Electromagnetism

Problem Set 1

Electric Fields, Potentials and the Principle of Superposition

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Electric Fields, Potentials and the Principle of Superposition

0 Background

An electric field at a point in space generated by a distribution of charges is the force per unit charge that a test charge would experience if placed on that point.

The electric potential at a point in space is the work done per unit charge to move a test charge from a reference point to that point.

We have the relationship between an electric field $\mathbf{E}(\mathbf{r})$ and its electric potential $\phi(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) \tag{1}$$

For both the electric field and the electric potential, the net effect of a distribution of charges is the sum of the effects of each individual charge, i.e.:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \sum_i \mathbf{E}_i(\mathbf{r}) \\ \phi(\mathbf{r}) &= \sum_i \phi_i(\mathbf{r}) \end{aligned} \tag{2}$$

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1 Assembly of point charges in the corners of a square

(a) By symmetry, we only consider the electric field along the line AC:

$$E = \frac{1}{4\pi\epsilon_0}\left(\frac{5q}{a^2/2} + \frac{q}{a^2/2}\right) = \frac{1}{4\pi\epsilon_0}\frac{12q}{a^2} \tag{3}$$

where the direction of the field is from A to C.

(b) The assembly energy is given by:

$$W = \frac{1}{2}\sum_i q_i\phi_i = -\frac{1}{4\pi\epsilon_0}(16 + \sqrt{2}/2)q^2 \tag{4}$$

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2 Electric dipole

(a) Under spherical coordinates, the potential V produced by the dipole is given by:

$$\begin{aligned} V(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}-\mathbf{d}/2|}-\frac{1}{|\mathbf{r}+\mathbf{d}/2|}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left[\left(r^2-\mathbf{r}\cdot\mathbf{d}+\frac{d^2}{4}\right)^{-1/2}-\left(r^2+\mathbf{r}\cdot\mathbf{d}+\frac{d^2}{4}\right)^{-1/2}\right] \\ &\approx \frac{q}{4\pi\epsilon_0}\frac{1}{r}\left[\left(1+\frac{\mathbf{r}\cdot\mathbf{d}}{r^2}\right)-\left(1-\frac{\mathbf{r}\cdot\mathbf{d}}{r^2}\right)\right] \\ &= \frac{\mathbf{p}\cdot\mathbf{r}}{2\pi\epsilon_0 r^3} \\ &= \frac{p\cos\theta}{2\pi\epsilon_0 r^2} \end{aligned} \tag{5}$$

where $\mathbf{p}=q\mathbf{d}$ is the dipole moment and θ is the polar angle.

(b) The electric field is given by:

$$\begin{aligned} E &= -\nabla V \\ &= -\frac{\partial V}{\partial r}\hat{r}-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} \\ &= \frac{p\cos\theta}{2\pi\epsilon_0 r^3}\hat{r}+\frac{p\sin\theta}{4\pi\epsilon_0 r^3}\hat{\theta} \end{aligned} \tag{6}$$

(c) The torque experienced by the dipole is $\mathbf{T}=\mathbf{p}\times\mathbf{E}_{ext}$ so that the energy can be defined as the work done against the torque in rotating the dipole from $\pi/2$ to α:

$$W(\alpha) \equiv \int_{\pi/2}^{\alpha} pE_{ext}\sin\theta\,\mathrm{d}\theta = -pE_{ext}\cos\alpha = -\mathbf{p}\cdot\mathbf{E}_{ext} \tag{7}$$

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3 Assembly of point charges on a line; multipoles

Under spherical coordinates, the potential V produced by the dipole is given by:

$$
\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\left(\frac{q_2}{r} - \frac{q_1}{|\mathbf{r}-\mathbf{a}|} - \frac{q_1}{|\mathbf{r}+\mathbf{a}|}\right) \\
&= \frac{q_2}{4\pi\epsilon_0}\frac{1}{r} - \frac{q_1}{4\pi\epsilon_0 r}\left[\left(1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2} + \left(1 + 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2}\right]
\end{aligned} \tag{8}
$$

The term $[1 \mp 2(a/r)\cos\theta + (a/r)^2]^{-1/2}$ can be treated as a binomial expansion and further expansion of the resulting items can yield the desired result. Alternatively, we note that the term is the generating function of the Legendre polynomials:

$$
\left(1 \mp 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}\right)^{-1/2} = \sum_{i=0}^{\infty} P_i(\cos\theta)\left(\mp\frac{a}{r}\right)^i \tag{9}
$$

where $P_i(\cos\theta)$ is the ith Legendre polynomial.

The first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = 3x^2/2 - 1/2$. Expanding the terms up to $P_2(\cos\theta)$ and simplifying:

$$
V(\mathbf{r}) = \frac{q_2}{4\pi\epsilon_0 r} - \frac{q_1}{2\pi\epsilon_0 r} - \frac{q_1}{4\pi\epsilon_0 r}(3\cos^2\theta - 1)\frac{a^2}{r^2} \tag{10}
$$

where the terms are monopole, dipole and quadrupole contributions, respectively.

