $p(t) := (\lambda(t), \mu(t))$ be the solution of the equations (here and only here $\phi(q) := \|q\|$):*

$$
\begin{cases}
\phi(-\lambda(t), 1-\mu(t)) - \phi(0,1) = t \\
\phi(a-\lambda(t), b-\mu(t)) - \phi(a,b) = 0
\end{cases}
\tag{65}
$$

belonging to the 0-component. Then

$$
\begin{aligned}
&diamcomp(V_x \cup V_y) \sim \int_{-\frac{1}{n}}^{\frac{1}{n}} (|\dot{\lambda(t)}| + |\dot{\mu(t)}|) dt \\
&\langle V_x, V_y \rangle \sim \frac{1}{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} (|\dot{\lambda(t)}| + |\dot{\mu(t)}|) dt \\
&|\langle \tilde{V_x}, \tilde{V_y} \rangle| \le \frac{C}{n^2} \int_{-\frac{1}{n}}^{\frac{1}{n}} (|\ddot{\lambda(t)}| + |\ddot{\mu(t)}|) dt
\end{aligned}
\tag{66}
$$

Notice $(\partial_x \phi(-\lambda, 1-\mu), \partial_y \phi(-\lambda, 1-\mu))$ and $(\partial_x \phi(a-\lambda, b-\mu), \partial_y \phi(a-\lambda, b-\lambda))$ give the normal directions at the respective points $(-\lambda, 1-\mu)$ and $(a-\lambda, b-\mu)$. Hence, by the curvature hyothesis

$$
\begin{vmatrix}
\partial_x \phi(-\lambda, 1-\mu) & \partial_y \phi(-\lambda, 1-\mu) \\
\partial_x \phi(a-\lambda, b-\mu) & \partial_y \phi(a-\lambda, b-\mu)
\end{vmatrix}
\sim
\begin{vmatrix}
-\lambda & 1-\mu \\
a-\lambda & b-\mu
\end{vmatrix}
= |(1-b)\lambda + a\mu - a| \tag{67}
$$

For a picture of this differential equation, please see (2). We omit the detail since after understanding its picture, the proof is regular calculation.

Proof. *We prove (1) under assumption $na^2 < C|x-y|$:*
Since

$$
\|x-y\| - |r_x - r_y| < Cn^{-1} \tag{68}
$$

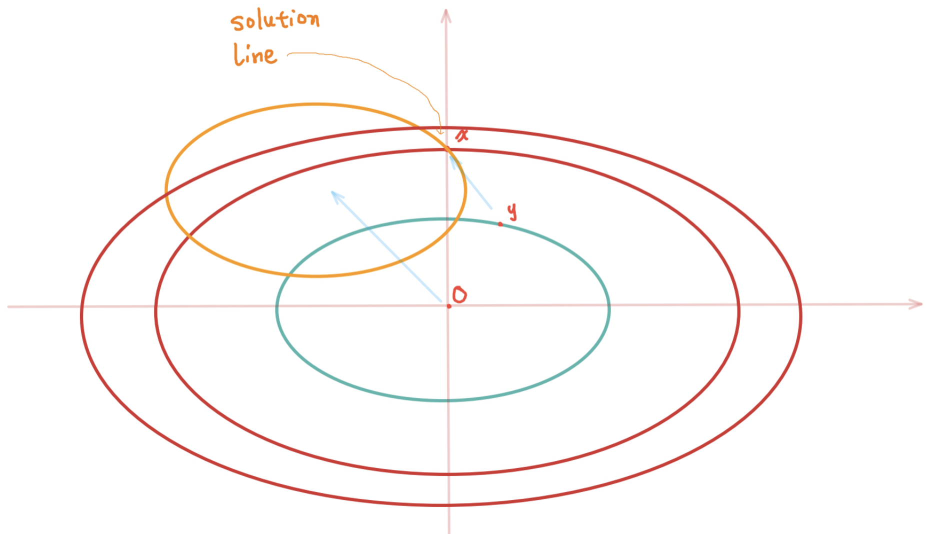

Figure 2: The small red circle is the rescaled curve$x \in \alpha\Gamma$, the blue circle is the rescaled curve $y \in \beta\Gamma$. Then second equation in the group means the solution curve should preserve the norm form y, so obviously it's of the shape of the curve $y \in \beta\Gamma$. And to ensure the solution being in the 0-component of $V_x \cap V_y$, we use the big red circle to restrain the solution curve. The equations in the lemma follow naturally.

