Vectors & Matrices

Problem Set 3

Scalar Products and Determinants

Xin, Wenkang

November 15, 2022

Scalar products and determinants

1

(a)

$$\det(A) = i(1) + i(-1) = 0 \tag{1}$$

The matrix is antisymmetric, singular and Hermitian.

(b)

$$\det(A) = \frac{1}{\sqrt{8}}\left[2(\sqrt{2}+\sqrt{18}) + 2(\sqrt{18}+\sqrt{2})\right] = 8 \tag{2}$$

$$\det(A) = \frac{1}{\sqrt{8}^3}\left[2(\sqrt{2}+\sqrt{18}) + 2(\sqrt{18}+\sqrt{2})\right] = 1 \tag{3}$$

The matrix is real and orthogonal/unitary/Hermitian.

•

2

We first have:

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{1}{\sqrt{2}}(1,1,0)^\intercal \tag{4}$$

Then $\mathbf{v}_2$ minus its component along $\mathbf{e}_1$ is:

$$\mathbf{v}_2' = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{e}_1)\mathbf{e}_1 = (1/2, -1/2, 2)^\intercal \tag{5}$$

Then:

$$\mathbf{e}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|} = \frac{1}{3\sqrt{2}}(1,-1,4)^\intercal \tag{6}$$

And:

$$\begin{aligned} \mathbf{v}_3' &= \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{v}_3 \cdot \mathbf{e}_2)\mathbf{e}_2 = (-2/3, 2/3, 1/3)^\intercal \\ \mathbf{e}_3 &= \frac{\mathbf{v}_3}{|\mathbf{v}_3|} = \frac{1}{3}(-2,2,1)^\intercal \end{aligned} \tag{7}$$

Note that:

$$
\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{e}_2 &= \frac{1}{\sqrt{2}}\frac{1}{3\sqrt{2}}(1,1,0)^\intercal \cdot (1,-1,4)^\intercal = 0 \\
\mathbf{e}_1 \cdot \mathbf{e}_3 &= \frac{1}{\sqrt{2}}\frac{1}{3}(1,1,0)^\intercal \cdot (-2,2,1)^\intercal = 0 \\
\mathbf{e}_2 \cdot \mathbf{e}_3 &= \frac{1}{3\sqrt{2}}\frac{1}{3}(1,-1,4)^\intercal \cdot (-2,2,1)^\intercal = 0
\end{aligned} \tag{8}
$$

•

3

(a) This is because $A^{-1} = \operatorname{adj}(A)/\det(A)$ and if $\det(A) = 0$, then A^{-1} is not defined.

(b)

$$\det(A) = a(ba - 1) - (a + a) = a(ab - 3) \tag{9}$$

Thus, the matrix is not invertible if $a = 0$ or $ab = 3$.

•

4

(a) Note that:

$$\langle \mathbf{w}, \mathbf{v}_a \rangle = \sum_{i=1}^{n} w_i v_{ai} = \sum_{i=1}^{n} \det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{e}_i) v_{ai} \tag{10}$$

But the expansion of $\mathbf{v}_a$ is just $\mathbf{v}_a = \sum_{i=1}^{n} v_{ai}\mathbf{e}_i$. By the multilinear property of the determinant:

$$\langle \mathbf{w}, \mathbf{v}_a \rangle = \det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_a) = 0 \tag{11}$$

for any $\mathbf{v}_a$.

(b)

$$\begin{aligned}
\det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{n}) &= \frac{1}{|\mathbf{w}|} \det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{w}) \\
&= \frac{1}{|\mathbf{w}|} \sum_{i=1}^{n} \det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{e}_i) w_i \\
&= \frac{1}{|\mathbf{w}|} \sum_{i=1}^{n} w_i^2 \\
&= |\mathbf{w}|
\end{aligned} \tag{12}$$

(c)

$$w_i = \det(\mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \mathbf{e}_i) \tag{13}$$

which is zero for $i \neq n$ and unity for $i = n$.

Thus, $\mathbf{w} = \mathbf{e}_n$.

(d) Explicitly writing the components of $\mathbf{w}$:

$$w_1 = \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1) = \det \begin{pmatrix} x_1 & x_2 & 1 \\ y_1 & y_2 & 0 \\ z_1 & z_2 & 0 \end{pmatrix} = y_1 z_2 - y_2 z_1 \tag{14}$$

$$w_2 = \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2) = \det \begin{pmatrix} x_1 & x_2 & 0 \\ y_1 & y_2 & 1 \\ z_1 & z_2 & 0 \end{pmatrix} = z_1 x_2 - z_2 x_1 \tag{15}$$

$$w_3 = \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3) = \det \begin{pmatrix} x_1 & x_2 & 0 \\ y_1 & y_2 & 0 \\ z_1 & z_2 & 1 \end{pmatrix} = x_1 y_2 - x_2 y_1 \tag{16}$$

Hence, $\mathbf{w} = \mathbf{v}_1 \times \mathbf{v}_2$.

