Symmetry and Relativity

Problem Set 4

Dynamics and Electromagnetism in Special Relativity

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1 Sticky collision

Initially, we have two 4-momenta $P_1^\mu = (4mc, p_0)$ and $P_2^\mu = (2mc, 0)$, where $p_0 = \sqrt{3}mc$. Suppose that after the collision, the combined system has a 4-momentum $P^\mu = (E_f/c, p_f) = (6mc, p_0)$, since we demand conservation of 4-momentum. We have:

$$\frac{E_f^2}{c^2} - p_f^2 = m_f^2 c^2 \tag{1}$$

On the other hand, p_0 satisfies $16m^2c^2 - p_0^2 = m^2c^2$, so $p_0 = \sqrt{15}mc$. We can then solve for m_f:

$$m_f = \sqrt{21}m \tag{2}$$

To find the velocity of the combined system, consider the relation $E_f = \gamma m_f c^2$, leading to $\gamma = 6/\sqrt{21}$. The velocity is then:

$$v = \beta c = \sqrt{\frac{5}{12}}c \tag{3}$$

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2 Pair production

In the centre of mass frame, the two photons have the 4-momenta $P_1^\mu = (E'/c, E'/c)$ and $P_2^\mu = (E'/c, -E'/c)$. After production of the electron-positron pair, the 4-momenta are $P_\pm^\mu = (mc, 0)$. Thus, we require $2E'/c = 2mc$ so the threshold energy is $E' = mc^2$.

On the other hand, the velocity of the centre of mass frame is:

$$v_{\text{cm}} = \frac{(E_0/c - E/c)c^2}{E_0/c + E/c} = \frac{E_0 - E}{E_0 + E}c \tag{4}$$

Thus, we find the relation between E' and E_0:

$$E' = \gamma\frac{E_0}{c} - \gamma\beta\frac{E_0}{c} = \frac{E_0}{c}\sqrt{\frac{1-\beta}{1+\beta}} = \sqrt{E_0 E} \tag{5}$$

where $\beta = (E_0 - E)/(E_0 + E)$.

Since $E' = mc^2$, we have:

$$E = \frac{mc^4}{E_0} \tag{6}$$

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3 Two-body decay

In the centre of mass frame, the initial 4-momenta $(Mc, 0)$ splits into $P_1^\mu = (E_1'/c, p')$ and $P_2^\mu = (E_2'/c, -p')$. We have the equations:

$$\begin{aligned} E_1' + E_2' &= Mc^2 \\ E_1'^2/c^2 - p'^2 &= m_1^2c^2 \\ E_2'^2/c^2 - p'^2 &= m_2^2c^2 \end{aligned} \tag{7}$$

Solving for E_1' and E_2':

$$\begin{aligned} E_1' &= \frac{c^2}{2}\left(M + \frac{m_1^2 - m_2^2}{M}\right) \\ E_2' &= \frac{c^2}{2}\left(M + \frac{m_2^2 - m_1^2}{M}\right) \end{aligned} \tag{8}$$

which leads to an expression for p':

$$p' = \frac{c}{2M}\left[(m_1^2 + m_2^2 - M^2)^2 - 4m_1^2m_2^2\right]^{1/2} \tag{9}$$

To find the Lorentz factor, first note that in the lab frame, $\gamma = E/Mc^2$ so $\beta = \sqrt{1 - M^2c^4/E^2}$. E_1 and E_1' are related via Lorentz transformations. If the products are emitted along the line of motion, we have:

$$E_1 = \gamma(E_1' + vp') \tag{10}$$

If the products are emitted perpendicular to the line of motion, we have:

$$E_1 = \gamma E_1' \tag{11}$$

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4 Motion in an electromagnetic field

(a) Following transformation rules for the electromagnetic field, we have:

$$\begin{aligned} \mathbf{E}' &= \gamma(E\hat{\mathbf{x}} - vB\hat{\mathbf{x}}) \\ \mathbf{B}' &= \gamma(B\hat{\mathbf{y}} - \frac{1}{c^2}vE\hat{\mathbf{y}}) \end{aligned} \tag{12}$$

For $\mathbf{E}' = 0$, we need $v = E/B$, which leads to $\mathbf{B}' = \gamma B\hat{\mathbf{y}}$. We must have $v = E/B < c$ for this to be possible.

