Ordinary Differential Equations

Problem Set 6

Masses on Springs and Other Things

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Minimal Set

6.1 Coupled Pendula

(a) Consider the vector $Y(t) = (y_1, y_2)^\intercal$, with the following system of equations:

$$\ddot{Y} = -\begin{pmatrix} \omega_0^2 + k/m & -k/m \\ -k/m & \omega_0^2 + k/m \end{pmatrix} Y \tag{1}$$

Solving for the eigenvalues yields $\lambda_1 = \omega_0^2$, $\Omega_1 = \omega_0$ and $\mathbf{e}_1 = (1,1)^\intercal$; $\lambda_2 = \omega_0^2 + 2k/m$, $\Omega_2 = \sqrt{\omega_0^2 + 2k/m}$ and $\mathbf{e}_2 = (1,-1)^\intercal$.

The solutions are then given by:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A \sin(\Omega_1 t + \phi_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B \sin(\Omega_2 t + \phi_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2}$$

Substituting the initial conditions yields $A = v/(2\Omega_1)$, $A = v/(2\Omega_2)$, $\phi_1 = \phi_2 = 0$. Therefore:

$$\begin{aligned} y_1(t) &= \frac{v}{2}\left(\frac{1}{\Omega_1}\sin\Omega_1 t + \frac{1}{\Omega_2}\sin\Omega_2 t\right) \\ y_2(t) &= \frac{v}{2}\left(\frac{1}{\Omega_1}\sin\Omega_1 t - \frac{1}{\Omega_2}\sin\Omega_2 t\right) \end{aligned} \tag{3}$$

(b) With $k/m = 2\epsilon g/l$, we have $\Omega_2 = \sqrt{1+4\epsilon}\omega_0 \approx (1+2\epsilon)\Omega_1$, so that $\Delta = (\Omega_2 - \Omega_1)/2 = \epsilon\omega_0$

(c) Consider the $\sin\Omega_2 t$ term:

$$\begin{aligned} \sin\Omega_2 t &= \sin(\Omega_1 t + 2\epsilon\Omega_1 t) \\ &= \sin(\Omega_1 t)\cos(2\epsilon\Omega_1 t) + \cos(\Omega_1 t)\sin(2\epsilon\Omega_1 t) \\ &\approx \sin(\Omega_1 t) + 2\epsilon\Omega_1 t\cos(\Omega_1 t) \end{aligned} \tag{4}$$

Thus the minimal amplitude is at first order of ϵ.

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6.2 Hanging Masses

Let the point of contact with the wall be the origin and take downwards as positive. Let y_1 be the coordinate of the upper mass and y_2 be the coordinate of the lower mass. Let the original length of the spring be L. Consider the vector $Y(t) = (y_1, y_2)^\intercal$, with the following system of equations:

$$\ddot{Y} = -\frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Y + \frac{kL}{m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5}$$

We may disregard the constant term as it manifests as a constant offset in the solution. Define $\omega_0 = \sqrt{k/m}$. Solving for the eigenvalues yields $\lambda_1 = (3+\sqrt{5})\omega_0^2/2$ and $\mathbf{e}_1 = ((-1-\sqrt{5})/2, 1)^\intercal$; $\lambda_2 = (3-\sqrt{5})\omega_0^2/2$ and $\mathbf{e}_2 = ((-1+\sqrt{5})/2, 1)^\intercal$. Thus, the normal modes are:

$$q_1 = \frac{-1-\sqrt{5}}{2} y_1 + y_2 \tag{6}$$

corresponding to the eigenfrequency $\Omega_1 = \sqrt{\frac{3+\sqrt{5}}{2}}\omega_0$, and:

$$q_2 = \frac{-1+\sqrt{5}}{2} y_1 + y_2 \tag{7}$$

corresponding to the eigenfrequency $\Omega_2 = \sqrt{\frac{3-\sqrt{5}}{2}}\omega_0$.

