YOU NEED 27 TICKETS TO GUARANTEE A WIN ON THE UK NATIONAL LOTTERY

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ABSTRACT. In the UK National Lottery, players purchase tickets comprising their choices of six different numbers between 1 and 59. During the draw, six balls are randomly selected without replacement from a set numbered from 1 to 59. A prize is awarded to any player who matches at least two of the six drawn numbers. We identify 27 tickets that guarantee a prize, regardless of which of the 45,057,474 possible draws occurs. Moreover, we determine that 27 is the optimal number of tickets required, as achieving the same guarantee with 26 tickets is not possible.

1. INTRODUCTION

A lottery is a popular method of gambling in which players try to guess random numbers in advance of their generation. For our purposes, a lottery revolves around sampling random draws of size p from the set of numbers $\{1, \ldots, n\}$; participants purchase tickets containing their choice of a subset of k of these numbers and will win a prize if one of their tickets has at least t numbers in common with those drawn. A well-studied question is to determine the minimum number of tickets, denoted $j = L(n, k, p, t)$, that must be purchased to guarantee that no matter which numbers appear in the random draw, at least one of the j tickets will have at least t numbers in common with the draw.

Calculating these numbers has proven to be challenging, particularly for lotteries that are actively played worldwide. One common set of parameters is $(n, k, p) = (49, 6, 6)$, an example of which was the UK national lottery upon its introduction in 1994. In October 2015, the UK national lottery underwent a change in which n was increased to 59. By way of compensation, a prize was introduced for matching 2 balls on a ticket. Consequently, the value of $L(59, 6, 6, 2)$ is somewhat significant.

Main Theorem. *We have $L(59, 6, 6, 2) = 27$.*

In Section 2 we give an explicit list of 27 tickets which are always guaranteed to win some prize; we are pleased that we could describe this rather elegantly using some basic structures coming from finite geometry. The hard part in proving the theorem is to show that there does not also exist a set of 26 tickets that work. To this end, the best available lowest bound in the literature is to be found in [FSZ96]:

Theorem 1.1 (Furedi–Székely–Zubor). *We have*

$$L(n, k, p, 2) \geq \frac{1}{k} \cdot \min_{\sum_{i=1}^{p-1} a_i = n} \left(\sum_{i=1}^{p-1} a_i \left\lceil \frac{a_i - 1}{k - 1} \right\rceil \right).$$

Applied to our situation this yields only $L(59, 6, 6, 2) \geq 23$, which is some distance from the true value. In the foundational paper, [BR98], the authors compute $L(n, 6, 6, 2)$ for $n \leq 54$. We bolster the techniques of *op. cit.* theoretically and computationally, recovering

and extending the results therein to calculate all values of $L(n, 6, 6, 2)$ for $n \leq 61$; see Theorem 6.1. Notably, we make substantial use of the constraint programming library [COC97] in SICStus Prolog[1] [CM12]. In that respect, this paper continues a programme of work the authors began in [CSS22] to apply Prolog and constraint programming to questions arising in pure mathematics. See *op. cit.* for a discussion of Prolog.

Remark 1.2. Having observed that the set of tickets we describe below would have netted the authors £1810 in the lottery draw of 21 June 2023, the authors were motivated to road-test the tickets in the lottery draw of 1 July 2023; they matched just two balls on three of the tickets, the reward being three lucky dip tries on a subsequent lottery, each of which came to nothing. Since a ticket costs £2, the experiment represented a loss to the authors of £54.

This unfortunate incident therefore serves both as a verification of our result and of the principle that one should expect to lose money when gambling.

For a more philosophical discussion of the National Lottery and its implementation for supporting charitable causes, we recommend David Runciman's article in the London Review of Books [Run96].

2. THE 27 TICKETS

An appealing way to construct 27 tickets that will match two numbers no matter which balls are drawn is through labelling the vertices of some finite geometrical structures as in Fig. 1.

Perhaps the most well-known finite geometry is the projective plane of order 2—also known as a Fano plane—which is depicted by the three diagrams in Fig. 1 which have 7 vertices. There are 7 'lines' (one being represented by a circle) that satisfy the property that any two points lie in exactly one line, and two lines intersect in exactly one point. Tickets may be read off the Fano plane diagrams by concatenating the labels on the points in each of the 7 lines. (Together this accounts for 21 of the tickets.)

The triangle can be viewed as the projective plane of order 1; it supplies 3 tickets by writing down each of the two labels on its three lines.

Lastly, the triangle with a point in the middle is a sort of 'decompactified' projective plane of order 1: every line should be considered also to go through the point $\varnothing$; this gives us the final 3 tickets.

Explicity, we get:

$1, 2, 3, 4, 5, 6$	$9, 10, 11, 12, 13, 14$	$18, 19, 20, 21, 26, 27$	$32, 33, 34, 35, 40, 41$	$46, 47, 48, 49, 54, 55$
$1, 2, 3, 4, 7, 8$	$9, 10, 11, 15, 16, 17$	$18, 19, 22, 23, 30, 31$	$32, 33, 36, 37, 44, 45$	$46, 47, 50, 51, 58, 59$
$1, 2, 5, 6, 7, 8$	$12, 13, 14, 15, 16, 17$	$18, 19, 24, 25, 28, 29$	$32, 33, 38, 39, 42, 43$	$46, 47, 52, 53, 56, 57$
		$20, 21, 22, 23, 28, 29$	$34, 35, 36, 37, 42, 43$	$48, 49, 50, 51, 56, 57$
		$20, 21, 24, 25, 30, 31$	$34, 35, 38, 39, 44, 45$	$48, 49, 52, 53, 58, 59$
		$22, 23, 24, 25, 26, 27$	$36, 37, 38, 39, 40, 41$	$50, 51, 52, 53, 54, 55$
		$26, 27, 28, 29, 30, 31$	$40, 41, 42, 43, 44, 45$	$54, 55, 56, 57, 58, 59$

The reader may check for themselves that any draw of 6 numbers from 1 to 59 will match at least two numbers with at least one of the above tickets.

[1] A free evaluation copy of SICStus can be downloaded from `https://sicstus.sics.se/`
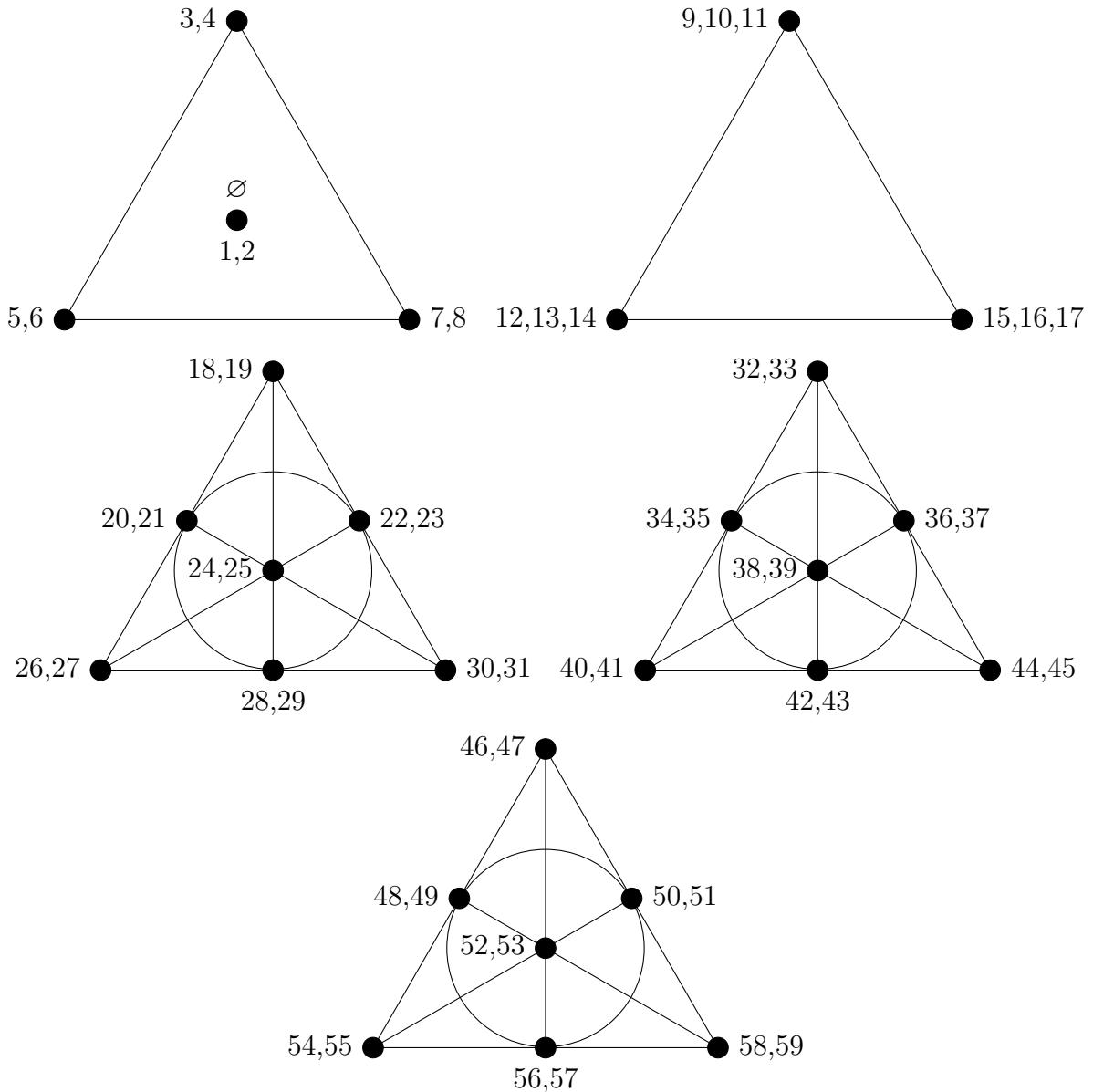
FIGURE 1. 27 tickets through finite geometry

3. DEFINITIONS AND NOTATION

A *hypergraph H* is a pair $(X, \mathcal{B})$ with X a set and $\mathcal{B}$ a set of subsets of X. We will refer to the elements $x \in X$ as *vertices* and the elements $B \in \mathcal{B}$ as *blocks*. The *order* of H is the cardinality of the set X. The *size* of H is the number of blocks in $\mathcal{B}$; i.e. the cardinality of the set $\mathcal{B}$. A hypergraph is said to be *k-uniform* if each $B \in \mathcal{B}$ has size k. Note that a 2-uniform hypergraph is a graph: a block $B = \{x, y\}$ identifies with an edge $x - y$.

