CHAPTER 4

Simultaneous Linear Algebraic Equations

4.1 INTRODUCTION

Simultaneous linear algebraic equations are very common in various fields of Engineering and Science. We use matrix inversion method or Cramer's rule to solve these equations in general. But these methods prove to be tedious when the system of equations contain a large number of unknowns. To solve such equations there are other numerical methods, which are particularly suited for computer operations. These are of two types: *direct* and *iterative*.

Gauss elimination method, Gauss–Jordan method, Triangularisation method and Crout's method are *direct* methods whereas Gauss–Jacobi method, Gauss–Seidel iterative method and Relaxation method are *iterative* methods. These methods are explained in the following sections.

4.2 GAUSS ELIMINATION METHOD

In this method, the unknowns are eliminated successively by transforming the given system into an equivalent system with upper triangular coefficient matrix (i.e. a matrix in which all the elements below the principal diagonal are zero) by means of elementary row operations, from which the unknowns are found by back substitution. Here, we shall explain it by considering a system of three equations in three unknowns.

Consider the system

$$a_1x + b_1y + c_1z = d_1 \quad (4.1a)$$
$$a_2x + b_2y + c_2z = d_2 \quad (4.1b)$$
$$a_3x + b_3y + c_3z = d_3 \quad (4.1c)$$

Scanned with CamScanner

where x, y, z are unknowns. The system in matrix form is $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Consider the argumented matrix $[A|B]$

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \tag{4.2}$$

Now Eqn (4.2) is to be reduced to an upper triangular matrix. Let $a_1 \neq 0$. Then

$$\begin{matrix} R_2 \to R_2 - \dfrac{a_2}{a_1} R_1 \\ R_3 \to R_3 - \dfrac{a_3}{a_1} R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right] \tag{4.3}$$

Here, a_1 is called the first pivot and $b_2', c_2', d_2', b_3', c_3', d_3'$ are transformed elements.

Now take b_2' as the pivot ($b'_2 \neq 0$). Then

$$R_3 \to R_3 - \frac{b_3'}{b_2'} R_2 \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3'' & d_3'' \end{array}\right] \tag{4.4}$$

Now, if $c_3'' \neq 0$, from Eqn (4.4), the given system of linear equations is equivalent to $a_1x + b_1y + c_1z = d_1$

$$\begin{aligned} b_2'y + c_2'z &= d_2' \\ d_3'z &= d_3'' \end{aligned}$$

Using back substitution, $z = \dfrac{d_3''}{c_3''}$

$$y = \frac{1}{b_2'c_3''}\{d_2'c_3'' - c_2'd_3''\} \text{ and}$$

$$x = \frac{1}{a_1b_2'c_3''}\{d_1b_2'c_3'' - b_1d_2'c_3'' + b_1c_2'd_3'' - b_2'c_1d_3''\}$$

Scanned with CamScanner

Note:

1. This method fails if any one of the pivots a_1, b_2', or c_3'' becomes zero. In such cases, by interchanging the rows we can get the non-zero pivots.

2. *Partial pivoting*: From the first column of Eqn (4.2) [called the pivot column if $a_i \neq 0$, $i = 1, 2, 3$], select the component with the largest absolute value. This is called the pivot. Then at the second stage, i.e. from the second column of Eqn (4.3), select once again the component with the largest absolute value as the pivot. Continue this process. This procedure is called partial pivoting [refer to Example 1.1].

3. *Complete pivoting*: If we are not interested in the elimination of x, y, z in a particular order, then we can choose at each stage the numerically largest coefficient of the entire coefficient matrix. This requires an interchange of equations and also an interchange of position of the variables.

Example 4.1 Solve the system of equations $3x + y - z = 3$, $2x - 8y + z = -5$, $x - 2y + 9z = 8$ using Gauss elimination method.

(*M.U, B.E., 1992*)

Solution The given system is equivalent to

$$\underbrace{\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}}_{B}$$

$\therefore$ The argumented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array}\right]$$

Now we will make A as upper triangular choosing '3' as pivot,

$$\begin{array}{l} R_2 \to R_2 - \frac{2}{3}R_1 \\ R_3 \to R_3 - \frac{1}{3}R_1 \end{array} \sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & \frac{-26}{3} & \frac{5}{3} & -7 \\ 0 & \frac{-7}{3} & \frac{28}{8} & 7 \end{array}\right]$$

3

Now choosing $-\frac{26}{3}$ as the pivot from the second column,

Scanned with CamScanner

$$R_3 \to R_3 - \frac{7}{26}R_2 \sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & -\frac{26}{3} & \frac{5}{3} & -7 \\ 0 & 0 & \frac{693}{78} & \frac{231}{26} \end{array}\right]$$

From this we get, $3x + y - z = 3$,

$$-\frac{26}{3}y + \frac{5}{3}z = -7 \text{ and } \frac{693}{78}z = \frac{231}{26}$$

Now by back substitution, $z = 1$,

$$-\frac{26}{3}y = -7 - \frac{5}{3}z = -7 - \frac{5}{3} = -\frac{26}{3} \quad \therefore y = 1$$

and $$x = \frac{1}{3}[3 - y + z] = \frac{1}{3}[3 - 1 + 1] = 1$$

$$\therefore x = 1,\ y = 1,\ z = 1$$

Example 4.2 Solve the system of equations,
$28x + 4y - z = 32$, $x + 3y + 10z = 24$ and $2x + 17y + 4z = 35$ by Gauss elimination method.

(M.U, 1990,1992)

Solution The given system is equivalent to

$$\begin{bmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 24 \\ 35 \end{bmatrix}$$

$$A \qquad X = B$$

The argumented matrix is

$$[A/B] = \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{array}\right]$$

Now we will make A as upper triangular choosing 28 as pivot.

$$\begin{array}{l} R_2 \to R_2 - \frac{R_1}{28} \\ R_3 \to R_3 - \frac{R_1}{14} \end{array} \sim \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 0 & \frac{80}{28} & \frac{281}{28} & \frac{640}{28} \\ 0 & \frac{234}{14} & \frac{57}{14} & \frac{458}{14} \end{array}\right]$$

Scanned with CamScanner

Now the pivot is $\dfrac{234}{14}$

$$\therefore\ R(2,3) \sim \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 0 & \dfrac{234}{14} & \dfrac{57}{14} & \dfrac{458}{14} \\ 0 & \dfrac{80}{28} & \dfrac{281}{28} & \dfrac{640}{28} \end{array}\right]$$

$$R_3 \to R_3 - \frac{20}{117} R_2 \sim \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 0 & \dfrac{234}{14} & \dfrac{57}{14} & \dfrac{458}{14} \\ 0 & 0 & \dfrac{30597}{1638} & \dfrac{56560}{1638} \end{array}\right]$$

From this, we get

$$28x + 4y - z = 32$$
$$234y + 57z = 458$$

and

$$30597z = 56560$$

Now by back substitution, we get

$$\therefore \quad z = \frac{56560}{30597} = 1.8485472$$

$$y = \frac{458 - 57z}{234} = 1.5069778$$

and

$$x = \frac{-4y + z + 32}{28} = 0.9935941$$

Example 4.3 Using Gauss elimination method, solve the system

$$3.15\,x - 1.96\,y + 3.85\,z = 12.95$$
$$2.13\,x + 5.12\,y - 2.89\,z = -8.61$$
$$5.92\,x + 3.05\,y + 2.15\,z = 6.88$$

(M.K.U,1981)

Scanned with CamScanner

Solution The given system is equivalent to

$$\begin{bmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}$$

$$A \qquad X \quad = \quad B$$

$$\therefore [A/B] = \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{array}\right]$$

Now we will make A as upper triangular, choosing 3.15 as pivot

$$\begin{matrix} R_2 \to R_2 - \dfrac{2.13}{3.15} R_1 \\ R_3 \to R_3 - \dfrac{5.92}{3.15} R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3667 \\ 0 & 6.7335 & -5.0855 & -17.4578 \end{array}\right]$$

Choosing 6.4453 as pivot

$$R_3 \to R_3 - \frac{6.7335}{6.4453} R_2 \sim \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3667 \\ 0 & 0 & 0.6534 & 0.6853 \end{array}\right]$$

From this, we get

$$3.15\,x - 1.96\,y + 3.85\,z = 12.95$$
$$6.4453\,y - 5.4933\,z = -17.3667$$
$$0.6534\,z = 0.6853$$

By back substitution

$$z = \frac{0.6853}{0.6534} = 1.0488215$$

$$y = \frac{5.4933z - 17.3667}{6.4453} = -1.8005692$$

and $$x = \frac{1.96y - 3.85z + 12.95}{3.15} = 1.708864$$

Example 4.4 Solve the system of equations,

$x_1 + x_2 + x_3 + x_4 = 2,$ $\quad x_1 + x_2 + 3x_3 - 2x_4 = -6$

$2x_1 + 3x_2 - x_3 + 2x_4 = 7,$ $\quad x_1 + 2x_2 + x_3 - x_4 = -2$

by Gauss elimination method. *(M.U., B.E., 1986)*

Scanned with CamScanner

Solution The given system in matrix form is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \\ -2 \end{bmatrix}$$

$$A \qquad X \quad = \quad B$$

The argumented matrix $[A|B]$ is

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & -2 & -6 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 2 & 1 & -1 & -2 \end{array}\right]$$

Choose 1 of first column as pivot. Then

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & 0 & -2 & -4 \end{array}\right]$$

Since the element in 2nd row, 2nd column is zero, interchange 2nd and 3rd rows to get pivot element 1. That is,

$$R(2,3) \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & 0 & -2 & -4 \end{array}\right]$$

$$R_4 \to R_4 - R_2 \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 0 & 3 & -2 & -7 \end{array}\right]$$

Now the pivot is 2, therefore,

$$R_4 \to R_4 - \frac{3}{2}R_3 \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 0 & 0 & \frac{5}{2} & 5 \end{array}\right]$$

Scanned with CamScanner

From this we get,

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_2 - 3x_3 = 3$$
$$2x_3 - 3x_4 = -8$$
$$\left(\frac{5}{2}\right)x_4 = 2$$

Now, from Eqn (iv), $x_4 = 2$

from Eqn (iii),

$$x_3 = \frac{1}{2}(-8 + 3x_4) = \frac{1}{2}(-8 + 6) = -1$$

from Eqn (ii),

$$x_2 = 3 + 3x_3 = 3 - 3 = 0$$

and from Eqn (i),

$$x_1 = 2 - x_2 - x_3 - x_4 = 2 - 0 - (-1) - 2 = 1$$

$\therefore x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 2$

4.3 GAUSS–JORDAN METHOD

This method is a modified form of Gauss elimination method. In this method, the coefficient matrix A of $AX = B$ is reduced to a diagonal matrix or unit matrix by making all the elements above and below to the principal diagonal of A as zero. The labour of back substitution is saved here even though it involves additional computations.

