Math Preliminaries

1. Infimum and its properties regarding subsets
2. Limit point and infimum
3. Order Properties of a function
4. Right continuity of a function

mathematical preliminaries:

1. **infimum (**inf**):** Greatest lower bound

Example

1. $\inf\{1, 2, 3, \dots\} = 1$ (in the set)
2. $\inf\{x \in \mathbb{R} \mid 0 < x < 1\} = 0$ (not in the set)

Proposition: Suppose that A, B are subsets of $\mathbb{R}$ such that $A \subset B$. If $\inf A, \inf B$ both exist, then

$$\inf A \geq \inf B$$

mathematical preliminaries:

2. A point a is a *limit* (or accumulation or cluster) point of a set A if every $\epsilon-$ neighborhood $V_\epsilon(a)$ of a intersects the set A in some point other than a.

3. If $A \subset \mathbb{R}$, $a = \inf\{A\}, a \notin A$, then the infimum a is a limit point of A.

4. **Theorem** A point a is a limit point of a set A if and only if $a = \lim x_i$ for some sequence x_i contained in A such that $x_i \neq a$ for all $i \in \mathbb{N}$.

mathematical preliminaries: The above theorems and definitions together imply that for an **infimum**

There exists a sequence $\{x_i\}_{i=1}^{\infty} \in A$ with $\lim_{i \to \infty} x_i = x^0$ and $x_i > x_0$ for all i.

mathematical preliminaries:

1. **Order Properties of Functions:** Suppose that $f, g : A \to \mathbb{R}$ and x^0 is a limit point of A. If

$$f(x) \geq g(x) \quad \forall x \in A$$

and $\lim_{x \to x^0} f(x)$ and $\lim_{x \to x^0} g(x)$ exist, then

$$\lim_{x \to x^0} f(x) \geq \lim_{x \to x^0} g(x)$$

2. **Right Continuity** If $f : A \to \mathbb{R}$ is right continuous at a point x^0 if

$$\lim_{x \to (x^0)^+} f(x) = f(x^0)$$

mathematical preliminaries:

Order properties and **Right continuity** imply that if $f : A \to \mathbb{R}$ is right continuous at a point x^0 that is an limit point of A, and $f(x) \geq \tau = g(x) \quad \forall x \in A$, then

$$f(x^0) = \lim_{x \to x^0} f(x) \geq \tau \quad (= \lim_{x \to x^0} \tau)$$
$$f(x^0) \geq \tau$$