An $(n, k, p, t; j)$-*lottery design* is a hypergraph $H = (X, \mathcal{B})$, such that:

   (i) there are n vertices in X—i.e. H has order n;
   (ii) there are j elements $B \in \mathcal{B}$—i.e. H has size j;
   (iii) each $B \in \mathcal{B}$ contains k vertices—i.e. H is k-uniform;

(iv) for any subset D of vertices with $|D| = p$, there is at least one $B \in \mathcal{B}$ such that $|B \cap D| \geq t$.

An (n, k, p, t)-*lottery design* is an $(n, k, p, t; j)$-lottery design in which j is minimal; we denote this minimal integer by $L(n, k, p, t)$.

Remark 3.1. Thinking of X as a set of balls, $\mathcal{B}$ as a set of tickets, and D as a draw, condition (iv) says that at least one ticket in $\mathcal{B}$ matches at least t of the balls in D.

In order to avoid vacuous or incorrect statements we shall always insist $n \geq k \geq t \geq 2$.

We say a block B is an x-*block*—or that B is *incident* with a vertex x—if $x \in B$. The set $\mathcal{B}_x$ of all x-blocks is the *star* of x. We define the function

$$d : X \to \mathbb{Z}_{\geq 0}; \ x \mapsto |\mathcal{B}_x|$$

so that $d(x)$ is the *degree* of x; i.e. the number $|\mathcal{B}_x|$ of blocks incident with x. More generally, if I is any subset of the vertices, then B is an I-block if it is an x-block for some $x \in I$. We let $\mathcal{B}_I = \bigcup_{x \in I} \mathcal{B}_x$, and $d(I) = \bigcup_{x \in I} d(x)$ its degree. We will denote by $d_i := |d^{-1}(i)|$ the number of vertices of degree i in H. We also need to analyse the multiset of degrees of vertices in a subset $I \subseteq X$, so we define the function from the power set $\mathcal{P}(X)$ of X to non-decreasing sequences of non-negative integers:

$$\delta : \mathcal{P}(X) \to \bigcup_{r=0}^{n} (\mathbb{Z}_{\geq 0})^r; \ I \mapsto (d(x_1), \ldots, d(x_n)), \text{ such that } d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n).$$

It will be convenient to label the elements of I consistent with the output of δ, so for example we may have $I = \{x_1, \ldots, x_5\}$ such that $\delta(I) = (2, 2, 3, 3, 4)$ indicating $d(x_1) = 2, \ldots, d(x_5) = 4$.

Two vertices $x, y \in X$ are *adjacent* if they are contained in a common block. The set of all vertices adjacent to x is its *neighbourhood* $N(x)$. A subset I of X is an *independent set* or *coclique* if no pair of elements in I are adjacent. An independent set is *maximal* if there is no independent set $J \subset X$ with $I \subsetneq J$.

Remarks 3.2. (i) If I is an independent set in an $(n, k, p, t; x)$-lottery design, then $|I| < p$ or else there would be a draw $D = I$ matching at most $1 < t$ vertex in any block.

(ii) If I is a maximal independent set then we must have $\mathcal{B}_I = X$ or else there is another element $y \in X$ which is not adjacent to any $x \in I$; which would imply there were a larger maximal independent set $X \cup \{y\}$.

(iii) It is easy to see that maximal independent sets need not have the same order. For example, in the graph $x - y - z$, $\{y\}$ is maximal independent, but $\{x, z\}$ is a maximal independent set of higher cardinality.

A set $\mathcal{B}' \subset \mathcal{B}$ of blocks is *disjoint* if $B_1 \cap B_2 = \emptyset$ for any $B_1, B_2 \in \mathcal{B}'$. In the literature, such a $\mathcal{B}'$ is also referred to as a *matching*. A block $B \in \mathcal{B}$ is *isolated* if B is disjoint from any other block in $\mathcal{B}$. (Of course, each vertex of an isolated block has degree 1.) If $d_0 = 0$ and each vertex of degree 1 is contained in an isolated block, then we say H is *segregated*.

We will show later that H may be assumed segregated under the hypotheses of our main theorem. In that case, the interesting analysis reduces to the full subhypergraph induced by the vertices of degree at least 2. With that in mind, we give a further couple of definitions which will prove central to our argument. Let I be an independent set. Then an I-*toe* is a vertex of degree at least 2 appearing in just one block of $\mathcal{B}_I$. The I-*foot*

F_I is the set of I-toes. If $I = \{x\}$ we refer more succinctly to x-toes. (Note that an x-toe may fail to be an J-toe for $\{x\} \subsetneq J$.) If the independent set I is clear from the context, then we will simply refer to toes and feet. So suppose $I = \{x_1, \ldots, x_\ell\}$, with $\delta(I) = (d(x_1), \ldots, d(x_\ell))$, where $d(x_1) > 1$. Then set

$$\tau(I) = \left(\left| F_I \cap \bigcup \mathcal{B}_{x_1} \right|, \ldots, \left| F_I \cap \bigcup \mathcal{B}_{x_\ell} \right| \right),$$

which gives the distribution of the toes among the x-blocks as x ranges over the elements of I.

4. Preliminaries

We use this section to collect a miscellany of results on hypergraphs and lottery designs of varying generality, which we use in the sequel. Our strategy is heavily influenced by the paper [BR98] and we use most of its results in one form or another. However, we noticed a number of infelicities among the statements and proofs in *op. cit.* and so we take the opportunity here to correct and simplify the statements and proofs therein. Happily, it follows this paper is largely self-contained.

4.1. **Upper bounds.** An (n, k, t)-*covering design* is a k-uniform hypergraph $H = (X, \mathcal{B})$ such that every subset of X of size t appears as a subset in at least one block of $\mathcal{B}$; we assume $n \geq k \geq t$. Define $C(n, k, t)$ as the minimal size of an (n, k, t)-covering design. Obviously, an (n, k, t)-covering design is an (n, k, t, t)-lottery design, so $L(n, k, p, t) \leq C(n, k, t)$.

Lemma 4.1. *We have*

$$L(n, k, p, 2) \leq \min_{\sum_{i=1}^{p-1} a_i = n} \left(\sum_{i=1}^{p-1} C(a_i, k, 2) \right).$$

Proof. Let $H_i = (X_i, \mathcal{B}_i)$ be $(a_i, k, 2)$-covering designs of size $C(a_i, k, 2)$ such that $X = \bigsqcup(X_i)$, and let D be a draw of p elements. Then at least two elements of D lie in at least one X_j, so one block B of $\mathcal{B}_i$ contains those two elements. $\square$

Lemma 4.1 can be deployed using the entries from Table 1. The table lists upper bounds for $C(n, 6, 2)$ for $n \geq 6$ which were harvested from `https://ljcr.dmgordon.org/cover.php`; these upper bounds are all known to be sharp except when $n = 23$ or 24.

Secondly, it is useful to know that lottery numbers increase mononotonically with n.

Lemma 4.2. *We have* $L(n, k, p, t) \leq L(n + 1, k, p, t)$.

Proof. Take an $(n + 1, k, p, t; j)$-lottery design $H = (X, \mathcal{B})$ with $|\mathcal{B}| = j$. Pick $x \in X$ and construct j subsets $\mathcal{C}$ of $X \setminus \{x\}$ from those of $\mathcal{B}$ by replacing $x \in B$ with any other vertex of $X \setminus B$, where necessary. Then it is clear that the hypergraph $H_0 = (X \setminus \{x\}, \mathcal{C})$ is an $(n, k, p, t; j)$-lottery design. $\square$

4.2. **Reductions and constraints.**

Lemma 4.3. *Let* $H = (X, \mathcal{B})$ *be an* $(n, k, p, t; j)$-*lottery design with* $j \geq n/k$. *Then there exists an* $(n, k, p, t; j)$-*lottery design* $H_0 = (X, \mathcal{B}_0)$ *with* $\bigcup \mathcal{B}_0 = X$; *i.e. there are no elements of degree* 0.

n	$C(n,6,2) \leq$	n	$C(n,6,2) \leq$
6	1	17	12
7	3	18	12
8	3	19	15
9	3	20	16
10	4	21	17
11	6	22	19
12	6	23	21
13	7	24	22
14	7	25	23
15	10	26	24
16	10	27	27

TABLE 1. Upper bounds for $C(n,6,2)$

Proof. Let H be a counterexample with $d_0 > 0$ minimal. Suppose $x \in X$ has degree 0. Since $jk \geq n$, there must be $y \in X$ with $d(y) \geq 2$. Suppose $y \in B$ for some block B of $\mathcal{B}$ and set $B_0 = (B \setminus \{y\}) \cup \{x\}$. Now set $H_0 = (X, \mathcal{B}_0)$, where $\mathcal{B}_0$ is $\mathcal{B}$ with the block B replaced by B_0. Then H_0 is a k-uniform hypergraph of order n and size j. If D is any subset of X of order p, then either there is $C \in \mathcal{B}_0$ with $|C \cap D| \geq t$ or we may assume $|B \cap D| = t$, $y \in D$ and $|B_0 \cap D| = t - 1$. This implies also $x \notin D$; but then the alternative draw $(D \setminus \{y\}) \cup \{x\}$ cannot intersect a block of $\mathcal{B}$ in t elements. $\square$

Lemma 4.4. *If $H = (X, \mathcal{B})$ is an $(n, k, p, t; j)$-lottery design with $n \geq k(p - 1)$ and $d_0 = 0$, then there exists a segregated $(n, k, p, t; j)$-lottery design.*

Proof. Suppose there are r isolated blocks in H. Then taking one vertex from each yields an independent set I, so $r \leq p - 1$ by Remarks 3.2(i). If $r = p - 1$ then the isolated blocks supply all $kr = k(p - 1)$ vertices of X and the statement holds.

