$$E_{(f_{true}|f_{observed}=m|n)} = \frac{Beta(m+2, n-m+1)}{Beta(m+1, n-m+1)} = \frac{\frac{\Gamma(m+2)\Gamma(n-m+1)}{\Gamma(n+3)}}{\frac{\Gamma(m+1)\Gamma(n-m+1)}{\Gamma(n+2)}} = \frac{(m+1)!}{(m)!}\frac{(n+1)!}{(n+2)!} = \frac{m+1}{n+2}$$

Let $P_{true,\,obs}$ (x, y) be the probability density function of the true frequency (x) and observed frequency (y) in the rectangle space < x ∈ [0,1] , y ∈ [0,1] > .
Thus,

$$E(f_{true} \mid f_{obs} = m \mid n\ ) = E(x \mid_{\ y\ =\ m\,|\,n}\ ) = \int P_{true}(x \mid_{\ y\ =\ m\,|\,n}\ )\ x\ dx$$

$$= \int \frac{P_{obs}(m/n \mid_{true=x}\ ) * P_{true}(x)}{P_{obs}(m/n)} x\,dx \quad ①$$

$$= \int \frac{P_{obs}(m/n \mid_{true=x}\ )}{P_{obs}(m/n)} x\,dx \quad ②$$

$$= \int \frac{P_{obs}(m/n \mid_{f_{true}=x}\ )}{\int P_{true,obs}(x, m/n)\,dx} x\,dx = \frac{\int P_{obs}(m/n \mid_{true=x}\ )\ x\,dx}{\int P_{obs}(m/n \mid_{true=x}\ )\,dx}$$

$$= \frac{\int \begin{pmatrix} m \\ n \end{pmatrix} * x^{m} * (1-x)^{n}\, x\,dx}{\int \begin{pmatrix} m \\ n \end{pmatrix} * x^{m} * (1-x)^{n}\,dx} \quad ③$$

$$= \frac{\int \begin{pmatrix} m \\ n \end{pmatrix} * x^{m+1} * (1-x)^{n}\,dx}{\int \begin{pmatrix} m \\ n \end{pmatrix} * x^{m} * (1-x)^{n}\,dx} = \frac{\int x^{m+1} * (1-x)^{n-m}\,dx}{\int x^{m} * (1-x)^{n-m}\,dx}$$

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① : This is from the definition: $f_Y(y \mid X = x) f_X(x) = f_{X,Y}(x,y) = f_X(x \mid Y = y) f_Y(y).$

② : Without prior data and assumption on the distribution, $\int P_{true}(x)\,dx = \int_0^1 P_{true}(x)\,dx = 1$, thus $P_{true}(x)$ =1.

③ : This is by Binomial Distribution.

$$=\frac{\int x^{n-m}*(1-x)^{m+1}dx}{\int x^{n-m}*(1-x)^{m}dx} \quad ④$$

$$=\frac{\int x^{n-m}*\sum_{k\in(Z\cap[0,m+1])}\binom{m+1}{k}(-x)^{k}dx}{\int x^{n-m}*\sum_{k\in(Z\cap[0,m])}\binom{m}{k}(-x)^{k}dx} \quad ⑤$$

$$=\frac{\int x^{n-m}*\sum_{k\in(Z\cap[0,m+1])}\binom{m+1}{k}(-x)^{k}dx}{\int x^{n-m}*\sum_{k\in(Z\cap[0,m])}\binom{m}{k}(-x)^{k}dx}=\frac{\sum_{k\in(Z\cap[0,m+1])}(-1)^{k}\binom{m+1}{k}\int x^{n-m+k}dx}{\sum_{k\in(Z\cap[0,m])}(-1)^{k}\binom{m}{k}\int x^{n-m+k}dx}$$

$$=\frac{x^{n-m}*\sum_{k\in(Z\cap[0,m+1])}(-1)^{k}\binom{m+1}{m+1-k}\int x^{m+1-k}dx}{x^{n-m}*\sum_{k\in(Z\cap[0,m])}(-1)^{k}\binom{m}{m-k}\int x^{m-k}dx} \quad ⑥$$

$$=\frac{\sum_{k\in(Z\cap[0,m+1])}(-1)^{k}\binom{m+1}{m+1-k}\int x^{n-k+1}dx}{\sum_{k\in(Z\cap[0,m])}(-1)^{k}\binom{m}{m-k}\int x^{n-k}dx}=\frac{\sum_{k\in(Z\cap[0,m+1])}(-1)^{k}\binom{m+1}{m+1-k}\frac{1}{n-k+2}}{\sum_{k\in(Z\cap[0,m])}(-1)^{k}\binom{m}{m-k}\frac{1}{n-k+1}}$$

