

# NUI projection recapture model

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To estimate the diversities among populations and to predict future sequencing effectiveness, we project our current data on bigger sample sizes base on a computational resampling model we inferred. We first expand our dataset by down-sampling to obtain sufficient points to generate distribution curves, as explained in the following subsection. Then, based on the resampling model and mathematical reasoning on the current dataset, we propose an recapturing model to simulate sequencing process, which is also explained in another subsection. Projection is performed by fitting the last 10 points of our expanded dataset on the model. The package can be found at url: [https://github.com/WalfredMA/NUI\\_Projection](https://github.com/WalfredMA/NUI_Projection).

## Down-sampling:

We down sample the current SVs based on the mean of all possible subsamples at certain lower sample size.

*Let  $S$  denotes the set of all sequenced samples. ( $|S| = s$ )*

*$X_i$  denotes the SV set the sample # $i$  has and  $\overline{X}_i$  denotes the SV set the sample # $i$  does not have.*

*$V_T^f$  denotes the number of SVs at frequency =  $f$  in sample set  $T$ .*

*$V_t^f$  denotes mean number of SVs at frequency =  $f$  when sample size =  $t$ . Thus,  $Mean(V_{T|T \subset S}^f) = V_t^f$ .*

*For any not intersect sample set pair  $M, N$  ( $|M| = m, |N| = n, M \cup N = T, m + n = |T| = t$ ).*

*By the defination,  $V_T^f = \sum_{N \subset T} \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M} \overline{X}_j \right|$*

Let  $k$  be additional sample  $\notin M \cup N$ ,

Let  $M'$  denotes  $M \cup \{k\}$ ,  $N'$  denotes  $N \cup \{k\}$ ,  $T'$  denotes  $T \cup \{k\}$

$$V_T^f = \sum_{N \subset T} \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M} \overline{X}_j \right| = \sum_{N \subset T} \left( \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M} \overline{X}_j \cap X_k \right| + \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M} \overline{X}_j \cap \overline{X}_k \right| \right)$$

$$\text{So, } \sum_{k \in T} V_T^f = \sum_{k \in T} \sum_{N \subset T} \left( \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M} \overline{X}_j \cap X_k \right| + \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M} \overline{X}_j \cap \overline{X}_k \right| \right)$$

Thus for all subsets whose module =  $t$  in set  $S$ .

$$\sum_{T \subset S} \sum_{k \in T} V_T^f = \sum_{k \in T} \sum_{T \subset S} \sum_{N \subset T} \left( \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M} \overline{X}_j \cap X_k \right| + \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M} \overline{X}_j \cap \overline{X}_k \right| \right)$$

$$= \sum_{T' \subset S} \sum_{N' \subset T'} \sum_{k \in N'} \left( \left| \bigcap_{i \in N'} X_i \cap \bigcap_{j \in M} \overline{X}_j \right| \right) + \sum_{T' \subset S} \sum_{M' \subset T'} \sum_{k \in M'} \left( \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M'} \overline{X}_j \right| \right)$$

$$= |N'| \sum_{T' \subset S} \sum_{N' \subset T'} \left( \left| \bigcap_{i \in N'} X_i \cap \bigcap_{j \in M} \overline{X}_j \right| \right) + |M'| \sum_{T' \subset S} \sum_{M' \subset T'} \left( \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M'} \overline{X}_j \right| \right)$$

$$\text{Considering } |T| = \binom{|S|}{|T|} = \binom{s}{t}, \quad \sum_{T \subset S} V_T^f = \binom{s}{t} * \text{Mean}(V_T^f) = \binom{s}{t} V_t^f$$

$$= (n+1) \binom{s}{t+1} \text{Mean} \left( \sum_{T' \subset S} \sum_{N' \subset T'} \left( \left| \bigcap_{i \in N'} X_i \cap \bigcap_{j \in M} \overline{X}_j \right| \right) \right) + (m+1) \binom{s}{t+1} \text{Mean} \left( \sum_{T' \subset S} \sum_{M' \subset T'} \left( \left| \bigcap_{i \in N} X_i \cap \bigcap_{j \in M'} \overline{X}_j \right| \right) \right)$$