Example 4.5 Solve the equations
$10x + y + z = 12$, $2x + 10y + z = 13$ and
$x + y + 5z = 7$ by Gauss–Jordan method. (M.U

Solution The given system in matrix form is

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

$$A \quad X \quad = \quad B$$

$$\therefore [A/B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

Scanned with CamScanner

$$R_1 \to R_1 - 9R_3 \sim \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

$$\begin{array}{c} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

$$R_2 \to -(R_2 - 3R_3) \sim \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 1 & 58 & 59 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

$$\begin{array}{c} R_1 \to R_1 + 8R_2 \\ R_3 \to R_3 - 9R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & -473 & -473 \end{array}\right]$$

$$\begin{array}{c} R_3 \to -\dfrac{1}{473} R_3 \\ R_1 \to R_1 - 420R_3 \\ R_2 \to R_2 - 58R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore$ The system AX = B reduces to the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

i.e., $x = y = z = 1$

Example 4.6 Solve the equations
$10x_1 + x_2 + x_3 = 12$, $x_1 + 10x_2 - x_3 = 10$ and
$x_1 - 2x_2 + 10x_3 = 9$ By Gauss–Jordan method. *(M.U. 1997)*

Solution The matrix form of the given systems is

$$\underset{A}{\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}}$$

Scanned with CamScanner

$$[A/B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{array}\right]$$

$$R_1 \to R_1 - 9R_2 \sim \begin{bmatrix} 1 & -89 & 10 & -78 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 99 & -11 & 88 \\ 0 & 87 & 0 & 87 \end{array}\right]$$

$$\begin{matrix} R_2 \to \dfrac{R_2}{99} \\ R_3 \to \dfrac{R_3}{87} \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 & 8 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 8R_3 \sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{matrix} R_1 \to R_1 + 89R_2 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -79 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{matrix} R_1 \to R_1 + 79R_3 \\ R_2 \to R_2 + R_3 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore$ The system $AX = B$ reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow x_1 = x_2 = x_3 = 1$

Scanned with CamScanner

Example 4.7 Solve the following equations by Gauss–Jordan method.

$x + 2y + z - w = -2;$ $\quad 2x + 3y - z + 2w = 7$

$x + y + 3z - 2w = -6;$ $\quad x + y + z + w = 2$

(M.U. B.E., 1986)

Solution The given system in matrix form is

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -6 \\ 2 \end{bmatrix}$$

$$A \qquad X \quad = \quad B$$

The argumented matrix is

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 1 & 3 & -2 & -6 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\begin{array}{lll} R_2 & \to & -(R_2 - 2R_1) \\ R_3 & \to & R_3 - R_1 \\ R_4 & \to & R_4 - R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 0 & 1 & 3 & -4 & 11 \\ 0 & -1 & 2 & -1 & -4 \\ 0 & -1 & 0 & 2 & 4 \end{array}\right]$$

$$\begin{array}{lll} R_1 & \to & R_1 - 2R_2 \\ R_3 & \to & \frac{1}{5}(R_3 + R_2) \\ R_4 & \to & R_4 + R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 7 & 20 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 3 & -2 & -7 \end{array}\right]$$

$$\begin{array}{lll} R_1 & \to & R_1 + 5R_3 \\ R_2 & \to & R_2 - 3R_3 \\ R_4 & \to & R_4 - 3R_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{lll} R_1 & \to & R_1 - 2R_4 \\ R_2 & \to & R_2 + R_4 \\ R_3 & \to & R_3 + R_4 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Scanned with CamScanner

$\therefore$ The system $AX = B$ reduces to the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

i.e. $x = 1, y = 0, z = -1$ and $w = 2$.

4.4 INVERSE OF A MATRIX USING GAUSS ELIMINATION METHOD

Let A be a square matrix of order 3, $|A| \neq 0$ and X be its inverse. Then we know that $AX = I$, where I is a unit matrix of order 3.

$$\therefore \text{ If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

then

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.5}$$

This equation is equivalent to the three equations given below

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{4.6}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{4.7}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.8}$$

Scanned with CamScanner

Each of the systems given in Eqns (4.6)–(4.8) can be solved for

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}, \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}, \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix}$$

using Gauss elimination method. The solution set of each system will be the corresponding column of the inverse matrix X. Since the coefficient matrix A is the same in all the three systems, we can solve them simultaneously by considering $[A|I]$.

Example 4.8 Find the inverse of

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$ using Gauss elimination method.

Solution We have the argumented system as

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - \left(\frac{3}{2}\right) R_1 \\ R_3 \to R_3 - \left(\frac{1}{2}\right) R_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{-3}{2} & 1 & 0 \\ 0 & \frac{7}{2} & \frac{17}{2} & \frac{-1}{2} & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - 7R_2 \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right] \quad \text{(i)}$$

Now if

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

Scanned with CamScanner

is the inverse of the given matrix, then the system (i) is equivalent to the following three systems:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 10 \end{bmatrix} \qquad \text{(ii)}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix} \qquad \text{(iii)}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(iv)}$$

By back substitution, the three systems of equations, i.e. (ii)–(iv) give the three columns of the inverse matrix.

$\therefore \quad x_{11} = -3; \qquad x_{21} = 12; \qquad x_{31} = -5$

$x_{12} = \frac{5}{2}; \qquad x_{22} = -\frac{17}{2}; \quad x_{23} = \frac{7}{2}$

$x_{13} = -\frac{1}{2}; \qquad x_{23} = \frac{3}{2}; \qquad x_{33} = -\frac{1}{2}$

and the inverse matrix is

$$= \frac{1}{2}\begin{bmatrix} -6 & 5 & -1 \\ 24 & -17 & 3 \\ -10 & 7 & -1 \end{bmatrix}$$

Example 4.9 Find the inverse of the matrix

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$ using Gauss elimination method. *(M.U 1997)*

Scanned with CamScanner

Solution We have the argumented system

$$[A/I] = \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - \frac{1}{2}R_1 \\ R_3 \to R_3 - \frac{1}{4}R_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -2 & \frac{-1}{2} & 1 & 0 \\ 0 & \frac{-9}{4} & \frac{3}{2} & \frac{-1}{4} & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 + \frac{9}{10}R_2 \sim \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{3}{10} & -\frac{14}{20} & \frac{9}{10} & 1 \end{array}\right] \qquad \text{(i)}$$

Now if $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ is the inverse of the given matrix, then the system (1) is equivalent to three systems

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{14}{20} \end{bmatrix} \qquad \text{(ii)}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{9}{10} \end{bmatrix} \qquad \text{(iii)}$$

Scanned with CamScanner

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

By back substitution, the three systems i.e., (ii), (iii) and (iv) give three columns of the inverse matrix. i.e.,

$$x_{11} = \frac{4}{3}, \quad x_{21} = \frac{5}{3}, \quad x_{31} = \frac{7}{3}$$

$$x_{12} = 2, \quad x_{22} = -2, \quad x_{23} = -3$$

$$x_{31} = \frac{7}{3}, \quad x_{11} = -\frac{8}{3}, \quad x_{33} = -\frac{10}{3}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{-4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & \frac{-8}{3} \\ \frac{7}{3} & -3 & \frac{-10}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -4 & 6 & 7 \\ 5 & -6 & -8 \\ 7 & -9 & -10 \end{bmatrix}$$

EXERCISE 4.1

Solve the following equations by Gauss elimination method

1. $3x + 4y - z = 8,\ -2x + y + z = 3,\ x + 2y - z = 2$
2. $x - y + z = 1,\ -3x + 2y - 3z = -6,\ 2x - 5y + 4z = 5$
3. $10x + y + z = 12,\ 2x + 10y + z = 13,\ 2x + 2y + 10z = 14$
   (M.U. B.E., 1991)
4. $2x - y + 2z = 2,\ x + 10y - 3z = 5,\ x - y - z = 3$
   (Ranchi. B. Tech. 1987)
5. $10x_1 + x_2 + x_3 = 18.141,\ x_1 + x_2 + 10x_3 = 38.139,\ x_1 + 10x_2 + x_3 = 28.140$
   (M.U. B.E., 1991)
6. $x + y + z = 6.6,\ x - y + z = 2.2,\ x + 2y + 3z = 15.2$
   (North Bengal B Tech. 1987)
7. $2x + 4y + 2z = 15,\ 2x + y + 2z = -5,\ 4x + y - 2z = 0$
   (M.U.

Scanned with CamScanner

8. $2x + y + z = 10,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16$

(*Bangalore, B.E., 1990*)

9. $2x_1 + 2x_2 + x_3 = 12,\ 3x_1 + 2x_2 + 2x_3 = 8,\ 5x_1 + 10x_2 - 8x_3 = 10$

(*S.Gujarat, B.E., 1990*)

10. $x + 2y - 12z + 8w = 27,\ 5x + 4y + 7z - 2w = 4,$
$-3x + 7y + 9z + 5w = 11,\ 6x - 12y - 8z + 3w = 49$

(*M.U, B.E., 1987*)

Solve the following equations by Gauss – Jordan method

11. $2x - 3y + z = -1,\ x + 4y + 5z = 25,\ 3x - 4y + z = 2$

(*M.U, B.E., 1993*)

12. $2x + y + z = 12,\ 3x + 2y + 3z = 24,\ x + 4y + 9z = 34$

13. $10x + y + z = 12,\ x + 10y + z = 12,\ x + y + 10z = 12$

(*Bhopal B.E., 1991*)

14. $x + 2y + z = 8,\ 2x + 3y + 4z = 20,\ 4x + 3y + 2z = 16$

(*Punjab, B.E., 1987*)

15. $4x - y - z = -7,\ x - 5y + z = -10,\ x + 2y + 6z = 9$

16. $2x + 2y - z + t = 4,\ 4x + 3y - z + 2t = 6,$
$8x + 5y - 3z + 4t = 12,\ 3x + 3y - 2z + 2t = 6$

17. $5x_1 + x_2 + x_3 + x_4 = 4;\ x_1 + 7x_2 + x_3 + 4x_4 = 2$
$x_1 + x_2 + 6x_3 + x_4 = -5;\ x_1 + x_2 + x_3 + x_4 = -6$

Find the inverse of the following matrices using Gauss elimination method.

18. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ 19. $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ 20. $\begin{bmatrix} 3 & -1 & 10 & 2 \\ 5 & 1 & 20 & 3 \\ 9 & 7 & 39 & 4 \\ 1 & -2 & 2 & 1 \end{bmatrix}$

ANSWERS

1. 1, 2, 3
2. −2, 3, 6
3. 1, 1, 1
4. 2, 0, −1
5. 1.234, 2.348, 3.455
6. 1.2, 2.2, 3.2
7. −3.0556, 6.6667, −2.778
8. 7, −9, 5

Scanned with CamScanner

9. $-12.75, 14.375, 8.75$

10. $3.0016, -1.999, 0.9998, 5.0001$

11. $8.7, 5.7, -1.3$

12. $7, -9, 7$

13. $1, 1, 1$

14. $1, 2, 3$

15. $-1, 2, 1$

16. $1, 1, -1, -1$

17. $1, 2, -1, -2$

18. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

19. $\frac{1}{5}\begin{bmatrix} 2 & 2 & -3 \\ -2 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 7 & -3 & 0 & -5 \\ 8 & 10 & -2 & 11 \\ -5 & 0 & 1 & 6 \\ 19 & 5 & -6 & -28 \end{bmatrix}$

4.5 METHOD OF FACTORISATION OR TRIANGULARISATION

Consider the following system of equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These equations can be written in matrix form as

$$AX = B \tag{4.9}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

In this method we use the fact that the square matrix A can be factorised into the form LU, where L is a unit lower triangular matrix and U, an upper triangular matrix if all the minors of A are non-singular.

Let

$$A = LU \tag{4.10}$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Scanned with CamScanner

Now Eqn (4.9) becomes

$$LUX = B \tag{4.11}$$

Setting

$$UX = Y \tag{4.12}$$

Eqn (4.11) implies

$$LY = B \tag{4.13}$$

i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

giving

$$y = b_1,$$

$$l_{21} y_1 + y_2 = b_2$$

and

$$l_{31} y_1 + l_{32} y_2 + y_3 = b_3.$$

From these, y_1, y_2 and y_3 can be solved by forward substitution. Now, from Eqn (4.12) we have

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

i.e. $u_{11} x_1 + u_{12} x_2 + u_{13} x_3 = y_1$; $\quad u_{22} x_2 + u_{23} x_3 = y_2$ and $u_{33} x_3 = y_3$.

Solving these, we get, by back substitution, x_1, x_2, x_3.

Now L and U can be found from $LU = A$, i.e

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating the corresponding coefficients, we get

$u_{11} = a_{11}$, $u_{12} = a_{12}$, $u_{13} = a_{13}$

$l_{21} u_{11} = a_{21}$ or $l_{21} = \dfrac{a_{21}}{a_{11}}$; $l_{31} u_{11} = a_{31}$ or $l_{31} = \dfrac{a_{21}}{a_{11}}$

$l_{21} u_{12} + u_{22} = a_{22}$ or $u_{22} = \dfrac{1}{a_{11}} (a_{11} a_{22} - a_{21} a_{12})$

Scanned with CamScanner

$$l_{21}\, u_{13} + u_{23} = a_{23} \text{ or } u_{23} = \frac{1}{a_{11}}(a_{11}\, a_{23} - a_{21}\, a_{13})$$

$$l_{31}\, u_{12} + l_{32}\, u_{22} = a_{32} \text{ or } l_{32} = \frac{1}{u_{22}}(a_{32} - l_{31}\, u_{12})$$

and $l_{31}\, u_{13} + l_{32}\, u_{23} + u_{33} = a_{33}$ or $u_{33} = a_{33} - l_{31}\, u_{13} - l_{32}\, u_{23}$

Thus, L and U are known.

Example 4.10 Solve the following system by the method of triangularisation:

$$2x - 3y + 10z = 3,\ -x + 4y + 2z = 20,\ 5x + 2y + z = -12$$

Solution The given system is $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Let $LU = A$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

$u_{11} = 2,\ u_{12} = -3,\ u_{13} = 10.$

$l_{21}u_{11} = -1, \qquad \therefore l_{21} = -\frac{1}{2},$

$l_{31}u_{11} = 5, \qquad \therefore l_{31} = \frac{5}{2}$

$l_{21}u_{12} + u_{22} = 4, \qquad \therefore u_{22} = 4 - l_{21}u_{12} = 4 - (-\frac{1}{2})(-3) = \frac{5}{2}$

$l_{21}u_{13} + u_{23} = 2, \qquad \therefore u_{23} = 2 - l_{12}u_{13} = 2 - (-\frac{1}{2})(10) = 7$

Scanned with CamScanner

$l_{31}u_{12} + l_{32}u_{22} = 5,$ $\quad \therefore l_{32} = \frac{1}{u_{22}}[5 - l_{31}u_{12}] = \frac{19}{5}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1,$ $\quad \therefore u_{33} = 1 - l_{31}u_{13} - l_{32}u_{23} = -\frac{253}{5}$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & -\frac{253}{5} \end{bmatrix}$$

Let $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then $LY = B$,

i.e.
$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix} \qquad \text{(i)}$$

and
$$\begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & -\frac{253}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \text{(ii)}$$

Now Eqn (i) implies

$y_1 = 3;$ $\quad -\frac{1}{2}y_1 + y_2 = 20;$ $\quad \frac{5}{2}y_1 + \frac{19}{5}y_2 + y_3 = -12$

$\therefore y_1 = 3,$ $\quad y_2 = \frac{43}{2}$ $\quad y_3 = -\frac{506}{5}$

Scanned with CamScanner

$$2x - 3y + 10z = 3; \qquad \frac{5}{2}y + 7z = \frac{43}{2}, \qquad -\frac{253}{5}z = -\frac{506}{3}$$

Solving these, we get, by back substitution, $x = -4$, $y = 3$ and $z = 2$.

Example 4.11 Solve the following system by the method of factorisati

$x + 3y + 8z = 4$, $x + 4y + 3z = -2$ and
$x + 3y + 4z = 1$ (M.U, 19

Solution The given system is $AX = B$, where

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Let $LU = A$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$\Rightarrow u_{11} = 1,\ u_{12} = 3,\ u_{13} = 8$

$l_{21}u_{11} = 1 \Rightarrow l_{21} = 1;$ $\qquad l_{31}u_{11} = 1 \Rightarrow l_{31} = 1$

$l_{21}u_{12} + u_{22} = 4$ $\qquad \Rightarrow u_{22} = 4 - l_{21}u_{12} = 4 - 1(3) = 1$

$l_{21}u_{13} + u_{23} = 3$ $\qquad \Rightarrow u_{23} = 3 - l_{21}u_{13} = 3 - 1(8) = -5$

$l_{31}u_{12} + l_{32}u_{22} = 3$ $\qquad \Rightarrow l_{32} = \dfrac{1}{u_{22}}[3 - l_{31}u_{12}] = 0$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$

$\therefore u_{33} = 4 - l_{31}u_{13} - l_{32}u_{23}$

$= 4 - 1.8 - 0.(-5) = -4$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Scanned with CamScanner

Let $UX = Y$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then $LY = B$

$$\therefore \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad \text{(i)}$$

and

$$\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \text{(ii)}$$

Now (i) $\Rightarrow y_1 = 4, y_2 = -2$

and $y_1 + y_3 = 1 \Rightarrow y_3 = 1 - y_1 = 1 - 4 = -3$

From (ii) $x + 3y + 8z = 4$

$$y - 5z = -2$$

$$-4z = -3$$

By back substitution,

$$z = \frac{3}{4}$$

$$y = -4 + 5z = -2 + 5\left(\frac{3}{4}\right) = \frac{7}{4}$$

and

$$x = 4 - 3y - 8z = 4 - 3\left(\frac{7}{4}\right) - 8\left(\frac{3}{4}\right) = \frac{29}{4}$$

$$\therefore x = -\frac{29}{4}, y = \frac{7}{4}, z = \frac{3}{4}$$

4.6 CROUT'S METHOD

This method is superior to the Gauss elimination method because it requires less calculation. It is based on the fact that every square matrix A can be expressed as the product of a lower triangular matrix and a unit upper triangular matrix. Let

$$AX = B \qquad (4.12)$$

be the given system [Ref 4.5] and let

$$A = LU \qquad (4.13)$$

Scanned with CamScanner

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 1 & 1 \end{bmatrix}$$

Then Eqn (4.12) becomes

$$LUX = B \tag{4.14}$$

Let

$$UX = Y \tag{4.15}$$

so that Eqn (4.14) becomes

$$LY = B \tag{4.16}$$

i.e.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\therefore l_{11} = y_1 = b_1;$ $\quad l_{21} y_1 + l_{22} y_2 = b_2;$ $\quad l_{31} y_1 + l_{32} y_2 + l_{33} y_3 = b_3$

By forward substitution, y_1, y_2 and y_3 can be found out if L is know

From Eqn (4.15),

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4.1}$$

i.e. $x_1 + u_{12} x_2 + u_{13} x_3 = y_1;$ $\quad x_2 + u_{23} x_3 = y_2;$ $\quad x_3 = y_3$

By back substitution, x_1, x_2 and x_3 can be found out if U is known.

Now L and U can be found from $LU = A$.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

or

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating the corresponding elements on both sides, we get

$l_{11} = a_{11}, l_{21} = a_{21}, l_{31} = a_{31}$

$l_{11} u_{12} = a_{12}$ $\qquad \therefore u_{12} = a_{12}/a_{11}$

$l_{11} u_{13} = a_{13}$ $\qquad \therefore u_{13} = a_{13}/a_{11}$

Scanned with CamScanner

$l_{21}\, u_{12} + l_{22} = a_{22}$ $\quad\therefore l_{22} = a_{22} - l_{21}u_{12}$

$l_{31}\, u_{12} + l_{32} = a_{32}$ $\quad\therefore l_{32} = a_{32} - l_{31}u_{12}$

$l_{21}\, u_{13} + l_{22}\, u_{23} = a_{23}$ $\quad\therefore u_{32} = \{a_{23} - l_{21}u_{13}\}/u_{23}$

$l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33} = a_{33}$ $\quad\therefore l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$

Thus all the 12 unknowns are determined and the solution, as shown already, is obtained from Eqn (4.17).

Crout, during the above decomposition of the coefficient matrix A, devised a technique to determine the 12 unknowns systematically.

Computation Scheme by Crout's Method

The argumented matrix of the system given by Eqn (4.12) ($AX = B$) is

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

The matrix of 12 unknowns, so called *derived matrix or auxiliary matrix*, is

$$\begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$$

and is to be calculated as follows.

Step 1 The *first column* of the derived matrix (DM) is identical with the first column of $[A|B]$.

Step 2 The *first row* to the right of the first column of the DM is obtained by dividing the corresponding element in $[A|B]$ by the leading diagonal element of that row.

Step 3 *Remaining second column of DM.*

$$\because l_{22} = a_{22} - l_{21}u_{12}\,;\; l_{32} = a_{32} - l_{31}u_{12}$$

{Each element on or below the diagonal} = {(Corresponding element in $[A|B]$) – (The product of the first element in that row and in that column)}

Scanned with CamScanner

Step 4 *Remaining elements of second row of DM.*

Each element = {(Corresponding element in $[A|B]$) – (the product of the first element in that row and in that column divided by leading diagonal element in that row)

i.e. $u_{23} = \dfrac{a_{23} - l_{21}u_{13}}{l_{22}}$; $y_2 = \dfrac{b_2 - l_{21}y_1}{l_{22}}$

Step 5 *Remaining elements of third column of DM.*

Each element = {(Corresponding element in $[A|B]$) – (sum of the inner products of the previously calculated elements in the same row and column)}

i.e. $l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$

Step 6 *Remaining elements of third row of DM.*

Each element = {[Corresponding element in $[A|B]$) – Sum of the inner products of the previously calculated elements in the same row and column] ÷ [The leading diagonal element in that row]}

i.e. $y_3 = \dfrac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}}$

Now the matrices L, U, and Y can be written and hence, X can be obtained from $UX = Y$.