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④ :Replacing x with (1-v), v $\in$ [0,1], thus

$$\left.\frac{\int x^{m+1}*(1-x)^{n-m}dx}{\int x^{m}*(1-x)^{n-m}dx}\right|_{0}^{1}=\left.\frac{\int(1-v)^{m+1}*(v)^{n-m}d(1-v)}{\int(1-v)^{m}*(v)^{n-m}d(1-v)}\right|_{0}^{1}=\left.\frac{-\int(1-v)^{m+1}*(v)^{n-m}d(v)}{-\int(1-v)^{m}*(v)^{n-m}d(v)}\right|_{0}^{1}=\left.\frac{\int(1-v)^{m+1}*(v)^{n-m}d(v)}{\int(1-v)^{m}*(v)^{n-m}d(v)}\right|_{0}^{1}=\left.\frac{\int(1-x)^{m+1}*(x)^{n-m}d(x)}{\int(1-x)^{m}*(x)^{n-m}d(x)}\right|_{0}^{1}$$

⑤ : , $(1-x)^{m}=\sum_{k\in(Z\cap[0,m+1])}\binom{m+1}{k}(-x)^{k}$ Binomial theorem.

⑥ : Similar to ④, using m+1-k (for the numerator) and m-k (for the Denominator) to replace k.

$$=\frac{\sum_{k\in(Z\cap[0,m])}(-1)^k\begin{pmatrix}m+1\\m-k\end{pmatrix}\frac{1}{n-k+1}+\frac{1}{n+2}}{\sum_{k\in(Z\cap[0,m])}(-1)^k\begin{pmatrix}m\\m-k\end{pmatrix}\frac{1}{n-k+1}} \quad ⑦$$

$$=\frac{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m+1)!}{(m-k+1)!k!}\frac{1}{n-k+1}+\frac{1}{n+2}}{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k)!k!}\frac{1}{n-k+1}}=\frac{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k+1)!k!}\frac{m+1}{n-k+1}+\frac{1}{n+2}}{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k)!(k+1)!}\frac{k+1}{n-k+1}}$$

$$=\frac{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k+1)!k!}\frac{m+1}{n-k+1}+\frac{1}{n+2}}{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k)!(k+1)!}\frac{k+1}{n-k+1}+\sum_{k\in(Z\cap[0,m+1])}(-1)^k\frac{(m)!}{(m-k+1)!k!}} \quad ⑧$$

$$=\frac{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k+1)!k!}\frac{m+1}{n-k+1}+\frac{1}{n+2}}{\sum_{k\in(Z\cap[0,m])}\left((-1)^k\frac{(m)!}{(m-k+1)!k!}\left(\frac{k+1}{n-k+1}+1\right)\right)+\frac{1}{m+1}} \quad ⑨$$

$$=\frac{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k+1)!k!}\frac{m+1}{n-k+1}+\frac{1}{n+2}}{\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k+1)!k!}\left(\frac{n+2}{n-k+1}\right)+\frac{1}{m+1}}=\frac{\left(\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k+1)!k!}\frac{1}{n-k+1}+\frac{1}{(n+2)(m+1)}\right)(m+1)}{\left(\sum_{k\in(Z\cap[0,m])}(-1)^k\frac{(m)!}{(m-k+1)!k!}\left(\frac{1}{n-k+1}\right)+\frac{1}{(n+2)(m+1)}\right)(n+2)}$$

$$=\frac{m+1}{n+2}$$

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⑦: Similar to ⑥ and ④, replacing k with k+1.

⑧: $\sum_{k\in(Z\cap[0,m+1])}(-1)^k\frac{(m)!}{(m-k+1)!k!}=\frac{1}{m+1}\sum_{k\in(Z\cap[0,m+1])}(-1)^k\frac{(m+1)!}{(m-k+1)!k!}=\frac{1}{m+1}(1-1)^{m+1}=0$

⑨: Combining both two summations.