Assignment [1] CI

TASK [1]

1.1 To check intersection let $r_1 = r_2$

$$P_1 + t_1 v_1 + u_1 w_1 = P_2 + t_2 v_2 + u_2 w_2$$

$$P_1 - P_2 = t_2 v_2 + u_2 w_2 - t_1 v_1 - u_1 w_1$$

$$\therefore \quad P_1 - P_2 = \underbrace{\begin{bmatrix} v_2 & w_2 & -v_1 & -w_1 \end{bmatrix}}_{A \; 3\times 4} \begin{bmatrix} t_2 \\ u_2 \\ t_1 \\ u_1 \end{bmatrix}_{4\times 1} \quad \boxed{1}$$

multiply $\boxed{1}$ by A^T

$$A^T A \begin{bmatrix} t_2 \\ u_2 \\ t_1 \\ u_1 \end{bmatrix} = A^T (P_1 - P_2)$$

$$\Rightarrow \begin{bmatrix} t_2 \\ u_2 \\ t_1 \\ u_1 \end{bmatrix} = (A^T A)^{-1} A^T (P_1 - P_2) = A^+ (P_1 - P_2) \quad \boxed{2}$$

From ①, ② $\Rightarrow$ $P_1 - P_2 = A A^+ (P_1 - P_2)$

$$\therefore \quad [I - AA^+](P_1 - P_2) = 0$$

The condition for intersection is $AA^+ = I$
which means $(A^T A)^{-1}$ should be full ranks.

or $P_1 = P_2$ this is also statisfy the condition
(See Matlab File Task1-1)

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1.2 To get the normal vector:

1. v, w span the space of the plane (Combinations)

$$a_1 V + a_2 w = P$$

$$\underbrace{[V \quad w]}_{A} \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{b} = P \qquad Ab = P$$

Since v, w span the Column Space, so left null space of A can get the normal vector. $N(A^T)$

$$A = \begin{bmatrix} 1 & 2 \\ 7 & 8 \\ 4 & -5 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 7 & 4 \\ 2 & 8 & -5 \end{bmatrix} \xrightarrow[R_2]{-2R_1 + R_2}$$

$$A^T = \begin{bmatrix} 1 & 7 & 4 \\ 0 & -6 & -13 \end{bmatrix} \qquad \begin{aligned} x_1 + 7x_2 + 4x_3 &= 0 \\ -6x_2 - 13x_3 &= 0 \end{aligned}$$

$$x_2 = -\frac{13}{6} x_3, \quad x_3 = 6 \Rightarrow x_2 = -13 \Rightarrow x_1 = 67$$

$$N(A^T) = \left\{ \begin{bmatrix} 67 \\ -13 \\ 6 \end{bmatrix} \right\} \xrightarrow{\text{normalize}} = \left\{ \begin{bmatrix} 0.9779 \\ -0.1897 \\ 0.0876 \end{bmatrix} \right\} \rightarrow n$$

norm = 68.512

To Check $\vec{n} \cdot (\vec{v} - \vec{w}) = 0$ see matlab file. Task1-2

Representation:

$$\begin{bmatrix} 0.9779 & -0.1897 & 0.0876 \end{bmatrix} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] = 0$$

$$\boxed{0.9779x - 0.1897y + 0.0876z + 0.1897 = 0}$$

$$ax + by + cz + d = 0$$

② Again

$A = \begin{bmatrix} -2 & 5 \\ -2 & 5 \\ 1 & -5 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} -2 & -2 & 1 \\ 5 & 5 & -5 \end{bmatrix}$ $\frac{5}{2}R_1 + R_2 \rightarrow R_2$