$$= (n+1) \binom{s}{t+1} V_{t+1}^{n+1} + (t-n+1) \binom{s}{t+1} V_{t+1}^n$$

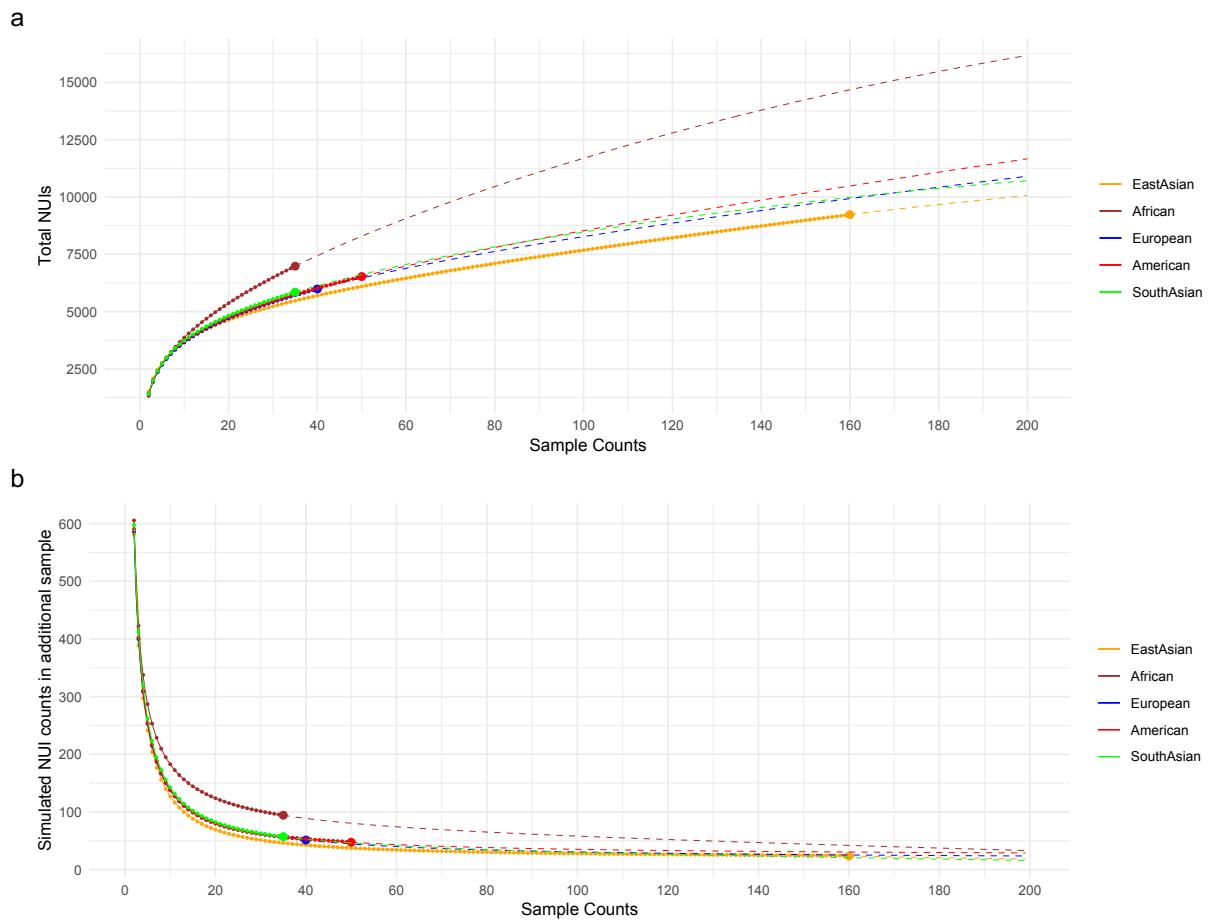
$$\text{Similarly, } \sum_{T \subset S} \sum_{k \in T} V_T^f = |\overline{T}| \binom{s}{t} * \text{Mean}(V_T^f) = (s-t) \binom{s}{t} V_t^n$$

$$\text{Hence, } (s-t) \binom{s}{t} V_t^n = (n+1) \binom{s}{t+1} V_{t+1}^{n+1} + (t-n+1) \binom{s}{t+1} V_{t+1}^n$$

$$\frac{s!}{t!(s-t-1)!} V_t^n = (n+1) \frac{s!}{(t+1)!(s-t-1)!} V_{t+1}^{n+1} + (t-n+1) \frac{s!}{(t+1)!(s-t-1)!} V_{t+1}^n$$

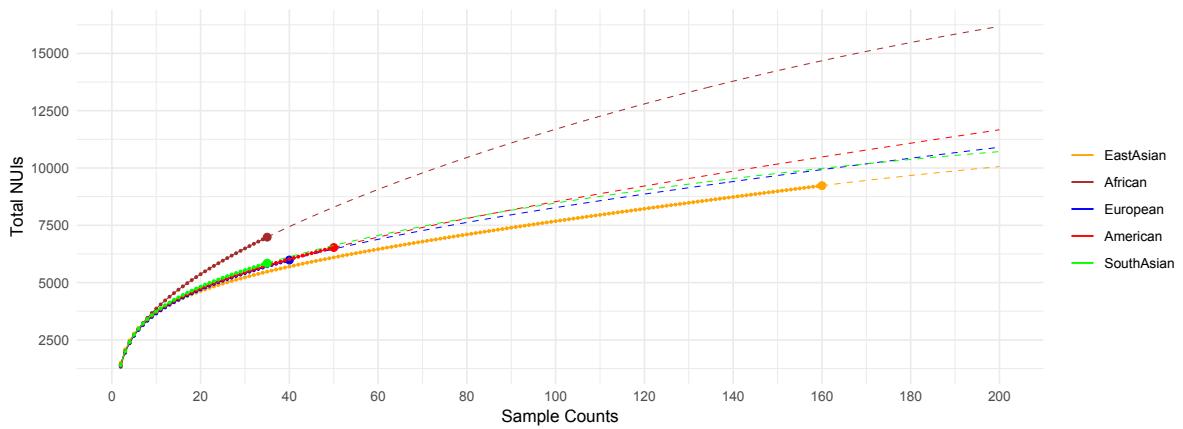
$$V_t^n = \frac{n+1}{t+1} V_{t+1}^{n+1} + \frac{t-n+1}{t+1} V_{t+1}^n$$

