ECON 5213: ADVANCED ECONOMETRICS

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Problem Set #5

Part I: Analytical Questions

There may appear to be many questions, but as you will see, the answers to each question is relatively short. I also provide you with many hints to get you started. Just try your best.

Question 1. [From Joint to Marginal Distribution]. Given that $F(x,y) = \Pr[X \leq x, Y \leq y] = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)dudv$, show that

$$f(x) = \frac{d}{dx}F(x) = \int_{-\infty}^{\infty} f(x,v)dv$$

Hint: As discussed in class, write down $F(x)$ based on the joint CDF first and then take the derivative with respect to x. The answer would be a straightforward application of **Leibniz Integral Rule.**

Question 2. [Properties of Covariance]. Show that the following properties hold:

(1) $Cov(aX + bY, cZ) = acCov(X,Z) + bcCov(Y,Z)$
(2) $Cov(X,Y) = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$
(3) Suppose $Y = X \cdot \beta + Z \cdot \gamma + \epsilon$, where β is the direct relationship between X and Y, γ is the direct relationship between Z and Y, and ϵ is some variable that we can never observe. Tthe observed relationship between X and Y, $Cov(X,Y)$ is equal to what? Interpret your result. Later this is nothing but the formula of omitted variable bias.

Question 3. [Zero Correlation not equal to Indepdence]. Let $y = x^2$, where x is symmetrically distributed around 0 (in other words, all the (central) odd-order moments such as $\mathbb{E}[x] = \mathbb{E}[x^3] = 0$, as we have shown before). Show that the correlation ρ between x and y is equal to 0, assuming the second moment of y exists (in other words, $\mathbb{E}[x^j]$ exists for $j \leq 4$; $\mathbb{E}[x]$,$\mathbb{E}[x^2]$,$\mathbb{E}[x^3]$,$\mathbb{E}[x^4]$ definitely exist). **Hint:** Show that $cov(x,y) = 0$.

Question 4. [Independence and expectation of nonlinear functions of variables]. Show that under independence ($x \perp y$) for continuous variables (x, y) and any nonlinear functions $h(\cdot), g(\cdot)$,

$$\mathbb{E}[h(x)g(y)] = \mathbb{E}[g(x)]\mathbb{E}[h(x)]$$

Question 5. [Independence]. Show that if $x \perp y$ (i.e., x is independent of y), then

$$F(x, y) = F(x) \cdot F(y)$$

Question 6. [Independence]. We know that the following joint distribution of X and Y are bivariate normal:

$$f(x,y) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)\right\}$$

Further, both X and Y are distributed from a normal distribution with different means (μ_1, μ_2) and variances (σ_1^2, σ_2^2). Under what condition(s), X and Y are independent? Show that your result is indeed true. Remember that the key is to show $f(x, y) = f(x) \cdot f(y)$.