$A^T = \begin{bmatrix} -2 & -2 & 1 \\ 0 & 0 & -5/2 \end{bmatrix}$

$-5/2\, x_3 = 0$ $\boxed{x_3 = 0}$

$-2x_1 - 2x_2 + x_3 = 0$

$x_1 = -x_2$ $\boxed{x_2 = -1} \Rightarrow \boxed{x_1 = 1}$

$N(A^T) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ $n = \begin{bmatrix} 0.7071 \\ -0.7071 \\ 0 \end{bmatrix}$

normalize

norm $= \sqrt{2}$

check $\vec{n} \cdot (\vec{v} - \vec{w}) = 0$ ([matlab] TASK1-2)

representation $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$

$[0.7071 \quad -0.7071 \quad 0] \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right] = 0$

$\boxed{0.7071\, x - 0.7071\, y + 0.7071 = 0}$

$ax + by + cz + d = 0$

$a = 0.7071$ $b = -0.7071$ $c = 0$ $d = +0.7071$

plotting → Matlab

To check the equation, let t, u any values

get (r) point in plane <x, y, z> substitute → 0

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(1.3) First of all get the normal vector to the plane:

as previous

$$A = \begin{bmatrix} v & w \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & 1 \\ 3 & -5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -2 & 3 & 3 \\ 1 & 1 & -5 \end{bmatrix}$$ $2R_2 + R_1 \longrightarrow R_2$

Solve for $N(A^T)$

$$A^T = \begin{bmatrix} -2 & 3 & 3 \\ 0 & 5 & 7 \end{bmatrix}$$

$5x_2 = -7x_3$

$x_3 = 5$, $x_2 = 7$, $x_1 = 18$

$$N(A^T) = \left\{ \begin{bmatrix} 18 \\ 7 \\ 5 \end{bmatrix} \right\}$$

normalize

norm = 19.949

$$n = \begin{bmatrix} 0.9023 \\ 0.3509 \\ 0.2506 \end{bmatrix}$$

The equation of a line $r = P_0 + tV$

point on line (origin)

vector parallel to the line (n)

$\boxed{V \parallel n}$, $P_0 = \langle 0, 0, 0 \rangle$

$$\Rightarrow \boxed{r = t \begin{bmatrix} 0.9023 \\ 0.3509 \\ 0.2506 \end{bmatrix}}$$

equation of line $\perp$ plane S

projection of g onto line r

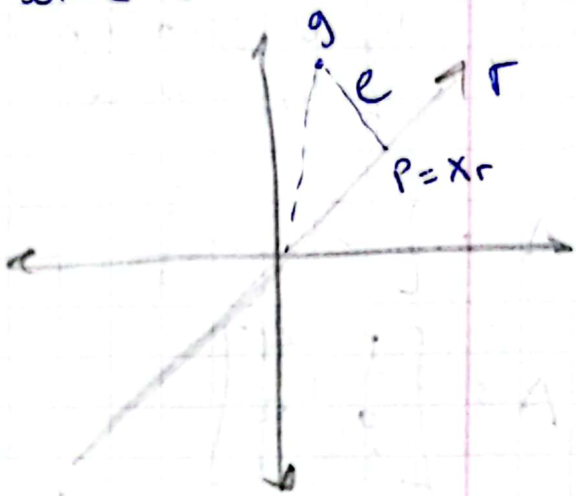
$P = xr$

$e = g - P = g - xr$

$r^T e = 0$

$\Rightarrow r^T(g - xr) = 0$

$x r^T r = r^T g \quad x = \frac{r^T g}{r^T r}$



$P = Xr \Rightarrow \boxed{p = \frac{r r^T g}{r^T r}}$, $P_{\text{projection matrix}} = \frac{r r^T}{r^T r}$

$p_{\text{projection of g onto Line r}} = P_{\text{matrix}} * g$

for $t = 1$ $\quad r = \begin{bmatrix} 0.9023 \\ 0.3509 \\ 0.2506 \end{bmatrix}$

Calculate $p = Pg$ see matlab file (task1_3)

$$p = \begin{bmatrix} -7.9598 \\ -3.0955 \\ -2.2111 \end{bmatrix}$$

To check cross product of line r and projection point p must equal zero [check done in matlab file]

