Special Relativity

Problems 1

Collision Problems, Threshold energies, Decays, Recoils

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1

The energies of the mesons are given by:

$$2E = 2\gamma m_\pi c^2 = m_K c^2 \tag{1}$$

Solving for v via γ yields:

$$v = \sqrt{1 - \left(\frac{2m_\pi}{m_K}\right)^2} c = 0.83c \tag{2}$$

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(a)

$$\frac{T}{E} = 1 - \frac{1}{\gamma} = 1 - \left(\frac{mc^2}{E}\right) = 0.99999 \tag{3}$$

(b)

$$p = \frac{1}{c}\sqrt{E^2 - m^2c^4} = 2.63 \times 10^{-17}\,\mathrm{kgms^{-1}} \tag{4}$$

(c)

$$v = \sqrt{1 - \left(\frac{mc^2}{E}\right)^2} \approx \left[1 - \frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right] = \left[1 - (5 \times 10^{-11})\right] c \tag{5}$$

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Let the two photons have energies E_1 and E_2 respectively. Then by conservation of energy and momentum:

$$\begin{aligned} E &= m_0c^2 + T = E_1 + E_2 \\ pc &= \sqrt{2m_0c^2T + T^2} = E_1 - E_2 \end{aligned} \tag{6}$$

Solving for E_1 and E_2 yields:

$$
\begin{aligned}
E_1 &= 1131\,\mathrm{MeV} \\
E_2 &= 4\,\mathrm{MeV}
\end{aligned} \tag{7}
$$

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In the laboratory frame, the total energy is $m_p c^2 + E$. In the CM frame:

$$E_{\mathrm{CM}}^2 = (m_p c^2 + E)^2 - (pc)^2 = 2m_p c^2 E + 2m_p^2 c^4 \tag{8}$$

For threshold energy, the products in the CM frame are at rest. Thus:

$$E_{\mathrm{CM}}^2 = 2m_p c^2 E + 2m_p^2 c^4 = (2m_p c^2 + m_\pi c^2)^2 \tag{9}$$

Solving for E leads to a formula for the threshold kinetic energy:

$$T = E - m_p c^2 = 289\,\mathrm{MeV} \tag{10}$$

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Following the above procedure, the energy in the CM frame is given by:

$$E_{\mathrm{CM}}^2 = 2m_e c^2 E + 2m_e^2 c^4 = (4m_e c^2)^2 \tag{11}$$

This leads to $E = 7m_e c^2$ and $T = 6m_e c^2$.

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Again, the energy in the CM frame is given by:

$$E_{\mathrm{CM}}^2 = 2m_p c^2 E + 2m_p^2 c^4 = (2E_0)^2 \tag{12}$$

Solving for E yields:

$$E = \frac{2E_0^2}{m_p c^2} - m_p c^2 = 2.13 \times 10^3 \,\mathrm{TeV} \tag{13}$$

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(a) The electron cannot emit a photon as it must obey conservation of energy.

(b) This is because the electron in a hydrogen atom has orbital angular momentum.

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In the lab frame, by conservation of energy and momentum, we have:

$$\begin{aligned} E &= Mc^2 + Q \\ \gamma_v M v &= \frac{Q}{C} \end{aligned} \tag{14}$$

where E is the energy of the excited atom observed in the lab frame and v is its speed in the lab frame.

In the atom's frame, the energy is $Mc^2 + Q_0$. By Lorentz invariant, we have:

$$(Mc^2 + Q_0)^2 = E^2 - (\gamma_v M v c)^2 = (Mc^2 + Q)^2 - Q^2 \tag{15}$$

Solving for Q leads to:

$$Q = Q_0 \left(1 + \frac{Q_0}{2Mc^2}\right) \tag{16}$$

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The red light has a frequency f_0 450 Hz and the green has a frequency f 550 Hz. The Doppler shift is given by:

$$\frac{1+\beta}{1-\beta} = \frac{f^2}{f_0^2} \tag{17}$$

This leads to $\beta \approx 0.2$. Therefore, driver must be travelling at around $6 \times 10^4\,\mathrm{kms^{-1}}$ for him to mistake red for green. His speed is way too high for any human vehicle (alien technology).

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Past Prelims Questions

11

Consider a stationary rod in frame S' along the of proper length L_0 with one end at the origin along the x-axis. The world lines of the two ends of the rod in S' are given by:

$$\begin{aligned} X_1' &= (ct', 0, 0, 0)^\intercal \\ X_2' &= (ct', L_0, 0, 0)^\intercal \end{aligned} \tag{18}$$

In frame S, the world lines are transformed according to $X_i = \Lambda^{-1} X_i'$:

$$\begin{aligned} X_1 &= \gamma_v (ct', \beta ct', 0, 0)^\intercal = (ct_1, x_1, 0, 0)^\intercal \\ X_2 &= \gamma_v (ct' + \beta L_0, \beta ct' + L_0, 0, 0)^\intercal = (ct_2, x_2, 0, 0)^\intercal \end{aligned} \tag{19}$$

Choose $t_1 = t_2$ and compute $x_2 - x_1$ yields:

$$x_2 - x_1 = (1 - \beta^2)\gamma_v L_0 = \frac{L_0}{\gamma_v} \tag{20}$$

which is the length of the rod in frame S.

