Statistical Mechanics

Problem Sheet 4

Relativistic and Fermi Gases

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Relativistic and Fermi gases

4.1

(i) For relativistic gas, the energy of a particle is given by:

$$E = \sqrt{p^2c^2 + m^2c^4} = \sqrt{\hbar^2k^2c^2 + m^2c^4} \tag{1}$$

where $k_i = \pi n_i / L$ are quantised wave numbers.

Proceeding as in the ordinary ideal gas case, we have the single particle partition function:

$$\begin{aligned} Z_1 &= \sum_n e^{-\beta E_n} \\ &\approx \int e^{-\beta\sqrt{\hbar^2k^2c^2+m^2c^4}} g(k)\,\mathrm{d}k \end{aligned} \tag{2}$$

where $g(k) = Vk^2/2\pi^2$ is the usual density of states.

The overall partition function is still $Z = Z_1^N/N!$ such that its logarithm is:

$$\begin{aligned} \ln Z &= N \ln Z_1 - \ln N! \\ &\approx N(\ln Z_1 - \ln N - 1) \end{aligned} \tag{3}$$

We may compute the pressure from the free energy:

$$\begin{aligned} p &= -\left(\frac{\partial F}{\partial V}\right)_{T,N} \\ &= k_BT\left(\frac{\partial \ln Z}{\partial V}\right)_{T,N} \\ &= Nk_BT\left(\frac{\partial \ln Z_1}{\partial V}\right)_{T,N} \\ &= \frac{Nk_BT}{V} \end{aligned} \tag{4}$$

where the last equality follows because Z_1 is proportional to V.

We see that the equation of state $pV = Nk_BT$ is the same as for the non-relativistic case.

(ii) From thermodynamic arguments, we have the general result that during an adiabatic process, $pV^\gamma = \text{const}$ where $\gamma = C_p/C_V$. Note also the following relation:

$$C_p - C_V = \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p = \frac{U}{T} + Nk_B \tag{5}$$

Let us write the internal energy explicitly:

$$U = \frac{N}{Z_1}\int E(k)e^{-\beta E(k)}g(k)\,\mathrm{d}k \tag{6}$$

where $E(k) = \sqrt{\hbar^2 k^2 c^2 + m^2 c^4}$.

We may compute C_V from the internal energy:

$$\begin{aligned} C_V &= \left(\frac{\partial U}{\partial T}\right)_V \\ &= \left(\frac{\partial U}{\partial \beta}\right)_V \left(\frac{-1}{k_B T^2}\right) \\ &= \frac{N}{k_B T^2 Z_1}\int E(k)^2 e^{-\beta E(k)}g(k)\,\mathrm{d}k - \frac{N}{k_B T^2 Z_1^2}\left[\int E(k)e^{-\beta E(k)}g(k)\,\mathrm{d}k\right]^2 \end{aligned} \tag{7}$$

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4.2

We have the expression for Fermi energy:

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{2s+1}\right)^{2/3} \tag{8}$$

and we define the Fermi temperature as:

$$T_F = \frac{\varepsilon_F}{k_B} \tag{9}$$

(a) For liquid helium, we have $m = 3u$, $s = 1/2$ and $n = \rho/m$. Thus:

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 \rho}{2m}\right)^{2/3} = 6.87 \times 10^{-27}\,\mathrm{J} \tag{10}$$

and:

$$T_F = \frac{\varepsilon_F}{k_B} = 4.98 \times 10^{-4}\,\mathrm{K} \tag{11}$$

(b) For electrons in aluminium, we have $m = 9.11 \times 10^{-31}$ kg, $s = 1/2$ and $n = \rho/m$:

$$\varepsilon_F = 1.66 \times 10^{-19}\,\text{J} \tag{12}$$

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4.3

(i) Consider the Fermi-Dirac distribution:

$$\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1} \tag{13}$$

At absolute zero, the distribution function is a step function that is zero for $\varepsilon_i > \mu$ and unity for $\varepsilon_i < \mu$. We define the Fermi energy as $\varepsilon_F = \mu(T = 0)$. The total energy is then:

$$\begin{aligned} U &= \sum_i \varepsilon_i \bar{n}_i \\ &\approx \int g(\varepsilon) \frac{\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1}\,\mathrm{d}\varepsilon \\ &= \int_0^{\varepsilon_F} g(\varepsilon)\varepsilon\,\mathrm{d}\varepsilon \end{aligned} \tag{14}$$

The mean energy per particle is then:

$$\begin{aligned} u &= \frac{U}{N} \\ &= \frac{\int_0^{\varepsilon_F} g(\varepsilon)\varepsilon\,\mathrm{d}\varepsilon}{\int_0^{\varepsilon_F} g(\varepsilon)\,\mathrm{d}\varepsilon} \\ &= \frac{3}{5}\varepsilon_F \end{aligned} \tag{15}$$

since $g(\varepsilon) \propto \varepsilon^{1/2}$.