For $\mathbf{E}' = 0$, we need $v = E/B$. Then we have $B' = \gamma B(1 - E^2/B^2c^2)$. On the other hand, note that $E^2 - B^2c^2$ is an invariant, so:

$$E^2 - B^2c^2 = -B'^2c^2 \tag{13}$$

where the right hand side has no E' term as it is zero.
This leads to $B' = B/\gamma$.

(b) In frame S', the particle has velocity:

$$\mathbf{u}' = \frac{1}{1 + vu_z/c^2}\left[(u_z - v)\hat{\mathbf{z}} + \sqrt{1-\beta^2}u_x\hat{\mathbf{x}}\right] \tag{14}$$

As in S', the magnetic field is along $\hat{\mathbf{y}}$, the particle undergoes a circular motion in the xz-plane. The radius of the circle is:

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5 Interactions between two charged beams in a magnetic field

(a) In the lab frame, the two particle beams can be represented by the 4-currents $J_1^\mu = J_2^\mu = (c\lambda/A, v\lambda/A)$. The electric field produced by one of the beam at the other is $E = \lambda/(2\pi\epsilon_0 d)$. The balance of forces is:

$$E\lambda = Bv\lambda \tag{15}$$

which leads to:

$$B = \frac{E}{v} = \frac{\lambda}{2\pi\epsilon_0 vd} \tag{16}$$

The magnetic field on one beam by the other is $\mu_0 v\lambda/(2\pi d)$. The net force must be compensated by an external magnetic field B so we have:

$$Bv\lambda = \frac{\lambda^2}{2\pi\epsilon_0 d} - \frac{\mu_0\lambda^2v^2}{2\pi d} \tag{17}$$

which can be solved for B to give:

$$B = \left(1 - \frac{v^2}{c^2}\right)\frac{\lambda}{2\pi\epsilon_0 vd} \tag{18}$$

(b) In the rest frame of the beams, we transform the 4-currents to:

$$(J_1')^\mu = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} c\lambda/A \\ v\lambda/A \end{pmatrix} = \gamma\frac{\lambda}{A}\begin{pmatrix} c - \beta v \\ -\beta c + v \end{pmatrix} = \begin{pmatrix} c\rho' \\ j' \end{pmatrix} \tag{19}$$

This gives the linear charge density $\lambda' = A\rho' = \sqrt{1-\beta^2}\lambda$ and the current density $j' = 0$. The force by one beam on the other is:

$$f' = E'\lambda' = \frac{\lambda'^2}{2\pi\epsilon_0 d} = (1-\beta^2)\frac{\lambda^2}{2\pi\epsilon_0 d} \tag{20}$$

Now transform this force back to the lab frame:

$$f = \gamma f' = \sqrt{1-\beta^2}\frac{\lambda^2}{2\pi\epsilon_0 d} \tag{21}$$

This must be equal to the magnetic force $Bv\lambda$ in the lab frame, so we have:

$$B = \frac{1}{\gamma}\frac{\lambda}{2\pi\epsilon_0 v d} \tag{22}$$

We transform the pure electric field in the rest frame to the lab frame to obtain the extra magnetic field B:

$$E' = \frac{\lambda'}{2\pi\epsilon_0 d} = \gamma v B \tag{23}$$

which gives:

$$B = \frac{\lambda}{2\pi\epsilon_0 v d}\left(1 - \frac{v^2}{c^2}\right) \tag{24}$$

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6 Covariant generalisation of Ohm's law