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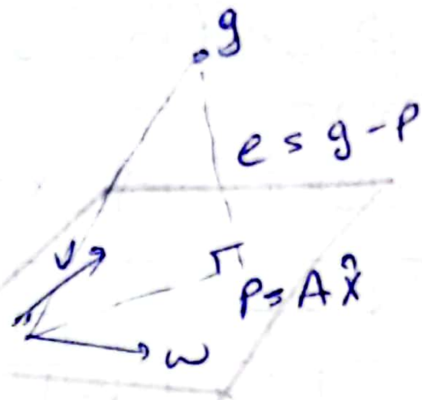
(1.4) since
v, w span the column space
of the plane



$$A = \begin{bmatrix} v & w \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$e \perp$ plane $\Rightarrow A^T e = 0$

$\therefore e$ in $N(A^T)$, $e \perp C(A)$

$$\therefore \quad A^T(g-p) = A^T(g - A\hat{x}) = 0$$

$$A^T A \hat{x} = A^T g \rightarrow \hat{x} = (A^T A)^{-1} A^T g$$

$$p = A\hat{x} \quad \Rightarrow \quad p = A(A^T A)^{-1} A^T g = \underbrace{A A^+}_{\text{projection matrix}} g$$

projection matrix $= AA^+ = CC^T$

But this is the projection in case the plane pass through the origion.

For a plane passes through point P_0

$\therefore$ proj of g on plane $= \boxed{g^* = CC^T(g - P_0) + P_0}$

$$g^* = \begin{bmatrix} -10 \\ -2 \\ 7 \end{bmatrix}$$ See matlab file task1_4.

To check that g^* on the plane $\vec{n} \cdot (\vec{g^*} - \vec{P_0}) = 0$

$$\begin{pmatrix} 0 & 0.4472 & 0.8944 \end{pmatrix} \left[\begin{pmatrix} -10 \\ -2 \\ 7 \end{pmatrix} - \begin{pmatrix} -5 \\ 11 \\ 0.5 \end{pmatrix} \right] =$$

$0.4472 \times -13 + 0.8944 \times 6.5 =$ Zero #

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The projection of g^* symmetric to g

$$g^* = (P - g) + P \quad \# \quad g^* = \begin{bmatrix} -10 \\ -1 \\ 9 \end{bmatrix}$$

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TASK 2

2.1

$$\begin{array}{c} \\ \end{array}\overset{V}{\begin{bmatrix} 3 & 1 & 1 \\ 6 & 2 & 2 \\ -9 & -3 & -3 \end{bmatrix}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Space of Solution $N(V)$ $\xrightarrow{\text{rref}}$ $-2R_1 + R_2 \rightarrow R_2$

$+3R_1 + R_3 \rightarrow R_3$

$$V = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1]{\frac{1}{3}R_1 + R_2} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot free variable

$x_1 = (-x_2 - x_3)/3$

$\boxed{x = -y/3 - z/3}$

$$N(V) = c\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$$

Special solutions

Basis for $N(V) = \left\{ \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

(See matlab Task 2-1)

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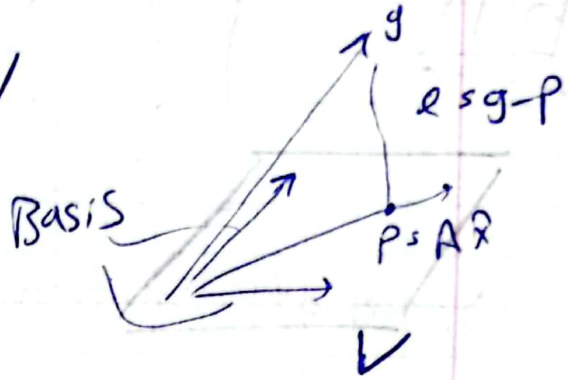
(2.2) Since we obtain the basis of the space V

$$A = \begin{bmatrix} -1/3 & -1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ Basis of } V$$