## Sample results

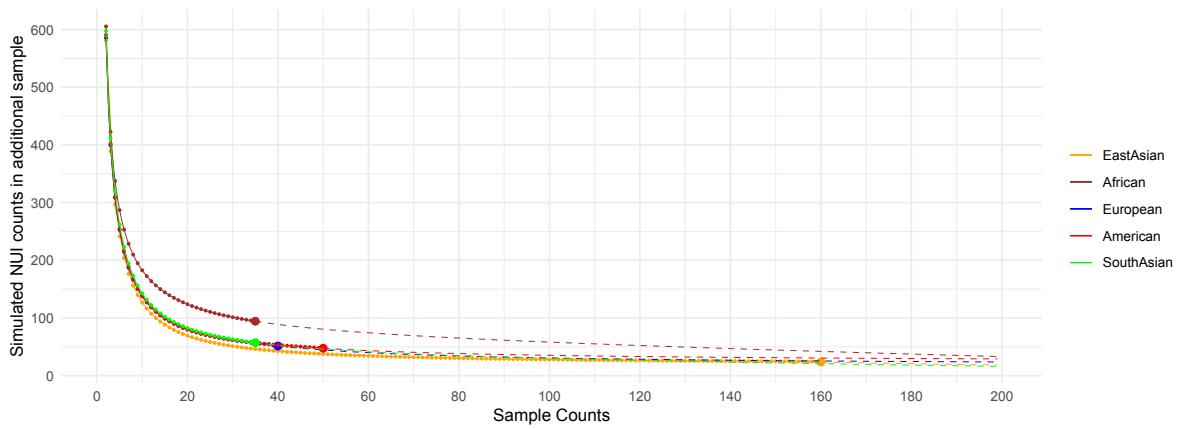


## Sample results

a

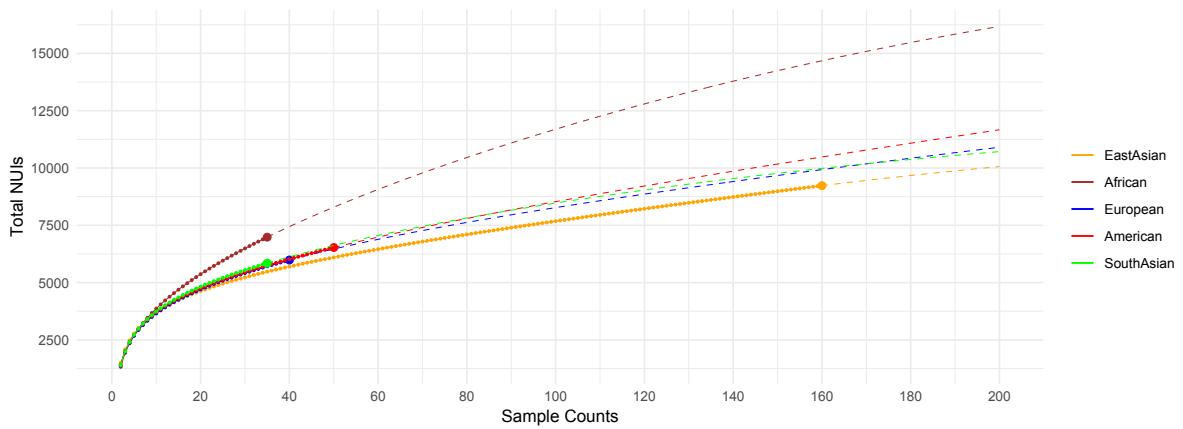


b

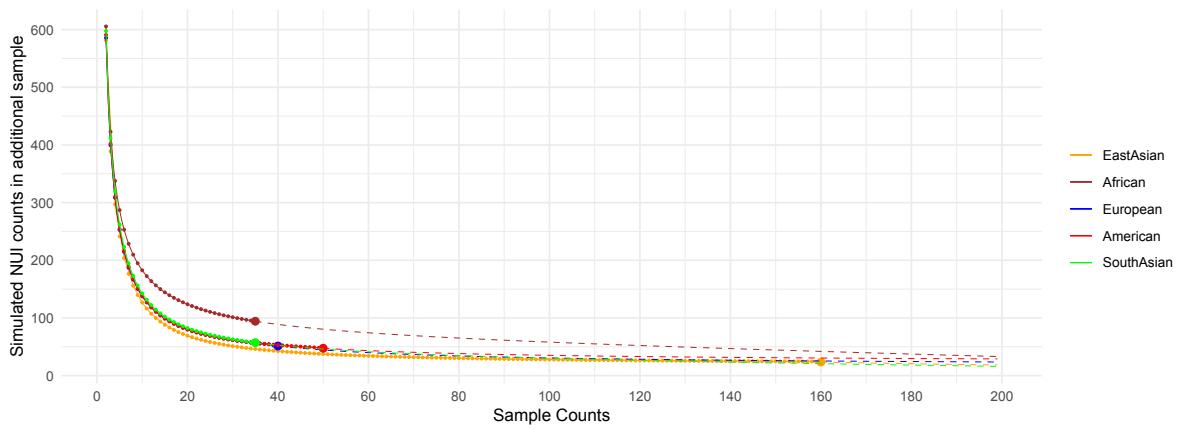


## Sample results

a



b



Let  $A_T^N$  ( $N \subseteq T$ ) represents the set of SVs which are exclusive in set  $N$ , but not in any other sample.

$A_t^n$  represents the mean of set  $\{A_T^N \mid N \subseteq T \text{ and } |N| = n\}$ .

Apprely,  $A_T^{N1} \cap A_T^{N2} = \emptyset$  and  $V_T^n = \left| \bigcup \{A_T^N \mid N \subseteq T \text{ and } |N| = n\} \right|$ . We have

$V_T^n = \text{sum}(\{A_T^N \mid N \subseteq T \text{ and } |N| = n\})$ .

$$V_T^n = A_t^n * \left| \{A_T^N \mid N \subseteq T \text{ and } |N| = n\} \right| = \binom{t}{n} A_t^n$$

Replacing  $V_t^n$  with  $A_t^n$ ,

