Electromagnetism

Problem Set 5

Motion of Charged Particles & Electro-Magnetic Fields and Maxwell's Equations

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Motion of Charged Particles

0 Background

The Lorentz force is given by:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{1}$$

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1 Bainbridge mass spectrometer

(a)

(b) We need $Eq = Bvq$, or:

$$v = \frac{E}{B} = 500\,\mathrm{ms}^{-1} \tag{2}$$

(c)

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2 Charged particles moving in a constant magnetic field

(a) The equation of motion is:

$$m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B} \tag{3}$$

where we take $\mathbf{B} = (0, 0, B)^\intercal$ without loss of generality as we can always align the z-axis with the magnetic field.

Writing out the components:

$$\begin{aligned} m\ddot{x} &= q\dot{y}B \\ m\ddot{y} &= -q\dot{x}B \\ m\ddot{z} &= 0 \end{aligned} \tag{4}$$

Thus the speed in the z-direction is constant and in the x-y plane, $\ddot{x}^2 + \ddot{y}^2 = \frac{q^2B^2}{m^2}(\dot{x}^2 + \dot{y}^2)$, which is uniform circular motion. This demonstrates that the particle moves in a helical path.

(b) Let the angle between the magnetic field and the velocity be $\delta\theta$. The velocity component parallel to the magnetic field is $v_{\parallel} = v\cos\delta\theta$, while the perpendicular component determines the radius of the helix:

$$R = \frac{mv_{\perp}}{qB} = \frac{mv\sin\delta\theta}{qB} \tag{5}$$

For every revolution, the time taken is $T = 2\pi R/v_{\perp} = 2\pi m/qB$. In this time, the particle has travelled a distance d in the perpendicular direction given by:

$$D = v_{\parallel}T = \frac{2\pi m}{qB}v\cos\delta\theta \approx 2\pi\frac{mv}{Be} \tag{6}$$

where q is taken as e.

(c) The total number of revolutions is:

$$N = \frac{T}{2\pi m/qB} = \frac{100\,\mathrm{y}}{0.0357\,\mathrm{s}} = 8.8\times 10^{10} \tag{7}$$

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3 Magnetic quadrupole lens

(a) The equations of motion are:

$$\begin{aligned} m\ddot{x} &= -qAx\dot{z} \\ m\ddot{y} &= qAy\dot{z} \\ m\ddot{z} &= qA(x\dot{x} - y\dot{y}) \end{aligned} \tag{8}$$

Now let $\dot{x}$ and $\dot{y}$ be very small so that we ignore the z-direction acceleration and $\dot{z} = v$ is constant. We have $\ddot{x} = -\kappa^2 x$ and $\ddot{y} = \kappa^2 y$, where $\kappa^2 = qA/mv$ is a positive constant.

(b) Obviously, the equation of motion in x-direction gives a harmonic oscillator (focusing), while y-direction is unstable as any small perturbation will grow exponentially (defocusing).

(c) The period of the x-direction is $2\pi/\kappa$, so that after one quarter period, the distance travelled is:

$$D = \frac{\pi}{2\kappa}v = \frac{\pi}{2}\sqrt{\frac{mv}{|qA|}} \tag{9}$$

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Electro-Magnetic Fields and Maxwell's Equations

0 Background

The Maxwell equations are:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \end{aligned} \tag{10}$$

The first one is Gauss's law for electric field while the second one is Gauss's law for magnetic field, which implies the non-existence of magnetic monopoles. The third one is Faraday's law of induction, which implies that a changing electric field induces a magnetic field. The fourth one is Ampere's law, which implies that a current induces a magnetic field.

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1 Displacement current

(a) For a long wire, Ampere's law gives:

$$B = \mu_0 \frac{I}{2\pi r} \tag{11}$$

This cannot be used for a short wire because a current flowing in a short wire causes a build-up of charges at the ends, which causes a time-varying electric field that induces a magnetic field.