Suppose $r = p - 2$; then there are $n - k(p - 2)$ elements not in isolated blocks. A draw of order p containing one vertex from each isolated block together with two non-isolated vertices can only intersect a non-isolated block in at least t places. Thus the non-isolated blocks must between them contain every pair of the non-isolated vertices. This means that each appears at least twice, or $n - k(p - 2) = 2$ and they both appear exactly once. But the latter says they are themselves in an isolated block, a contradiction.

Hence we may assume $r \leq p - 3$, leaving at least $2k$ elements not in isolated blocks. Let B be a non-isolated block and assume that there are $x, y \in B$ with $d(x) = 1$ and $d(y) > 1$. We modify H to give a lottery design H_0 with $d(y) = 1$. By an evident induction, this implies the existence of the required design.

Let $B = B_1, \ldots, B_\ell$ be the blocks containing y. For $2 \leq i \leq \ell$, find a non-isolated element z_i such that $z_i \notin B_i$; this is possible since there are at least $2k$ non-isolated elements. Form C_i by replacing y with z_i and let $C_1 = B_1$. Then we claim we get a new $(n, k, p, t; j)$-lottery design $H_0 = (X, \mathcal{C})$ with $d_0 = 0$ by letting $\mathcal{C} = (\mathcal{B} \setminus \{B_1, \ldots, B_\ell\}) \cup \{C_1, \ldots, C_\ell\}$.

To prove the claim, take a draw D and assume D does not intersect any block $C \in \mathcal{C}$ in at least t elements. Then we may assume D contains y, $|D \cap B_i| = t$ for some $2 \leq i \leq l$ and $|D \cap C_i| = t - 1$. Furthermore, since $d(x) = 1$, if $x \in D$, then $D \cap B_1 = D \cap C_1$ has at least t elements, so we may assume $x \notin D$. Then replacing y with x in D gives a draw D_0 which intersects no block of $\mathcal{B}$ in at least t elements. $\square$

The above results are essentially the same as [BR98, Lem. 3.2, Thm. 3.5]. In between these is [BR98, Lem. 3.4] which shows (correctly) that given an $(n, k, p, 2)$-lottery design with $n > k(p-2)$, then there is another with a maximal independent set of size $p-1$. It is combined with the above two results to claim the existence of a lottery design satisfying the conclusions of all three results. Unfortunately the methods of proof go by altering the degrees of vertices in the design, and it is unclear whether this can be done compatibly.

We resolve this issue in Proposition 4.6 below, with a stronger result. As in *op. cit.*, we make use of the following lemma, which is an application of the main theorem of [Sha49]. One should also note that [BR98] implicity assumes, without proof, that the independent set guaranteed by Lemma 4.5 can be enlarged to one of maximal cardinality; in fact, this can actually fail, and results in a separate case to check—*viz.* Case (i) in Section 5.2 below.

Lemma 4.5. *Let $(X, \mathcal{B})$ be a k-uniform hypergraph. Then there exists an independent set I containing at least $\frac{d_2}{\lceil 3k/2 \rceil}$ vertices of degree 2.*

Proposition 4.6. *Suppose $H = (X, \mathcal{B})$ is a segregated k-uniform hypergraph with $|X| = n$, $|\mathcal{B}| = j$, $d_1 = kr$ and π is the maximal order of an independent set in H. Then if*

$$(*) \qquad 4n - jk - (3k-2)(\pi-1) + r(\lfloor k/2 \rfloor - 1) - (\pi-1) \cdot (\lfloor k/2 \rfloor + 1) > 0,$$

there is a (maximal) independent set $I \subseteq X$ of order π consisting entirely of vertices whose degrees are 1, 2 or 3 and containing at least $\lceil \frac{d_2}{\lfloor \frac{3k}{2} \rfloor} \rceil$ vertices of degree 2.

Furthermore if the left-hand side of the inequality (*) is instead equal to zero then there is an independent set I of order $\pi - 1$ with $d(I) \in \{1, 2\}$, all vertices have degree at most 4, and $d_2 = (\pi - 1 - r) \lfloor \frac{3k}{2} \rfloor$.

Before giving the proof of the proposition, let us make some elementary observations that are used several times in the sequel.

Let $H = (X, \mathcal{B})$ be a k-uniform hypergraph with $|X| = n$ and $|\mathcal{B}| = j$. Recall d_i is the number of vertices having degree i. Clearly

$$(1) \qquad \sum_{i \geq 0} d_i = n.$$

Let $\mathscr{B}$ denote the multiset $\bigsqcup B_i$, which contains the vertices of the hypergraph counted with their multiplicities in the blocks $\mathcal{B}$. As there are j blocks we have $|\mathscr{B}| = jk$. Of course a vertex x will occur in exactly $d(x)$ of the blocks, and so:

$$(2) \qquad \sum_{i \geq 1} i d_i = jk.$$

Proof of Proposition 4.6. Suppose for a contradiction that there is no independent set I of order π with $d(I) \subset \{1, 2, 3\}$. We will show this leads to a contradiction.

For a given independent set I, we denote by $\delta_i = d^{-1}(i) \cap I$ the number of elements of degree i in I. By Lemma 4.5 there exists an independent set with

$$(3) \qquad \delta_2 \geq \frac{d_2}{\lfloor \frac{3k}{2} \rfloor}.$$

Let J be the union of this set together with one element chosen from each of the r isolated blocks. Now take $I \supseteq J$ maximal subject to $d(I) \leq 3$; therefore $\delta_1 + \delta_2 + \delta_3 \leq \pi - 1$. Note that by segregation, the $\delta_1 = r$ isolated I-blocks account for all elements in X of degree 1.

We now bound from above and below the number $d_2 + d_3$ of vertices in X having degrees 2 or 3.

Since I is maximal, we have $X = \bigcup \mathcal{B}_I$. By segregation, no vertex of degree 2 or 3 is adjacent to a vertex of degree 1 and so it must live in $\bigcup \mathcal{B}_x$, with x running over the elements of I of degree 2 or 3. The latter contains at most $(2k-1)\delta_2 + (3k-2)\delta_3$ distinct elements. Using $\delta_1 + \delta_2 + \delta_3 \leq \pi - 1$, we get

$$(4) \qquad d_2 + d_3 \leq (2k-1)\delta_2 + (3k-2)\delta_3$$

$$(5) \qquad\qquad \leq (2k-1)\delta_2 + (3k-2)(\pi - 1 - r - \delta_2)$$

$$(6) \qquad\qquad \leq (1-k)\delta_2 + (3k-2)(\pi - 1 - r)$$

On the other hand, consider the multiset $\mathscr{B} = \bigsqcup_{B \in \mathcal{B}} B$, of order jk. The I-blocks contribute exactly $\delta_1 k + 2\delta_2 k + 3\delta_3 k$ of these elements. Since there are $2d_2 + 3d_3$ elements of degrees 2 or 3, there are $2\delta_2 k + 3\delta_3 k - 2d_2 - 3d_3$ elements of $\mathscr{B}$ of order at least 4 coming from the I-blocks. The remaining $(j - \delta_1 - 2\delta_2 - 3\delta_3)$-blocks all contain elements of order at least 4. Thus the multiset $|\mathscr{B}_{\geq 4}|$ of all elements of degrees 4 and above has order

$$|\mathscr{B}_{\geq 4}| = (j - \delta_1 - 2\delta_2 - 3\delta_3)k + 2\delta_2 k + 3\delta_3 k - 2d_2 - 3d_3 = jk - rk - 2d_2 - 3d_3.$$

Since $|\mathscr{B}_{\geq 4}| = \sum_{i \geq 4} i d_i$, we get

$$(7) \qquad\qquad \sum_{i \geq 4} d_i \leq \frac{jk - rk - 2d_2 - 3d_3}{4}.$$

Now $n = \sum d_i \leq rk + d_2 + d_3 + \frac{jk - rk - 2d_2 - 3d_3}{4}$.

Rearranging, we get:

$$(8) \qquad\qquad d_2 + d_3 \geq 4n - jk - d_2 - 3rk,$$

which combines with (6) to give

$$(1 - k)\delta_2 + (3k - 2)(\pi - 1 - r) \geq 4n - jk - d_2 - 3rk.$$

Now using Eq. (3) we get

$$\delta_2 \cdot \lfloor 3k/2 \rfloor - (k-1)\delta_2 \geq 4n - jk - (3k-2)(\pi - 1) - 2r.$$

Using $\delta_1 + \delta_2 \leq \pi - 1$, with equality if and only if $\delta_3 = 0$, then after some manipulation, we get

$$4n - jk - (3k-2)(\pi - 1) + r(\lfloor k/2 \rfloor - 1) - (\pi - 1) \cdot (\lfloor k/2 \rfloor + 1) \leq 0,$$

which is a contradiction, proving the first statement of the proposition.