The solutions are then given by:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\cos(\Omega_1 t + \phi_1)\begin{pmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{pmatrix} + B\cos(\Omega_2 t + \phi_2)\begin{pmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \tag{8}$$

If $y_2 = 2a$ at rest, by symmetry, for the upper mass to be at rest, $y_1 = a$. This constitutes the initial conditions $Y(0) = (a, 2a)$ and $\dot{Y}(0) = (0, 0)$. Substituting the initial conditions yields $A = \frac{5-3\sqrt{5}}{10}a$, $B = \frac{5+3\sqrt{5}}{10}a$, $\phi_1 = \phi_2 = 0$. Therefore:

$$\begin{aligned} y_1(t) &= \frac{5-\sqrt{5}}{10} a\cos\Omega_1 t + \frac{5+\sqrt{5}}{10} a\cos\Omega_2 t \\ y_2(t) &= a\cos\Omega_1 t + a\cos\Omega_2 t \end{aligned} \tag{9}$$

By conservation of energy, the total energy is given by:

$$E = \frac{1}{2}ky_1(0)^2 + \frac{1}{2}ky_2(0)^2 = ka^2 \tag{10}$$

The ratio is given by:

$$E_{q_1}/E_{q_2} = \frac{\left(\frac{-1-\sqrt{5}}{2}\right)^2 + 1}{\left(\frac{-1+\sqrt{5}}{2}\right)^2 + 1} \tag{11}$$

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6.3 Sliding Masses

(a) Let y_1 be the displacement of the left mass, with rightwards taken as positive, and let y_2 be the displacement of the right mass, with leftwards taken as positive. Consider the vector $Y(t) = (y_1, y_2)^\intercal$, with the following system of equations:

$$\ddot{Y} = -\frac{1}{m}\begin{pmatrix} k_0 + k_1 & k_1 \\ k_1 & k_0 + k_1 \end{pmatrix} Y \tag{12}$$

Define $\omega_0 = \sqrt{k_0/m}$ and $\omega_1 = \sqrt{k_1/m}$. Solving for the eigenvalues yields $\lambda_1 = \omega_0^2$, $\Omega_1 = \omega_0$ and $\mathbf{e}_1 = (1, -1)^\intercal$; $\lambda_2 = \omega_0^2 + 2\omega_1^2$, $\Omega_2 = \sqrt{\omega_0^2 + 2\omega_1^2}$ and $\mathbf{e}_2 = (1, 1)^\intercal$. We also know that $\Omega_0 = \sqrt{(k_0 + k_1)/m} = \sqrt{\omega_0^2 + \omega_1^2}$. Therefore:

$$\Omega_2 = \sqrt{2\Omega_0^2 - \Omega_1^2} \tag{13}$$

and:

$$\frac{k_1}{k_0} = \frac{\sqrt{\Omega_0^2 - \Omega_1^2}}{\Omega_0} \tag{14}$$

(b) The normal modes are $p_1 = y_1 - y_2$ and $p_2 = y_1 + y_2$. p_1 corresponds to the masses moving in same directions with the same displacement (lower effective spring constant), and p_2 corresponds to the masses moving in the opposite direction with the same displacement (higher spring constant).

(c) With friction taken into account, the system of equations becomes:

$$\ddot{Y} = -\frac{1}{m}\begin{pmatrix} k_0 + k_1 & k_1 \\ k_1 & k_0 + k_1 \end{pmatrix} Y - 2\gamma\dot{Y} \tag{15}$$

The general solution has the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (Ae^{r_{1+}t} + Be^{r_{1-}t})\begin{pmatrix} 1 \\ -1 \end{pmatrix} + (Ce^{r_{2+}t} + De^{r_{2-}t})\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{16}$$

where $r_{1,2}$ satisfy the characteristic equation:

$$r_{1,2}^2 + 2\gamma r_{1,2} + \Omega_{1,2}^2 = 0 \tag{17}$$

Depending on the sign of $\gamma^2 - \Omega_{1,2}^2$, the solution may be overdamped, critically damped, or underdamped.

(d) In the limit $\omega_0 \ll \gamma \ll \omega_1$, we have $r_{1\pm} = -\gamma \pm w_1$, where $w_1^2 = \gamma^2 - \Omega_1^2$, and $r_{2\pm} = -\gamma \pm i w_2$, where $w_2^2 = \Omega_2^2 - \gamma^2$. The solution associated with p_1 is overdamped, and the solution associated with p_2 is underdamped. Given the initial conditions $Y(0) = (a, 0)^\intercal$ and $\dot{Y}(0) = (0, 0)^\intercal$, the solutions are:

$$\begin{aligned} y_1(t) &= \frac{a}{4} e^{-\gamma t} \left(\frac{w_1 + \gamma}{w_1} e^{w_1 t} + \frac{w_1 - \gamma}{w_1} e^{-w_1 t} \right) + \frac{a}{2} e^{-\gamma t} \cos w_2 t \\ y_2(t) &= -\frac{a}{4} e^{-\gamma t} \left(\frac{w_1 + \gamma}{w_1} e^{w_1 t} + \frac{w_1 - \gamma}{w_1} e^{-w_1 t} \right) + \frac{a}{2} e^{-\gamma t} \cos w_2 t \end{aligned} \tag{18}$$

Essentially, this tells us that p_1 component (moving in same directions) decays much faster than the p_2 component (moving in opposite direction). This is because in the p_1 mode, the central spring is not stretched, and the damping effect is much stronger than the spring effect.