(b) Given a steady current I flowing in a wire of length $2b$, charges accumulate at the ends according to $q = It$ (the end where the current is flowing into). Let the current be flowing in the z-direction. By symmetry, the electric field created by the charges at the centre is:

$$\mathbf{E}(r) = -\frac{q}{2\pi\epsilon_0} \frac{b}{(r^2 + b^2)^{3/2}} \hat{z} \tag{12}$$

The modified Ampere's law reads:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{encl}} + \mu_0 \int_S \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{A} \tag{13}$$

The surface integral evaluates to:

$$\int_S \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{A} = -\int_0^r \frac{I}{2\pi} \frac{b}{(r^2+b^2)^{3/2}} 2\pi r \mathrm{d}r = -I\left(1 - \frac{b}{\sqrt{r^2+b^2}}\right) \tag{14}$$

Combining this with the first term, we have:

$$\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r} \frac{b}{\sqrt{r^2+b^2}} \hat{\phi} \tag{15}$$

which agrees with the result from Biot-Savart law.

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2 Electro-magnetic waves in vacuo

(a) In vacuum, we have the Maxwell equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{16}$$

Taking the curl of the curl terms, we can decouple the equations into two sets:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned} \tag{17}$$

Since the divergence terms are zero:

$$\begin{aligned} \nabla^2 \mathbf{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla^2 \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned} \tag{18}$$

Written in components form in Cartesian coordinates, we have:

$$\nabla^2 E_i = \mu_0 \epsilon_0 \frac{\partial^2 E_i}{\partial t^2}$$
$$\nabla^2 B_i = \mu_0 \epsilon_0 \frac{\partial^2 B_i}{\partial t^2} \tag{19}$$

which are the wave equations.

(b) We can check that the following solutions satisfy the wave equations:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$$
$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)} \tag{20}$$

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are some amplitudes, $\omega = ck$ and $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of the wave (speed of light in vacuum).

They are called plane waves for they do not have x- or y-dependence. Since the divergence of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are zero, we need $\tilde{E}_{0z} = \tilde{B}_{0z} = 0$. By Faraday's law, we need $\nabla \times \tilde{\mathbf{E}} = -\partial \tilde{\mathbf{B}}/\partial t$, which gives $-k\tilde{E}_{0y} = \omega \tilde{B}_{0x}$ and $k\tilde{E}_{0x} = \omega \tilde{B}_{0y}$. This implies:

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0 \tag{21}$$

Therefore, given an electric field of the form $\mathbf{E} = E_0(\sin(kz - \omega t), 0, 0)^\intercal$, the magnetic field is given by:

$$\mathbf{B} = \frac{1}{c} E_0 (0, \sin(kz - \omega t), 0)^\intercal \tag{22}$$

(c) The characteristic impedance of free space is just $E_0/B_0 = c$.

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3 Poynting vector of an electro-magnetic wave

(a) The Poynting vector in a plane electromagnetic wave is given by:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{1}{\mu_0 c} E_0^2 \sin^2(kz - \omega t + \phi) \hat{z} = c\epsilon_0 E_0^2 \sin^2(kz - \omega t + \phi) \hat{z} \tag{23}$$

the average magnitude of which is $c\epsilon_0 E_0^2/2$.

(b) The distance from the sun to the earth is cT. The total radiative power is distributed over the sphere of radius cT

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4 Poynting vector for a long resistive rod

The electric field in the rod is given by:

$$\mathbf{E} = \frac{V}{l}\hat{z} = \frac{IR}{l}\hat{z} \tag{24}$$

where the z-direction is defined as the direction of the current.

The magnetic field is given by:

$$\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r}\hat{\phi} \tag{25}$$

The Poynting vector (outside or on the surface of the rod) is thus given by:

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = -\frac{I^2 R}{2\pi r l}\hat{r} \tag{26}$$

which points radially inwards.

Integrating the Poynting vector over the surface of the rod, we get:

$$-\int_S \mathbf{S} \cdot \mathrm{d}\mathbf{A} = I^2 R \tag{27}$$

which is the power radiated by the rod as expected.

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