If $4n - jk - (3k-2)(\pi - 1) + r(\lfloor k/2 \rfloor - 1) - (\pi - 1) \cdot (\lfloor k/2 \rfloor + 1) = 0$ then we must have had equality in all the previous inequalities that were used, including Eq. (3). Thus, $\delta_2 \lfloor 3k/2 \rfloor = d_2$, $\delta_3 = 0$, $\pi - 1 = \delta_1 + \delta_2$ and all vertices are degree at most 4. $\qquad \square$

The following results are all used to give constraints to Prolog. The first two respectively bound above and below the number of isolated blocks.

Lemma 4.7. *Let $(X, \mathcal{B})$ be a segregated $(n, k, p, 2; j)$-lottery design. Suppose that $\mathcal{B}$ has r isolated blocks. Then,*

$$r \geq \left\lceil \frac{2n - jk}{k} \right\rceil.$$

Proof. We have $jk = d_1 + \sum_{i \geq 2} id_i \geq d_1 + 2(n - d_1)$. Write $d_1 = rk$ and rearrange to get the formula above. $\quad\square$

Lemma 4.8. *Let $H = (X, \mathcal{B})$ be a segregated $(n, k, p, t; j)$-lottery design with at least r isolated blocks. Then*

$$r \leq j - L(n - rk, k, p - r, t).$$

Proof. Suppose $B_1, \ldots, B_r$ are isolated and let $Y = X \setminus \bigcup_{1 \leq i \leq r} B_i$ be the set of non-isolated vertices and $\mathcal{C} = \mathcal{B} \setminus \{B_1, \ldots, B_r\}$; so $(Y, \mathcal{C})$ is a hypergraph with order $n - rk$ and size $j - r$. Fix one vertex $v_i \in B_i$ for each i. For any choice $\{v_{r+1}, \ldots, v_p\}$ of vertices from Y, the draw $D = \{v_1, \ldots, v_p\}$ intersects with some $B \in \mathcal{B}$ in at least t vertices. Clearly we cannot have $B = B_i$ for $1 \leq i \leq r$ and so $B \in \mathcal{C}$. But this means no $v_1, \ldots, v_r$ is in B, so B must match t of the remaining $p - r$ elements of D. Hence $(Y, \mathcal{C})$ is an $(n - rk, k, p - r, t; j - r)$-lottery design, which implies the inequality as shown. $\quad\square$

Proposition 4.9. *Let H be a segregated k-uniform hypergraph. Then*

$$d_2 \geq 3n - 2rk - jk$$

Proof. We recall $\sum d_i = n$ and $\sum id_i = jk$, which implies

$$jk - n = d_2 + \sum_{i \geq 3}(i - 1)d_i \geq d_2 + 2\sum d_i = -2d_1 - d_2 + 2n. \quad\square$$

Lemma 4.10. *Let $(X, \mathcal{B})$ be a k-uniform hypergraph with maximal independent set I. Then*

$$|\mathcal{B}_I| \geq \left\lceil \frac{n - p + 1}{k - 1} \right\rceil.$$

Proof. We must have $\bigcup \mathcal{B}_I = X$ or I is not maximal. For $x \in I$, two blocks $B, C \in \mathcal{B}_x$ intersect in at least x so $|\bigcup \mathcal{B}_x| \leq (k - 1)|\mathcal{B}_x| - 1$. Since $|I| \leq p - 1$, then summing over $x \in I$ yields the statement of the lemma. $\quad\square$

4.3. **Excess, toes and webbings.** For the values of (n, k, p) under consideration, one tends to find $L(n, k, p, 2) \sim n/2$. Thus, on average, the degree of a vertex in an $(n, k, p, 2)$-lottery design is about 2. The next definition follows [BR98] with a view to bounding the extent of departure from this average value.

Definition 4.11. *For a set of vertices $Y \subset X$, the* excess *of Y is the sum*

$$\mathfrak{E}(Y) = \sum_{i > 2}(i - 2) \cdot |d^{-1}(i) \cap Y|.$$

Note that if $Y = X$, we get $\mathfrak{E}(X) = \sum_{i > 2}(i - 2) \cdot d_i$. The following gives an easy characterisation of $\mathfrak{E}(X)$.

Lemma 4.12. *Let $H = (X, \mathcal{B})$ be a segregated hypergraph with r isolated blocks. Then*
$$\mathfrak{E}(X) = jk + rk - 2n.$$

Proof. We have $jk = \sum i d_i$. Since $n = \sum d_i$, we get
$$jk - 2n = -d_1 + \sum_{i \geq 3}(i - 2)d_i = -d_1 + \mathfrak{E}(X). \qquad \square$$

The possible number of toes are constrained by the value of the excess $\mathfrak{E}(X)$, in a manner we now describe. First, the following is [BR98, Lem. 4.7] after the removal of a significant typo.

Lemma 4.13. *Let $(X, \mathcal{B})$ be a segregated $(n, k, p, 2; j)$-lottery design with maximal independent set I and F_I its I-foot. Suppose further that $\mathcal{B}$ has r isolated blocks. Then,*
$$|F_I| \geq 2n - 2p + 2 - (k - 1)(|\mathcal{B}_I| + r).$$

Proof. As I is maximal, we have $\bigcup \mathcal{B}_I = X$. The multiset
$$\mathscr{R} := \bigsqcup_{x \in I, d(x) > 1, B \in \mathcal{B}_x} B \setminus \{x\}$$
contains the $kj - rk - |I|$ vertices which are neither isolated nor contained in I, and by definition, the toes are the elements of multiplicity 1 in $\mathscr{R}$. Therefore each of the remaining $kj - rk - |I| - |F_I|$ vertices appears at least twice in $\mathscr{R}$. Since $|\mathscr{R}| = (k - 1)(|\mathcal{B}_I| - r)$, we have
$$|F_I| + 2(n - rk - (|I| - r) - |F_I|) \leq (k - 1)(|\mathcal{B}_I| - r).$$
Rearranging and using $|I| \leq p - 1$, we get the inequality as claimed. $\qquad \square$

Now suppose $x \in I$ for I a maximal independent set, with F_x a set of x-toes. Let us assume

(*) $$|F_x| \geq k.$$

Then it follows there are at least two blocks $B, C \in \mathcal{B}_x$, say, containing (necessarily distinct) x-toes; say $y \in B$ and $z \in C$. If y and z were not adjacent, then replacing x with y, z in I would yield a larger independent set, a contradiction. Thus there is a block W with $y, z \in W$. We refer to such blocks as *webbings*. Formally, W is an *x-webbing* if $W \notin \mathcal{B}_I$ and W contains distinct x-toes; the set of x-webbings is later denoted $\mathcal{W}(x)$. Note that under the assumption (*), each toe appears at least once in a webbing, so that it must have degree at least 2.

More precisely, suppose there are τ_i toes in distinct x-blocks B_i with $\tau_1 \geq \tau_2 \cdots \geq \tau_s \geq 1$. Then each of the $\sum_{i < j} \tau_i \tau_j$ pairs of toes must appear in some webbing. In the case $k = 6$ and $|F_x| \geq 7$, one can see that some toes must have degree at least 3, for example. This implies non-trivial lower bounds on the excess $\mathfrak{E}(F_x) \leq \mathfrak{E}(X)$.

Lemma 4.14. *Let $x \in X$ with x of degree 2 or 3 and suppose F_x is its x-foot; that is, F_x the set of all vertices occuring just once in the star of x. The table below gives minimum values of $\mathfrak{E}(F_x)$ in terms of $|F_x|$.*

$|F_x|$	≤ 5	6	7	8	9	10	11	12	13	14	15
$\min(\mathfrak{E}(F_x))$	0	0	2	3	7	10	11	12	20	25	27

If, moreover, F_x is known to contain no elements of degree 2, then $\min(\mathfrak{E}(F_x)) \geq |F_x|$.

Proof. For the second statement of the lemma, just observe that any x-toe of degree at least 3 contributes at least 1 to $\mathfrak{E}(F_x)$.

For the table itself, suppose $|F_x| = \tau_1 + \tau_2 + \tau_3$ is a partition of $|F_x|$ into summands of size at most 6. If $|F_x| \leq 6$, then one webbing W suffices to cover all toes, and so the minimum possible excess of 0 is a achieved by a configuration of x-blocks and one webbings in which each toe appears just twice. Otherwise suppose there are $w > 1$ webbings, containing each of the $\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3$ pairs of toes, so that

$$\left\lceil \frac{\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3}{3} \right\rceil.$$

is an upper limit for w.

We used the immensely powerful linear programming solver Gurobi to solve the following problem. let M be a $w \times |F_x|$-matrix of variables taking values in 0 and 1, with the rows representing webbings and a 1 appearing in the (i,j)-th entry if a toe labelled j is in the ith webbing. Since $|F_x| > 6$, we know that toe j appears once in the x-blocks and at least once in the webbings, so $\mathfrak{E}(F_x) + |F_x|$ is the sum S of all entries of the M. Furthermore, the rows of M must all sum up to integers less than or equal to $k = 6$, and columns j_1 and j_2 must have scalar product at least 1 whenever j_1 and j_2 come from different parts of the partition

$$\{1, \ldots, n\} = \{1, \ldots, \tau_1\} \cup \{\tau_1 + 1, \ldots, \tau_1 + \tau_2\} \cup \{\tau_1 + \tau_2 + 1, \ldots, \tau_1 + \tau_2 + \tau_3\}.$$

We ask Gurobi to minimise the sum S subject to these constraints, and it results in the table in the lemma.