$$\frac{t!}{n!(t-n)!} A_t^n = \frac{n+1}{t+1} \frac{(t+1)!}{(n+1)!(t-n)!} A_{t+1}^{n+1} + \frac{t-n+1}{t+1} \frac{(t+1)!}{n!(t+1-n)!} A_{t+1}^n = \frac{t!}{n!(t-n)!} A_{t+1}^{n+1} + \frac{(t)!}{n!(t-n)!} A_{t+1}^n$$

$$\text{Hence, } A_t^n = A_{t+1}^{n+1} + A_{t+1}^n \quad (1)$$

$A_t^n$  follows binomial distribution. Thus,

$$A_{s-x}^n = \sum_{0 \leq i \leq x} \binom{x}{i} A_s^{n+i}$$

$$\text{Hence, } V_{s-x}^n = \sum_{0 \leq i \leq x} \binom{\binom{x}{i} \binom{s-x}{n}}{\binom{s}{n+i}} V_s^{n+i} \quad (2)$$

## Up-sampling:

let  $T_t$  denotes total number of non-singleton SVs in all samples when sample size =  $t$ .

$S_t$  denotes mean number of non-singleton SVs in each sample when sample size =  $t$ .

Obvious,

$$T_t = \sum_{2 \leq n \leq s} V_s^n$$

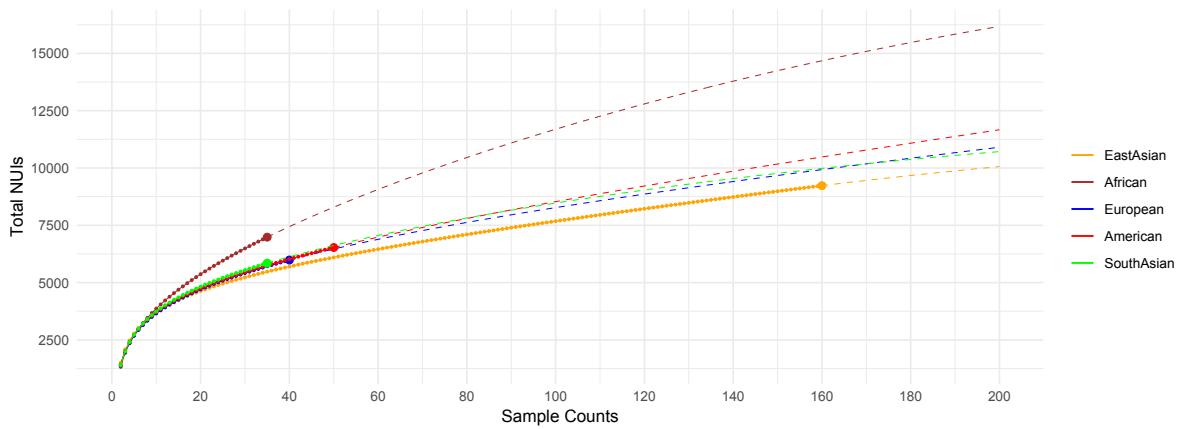
$$S_t = \frac{1}{s} \sum_{2 \leq n \leq s} n V_s^n$$

