Circuit Theory

Problem Set 3

Complex Impedances and Response of Linear Circuits to AC

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(a)

$$V_{\text{rms}} = V_C \tag{1}$$

(b)

$$V_{\text{rms}} = \frac{V_C}{\sqrt{2}} \tag{2}$$

(c)

$$V_{\text{rms}} = \frac{V_C}{2} \tag{3}$$

(b) and (c) are different because (b) has a higher extreme absolute value.

(d)

$$V_{\text{rms}} = \sqrt{\frac{1}{2T}\int_{-T}^{T}\left(\frac{V_0}{T}\right)^2 t^2\,\mathrm{d}t} = \frac{V_0}{\sqrt{3}} \tag{4}$$

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For all the networks we have $\tilde{V} = V_0 e^{i(\omega t - \pi/2)}$

(a)

$$Z = R - \frac{j}{\omega C} \tag{5}$$

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0}{\sqrt{R^2 + (1/\omega C)^2}} e^{i(\omega t - \pi/2 + \phi)} \tag{6}$$

where $\phi = \tan^{-1}[1/(\omega C R)] = 0.016\,\text{rad}$.

The current leads the voltage by ϕ.

$$\tilde{V_R} = R\tilde{I} = V_0 \frac{R}{\sqrt{R^2 + (1/\omega C)^2}} e^{i(\omega t - \pi/2 + \phi)} \tag{7}$$

Thus, $V_{R,\max} = V_0 R/(\sqrt{R^2 + (1/\omega C)^2}) = 10\,\text{V}$.

(b)

$$Z = j\omega L - \frac{j}{\omega C} \tag{8}$$

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0}{\sqrt{(\omega L)^2 + (1/\omega C)^2}} e^{i(\omega t - \pi)} \tag{9}$$

The current lags the voltage by $\pi/2$.

$$V_{C,\max} = V_0 \frac{(1/\omega C)}{\sqrt{(\omega L)^2 + (1/\omega C)^2}} = 10\,\mathrm{V} \tag{10}$$

(c)

$$Z = R + j\omega L - \frac{j}{\omega C} \tag{11}$$

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0}{\sqrt{R^2 + (\omega L)^2 + (1/\omega C)^2}} e^{i(\omega t - \pi/2 + \phi)} \tag{12}$$

where:

$$\phi = \tan^{-1}\left(\frac{1/(\omega C) - \omega L}{R}\right) = 5.5 \times 10^{-3}\,\mathrm{rad} \tag{13}$$

The current leads the voltage by ϕ.

$$V_{L,\max} = V_0 \frac{\omega L}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} = 1.3 \times 10^{-3}\,\mathrm{V} \tag{14}$$

(d)

$$Z = \frac{1}{1/R + 1/(j\omega L)} = X e^{i\pi/2} \tag{15}$$

where $X = 3.95\,\Omega$.

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0}{X} e^{i(\omega t - \pi)} \tag{16}$$

The current lags the voltage by $\pi/2$.

$$V_{R,\max} = V_0 = 10\,\mathrm{V} \tag{17}$$

(e)

$$Z = \frac{1}{1/(j\omega L) + j\omega C} = Xe^{i\pi/2} \tag{18}$$

where $X = 39.5\,\Omega$.

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0}{X}e^{i(\omega t - \pi)} \tag{19}$$

The current lags the voltage by $\pi/2$.

$$V_{C,\max} = V_0 = 10\,\mathrm{V} \tag{20}$$

(f)

$$Z = \frac{1}{1/R + \frac{1}{j\omega L - j/(\omega C)}} = \frac{R(\omega L - 1/(\omega C))}{\sqrt{R^2 + (\omega L - 1/(\omega C))^2}}e^{i\phi} = Xe^{i\phi} \tag{21}$$

where:

$$\phi = \tan^{-1}\left(\frac{R}{1/(\omega C) - \omega L}\right) = -1.54\,\mathrm{rad} \tag{22}$$

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0}{X}e^{i(\omega t - \pi/2 - \phi)} \tag{23}$$

The current lags the voltage by ϕ.

$$V_{L,\max} = V_0\frac{\omega L}{\omega L - 1/(\omega C)} = 10.3\,\mathrm{V} \tag{24}$$

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For both circuits, $\tilde{V_1} = V_0e^{i(\omega t - \pi/2)}$.