Since Gurobi only gives answers up to a percentage accuracy, its output does not amount to a proof of optimality, and so we wrote some Prolog code to check that the values of $\min(F_x)$ one below those in the table are infeasible. $\qquad\square$

Remark 4.15. This improves the bound in [BR98, Table 1] for $\min(|F_x|)$ when $|F_x| = 13$ from 16 to 20. Since Gurobi outputs a feasible configuration of webbings for each value of $|F_x|$, we know that the minima can be achieved.

5. Proof of the theorem

5.1. **SICStus Prolog code.** The code in Appendix Section A—also available at `github.com/cushydom`
implements the constraints described in the previous section. We give a description of the strategy and its functionality.

One first loads SICStus Prolog and consults the file `lottery.pl` through the command

```
?- ['$PATH_TO_DIRECTORY/lottery.pl'].
```

Then one queries Prolog at the command line by asking it to solve for variables in a (conjunction) of predicates. The following is an example of the output produced from the main predicate:

The principal predicates are:

```
| ?- lottery_numbers_in_range(50,52).
% L(50,6,6,2) = 19
% L(51,6,6,2) = 20
% We conjecture that L(52,6,6,2) = 21 and must rule out the following cases
% d_2 <= 36 and \delta(I) = [2,2,2,2,3]
```

FIGURE 2. Example output

lottery_numbers_in_range(NMin,NMax). This predicate writes output to the terminal of the form given in Fig. 2; i.e. it works sequentially with n from the value of NMin to the value of NMax either outputting a line stating the value of $L(n, 6, 6, 2)$ or a conjectured value of it, which is correct modulo a list of cases to be checked by hand. The first case is computed from scratch, using a lower bound of 1 but then in order to reduce unnessary computation, it passes the (conjectured) value of $L(n, 6, 6, 2)$ as a lower bound for $L(n + 1, 6, 6, 2)$, using Lemma 4.2.

upper_bound(N,Guess,UB) is called by the previous predicate, and it holds when Guess equals $L(N-1, 6, 6, 2)$ and UB is an upper bound for $L(N, 6, 6, 2)$. The predicate considers the list of upper bounds for sizes of covering designs in Table 1 searching for, initially whether Guess can be achieved as a sum of such values according to Lemma 4.1, or increases Guess until it can.

In many cases, one finds that the (conjectured) value of $L(n - 1, 6, 6, 2)$ coincides with the value UB. In that case, by Lemma 4.2, we conclude immediately $L(n, 6, 6, 2) = L(n - 1, 6, 6, 2)$ modulo any previous cases that require checking by hand.

Otherwise, further predicates are engaged to establish when the hypotheses of Proposition 4.6 can fail and otherwise compute possible vectors $\delta(I)$ satisfying the conclusions of Proposition 4.6. We start by using the predicate r_bound to give lower and upper bounds for $r = d_1/k$ using, respectively, Lemma 4.7 and Theorem 1.1 together with Lemma 4.8.

get_slim_I_exceptions(N,UB,R,PossSlimIException). For brevity, say an independent set is *slim* if it is of maximal cardinality and its component vertices have degrees at most 3. Then get_slim_I_exceptions is true when R is a possible value of r from rbound and Proposition 4.6 does not imply the existence of a slim independent set. In that case, the variable PossSlimIException is bound to a pair $(r, (a, b))$, indicating the value of r and an interval $a \leq d_2 \leq b$ where a is taken from Proposition 4.6 and b comes from Eq. (3). Any feasible values for PossSlimIException are sent to user output. Otherwise get_min_num_Iblocks(N,MinNumIBlocks) binds MinNumIBlocks to the value implied by Lemma 4.10 and it is passed to the next predicate along with n, the upper bound from above, and queried for each feasible value of r.

get_deltaI_exceptions(N,UB,MinNumIBlocks,R,Deltas) binds Deltas to a list of possible tuples $\delta(I)$ which are constrained by the following:

(1) the first r elements of $\delta(I)$ are 1;
(2) the next s elements of $\delta(I)$ are each 2, where s is determined by the lower bound Proposition 4.9 combined with Eq. (3).
(3) the remaining $p - 1 - r - s$ elements are either 2 or 3;
(4) the number of I-blocks is at least the minimum already described.

Finally, for each feasible value of r and feasible sequence $\delta(I)$ in Deltas, we solve a constraint problem to determine the number and distribution of toes among I-blocks

subject to their contribution to the excess $\mathfrak{E}(X)$ as implied by Lemma 4.14. Since r is known, the value $\mathfrak{E}(X)$ can be computed from Lemma 4.12 and `get_base_excess` binds `BaseExcess` to this value.

`can_populate_toes_in_Iblocks(N,R,BaseExcess,Delta)`. This creates and a binds a variable `MinToes` to the lower bound supplied by Lemma 4.13. Since any vertex in I of degree at least 3 contributes 1 to $\mathfrak{E}(X)$, the contribution to the excess $\mathfrak{E}(X)$ from toes must be at most $\mathfrak{E}(X) - \delta_3$, where $\delta_3 = |d^{-1}(3) \cap I|$. Hence a variable `Excess` is created and bound to `BaseExcess-NumThrees`. This information is passed to the final predicate with the 1s stripped from Δ to give the sequence `DeltaNoOnes`.

`populate_toes_in_Iblocks( DeltaNoOnes,MinToes,Excess,Vs )` solves the following constraint problem:

(1) $J = \{x_{r+1}, \ldots, x_{p-1}\} \subseteq I$ is the subset of I of vertices of degree at least 2, such that $\delta(J)$ is bound to `DeltaNoOnes`;
(2) `Vs` is a sequence of variables representing $\tau(J)$; so the total number of toes $\sum \tau(J)$ represented by `sum(Vs)` is at least the calculated bound `MinToes`;
(3) given the values of $d(x_i)$ and $\tau(x_i)$, the sum over $r+1 \leq i \leq p-1$ of the minimum excesses implied by Lemma 4.14 is at most `Excess`.

Any solutions for the variables `Vs` together with the choice of `Delta` are outputted for manual checking.

5.2. **The remaining cases.** The output from `compute_lottery_numbers_in_range(30,61)` tells us that we must consider separately the following cases:

(1) $n = 54$, $j = 22$, $d_2 \leq 36$, $\delta(I) = (2,2,2,2,3)$;
(2) $n = 54$, $j = 22$, $d_2 \leq 18$, $\delta(I) = (1,2,2,3,3)$;
(3) $n = 57$, $j = 24$, $d_2 \leq 27$, $\delta(I) = (2,2,2,3,3)$;
(4) $n = 58$, $j = 25$, $d_2 \leq 27$, $\delta(I) = (2,2,2,3,3)$;

And for $n = 59$, we must rule out the following cases for $j = 26$:

(i) $d_1 = 0$, $d_2 = 36$;
(ii) $d_2 \leq 9$, $\delta(I) = (1,2,3,3,3)$;
(iii) $d_2 \leq 27$, $\delta(I) = (2,2,2,3,3)$;

In the next three subsections, we treat the cases for $n = 59$; the cases where $n < 59$ are similar and easier.

Case (i). In this case, we have $d_2 = 36$ and so we cannot apply Proposition 4.6 in order to get a maximal independent set of order 5. However, by Eq. (3), we do get an independent set $I_0 = \{x_1, \ldots, x_4\}$ with $\delta(I_0) = (2,2,2,2)$. But then the 8 blocks of B_{I_0} supply at most 44 elements, so we can enlarge I_0 to a maximal independent set $I = I_0 \cup \{x_5\}$. Since the Prolog query did not flag up the cases where $\delta(I) = (2,2,2,2,2)$ or $(2,2,2,2,3)$, we may assume $d(x_5) \geq 4$. Furthermore, all vertices of degree 2 or 3 must appear in a block of $\mathcal{B}_{I_0}$, or we could have chosen x_5 of degree 2 or 3. Since there are at most 44 distinct vertices in these blocks, we have $d_3 \leq 8$.

Now, the number of elements $d_{\geq 4}$ of degree at least 4 is $d_{\geq 4} = 59 - 36 - d_3 = 23 - d_3$. Hence,

$$156 = \sum i d_i \geq 2 \cdot 36 + 3 \cdot 8 + 4 \cdot (23 - d_3),$$
$$\text{whence} \quad 84 \geq 3 \cdot d_3 + 4 \cdot (23 - d_3).$$

Therefore $d_3 \geq 8$, which gives $d_3 = 8$, $d_4 = 15$ and $d(x_5) = 4$. Consequently, the elements of order 4 cannot appear in any blocks of B_{I_0} so must each appear at least once in the blocks B_{x_5}, which can contain a maximum of 21 distinct elements, so that there are at most 6 vertices in B_{x_5} with degree 2 or 3. This shows that there must be at least 34 I-toes coming from vertices in blocks of B_{I_0}. Checking Lemma 4.14, it is easy to see that $\mathfrak{E}(\bigcup B_{I_0}) > 8$ which implies there are at least 9 vertices of degree 3, a contradiction.

Case (ii). Here we assume

$$\delta(I) = (1, 2, 3, 3, 3),$$

with $I = \{x_1, \ldots, x_5\}$ and that there is no independent set J with $\delta_2(J) > 1$. In particular, every vertex of degree 2 appears in a block with every other.

By Proposition 4.9, we have $d_2 \geq 9$, hence $d_2 = 9$. Furthermore we have $d_1 = 6$ so

$$156 \geq 6 + 2 \cdot 9 + 3 \cdot d_{\geq 3}$$

with $d_{\geq 3} = 44$. Hence we have equality above and $d_{\geq 3} = d_3 = 44$. If $y \in B \in \mathcal{B}_{x_i}$ with $i > 2$, $d(y) = 2$, then there is another independent set $J \subset X$ with $\delta(J) = (1, 2, 2, 3, 3)$, a contradiction. So all elements of degree 2 are contained in the blocks $\{B_1, B_2\} = \mathcal{B}_{x_2}$. The multiset $\mathscr{B}_{x_2} = B_1 \sqcup B_2$ has 10 elements not equal to x_2. Now, 8 of these must be the remaining vertices of degree 2, so there are at least 6 occurring just once. In other words F_{x_2} has at least 6 toes of degree 2. Suppose τ_1 of these appear in B_1 and τ_2 appear in B_2. Since $\tau_1 + \tau_2 \geq 6$ we see $\tau_1, \tau_2 > 0$. This tells us that all these toes must lie in a single webbing. It follows that $\tau_1 + \tau_2 = 6$ and so there are exactly 6 toes in F_{x_2}, meaning that the remaining 2 vertices of degree 2 both appear twice.

