Mathematical Methods

Problem Sheet 2

Fourier Series and Fourier Integrals

Xin, Wenkang

October 30, 2023

Fourier Series and Fourier Integrals

1 Fourier Series

(a) We have the coefficients for the cosine series as:

$$a_r = \frac{2}{2\pi}\int_0^\pi \sin x \cos rx \,\mathrm{d}x \tag{1}$$

where apparently $a_0 = 2/\pi$.

For $r \geq 1$, the integral denoted as I_r can be evaluated by parts:

$$\begin{aligned} I_r &= [-\cos x \cos rx]_0^\pi - r\int_0^\pi \cos x \sin rx \,\mathrm{d}x \\ &= [\cos x \cos rx]_\pi^0 - r\left\{[\sin x \sin rx]_0^\pi - r\int_0^\pi \sin x \cos rx \,\mathrm{d}x\right\} \\ &= 1 + \cos \pi r + r^2 I_r \end{aligned} \tag{2}$$

Therefore, the coefficients are:

$$a_r = \frac{1}{\pi}\frac{1+\cos\pi r}{1-r^2} = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 2/\pi(1-r^2) & \text{if } r \text{ is even} \end{cases} \tag{3}$$

On the other hand, the coefficients for the sine series are:

$$b_r = \frac{2}{2\pi}\int_0^\pi \sin x \sin rx \,\mathrm{d}x \tag{4}$$

which are zero except for $r = 1$:

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin^2 x \,\mathrm{d}x = \frac{1}{2} \tag{5}$$

Hence, the Fourier series is:

$$f(x) = \frac{1}{2}\sin x + \frac{2}{\pi}\sum_{\text{even } r\geq 0}^{\infty} \frac{1}{1-r^2}\cos rx \tag{6}$$

(b) Since the function is even, we only need the coefficients for the cosine series:

$$\begin{aligned}
a_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos rx \,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos rx \,\mathrm{d}x \\
&= \frac{2}{\pi}\left\{\left[x^2\frac{1}{r}\sin rx\right]_0^{\pi} - \frac{2}{r}\int_0^{\pi} x\sin rx \,\mathrm{d}x\right\} \\
&= -\frac{4}{r\pi}\left\{\left[-x\frac{1}{r}\cos rx\right]_0^{\pi} + \frac{1}{r}\int_0^{\pi} \cos rx \,\mathrm{d}x\right\} \\
&= \frac{4}{r^2}\cos r\pi = \frac{4}{r^2}(-1)^r
\end{aligned} \tag{7}$$

for $r \geq 1$.

Apparently, $a_0 = 2\pi^2/3$ and the Fourier series is:

$$f(x) = \frac{\pi^2}{3} + \sum_{r=1}^{\infty}(-1)^r\frac{4}{r^2}\cos rx \tag{8}$$

(c) Consider the norm of the function $f(x) = x^2$ on the interval $[-\pi, \pi]$:

$$\|f\|^2 = \int_{-\pi}^{\pi} x^4 \,\mathrm{d}x = \frac{2\pi^5}{5} \tag{9}$$

By Parseval's equation, we have:

$$\frac{\|f\|^2}{\pi} = \frac{1}{2}\left(\frac{\pi^2}{3}\right)^2 + \sum_{r=1}^{\infty}\frac{16}{r^4} \tag{10}$$

so that:

$$\sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{1}{16}\left(\frac{2\pi^4}{5} - \frac{\pi^4}{18}\right) = \tag{11}$$

2 Sine and cosine Fourier series

(a) The coefficients for the cosine series are:

$$a_r = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos rx \,\mathrm{d}x \tag{12}$$

For the case $r = 0$ and $r = 1$, we have:

$$a_0 = \frac{2}{\pi}\int_0^\pi x\sin x\,\mathrm{d}x = \frac{2}{\pi}\left\{[-x\cos x]_0^\pi + \int_0^\pi \cos x\,\mathrm{d}x\right\} = 2 \tag{13}$$

and:

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi x\sin 2x\,\mathrm{d}x = \frac{1}{2\pi}\left\{[-x\cos 2x]_0^\pi + \int_0^\pi \cos 2x\,\mathrm{d}x\right\} = -\frac{1}{2} \tag{14}$$