$e \perp$ plane, $e \in N(A^T)$

$e \perp C(A)$

$$\Rightarrow \quad A^T e = 0$$

$$A^T (g - p) = A^T (g - A\hat{x}) = 0$$

$$A^T A \hat{x} = A^T g \quad \rightarrow \quad \hat{x} = (A^T A)^{-1} A^T g$$

$$p = A\hat{x} \quad \rightarrow \quad p = A(A^T A)^{-1} A^T g = \boxed{\underbrace{AA^+}_{\text{projection matrix}} g}$$

Note that

$$AA^+ = NN^T$$

using null(V) in matlab to get the orthonormal basis N

$$P_V = AA^+ g = NN^T g = \begin{bmatrix} -0.7273 \\ -0.9091 \\ 3.0909 \end{bmatrix}$$

See Matlab file Task2_2

projection onto $V^{\perp}$ is projection on the orthogonal component to the plane V

$$e \perp V$$

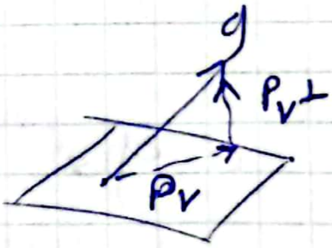
$$e = g - p = g - AA^{+}g = (I - AA^{+})g$$

$$P_{V^{\perp}} = (I - AA^{+})g = (I - NN^{T})g$$

using Matlab

$$P_{V^{\perp}} = \begin{bmatrix} -0.2727 \\ -0.0909 \\ -0.0909 \end{bmatrix}$$

To prove that the procedure is correct:



check $g = P_V + P_{V^{\perp}}$

$$g = \begin{bmatrix} -0.7273 \\ -0.9091 \\ 3.0909 \end{bmatrix} + \begin{bmatrix} -0.2727 \\ -0.0909 \\ -0.0909 \end{bmatrix}$$

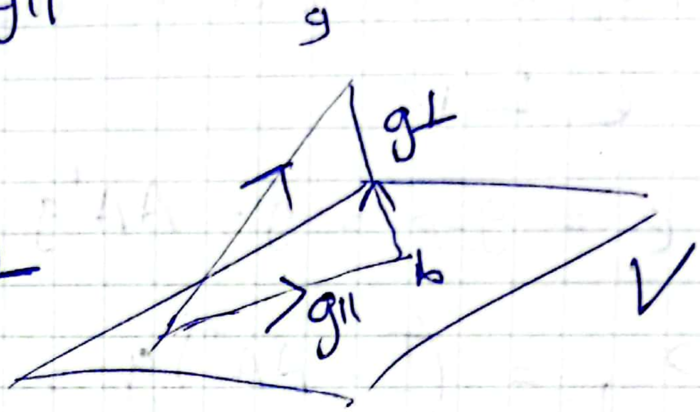
$$g = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \quad \#$$

projection visualization $\longrightarrow$ Matlab.

2.3

Since we have $g^{\parallel}$ and $g^{\perp}$

$$\Rightarrow g = g^{\parallel} + g^{\perp}$$



as proved previously, projection of vector onto column space can be achieved as follows,

$$g^{\parallel} = AA^{+}g = Pg$$

the projection onto left null space

$$g^{\perp} = g - b = g - Pg = (I - AA^{+})g$$

$$\therefore g = g^{\parallel} + g^{\perp} = \cancel{AA^{+}g} + Ig - \cancel{AA^{+}g}$$

$$\boxed{g^{\parallel} + g^{\perp} = g}$$

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TASK (3)

(3.1)

$$\min_{x_1, x_2} \ \frac{1}{2}x_1^2 + 4x_2^2 - 32x_2 + 60$$

s.t.

$$x_1 + x_2 \leq 6$$

$$x_1 + 2x_2 \leq 8$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_2 \leq 9$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$f(X) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix}}_{H} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -32 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{60}_{C_0}$$

$$\therefore f(X) = \frac{1}{2} X^T H X + C X + C_0 \quad \#$$

The ==constraints==

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} \leq \begin{bmatrix} 6 \\ 8 \\ 0 \\ 0 \\ 9 \end{bmatrix}$$

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