Returning to the output of the Prolog code

```
?- populate_toes_in_Iblocks([2,3,3,3], 43, 41, Vs ), labeling([], Vs).
Vs = [7,12,12,12] ? ;
Vs = [8,11,12,12] ? ;
Vs = [8,12,12,12] ? ;
Vs = [9,10,12,12] ? ;
Vs = [9,11,11,12] ? ;
no
```

we conclude only the first possibility can hold. Hence the 43 toes are divided between the x_i-blocks in proportion 6, 12, 12 and 13 for $2 \leq i \leq 5$. Now from Lemma 4.14 we see that if $F_x = 13$ then $\mathfrak{E}(F_x) \geq 16$. But if every $y \in F_x$ had $d(y) \leq 3$ then $\mathfrak{E}(F_x) \leq 13$. Thus some $d(y) \geq 4$, a contradiction.

Case (iii). Here we assume $d_2 \leq 27$ and

$$\delta(I) = (2, 2, 2, 3, 3),$$

with $I = \{x_1, \ldots, x_5\}$ and that any independent set J with degrees in $\{2, 3\}$ has $\delta(J) = \delta(I)$. In particular, every vertex of degree 2 appears in $\bigcup \mathcal{B}_{\{x_1, x_2, x_3\}}$ and appears in no block of $\mathcal{B}_{\{x_4, x_5\}}$.

By Proposition 4.9, we have $d_2 \geq 21$, and Lemma 4.13 implies $|F_I| \geq 48$. Prolog informs us of the following possibilities for $\tau(I)$:

```
?- populate_toes_in_Iblocks([2,2,2,3,3], 48, 36, Vs ), labeling([], Vs).
Vs = [7,8,9,12,12] ? ;
Vs = [8,8,8,12,12] ? ;
Vs = [8,8,9,11,12] ? ;
no
```

These solutions lead to $\mathfrak{E}(F_I) \geq 36$ except for $\tau(I) = (8, 8, 8, 12, 12)$ which implies only $\mathfrak{E}(F_I) \geq 33$. Let us rule out the cases where $\mathfrak{E}(F_I) \geq 36$.

Any toe $y \in F_I$ occurs once in the blocks of $\mathcal{B}_I$ and all other occurrences are in blocks not in $\mathcal{B}_I$. Let $d_{i,\tau}$ denote the number of toes having degree i. Then counted with multiplicy in $\mathscr{B}$, the toes account for $T = \sum_{i \geq 2} i d_{i,\tau}$ elements of $\mathscr{B}$. We have $\mathfrak{E}(F_I) = \sum(i - 2)d_{i,\tau}$, and $\sum d_{i,\tau} =: n_\tau$ is the number of toes. So $T = \mathfrak{E}(F_I) + 2 \cdot n_\tau$ and exactly n_τ of these entries come from F_I, so there are $\mathfrak{E}(F_I) + n_\tau \geq 48 + 36 = 84$ elements lying in blocks not in $\mathcal{B}_I$. But $84/6 = 14$ so there are at least 14 blocks not lying in $\mathcal{B}_I$. Since $|\mathcal{B}_I| = 12$ and $|\mathcal{B}| = 26$, this exhausts $\mathcal{B}$, implies $n_\tau = 48$, that

$$(\dagger) \qquad \mathfrak{E}(F_I) = 36 = \sum_{1 \leq i \leq 5} \mathfrak{E}(F_{x_i}),$$

and that the blocks $\mathcal{B} \setminus \mathcal{B}_I$ contain only toes.

Consider F_{x_2}, which, by $(\dagger)$ contributes exactly 3 to $\mathfrak{E}(F_I)$. It is easy to see that this means there must be 5 toes in B and 3 toes in C, say, where $\{B, C\} = \mathcal{B}_{x_2}$; also that there are exactly two x_2-webbings B' and C'. Let us label the vertices of B and C as $B = \{x_2, 1, 2, 3, 4, 5\}$, $C = \{x_2, 6, 7, 8, \_, \_\}$. Then up to an action of $S_5 \times S_3$ permuting the toes, we may write $B' = \{1, 2, 3, 6, 7, 8\}$ and $C' = \{4, 5, 6, 7, 8, z\}$, where z is as yet undetermined. But since $C' \notin \mathcal{B}_I$ it follows that z is a toe. We cannot have $z \in \{1, \ldots, 8\}$ since that would imply $E(F_{x_2}) > 3$. Therefore $z \in F_{x_i}$ for $i \neq 2$. But as $|F_{x_i}| \geq k = 6$, each x_i-toe must already appear once in $\mathcal{B}_{x_i}$ and again in some $D \in \mathcal{B} \setminus \mathcal{B}_i$ with some other x_i-toe. We conclude $D \neq C'$ and so C' is not a webbing of x_i-toes. Since the webbings of x_i-toes account entirely for the occurrences of the x_i-toes outside of $\mathcal{B}_I$, we reach a contradiction.

Therefore we must have $\mathfrak{E}(X) < 36$ and so $\tau(I) = (8, 8, 8, 12, 12)$. We use a similar counting argument as above: for $x = x_4$ or x_5, we have $|F_x| = 12$ and these incur an excess $\mathfrak{E}(F_x) \geq 12$. It follows that these 12 x-toes must appear at least 24 times in the webbings $\mathcal{W}(x)$. This implies that $|\mathcal{W}(x)| \geq 4$ and if equality holds, then $\bigcup \mathcal{W}(x)$ consists entirely of x-toes; moreover a closer inspection of the solutions found by Gurobi implies that the toes are distributed in such a way as to have 4 in each x-block. Similarly, if $x \in \{x_1, x_2, x_3\}$ then $|\mathcal{W}(x)| \geq 2$ and in the case of equality, there is at most 1-non-toe in $\bigcup(\mathcal{W}(x))$. Since $\sum_{x \in I} |W(x)| \geq 14$ and there are 12 I-blocks, we do indeed have equality everywhere.

Consequently, there are $59 - 5 - 48 = 6$ vertices which are neither in I nor I-toes. Consider the draw D comprising these 6 elements. We have by assumption that $D \cap B$ has at least 2 elements for some $B \in \mathcal{B}$. But by the above remarks, we cannot have $B \in \mathcal{W}(x)$ for any $x \in I$, since such a B contains at most 1 non-x-toe; equally we cannot have $B \in B_x$ for $x \in \{x_4, x_5\}$ as $B \setminus \{x\}$ has at most 1 non-x-toe. Therefore $B \in \mathcal{B}_{x_1} = \{E, F\}$, say and that there are 3 toes and 2-non-toes in E and 5 toes in F. Let us therefore

label $E = \{x_1, 1, 2, 3, -1, -2\}$, $F = \{x_1, 4, 5, 6, 7, 8\}$, $D = \{-1, -2, -3, -4, -5, -6\}$, accordingly. Up to $S_3 \times S_5$ symmetry, we have $\mathcal{W}(x) = \{B', C'\}$ with $B' = \{1, 2, 3, 4, 5, 6\}$, $C' = \{1, 2, 3, 7, 8, \_\}$. Therefore all the pairs $\{-1, -2\} \times \{4, 5, 6\}$ are non-adjacent. Being toes, none of $4, 5$ or 6 can appear in a block of $\mathcal{B}_{x_2}$ or $\mathcal{B}_{x_3}$ and furthermore, there is just one space in $\mathcal{W}(x_2)$ and $\mathcal{W}(x_3)$ where any of $\{4, 5, 6\}$ could appear. Without loss of generality, then, 4 is no block in $\mathcal{B}_{x_2}, \mathcal{B}_{x_3}, \mathcal{W}(x_2), \mathcal{W}(x_3)$.

Now consider the draw $D_1 = \{4, -2, -3, -4, -5, -6\}$, which must intersects a block B_1 in $\mathcal{B}_{x_2} = \{B_1, C_1\}$ say, in at least 2 places. Without loss of generality, we have either $B_1 = \{x_2, 9, 10, 11, -2, -3\}$ or $B_1 = \{x_2, 9, 10, 11, -3, -4\}$ and $C_1 = \{x_2, 12, 13, 14, 15, 16\}$. In a similar to manner to before, we may assume 12 does appear in any of $\mathcal{B}_{x_i}, \mathcal{W}(x_i)$ for $i \in \{1, 3\}$.

Now let $D_2 = \{4, -2, 12, -4, -5, -6\}$. Again, D_2 intersects a block which must be in $\mathcal{B}_{x_3} = \{B_3, C_3\}$, say, in a similar configuration of containing 3 and 5 toes, respectively. Label $B_3 = \{x_3, 17, 18, 19, -5, -6\}$: the other cases are similar or easier. Writing $C_3 = \{x_3, 20, 21, 22, 23, 24\}$, and assuming 20 does not appear in $\mathcal{B}_{x_i}$ or $\mathcal{W}(x_i)$ for $i \in \{1, 2\}$, we finally set $D_3 = \{4, -2, 12, -4, 20, -6\}$. By construction this does not intersect any other block in at least 2 places, and leads to a final contradiction.

This completes the proof of the main theorem.