Thus the total energy is just $U = 3N\varepsilon_F/5$.

(ii) The partition function of a Fermi gas is:

$$\ln \mathcal{Z} = \sum_i \ln\left[1 \pm e^{-\beta(\varepsilon_i - \mu)}\right] \tag{16}$$

We could compute the grand potential Φ from the partition function $\mathcal{Z}$:

$$
\begin{aligned}
\Phi &= -k_B T \ln \mathcal{Z} \\
&= -\frac{1}{\beta} \sum_i \ln \left[1 + e^{-\beta(\varepsilon_i - \mu)}\right] \\
&= -\frac{1}{\beta} \int_0^\infty g(\varepsilon) \ln \left[1 + e^{-\beta(\varepsilon - \mu)}\right] \mathrm{d}\varepsilon \\
&= -\frac{2(2s+1)}{\sqrt{\pi}} \frac{V}{\lambda_{\mathrm{th}}^3} \sqrt{\beta} \int_0^\infty \sqrt{\varepsilon} \ln \left[1 + e^{-\beta(\varepsilon - \mu)}\right] \mathrm{d}\varepsilon
\end{aligned}
\tag{17}
$$

The integral can be solved by parts:

$$
\begin{aligned}
\int_0^\infty \sqrt{x} \ln \left[1 + e^{-x+\mu}\right] \mathrm{d}x &= \left[\frac{2}{3} x^{3/2} \ln \left[1 + e^{-x+\mu}\right]\right]_0^\infty - \int_0^\infty \frac{2}{3} x^{3/2} \frac{-e^{-x+\mu}}{1 + e^{-x+\mu}} \mathrm{d}x \\
&= \frac{2}{3} \int_0^\infty x^{3/2} \frac{e^{-x+\mu}}{1 + e^{-x+\mu}} \mathrm{d}x \\
&= \frac{2}{3} \int_0^\infty x^{3/2} \frac{1}{e^{x-\mu} + 1} \mathrm{d}x
\end{aligned}
\tag{18}
$$

so that we finally obtain:

$$
\Phi = -\frac{2}{3} \frac{2(2s+1)}{\sqrt{\pi}} \frac{V}{\lambda_{\mathrm{th}}^3} \frac{1}{\beta} \int_0^\infty \frac{x^{3/2}}{e^{x-\beta\mu} \pm 1} \mathrm{d}x
\tag{19}
$$

This turns out to be simply $-2U/3$. Since pressure is grand potential per unit volume, we have:

$$
P = -\frac{\Phi}{V} = \frac{2U}{3V} = \frac{2}{3} \frac{N\varepsilon_F}{V}
\tag{20}
$$

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4.4

(i) For non-relativistic gas, the energy of a particle is given by:

$$
E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}
\tag{21}
$$

where for each energy, the degeneracy is $(2s+1)$ due to the spin.

The sum over single particle states can be approximated by an integral:

$$
\begin{aligned}
\sum_i &= (2s+1)\sum_k \\
&\approx (2s+1)\frac{V}{(2\pi)^3}\int \mathrm{d}^3k \\
&= (2s+1)\frac{V}{(2\pi)^3}\int_0^\infty 4\pi k^2\,\mathrm{d}k \\
&= \int_0^\infty g(k)\,\mathrm{d}k
\end{aligned}
\tag{22}
$$

where we define:

$$
g(k) \equiv (2s+1)\frac{Vk^2}{2\pi^2} \tag{23}
$$

For electrons, we have $s = 1/2$ and $g(k)\,\mathrm{d}k = 2 \times Vk^2\,\mathrm{d}k/2\pi^2$.