With $q_2 = 2q_1$, we can write the potential as:

$$
V(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0 r}(1 - 3\cos^2\theta)\frac{a^2}{r^2} \tag{11}
$$

and the associated electric field is:

$$
\mathbf{E}(\mathbf{r}) = -\nabla V = \frac{3q_1 a^2}{4\pi\epsilon_0 r^4}\left[(3\cos^2\theta - 1)\hat{r} + (6\cos\theta\sin\theta)\hat{\theta}\right] \tag{12}
$$

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4 Uniformly charged rod

(a) By symmetry, the electric field only has a z-component given by:

$$
\mathbf{E}(z) = \hat{z}\int_{-l}^{l} \frac{1}{4\pi\epsilon_0}\frac{\lambda}{(z-h)^2}\,\mathrm{d}h = \frac{\lambda}{4\pi\epsilon_0}\frac{2l}{z^2 - l^2}\hat{z} \tag{13}
$$

(b) By symmetry, the electric field only has an x-component given by:

$$\mathbf{E}(x) = \hat{x}\int_{-l}^{l} \frac{1}{4\pi\epsilon_0}\frac{\lambda}{x^2+h^2}\,\mathrm{d}h = \frac{\lambda}{2\pi\epsilon_0 x}\tan^{-1}(h/x)\hat{x} \tag{14}$$

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5 Uniformly charged disk

(a) Considering the contributions of infinitesimal ring elements:

$$\begin{aligned}\mathbf{E}(z) &= \hat{z}\int \mathrm{d}E_z = \hat{z}\int_0^b \frac{1}{4\pi\epsilon_0}\frac{\sigma 2\pi r\mathrm{d}r}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}} \\ &= \hat{z}\frac{\sigma z}{4\epsilon_0}\int_0^b \frac{2r}{(r^2+z^2)^{3/2}}\,\mathrm{d}r \\ &= \frac{\sigma}{2\epsilon_0}\left[1-\left(1+\frac{b}{z}\right)^{-1/2}\right]\hat{z}\end{aligned} \tag{15}$$

(b) In the limit $z \ll b$:

$$E(z) = \frac{\sigma}{2\epsilon_0}\left[1-\sqrt{\frac{z}{b}}\left(1+\frac{z}{b}\right)^{-1/2}\right] \approx \frac{\sigma}{2\epsilon_0}\left(1-\sqrt{\frac{z}{b}}\right) \tag{16}$$

The first term is expected as it is the perpendicular component of the boundary electric field close to a thin charge distribution.

In the limit $z \gg b$:

$$E(z) \approx \frac{\sigma b}{2\epsilon_0 z} \tag{17}$$

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6 Uniformly charged sphere

(a) The electric field along the axis of the thin ring is given by:

$$E(z) = \frac{q}{4\pi\epsilon_0}\frac{z}{(a^2+z^2)^{3/2}} \tag{18}$$

For a maximum, we compute the first derivative:

$$\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{q}{4\pi\epsilon_0}\frac{(a^2+z^2)^{3/2}-(a^2+z^2)^{1/2}z^2}{(a^2+z^2)^{3/2}} = 0 \tag{19}$$

The solutions are $z = \pm a/\sqrt{2}$.

(b) The force exerted on an electron is given by:

$$F(z) = -eE(z) = -\frac{eqz}{4\pi\epsilon_0 a^3}\left[1+\left(\frac{z}{a}\right)^2\right]^{-3/2} \approx -\frac{eqz}{4\pi\epsilon_0 a^3} \tag{20}$$

if $z \ll a$ and terms beyond the first order are neglected.

Thus the motion is approximately simple harmonic with frequency:

$$f = \frac{1}{2\pi}\sqrt{\frac{eq}{4\pi\epsilon_0 a^3 m}} = \sqrt{\frac{eq}{16\pi^3\epsilon_0 a^3 m}} \tag{21}$$

The oscillation is possible because for th electron, the point $z = 0$ is a stable equilibrium. In fact, any small oscillation around a stable equilibrium is simple harmonic.

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7 Uniformly charged hollow sphere

In spherical coordinates, the infinitesimal area element can be expressed as $\mathrm{d}A = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$, where r can be treated as a constant a. Orienting the axis so that $\mathbf{P}$ lies on the z-axis, the potential is given by:

$$\begin{aligned}\phi(\mathbf{P}) &= \iint_S \mathrm{d}\phi = \int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\epsilon_0}\frac{\sigma a^2\sin\theta}{\sqrt{a^2+p^2-2ap\cos\theta}}\mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{\sigma a^2}{2\epsilon_0}\int_0^{\pi}\frac{\sin\theta}{\sqrt{a^2+p^2-2ap\cos\theta}}\mathrm{d}\theta \\ &= \frac{\sigma a}{\epsilon_0}\end{aligned} \tag{22}$$

This demonstrates that the potential inside a conductor is constant so that the field is zero.

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