(I)

$$\tilde{V_2} = \tilde{V_1}\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{V_0}{\sqrt{1 + (\omega CR)^2}}e^{i(\omega t - \pi/2 - \phi)} \tag{25}$$

where:

$$\phi = \tan^{-1}(\omega CR) \tag{26}$$

$$\frac{V_{2,\max}}{V_{1,\max}} = \frac{1}{\sqrt{1+(\omega CR)^2}} \tag{27}$$

(II)

$$\tilde{V_2} = \tilde{V_1}\frac{j\omega L}{R+j\omega L} = \frac{V_0}{\sqrt{1+(R/\omega L)^2}}e^{i(\omega t-\pi/2+\phi)} \tag{28}$$

where:

$$\phi = \tan^{-1}\left(\frac{R}{\omega L}\right) \tag{29}$$

$$\frac{V_{2,\max}}{V_{1,\max}} = \frac{1}{\sqrt{1+(R/\omega L)^2}} \tag{30}$$

For both circuits, the ratio is given by:

$$\frac{1}{\sqrt{1+(\omega/\omega_0)^2}} \tag{31}$$

and the phase is given by:

$$\tan^{-1}\left(\frac{\omega}{\omega_0}\right) \tag{32}$$

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We have $\tilde{I} = I_0 e^{i(\omega t-\pi/2)}$.

(a)

$$\tilde{V_C} = \frac{\tilde{I}}{j\omega C} = \frac{I_0}{\omega C}e^{i(\omega t-\pi)} \tag{33}$$

where $\frac{I_0}{\omega C} = 10\,\mathrm{V}$.

$$\tilde{V_L} = \tilde{I}j\omega L = I_0\omega L e^{i\omega t} \tag{34}$$

where $I_0\omega L = 20\,\mathrm{V}$.

$$\tilde{V_R} = \tilde{I}R = I_0 R e^{i(\omega t-\pi/2)} \tag{35}$$

where $I_0 R = 10\,\mathrm{V}$.

$$\tilde{V} = I_0 X e^{i(\omega t - \pi/2 + \phi)} \tag{36}$$

where:

$$X = \sqrt{R^2 + (\omega L - 1/(\omega C))^2} = 141\,\Omega \tag{37}$$

and:

$$\phi = \tan^{-1}\left(\frac{\omega L - 1/(\omega C)}{R}\right) = \pi/2 \tag{38}$$

and $I_0 X = 14.1\,\mathrm{V}$

(b)

$$P = \mathrm{Re}(\tilde{V})\,\mathrm{Re}(\tilde{I}) = I_0^2 X \sin(\omega t + \phi) \sin \omega t \tag{39}$$

(c)

$$W_L = \frac{1}{2} L\,\mathrm{Re}(\tilde{I})^2 = \frac{1}{2} L I_0^2 \sin^2 \omega t \tag{40}$$

$$W_C = \frac{1}{2} C\,\mathrm{Im}(\tilde{V_C})^2 = \frac{1}{2}\frac{1}{\omega^2 C} I_0^2 \cos^2 \omega t \tag{41}$$

(d) Their maximum values are $\frac{1}{2} L I_0^2$ and $\frac{1}{2}\frac{1}{\omega^2 C} I_0^2$ respectively.

(f) For the sum to be constant, $L = 1/(\omega^2 C) = 0.1\,\mathrm{mH}$.