Considering the equation (2),

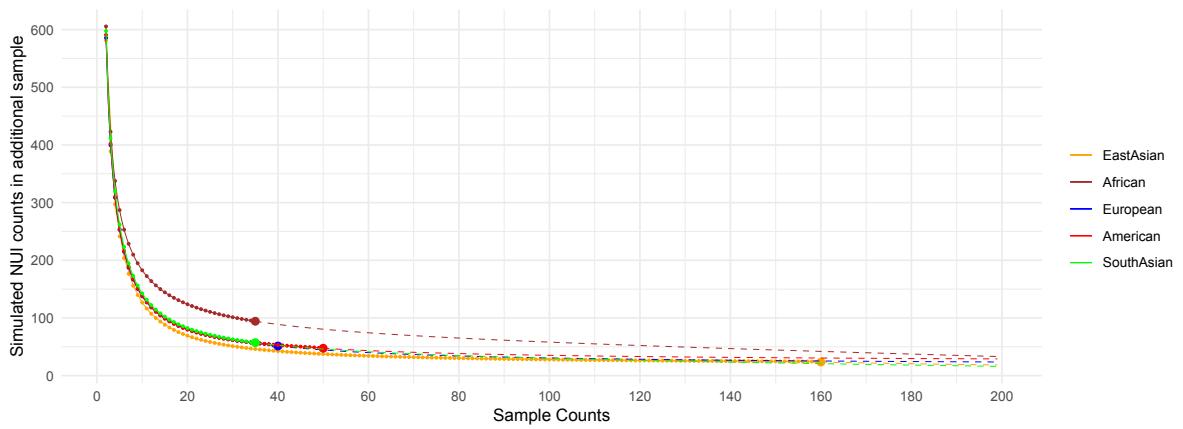
$$V_{s-1}^n = \left( \frac{n+1}{s} \right) V_s^{n+1} + \left( \frac{s-n}{s} \right) V_s^n, \text{ thus,}$$