Note: The above procedure holds good for any order square matrix A.

Example 4.12 Solve the system

$2x + y + 4z = 12$, $8x + 3y + 2z = 20$, $4x + 11y + z = 33$

by Crout's method. (M.U, B.E., 1991)

Solution Argumented matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 23 \end{array}\right]$$

Let the derived matrix (DM)

$$= \begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$$

(1) Elements of the first column of DM are $l_{11} = 2, l_{21} = 8, l_{31} = 4$

Scanned with CamScanner

(2) Elements of the first row to the right of the first column are

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{2};\ u_{13} = \frac{a_{13}}{l_{11}} = \frac{y}{2} = 2;\ y_1 = \frac{b_1}{l_{11}} = \frac{12}{2} = 6$$

(3) Elements of the remaining second column are

$$l_{22} = a_{22} - u_{12}\, l_{21} = (-3) - \frac{1}{2}(8) = -7$$

$$l_{32} = a_{32} - u_{12}\, l_{31} = 11 - \frac{1}{2}(4) = 9$$

(4) Elements of the remaining second row are

$$u_{23} = \frac{a_{23} - u_{13} l_{21}}{l_{22}} = -\frac{1}{7}\{2 - (-2)(8)\} = 2$$

$$y_2 = \frac{b_2 - l_{21} y_1}{l_{22}} = \frac{20 - (8)(6)}{-7} = 4$$

(5) For the remaining third column,

$$l_{33} = a_{33} - (l_{31}\, u_{13} + l_{32}\, u_{23})$$
$$= -1 - (4)(2) - (9)(2) = -27$$

(6) For the remaining third row,

$$y_3 = \frac{b_3 - (l_{31} y_1 + l_{32} y_2)}{l_{33}} = \frac{33 - (4)(6) - (9)(4)}{-27} = -1$$

$$\therefore DM = \begin{bmatrix} 2 & \frac{1}{2} & 2 & 6 \\ 8 & -7 & 2 & 4 \\ 4 & 9 & -27 & 1 \end{bmatrix}$$

Now the solution is obtained from the system $UX = Y$,

i.e.

$$\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

Scanned with CamScanner

which is equivalent to

$$x + \frac{1}{2}y + 2z = 6;\ y + z = 4,\ z = 1$$

By back substitution, $x = 3$, $y = 2$ and $z = 1$

Example 4.13 Solve the following system by Crout's method.

$x + y + z = 3,\ 2x - y + 3z = 16,\ 3x + y - z = -3$ (*M.U. 1997, M.K.U, 198*

Solution Argumented matrix of the system is

$$[A/B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 16 \\ 3 & 1 & -1 & -3 \end{array}\right]$$

Let the derived matrix $(DM) = \left[\begin{array}{ccc|c} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{array}\right]$

1. Elements of the first column of DM are $l_{11} = 1$, $l_{21} = 2$, $l_{31} = 3$
2. Elements of the first row to the right of first column

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{1} = 1;\ u_{13} = \frac{a_{13}}{l_{11}} = \frac{1}{1} = 1;\ y_1 = \frac{b_1}{l_{11}} = \frac{3}{1} = 3$$

3. Elements of the remaining second column

$$l_{22} = a_{22} - u_{12}\, l_{21} = -1 - 1(2) = -3$$
$$l_{32} = a_{32} - u_{12}\, l_{31} = 1 - 1(3) = -2$$

4. Elements of the remaining second row

$$u_{23} = \frac{a_{23} - u_{13} l_{21}}{l_{22}} = \frac{3 - 1(2)}{-3} = -\frac{1}{3}$$

$$y_2 = \frac{b_2 - l_2 y_1}{l_{22}} = \frac{16 - 3(2)}{-3} = -\frac{10}{3}$$

5. Remaining third column

$$l_{33} = a_{33} - (l_{31}\, u_{13} + l_{32}\, u_{23})$$
$$= -1 - 1(3) - (-\frac{1}{3})(-2) = -\frac{14}{3}$$

Scanned with CamScanner

6. Remaining third row

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}} = \frac{-3 - 3.3 - (-2) - (10/3)}{-(14/3)} = 4$$

$$\therefore DM = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -1/3 & -10/3 \\ 3 & -2 & -14/3 & 4 \end{bmatrix}$$

Now the solution is got from the system UX = Y

i.e.
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -10/3 \\ 4 \end{bmatrix}$$

$\therefore$
$$x + y + z = 3$$
$$y - z/3 = -(10/3)$$
$$z = 4$$

By back substitution,

$z = 4,\ y = -2$ and $x = 1$

Example 4.14 Solve the following system by Crout's method.

$$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6;$$
$$-6x_1 - 8x_2 + x_3 + 4x_4 = 5;$$
$$3x_1 - x_2 + 4x_3 + 11x_4 = 2;$$
$$5x_1 - 9x_2 + 2x_3 + 4x_4 = 7.$$

Solution Argumented matrix of the given system is

$$[A|B] = \begin{bmatrix} 10 & -7 & 3 & 5 & 6 \\ -6 & 8 & -1 & -4 & 5 \\ 3 & 1 & 4 & 11 & 2 \\ 5 & -9 & -2 & 4 & 7 \end{bmatrix}$$

The derived matrix is,

$$DM = \begin{bmatrix} l_{11} & u_{12} & u_{13} & u_{14} & y_1 \\ l_{21} & l_{22} & u_{23} & u_{24} & y_2 \\ l_{31} & l_{32} & l_{33} & u_{34} & y_3 \\ l_{41} & l_{42} & l_{43} & l_{44} & y_4 \end{bmatrix}$$

Scanned with CamScanner

(1) Elements of the first column of DM.

$$l_{11} = 10,\ l_{21} = -6,\ l_{31} = 3,\ l_{41} = 5$$

(2) Elements of the first row right to the first column:

$$u_{12} = -\frac{7}{10},\ u_{13} = \frac{3}{10},\ u_{14} = \frac{5}{10},\ y_1 = \frac{6}{10}$$

(3) Remaining second column:

$$l_{22} = a_{22} - l_{21}\, u_{12} = 8 - (-6)\,(-\frac{7}{10}) = \frac{38}{10}$$

$$l_{32} = a_{32} - l_{31}\, u_{12} = 1 - (3)\,(-\frac{7}{10}) = \frac{31}{10}$$

$$l_{42} = a_{42} - l_{41}\, u_{12} = -9 - (5)\,(-\frac{7}{10}) = -\frac{55}{10}$$

(3) Remaining second row:

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}} = \frac{-1-(-6)\left(\frac{3}{10}\right)}{\left(\frac{38}{10}\right)} = \frac{8}{38}$$

$$u_{24} = \frac{a_{14} - l_{21}u_{14}}{l_{22}} = \frac{-4-(-6)\left(\frac{5}{10}\right)}{\left(\frac{38}{10}\right)} = -\frac{10}{38}$$

$$y_2 = \frac{b_2 - l_{21}y_1}{l_{22}} = \frac{5-(-6)\left(\frac{6}{10}\right)}{\left(\frac{38}{10}\right)} = \frac{86}{38}$$

(5) Remaining third column:

$$l_{33} = a_{33} - (l_{31}\, u_{13} + l_{32}\, u_{23})$$

$$= 4 - \{\,(3)\,(\frac{3}{10}) + (\frac{31}{10})\,(\frac{8}{38})\} = \frac{93}{38}$$

$$l_{43} = a_{43} - (l_{41}\, u_{13} + l_{42}\, u_{23})$$

$$= -2 - \{(5)\,(\frac{3}{10}) - (\frac{55}{10})\,(\frac{8}{38})\} = -\frac{89}{38}$$

Scanned with CamScanner

(6) **Remaining third row:**

$$u_{34} = \frac{a_{34} - (l_{31}u_{14} + l_{32}u_{24})}{l_{33}} = \frac{11 - \left\{3\left(\frac{5}{10}\right) + \left(\frac{31}{10}\right)\left(-\frac{10}{38}\right)\right\}}{\left(\frac{93}{38}\right)}$$

$$= \frac{392}{93}$$

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}} = \frac{2 - \left\{3\left(\frac{6}{10}\right) + \left(\frac{31}{10}\right)\left(\frac{86}{38}\right)\right\}}{\left(\frac{93}{38}\right)}$$

$$= -\frac{259}{93}$$

(7) **Remaining fourth column:**

$$l_{44} = a_{44} - (l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34})$$

$$= 4 - \left\{5\left(\frac{5}{10}\right) + \left(\frac{55}{10}\right)\left(\frac{10}{38}\right) - \left(\frac{89}{38}\right)\left(\frac{392}{93}\right)\right\}$$

$$= \frac{35074}{3534}$$

(8) **Remaining fourth row:**

$$y_4 = \frac{b_4 - \{l_{41}y_1 + l_{42}y_2 + l_{43}y_3\}}{l_{44}}$$

$$= \frac{7 - \left\{5\left(\frac{6}{10}\right) - \left(\frac{55}{10}\right)\left(\frac{86}{38}\right) + \left(\frac{89}{38}\right)\left(\frac{259}{93}\right)\right\}}{\left(\frac{35074}{3534}\right)} = 1.$$

Scanned with CamScanner

Now the solution to the given system is obtained by $UX = Y$, i.e.

$$\begin{bmatrix} 1 & -\frac{7}{10} & \frac{3}{10} & \frac{5}{10} \\ 0 & 1 & \frac{8}{38} & -\frac{10}{38} \\ 0 & 0 & 1 & \frac{392}{93} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{6}{10} \\ \frac{86}{38} \\ -\frac{259}{93} \\ 1 \end{bmatrix}$$

$$x_1 - \frac{7}{10}x_2 + \frac{3}{10}x_3 + \frac{5}{10}x_4 = \frac{6}{10}$$

$$x_2 + \frac{8}{38}x_3 - \frac{10}{38}x_4 = \frac{86}{38}$$

$$x_3 + \frac{392}{93}x_4 = -\frac{259}{93}$$

$$x_4 = 1$$

$\therefore$ By back substitution, we get

$x_4 = 1$, $x_3 = -7$, $x_2 = 4$, and $x_1 = 5$,

4.7 INVERSE OF MATRIX BY CROUT'S METHOD

We have seen that by crout's method the square matrix A can be decomposed into $A = LU$, where L is the lower triangular matrix and U, the unit upper triangular matrix.

$$\therefore \quad A^{-1} = (LU)^{-1} = U^{-1} L^{-1} \tag{4.18}$$

where U^{-1}, L^{-1} are also unit upper triangular, lower triangular matrices.

