Problem Set 2

Dynamics of Fluids

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1 Poiseuille flow in a cylindrical tube

(a) For the current problem, we the Navier-Stokes equation:

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \tag{1}$$

We only have $\mathbf{v} = v_z(r)\hat{z}$ and consider steady-state flow, so the equation reduces to:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\nabla^2\mathbf{v})_z \tag{2}$$

Note the vector calculus identity:

$$\begin{aligned}\nabla^2\mathbf{v} &= \nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}) \\ &= -\nabla\times(\nabla\times\mathbf{v}) \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\hat{z}\end{aligned} \tag{3}$$

Letting $\partial p/\partial z = \Delta p/L$ and solving the equation, we have:

$$v_z(r) = \frac{\Delta p}{L}\frac{1}{4\eta}(R^2 - r^2) \tag{4}$$

where we have used the boundary condition $v_z(R) = 0$.

(b) The flow rate across a cross section of the tube is given by:

$$\begin{aligned}Q &= \oint v_z \mathrm{d}A \\ &= \int_0^R \frac{\Delta p}{L}\frac{1}{4\eta}(R^2 - r^2)2\pi r\mathrm{d}r \\ &\propto R^4\Delta p\end{aligned} \tag{5}$$

which implies $\Delta p \propto QR^{-4}$.

If the flow rate reduces to half $Q \to Q/2$, the radius should change according to $R \to 2^{-1/4}R$ with constant pressure difference; the pressure should decrease according to $\Delta p \to 2^{-1}\Delta p$ with constant radius.

(c) The viscous friction force is given by:

$$\begin{aligned} f_z &= \frac{\mathrm{d}\sigma_{zr}}{\mathrm{d}r} \\ &= \eta\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial v_z}{\partial r}\right) \\ &= \eta\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{\Delta p}{L}\frac{1}{2\eta}r\right) \\ &= -\frac{\Delta p}{2L} \end{aligned} \tag{6}$$

The energy loss due to viscous friction is given by the integral:

$$\begin{aligned} D &= \int_V \sigma_{ij}\frac{\partial v_i}{\partial x_j}\mathrm{d}V \\ &= \int_0^L\int_0^{2\pi}\int_0^R \sigma_{zr}\frac{\partial v_z}{\partial r} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\ &= \eta\left(\frac{\Delta p}{L}\frac{1}{2\eta}\right)^2\frac{R^4}{4}2\pi L \\ &= 2\pi L v_{\mathrm{max}}^2\eta \end{aligned} \tag{7}$$

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2 Couette ow between rotating cylinders

(a) Since there is longitudinal symmetry, we can write the velocity field as $\mathbf{v} = v_r(r)\hat{r} + v_\theta(r)\hat{\theta}$. Neglecting any external pressure, the Navier-Stokes equation reduces to $\nabla^2\mathbf{v} = 0$. This gives:

$$\begin{aligned} \nabla^2\mathbf{v} &= -\nabla\times(\nabla\times\mathbf{v}) \\ &= \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]\hat{\theta} \\ &= \mathbf{0} \end{aligned} \tag{8}$$

This can be solved by integrating the equation to yield $v_\theta(r) = C_1 r + C_2/r$ and $\Omega = C_1 + C_2/r^2$. The boundary conditions are $v_\theta(R_{1,2})/R_{1,2} = \Omega_{1,2}$, which gives:

$$\begin{aligned} C_1 &= \frac{\Omega_1 - \Omega_2}{1/R_1^2 - 1/R_2^2} \\ C_2 &= \frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_1^2 - R_2^2} \end{aligned} \tag{9}$$

For the case $\Omega_1 = \Omega_2 = \Omega$, we have $C_1 = 0$ and $C_2 = \Omega$. This is just co-rotation with the two cylinders rotating at the same angular velocity. For the case $\Omega_2 = 0$ and $R_2 \to \infty$, we have

$C_1 = \Omega_1 R_1^2$ and $C_2 = 0$. This is just the case of a rotating cylinder in an otherwise stationary fluid. For $R_{1,2} \gg R_2 - R_1$

