**Calculus**

# Probelm Sheet D

Partial Differentiation

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# Partial Differentiation

## D2 Getting used to partial differentiation

### (a)

**i**

$$\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} \tag{1}$$

**ii**

$$\frac{\partial}{\partial x}\tan^{-1}(y/x) = -\frac{y}{x^2+y^2} \tag{2}$$

**iii**

$$\begin{aligned} \ln f &= x\ln y \\ \frac{1}{f}\frac{\partial f}{\partial x} &= \ln y \\ \frac{\partial f}{\partial x} &= y^x \ln y \end{aligned} \tag{3}$$

### (b)

**i**

$$\begin{aligned} \frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial y}\left[2x\sin(x+y) + (x^2+y^2)\cos(x+y)\right] \\ &= 2x\cos(x+y) + 2y\cos(x+y) - (x^2+y^2)\sin(x+y) \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial x}\left[2y\sin(x+y) + (x^2+y^2)\cos(x+y)\right] \\ &= 2x\cos(x+y) + 2y\cos(x+y) - (x^2+y^2)\sin(x+y) \end{aligned} \tag{5}$$

**ii**

$$\frac{\partial^2 f}{\partial x\partial y} = mnx^{m-1}y^{n-1} = \frac{\partial^2 f}{\partial y\partial x} \tag{6}$$

**(c)** We have:

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= 0 \\
\frac{\partial f}{\partial x} &= G(y) \\
f(x,y) &= F(x) + G(y) \\
\frac{\partial f}{\partial y} &= F(x)
\end{aligned} \tag{7}$$

**(d)**

$$V_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial (x-ct)}\frac{\partial (x-ct)}{\partial x}\right] = \frac{\partial^2 f}{\partial (x-ct)^2} \tag{8}$$

$$V_{tt} = \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial (x-ct)}\frac{\partial (x-ct)}{\partial t}\right] = (-c)^2\frac{\partial^2 f}{\partial (x-ct)^2} \tag{9}$$

Thus, $V_{xx} - V_{tt}/c^2 = 0$.

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## D3 Error estimates

We have $g(l,T) = 4\pi^2 l T^{-2}$. Thus:

$$\begin{aligned}
(\Delta g)^2 &= (\frac{\partial g}{\partial l}\Delta l)^2 + (\frac{\partial g}{\partial T}\Delta T)^2 \\
&= (4\pi^2 T^{-2}\Delta l)^2 + (-8\pi^2 l T^{-3}\Delta T)^2 \\
\left(\frac{\Delta g}{g}\right)^2 &= \left(\frac{\Delta l}{l}\right)^2 + \left(2\frac{\Delta T}{T}\right)^2 \\
&= (5\%)^2 + (4\%)^2
\end{aligned} \tag{10}$$

Therefore, $\Delta g/g = \sqrt{41}\%$.

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## D4 Total derivatives

**(a)**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos^n at \cos^n at\right) = na\left(\cos^{n+1} at \sin^{n-1} at - \cos^{n-1} at \sin^{n+1} at\right) \tag{11}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} &= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= ny^n x^{n-1}(-a\sin at) + ny^{n-1}x^n(a\cos at) \\ &= na\left(\cos^{n+1} at \sin^{n-1} at - \cos^{n-1} at \sin^{n+1} at\right)\end{aligned} \tag{12}$$

**(b)**

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\ln x + \frac{1}{\ln x}\right) = 2x\ln x + x - \frac{1}{x(\ln x)^2} \tag{13}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} &= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= 2yx + (x^2 - \frac{1}{y^2})\frac{1}{x} \\ &= 2x\ln x + x - \frac{1}{x(\ln x)^2}\end{aligned} \tag{14}$$

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## D5 Chain rule

$\omega = e^{-r^2}$, so $\partial\omega/\partial r = -2re^{-r^2}$ and $\partial\omega/\partial\theta = 0$. Alternatively:

$$\begin{aligned}\frac{\partial\omega}{\partial r} &= \frac{\partial\omega}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\omega}{\partial y}\frac{\partial y}{\partial r} \\ &= -e^{x^2+y^2}(2x\cos\theta + 2y\sin\theta) \\ &= -2re^{-r^2}\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\partial\omega}{\partial\theta} &= \frac{\partial\omega}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\omega}{\partial y}\frac{\partial y}{\partial\theta} \\ &= -2e^{x^2+y^2}(-xr\sin\theta + yr\cos\theta) \\ &= 0\end{aligned} \tag{16}$$