Since L and U are known, their inverses i.e. L^{-1} and U^{-1}, can be determined using $LL^{-1} = I$ and $UU^{-1} = I$ and hence, A^{-1} from Eqn (4.18).

Example 4.15 Find the inverse of $\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ by Crout's method.

Solution Let $A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

Scanned with CamScanner

and the derived matrix be $\begin{bmatrix} l_{11} & u_{12} & u_{13} \\ l_{21} & l_{22} & u_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

Then $l_{11} = 2, l_{21} = 2, l_{31} = -1$

$u_{12} = -1, u_{13} = 2, l_{22} = 5, l_{32} = 0, u_{23} = -\frac{2}{5}$ and $l_{33} = 1$ [using Crout's scheme].

$$\therefore \qquad L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Let $\quad L^{-1} = \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ and $U^{-1} = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Using $LL^{-1} = I$, we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow 2x_{11} = 1; \quad 2x_{11} + 5x_{21} = 0; \quad 5x_{22} = 1;$

$-x_{11} + x_{31} = 0; \quad x_{32} = 0; \quad x_{33} = 1$

$\Leftrightarrow x_{11} = \frac{1}{2}, \quad x_{21} = -\frac{1}{5}, \quad x_{31} = \frac{1}{2}, \quad x_{22} = \frac{1}{5}, \quad x_{32} = 0, \quad x_{33} = 1$

Similarly, using $UU^{-1} = I$, i.e.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & 1 & y_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scanned with CamScanner

We get $y_{12} = 1, y_{13} = -8/5; y_{23} = 2/5$

$$\therefore L^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \text{ and } U^{-1} = \begin{bmatrix} 1 & 1 & -\frac{8}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

EXERCISE 4.2

Solve the following equations by Factorisation (or Triangularisation) method.

1. $3x + y + 2z = 16;\ 2x - 6y + 8z = 24;\ 5x + 4y - 3z = 2$
2. $3x + 2y + 7z = 32;\ 2x + 3y + z = 40;\ 3x + 4y - z = 56$
3. $10x + y + z = 12;\ 2x + 10y + z = 13;\ x + y + 5z = 7$

   (M.U, B.E., 199
4. $28x + 4y - z = 32;\ x + 3y + 10z = 24;\ 2x + 17y + 4z = 35$
5. $2x - y + z = 0.3;\ -4x + 3y - 2z = -1.4;\ 3x - 8y + 3z = 0.1$
6. $10x + 7y + 8z + 7w = 32;\ 7x + 5y + 6z + 5w = 23,$
   $8x + 6y + 10z + 9w = 33;\ 7x + 5y + 9z + 10w = 31$

Solve the following equations by Crout's method

7. $x + 3y + 8z = 4,\ x + 4y + 3z = -2;\ x + 3y + 4z = 1$ (M.U, B.E., 19
8. $2x - 6y + 8z = 24,\ 5x + 4y - 3z = 2;\ 3x + y + 2z = 16$ (M.U, B.E., 19
9. $10x + y + 2z = 13;\ 3x + 10y + z = 14,\ 2x + 3y + 10z = 15$

   (Madurai, B.E., 19
10. $9x - 2y + z = 50,\ x + 5y - 3z = 18,\ -2x + 2y + 7z = 19$
11. $10x + y + z = 12,\ 2x + 10y + z = 13,\ 2x + 2y + 10z = 14$

    (Coimbatore, B.Tech., 19

Scanned with CamScanner

12. $3x+y+2z=3,\ 2x-3y-z=-3,\ x-2y+z=4$

13. $10x_1+9x_2+6x_3+x_4=26,\ 11x_1+6x_2-x_3+2x_4=18$
$x_1-7x_2+3x_3+6x_4=3,\ 7x_1+x_2+x_3+x_4=10$

14. $5x+y+z+w=4,\ x+7y+z+4w=12,$
$x+y+6z+w=-5,\ x+y+z+4w=-6$

Find the inverse of the following matrices using Crout's method.

(15) $\begin{bmatrix} -2 & 4 & 8 \\ -4 & 18 & -16 \\ -6 & 2 & -20 \end{bmatrix}$ (16) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

(17) $\begin{bmatrix} 13 & 14 & 6 & 4 \\ 8 & -1 & 13 & 9 \\ 6 & 7 & 3 & 2 \\ 9 & 5 & 16 & 11 \end{bmatrix}$

ANSWERS

(1) 1, 3, 5
(2) 7, 9, −1
(3) 1, 1, 1
(4) 0.996, 1.5070, 1.8485
(5) 1.6, −0.8, −3.71
(6) 1, 1, 1, 1
(7) $\frac{19}{4}, -\frac{9}{4}, \frac{3}{4}$
(8) 1, 3, 5
(9) 1, 1, 1
(10) 6.13, 4.31, 3.23
(11) 1, 1, 1
(12) 1, 2, −1
(13) 1, 1, 1, 1
(14) 1, 2, −1, −2

(15) $\frac{1}{190}\begin{bmatrix} -41 & 12 & -26 \\ 2 & 11 & -8 \\ 12.5 & -2.5 & -2.5 \end{bmatrix}$ (16) $\frac{1}{12}\begin{bmatrix} -5 & 3 & 4 \\ -7 & 3 & -8 \\ 1 & -3 & 4 \end{bmatrix}$

(17) $\begin{bmatrix} 1 & 0 & -2 & 0 \\ -5 & 1 & 11 & -1 \\ 287 & -67 & -630 & 65 \\ -416 & 97 & 913 & -94 \end{bmatrix}$

Scanned with CamScanner

4.8 ITERATIVE METHODS

So far we have studied some direct methods which yield, after a certa amount of fixed computation, the solution to simultaneous linear equation Now we shall discuss the *iterative* or *indirect methods*. In these metho we start from an approximation to the true solution and, if converge derive a sequence of closer approximations. We repeat the cycle computations till the required accuracy is obtained.

But, the method of iteration is not applicable to all systems of equation For this, each equation of the system must contain one large coefficie (much larger than the others in that equation) and the large coefficie must be attached to a different unknown in that equation.

In other words, the solution to a system of linear equations will exist b iterative procedure *if the absolute value of the largest coefficient is great than the sum of the absolute values of all remaining coefficients in eac equation* (condition for convergence).

Now let us see some of such methods in detail.

4.9 JACOBI METHOD OF ITERATION

This method is also known as Gauss–Jacobi method. Here, consider th following system of equations.

$$a_1x + b_1y + c_1z = a_1' \quad (4.19a)$$
$$a_2x + b_2y + c_2z = d_2 \quad (4.19b)$$
$$a_3x + b_3y + c_3z = d_3 \quad (4.19c)$$

Let $|a_1| > |b_1| + |c_1|$; $|b_2| > |a_2| + |c_2|$; $|c_3| > |a_3| + |b_3|$

That is, in each equation the coefficients of the diagonal terms are larg Hence the system (4.19) is ready for iteration. Solving for x, y and respectively, we get

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \quad (4.20a)$$

$$y = \frac{1}{b_2}(d_2 - a_1x - c_2z) \quad (4.20b)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \quad (4.20c)$$

Let x_0, y_0 and z_0 be the initial approximations of the unknowns x, y and z. Substituting these on RHS of Eqns (4.20) the first approximations are given by

Scanned with CamScanner

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_3 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

Substituting the values x_1, y_1 and z_1 in the RHS of Eqn (4.20), the second approximations are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

Proceeding in the same way, if x_r, y_r, z_r are the rth iterates then

$$x_{r+1} = \frac{1}{a_1}(d_1 - b_1 y_r - c_1 z_r)$$

$$y_{r+1} = \frac{1}{b_2}(d_2 - a_2 x_r - c_2 z_r)$$

$$z_{r+1} = \frac{1}{c_3}(d_3 - a_3 x_r - b_3 y_r)$$

The process is continued till convergency is secured.

Note: In the absence of any better estimates, the initial approximations are taken as $x_0 = 0, y_0 = 0, z_0 = 0$

Example 4.16 Solve by Jacobi iteration method the system $8x - 3y + 2z = 20$; $6x + 3y + 12z = 35$; and $4x + 11y - z = 33$.

(Bangalore, B.E., 1994, M.U, B.E., 1981)

Solution Consider the given system as

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35$$

Scanned with CamScanner

so that the diagonal elements are dominant in the coefficient matrix. No
we write the equations in the form

$$x = \frac{1}{8}(20 + 3y - 2z) \quad \text{(i}$$

$$y = \frac{1}{11}(33 - 4x + z) \quad \text{(i}$$

$$z = \frac{1}{12}(35 - 6x - 3y) \quad \text{(i}$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.
Substituting these on RHS of Eqns (i), we get

First approximation as

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y_1 = \frac{1}{11}[33 - 4(0) + 0] = 3$$

$$z_1 = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.9166667$$

Second approximation:

Substituting x_1, y_1, z_1 on RHS of Eqns (i), we get

$$x_2 = \frac{1}{8}[20 + 3(3) - 2(2.9166667)] = 2.895833$$

$$y_2 = \frac{1}{11}[33 - 4(2.5) + 2.9166667] = 2.3560606$$

$$z_2 = \frac{1}{12}[35 - 6(2.5) - 3(3)] = 0.9166666$$

Third approximation:

Substituting x_2, y_2, z_2 on RHS of Eqns (i), we get

$$x_3 = \frac{1}{8}[20 + 3(2.3560606) - 2(0.9166666)] = 3.1543561$$

$$y_3 = \frac{1}{11}[33 - 4(2.8958333) + 0.9166666] = 2.030303$$

$$z_3 = \frac{1}{12}[35 - 6(2.8958333) - 3(2.3560606)] = 0.8797348$$

Scanned with CamScanner

Fourth approximation:

Substituting x_3, y_3, z_3 on RHS of Eqns (i), we get

$$x_4 = \frac{1}{8}[20 + 3(2.030303) - 2(0.8797348)] = 3.0419299$$

$$y_4 = \frac{1}{11}[33 - 4(3.1543561) + 0.8797348] = 1.9329373$$

$$z_4 = \frac{1}{12}[35 - 6(3.1543561) - 3(2.030303)] = 0.8319128$$

Fifth approximation:

Substituting x_4, y_4, z_4 on RHS of Eqns (i), we get

$$x_5 = \frac{1}{8}[20 + 3(1.9329373) - 2(0.8319128)] = 3.0168733$$

$$y_5 = \frac{1}{11}[33 - 4(3.0414299) + 0.8319128] = 1.9696539$$

$$z_5 = \frac{1}{12}[35 - 6(3.0414299) - 3(1.9329373)] = 0.9127173$$

Sixth approximation:

Substituting x_5, y_5, z_5 on RHS of Eqns (i), we get

$$x_6 = \frac{1}{8}[20 + 3(1.9696539) - 2(0.9127173)] = 3.0104409$$

$$y_6 = \frac{1}{11}[33 - 4(3.0168733) + 0.9127173] = 1.9859295$$

$$z_6 = \frac{1}{12}[35 - 6(3.0168733) - 3(1.9696539)] = 0.9158165$$

Seventh approximation:

Substituting x_6, y_6, z_6 on RHS of Eqns (i), we get

$$x_7 = \frac{1}{8}[20 + 3(1.9859295) - 2(0.9158165)] = 3.0157694$$

$$y_7 = \frac{1}{11}[33 - 4(3.0104409) + 0.9158165] = 1.9885503$$

$$z_7 = \frac{1}{12}[35 - 6(3.0104409) - 3(1.9859295)] = 0.9149638$$

Scanned with CamScanner

Eighth approximation:

Substituting x_7, y_7, z_7 on RHS of Eqns (i), we get

$$x_8 = \frac{1}{8}[20 + 3(1.9885503) - 2(0.9149638)] = 3.0169654$$

$$y_8 = \frac{1}{11}[33 - 4(3.0157694) + 0.9149638] = 1.9865351$$

$$z_8 = \frac{1}{12}[35 - 6(3.0157694) - 3(1.9885503)] = 0.9116443$$

Ninth approximation:

Substituting x_8, y_8, z_8 on RHS of Eqns (i), we get

$$x_9 = \frac{1}{8}[20 + 3(1.9865351) - 2(0.9116443)] = 3.0170396$$

$$y_9 = \frac{1}{11}[33 - 4(3.0169654) + 0.9116443] = 1.9857984$$

$$z_9 = \frac{1}{12}[35 - 6(3.0169654) - 3(1.9865351)] = 0.9115501$$

Tenth approximation:

Substituting x_9, y_9, z_9 on RHS of Eqns (i), we get

$$x_{10} = \frac{1}{8}[20 + 3(1.9857984) - 2(0.9115501)] = 3.0167869$$

$$y_{10} = \frac{1}{11}[33 - 4(3.0170396) + 0.9115501] = 1.9857629$$

$$z_{10} = \frac{1}{12}[35 - 6(3.0170396) - 3(1.9857984)] = 0.9116972$$

Eleventh approximation:

Substituting x_{10}, y_{10}, z_{10} on RHS of Eqns (i), we get

$$x_{11} = \frac{1}{8}[20 + 3(1.9857629) - 2(0.9116972)] = 3.0167368$$

$$y_{11} = \frac{1}{11}[33 - 4(3.0167869) + 0.9116972] = 1.9858681$$

$$z_{11} = \frac{1}{12}[35 - 6(3.0167869) - 3(1.9857629)] = 0.9118326$$

Scanned with CamScanner

Twelfth approximation:

Substituting x_{11}, y_{11}, z_{11} on RHS of Eqns (i), we get

$$x_{12} = \frac{1}{8}[20 + 3(1.9858681) - 2(0.91183226)] = 3.0167424$$

$$y_{12} = \frac{1}{11}[33 - 4(3.0167368) + 0.9118326] = 1.9858987$$

$$z_{12} = \frac{1}{12}[35 - 6(3.0167368) - 3(1.9858681)] = 0.9118312$$

$\therefore$ From the 11th and 12th approximations, the values of x, y, z are same correct to four decimal places. Stopping at this stage, we get $x = 3.0167$, $y = 1.9858$, $z = 0.9118$.

Example 4.17 Solve the following system of equations by Jacobi iteration method:

$3x + 4y + 15z = 54.8$, $x + 12y + 3z = 39.66$

and $10x + y - 2z = 7.74$

(*M.U, 1990*)

Solution The coefficient matrix of the given system is not diagonally dominant. Hence we rearrange the equations, as follows, such that the elements in the coefficient matrix are diagonally dominant

$$10x + y - 2z = 7.74$$
$$x + 12y + 3z = 39.66$$
$$3x + 4y + 15z = 54.8$$

Now we write the equations in the form

$$x = \frac{1}{10}(7.74 - y + 2z) \qquad \text{(ia)}$$

$$y = \frac{1}{12}(39.66 - x - 3z) \qquad \text{(ib)}$$

$$z = \frac{1}{15}(54.8 - 3x - 4y) \qquad \text{(ic)}$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on RHS of (i), we get

First approximation as

$$x_1 = \frac{1}{10}[7.74 - 0 + 2(0)] = 0.774$$

Scanned with CamScanner

$$y_1 = \frac{1}{12}[39.66 - 0 + 3(0)] = 1.1383333$$

$$z_1 = \frac{1}{15}[54.8 - 3(0) + 4(0)] = 3.6533333$$

Second approximation

$$x_2 = \frac{1}{10}[7.74 - 1.1383333 + 2(3.6533333)] = 1.3908333$$

$$y_2 = \frac{1}{12}[39.66 - 0.774 - 3(3.6533333)] = 2.3271667$$

$$z_2 = \frac{1}{15}[54.8 - 3(0.774) + 4(1.1383333)] = 3.1949778$$

Third approximation

$$x_3 = \frac{1}{10}[7.74 - 2.3271667 + 2(3.1949778)] = 1.1802789$$

$$y_3 = \frac{1}{12}[39.66 - 1.3908333 - 3(3.1949778)] = 2.3903528$$

$$z_3 = \frac{1}{15}[54.8 - 3(1.3908333) - 4(2.3271667)] = 2.7545889$$

Fourth approximation

$$x_4 = \frac{1}{10}[7.74 - 2.3903528 + 2(2.7545889)] = 1.0858825$$

$$y_4 = \frac{1}{12}[39.66 - 1.1802789 - 3(2.7545889)] = 2.5179962$$

$$z_4 = \frac{1}{15}[54.8 - 3(1.1802789) - 4(2.3903528)] = 2.7798501$$

Fifth approximation

$$x_5 = \frac{1}{10}[7.74 - 2.5179962 + 2(2.7798501)] = 1.0781704$$

$$y_5 = \frac{1}{12}[39.66 - 1.0858825 - 3(2.7798501)] = 2.5195473$$

$$z_5 = \frac{1}{15}[54.8 - 3(1.0858825) - 4(2.5179962)] = 2.7646912$$

Scanned with CamScanner

Sixth approximation

$$x_6 = \frac{1}{10}[7.74 - 2.5195473 + 2(2.7646912)] = 1.0749835$$

$$y_6 = \frac{1}{12}[39.66 - 1.0781704 - 3(2.7646912)] = 2.5239797$$

$$z_6 = \frac{1}{15}[54.8 - 3(1.0781704) + 4(2.5195473)] = 2.76582$$

Seventh approximation

$$x_7 = \frac{1}{10}[7.74 - 2.5239797 + 2(2.76582)] = 1.074766$$

$$y_7 = \frac{1}{12}[39.66 - 1.0749835 - 3(2.76582)] = 2.523963$$

$$z_7 = \frac{1}{15}[54.8 - 3(1.0749835) + 4(2.5239797)] = 2.7652754$$

$\therefore$ From the sixth and seventh approximations
$x = 1.075$, $y = 2.524$ and $z = 2.765$ correct to three decimals.

4.10 GAUSS–SEIDEL ITERATION METHOD

This is a modification of Gauss–Jacobi method. As before, the system of the linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is written as

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z) \tag{4.21a}$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \tag{4.21b}$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \tag{4.21c}$$

Scanned with CamScanner

and we start with the initial approximation x_0, y_0, z_0. Substituting y_0 and z_0 in Eqn (4.21a), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

Now substituting $x = x_1$, $z = z_0$ in Eqn (4.21b), we get

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

Substituting $x = x_1$, $y = y_1$, in Eqn (4.21c), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is continued till the values of x, y, z are obtained to the desired degree of accuracy. The general algorithm is as follows:

If x_k, y_k, z_k are the kth iterates, then

$$x_{k+1} = \frac{1}{a_1}(d_1 - b_1 y_k - c_1 z_k)$$

$$y_{k+1} = \frac{1}{b_2}(d_2 - a_2 x_{k+1} - c_2 z_k)$$

and

$$z_{k+1} = \frac{1}{c_3}(d_3 - a_3 x_{k+1} - b_3 y_{k+1})$$

Since the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration, this method is more rapid in convergence than Gauss–Jacobi method.

The rate of convergence of Gauss–Seidel method is roughly twice that of Gauss–Jacobi and the condition of convergence is same as we saw earlier in Section 4.8.

Note: Gauss–Seidel iteration method converges only for special systems of equations. In general, the round of errors will be small in iteration methods. Moreover, these are self-correcting methods; that is, any error made in computation will be corrected in the subsequent iteration.

Example 4.18 Solve the equations given in Example 4.9 by Gauss–Seidel iteration method.

(B.U, B.E., 1994)

Scanned with CamScanner

Solution From the given equations, we have

$$x = \frac{1}{8}(20 + 3y - 2z) \qquad \text{(i)}$$

$$y = \frac{1}{11}(33 - 4x + z) \qquad \text{(ii)}$$

$$z = \frac{1}{12}(35 - 6x - 3y) \qquad \text{(iii)}$$

Putting $y = 0$, $z = 0$ in RHS of (i), we get $x = \frac{20}{8} = 2.5$

Putting $x = 2.5$, $z = 0$ in RHS of (ii), we get

$$y = \frac{1}{11}[33 - 4(2.5)] = 2.0909091$$

Putting $x = 2.5$, $y = 2.0909091$ in RHS of (iii), we get

$$z_1 = \frac{1}{12}[35 - 6(2.5) - 3(2.0909091)] = 1.1439394$$

For the *second approximation*,

$$x_2 = \frac{1}{8}[20 + 3y_1 - 2z_1]$$

$$= \frac{1}{8}[20 + 3(2.0909091) - 2(1.1439394)] = 2.9981061$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1]$$

$$= \frac{1}{11}[33 - 4(2.9981061) + 1.1439394] = 2.0137741$$

$$z_2 = \frac{1}{12}[35 - 6x_2 - 3y_2]$$

$$= \frac{1}{12}[35 - 6(2.9981061) - 3(2.0137741)] = 0.9141701$$

Third approximation:

$$x_3 = \frac{1}{8}[20 + 3(2.0137741) - 2(0.9141701)] = 3.0266228$$

Scanned with CamScanner

$$y_3 = \frac{1}{11}[33 - 4(3.0266228) + 0.9141701)] = 1.9825163$$

$$z_3 = \frac{1}{12}[35 - 6(3.0266228) - 3(1.9825163)] = 0.9077262$$

Fourth approximation:

$$x_4 = \frac{1}{8}[20 + 3(1.9825163) - 2(0.9077262)] = 3.0165121$$

$$y_4 = \frac{1}{11}[33 - 4(3.0165121) + 0.9077262)] = 1.9856071$$

$$z_4 = \frac{1}{12}[35 - 6(3.0165121) - 3(1.9856071)] = 0.9120088$$

Fifth approximation:

$$x_5 = \frac{1}{8}[20 + 3(1.9856071) - 2(0.9120088)] = 3.0166005$$

$$y_5 = \frac{1}{11}[33 - 4(3.0166005) + 0.9120088] = 1.9859643$$

$$z_5 = \frac{1}{12}[35 - 6(3.0166005) - 3(1.9859643)] = 0.9118753$$

Sixth approximation:

$$x_6 = \frac{1}{8}[20 + 3(1.9859643) - 2(0.9118753)] = 3.0167678$$

$$y_6 = \frac{1}{11}[33 - 4(3.0167678) + 0.9118753] = 1.9858913$$

$$z_6 = \frac{1}{12}[35 - 6(3.0167678) - 3(1.9858913)] = 0.9118099$$

Seventh approximation:

$$x_7 = \frac{1}{8}[20 + 3(1.9858913) - 2(0.9118099)] = 3.0167568$$

$$y_7 = \frac{1}{11}[33 - 4(3.0167568) + 0.9118099] = 1.9858894$$

$$z_7 = \frac{1}{12}[35 - 6(3.0167568) - 3(1.9858894)] = 0.9118159$$

Scanned with CamScanner

Since at the sixth and seventh approximations, the values of x, y, z are the same, correct to four decimal places, we can stop the iteration process.