Consider further two events A and B in S' given by the coordinates:

$$\begin{aligned} X_A' &= (0, 0, 0, 0)^\intercal \\ X_B' &= (c\Delta t, 0, 0, 0)^\intercal \end{aligned} \tag{21}$$

In frame S:

$$\begin{aligned} X_A &= \gamma_v (0, 0, 0, 0)^\intercal = (ct_A, x_A, 0, 0)^\intercal \\ X_B &= \gamma_v (c\Delta t, \beta c\Delta t, 0, 0)^\intercal = (ct_B, x_B, 0, 0)^\intercal \end{aligned} \tag{22}$$

The time interval between events A and B in frame S is given by:

$$t_B - t_A = \gamma_v \Delta t \tag{23}$$

(a)

$$\Delta t_E = \gamma \Delta t_R = 50\,\text{min} \tag{24}$$

(b)

$$\begin{aligned} D_E &= \Delta t_E v = 7.2 \times 10^{11}\,\mathrm{m} \\ D_R &= \frac{D_E}{\gamma} = 4.32 \times 10^{11}\,\mathrm{m} \end{aligned} \tag{25}$$

(c)

$$T = \Delta t_E + \frac{D_E}{c} = 90\,\mathrm{min} \tag{26}$$

(d) In Earth's frame, the event of the signal reaching the rocket happens at:

$$T + \Delta T = T + \frac{D_E + \frac{D_E}{c}0.8c}{0.2c} = 360\,\mathrm{min} \tag{27}$$

In rocket's frame, the time is thus:

$$360\,\mathrm{min}/\gamma = 216\,\mathrm{min} \tag{28}$$

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$$\begin{aligned} p &= \gamma_v m_0 v \\ E &= \gamma_v m_0 c^2 \end{aligned} \tag{29}$$

so that:

$$p^2c^2 + m_0^2c^4 = m_0^2c^2\left(\frac{c^2}{1 - v^2/c^2}\right) = E^2 \tag{30}$$

(a) Treating electrons and positrons as massless particles, the total energy-momentum 4-vector in the lab frame is:

$$P = (E_e/c, p_e, 0, 0)^\intercal + (E/c, -p_p, 0, 0)^\intercal \tag{31}$$

where $p_e = E_e/c$ is the momentum of the electron and $p_p = E/c$ is the (magnitude of) the momentum of the positron.

In the CM frame, we demand all product to be stationary after the collision. The total energy is $E_{\mathrm{CM}} = 2m_Bc^2$. By Lorentz invariant:

$$E_{\mathrm{CM}}^2 = (E_e + E)^2 - (p_e - p_p)^2c^2 = (E_e + E)^2 - (E_e - E)^2 = 4E_eE = 4m_B^2c^4 \tag{32}$$

This leads to:

$$E = \frac{m_B^2 c^4}{E_e} = 3.1\,\mathrm{GeV} \tag{33}$$

(b) By Lorentz transformation, we have $E + E_e = \gamma E_{\mathrm{CM}}$ so that:

$$\gamma = \frac{E + E_e}{E_{\mathrm{CM}}} \tag{34}$$

The mean distance travelled in the lab frame is given by:

$$D = v\frac{\tau}{\gamma} = c\sqrt{1 - \frac{1}{\gamma^2}}\frac{\tau}{\gamma} = 1.9 \times 10^{-4}\,\mathrm{m} \tag{35}$$

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In frame S, $X = (ct, u_x t, 0, 0)^\intercal$ and this becomes $X' = \gamma(ct - \beta u_x t, -\beta ct + u_x t, 0, 0)^\intercal = (ct', x', 0, 0)$ in frame S'. Thus, by the definition of speed:

$$u'_x \equiv \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x'}{\mathrm{d}t}\Big/\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{\gamma(-\beta c + u_x)}{\gamma(1 - \beta u_x/c)} = \frac{u_x - v}{1 - v u_x/c^2} \tag{36}$$

(a)

$$\begin{aligned} L_1 &= \frac{L_0}{\gamma_1} = 71\,\mathrm{m} \\ L_2 &= \frac{L_0}{\gamma_2} = 60\,\mathrm{m} \end{aligned} \tag{37}$$

(b) The velocity of the second spaceship as measured by the first is:

$$v = \frac{-v_2 - v_1}{1 + v_1 v_2/c^2} = -0.961c \tag{38}$$

so that the measured length of the second spaceship is:

$$L_2 = \frac{L_0}{\gamma} = 27\,\mathrm{m} \tag{39}$$

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