(a) In the rest frame of the conductor, the product $U_\nu J^\nu$ evaluates to $-\rho_0 c^2$. Thus:

$$\begin{aligned}(J_0)^\mu &= -\frac{1}{c^2}(U_\nu J^\nu)(U_0)^\mu + \sigma_0 (F_0)^{\mu\nu}(U_0)_\nu \\ &= \rho_0 (U_0)^\mu + \sigma_0 (F_0)^{\mu\nu}(U_0)_\nu\end{aligned} \tag{25}$$

Noting that the only non-zero component of U_0 is its time component, we have:

$$\mathbf{j} = (J_0)^i = \sigma_0 (F_0)^{i0}(-c) = \sigma_0 \mathbf{E}_0 \tag{26}$$

(b) In an arbitrary frame, we have:

$$J^\mu = \rho_0 U^\mu + \sigma_0 F^{\mu\nu} U_\nu \tag{27}$$

since the product $U_\nu J^\nu$ is a scalar invariant.

Isolating the spatial components of J^μ, we have:

$$\begin{aligned} J^i &= \rho_0\gamma v_i + \gamma\sigma_0(-F^{i0}c + F^{ij}v_j) \\ &= \gamma\rho_0 v_i + \gamma\sigma_0(E_i + \epsilon_{ijk}v_j B_k) \\ &= \gamma\rho_0\mathbf{v} + \gamma\sigma_0(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \end{aligned} \tag{28}$$

On the other hand, note the temporal component of J^μ:

$$\begin{aligned} J^0 &= \rho_0\gamma c + \sigma_0 F^{0\nu}U_\nu \\ &= \gamma\rho_0 c + \gamma_v\sigma_0\frac{E_i v_i}{c} \\ &= \rho c \end{aligned} \tag{29}$$

which means:

$$\gamma\rho_0 = \rho - \gamma\sigma_0\frac{\mathbf{E}\cdot\mathbf{v}}{c^2} \tag{30}$$

Combining the two results, we have:

$$\mathbf{j} = \rho\mathbf{v} + \gamma\sigma_0\left(\mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{\mathbf{E}\cdot\mathbf{v}}{c^2}\mathbf{v}\right) \tag{31}$$

(c) If $\rho_0 = 0$, from the above results, we have:

$$\rho = \gamma\sigma_0\frac{\mathbf{E}\cdot\mathbf{v}}{c^2} \tag{32}$$

and:

$$\mathbf{j} = \gamma\sigma_0\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) \tag{33}$$

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7 Angular momentum of the electromagnetic field

(a) We have:

$$\begin{aligned} \partial_\alpha M^{\alpha\beta\gamma} &= \partial_\alpha(X^\gamma T^{\alpha\beta}) - \partial_\alpha(X^\beta T^{\alpha\gamma}) \\ &= (\partial_\alpha X^\gamma)T^{\alpha\beta} + X^\gamma(\partial_\alpha T^{\alpha\beta}) - (\partial_\alpha X^\beta)T^{\alpha\gamma} - X^\beta(\partial_\alpha T^{\alpha\gamma}) \\ &= \delta^\gamma_\alpha T^{\alpha\beta} - \delta^\beta_\alpha T^{\alpha\gamma} \\ &= T^{\alpha\beta} - T^{\beta\gamma} \\ &= 0 \end{aligned} \tag{34}$$

where the third equality follows because we are relabelling the indices and the final equality follows from the symmetry of $T^{\alpha\beta}$.

(b)

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\begin{aligned}
\partial_\alpha M^{\alpha ij} &= \partial_\alpha(X^j T^{\alpha i}) - \partial_\alpha(X^i T^{\alpha j}) \\
&= T^{ji} + X^j(0 - \partial_\alpha T^{\alpha 0}) - T^{ij} - X^i(0 - \partial_\alpha T^{\alpha 0}) \\
&= X^i \partial_\alpha T^{\alpha 0} - X^j \partial_\alpha T^{\alpha 0} \\
&= cX^i \nabla \cdot \mathbf{g} - cX^j \nabla \cdot \mathbf{g}
\end{aligned}
\tag{35}
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