(b) The viscous friction force is given by:

$$\begin{aligned} f_\theta &= \frac{\mathrm{d}\sigma_{\theta r}}{\mathrm{d}r} \\ &= \eta \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial v_\theta}{\partial r}\right) \\ &= \frac{2C_2\eta}{r^3} \end{aligned} \tag{10}$$

This is a force per unit area, so the viscous torque on either cylinder is given by:

$$\begin{aligned} \tau &= R\int f_\theta \,\mathrm{d}A \\ &= RL\int f_\theta R\,\mathrm{d}\theta \\ &= 2\pi R^2 L f_\theta(R) \\ &= 4\pi C_2 \eta \frac{L}{R} \end{aligned} \tag{11}$$

where $R = R_1$ or R_2.

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3 Motion of a sphere in a very viscous fluid: Stokes law

(a) Given the velocity field $\mathbf{v} = \mathbf{v}_r \hat{r} + \mathbf{v}_\theta \hat{\theta}$, where:

$$\begin{aligned} v_r &= \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \\ v_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \end{aligned} \tag{12}$$

we have the curl of the velocity field:

$$\begin{aligned} \omega_\phi &= \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right] \\ &= \frac{1}{r\sin\theta}\left[-\frac{\partial^2\psi}{\partial x^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\theta}\frac{\partial\psi}{\partial\theta}\right)\right] \\ &= \frac{1}{r\sin\theta}\left[-\mathcal{O}_1(\psi) + \mathcal{O}_2(\psi)\right] \end{aligned} \tag{13}$$

The Navier-Stokes equation gives $0 = -\nabla p + \eta \nabla^2 \mathbf{v}$. But:

$$\nabla^2 \mathbf{v} = -\nabla \times (\nabla \times \mathbf{v}) = -\nabla \times \omega \tag{14}$$

which gives:

$$\nabla p = -\eta \nabla \times \omega \tag{15}$$

This implies that the curl of this equation is zero. Consider the ϕ component of $\nabla \times (\nabla \times \omega)$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial r}\left[-\frac{\partial(r\omega_\phi)}{\partial r} - \frac{\partial}{\partial \theta}\left(\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\omega_\phi)}{\partial\theta}\right)\right] \\ &= \mathcal{O}_1(r\omega_\phi) - \frac{1}{\sin\theta}\mathcal{O}_2(\sin\theta\omega_\phi) \end{aligned} \tag{16}$$

But given the previous results, we have:

$$\mathcal{O}_1(r\omega_\phi) = \frac{1}{\sin\theta}\left[-\mathcal{O}_1\mathcal{O}_1(\psi) + \mathcal{O}_1\mathcal{O}_2(\psi)\right] \tag{17}$$

and:

$$\mathcal{O}_2(\sin\theta\omega_\phi) = \frac{1}{r}\left[-\mathcal{O}_2\mathcal{O}_1(\psi) + \mathcal{O}_2\mathcal{O}_2(\psi)\right] \tag{18}$$

which eventually gives:

$$\left[\mathcal{O}_1 + \mathcal{O}_2\right]^2 \psi = 0 \tag{19}$$

Consider a solution of the form $\psi(r) = f(r)\sin^2\theta$. We have:

$$(\mathcal{O}_1 + \mathcal{O}_2)f = \sin^2\theta(n^2 - n - 2)r^{n-2} \tag{20}$$

and thus:

$$(\mathcal{O}_1 + \mathcal{O}_2)^2 f = \sin^2\theta(n^2 - n - 2)^2(n^2 - 5n + 4)r^{n-4} \tag{21}$$

This means that $n = -1, 1, 2, 4$ are the only possible values for n, meaning:

$$\psi = \sin^2\theta\left(Ar^{-1} + Br + Cr^2 + Dr^4\right) \tag{22}$$

For finite velocity, $D = 0$, and we have, upon subsituting $v_r(R) = v_\theta(R) = 0$:

$$\begin{aligned} v_r &= U\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)\cos\theta \\ v_\theta &= -U\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)\sin\theta \end{aligned} \tag{23}$$

