From Newton's Laws to Modelling Black Holes

The Power of Numerical Methods

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What is the **greatest** achievement of science?

Quantum mechanics? General relativity? The standard model?

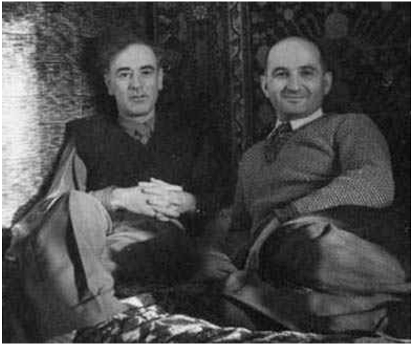
What is the **greatest** achievement of science?

Ability to predict the future,

using, for example, Newton's laws.

Contents

Newton's Laws



If all the co-ordinates and velocities (of a system) are simultaneously specified, it is known from experience that the state of the system is completely determined and that its subsequent motion can, **in principle**, *be calculated.*

[0] L.D. Landau and E.M. Lifshitz.

Newton's Laws

In principle, we can predict the future using Newton's second law:

$$\mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \tag{1}$$

Once we know the force, we can solve the differential equation.

Newton's Laws

Imagine you are a rocket moving in space towards Mother Earth. Your dynamics is a constant updating of the state vector:

$$\begin{pmatrix} x \\ v \end{pmatrix} \xrightarrow{\delta t} \begin{pmatrix} x + v\delta t \\ v + a\delta t \end{pmatrix} \tag{2}$$

How do we know the acceleration a?

Numerical Methods

What we just did is called the **Euler's method**.

It is a general method of solving ordinary differential equations.

[0]Leonhard Euler (1707-1783).

Numerical Methods

Consider the following differential equation:

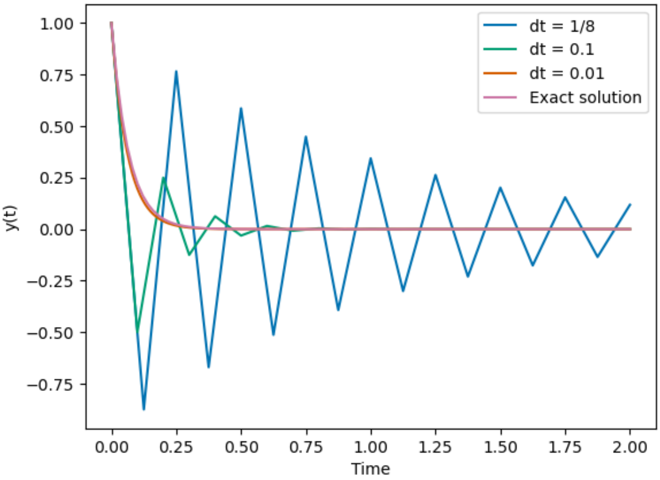
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -15y \quad y(0) = 1 \tag{3}$$

We know the solution is:

$$y(t) = e^{-15t} \tag{4}$$

Let's see how Euler's method performs with different step sizes.

Numerical Methods



Numerical Methods

There are at least two problems with the naive Euler's method:

1. It is (very) **inaccurate** for large step sizes.
2. It becomes (very) **slow** for small step sizes.

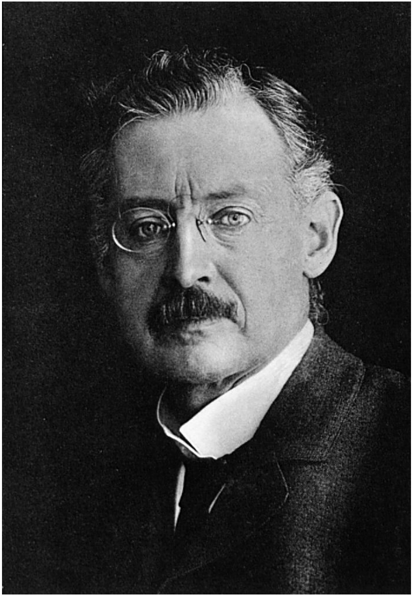
Numerical Methods - Go to Higher Orders

The Euler's method is naive in a sense that it is too ‘local’.

We could have 'scouted' ahead a bit and use the average of acceleration there and our current position.

If we use the immediate future, we make the error $\mathcal{O}(\delta t^2)$ instead of $\mathcal{O}(\delta t)$.

Numerical Methods - Go to Higher Orders



Let us go as far as four steps ahead!

Runge-Kutta methods are a family of numerical methods for solving ordinary differential equations.

The most famous is the RK4 method.

[0] Carl David Tolmé Runge (1856-1927).

Foraging to Black Holes

Instead of a rocket, what if we are photons travelling towards a black hole?

What even is a black hole?

John Michell first to proposed the existence of 'black holes'.

Alas, he was too far ahead of his time.

Scientists then did not have the tools to investigate his ideas.

[0] John Michell (1724-1793).

Foraging to Black Holes

Albert Einstein published the general theory of relativity in 1915 along with his field equations.

Karl Schwarzschild found the first solution to the equations in 1916.

From his solutions, the concept of a black hole emerged.

[0] Karl Schwarzschild (1873-1916).

Foraging to Black Holes

It was soon realised that BHs are very simple objects with only three properties:

1. Mass M
2. Charge Q (theorised to be zero)
3. Spin J

Foraging to Black Holes

BHs are often found in binary systems and a process called **accretion** occurs to give rise to **accretion disks**.

These disks get so hot that they emit some of the most energetic radiation (photons!) in the universe.