(e) In the limit $\gamma \ll \omega_0$, we have $r_{1\pm} = -\gamma \pm i w_1$, where $w_1^2 = \Omega_1^2 - \gamma^2$, and $r_{2\pm} = -\gamma \pm i w_2$, where $w_2^2 = \Omega_2^2 - \gamma^2$. Both modes are underdamped. If the right wall oscillates with $y_0 = a \sin \Omega_2 t$, the equation of motion becomes:

$$\ddot{Y} = -\frac{1}{m} \begin{pmatrix} k_0 + k_1 & k_1 \\ k_1 & k_0 + k_1 \end{pmatrix} Y - 2\gamma \dot{Y} - \frac{k_0}{m} y_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{19}$$

If we transform to the normal coordinates, the following equation results:

$$\ddot{Y}' = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Y' - 2\gamma \dot{Y}' - \frac{k_0}{m} y_0 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{20}$$

The long term behaviour would be the particular solution to the set of equations:

$$\begin{aligned} \ddot{p}_1 + 2\gamma \dot{p}_1 + \Omega_1^2 p_1 &= \omega_0^2 a \sin \Omega_2 t \\ \ddot{p}_2 + 2\gamma \dot{p}_2 + \Omega_2^2 p_2 &= -\omega_0^2 a \sin \Omega_2 t \end{aligned} \tag{21}$$

The particular solutions are of the form:

$$\begin{aligned} p_{1p}(t) &= C_1 \sin\left(\Omega_2 t + \phi_1\right) \\ p_{2p}(t) &= C_2 \sin\left(\Omega_2 t + \phi_2\right) \end{aligned} \tag{22}$$

The constants can be solved by substitution, although the algebra is too nasty to carry out. $y_1(t)$ and $y_2(t)$ can be obtained by noting $y_1 = (p_1 + p_2)/2$ and $y_2 = (p_1 - p_2)/2$. It is worth noting that C_2 would likely be much higher as the driving frequency is close to the eigenfrequency of q_2.

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6.4 Sliding Masses Unbound

Let the the coordinate of the masses be y_1 and y_2 respectively (measured in the same direction) and let the original length of the spring be l. The system of equations is:

$$\ddot{Y} = -\begin{pmatrix} k/m_1 & -k/m_1 \\ -k/m_2 & k/m_2 \end{pmatrix} Y + \begin{pmatrix} -lk/m_1 \\ lk/m_2 \end{pmatrix} \tag{23}$$

Solving for the eigenvalues and eigenvectors, we have $\lambda_1 = 0$ with eigenvector $\mathbf{e}_1 = (1, 1)^\intercal$, and $\lambda_2 = k(1/m_1 + 1/m_2)$ with eigenvector $\mathbf{e}_2 = (-m_2/m_1, 1)^\intercal$.

The normal modes are $p_1 = y_1 + y_2$ and $p_2 = -(m_2/m_1)y_1 + y_2$. p_1 is the normal mode where the masses have the same displacement so that the spring is not stretched; p_2 is the normal mode where the masses have opposite displacements such that the centre of mass is stationary.

Disregarding the inhomogeneous term, the general solution is:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (A\cos\omega t + B\sin\omega t)\begin{pmatrix} -m_2/m_1 \\ 1 \end{pmatrix} + (Ct + D)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{24}$$

where $\omega = \sqrt{k(1/m_1 + 1/m_2)}$.

Consider the initial conditions $Y(0) = (0, l)$ and $\dot{Y}(0) = (v, 0)$. The full solutions are:

$$\begin{aligned} y_1(t) &= \frac{m_2}{m_1 + m_2}\left(\frac{v}{\omega}\sin\omega t - l\cos\omega t\right) + \frac{m_1}{m_1 + m_2}vt + \frac{m_2}{m_1 + m_2}l \\ y_2(t) &= \frac{m_1}{m_1 + m_2}\left(l\cos\omega t - \frac{v}{\omega}\sin\omega t\right) + \frac{m_1}{m_1 + m_2}vt + \frac{m_2}{m_1 + m_2}l \end{aligned} \tag{25}$$

The total energy is $mv^2/2$.

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