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## D6 Exact differetials

Note that:

$$P = \frac{RT}{V}, V = \frac{RT}{P}, T = \frac{VP}{R} \tag{17}$$

**(a)**

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = \left(-\frac{RT}{V^2}\right)\left(\frac{R}{P}\right)\left(\frac{V}{R}\right) = -1 \tag{18}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{RT}{V^2} = -\frac{P}{V} \tag{19}$$

$$\left(\frac{\partial V}{\partial P}\right)_T = -\frac{RT}{P^2} = -\frac{V}{P} \tag{20}$$

**(b)** We have:

$$\frac{\partial f}{\partial P}\mathrm{d}P + \frac{\partial f}{\partial V}\mathrm{d}V + \frac{\partial f}{\partial T}\mathrm{d}T = 0 \tag{21}$$

Consider $\partial P/\partial V$. This is the same as setting $\mathrm{d}T = 0$ and take the ordinary derivative using the above equation. Applying to all three case:

$$\begin{aligned} \left(\frac{\partial P}{\partial V}\right)_T &= \frac{\mathrm{d}P}{\mathrm{d}V} = -\frac{\partial f}{\partial V}\Big/\frac{\partial f}{\partial P} \\ \left(\frac{\partial V}{\partial T}\right)_T &= \frac{\mathrm{d}V}{\mathrm{d}T} = -\frac{\partial f}{\partial T}\Big/\frac{\partial f}{\partial V} \\ \left(\frac{\partial T}{\partial P}\right)_T &= \frac{\mathrm{d}T}{\mathrm{d}P} = -\frac{\partial f}{\partial P}\Big/\frac{\partial f}{\partial T} \end{aligned} \tag{22}$$

Multiplying the above three equations together, we recover the previous results.

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## D7 Change of variable

**(a)**

$$\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)_y = 2x\left(\frac{\partial z}{\partial u}\right)_v + 2y\left(\frac{\partial z}{\partial v}\right)_u \tag{23}$$

**(b)**

$$\left(\frac{\partial z}{\partial u}\right)_v = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)_v = \frac{1}{2x}\left(\frac{\partial z}{\partial x}\right)_y + \frac{1}{2y}\left(\frac{\partial z}{\partial y}\right)_x \tag{24}$$

$$\left(\frac{\partial z}{\partial u}\right)_v = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)_v \tag{25}$$

On the other hand, we have:

$$\begin{aligned} u = x^2 + y^2 &= x^2 + \frac{v^2}{4x^2} \\ \left(\frac{\partial u}{\partial x}\right)_v &= 2x - \frac{v^2}{2x^3} = \frac{2(x^2-y^2)}{x} \\ \left(\frac{\partial x}{\partial u}\right)_v &= \frac{x}{2(x^2-y^2)} \end{aligned} \tag{26}$$

By symmetry:

$$\left(\frac{\partial y}{\partial u}\right)_v = -\frac{y}{2(x^2-y^2)} \tag{27}$$

Thus:

$$\left(\frac{\partial z}{\partial u}\right)_v = \frac{1}{2(x^2-y^2)}\left[x\left(\frac{\partial z}{\partial x}\right)_y - y\left(\frac{\partial z}{\partial y}\right)_x\right] \tag{28}$$

**(c)**

$$\left(\frac{\partial z}{\partial v}\right)_v = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)_v = \frac{1}{2y}\left(\frac{\partial z}{\partial x}\right)_y + \frac{1}{2x}\left(\frac{\partial z}{\partial y}\right)_x \tag{29}$$

Thus:

$$\left(\frac{\partial z}{\partial u}\right)_v - \left(\frac{\partial z}{\partial v}\right)_v = \left(\frac{1}{2x} - \frac{1}{2y}\right)\left[\left(\frac{\partial z}{\partial x}\right)_y - \left(\frac{\partial z}{\partial y}\right)_x\right] \tag{30}$$

$$\left(\frac{\partial z}{\partial v}\right)_u = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)_u \tag{31}$$