If we instead consider a two-dimensional gas, the approximation becomes:

$$
\begin{aligned}
\sum_i &= (2s+1)\sum_k \\
&\approx (2s+1)\frac{A}{(2\pi)^2}\int \mathrm{d}^2k \\
&= (2s+1)\frac{A}{(2\pi)^2}\int k\,\mathrm{d}k\mathrm{d}\theta \\
&= \int h(k)\,\mathrm{d}k
\end{aligned}
\tag{24}
$$

where the new density of states is:

$$
h(k) \equiv (2s+1)\frac{Ak}{2\pi} \tag{25}
$$

Consider the change of variable from k to $\epsilon = \hbar^2k^2/2m$:

$$
h(k)\,\mathrm{d}k = \tilde{h}(k)\,\mathrm{d}\epsilon \tag{26}
$$

where:

$$
\tilde{h}(\epsilon) = (2s+1)\frac{Am}{2\pi\hbar^2} \tag{27}
$$

For this two-dimensional gas, we can compute the total particle number:

$$
\begin{aligned}
N &= \int \bar{n}\tilde{h}(\epsilon)\,\mathrm{d}\epsilon \\
&= (2s+1)\frac{Am}{2\pi\hbar^2}\int_0^{\epsilon_F}\frac{1}{e^{\beta(\epsilon-\mu)}+1}\,\mathrm{d}\epsilon \\
&= (2s+1)\frac{Am}{2\pi\hbar^2}\epsilon_F
\end{aligned}
\tag{28}
$$

which gives the Fermi energy for a two-dimensional electron gas with spin $s = 1/2$:

$$
\epsilon_F = \frac{\pi\hbar^2 N}{mA}
\tag{29}
$$

(ii) With $n = 4\times10^{17}\,\mathrm{m}^{-2}$ and $m = 0.15\times9.11\times10^{-31}\,\mathrm{kg}$, we have the Fermi energy:

$$
\epsilon_F = 1.01\times10^{-19}\,\mathrm{J}
\tag{30}
$$

(iii) For an one-dimensional gas, the approximation becomes:

$$
\begin{aligned}
\sum_i &= (2s+1)\sum_k \\
&\approx (2s+1)\frac{L}{2\pi}\int \mathrm{d}k \\
&= \int u(k)\,\mathrm{d}k
\end{aligned}
\tag{31}
$$

where the density of state is $u(k) = (2s+1)L/2\pi$.

Changing variable to $\epsilon = \hbar^2k^2/2m$, we have:

$$
\tilde{u}(\epsilon) = (2s+1)\frac{L\sqrt{2m}}{4\pi\hbar}\epsilon^{-1/2}
\tag{32}
$$

so that the total particle number is:

$$
\begin{aligned}
N &= \int \bar{n}\tilde{u}(\epsilon)\,\mathrm{d}\epsilon \\
&= (2s+1)\frac{L\sqrt{2m}}{4\pi\hbar}\int_0^{\epsilon_F}\frac{1}{e^{\beta(\epsilon-\mu)}+1}\epsilon^{-1/2}\,\mathrm{d}\epsilon \\
&= (2s+1)\frac{L\sqrt{2m}}{2\pi\hbar}\epsilon_F^{1/2}
\end{aligned}
\tag{33}
$$

which gives the Fermi energy for a one-dimensional electron gas:

$$\epsilon_F = \frac{\pi^2\hbar^2}{2m}\left(\frac{N}{L}\right)^2 \tag{34}$$

(iv) For the given long-chain molecule, we have $n = 0.5/1\times10^{-10}\,\mathrm{m} = 5\times10^9\,\mathrm{m}^{-1}$. Thus the Fermi energy is:

$$\epsilon_F = 9.95\times10^{-18}\,\mathrm{J} \tag{35}$$

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4.5

(i) If most of the particles are ultra-relativistic, we change the energy of a particle to:

$$\epsilon = pc = \hbar kc \tag{36}$$

The density of states in k-space is still $g(k) = (2s+1)Vk^2/2\pi^2$, but the switch to ϵ-space is altered:

$$\tilde{g}(\epsilon) = (2s+1)\frac{V}{2\pi^2(\hbar c)^3}\epsilon^2 \tag{37}$$

The total number of particles is:

$$\begin{aligned} N &= \int \bar{n}\tilde{g}(\epsilon)\,\mathrm{d}\epsilon \\ &= (2s+1)\frac{V}{2\pi^2(\hbar c)^3}\int_0^{\epsilon_F}\frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}+1}\,\mathrm{d}\epsilon \\ &= (2s+1)\frac{V}{2\pi^2(\hbar c)^3}\frac{1}{3}\epsilon_F^3 \end{aligned} \tag{38}$$

so that the Fermi energy for an ultra-relativistic electron gas is:

$$\epsilon_F = \hbar c(3n\pi^2)^{1/3} = hc\left(\frac{3n}{8\pi}\right)^{1/3} \tag{39}$$