For $r \geq 2$, the integral, which is denoted as I_r, can be evaluated by parts:

$$I_r = [-x\cos x\cos rx]_0^\pi + \int_0^\pi \cos x\cos rx\,\mathrm{d}x - r\int_0^\pi x\cos x\sin rx\,\mathrm{d}x \tag{15}$$

For the middle term, the only non-zero contribution is when $r = 1$ as $\cos rx$ are orthogonal. Therefore we may neglect the middle term for $r \geq 2$:

$$\begin{aligned} I_r &= \pi\cos r\pi - r\left\{[x\sin x\sin rx]_0^\pi - \int_0^\pi \sin x\sin rx\,\mathrm{d}x - r\int_0^\pi x\sin x\cos rx\,\mathrm{d}x\right\} \\ &= \pi\cos r\pi + r^2 I_r \end{aligned} \tag{16}$$

where in the last step we have again used the orthogonality of $\sin rx$.

This means that for $r \geq 2$, the coefficients are:

$$a_r = (-1)^r\frac{2}{1-r^2} \tag{17}$$

Hence the cosine Fourier series is:

$$f(x) = 2 - \frac{1}{2}\cos x + \sum_{r=2}^{\infty}(-1)^r\frac{2}{1-r^2}\cos rx \tag{18}$$

(b) The coefficients for the sine series are:

$$b_r = \frac{2}{\pi}\int_0^\pi x\sin x\sin rx\,\mathrm{d}x \tag{19}$$

where $b_1 = \pi/2$.

For $r \geq 2$, the integral, which is denoted as I_r, can be evaluated as:

$$\begin{aligned} I_r &= \frac{1}{2}\int_0^{\pi} x\cos{(1+r)x}\,\mathrm{d}x - \frac{1}{2}\int_0^{\pi} x\cos{(1-r)x}\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\cos{(1+r)\pi}-1}{(1+r)^2} - \frac{\cos{(1-r)\pi}-1}{(1-r)^2}\right] \end{aligned} \tag{20}$$

where we used the integral result:

$$\int_0^{\pi} x\cos kx\,\mathrm{d}x = \frac{\cos k\pi - 1}{k^2} \tag{21}$$

This means that the coefficients are:

$$b_r = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 2\left[(1+r)^{-2} - (1-r)^{-2}\right]/\pi & \text{if } r \text{ is even} \end{cases} \tag{22}$$

Hence the sine Fourier series is:

$$f(x) = \frac{\pi}{2}\sin x + \sum_{\text{even } r\geq 2}^{\infty} \frac{2}{\pi}\left[\frac{1}{(1+r)^2} - \frac{1}{(1-r)^2}\right]\sin rx \tag{23}$$

(c) The coefficients for the cosine series are:

$$a_r = \frac{2}{\pi}\int_0^{\pi} x\cos rx\,\mathrm{d}x = \frac{2}{\pi}\frac{\cos k\pi - 1}{k^2} = \begin{cases} -4/\pi r^2 & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases} \tag{24}$$

except for $r = 0$ where $a_0 = \pi$.

Hence the cosine Fourier series is:

$$f(x) = \pi - \frac{4}{\pi}\sum_{\text{odd } r\geq 1}^{\infty} \frac{1}{r^2}\cos rx \tag{25}$$

(d) The coefficients for the sine series are:

$$b_r = \frac{2}{\pi}\int_0^{\pi} x\sin rx\,\mathrm{d}x = -\frac{2}{\pi}\frac{\pi\cos r\pi}{r} = \begin{cases} 2/r & \text{if } r \text{ is odd} \\ -2/r & \text{if } r \text{ is even} \end{cases} \tag{26}$$

Hence the sine Fourier series is:

$$f(x) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{2}{r} \sin rx \tag{27}$$

The cosine series does not converge to $f(x)$ near zero because $f(x)$ is not even, whereas the sine series converges to zero.

