A brief literature review on probabilistic approach for infinite times renewal equation

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November 2022

Abstract Hawkes process seems to be a particular kind of infinite times renewal process. There are presently two directions in Probability field: one is to study the mean-field limit of Hawkes process, which leads to a history-dependent 1-time renewal equation; One is to study linear Hawkes process, and there has been several results on the long time convergence of this process. Seeing Hawkes process as a particular infinite times process, we may wonder: Is there a rigorous correspondence between Hawkes process and infinite times renewal equation? What is the PDE-expression of this probabilistic convergence

1 Microscopic approach of a time elapsed neural model [3]

This article constructed several weak solutions of certain PDEs related seperately to Wold process and Hawkes process.

We first summarize the relevant notations and definitions used in these literature.

- We use N to denote a general point process in $\mathbb{R}$, understood as a random set of points $N := \{T_i\}_{i\in\mathbb{Z}}$.
  - Seeing N as a random variable, we use $\mathcal{F}_t^N$ to denote the σ-algebra generated by the random variable(as a set) $N \cap (-\infty, t]$,
  - and use $\mathcal{F}_{t^-}^N$ to denote the σ-algebra $\bigcup_{t'<t} \mathcal{F}_{t'}^N$.
- We associate a counting process N_t to a point process N, defined as $t \mapsto N_t := \Big|N \cap (0,t]\Big|$ for $t \in \mathbb{R}_+$.
  - We define $S_{t^-} := t - T_{N_{t^-}}$ and $A_t^k := T_{[N_{t^-}-(k-1)]} - T_{[N_{t^-}-k]}$.
- Informally speaking, we define $\lambda(t, \mathcal{F}_{t^-}^N)$ as a random variable, which means the probability of a point process N_t to have a new point at t.
  - Formally speaking, we define it as $\mathbb{E}\big(N_t - N_{t^-}\big|\mathcal{F}_{t^-}^N\big)$, where $\mathbb{E}$ is understood as the conditional expectation (Radon-Nikodym derivative) on $\mathcal{F}_{t^-}^N$.

– The term $\mathcal{F}_{t^-}^N$ in $\lambda(t, \mathcal{F}_{t^-}^N)$ means that the renewal rate depends on the past.

– Intuitively, in the context of time elapsed model, it can be understood as the $p_N([s]_N)$.

We recognize that, give the past, $N \cap (-\infty, 0]$, of point process N, and its $\lambda(t, \mathcal{F}_{t^-}^N)$, the behaviour of N would be completely characterised. (To construct a point process, we can use Ogata's thining)

Definition 1. (Wold process) *We call a point process N as a Wold process, if for some $k \geq 0$*

$$\lambda(t, \mathcal{F}_{t^-}^N) = f(S_{t^-}, A_t^1, ..., A_t^k) \tag{1}$$

where f is a function. A Wold process can be understood as the microscopic model of finite times renewal equation.

It seems to me that we can generalize Wold process to a ∞-Wold process, defined as

$$\lambda(t, \mathcal{F}_{t^-}^N) = f(S_{t^-}, A_t^1, ..., A_t^k, ...) \tag{2}$$

Definition 2. (Hawkes Process) *We call a point process N as a Hawkes process, if*

$$\lambda(t, \mathcal{F}_{t^-}^N) = \mu + \int_{-\infty}^{t^-} h(t-u)N(du) \tag{3}$$

where $\mu \in \mathbb{R}$ and h is a function defined on $\mathbb{R}_+$. We see $N(du)$ as a summation of dirac measures $\sum_{i\in\mathbb{Z}} \delta_{T_i}(du)$.

It seems to me that, Hawkes process seems to be a certain sub-class of microscopic model for infinite times renewal equation, since given $\lambda(t, \mathcal{F}_{t^-}^N) = \mu + \int_{-\infty}^{t^-} h(t-u)N(du)$, we can define the firing rate for infinite times renewal equation as

$$p_\infty([s]_\infty) = \mu + \sum_{i=1}^{+\infty} h(s_i) \tag{4}$$

However, this may not be the direction that present probabilistic literature is taking. In the narrative of [3], the authors goes from PDE to stochastic process in this way: take the non-linear (PPS) equation (see [6]) as an example

$$\begin{cases} \partial_t n(s,t) + \partial_s n(s,t) + p(s, X(t))n(s,t) = 0 \\ m(t) := n(0,t) = \displaystyle\int_0^{+\infty} p(s, X(t))n(s,t)ds \\ X(t) = \displaystyle\int_0^t h(u)m(t-u)du \end{cases} \tag{5}$$

The term $X(t)$ poses a global-in-time effect, which seems to be the effect that probabilists want to describe, using Hawkes process to handle the history-dependency.