(ii) The total energy is:

$$
\begin{aligned}
U &= \int \epsilon \bar{n} \tilde{g}(\epsilon)\,\mathrm{d}\epsilon \\
&= (2s+1)\frac{V}{2\pi^2(\hbar c)^3}\int_0^{\epsilon_F} \frac{\epsilon^3}{e^{\beta(\epsilon-\mu)}+1}\,\mathrm{d}\epsilon \\
&= (2s+1)\frac{V}{2\pi^2(\hbar c)^3}\frac{1}{4}\epsilon_F^4
\end{aligned} \tag{40}
$$

which means that the energy per particle is:

$$
u = \frac{U}{N} = \frac{3}{4}\epsilon_F \tag{41}
$$

and the energy density is just $3n\epsilon_F/4$.

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4.6

(i) For a sphere of mass M and radius R, the gravitational field inside the sphere is linear due to Gauss' law:

$$
g = -\frac{GM}{R^3}r \tag{42}
$$

whereas outside the sphere, the field is:

$$
g = -\frac{GM}{r^2} \tag{43}
$$

The potential energy of the star is found by integrating the energy density $u = -g^2/8\pi G$ over the entire space:

$$
\begin{aligned}
U_{\text{grav}} &= \int_{\text{all space}} u\,\mathrm{d}V \\
&= -\frac{1}{8\pi G}\left[\int_0^R \left(\frac{GM}{R^3}\right)^2 r^2 4\pi r^2\,\mathrm{d}r + \int_R^\infty \left(\frac{GM}{r^2}\right)^2 4\pi r^2\,\mathrm{d}r\right] \\
&= -\frac{3}{5}\frac{GM^2}{R}
\end{aligned} \tag{44}
$$

(ii) Suppose that in a white dwarf star, there are N electrons, N protons and N neutrons. The total mass of the star is $M = 2Nm_p$ as we neglect the mass of the electrons. The total Fermi energy of the electrons is:

$$\begin{aligned}\frac{3}{5}N\varepsilon_F &= \frac{3}{5}N\left[\frac{\hbar^2}{2m_e}\left(3\pi^2 n\right)^{2/3}\right] \\ &= \frac{3}{5}N\left[\frac{\hbar^2}{2m_e}\left(\frac{9\pi^2 N}{4\pi R^3}\right)^{2/3}\right] \\ &= 0.0088\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2}\end{aligned} \tag{45}$$

(iii) The total energy of the star is an inverse square function of the radius R minus an inverse function:

$$U_{\text{tot}} = 0.0088\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2} - \frac{3}{5}\frac{GM^2}{R} \tag{46}$$

To minimise the energy, we differentiate with respect to R and set the result to zero:

$$R_{\min} = 0.0088 \times \frac{10h^2}{3m_e m_p^{5/3} G} M^{-1/3} \tag{47}$$

(iv) With $M = 2 \times 10^{30}$ kg, we have $R = 2.34 \times 10^3$ km which is of the same order as the radius of Earth 6.37×10^3 km.

(v) At $R_{\min}$, the Fermi energy is $\varepsilon_F = 1 \times 10^{44}$ J which is too large for the non-relativistic approximation to be valid. Rather, the ultra-relativistic approximation would be more appropriate.

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4.7

(i) In the ultra-relativistic case, the Fermi energy is:

$$\begin{aligned}\varepsilon_F &= hc\left(\frac{3n}{8\pi}\right)^{1/3} \\ &= hc\left(\frac{9N}{32\pi^2 R^3}\right)^{1/3}\end{aligned} \tag{48}$$

which scales as R^{-1}.

(ii) If the Fermi energy is the same order as the rest energy of an electron, we need:

$$\begin{aligned} \varepsilon_F &= m_e c^2 \\ hc\left(\frac{9M}{64\pi^2 m_p R^3}\right)^{1/3} &= m_e c^2 \\ M &= \frac{64\pi^2 m_p R^3}{9}\left(\frac{m_e c}{h}\right)^3 \end{aligned} \tag{49}$$

Taking $R \approx 2000\,\text{km}$, we have $M \approx 10^{30}\,\text{kg}$, which is of the same order as the mass of the Sun.