Consider the norm of the function $f(x) = x$ on the interval $[0, \pi]$:

$$\|f\|^2 = \int_0^{\pi} x^2 \,\mathrm{d}x = \frac{\pi^3}{3} \tag{28}$$

The Parseval's equation gives:

$$\frac{2\|f\|^2}{\pi} = \sum_{r=1}^{\infty} \frac{4}{r^2} \tag{29}$$

so that:

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6} \tag{30}$$

•

3 Legendre polynomials as an orthogonal basis

(a) We know that the Legendre polynomials can be defined as:

$$P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} (x^2 - 1)^l \tag{31}$$

Consider the inner product $\left\langle P_n, x^k \right\rangle$ for $k < n$:

$$\begin{aligned} \left\langle P_n, x^k \right\rangle &\propto \int_{-1}^{1} x^k \frac{\mathrm{d}^l}{\mathrm{d}x^l} (x^2 - 1)^l \,\mathrm{d}x \\ &= \left[x^k \frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}} (x^2 - 1)^l \right]_{-1}^{1} - \int_{-1}^{1} k x^{k-1} \frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}} (x^2 - 1)^l \,\mathrm{d}x \\ &= \dots \\ &\propto (-1)^k \int_{-1}^{1} \frac{\mathrm{d}^{l-k}}{\mathrm{d}x^{l-k}} (x^2 - 1)^l \,\mathrm{d}x \\ &= 0 \end{aligned} \tag{32}$$

where we have used the fact that $k < n$ and $l - k \geq 0$.

Any polynomial p of degree $k < n$ can be written as $p(x) = \sum_{r=0}^{k} a_r x^r$, so that:

$$\langle P_n, p \rangle = \sum_{r=0}^{k} a_r \langle P_n, x^r \rangle = 0 \tag{33}$$

(b) Using the above definition, the first five Legendre polynomials are:

$$\begin{aligned}
P_0 &= 1 \\
P_1 &= x \\
P_2 &= \frac{1}{2}(3x^2 - 1) \\
P_3 &= \frac{1}{2}(5x^3 - 3x) \\
P_4 &= \frac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned} \tag{34}$$

(c) Decomposing the function $f(x) = x^4$ in terms of the Legendre polynomials:

$$\begin{aligned}
a_0 = \langle P_0, f \rangle &= \int_{-1}^{1} x^4 \,\mathrm{d}x = \frac{2}{5} \\
a_1 = \langle P_1, f \rangle &= \int_{-1}^{1} x^5 \,\mathrm{d}x = 0 \\
a_2 = \langle P_2, f \rangle &= \int_{-1}^{1} \frac{3x^6 - x^4}{2} \,\mathrm{d}x = \frac{1}{35} \\
a_3 = \langle P_3, f \rangle &= \int_{-1}^{1} \frac{5x^7 - 3x^5}{2} \,\mathrm{d}x = 0 \\
a_4 = \langle P_4, f \rangle &= \int_{-1}^{1} \frac{35x^8 - 30x^6 + 3x^4}{8} \,\mathrm{d}x =
\end{aligned} \tag{35}$$

(d) The given function can be written as a generating function of the Legendre polynomials:

$$f(x) = \frac{1}{\sqrt{(1/2)^2 - x + 1}} = \sum_{r=0}^{\infty} P_r(x) \left(\frac{1}{2}\right)^r \tag{36}$$

•

4 Examples of Fourier transforms

(a) For the given square pulse, its Fourier transform is:

$$\hat{\chi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx} \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \frac{e^{-ik} - e^{ik}}{-ik} = \sqrt{\frac{2}{\pi}} \frac{\sin k}{k} \tag{37}$$

(c) The convolution is:

$$f(\mu) = \chi * \chi = \int_{\mathbb{R}} \chi(x)\chi(\mu - x) \, \mathrm{d}x \tag{38}$$

But χ is symmetric about the origin and $\chi(\mu - x) = \chi(x - \mu)$, so that:

$$f(\mu) = \int_{-1}^{1} \chi(x)\chi(x - \mu) \, \mathrm{d}x \tag{39}$$

where the range of integration is restricted to $[-1, 1]$ because $\chi(x) = 0$ otherwise.

Consider the case $0 < \mu < 2$, the integrand is non-zero only when $x \in [\mu - 1, 1]$. Therefore:

$$f(\mu) = \int_{\mu-1}^{1} \mathrm{d}x = 2 - \mu \tag{40}$$

Similarly, for $-2 < \mu \leq 0$:

$$f(\mu) = \int_{-1}^{\mu+1} \mathrm{d}x = \mu + 2 \tag{41}$$

Thus, the convolution is:

$$f(\mu) = \begin{cases} 2 - \mu & \text{if } 0 < \mu < 2 \\ \mu + 2 & \text{if } -2 < \mu \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

which is a triangular pulse.