we only need to prove

$$diam\ comp(V_x \cap V_y) < Cn^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}} \tag{69}$$

We assume $|a| < |1-b|$, then by curvature hypothesis we have

$$\begin{aligned} |\dot{\lambda}| + |\dot{\mu}| &\leq C|(1-b)\lambda + a\mu - a|^{-1} \\ |(1-b)\lambda + a\mu - a|(|\dot{\lambda}| + |\dot{\mu}|) &\leq C \\ |(1-b)\lambda + a\mu - a||\dot{\lambda}| &\leq C \\ |(1-b)\lambda||\dot{\lambda}| &\leq C + (1+|\mu|)|a||\dot{\lambda}| \\ |(1-b)||\lambda\dot{\lambda}| &\leq 2|a||\dot{\lambda}| + C, \quad by\ \mu \in [-1, \frac{2}{n}] \subset [-1,1] \end{aligned} \tag{70}$$

It should be noticed that the boundedness of μ comes from (2) since $\mu(t)$ in the solution curve is restrained in the 0-component, whose width is about $\frac{1}{n}$.
Hence we get

$$|(1-b)||\lambda\dot{\lambda}| \leq 2|a||\dot{\lambda}| + C \tag{71}$$

Since $\dot{\lambda^2} = 2\lambda\dot{\lambda}$, integrating $|t| < \frac{1}{n}$, we have

$$|1-b|\lambda(t)^2 \leq C(\frac{1}{n} + |a||\lambda(t)|) \tag{72}$$

thus

$$\begin{aligned} |1-b|\lambda(t)^2 &\le C\frac{1}{n} \\ |\lambda(t)| &\le Cn^{-\frac{1}{2}}|1-b|^{-\frac{1}{2}} \end{aligned} \tag{73}$$

or

$$\begin{aligned} |1-b|\lambda(t)^2 &\le C|a||\lambda(t)| \\ |\lambda(t)| &\le C|a||1-b|^{-1} \\ |\lambda(t)| &\le Cn^{-\frac{1}{2}}|x-y|^{\frac{1}{2}}|1-b|^{-1}, \quad by\ assumption\ na^2 < C|x-y| \\ |\lambda(t)| &\le Cn^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}, \quad by\ |a| < |1-b|\ hence\ |1-b| \sim |x-y| \end{aligned} \tag{74}$$

For the 'or' relation, see

$$\frac{1}{2}|1-b|\lambda(t)^2 \le C(\frac{1}{n} + |a||\lambda(t)|) \tag{75}$$

or

$$\frac{1}{2}|1-b|\lambda(t)^2 \le C|a||\lambda(t)| \tag{76}$$

By geometrical estimation we can show that $|\mu(t)| \gg |\lambda(t)|$. Hence we only need to prove

$$\sup_{|t|<\frac{1}{n}} |\lambda(t)| < Cn^{-\frac{1}{2}}(|x-y| + \frac{1}{n})^{-\frac{1}{2}} \tag{77}$$

and by $na^2 < C|x-y|$, we just need to prove

$$|\lambda(t)| \le C|a||1-b|^{-1} \le Cn^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}} \tag{78}$$

which has been proved since $Cn^{-\frac{1}{2}}|1-b|^{-\frac{1}{2}} \sim Cn^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}$. □

Proof. We prove (1) under assumption $na^2 \gg |x-y|$:
Since we have

$$|x-y| \sim |a| + |1-b| \gg n^{-1} \tag{79}$$

and

$$\|x-y\| - |r_x - r_y| \gg n^{-1} \tag{80}$$

we only need to prove

$$diam(V_x \cap V_y) \le \frac{C}{n}|x-y|^{-\frac{1}{2}} \left(|\|x-y\| - |r_x - r_y||\right)^{-\frac{1}{2}} \tag{81}$$

and since

$$|a| \sim dist(y, \mathbb{R}x) \sim (\|x-y\| - |r_x - r_y|)^{\frac{1}{2}} \|x-y\|^{\frac{1}{2}} \tag{82}$$

we only need to prove that

$$diam(V_x \cap V_y) \le \frac{C}{n|a|} \tag{83}$$

which means we need to prove

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} (|\dot{\lambda}| + |\dot{\mu}|)dt \sim \frac{1}{na} \tag{84}$$