$\therefore x = 3.0167, \quad y = 1.9858, \quad z = 0.9118$

We find that 12 iterations are necessary in Gauss–Jacobi method to get the same accuracy as achieved by 7 iterations in Gauss–Seidel method.

Example 4.19 Solve by Gauss–Seidel method, the following system of equations

$28x + 4y - z = 32, \quad x + 3y + 10z = 24,$

and $2x + 17y + 4z = 35$ *(B.U. 1997, M.U. 1991)*

Solution The coefficient matrix of the given system is not diagonally dominant. Hence, we rearrange the equations as follows, such that the elements in the coefficient matrix are diagonally dominant.

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Hence we can apply Gauss–Seidal iteration method.

From the above equations

$$x = \frac{1}{28}[32 - 4y + z] \qquad \text{(i)}$$

$$y = \frac{1}{17}[35 - 2x - 4z] \qquad \text{(ii)}$$

$$z = \frac{1}{10}[24 - x - 3y] \qquad \text{(iii)}$$

First approximation

Putting $y = z = 0$ in (i), we get

$$x_1 = \frac{1}{28}(32) = 1.1428571$$

Putting $x = 1.1428571, \; z = 0$ in (ii), we get

$$y_1 = \frac{1}{17}[35 - 2(1.1428571)] = 1.9243697$$

Putting $x = 1.1428571, \; y = 1.9243697$ in (iii), we get

$$z_1 = \frac{1}{10}[24 - 1.1428571 - 3(1.9243697)] = 1.7084034$$

Scanned with CamScanner

Second approximation

$$x_2 = \frac{1}{28}[32 - 4(1.9243697) + 1.7084034] = 0.9289615$$

$$y_2 = \frac{1}{17}[35 - 2(0.9289615) - 4(1.7084034)] = 1.5475567$$

$$z_2 = \frac{1}{10}[24 - 0.9289615 - 3(1.5475567)] = 1.8428368$$

Third approximation

$$x_3 = \frac{1}{28}[32 - 4(1.5475567) + 1.8428368] = 0.9875932$$

$$y_3 = \frac{1}{17}[35 - 2(0.9875932) - 4(1.8428368)] = 1.5090274$$

$$z_3 = \frac{1}{10}[24 - 0.9875932 - 3(1.5090274)] = 1.8485325$$

Fourth approximation

$$x_4 = \frac{1}{28}[32 - 4(1.5090274) + 1.8485325] = 0.9933008$$

$$y_4 = \frac{1}{17}[35 - 2(0.9933008)] = 4(1.8485325)] = 1.5070158$$

$$z_4 = \frac{1}{10}[24 - 0.9933008 - 3(1.5070158)] = 1.8485652$$

Fifth approximation

$$x_5 = \frac{1}{28}[32 - 4(1.5070158) + 1.8485652] = 0.9935893$$

$$y_5 = \frac{1}{17}[35 - 2(0.9935893) - 4(1.8485652)] = 1.5069741$$

$$z_5 = \frac{1}{10}[24 - 0.9935893 - 3(1.5069741)] = 1.8485488$$

Sixth approximation

$$x_6 = \frac{1}{28}[32 - 4(1.5069741) + 1.8485488] = 0.9935947$$

$$y_6 = \frac{1}{17}[35 - 2(0.9935947) - 4(1.8485488)] = 1.5069774$$

Scanned with CamScanner

$$z_6 = \frac{1}{10}[24 - 0.9935947 - 3(1.5069774)] = 1.8485473$$

$\because$ the values of x, y, z in the fourth and fifth iteration are same upto four decimals, we stop the process here.

Hence $x = 0.9936,\ y = 1.5069,\ z = 1.8485$

Example 4.20 Using Gauss–Seidel iteration method, solve the system of equations.

$$10x - 2y - z - w = 3;\ -2x + 10y - z - w = 15;$$
$$-x - y + 10z - 2w = 27;\ -x - y - 2z + 10w = -9$$

(*M.U, B.E., 1987*)

Solution The coefficient matrix of the given system is diagonally dominant. Hence we can apply Gauss–Seidel iteration method.

From the given equations, we can write

$$x = \frac{1}{10}[3 + 2y + z + w] \quad \text{(i)}$$

$$y = \frac{1}{10}[15 + 2x + z + w] \quad \text{(ii)}$$

$$z = \frac{1}{10}[27 + x + y + 2w] \quad \text{(iii)}$$

$$w = \frac{1}{10}[-9 + x + y + 2z] \quad \text{(iv)}$$

First approximation:

Putting $y = z = w = 0$ on RHS of (i), we get $x_1 = \frac{3}{10} = 0.3$

Putting $x = 0.3,\ z = w = 0$ on RHS of (ii),

we get $y_1 = \frac{1}{10}[15 + 2(0.3)] = 1.56$

Putting $x = 0.3,\ y = 1.56,\ w = 0$ on RHS of (iii),

we get $z_1 = \frac{1}{10}[27 + 0.3 + 1.56] = 2.886$

Putting $x = 0.3,\ y = 1.56,\ z = 2.886$ on RHS of (iv),

we get $w_1 = \frac{1}{10}[-9 + 0.3 + 1.56 + 2(2.886)] = -0.1368$

Scanned with CamScanner

Second approximation:

$$x_2 = \frac{1}{10}[3 + 2(1.56) + 2.886 - 0.1368] = 0.88692$$

$$y_2 = \frac{1}{10}[15 + 2(0.88692) + 2.886 - 0.1368] = 1.952304$$

$$z_2 = \frac{1}{10}[27 + 0.88692 + 1.952304 + 2(-0.1368] = 2.9565624$$

$$w_2 = \frac{1}{10}[-9 + 0.88692 + 1.952304 + 2(2.9565624)] = -0.0247651$$

Third approximation:

$$x_3 = \frac{1}{10}[3 + 2(1.952304) + 2.9565624 - 0.0247651] = 0.9836405$$

$$y_3 = \frac{1}{10}[15 + 2(0.9836405) + 2.9565624 - 0.0247651] = 1.9899087$$

$$z_3 = \frac{1}{10}[27 + 0.9836405 + 1.9899087 + 2(-0.0247651)] = 2.9924019$$

$$w_3 = \frac{1}{10}[-9 + 0.983405 + 1.9899087 + 2(2.9924019)] = -0.0041647$$

Fourth approximation:

$$x_4 = \frac{1}{10}[3 + 2(1.9899087) + 2.9924019 - 0.0041647] = 0.9968054$$

$$y_4 = \frac{1}{10}[15 + 2(0.9968054) + 2.9924019 - 0.0041647] = 1.9981848$$

$$z_4 = \frac{1}{10}[27 + 0.9968054 + 1.9981848 + 2(-0.0041647)] = 2.9986661$$

$$w_4 = \frac{1}{10}[-9 + 0.9968054 + 1.9981848 + 2(2.9986661)] = -0.0007677$$

Fifth approximation:

$$x_5 = \frac{1}{10}[3 + 2(1.9981848) + 2.9986661 - 0.0007677] = 0.9994268$$

$$y_5 = \frac{1}{10}[15 + 2(0.9994268) + 2.998666 - 0.0007677] = 1.9996752$$

Scanned with CamScanner

$$z_5 = \frac{1}{10}[27 + 0.9994268 + 1.9996752 + 2(-0.0007677)] = 2.9997567$$

$$w_5 = \frac{1}{10}[-9 + 0.9994268 + 1.9996752 + 2(2.9997567)] = -0.0001384$$

Sixth approximation:

$$x_6 = \frac{1}{10}[3 + 2(1.9996752) + 2.9997567 - 0.0001384] = 0.9998968$$

$$y_6 = \frac{1}{10}[15 + 2(0.9998968) + 2.9997567 - 0.0001384] = 1.9999412$$

$$z_6 = \frac{1}{10}[27 + 0.9998968 + 1.9999412 + 2(-0.0001384)] = 2.9999561$$

$$w_6 = \frac{1}{10}[-9 + 0.9998968 + 1.9999412 + 2(2.9999561)] = -0.0002498$$

Seventh approximation:

$$x_7 = \frac{1}{10}[3 + 2(1.9999412) + 2.9999561 - 0.0002498] = 0.9999588$$

$$y_7 = \frac{1}{10}[15 + 2(0.9999588) + 2.9999561 - 0.0002498] = 1.9999624$$

$$z_7 = \frac{1}{10}[27 + 0.9999588 + 1.9999264 + 2(-0.0002498)] = 2.9999422$$

$$w_7 = \frac{1}{10}[-9 + 0.9999588 + 1.9999624 + 2(2.9999422)] = -0.0001945$$

Now, from sixth and seventh approximations the values of x, y, z and w correct to four decimal places are

$x = 0.9999$ $\quad y = 1.9999$ $\quad z = 2.9999$ $\quad w = 0.0002$

4.11 RELAXATION METHOD

Consider the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Scanned with CamScanner

We define the residuals r_1, r_2, r_3 by the relations

$$r_1 = d_1 - a_1 x - b_1 y - c_1 z \quad (4.22a)$$
$$r_2 = d_2 - a_2 x - b_2 y - c_2 z \quad (4.22b)$$
$$r_3 = d_3 - a_3 x - b_3 y - c_3 z \quad (4.22c)$$

To start with, we assume $x = y = z = 0$ and calculate the initial residuals. Then these residuals are reduced step by step giving increments to the variables. If we can find x, y, z such that residuals $r_1 = r_2 = r_3 = 0$ then those values of x, y, z are the exact values. Otherwise, we liquidate the residuals smaller and smaller and finally negligible to get better approximate values of x, y, z. For this purpose we construct an *operation table* as shown below.

x	y	z	r_1	r_2	r_3
1	0	0	$-a_1$	$-a_2$	$-a_3$
0	1	0	$-b_1$	$-b_2$	$-b_3$
0	0	1	$-c_1$	$-c_2$	$-c_3$

From Eqns (4.22), we can see that if x is increased by 1, keeping y and z constant, r_1, r_2 and r_3 decrease by a_1, a_2 and a_3, respectively.