## Sample results

a

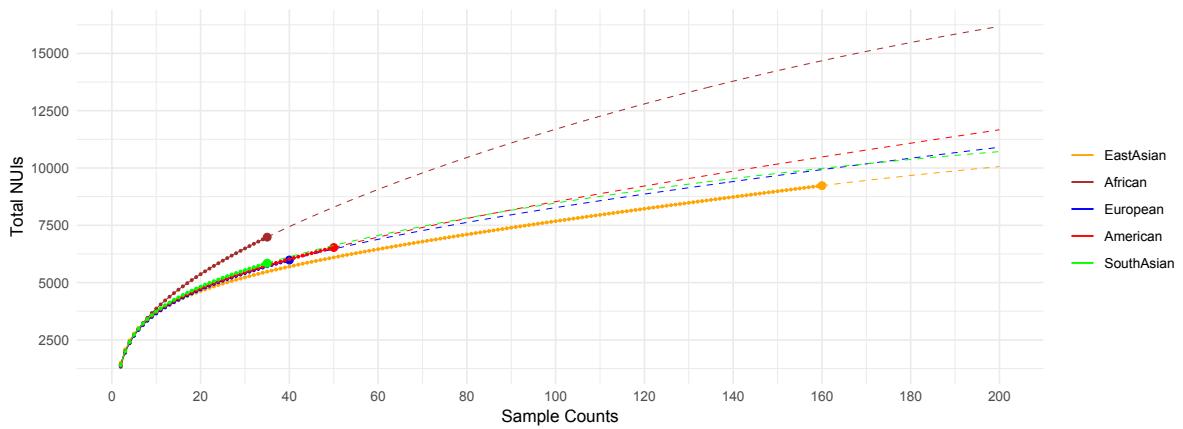


b

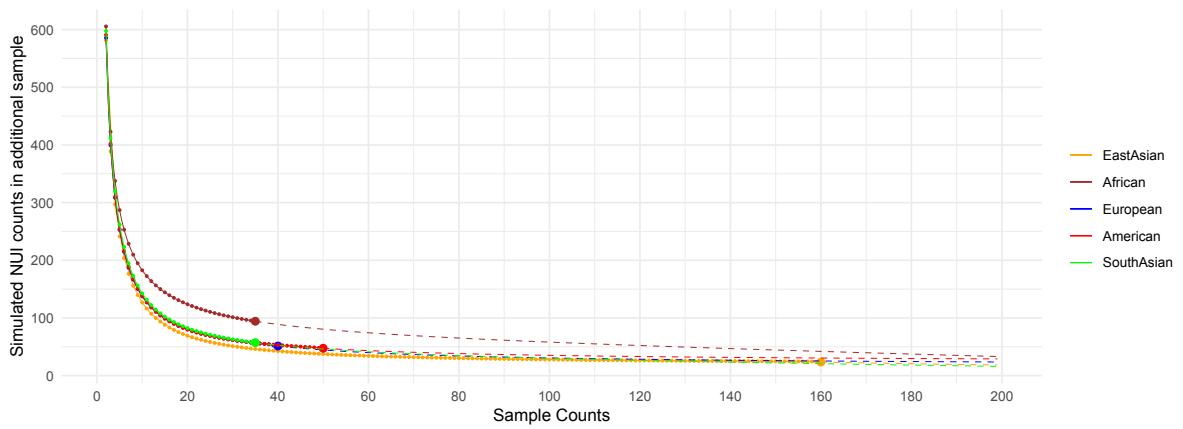


## Sample results

a

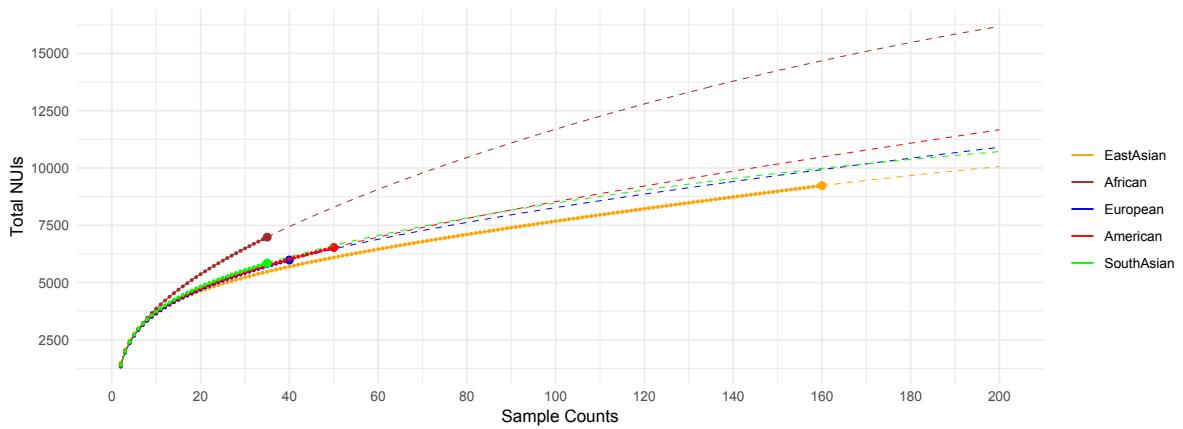


b

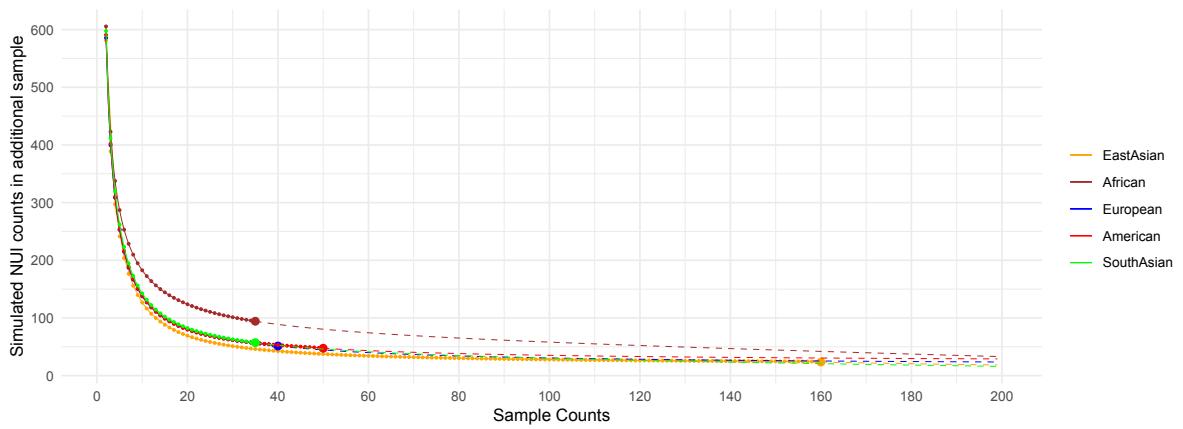


## Sample results

a



b



$$\sum_{2 \leq n \leq s-1} V_{s-1}^n = \frac{1}{S} \sum_{3 \leq n \leq s} n * V_s^n - \frac{1}{S} \sum_{2 \leq n \leq s-1} n * V_s^n + \sum_{2 \leq n \leq s-1} V_s^n$$

$$\sum_{2 \leq n \leq s-1} V_{s-1}^n = \frac{1}{S} \sum_{3 \leq n \leq s} n * V_s^n - \frac{1}{S} \sum_{2 \leq n \leq s} n * V_s^n + \sum_{2 \leq n \leq s} V_s^n$$

$$\sum_{2 \leq n \leq s-1} V_s^n - \sum_{2 \leq n \leq s-1} V_{s-1}^n = \frac{2}{S} * V_s^2 = (s-1) A_s^2$$