•

5

(a)

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{2} \begin{pmatrix} 1 & 1 & -2 \\ -7 & 1 & 4 \\ 5 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1 \\ -7/2 & 1/2 & 2 \\ 5/2 & -1/2 & -1 \end{pmatrix} \tag{17}$$

$$X = A^{-1}B = \begin{pmatrix} 1/2 & 1/2 & -1 \\ -7/2 & 1/2 & 2 \\ 5/2 & -1/2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 6 \end{pmatrix} = (-3, 7, -3)^\intercal \tag{18}$$

(b)

$$x_1 = \det \begin{pmatrix} 2 & 2 & 3 \\ 4 & 4 & 5 \\ 6 & 3 & 4 \end{pmatrix} / \det(A) = -3 \tag{19}$$

$$x_2 = \det \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 6 & 4 \end{pmatrix} / \det(A) = 7 \tag{20}$$

$$x_2 = \det \begin{pmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \\ 1 & 4 & 6 \end{pmatrix} / \det(A) = -3 \tag{21}$$

(c)

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 3 & 4 & 5 & 4 \\ 1 & 3 & 4 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4/3 & 5/3 & 4/3 \\ 0 & 1 & 7/5 & 14/5 \\ 0 & 0 & 1 & -3 \end{array}\right) \tag{22}$$

Therefore:

$$X = (-3, 7, -3)^\intercal \tag{23}$$

The row reduction method is least computationally demanding as only additional and subtraction operations are required.

•

6

(a) Note that:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x) g(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2} g(x) f(x) \, \mathrm{d}x = \langle g, f \rangle \tag{24}$$

$$\langle f, \alpha g + \beta h \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x) \left[\alpha g(x) + \beta h(x) \right] \, \mathrm{d}x = \alpha \langle f, g \rangle + \beta \langle f, h \rangle \tag{25}$$

$$\langle f, f \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x) f(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2} f(x)^2 \, \mathrm{d}x \geq 0 \tag{26}$$

$$\langle f, f\rangle = 0 \implies f(x) = 0 \quad \forall x \tag{27}$$

Therefore, $\langle f, g\rangle$ is a scalar product.

(b)

$$\langle p_0, p_1\rangle = \int_{-\infty}^{\infty} e^{-x^2} \frac{2}{n_0 n_1} \,\mathrm{d}x = 0 \tag{28}$$

as this is an odd function.

Further:

$$\langle p_0, p_2\rangle = \int_{-\infty}^{\infty} e^{-x^2} \frac{2}{n_0 n_2}(4x^2 - 2) \,\mathrm{d}x = \frac{2}{n_0 n_2} \left[-e^{x^2} x\right]_{-\infty}^{\infty} = 0 \tag{29}$$

and:

$$\langle p_1, p_2\rangle = \int_{-\infty}^{\infty} e^{-x^2} \frac{2}{n_1 n_2} 2x(4x^2 - 2) \,\mathrm{d}x = 0 \tag{30}$$

as this is an odd function.

Hence, p_i are orthogonal under the defined scalar product.

(c)

$$\langle p_0, p_0\rangle = \int_{-\infty}^{\infty} e^{-x^2} \frac{1}{n_0^2} \,\mathrm{d}x = \frac{1}{n_0^2}\sqrt{\pi} = 1 \tag{31}$$

so $n_0 = \pi^{1/4}$.

$$\langle p_1, p_1\rangle = \int_{-\infty}^{\infty} e^{-x^2} \frac{4}{n_1^2} x^2 \,\mathrm{d}x = \frac{2}{n_1^2}\sqrt{\pi} = 1 \tag{32}$$

so $n_1 = \sqrt{2}\pi^{1/4}$.

$$\langle p_2, p_2\rangle = \int_{-\infty}^{\infty} e^{-x^2} \frac{1}{n_2^2}(4x^2 - 2)^2 \,\mathrm{d}x = \frac{8}{n_2^2}\sqrt{\pi} = 1 \tag{33}$$

so $n_2 = 2\sqrt{2}\pi^{1/4}$.