6. Configurations for other lottery designs

Tickets of size 6 are represented in the diagrams of Fig. 3 by concatenating the labels over the vertices in each line. These sets of blocks can be combined to give $(n, 6, 6, 2; x)$-lottery designs. For example our $(59, 6, 6, 2; 27)$-lottery design from Section 2 has configuration (B, C, E, E, E).

Theorem 6.1. *Table 2 lists $j = L(n, 6, 6, 2)$ together with configurations described in Fig. 3 which afford an $(n, 6, 6, 2; j)$-lottery design.*

n	$L(n, 6, 6, 2)$	Configuration	n	$L(n, 6, 6, 2)$	Configuration
32	7	(A, A, A, A, B)	47	17	(A, C, C, C, E)
33	7	(A, A, A, A, C)	48	18	(A, C, C, D, E)
34	7	(A, A, A, A, D)	49	19	(A, A, C, E, E)
35	9	(A, A, A, B, C)	50	19	(A, A, A, A, E)
36	9	(A, A, A, C, C)	51	20	(A, A, A, D, E)
37	10	(A, A, A, C, D)	52	21	(A, C, C, E, E)
38	11	(A, A, A, A, E)	53	22	(A, C, D, E, E)
39	11	(A, A, C, C, C)	54	23	(A, A, E, E, E)
40	12	(A, A, C, C, D)	55	23	(C, C, C, E, E)
41	13	(A, A, A, C, E)	56	24	(C, C, D, E, E)
42	13	(A, C, C, C, C)	57	25	(A, C, E, E, E)
43	14	(A, C, C, C, D)	58	26	(A, D, E, E, E)
44	15	(A, A, C, C, E)	59	27	(B, C, E, E, E)
45	15	(C, C, C, C, C)	60	27	(C, C, E, E, E)
46	16	(C, C, C, C, D)	61	28	(C, D, E, E, E)

TABLE 2. Lottery numbers and their configurations

(A)

1,2,3   4,5,6

(1,2,3,4,5,6)

(B)   3,4

(1,2,3,4,5,6)
(1,2,3,4,7,8)
(1,2,5,6,7,8)

{∅}

1,2

5,6   7,8

(C)   1,2,3

(1,2,3,4,5,6)
(1,2,3,7,8,9)
(4,5,6,7,8,9)

4,5,6   7,8,9

(D)   3,4

5,6

1,2   9,10

7,8

(1,2,3,4,9,10)
(1,2,5,6,9,10)
(1,2,7,8,9,10)
(3,4,5,6,7,8)

(E)   1,2

3,4   5,6

7,8

9,10   13,14

11,12

(1,2,3,4,9,10)
(1,2,5,6,13,14)
(1,2,7,8,11,12)
(3,4,5,6,11,12)
(3,4,7,8,13,14)
(5,6,7,8,9,10)
(9,10,11,12,13,14)
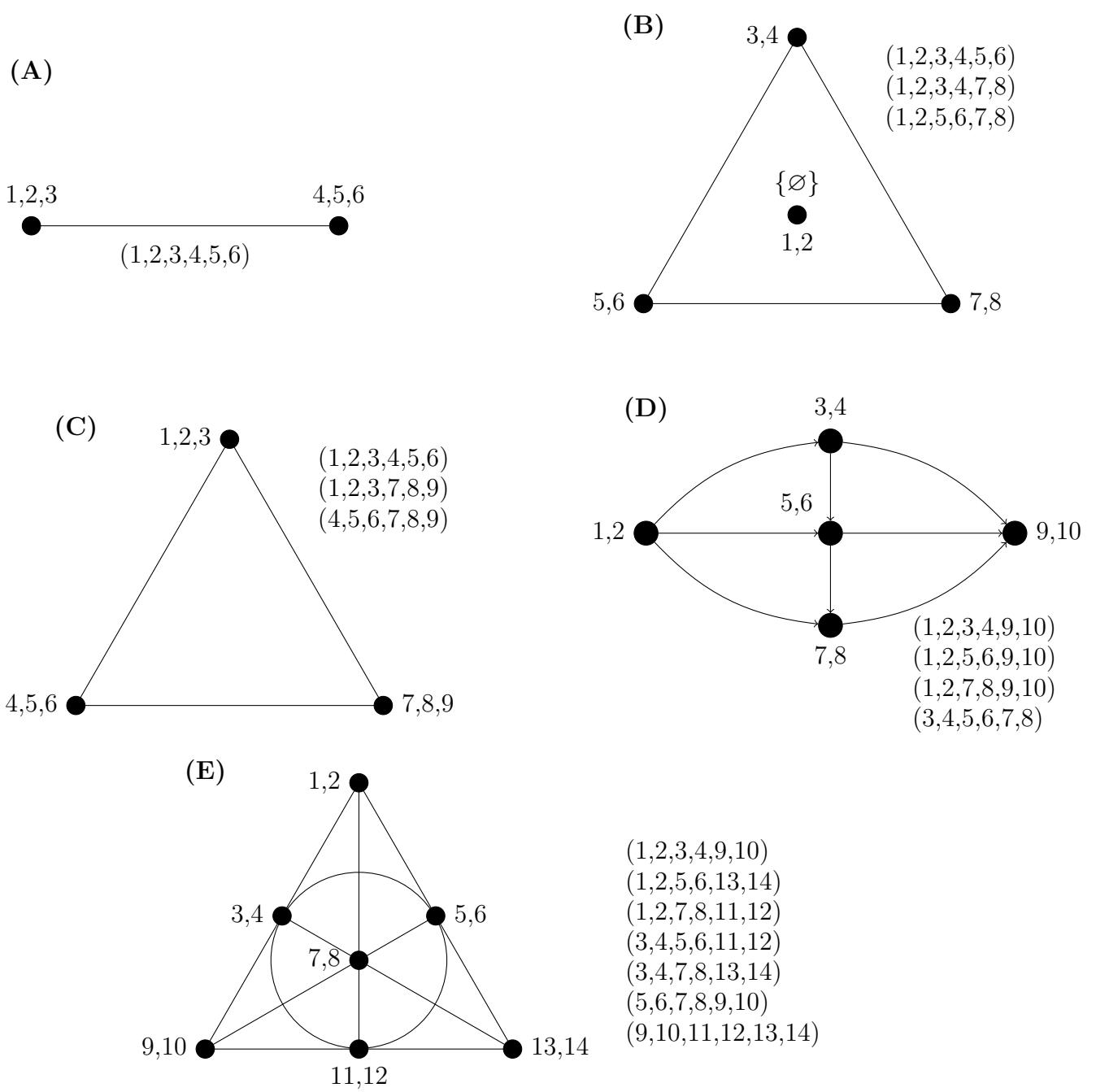
FIGURE 3. Ticket configurations