$$T_s - T_{s-1} = (s-1) A_s^2 = (s-1) V_s^2 / \binom{s}{2}$$

Similarly, considering the equation ①,

$$n V_{s-1}^n = \binom{n}{s} (n+1) V_s^{n+1} + \left(1 - \frac{n}{s}\right) n V_s^n = n V_s^n - \binom{n}{s} n V_s^n + \binom{n}{s} (n+1) V_s^{n+1}$$

$$n V_{s-1}^n = \binom{n+1}{s} (n+1) V_s^{n+1} - \binom{n}{s} n V_s^n + n V_s^n - \frac{1}{s} (n+1) V_s^{n+1}$$

$$\sum_{2 \leq n \leq s-1} n * V_{s-1}^n = s V_s^n - \frac{2}{S} 2 V_s^2 + \sum_{2 \leq n \leq s-1} n * V_s^n - \frac{1}{S} \sum_{3 \leq n \leq s} n * V_s^n$$

$$\sum_{2 \leq n \leq s-1} n * V_{s-1}^n = -\frac{2}{S} V_s^2 + \sum_{2 \leq n \leq s} n * V_s^n - \frac{1}{S} \sum_{2 \leq n \leq s} n * V_s^n$$

$$\sum_{2 \leq n \leq s-1} n * V_{s-1}^n = -\frac{2}{S} V_s^2 + \frac{s-1}{S} \sum_{2 \leq n \leq s} n * V_s^n$$

$$\frac{1}{s-1} \sum_{2 \leq n \leq s-1} n * V_{s-1}^n = -\frac{2}{s(s-1)} V_s^2 + \frac{1}{S} \sum_{2 \leq n \leq s} n * V_s^n$$

$$\frac{1}{S} \sum_{2 \leq n \leq s} n * V_s^n - \frac{1}{s-1} \sum_{2 \leq n \leq s-1} n * V_{s-1}^n = A_s^2$$

$$\text{Hence, } S_s - S_{s-1} = A_s^2 = V_s^2 / \binom{s}{2} = \frac{T_s - T_{s-1}}{s-1}$$

### Upsampling-projection:

Define the increment of  $T_s$ :

$$I_s = T_{s+1} - T_s$$

let  $p$  denotes the possibility of an SV in an arbitrary sample  $X$  to be find in another sample.

If the sampling is fully random, we believe it should be same in all samples.

Thus, the chance it only be captured twice is  $p(1-p)^s$  when sample set is  $s+2$

$$\text{We have } A_{s+2}^2 = \sum_{f \in X} p(1-p)^s$$

Thus,

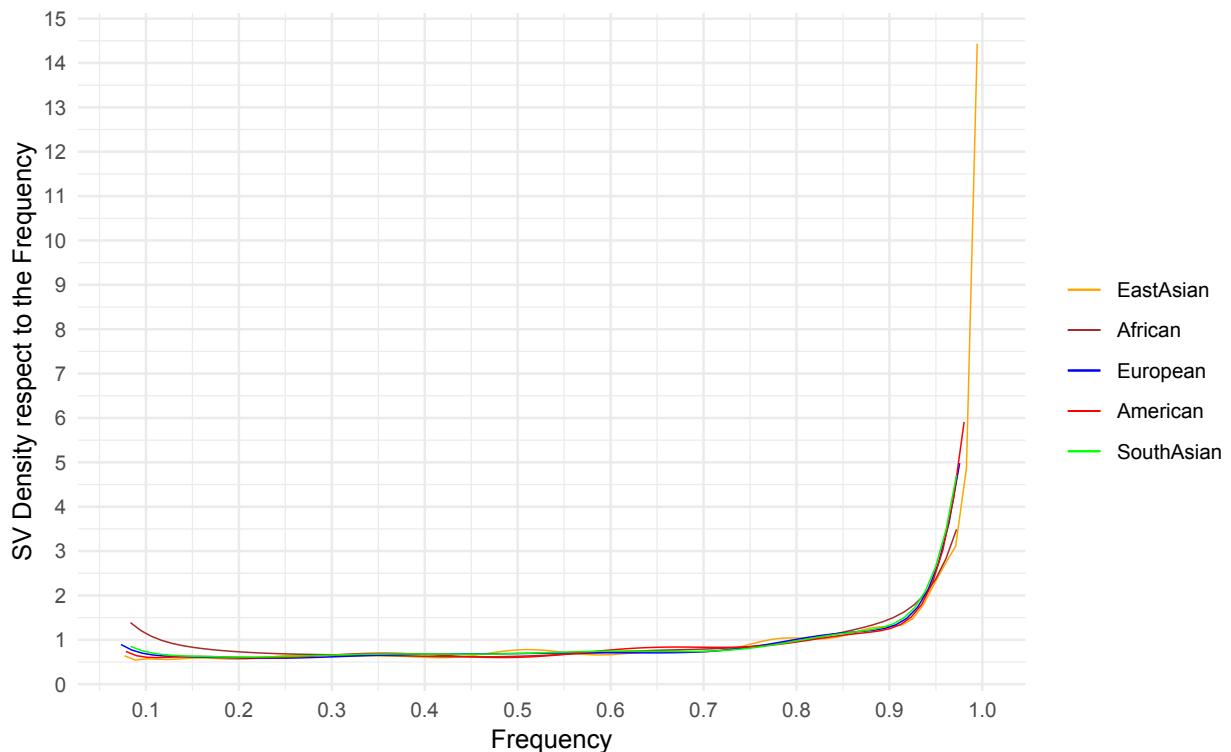
$$I_{s+2} = (s+1) \sum_{p \in X} p(1-p)^s$$