Since

$$|\dot{\lambda}| + |\dot{\mu}| \leq C|(1-b)\lambda + a\mu - a|^{-1} \tag{85}$$

we only need to prove that

$$|a - (1-b)\lambda - a\mu| \sim |a| \tag{86}$$

we have $|\dot{\lambda}| < C|a|^{-1}$, hence $|\lambda| < \frac{C}{na}$ for $|t| < \frac{1}{n}$.
Since by assumption $na^2 \gg |x-y|$, we have

$$\begin{aligned}
|1-b| &\ll n|a|^2, \quad by\ |1-b| < |x-y| \\
n^{-1}|a|^{-1} &\ll |a||1-b|^{-1} \\
|\lambda| &< C|a||1-b|^{-1} \\
|\lambda| &< \frac{1}{10}|a||1-b|^{-1}, \quad by\ choosing\ appropriate\ C
\end{aligned} \tag{87}$$

Hence we have

$$\begin{gathered}
|a||1-\mu| - |1-b||\lambda| < |a(1-\mu) - (1-b)\lambda| < |a||1-\mu| + |1-b||\lambda| \\
|a||1-\mu| - |1-b|\frac{1}{10}|1-b|^{-1}|a| < |a(1-\mu) - (1-b)\lambda| < |a||1-\mu| + |1-b|\frac{1}{10}|1-b|^{-1}|a| \\
|a(1-\mu) - (1-b)\lambda| \sim |a|
\end{gathered} \tag{88}$$

hence we have

$$\begin{gathered}
|\dot{\lambda}| + |\dot{\mu}| \sim |a|^{-1} \\
\int_{-\frac{1}{n}}^{\frac{1}{n}} (|\dot{\lambda}| + |\dot{\mu}|)dt \sim \frac{1}{na} \\
\int_{-\frac{1}{n}}^{\frac{1}{n}} (|\dot{\lambda}| + |\dot{\mu}|)dt \sim diam(V_x \cap V_y) \leq \frac{C}{n}|x-y|^{-\frac{1}{2}} \left(|\|x-y\| - |r_x - r_y||\right)^{-\frac{1}{2}} \sim \frac{1}{na}
\end{gathered} \tag{89}$$

□

For the proof of Lemma 4.2.(2), it can be analogously derived, and we omit this since it's clear in the original paper.

5 End of the Proof

Definition 12. *For $x \in \Omega$, define the set Ω_x as follows:*

$$\Omega_x := \{y \in \Omega : |\|x-y\| - |r_x - r_y|| < n^{-1+\varepsilon}\} \tag{90}$$

And we have the following inductively proved lemma:

Lemma 13. *We can obtain points $x_1, x_2, ..., x_J \in \Omega$ and a complementary set $\Omega_1 \subset \Omega$ such that for any $1 \le i \le J$, the following inequality holds:*

$$|\Omega_{x_i} \setminus \bigcup_{j<i} \Omega_{x_j}| > n^{-1+\delta} \tag{91}$$

and for $\Omega_1 := \Omega \setminus \bigcup_{i \le J} \Omega_{x_i}$ there is that for any $x \in \Omega$:

$$|\Omega_1 \cap \Omega_x| \le n^{-1+\delta} \tag{92}$$

Proof. We choose Ω_{x_i} inductively. Since $|\Omega| = 1$, we have that $J \le n^{1-\delta}$, which means the induction is finite. When the induction stop, if there is a $x \in \Omega$ such that $|\Omega_1 \cap \Omega_x| \ge n^{-1+\delta}$, then we can choose x as a new x_i, which is a contradiction. □

For our convenience, we use the following notation:

Definition 14. *Define for $j = 1, 2, ..., J$,*

$$\Omega_1' := \Omega_{x_1}, \quad and \quad \Omega_j' := \Omega_{x_j} \setminus \bigcup_{i \le j-1} \Omega_{x_i} \tag{93}$$

and

$$\Omega_0 := \bigcup_{1 \le j \le J} \Omega_j' \tag{94}$$

Now our final goal is to give the following estimates:

Theorem 15. *The following estimates hold:*

$$\begin{cases} \|\int_{\Omega_j'} \tilde{V}_x(y)dx\|_1 < Cn^{-1-\varepsilon}|\Omega_j'| \\ \|\int_{\Omega_1} \tilde{V}_x(y)dx\|_2 < Cn^{-1-\varepsilon}|\Omega_1|^{\frac{1}{2}} \end{cases} \tag{95}$$

5.1 The L^2-Estimate on Ω_1

To prove this, we first give the following lemma:

Lemma 16. *There holds that*

$$\|\int_{\Omega_1} \tilde{V}_x(y)dx\|_2^2 \le C \int\int_{\Omega_1 \times \Omega_1} \frac{\langle V_x, V_y \rangle}{1 + n|\|x-y\| - |r_x - r_y||} dxdy \tag{96}$$

Proof. We first define the dual function of $\tilde{V}_x(y)$ and the dual set in the following way:

$$\begin{cases} U_y(x) := \tilde{V}_x(y) \\ \Lambda_y = \Lambda_y^+ \cup \Lambda_y^- \\ \Lambda_y^+ := \{x \in \Omega : \tilde{V}_x(y) = 1\} \\ \Lambda_y^- := \{x \in \Omega : \tilde{V}_x(y) = -1\} \end{cases} \tag{97}$$

then we have the following calculation:

$$
\begin{aligned}
&\|\int_{\Omega_1} \tilde{V}_x(y)dx\|_2^2 \\
=&\int_{y\in\Omega_1}[\int_{x\in\Omega_1} \tilde{V}_x(y)dx]^2dy \\
=&\int_{y\in\Omega_1}[|\Lambda_y^+|-\Lambda_y^-]^2dy \\
=&\int_{y\in\Omega_1}(|\Lambda_y^+|^2+|\Lambda_y^-|^2-2|\Lambda_y^+||\Lambda_y^-|)dy
\end{aligned}
\tag{98}
$$

and

$$
\begin{aligned}
&\int_{\Omega_1^2}||\langle\tilde{V}_x,\tilde{V}_y\rangle|dxdy \\
=&\int_{\Omega_1^2}|\int\tilde{V}_x(z)\tilde{V}_y(z)dz|dxdy \\
=&\int_{\Omega_1^2}|\int(\mathbb{1}_{x\in\Lambda_z^+}-\mathbb{1}_{x\in\Lambda_z^-})(\mathbb{1}_{y\in\Lambda_z^+}-\mathbb{1}_{y\in\Lambda_z^-})dz|dxdy \\
\geq&\int_{\Omega_1^2}\int(\mathbb{1}_{x\in\Lambda_z^+}-\mathbb{1}_{x\in\Lambda_z^-})(\mathbb{1}_{y\in\Lambda_z^+}-\mathbb{1}_{y\in\Lambda_z^-})dzdxdy \\
=&\int_{\Omega_1^2}\int(\mathbb{1}_{x\in\Lambda_z^+}-\mathbb{1}_{x\in\Lambda_z^-})(\mathbb{1}_{y\in\Lambda_z^+}-\mathbb{1}_{y\in\Lambda_z^-})dxdydz \\
=&\int_{z\in\Omega_1}(|\Lambda_z^+|^2+|\Lambda_z^-|^2-2|\Lambda_z^+||\Lambda_z^-|)dz \\
=&\int_{y\in\Omega_1}(|\Lambda_y^+|^2+|\Lambda_y^-|^2-2|\Lambda_y^+||\Lambda_y^-|)dy
\end{aligned}
\tag{99}
$$

which means

$$
\begin{aligned}
&\|\int_{\Omega_1} \tilde{V}_x(y)dx\|_2^2 \\
\leq&\int_{y\in\Omega_1}(|\Lambda_y^+|^2+|\Lambda_y^-|^2-2|\Lambda_y^+||\Lambda_y^-|)dy \\
\leq&\int_{\Omega_1^2}||\langle\tilde{V}_x,\tilde{V}_y\rangle|dxdy \\
\leq& C\int\int_{\Omega_1\times\Omega_1}\frac{\langle V_x,V_y\rangle}{1+n|\|x-y\|-|r_x-r_y||}dxdy
\end{aligned}
\tag{100}
$$