On the other hand, we have:

$$\begin{aligned} v &= 2xy = 2x\sqrt{u-x^2} \\ \left(\frac{\partial v}{\partial x}\right)_u &= 2\left(\sqrt{u-x^2} - \frac{x^2}{\sqrt{u-x^2}}\right) \\ \left(\frac{\partial x}{\partial v}\right)_u &= \frac{y}{2(y^2-x^2)} \end{aligned} \tag{32}$$

By symmetry:

$$\left(\frac{\partial y}{\partial v}\right)_u = -\frac{x}{2(y^2-x^2)} \tag{33}$$

Thus:

$$\left(\frac{\partial z}{\partial v}\right)_u = \frac{1}{2(y^2-x^2)}\left[y\left(\frac{\partial z}{\partial x}\right)_y - x\left(\frac{\partial z}{\partial y}\right)_x\right] \tag{34}$$

and:

$$\left(\frac{\partial z}{\partial u}\right)_v - \left(\frac{\partial z}{\partial v}\right)_u = \frac{1}{2(x-y)}\left[\left(\frac{\partial z}{\partial x}\right)_y - \left(\frac{\partial z}{\partial y}\right)_x\right] \tag{35}$$

**(d)** We have $z = u + v = (x+y)^2$, so that:

$$\left(\frac{\partial z}{\partial u}\right)_v - \left(\frac{\partial z}{\partial v}\right)_v = \left(\frac{1}{2x} - \frac{1}{2y}\right)\left[\left(\frac{\partial z}{\partial x}\right)_y - \left(\frac{\partial z}{\partial y}\right)_x\right] = 0 \tag{36}$$

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## D8 Talor series in 2 variables

$$\begin{aligned} & f(x,y) \\ =& f(2,3) + \left[\frac{\partial f}{\partial x}(x-2) + \frac{\partial f}{\partial y}(y-3)\right] + \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(x-2)^2 + \frac{\partial^2 f}{\partial y^2}(y-3)^2 + \frac{\partial^2 f}{\partial x \partial y}(x-2)(y-3)\right] + \dots \\ =& e^6 + 3e^6(x-2) + 2e^6(y-3) + \frac{1}{2}\left[9e^6(x-2)^2 + 6e^6(y-3)^2 + 4e^6(x-2)(y-3)\right] + \dots \end{aligned} \tag{37}$$

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## D9 Stationary Points

### (i)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{38}$$

This gives us $(x, y) = (0, 0)$. At this point:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 2 \tag{39}$$

and

$$+\frac{\partial^2 f}{\partial x \partial y} = 0 \tag{40}$$

Thus this is a minimum.

### (ii)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{41}$$

This gives us the following equations:

$$\begin{aligned} 3x^2 - 4x + 3y = 0 \\ 3y^2 - 4y + 3x = 0 \end{aligned} \tag{42}$$

For real $x$ and $y$, the solutions are $(x, y) = (0, 0)$ and $(x, y) = (1/3, 1/3)$. For $(0, 0)$:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -4 \tag{43}$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = 3 \tag{44}$$

Thus $(0, 0)$ is a maximum. For $(1/3, 1/3)$:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -2 \tag{45}$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = 3 \tag{46}$$

Thus $f_{xy}^2 > f_{xx} f_{yy}$ and $(1/3, 1/3)$ is a saddle point.

**(iii)**

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \tag{47}$$

We have the following equations:

$$\begin{aligned} \sin y \left[\cos x \sin(x+y) + \sin x \cos(x+y)\right] = 0 \\ \sin x \left[\cos y \sin(x+y) + \sin y \cos(x+y)\right] = 0 \end{aligned} \tag{48}$$

This gives us the condition $\tan x = \tan y = -\tan(x+y)$. We have $x = 0$ or $y = 0$. Also:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} = 2 \sin y \left[\cos x \cos(x+y) - \sin x \sin(x+y)\right] \\ \frac{\partial^2 f}{\partial y^2} = 2 \sin x \left[\cos y \cos(x+y) - \sin y \sin(x+y)\right] \end{aligned} \tag{49}$$

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## D10 Exact differnetials

**(a)** (i) is exact. $f(x, y) = xy + C$ for an arbitrary constant $C$.

(ii) is inexact.

(iii) is exact. $f(x, y) = (x^2 + y^2 + z^2)/2 + C$ for an arbitrary constant $C$.

**(b)** As the integrand is an exact differential, the path integral evaluates to zero around a complete loop.

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