ECON 5213: ADVANCED ECONOMETRICS

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Problem Set #3

Part I: Analytical Questions

Note that many of the questions are long not because they are difficult, but simply because I want to clearly state some of the definitions and theorems. This way you do not have to worry that your answers may be affected by the definitions and theorems that you incorrectly copied down in class and . In some sense, these questions serve as my notes as well.

Question 1. [Moments of a Random Variable]. Show the following properties hold

(1) $Var(c) = 0$, where c is a constant.
(2) $Var(c + X) = Var(X)$.
(3) $Var(cX) = c^2 Var(X)$
(4) $Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

Question 2. [Odd Central Moments and Symmetry]. Let X be symmetric about its mean μ. In other words, $X - \mu$ and $\mu - X$ have the same distribution. Then for any odd number r, the r^{th} central moment, if it exists,

$$\mathbb{E}[(X - \mu)^m] = 0$$

Hint: Since $X - \mu$ and $\mu - X$ have the same distribution, every of their moments should be exactly equal to each other as well.

Question 3. [Moment Generating Functions]. Suppose that we have a random variable that can take on three different values, $\{1, 2, 3\}$, with equal probabilities $p_i = \frac{1}{3}$.

(1) Write down the moment generating function (MGF) for this variable.
(2) Derive the first two moments using MGF.
(3) Compare your results to those obtained using the definitions of mean and variance.

Question 4. [Quantile and Mean]. Show that the following relationship holds

$$\mathbb{E}[X] = \int_0^1 Q_X(u)du$$

Question 5. [Stochastic Dominance]. One of the important concepts in both micro theory, finance, and econometrics is stochastic dominance. The random variable $X_1$with CDF F_1 is said to first-order stochastically dominate the random variable X_2 if any of the following equivalent conditions holds:

(1) $F_1(x) \leq F_2(x)$, for all $x \in R$
(2) $F_1^{-1}(\tau) \geq F_2^{-1}(\tau)$ for all $\tau \in (0,1)$]

Show 1 implies 2 and then 2 implies 1.

Question 6. [Coninuity and Quantile Functions (Optional)]. Show that the following statements are equivalent

(1) $F(F^{-1}(\tau)) = \tau \quad \forall \tau \in (0,1)$
(2) F is continuous
(3) $F^{-1}(\tau)$ is strictly increasing.

Clearly define a continuous function first, and the state its properties. Reference to any math textbooks for necesary theorems.

Question 7. [Symmetric Distribution (Optional)].

Theorem. *Let X be distributed with f_x (the density function), a symmetric distribution around μ. Another way to express this definition is as follows,*

$$f_x(\mu + u) = f_x(\mu - u)$$

(1) $f(x) = f(2\mu - x)$
(2) $\mathbb{E}[X] = \mu$
(3) $q_{0.5} = \mu$ ($q_{0.5}$ is the 50^{th} percentile)
(4) $\mu - q_\tau = q_{1-\tau} - \mu$ (in other words, quantiles are symmetric around μ).

Show **the first result** is indeed true.