(d) The Fourier transform of the convolution is:

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-2}^{2}(\mu+2)e^{-ik\mu}\,\mathrm{d}\mu + \frac{1}{\sqrt{2\pi}}\int_{0}^{2}(2-\mu)e^{-ik\mu}\,\mathrm{d}\mu \\ &= \frac{2}{\sqrt{2\pi}}\int_{-2}^{2}(\mu+2)\cos(k\mu)\,\mathrm{d}\mu \\ &= \frac{4}{\sqrt{2\pi}}\frac{\sin^2 k}{k^2} \\ &= \sqrt{2\pi}\,[\hat{\chi}(k)]^2\end{aligned} \tag{43}$$

as expected.

(e) Consider the inner product $\langle \mathcal{F}(f), \mathcal{F}(f)\rangle$:

$$\begin{aligned}\langle \mathcal{F}(f), \mathcal{F}(f)\rangle &= \left\langle \mathcal{F}^\dagger[\mathcal{F}(f)], f\right\rangle \\ &= \langle f, f\rangle \\ &= \int_{\mathbb{R}} f(x)f(x)\,\mathrm{d}x \\ &= \int_{-2}^{0}(2+\mu)^2\,\mathrm{d}\mu + \int_{0}^{2}(2-\mu)^2\,\mathrm{d}\mu \\ &= \frac{16}{3}\end{aligned} \tag{44}$$

where the second step is due to the fact that $\mathcal{F}$ is unitary.

On the other hand, the norm of $\mathcal{F}(f)$ is just:

$$\|\mathcal{F}(f)\|^2 = \frac{8}{\pi}\int_{\mathbb{R}}\frac{\sin^4 k}{k^4}\,\mathrm{d}k \tag{45}$$

so that we have the result:

$$\int_{\mathbb{R}}\frac{\sin^4 k}{k^4}\,\mathrm{d}k = \frac{2\pi}{3} \tag{46}$$

•

5 Some properties of Fourier transforms

(a) Consider $\mathcal{F}\circ T_a$ acting on f:

$$\begin{aligned}
\mathcal{F} \circ T_a(f) &= \mathcal{F}[f(x-a)] \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-a) e^{-ikx} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-ik(y+a)} \, \mathrm{d}y \\
&= e^{-ika} \mathcal{F}(f) \\
&= E_{-a} \mathcal{F}(f)
\end{aligned} \tag{47}$$

where we have used the substitution $y = x - a$ in the third step.

Further consider $\mathcal{F} \circ E_a$:

$$\begin{aligned}
\mathcal{F} \circ E_a(f) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iax} f(x) e^{-ikx} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i(k-a)x} \, \mathrm{d}x \\
&= T_a(\hat{f})(k) \\
&= T_a \circ \mathcal{F}(f)
\end{aligned} \tag{48}$$

(b) Consider $\mathcal{F} \circ \mathcal{D}_a$ acting on f:

$$\begin{aligned}
\mathcal{F} \circ \mathcal{D}_a(f) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(ax) e^{-ikx} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{a} \int_{\mathbb{R}} f(y) e^{-iky/a} \, \mathrm{d}y \\
&= \frac{1}{a} \hat{f}(k/a) \\
&= \frac{1}{a} \mathcal{D}_{1/a} \circ \mathcal{F}(f)
\end{aligned} \tag{49}$$

(c) Consider $\mathcal{F} \circ D_x$ acting on f:

$$\begin{aligned}
\mathcal{F} \circ D_x(f) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathrm{d}f}{\mathrm{d}x} e^{-ikx} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left[f(x) e^{-ikx} \right]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} \, \mathrm{d}x \\
&= M_{ik} \circ \mathcal{F}(f)
\end{aligned} \tag{50}$$

where the boundary term vanishes because $f(x) \to 0$ as $x \to \pm\infty$ for f to be bounded.