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4.8

(i) Consider a neutron star with mass M, radius R and $N = M/m_n$ neutrons. The gravitational potential energy of the star is:

$$U_{\text{grav}} = -\frac{3}{5}\frac{GM^2}{R} \tag{50}$$

The total Fermi energy of the neutrons, assuming non-relativistic behaviour, is:

$$\begin{aligned} \frac{3}{5}N\varepsilon_F &= \frac{3}{5}N\left[\frac{\hbar^2}{2m_n}\left(3\pi^2 n\right)^{2/3}\right] \\ &= \frac{3}{5}N\left[\frac{\hbar^2}{2m_n}\left(\frac{9\pi^2 N}{4\pi R^3}\right)^{2/3}\right] \\ &= 0.0088(2)^{5/3}\frac{h^2 M^{5/3}}{m_n^{8/3}R^2} \end{aligned} \tag{51}$$

where the only difference from the white dwarf case is the $2^{5/3}$ factor.

We see that the results from the white dwarf star carries over to the neutron star with scaling $M \to 2M$. Thus, the radius-mass relation for a neutron star is:

$$R_{\min} = 0.0088 \times \frac{10h^2}{3m_n^{8/3}G}M^{-1/3} \times (2)^{-1/3} \tag{52}$$

(ii) With $M = 2 \times 10^{30}\,\text{kg}$, we have $R = 1034\,\text{m}$ which is very small for a celestial body.

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4.9

Fermi energy is the chemical potential of a Fermi gas at zero temperature, that is, it is the energy cost of adding one more Fermion to the system. Consider a three-dimensional harmonic oscillator with energy levels:

$$\varepsilon_n = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right) \tag{53}$$

where each energy level has total degeneracy $(2s+1)(n+1)(n+2)/2$ due to the spin and the three-dimensional nature of the oscillator.

For a Fermi gas of N non-interacting particles, the particles fill up the energy levels from the lowest to the highest, with each level allowing at most $(n+1)(n+2)$ particles. Therefore, for $1 \leq N \leq 50$, the Fermi energy is a piecewise function:

$$\frac{\varepsilon_F}{\hbar\omega} = \begin{cases} 3/2 & \text{if } N = 1 \\ 4/2 & \text{if } 2 \leq N \leq 5 \\ 7/2 & \text{if } 6 \leq N \leq 11 \\ 9/2 & \text{if } 12 \leq N \leq 19 \\ 11/2 & \text{if } 20 \leq N \leq 29 \\ 13/2 & \text{if } 30 \leq N \leq 41 \\ 15/2 & \text{if } 42 \leq N \leq 50 \end{cases} \tag{54}$$

Consider the sum of states denoted by $n \equiv (n_x + n_y + n_z)$:

$$\begin{aligned} \sum_{\text{states}} &= \sum_n (2s+1)\frac{(n+1)(n+2)}{2} \\ &= \sum_n (2s+1)\frac{(n+1)(n+2)}{2}\Delta n \\ &\approx \int (n+1)(n+2)\,\mathrm{d}n \end{aligned} \tag{55}$$

such that the density of states in the n-space is $g(n) = (n+1)(n+2)$. We may switch to the energy space by changing variable to $\varepsilon = \hbar\omega(n + 3/2)$:

$$\begin{aligned} \tilde{g}(\varepsilon) &= \frac{1}{\hbar\omega}\left(\frac{\varepsilon}{\hbar\omega} - \frac{1}{2}\right)\left(\frac{\varepsilon}{\hbar\omega} + \frac{1}{2}\right) \\ &\approx \frac{\varepsilon^2}{(\hbar\omega)^3} \end{aligned} \tag{56}$$

in the limit $\varepsilon \gg \hbar\omega$.

The total number of particles is:

$$\begin{aligned} N &= \int \bar{n}\tilde{g}(\varepsilon)\,\mathrm{d}\varepsilon \\ &= \int_0^{\varepsilon_F} \frac{\varepsilon^2}{(\hbar\omega)^3}\,\mathrm{d}\varepsilon \\ &= \frac{\varepsilon_F^3}{3(\hbar\omega)^3} \end{aligned} \tag{57}$$

which gives the Fermi energy:

$$\varepsilon_F = (3N)^{1/3}\hbar\omega = 9.51 \times 10^{-27}\,\mathrm{J} \tag{58}$$