□

We then introduce the following notation:

Definition 17. *We define the following set*

$$
\mathcal{D}:=\{(x,y)\in\Omega_1\times\Omega_1|y\in\Omega_x\}
\tag{101}
$$

we then have the following estimate

Lemma 18. *There holds that*

$$\int\int_{\Omega_1\times\Omega_1}\frac{\langle V_x,V_y\rangle}{1+n|\|x-y\|-|r_x-r_y||}dxdy\le n^{-\varepsilon}\int\int_{\Omega_1^2\setminus\mathcal{D}}\langle V_x,V_y\rangle dxdy+C\int\int_{\mathcal{D}}\frac{1}{n^{\frac{3}{2}}|x-y|^{\frac{1}{2}}}dxdy \tag{102}$$

Proof. We first decomposite Ω_1^2 using $\mathcal{D}$:

$$\begin{aligned}&\int\int_{\Omega_1\times\Omega_1}\frac{\langle V_x,V_y\rangle}{1+n|\|x-y\|-|r_x-r_y||}dxdy\\=&\int\int_{\Omega_1\times\Omega_1\setminus\mathcal{D}}\frac{\langle V_x,V_y\rangle}{1+n|\|x-y\|-|r_x-r_y||}dxdy+\int\int_{\mathcal{D}}\frac{\langle V_x,V_y\rangle}{1+n|\|x-y\|-|r_x-r_y||}dxdy\end{aligned} \tag{103}$$

For the first term, notice that there is by definition

$$|\|x-y\|-|r_x-r_y||\ge n^{-1+\varepsilon},\quad for\ (x,y)\in\Omega_1^2\setminus\mathcal{D} \tag{104}$$

so we get

$$\int\int_{\Omega_1\times\Omega_1\setminus\mathcal{D}}\frac{\langle V_x,V_y\rangle}{1+n|\|x-y\|-|r_x-r_y||}dxdy\le n^{-\varepsilon}\int\int_{\Omega_1^2\setminus\mathcal{D}}\langle V_x,V_y\rangle dxdy \tag{105}$$

And for the second term, we have that following calculation:

$$\begin{aligned}&\int\int_{\mathcal{D}}\frac{\langle V_x,V_y\rangle}{1+n|\|x-y\|-|r_x-r_y||}dxdy\\\le&\int\int_{\mathcal{D}}\frac{\frac{C}{n}\left(|x-y|+\frac{1}{n}\right)^{-\frac{1}{2}}\left(|\|x-y\|-|r_x-r_y||+\frac{1}{n}\right)^{-\frac{1}{2}}\frac{1}{n}(|x-y|+n^{-1+\varepsilon})}{1+n|\|x-y\|-|r_x-r_y||}dxdy\\\le&\int\int_{\mathcal{D}}C\frac{(|x-y|+n^{-1+\varepsilon})}{n^2(1+n|\|x-y\|-|r_x-r_y||)^{\frac{3}{2}}(n^{-1}+|x-y|)^{\frac{1}{2}}}dxdy\\\le&C\int\int_{\mathcal{D}}\frac{1}{n^{\frac{3}{2}}|x-y|^{\frac{1}{2}}}dxdy\end{aligned} \tag{106}$$

□

From the L^2-theory (see lemma 1 in Bourgain's paper), there is

$$n^{-\varepsilon}\|\int_{\Omega_1}V_xdx\|_2^2\le n^{-\varepsilon}\frac{|\Omega_1|}{n^2}(logn)^2 \tag{107}$$

It follows from the definitions of Ω_x and $\mathcal{D}$ that

$$|\mathcal{D}(x)|<n^{-1+\delta} \tag{108}$$

and then we prove the following lemma:

Lemma 19. *Given $a\in\mathbb{R}^2$ and $D\subset\mathbb{R}^2$, there is*

$$\int_D|a-y|^{-\frac{1}{2}}dy\le C|D|^{\frac{3}{4}} \tag{109}$$

Proof. We construct a two-dimensional ball D' centered at a with the same volume as D, which means $|D'| = |D|$. Obviously there is

$$\begin{aligned}&\int_D |a-y|^{-\frac{1}{2}}dy \\ \leq &\int_{D'} |a-y|^{-\frac{1}{2}}dy \\ \leq &C\int_0^{diam(D')} |y|^{\frac{1}{2}}dy \\ \leq &C\int_0^{|D'|^{\frac{1}{2}}} |y|^{\frac{1}{2}}dy \\ \leq &C(|D'|^{\frac{1}{2}})^{\frac{3}{2}} \\ \leq &C|D|^{\frac{3}{4}}\end{aligned} \tag{110}$$

□

hence there is

$$\begin{aligned}\int_{\mathcal{D}} \frac{1}{n^{\frac{3}{2}}|x-y|^{\frac{1}{2}}}dxdy \leq &Cn^{-\frac{3}{2}}\int_{\mathcal{D}(x)} |x-y|^{-\frac{1}{2}}dy \\ \leq &Cn^{-\frac{3}{2}}n^{-\frac{3}{4}+\frac{3}{4}\delta}|\Omega_1| \\ = &Cn^{-(\frac{9}{4})+\delta}|\Omega_1|\end{aligned} \tag{111}$$

Taking $\delta < \frac{1}{4}$, and the proof of the L^2-estimate follows.

5.2 The L^1-Estimate on Ω_0

We now aim to prove

$$\|\int_{\Omega'_j} \tilde{V}_x(y)dx\|_1 < Cn^{-1-\varepsilon}|\Omega'_j| \tag{112}$$

By hypothesis in the inductive construction, we have $|\Omega'_j| > n^{-1+\delta}$ and there is a point x_j with $\Omega'_j \subset \Omega_{x_j}$. For our convenience, we assume $x_j = 0$ and $r_{x_j} = 1$. Denoting $\Omega' := \Omega'_j$, we then have the property

$$||\|x\| - |r_x - 1|| < n^{-1+\varepsilon}, \quad for\ x \in \Omega' \tag{113}$$

Divide Ω' in the respective regions $[r_x \leq 1]$ and $[r_x > 1]$. The computations for both are analogous and we therefore only treat the second case, i.e.

$$|1 + \|x\| - r_x| < n^{-1+\varepsilon}, \quad for\ x \in \Omega' \tag{114}$$

And we can assume further $\|x\| \geq n^{-\frac{1}{2}}$ since we have the following lemma:

Lemma 20. *For regions of $\|x\| < n^{-\frac{1}{2}}$, there is*

$$\|\int_{\|x\|\leq n^{-\frac{1}{2}}} \tilde{V}_x(y)dx\|_1 \leq n^{-1-\delta}|\Omega'| \tag{115}$$

Proof. We have the following calculations:

$$\begin{aligned}\|\int_{\|x\|\le n^{-\frac{1}{2}}}\tilde{V}_x(y)dx\|_1 &\le \int_{y\in\Omega'}|\int_{\|x\|\le n^{-\frac{1}{2}}}\tilde{V}_x(y)dx|dy \\ &\le \int_{y\in\Omega'}\int_{\|x\|\le n^{-\frac{1}{2}}}|\tilde{V}_x(y)|dxdy \\ &\le C(1+n^{-\frac{1}{2}}+n^{-1+\varepsilon})^2|\Omega'| \\ &\le Cn^{-1-\delta}|\Omega'|\end{aligned} \tag{116}$$

□

Then we further decompose the region as

Definition 21. *For* $l = 0, 1, 2, ..., logn$*, define*

$$\Omega_l := \{x \in \Omega'; 2^l n^{-\frac{1}{2}} \le \|x\| \le 2^{l+1} n^{-\frac{1}{2}}\} \tag{117}$$

It will suffice to prove for fixed l and inequality

$$\|\int_{\Omega_l}\tilde{V}_x(y)dx\|_1 \le Cn^{-1-\varepsilon}|\Omega'| \tag{118}$$

The details left are quite clear in Bourgain's paper.