•

7

(a) Consider the equation $\mathbf{0} = \sum_i^k a_i \mathbf{w}_i$. Taking a scalar product of both sides with an arbitrary vector $\mathbf{v}$ gives:

$$\begin{aligned} \langle \mathbf{v}, \mathbf{0} \rangle &= \left\langle \mathbf{v}, \sum_i^k a_i \mathbf{w}_i \right\rangle \\ \sum_i^k a_i \langle \mathbf{v}, \mathbf{w}_i \rangle &= 0 \end{aligned} \tag{34}$$

As $\mathbf{v}$ is arbitrary, we can choose $\mathbf{v} = \mathbf{w}_j$ for some j. As $\mathbf{w}_i$ are orthogonal and non-zero:

$$\sum_i^k a_i \langle \mathbf{w}_j, \mathbf{w}_i \rangle = a_j \langle \mathbf{w}_j, \mathbf{w}_j \rangle = 0 \tag{35}$$

where $\langle \mathbf{w}_j, \mathbf{w}_j \rangle > 0$.

This implies $a_j = 0$ for all j. There is therefore only the trivial solution to the equation, and vectors $\mathbf{w}_i$ are linearly independent.

(b)

$$\langle \mathbf{e}_i, \mathbf{v} \rangle = \langle \mathbf{e}_i, v_j \mathbf{e}_j \rangle = v_j \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} v_j = v_i \tag{36}$$

(c)

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, v_i \mathbf{e}_i \rangle = v_i \langle \mathbf{u}, \mathbf{e}_i \rangle = \langle \mathbf{u}, \mathbf{e}_i \rangle \langle \mathbf{e}_i, \mathbf{v} \rangle \tag{37}$$

(d)

$$\left\langle \mathbf{e}_i, \mathbf{e}'_j \right\rangle = \langle \mathbf{e}_i, U_{kj} \mathbf{e}_k \rangle = U_{kj} \delta_{ik} = U_{ij} \tag{38}$$

Consider the product $U^\dagger U$:

$$\left(U^\dagger U\right)_{ij} = U^\dagger_{ik} U_{kj} = U^*_{ki} U_{kj} = \langle \mathbf{e}_k, \mathbf{e}'_i \rangle^* \left\langle \mathbf{e}_k, \mathbf{e}'_j \right\rangle = \langle \mathbf{e}'_i, \mathbf{e}_k \rangle \left\langle \mathbf{e}_k, \mathbf{e}'_j \right\rangle = \left\langle \mathbf{e}'_i, \mathbf{e}'_j \right\rangle = \delta_{ij} \tag{39}$$

This shows that U is unitary.

•

8

(a) We have:

$$\langle R\mathbf{v}, R\mathbf{w} \rangle = (R\mathbf{v})^\intercal R\mathbf{w} = \mathbf{v}^\intercal R^\intercal R \mathbf{w} = \mathbf{v}^\intercal \mathbf{w} \tag{40}$$

For this equation to be valid, $R^\intercal R = I$. Further note that:

$$\det(R^\intercal R) = \det(R)^2 = \det(I) = 1 \tag{41}$$

so $\det(R) = \pm 1$.

(b) Let the matrix have the form:

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{42}$$

We have:

$$\begin{aligned} R^{\intercal}R &= \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ a^2+c^2 &= b^2+d^2 = 1 \\ ab+cd &= 0 \end{aligned} \tag{43}$$

We immediately have $c = \pm\sqrt{1-a^2}$ and $d = \pm\sqrt{1-b^2}$ from the first equation. Then from the second equation:

$$\begin{aligned} ab+cd = ab \pm \sqrt{1-a^2}\sqrt{1-b^2} &= 0 \\ a^2b^2 &= (1-a^2)(1-b^2) \\ a^2+b^2 &= 1 \end{aligned} \tag{44}$$

Then we can rewrite R:

$$R = \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \sqrt{1-a^2} & \mp a \end{pmatrix} \tag{45}$$

Now we make the substitution $a = \cos\phi$, and we have:

$$R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tag{46}$$

The lower sign ($-\sin\phi$ and $+\cos\phi$) corresponds to two-dimensional rotations; the other sign represents the case where $\det(R) = -1$, which corresponds to a rotation followed by a reflection.

(c)

$$\begin{aligned} R(\phi_1)R(\phi_2) &= \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix} \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 & -\sin\phi_1\cos\phi_2 - \cos\phi_1\sin\phi_2 \\ \sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2 & \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2 \end{pmatrix} \\ &= R(\phi_1+\phi_2) \end{aligned} \tag{47}$$

(d) Let $z = Ae^{i\alpha}$, then $z' = Ae^{i(\alpha+\phi)}$.