This is shown in the above table along with the effects on the residuals when y and z are given unit increments. It can be noted that the operation table consists of the unit matrix I and transpose of the coefficient matrix.

At each step, the numerically largest residual is reduced almost to zero. To reduce a particular residual, the value of the corresponding variable is changed, i.e. to reduce, say, r_2 by α, y should be increased by $\frac{\alpha}{b_2}$. When all the residuals have been reduced to almost zero, then the increments in x, y and z are added separately to give the desired solution.

Note: After finding x, y and z, substitute them in Eqns (4.22), and check whether the residuals are negligible or not. If not, then there is some mistake and the entire process should be rechecked.

Convergency of Relaxation Method

This method can be applied successfully only if the diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row, i.e. if in Eqns (4.22)

$$|a_1| \geq |b_1| + |c_1|$$
$$|b_2| \geq |a_2| + |c_2|$$

and

$$|c_3| \geq |a_3| + |b_3|$$

with strict inequality for atleast one row.

Scanned with CamScanner

Example 4.21 Solve by relaxation method, the equations
$10x - 2y - 2z = 6;\ -x - 10y - 2z = 7;\ -x - y + 10z = 8$

(*M.U, B.E., 1991*)

Solution The residuals r_1, r_2 and r_3 are given by

$$r_1 = 6 - 10x + 2y + 2z$$
$$r_2 = 7 + x - 10y + 2z$$
$$r_3 = 8 + x + y - 10z$$

The operation table is as follows.

x	y	z	r_1	r_2	r_3		
1	0	0	−10	1	1	←	L_1
0	1	0	2	−10	1	←	L_2
0	0	1	2	2	−10	←	L_3

The relaxation table is as follows.

x	y	z	r_1	r_2	r_3		
0	0	0	6	7	8	←	L_4
0	0	1	8	9	−2	←	$L_5 = L_4 + L_3$
0	1	0	10	−1	−1	←	$L_6 = L_5 + L_2$
1	0	0	0	0	0	←	$L_7 = L_6 + L_1$

Explanation:

(i) In line L_4, the largest residual is 8. To reduce it, we give an increment of $\frac{8}{c_3} = \frac{8}{10} = 0.8 \simeq 1$. The resulting residuals are obtained by $L_4 + (1)\, L_3$, i.e.line L_5.

(ii) In line L_5, the largest residual is 9.

$$\therefore \text{ increment} = \frac{9}{b_2} = \frac{9}{10} = 0.9 \simeq 1$$

The resulting residuals $(= L_6) = L_5 + 1.L_2$

(iii) In line L_6, the largest residual is 10.

$$\therefore \text{ increment} = \frac{10}{a_1} = \frac{10}{10} = 1$$

Scanned with CamScanner

The resulting residuals $(= L_7) = L_6 + 1.\,L_3$, which are all zeros.

$\Rightarrow$ Exact solution is arrived and it is $x = 1, y = 1, z = 1$

Example 4.22 Solve by relaxation method, the equations

$$9x - y + 2z = 9,\ x + 10y - 2z = 15,\ 2x - 2y - 13z = -17$$

(M.U. B.E., 1996)

Solution The residuals r_1, r_2, r_3 are given by

$$r_1 = 9 - 9x + y - 2z\,;\ r_2 = 15 - x - 10y + 2z\,;\ r_3 = -17 - 2x + 2y + 13z \quad \text{(i)}$$

Operation table

x	y	z	r_1	r_2	r_3	
1	0	0	−9	−1	−2	$\leftarrow L_1$
0	1	0	1	−10	2	$\leftarrow L_2$
0	0	1	−2	2	13	$\leftarrow L_3$

Relaxation table

x	y	z	r_1	r_2	r_3	
0	0	0	9	15	−17	$\leftarrow L_4$
0	0	1	7	17	−4	$\leftarrow L_5 = L_4 + 1.L_3$
0	1	0	8	7	−2	$\leftarrow L_6 = L_5 + 1.L_2$
0.89	0	0	−0.01	6.11	−3.78	$\leftarrow L_7 = L_6 + 0.89L_1$
0	0.61	0	0.6	0.01	−2.56	$\leftarrow L_8 = L_7 + 0.61L_3$
0	0	0.19	0.22	0.39	−0.09	$\leftarrow L_9 = L_8 + 0.1L_3$
0	0.039	0	0.259	0	−0.012	$\leftarrow L_{10} = L_9 + 0.039L_2$
0.028	0	0	0.007	−0.028	−0.068	$\leftarrow L_{11} = L_{10} + 0.028L_1$
0	0	0.00523	−0.00346	−1.01754	−0.00001	$\leftarrow L_{12} = L_{11} + 0.00523L_3$

Thus, $x = 0.89 + 0.028 = 0.918$; $y = 1 + 0.61 + 0.039 = 1.649$ and

$z = 1 + 0.19 + 0.00523 = 1.19523$.

Now substituting the values of x, y, z in (i), we get

$$r_1 = 9 - 9\,(0.918) + 1.649 - 2(1.19523) = -0.00346$$

$$r_2 = 15 - 0.918 - 10\,(1.649) + 2\,(1.19523) = -0.1754$$

$$r_3 = -17 - 2\,(0.918) + 2\,(1.649) + 13\,(1.19523) = -0.00001$$

which are in agreement with the final residuals in the table.

Scanned with CamScanner

Example 4.23 Solve by relaxation method, the equations $10x - 2y + z = 12$, $x + 9y - z = 10$ and $2x - y + 11z = 20$

(M.U, 1997)

Solution The residuals r_1, r_2, r_3 are given by

$$r_1 = 12 - 10x + 2y - z$$
$$r_2 = 10 - x - 9y + z$$
$$r_3 = 20 - 2x + y - 11z$$

The operation table is

x	y	z	r_1	r_2	r_3	
1	0	0	−10	−1	−2	$\leftarrow L_1$
0	1	0	2	−9	1	$\leftarrow L_2$
0	0	1	−1	1	−11	$\leftarrow L_3$

The relation table is

x	y	z	r_1	r_2	r_3	
0	0	0	12	10	20	$\leftarrow L_4$
0	0	1.8	10.2	11.8	0.8	$\leftarrow L_5 = L_4 + 1.8\,L_3$
0	1.31	0	12.82	0.01	1.51	$\leftarrow L_6 = L_5 + 1.31\,L_2$
1.28	0	0	0.02	−1.27	−1.05	$\leftarrow L_7 = L_6 + 1.28\,L_1$
0.02	0	0	0	−1.272	−1.054	$\leftarrow L_8 = L_7 + 0.002\,L_1$
0	0	−0.1	0.1	−1.372	0.046	$\leftarrow L_9 = L_8 - 0.1\,L_3$
0.01	0	0	0	1.382	0.026	$\leftarrow L_{10} = L_9 + 0.01\,L_1$
0	0	0.0023	−0.0023	−1.3797	0.0007	$\leftarrow L_{11} = L_{10} + 0.0023$

Thus $x = 1.28 + 0.02 + 0.01 = 1.31$; $y = 1.31$ and
$z = 1.8 - 0.1 + 0.0023 = 1.7023$.

EXERCISE 4.3

Solve the following system of linear equations by (i) Gauss and (ii) Gauss Seidel iteration method.

1. $2x + y + z = 4$, $x + 2y + z = 4$; $x + y + 2z = 4$
2. $8x + y + z = 8$; $2x + 4y + z = 4$; $x + 3y + 5z = 5$
3. $5x + 2y + z = 12$, $x + 4y + 2z = 15$, $x + 2y + 5z = 20$

Scanned with CamScanner

4. $9x + 2y + 4z = 20, x + 10y + 4z = 6, 2x - 4y + 10z = -15$

5. $54x + y + z = 110, 2x + 15y + 6z = 72, -x + 6y + 27z = 85$ (M.U, B.E., 1993)

6. $28x - 4y - z = 32, x + 3y + 10z = 24, 2x + 17y + 4z = 35$

7. $5x - y + z = 10, 2x + 4y = 12, x + y + 5z = -1$ (Bangalore, B.E., 1990)

8. $10x_1 - 5x_2 - 2x_3 = 3, 4x_1 - 10x_2 + 3x_3 = -3, x_1 + 6x_2 + 10x_3 = -3$

9. $10x + 2y + z = 9, 2x + 20y - 2z = -44, -2x + 3y + 10z = 22$

10. $10x_1 + 7x_2 + 8x_3 + 7x_4 = 32, 7x_1 + 5x_2 + 6x_3 + 5x_4 = 23;$
$8x_1 + 6x_2 + 10x_3 + 9x_4 = 33, 7x_1 + 5x_2 + 9x_3 + 10x_4 = 31$

Solve by relaxation method the following equations:

11. $9x + 2y + z = 50, x + 5y - 3z = 18, -2x + 2y + 7z = 19$

12. $3x + 9y - 2z = 11, 4x + 2y + 13z = 24, 4x - 4y + 3z = -8$ (M.U, B.E., 1993)

13. $4.215x - 1.212y + 1.105z = 3.216$
$-2.120x + 3.505y - 1.632z = 1.247$
$1.122x - 1.313y + 3.986z = 2.112$

14. $10x - 2y - 3z = 305, -2x + 10y - 2z = 154, -2x - y + 10z = 120$

15. $8x_1 + x_2 + x_3 + x_4 = 14; 2x_1 + 10x_2 + 3x_3 + x_4 = -8$
$x_1 - 2x_2 - 20x_3 + 3x_4 = 111, 3x_1 + 2x_2 + 2x_3 + 19x_4 = 53$

ANSWERS

1. $x = 1, y = 1, z = 1$
2. $x = 0.876, y = 0.919, z = 0.574$
3. $x = 0.996, y = 1.95, z = 3.16$
4. $x = 2.733, y = 0.986, z = -1.652$
5. $x = 1.926, y = 3.573, z = 2.425$
6. $x = 0.994, y = 1.507, z = 1.849$
7. $x = 2.556, y = 1.722, z = -1.055$
8. $x_1 = 0.342, x_2 = 0.285, x_3 = -0.505$
9. $x = 1.013, y = -1.996, z = 3.001$
10. $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$
11. $x = 6.13, y = 4.31, z = 3.23$
12. $x = 1.35, y = 2.103, z = 2.845$
13. $x = 0.943, y = 1.239, z = 0.673$
14. $x = 32, y = 26, z = 21$
15. $x_1 = 2, x_2 = 0, x_3 = -5, x_4 = 3$

Scanned with CamScanner