Preliminary Examination 2023

CP 1

Classical Mechanics & Special Relativity

August 25, 2023

Section A

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(a) Let us set the origin of the coordinate system at the geometric centre of the rod, with the positive pointing at the $2m$ end. The new centre of mass is given by:

$$x_{\text{cm}} = \frac{2ml/2 - ml/2}{2m + m} = \frac{l}{6} \tag{1}$$

The momentum of inertia about the geometric centre is:

$$I = 3m\left(\frac{l}{2}\right)^2 = \frac{3}{4}ml^2 \tag{2}$$

so that by the parallel axis theorem, the momentum of inertia about the new centre of mass is:

$$I_{\text{cm}} = I - 3m\left(\frac{l}{6}\right)^2 = \frac{2}{3}ml^2 \tag{3}$$

By conservation of angular momentum, we have:

$$I_{\text{cm}}\omega = mv\left(\frac{l}{2} - \frac{l}{6}\right) \tag{4}$$

so that $\omega = v/2l$.

(b) The linear speed of the centre of mass is given by conservation of momentum:

$$mv = (2m + m)v_{\text{cm}} \tag{5}$$

so that $v_{\text{cm}} = v/3$.

After one half rotation, the linear speed due to ω is against v_{cm} for the $2m$ end so that the total speed is:

$$v_{\text{total}} = v_{\text{cm}} - \omega\left(\frac{l}{2} - \frac{l}{6}\right) = v/6 \tag{6}$$

Section B

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(a) Given the central potential of the form $V(r) = \beta/r^2$, the effective potential is:

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \frac{\beta}{r^2} \tag{7}$$

where $L \equiv mr^2\dot{\theta}$ is the angular momentum of the particle.

(b) With $\beta > -L^2/2m$, the effective potential is always positive. We have the energy conservation equation:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{\beta}{r^2} \tag{8}$$

where E is the conserved energy of the particle.

Differentiating with respect to time, we have:

$$0 = m\dot{r}\ddot{r} - \frac{L^2}{mr^3}\dot{r} - \frac{2\beta}{r^3}\dot{r} \tag{9}$$

or, assuming non-zero $\dot{r}$:

$$\ddot{r} = \frac{L^2}{mr^3} + \frac{2\beta}{r^3} \tag{10}$$

Returning to Equation (2), with the substitution $\dot{r} = \dot{\theta}\mathrm{d}r/\mathrm{d}\theta$, we have:

$$\dot{r} = \dot{\theta}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \pm\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - \frac{\beta}{r^2}\right)} \tag{11}$$

But $\dot{\theta} = L/mr^2$, so that:

$$\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \pm\frac{1}{L}\sqrt{2m\left(E - \frac{L^2}{2mr^2} - \frac{\beta}{r^2}\right)} \tag{12}$$

Now use the substitution $u = 1/r$, so that:

$$-\frac{\mathrm{d}u}{\mathrm{d}\theta} = \pm\frac{1}{L}\sqrt{2mE - (L^2 + 2m\beta)u^2} \tag{13}$$

This is a separable differential equation with the solution:

$$\frac{L}{\sqrt{L^2+2m\beta}}\sin^{-1}(r_0 u)=\pm\theta+\theta_0 \tag{14}$$

where $r_0=\sqrt{L^2/2mE+\beta/E}$.

The plus-minus sign corresponds to clock- and counter-clockwise orbits so let us choose the positive case for simplicity. We may set $\theta_0=0$ without loss of generality as this is just a rotation of the coordinate system. Further simplification gives:

$$\frac{1}{r}=\frac{1}{a}\sqrt{\frac{2mE}{L^2}}\sin(a\theta) \tag{15}$$

where $a^2=1+2m\beta/L^2$ as expected.

The minimum of r is apparently $r_{\text{min}}=\sqrt{2mE/L^2}/a$.

If $\beta=0$, $a=1$ and the equation becomes:

$$\frac{1}{r}=\sqrt{\frac{2mE}{L^2}}\sin\theta \tag{16}$$

which is a straight line as expected for a free particle.

(c) With $\beta=-L^2/2m$, the effective potential is zero and Equation (7) becomes:

$$\frac{\mathrm{d}u}{\mathrm{d}\theta}=\pm\frac{1}{L}\sqrt{2mE} \tag{17}$$

Taking the positive case, we have the solution:

$$r=\frac{1}{\theta}\sqrt{\frac{L^2}{2mE}} \tag{18}$$

Although for r to reach zero, θ must approach infinity, implying an infinite number of revolutions, this is still possible in finite time. To see this, consider Equation (4):

$$\ddot{r}=0 \tag{19}$$

which means $\dot{r}$ is constant and if $\dot{r}<0$ initially, r always reaches zero in finite time.

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(a) The Lagrangian of the system can be written as:

$$\mathcal{L} = \frac{1}{2}m\left(r^2\dot{\theta}^2 + \dot{r}^2\right) + \frac{1}{2}M\dot{r}^2 + Mg(l - r) \tag{20}$$

where the constant term Mgl can be ignored.