Appendices

A. Main Prolog code

```prolog
:-use_module(library(clpfd) ).
:-use_module(library(lists)).
:-use_module(library(ordsets)).
:-use_module(library(ugraphs)).
:-use_module(library(between)).

% ['lottery.pl'].

% compute_lottery_numbers_in_range(30,61).

%%% COMPUTE LOTTERY NUMBERS IN A RANGE %%%
% Given Nmin and Nmax try to compute L(n,6,6,2) for Nmin <= n <= Nmax.
% The value of L(n,6,6,2) is passed in to the computation of L(n+1,6,6,2)
compute_lottery_numbers_in_range( Nmin, Nmax ) :-
    PrevLottoNum #= 1,
    compute_lottery_numbers_in_range2( PrevLottoNum, Nmin, Nmax ).
compute_lottery_numbers_in_range2( PrevLottoNum, N, Nmax ) :-
    N #= Nmax,
    possible_lottery_number( N, PrevLottoNum, _ ).
compute_lottery_numbers_in_range2( PrevLottoNum, N, Nmax ) :-
    N #\= Nmax,
    possible_lottery_number( N, PrevLottoNum, NewLottoNum ),
    NewN #= N + 1,
    compute_lottery_numbers_in_range2( NewLottoNum, NewN, Nmax ).
%%%%%

%%% POSSIBLE LOTTERY NUMBER %%%
possible_lottery_number( N, PrevLottoNum, UB ) :-
    % first calculate the upper bound
    upper_bound( N, PrevLottoNum, UB ),
    possible_lottery_number2( N, PrevLottoNum, UB ).
possible_lottery_number2( N, PrevLottoNum, UB ) :-
    PrevLottoNum #= UB,
    print_message(informational, format(' L(~w,6,6,2) = ~w ',[N,UB]) ).
possible_lottery_number2( N, PrevLottoNum, UB ) :-
    PrevLottoNum #\= UB,
    % Calculate an upper bound on d1
    bound_isolated_blocks( N, UB, Rs ),
    get_min_num_Iblocks( N, MinNumIBlocks ),
    % get any exceptions for constructing a slim I
    maplist( get_slim_I_exceptions(N, UB), Rs, PossSlimIExceptions ),
    include( interval_check, PossSlimIExceptions, SlimIExceptions1 ),
    % get Delta(I) exceptions
    maplist( get_deltaI_exceptions(N, UB, MinNumIBlocks), Rs, DeltaExceptionsList )
        ,
    append( DeltaExceptionsList, DeltaExceptions),
    write_lotto_result( SlimIExceptions1, DeltaExceptions, N, UB ).
%%%%%

%%% D1 UPPER BOUND %%%
bound_isolated_blocks( N, UB, PossD1s ) :-
    get_r_min( N, UB, D1Min ),
```

```prolog
    include( bound_isolated_blocks_helper(N, UB, D1Min), [0,1,2,3,4], PossD1s ).
bound_isolated_blocks_helper(_, _, D1Min, 0) :-
    D1Min #< 1.
bound_isolated_blocks_helper(N, UB, D1Min, R) :-
    R #> 0,
    D1Min #=< 6*R,
    P #= 6 - R,
    LB #= UB - R,
    M #= N - 6 * R,
    furedi_lower_bound( Vs, M, 6, P, LB ),
    labeling([], Vs).
get_r_min( N, UB, D1Min ) :-
    T #= 2*N - 6*(UB - 1),
    T #=< 0,
    D1Min #= 0.
get_r_min( N, UB, D1Min ) :-
    T #= 2*N - 6*(UB - 1),
    T #> 0,
    D1Min #= T.
%%%%%

%%% COVERING DESIGN UPPER BOUND %%%
% Start with a guess for the upper bound and increase the guess until a solution
    is found
upper_bound( N, Guess, UB ) :-
    findall( Vs, ( test_upper_bound( Vs, N, Guess ), labeling([], Vs)), Sols ),
    upper_bound2( N, Guess, Sols, UB ).
upper_bound2( N, Guess, [], UB ) :-
    Guess2 #= Guess + 1,
    upper_bound( N, Guess2, UB ).
upper_bound2( _, Guess, Sols, UB ) :-
    length(Sols, KK),
    KK #>= 1,
    UB #= Guess.

test_upper_bound( Vs, N, UB ) :-
    length( Vs, 5 ),
    domain( Vs, 1, 65 ),
    M is N - 25,
    sum( Vs, #=, M ),
    get_covering_design_nums( Vs, Scores ),
    sum( Scores, #=<, UB ),
    sorting(Vs,[1,2,3,4,5],Vs).

get_covering_design_nums( Vs, Scores ) :-
    Vals = [1,3,3,3,4,
        6,6,7,7,10,10,12,12,15,16,
        17,19,21,22,23,24,27,28,30,31,
        31,38,39,40,42,47,50,51,54,55,
        59,63,65,67,70,73,79,80,82,87,
        90,96,98,99,105,110,114,117,119,128,
        132,135,140,142,143,143,157,160,163,172],
    maplist( get_covering_design_num(Vals), Vs, Scores ).
get_covering_design_num(Vals, I, Score ) :-
    nth1( I, Vals, Score ).
%%%%%
```

```prolog
%%% FUREDI LOWER BOUND %%%
furedi_lower_bound(Vs, N, K, P, LB ) :-
    LLB is K * LB,
    NP is P - 1,
    length( Vs, NP ),
    domain( Vs, 1, N ),
    sum( Vs, #=, N ),
    maplist( furedi_lower_bound_helper(K), Vs, Ws ),
    sum( Ws, #=<, LLB ).
furedi_lower_bound_helper( K, X, Y ) :-
    ((X -1 ) mod (K - 1 ) ) #= 0,
    Y #= X * ( (X-1) / (K - 1 ) ).
furedi_lower_bound_helper( K, X, Y ) :-
    ((X -1 ) mod (K - 1 ) ) #\= 0,
    Y #= X * ( 1 + ( (X-1) / ( K - 1 ) ) ).
%%%%%

%%% SLIM I EXCEPTIONS
get_slim_I_exceptions( N, UB, R, [R, [D2Lower, D2Upper] ] ) :-
    D2Upper #= 9 * (4 - R),
    D2Lower0 #= ( 9*( 4*N - 6*(UB-1) - 18*R - 16*(4-R) )) / 4,
    make_non_negative(D2Lower0,D2Lower).
interval_check([_, [A,B] ]) :-A #=< B.
make_non_negative(X, Y) :-
    X #< 0,
    Y #= 0.
make_non_negative(X, Y) :-
    X #>= 0,
    Y #= X.
%%%%%

%%% DELTA(I) EXCEPTIONS %%%
% Lower bound the number of blocks in B_I
get_min_num_Iblocks( N, Min) :-
    ( ( N - 5 ) mod 5 ) #= 0,
    Min #= (N - 5)/5.
get_min_num_Iblocks( N, Min ) :-
    ( ( N - 5 ) mod 5 ) #\= 0,
    Min #= 1 + ((N - 5)/5).

% Find the values of Delta(I) we cannot immediately rule out
get_deltaI_exceptions( N, UB, MinNumIBlocks, R, Deltas ) :-
    % Lower bound how many 2s are assumed to be in Delta(I)
    get_delta_two( N, UB, R, Delta2 ),
    % Fill Delta(I) with as many 1s and 2s that can be assumed
    % Then generate all ways to complete Delta(I) by adding 2s and 3s
    row_of_n_ms( R, 1, L1 ),
    row_of_n_ms( Delta2, 2, L2 ),
    append( L1, L2, L3 ),
    length( L3, K1),
    K2 #= 5 - K1,
    findall( Xs, ( length(Xs, 2), domain(Xs, 0, K2), sum(Xs, #=, K2), labeling([],
        Xs) ), VSizes ),
    maplist( make_delta_end, VSizes, DeltaEnds ),
    maplist( append(L3), DeltaEnds, Deltas1 ),
```

```prolog
    % Remove out Delta(I) where |B_I| is too small
    include( at_least_m_blocks(MinNumIBlocks), Deltas1, Deltas2 ),
    % Upper bound the excess of the whole design
    get_base_excess( N, R, UB, BaseExcess ),
    % For each possible Delta(I) consider the constraint problem of populating toes
         in the IBlocks
    include( can_populate_toes_in_Iblocks(N, R, BaseExcess), Deltas2, Deltas ).

get_delta_two( N, UB, R, Delta2 ) :-
    get_delta_two2( N, UB, R, PossDelta2 ),
    make_non_negative(PossDelta2, Delta2).
get_delta_two2( N, UB, R, Delta2 ) :-
    X #= (3*N - 12 * R - 6 * (UB - 1) ),
    (X mod 9) #= 0,
    Delta2 #= X / 9.
get_delta_two2( N, UB, R, Delta2 ) :-
    X #= (3*N - 12 * R - 6 * (UB - 1) ),
    (X mod 9) #\= 0,
    Delta2 #= 1 + (X / 9).
row_of_n_ms( N, M, Row ) :-
    length( Row, N ),
    maplist( #=(M), Row ).
make_delta_end( [A,B], DeltaEnd ) :-
    row_of_n_ms( A, 2, X1 ),
    row_of_n_ms( B, 3, X2 ),
    append(X1, X2, DeltaEnd).

at_least_m_blocks( MinNumIBlocks, PossDelta ) :-
    sum(PossDelta, #>=, MinNumIBlocks).

get_base_excess( N, R, UB, BaseExcess ) :-
    BaseExcess #= 6*(UB-1) + 6*R - 2*N.

can_populate_toes_in_Iblocks(N, R, BaseExcess, Delta) :-
    sum(Delta, #=, B),
    MinToes #= 2*N - 10 - 5*(B+R),
    delete( Delta, 3, Residue),
    length( Residue, NumOnesAndTwos ),
    NumThrees #= 5 - NumOnesAndTwos,
    Excess #= BaseExcess - NumThrees,
    delete( Delta, 1, DeltaNoOnes),
    populate_toes_in_Iblocks( DeltaNoOnes, MinToes, Excess, Vs ),
    labeling([], Vs).

populate_toes_in_Iblocks( DeltaNoOnes, MinToes, Excess, Vs ) :-
    same_length( Vs, DeltaNoOnes, N ),
    domain( Vs, 1, 15 ),
    maplist( adjust_domain, DeltaNoOnes, Vs ),
    maplist( get_ex, Vs, DeltaNoOnes, Excesses ),
    sum(Excesses, #=<, Excess ),
    sum(Vs, #>=, MinToes ),
    numlist(N,L),
    sorting(Vs,L,Vs).
adjust_domain( 2, V ) :-V #=< 10.
adjust_domain( 3, _).
get_ex( V, 2, Ex ) :-
```

```prolog
      element( V, [0,0,0,0,0,0,2,3,7,10], Ex ).
  get_ex( V, 3, Ex ) :-
      element( V, [1,2,3,4,5,6,7,8,9,10,11,12,20,25,27], Ex ).
  %%%%%

  %%% RESULT OUTPUT %%%
  write_lotto_result( [], [], N, UB ) :-
      print_message(informational, format(' L(~w,6,6,2) = ~w ',[N,UB]) ).
  write_lotto_result( SlimIExceptions, DeltaExceptions, N, UB ) :-
      append( SlimIExceptions, DeltaExceptions, TotalExceptions ),
      length(TotalExceptions, Ex),
      Ex #> 0,
      print_message(informational, format('We conjecture that L(~w,6,6,2) = ~w and
          must rule out the following cases',[N,UB]) ),
      write_slimI_exceptions( SlimIExceptions ),
      write_delta_exceptions( DeltaExceptions ).

  write_slimI_exceptions( Exceptions ) :-
      maplist( write_slimI_exception, Exceptions ).
  write_slimI_exception( [R, [D2, D2] ] ) :-
      D1 #= 6*R,
      print_message(informational, format(' d_1 = ~w and d_2 = ~w ',[D1,D2]) ).
  write_slimI_exception([R, [D2L, D2U] ] ) :-
      D1 #= 6*R,
      D2L #\= D2U,
      print_message(informational, format(' d_1 = ~w and d2 in the range [~w,~w] ',[
          D1,D2L, D2U]) ).

  write_delta_exceptions( Exceptions ) :-
      maplist( write_delta_exception, Exceptions ).
  write_delta_exception( Delta ) :-
      delete( Delta, 2, Residue),
      length( Residue, NumNoneTwos ),
      NumTwos #= 5 - NumNoneTwos,
      D2 #= NumTwos * 9,
      print_message(informational, format(' d_2 <= ~w and \\delta(I) = ~w ',[D2,
          Delta]) ).

  writeln( Stream ) :-
          write( Stream ),
          write('\n').
  %%%%%
```

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