But there is an essential difference. In (PPS) equation, the effect comes from the history of total firing rate

$$X(t) = \int_0^t h(u)m(t-u)du = \mathbb{E}\Big(\int_0^{t^-} h(t-u)N(du)\Big) \tag{6}$$

while in Hawkes process, the effect comes from the history of the particle's own firing

$$\lambda(t, \mathcal{F}^N_{t^-}) = \mu + \int_{-\infty}^{t^-} h(t-u)N(du) \tag{7}$$

Many authors are trying to use 1-time elapsed model to describe the distribution of Hawkes process. But since the process essentially depends on the particle's own history (not only the history of total firing rate), the 1-time elapsed model derived from Hawkes process is highly technical. **BUT**, when we are considering multivariate Hawkes processes

$$\lambda^i_t = \sum_{j=1}^{n} \int_0^{t^-} h_{j\to i}(t-u)N^j(du) \tag{8}$$

In this case a particle's own history's effect goes to zero, and we will have a mean field limit. In this direction, there has been several works about propagation of chaos, mean field limit and Gaussian approximation of Poisson noise. Please see [2] and [1] for the direction of going from Hawkes process to history-dependent 1-times renewal equation.

Possible directions:

- Generalize Wold process to ∞-Wold process, and see if we can using this process to build a weak solution for infinite times renewal equation
- Since Hawkes process can be seen as the microscopic model for a particular class of infinite times renewal equation, can we use Hawkes process to study this class of equations? Would there be some special phenomenon in this class?

2 Renewal in Hawkes processes with self-excitation and inhibition[4]

This article proved theorems about long time convergence in different senses for a quite general class of Hawkes process.

Assumption 3. *We define*

$$L(h) := \sup\{t > 0 : \ |h(t)| > 0\} \tag{9}$$

The signed function $h : (0, +\infty) \to \mathbb{R}$ is such that

$$L(h) < +\infty, \qquad \|h^+\|_1 := \int_{(0,+\infty)} h^+(t)dt < 1 \tag{10}$$

The distribution $\mathfrak{m}$ of the initial condition $N^0 := N \cap (-\infty, 0]$ is such that

$$\mathbb{E}_{\mathfrak{m}}\Big(N^0 \cap (-L(h), 0]\Big) < +\infty \tag{11}$$

Definition 4.
(i)Fix an $A \geq L(h)$. We introduce the stopping time τ as

$$\tau := \inf\Big\{t > 0 : N \cap [t-A, t) \neq \varnothing, N \cap (t-A, t] = \varnothing\Big\} \tag{12}$$

(ii) We use $\mathcal{N}(\mathbb{R})$ to denote the space of counting measures on $\mathbb{R}$, which means it is of the form

$$\mathcal{N}(\mathbb{R}) \ni N(du) = \sum_{i \in \mathbb{Z}} \delta_{T_i}(du) \tag{13}$$

The space is endowed with the weak topology $\sigma(\mathcal{N}(\mathbb{R}), \mathcal{C}_{bs}(\mathbb{R}))$ and the corresponding Borel σ-field, where $\mathcal{C}_{bs}$ denotes the space of continuous functions with bounded support.
(iii) We use $\mathcal{B}_{lb}\Big(\mathcal{N}\big((-A,0]\big)\Big)$ as the space of real Borel functions on $\mathcal{N}\big((-A,0]\big)$ which are locally bounded, which means

$$\text{Uniformly bounded on } \Big\{\nu \in \mathcal{N}\big((-A,0]\big) : \nu \cap (-A,0] \leq n\Big\} \tag{14}$$

(iv) We define π_A as the measure on $\mathcal{N}\big((-A,0]\big)$

$$\pi_A f := \frac{1}{\mathbb{E}_\varnothing} \mathbb{E}_\varnothing\Big(\int_0^\tau f\big(N \cap (-A+t, t]\big) dt\big)\Big) \tag{15}$$

Theorem 5. *Let N^h be a Hawkes process with immigration rate $\lambda > 0$, reproduction function h, which means*

$$\lambda(t, \mathcal{F}_{t^-}^N) = \lambda + \int_{-\infty}^t h(t-u) N(du) \tag{16}$$

and initial condition N^0 with law $\mathfrak{m}$ satisfying Assumption 3. Then
(i) If $f \in \mathcal{B}_{lb}\Big(\mathcal{N}\big((-A,0]\big)\Big)$ is non-negative or π_A-integrable, then

$$\frac{1}{T} \int_0^T f\Big(N^h \cap (-A+t, t]\Big) dt \to \pi_A f, \qquad T \to +\infty,\ \mathbb{P}_{\mathfrak{m}} - a.s. \tag{17}$$

(ii) Based on (i), assuming certain further conditions for f, we have

$$\sqrt{T}\Big[\frac{1}{T} \int_0^T f\Big(N^h \cap (-A+t, t]\Big) dt - \pi_A f\Big] \to \mathcal{N}\big(0, \sigma^2(f)\big), \qquad T \to +\infty, \text{ in law} \tag{18}$$

(iii) Convergence to equilibrium for large times holds in the following sense:

$$\mathbb{P}_{\mathfrak{m}}\Big(N^h \cap [t, +\infty) \in \cdot\Big) \to_{\text{TV}} \mathbb{P}_{\pi_A}\Big(N^h \cap [0, +\infty)\Big), \qquad t \to +\infty \tag{19}$$

Remark 6. *Notice all the theorems proved above are under the bounded memory assumption*

$$L(h) := \sup\{t > 0 : \ |h(t)| > 0\} < +\infty \tag{20}$$

For similar results in the case of unbounded memory, please see [5].

Possible directions:

- Can we build an equivalence between the distribution evolution of Hawkes process, and the solution of infinite times renewal equation?
- For the long time convergence in the language of Probability, what would it be when presented in the form of PDE?

References

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