The Euler-Lagrange equation gives the equations of motion:

$$\begin{aligned}(m + M)\ddot{r} &= mr\dot{\theta}^2 - Mg \\ L \equiv mr^2\dot{\theta} &= \text{constant}\end{aligned} \tag{21}$$

For circular motion, we impose the conditions $\dot{r} = 0$ and $\ddot{r} = 0$ for some $r = r_0$ and $\dot{\theta} = \omega$. The equation for r gives us:

$$mr_0\omega^2 - Mg = 0 \tag{22}$$

This means that given some initial radius r_0, $\dot{\theta}$ must satisfy the above equation for circular motion to occur. Under this circular motion, the angular momentum is:

$$L = mr_0^2\omega = \sqrt{mMgr_0^3} \tag{23}$$

which is a constant.

Returning to the equation for r, we use the substitution $\dot{\theta} = L/mr_0^2$ to obtain:

$$(m + M)\ddot{r} = \frac{L^2}{mr^3} - Mg \tag{24}$$

We can expand the right-hand side as a Taylor series about $r = r_0$:

$$\frac{L^2}{mr^3} - Mg = \frac{L^2}{m}\left[\frac{1}{r_0^3} - \frac{3(r - r_0)}{r_0^4} + \cdots\right] - Mg \tag{25}$$

We may set the origin at r_0 so that $r' \equiv r - r_0$ and $\ddot{r}' = \ddot{r}$. Collecting the coefficients of r' and ignoring any constant and higher-order terms, we have:

$$\begin{aligned}(m + M)\ddot{r}' &= -\frac{3L^2}{mr_0^4}r' \\ \ddot{r}' &= -\frac{3M}{m}\frac{g}{r_0}r'\end{aligned} \tag{26}$$

which is simple harmonic motion with angular frequency:

$$\Omega = \sqrt{\frac{3M}{m}\frac{g}{r_0}} \tag{27}$$

For $m \gg M$, this tends to zero as the effect of M can be ignored; for $m \ll M$, small oscillation approximation is no longer true; and for $M = 2m$, this becomes $\sqrt{6g/r_0}$.

(b) The coordinates of the mass are $X(t) = l\sin\theta + A\cos\omega t$ and $Y(t) = -l\cos\theta$ so that the Lagrangian is:

$$\frac{\mathcal{L}}{ml^2} = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\kappa^2\sin^2\omega t - \kappa\sin\omega t\cos\theta\dot{\theta} + \frac{g}{l}\cos\theta \tag{28}$$

where $\kappa \equiv \omega A/l$.

The equation of motion is:

$$\ddot{\theta} = \kappa\omega\cos\omega t\cos\theta - \frac{g}{l}\sin\theta \tag{29}$$

For small oscillations, we may approximate $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ so that:

$$\ddot{\theta} = \kappa\omega\cos\omega t - \frac{g}{l}\theta \tag{30}$$

This is a forced harmonic oscillator with the complementary solution:

$$\theta_c = C\cos\left(\sqrt{\frac{g}{l}}t + \phi\right) \tag{31}$$

and the particular solution:

$$\theta_p = \frac{\kappa\omega}{g/l - \omega^2}\cos\omega t \tag{32}$$

Assuming that $\omega \neq \sqrt{g/l}$, the general solution is then:

$$\theta(t) = C\cos\left(\omega_0 t + \phi\right) + \frac{A}{l}\frac{\omega^2}{\omega_0^2 - \omega^2}\cos\omega t \tag{33}$$

where $\omega_0 \equiv \sqrt{g/l}$ is the natural frequency and C and ϕ are constants determined by the initial conditions.

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(a) Consider the moment when the balloon has a mass $m(t)$ and velocity $v(t)$. In a short time δt, we can write the conservation of momentum as:

$$mv + (F - mg)\delta t = (m + \delta m)(v + \delta v) + (-\delta m)v \tag{34}$$

where $F = (M + m_0)g$ is the constant buoyancy force.

Suppose $\delta m = -\alpha \delta t$ for some positive constant α. Then, ignoring second order terms:

$$\frac{F - mg}{\alpha}\delta m = m\delta v \tag{35}$$

This is a separable equation with the solution:

$$v = \frac{1}{\alpha}\left[F \ln\left(\frac{m}{M}\right) - g(m - M)\right] \tag{36}$$

But $m(t) = M - \alpha t$, so that:

$$v(t) = \frac{F}{\alpha}\ln\left(1 - \frac{\alpha}{M}t\right) - gt \tag{37}$$

The height as a function of time is given by:

$$h(t) = \int_0^t v\,\mathrm{d}t = \frac{mF}{\alpha^2}\left(1 - \frac{\alpha}{M}t\right)\left[1 - \ln\left(1 - \frac{\alpha}{M}t\right)\right] - \frac{1}{2}gt^2 \tag{38}$$

(b) Consider the moment when the object has a mass m and velocity v. In a short time δt, we can write the conservation of momentum as:

$$mv + mg\sin\theta\delta t = (m + \delta m)(v + \delta v) \tag{39}$$

Ignoring any second order terms gives:

$$mg\sin\theta\delta t = m\delta v + \delta m v \tag{40}$$

By definition $\delta x = v\delta t$. We also have $m = \sigma x$ so that $\delta m = \sigma\delta x$. Substituting:

$$\sigma g\sin\theta\frac{x}{v}\delta x = \sigma x\delta v + \sigma v\delta x \tag{41}$$

Or:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \left(\frac{g \sin\theta}{v} - \frac{v}{x}\right) \tag{42}$$

Consider the trial solution $x = kt^2/2$. Substitution leads to:

$$\frac{1}{t} = \frac{1}{t}\left(\frac{g \sin\theta}{k} - 2\right) \tag{43}$$

which means $k = g \sin\theta/3$ as required.