Further consider $iD_k \circ \mathcal{F}(f)$:

$$\begin{aligned} iD_k \circ \mathcal{F}(f) &= \frac{i}{\sqrt{2\pi}} \frac{\mathrm{d}}{\mathrm{d}k} \left[\int_{\mathbb{R}} f(x) e^{-ikx} \, \mathrm{d}x \right] \\ &= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \frac{\mathrm{d}}{\mathrm{d}k} e^{-ikx} \, \mathrm{d}x \\ &= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)(-ix) e^{-ikx} \, \mathrm{d}x \\ &= M_x \circ \mathcal{F}(f) \end{aligned} \tag{51}$$

•

6 More Fourier transforms

(a) Consider the Fourier transform of the standard Gaussian:

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x^2/2+ikx)} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-((x+ik)^2-k^2)/2} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int_{\mathbb{R}} e^{-(x+ik)^2/2} \, \mathrm{d}x \\ &= e^{-k^2/2} \end{aligned} \tag{52}$$

so that the standard Gaussian is invariant under Fourier transform.

(b) We have $\mathcal{D}_a(f_a)(x) = \exp(-x^2/2)$ and:

$$\mathcal{F} \circ \mathcal{D}_a(f_a) = \frac{1}{a} \mathcal{D}_{1/a} \circ \mathcal{F}(f_a) \tag{53}$$

But we already established that $\mathcal{F}(f_a(ax)) = e^{-k^2/2}$, so that:

$$\mathcal{F}(f_a)(k) = \mathcal{D}_a \left(a e^{-k^2/2} \right) = a e^{-a^2k^2/2} \tag{54}$$

(c) We have $\mathcal{D}_a \circ T_{-c}(f_a)(x) = \exp(-x^2/2)$ and:

$$\mathcal{F} \circ \mathcal{D}_a \circ T_{-c}(f_a) = \frac{1}{a} \mathcal{D}_{1/a} \circ \mathcal{F} \circ T_{-c}(f_a) = \frac{1}{a} \mathcal{D}_{1/a} \circ E_c \circ \mathcal{F}(f_a) \tag{55}$$

Therefore:

$$\mathcal{F}(f_a)(k) = ae^{-a^2k^2/2+ick} \tag{56}$$

•

7 Hermite polynomials and Fourier transform

(a) We can rewrite the given Gaussian:

$$g(x) = \exp\left[-\left(\frac{x}{\sqrt{2}} - \sqrt{2}z\right)^2 + z^2\right] = \exp\left(z^2\right)\exp\left[-\frac{(x-\sqrt{2}z)^2}{2}\right] \tag{57}$$

so that its Fourier transform is:

$$\begin{aligned}\hat{g}(k) &= \exp\left(z^2\right)\mathcal{F}(f_{2,\sqrt{2}z}) \\ &= \exp\left(-\frac{k^2}{2} - 2ikz + z^2\right)\end{aligned} \tag{58}$$

where we have used the result from the previous problem.

(b) We know the generating function of the Hermite polynomials:

$$\exp\bigl(2yz - z^2\bigr) = \sum_{n=0}^{\infty} H_n(y)\frac{z^n}{n!} \tag{59}$$

Consider the Fourier transform of $h_n(x) = \exp(-x^2/2)H_n(x)$:

$$\begin{aligned}\mathcal{F}(h_n)(k) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-\frac{x^2}{2} - ikx\right)H_n(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(\frac{x^2}{2} - ikx\right)\exp\bigl(-x^2\bigr)H_n(x)\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left[\sum_j^{\infty} H_j(-ik/\sqrt{2})\frac{(x/\sqrt{2})^j}{j!}\right]\exp\bigl(-x^2\bigr)H_n(x)\,\mathrm{d}x \\ &= \sum_j^{\infty} H_j(-ik/\sqrt{2})\frac{1}{j!2^{j/2}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\bigl(-x^2\bigr)H_n(x)x^j\,\mathrm{d}x\end{aligned} \tag{60}$$

Consider the integral term. We can always write x^j as a linear combination of $H_n(x)$:

$$x^j = \sum_{r=0}^{j} a_r H_r(x) \tag{61}$$

so that:

$$\int_{\mathbb{R}} \exp\left(-x^2\right) H_n(x) x^j \, \mathrm{d}x = \sum_{r=0}^{j} a_r \int_{\mathbb{R}} \exp\left(-x^2\right) H_n(x) H_r(x) \, \mathrm{d}x = a_n \sqrt{\pi} 2^n n! \tag{62}$$

where $a_n = \langle H_n, x^j \rangle$.