We let  $d_p$  to represent the density of sv at frequency =  $p$ ,

Thus,

$$\text{Exp}(I_{s+2}) = \text{Exp}((s+1) \sum_{p \in X} p(1-p)^s) \approx (s+1) \int_0^1 d_p p(1-p)^s dp$$

We let  $s$  to be sufficiently big ( $>30$ ) as a precondition:



Based on estimated distribution of SV frequencies for individual of current data, we noticed that two peaks at both ends of the graph, which represent reference as major allele and minor allele and are believed to be somewhat symmetric.

We use  $\varphi$  to represent the upper boundary of those rare SVs' frequencies. Similarly, we also noticed that frequencies is relatively flat from  $\varphi$  to 0.6, thus the density between  $\varphi$ -0.6 can be treated as a constant, we use D to represent.

Based on what we observed:

$$\begin{aligned} & (s+1) \int_0^1 d_p p(1-p)^s dp \\ &= (s+1) \int_0^\varphi d_p p(1-p)^s dp + (s+1) \int_\varphi^1 d_p p(1-p)^s dp \\ &= (s+1) \int_0^\varphi d_p p(1-p)^s dp + (s+1) \int_\varphi^1 D p(1-p)^s dp + (s+1) \int_{0.4}^1 (d_p - D) p(1-p)^s dp \end{aligned}$$

for  $p$  is sufficiently big ( $> 0.4$ ),

$$\text{coefficient} = (s+1)(1-p)^s < 30 * 0.6^s < 30 * 0.6^30 = 2.210739 * 10^{-7},$$

Which is small enough compared to SV number (estimate to be  $< 4000$ ) to be ignored.

So,

$$it \approx (s+1) \int_0^\varphi (d_p - D) p(1-p)^s dp + (s+1) D \int_\varphi^1 p(1-p)^s dp$$

for  $\int p(1-p)^s dp$ , let  $q$  denote  $1-p$

$$\begin{aligned} &= \int (1-q) q^s d(1-q) \\ &= - \int (1-q) q^s dq \\ &= \frac{q_i^{s+1}}{s+1} - \frac{q_i^{s+2}}{s+2} \end{aligned}$$

$$\text{Thus } (s+1) \int_\varphi^1 p(1-p)^s dp = (s+1) \left( \frac{q_i^{s+1}}{s+1} - \frac{q_i^{s+2}}{s+2} \Big|_0^{1-\varphi} \right) = \frac{1+(s+1)\varphi}{s+2} (1-\varphi)^{s+1} \approx \frac{1}{s+2} (1-\varphi)^{s+1}$$

$$\text{when } \varphi \ll \frac{1}{s+1}, \text{ it } \approx \frac{1}{s+2} (1-\varphi)^{s+1}$$

(3)

Similarly, We let  $\varepsilon$  represents the peak closest to  $\frac{1}{s+2}$ , so

$$\begin{aligned} & (s+1) \int_0^\varphi (d_p - D) p(1-p)^s dp \\ & \approx (1-\varepsilon)^s * (s+1) \int_0^\varphi (d_p - D) p dp \\ & \quad \int_0^\varphi (d_p - D) p dp \text{ is a constant, we use R to represent.} \end{aligned}$$

$$\text{Thus, it } \approx (1-\varepsilon)^s * R$$

(4)

Hence, combining ③ and ④,

$$\text{Exp}(I_{s+2}) \approx R * (s+1)(1-\varepsilon)^s + \frac{D}{s+2} (1-\varphi)^{s+1}$$

$$\text{Hence, } \text{Exp}(I_s) \approx R * (s+1)(1-\varepsilon)^{s-2} + \frac{D}{s} (1-\varphi)^{s-1} \quad (5)$$

We fit last 10 increments on ⑤ to estimate constants R and D, which is used to project total number of non-singleton SVs on larger sample size.

## Reference

1. Crowley, P. (2003). Resampling Methods for Computation-Intensive Data Analysis in Ecology and Evolution. *Annual Review of Ecology and Systematics*. 23. 405-447. [10.1146/annurev.es.23.110192.002201](https://doi.org/10.1146/annurev.es.23.110192.002201).