•

9

(a) By analogy and symmetry, we have:

$$R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{48}$$

and:

$$R_2(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \tag{49}$$

Then:

$$\begin{aligned} R &= R_1(\alpha_3)R_2(-\alpha_2)R_3(\alpha_1) \\ &= \begin{pmatrix} \cos\alpha_2\cos\alpha_3 & \sin\alpha_1\sin\alpha_2\cos\alpha_3 - \cos\alpha_1\sin\alpha_3 & \cos\alpha_1\sin\alpha_2\cos\alpha_3 + \sin\alpha_1\sin\alpha_3 \\ \cos\alpha_2\sin\alpha_3 & \sin\alpha_1\sin\alpha_2\sin\alpha_3 + \cos\alpha_1\cos\alpha_3 & \cos\alpha_1\sin\alpha_2\sin\alpha_3 - \sin\alpha_1\cos\alpha_3 \\ -\sin\alpha_2 & \sin\alpha_1\cos\alpha_2 & \cos\alpha_1\cos\alpha_2 \end{pmatrix} \end{aligned} \tag{50}$$

(b) For small α_i:

$$R = \begin{pmatrix} 1 & -\alpha_3 & \alpha_2 \\ 1 & 1 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{51}$$

(c)

$$\delta\mathbf{x} = R\mathbf{x} - \mathbf{x} \approx \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha_2 x_3 - \alpha_3 x_2 \\ \alpha_3 x_1 - \alpha_1 x_3 \\ \alpha_1 x_2 - \alpha_2 x_1 \end{pmatrix} = \alpha \times \mathbf{x} \tag{52}$$

•

10

(a) This is because if $\mathbf{w} = I$, then $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{0}$ for all $\mathbf{v}$.

(b) We have:

$$\langle \Lambda\mathbf{v}, \Lambda\mathbf{w} \rangle = (\Lambda\mathbf{v})^\intercal \eta \Lambda\mathbf{w} = \mathbf{v}^\intercal (\Lambda^\intercal \eta \Lambda)\mathbf{w} = \mathbf{v}^\intercal \eta \mathbf{w} \tag{53}$$

Then:

$$\Lambda^\intercal \eta \Lambda = \eta \tag{54}$$

Since $\det(\eta) = 1$:

$$\det(\Lambda^\intercal \eta \Lambda) = \det(\Lambda)^2 = 1 \tag{55}$$

Thus, $\det\{\Lambda\} = \pm 1$.

(c) Suppose that Λ has the form:

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{56}$$

From $\Lambda^\intercal \eta \Lambda = \eta$ and $\det\{\Lambda\} = 1$, we have:

$$\begin{gathered} a^2 - c^2 = d^2 - b^2 = 1 \\ ab - cd = 0 \\ ac - bd = 1 \end{gathered} \tag{57}$$

We immediately have $c = \pm\sqrt{a^2 - 1}$ and $d = \pm\sqrt{b^2 + 1}$ from the first equation. Then from the second equation:

$$\begin{aligned}
ab - cd &= ab \pm \sqrt{a^2-1}\sqrt{b^2+1} = 0 \\
a^2b^2 &= (a^2-1)(b^2+1) \\
a^2 - b^2 &= 1
\end{aligned} \tag{58}$$

Then we can rewrite Λ:

$$\Lambda = \begin{pmatrix} a & \sqrt{a^2-1} \\ \sqrt{a^2-1} & a \end{pmatrix} \tag{59}$$

Now we make the substitution $a = \cosh\phi$, and we have:

$$\Lambda = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} = \sqrt{\frac{1}{1-\beta^2}} \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \tag{60}$$

where $\beta \equiv \tanh\phi$.

(d) We have:

$$\begin{aligned}
\Lambda(\phi_1)\Lambda(\phi_2) &= \begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} \\
&= \begin{pmatrix} \cosh\phi_1\cosh\phi_2 - \sinh\phi_1\sinh\phi_2 & \cosh\phi_1\sinh\phi_2 + \sinh\phi_1\cosh\phi_2 \\ \cosh\phi_1\sinh\phi_2 + \sinh\phi_1\cosh\phi_2 & \cosh\phi_1\cosh\phi_2 + \sinh\phi_1\sinh\phi_2 \end{pmatrix} \\
&= \Lambda(\phi_1+\phi_2)
\end{aligned} \tag{61}$$

This means:

$$\begin{aligned}
\Lambda(\beta_1)\Lambda(\beta_2) &= \Lambda(\beta) \\
\frac{1}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}} \begin{pmatrix} 1+\beta_1\beta_2 & \beta_1+\beta_2 \\ \beta_1+\beta_2 & 1+\beta_1\beta_2 \end{pmatrix} &= \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}
\end{aligned} \tag{62}$$

Solving this leads to the relationship:

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \tag{63}$$

which is the angle sum formula for hyperbolic tangent.

•