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For this circuit:

$$Z = R + j\omega L - \frac{j}{\omega C} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} e^{i\phi} \tag{42}$$

For the amplitude to be minimum:

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}\right] = \frac{2(\omega L - \frac{1}{\omega C})(L + \frac{1}{\omega^2 C})}{2\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} = 0 \tag{43}$$

This happens when $\omega L - \frac{1}{\omega C} = 0$ or $\omega = \frac{1}{\sqrt{LC}}$. Then $Z = Re^{i\phi}$, and $V_{C,\max} = V_0/(\omega CR) = 3.16\,\mathrm{V}$.

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$$Z = j\omega L - \frac{j}{2\omega C} + \frac{1}{j\omega C + \frac{1}{\frac{1}{2j\omega C} + j\omega L}} = j\left(\omega L - \frac{1}{2\omega C} - \frac{1}{\omega C + \frac{2\omega C}{1-2\omega^2 CL}}\right) = 0 \tag{44}$$

Solving this equation (avoiding the asymptote) gives $\omega = \sqrt{5/(2LC)}$. The other root $\omega = \sqrt{1/(2LC)}$ is obtained by substituting the asymptote value directly into the original expression.

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(a) For the bridge to be balanced:

$$Z_1 \frac{1}{\frac{1}{R_4} + j\omega C_4} = \frac{1}{j\omega C_3} R_2 \tag{45}$$

Solving for Z_1 yields:

$$Z_1 = R_2 \frac{C_4}{C_3} - j\frac{R_2}{R_4}\frac{1}{\omega C_3} \tag{46}$$

(b) For a series combination:

$$R_1 - j\frac{1}{\omega C_1} = R_2 \frac{C_4}{C_3} - j\frac{R_2}{R_4}\frac{1}{\omega C_3} \tag{47}$$

so that $R_1 = R_2 C_4/C_3$ and $C_1 = C_3 R_4/R_2$.

(c) For a parallel combination:

$$\frac{1}{\frac{1}{R_1} + j\frac{1}{\omega C_1}} = R_2 \frac{C_4}{C_3} - j\frac{R_2}{R_4}\frac{1}{\omega C_3} \tag{48}$$

Solving this equation yields:

$$\begin{aligned} R_1 &= \frac{R_2\left[1 + (\omega C_4 R_4)^2\right]}{\omega^2 C_3 C_4 R_4^2} \\ C_1 &= \frac{C_3 R_4}{R_2\left[1 + (\omega C_4 R_4)^2\right]} \end{aligned} \tag{49}$$

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(a) Note that $\omega_0 L = 1/(\omega_0 C) = \sqrt{3}R$

$$Z = R + \frac{1}{\frac{1}{R+j\omega_0 L} + \frac{1}{R-j/(\omega_0 C)}} = R + \frac{(R+j\sqrt{3}R)(R-j\sqrt{3}R)}{2R} = 3R \tag{50}$$

(b)

$$\tilde{V_{AX}} = \frac{R}{3R}\tilde{V_{AB}} = \frac{\tilde{V_{AB}}}{3} \tag{51}$$

$$\tilde{V_{XY}} = \frac{2}{3}\tilde{V_{AB}}\frac{R}{R+j\sqrt{3}R} = \frac{\tilde{V_{AB}}}{3}e^{i\pi/3} \tag{52}$$

$$\tilde{V_{XZ}} = \frac{1}{3}\tilde{V_{AB}}\frac{R}{R-j/(\sqrt{3}R)} = \frac{\tilde{V_{AB}}}{3}e^{-i\pi/3} \tag{53}$$

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We have:

$$V_{xy} = \left(\frac{1}{2} - \frac{1}{1+j\omega RC}\right)V = \frac{j\omega RC - 1}{j\omega RC + 1}\frac{V}{2} \tag{54}$$

so the amplitude is $|V_{xy} = V/2|$ which is independent of R and the phase is $\pi - 2\phi$ where $\phi = \tan^{-1}(\omega RC)$.

Therefore, when $R = 1/(\omega C)$, $\pi - 2\phi = \pi/2$.