(b) For an incompressible fluid, the stress tensor is given by:

$$\sigma_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{24}$$

The only non-zero component is:

$$\sigma_{r\theta} = -\eta U \sin\theta \frac{3}{2R} \tag{25}$$

(c) Balancing the viscous force with the gravitational force, we have:

$$U = \frac{1}{9}\frac{\rho g R^2}{\eta} = 6.48 \times 10^{-3}\,\mathrm{ms}^{-1} \tag{26}$$

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4 Coriolis force and vorticity

(a) In the fixed frame, the Navier-Stokes equation is given by:

$$\left(\frac{\mathrm{d}\mathbf{v}_f}{\mathrm{d}t}\right)_f = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}_f + \mathbf{g} \tag{27}$$

In the rotating frame, the Navier-Stokes equation is given by:

$$\left(\frac{\mathrm{d}\mathbf{v}_r}{\mathrm{d}t}\right)_r + 2\boldsymbol{\Omega}\times\mathbf{v}_r + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}_r + \mathbf{g} \tag{28}$$

Consider the third term on the left-hand side of the equation. We have:

$$\begin{aligned}\nabla(\boldsymbol{\Omega}\times\mathbf{r})^2 &= 2(\boldsymbol{\Omega}\times\mathbf{r})\cdot\nabla(\boldsymbol{\Omega}\times\mathbf{r}) + (\boldsymbol{\Omega}\times\mathbf{r})\times[\nabla(\boldsymbol{\Omega}\times\mathbf{r})] \\ &= (\boldsymbol{\Omega}\times\mathbf{r})\times(2\boldsymbol{\Omega})\end{aligned} \tag{29}$$

which gives:

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = -\frac{1}{2}\nabla(\boldsymbol{\Omega}\times\mathbf{r})^2 \tag{30}$$

so that the centrifugal force can be subsumed into the gravitational force.

(b) For inviscid flow, we have:

$$\left(\frac{\mathrm{d}\mathbf{v}_r}{\mathrm{d}t}\right)_r = -2\boldsymbol{\Omega}\times\mathbf{v}_r - \frac{1}{\rho}\nabla p + \mathbf{g}' \tag{31}$$

Consider the rate of change of circulation:

$$\begin{aligned}\frac{\partial\Gamma}{\partial t} &= \oint \frac{\partial\mathbf{v}_r}{\partial t}\cdot\mathrm{d}\mathbf{l} \\ &= \oint\left[-2\boldsymbol{\Omega}\times\mathbf{v}_r - \frac{1}{\rho}\nabla p + \mathbf{g}' - (\mathbf{v}_r\cdot\nabla)\mathbf{v}_r\right]\cdot\mathrm{d}\mathbf{l}\end{aligned} \tag{32}$$

The curl of a gradient is zero, so the pressure term does not contribute to the circulation. $(\mathbf{v}_r \cdot \nabla)\mathbf{v}_r$ is along the direction of the flow, so its curl vanishes too. The contributing term is the Coriolis force.

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5 Rankine vortex

(a) Given the velocity field:

$$v_\theta = \begin{cases} \Omega r & r < R \\ \frac{\Omega R^2}{r} & r > R \end{cases} \tag{33}$$

we have the vorticity:

$$\omega_z = \begin{cases} 2\Omega & r < R \\ 0 & r > R \end{cases} \tag{34}$$

(b) By Bernoulli's equation and ignoring gravity, we have $p + \frac{1}{2}\rho v^2 = \text{const}$. This gives:

$$p_\infty = p(r) + \frac{1}{2}\rho v_\theta^2 = p_0 \tag{35}$$

Thus, the pressure is given by:

$$p(r) = \begin{cases} p_\infty - \frac{1}{2}\rho\Omega^2 r^2 & r < R \\ p_\infty - \frac{1}{2}\rho\frac{\Omega^2 R^4}{r^2} & r > R \end{cases} \tag{36}$$

(c)