Classical Mechanics

Problem Set 1

Introductory Problems & Collisions in One Dimension

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Introductory Problems

1 Vectors in two dimensions

(a) The vector $\mathbf{r}$ makes an angle $\tan^{-1}(3/4)$ with the x-axis. Therefore the unit vector $\hat{\mathbf{u}}$ can make an angle $\theta = \tan^{-1}(3/4) + 20°$ with the x-axis so that:

$$\hat{\mathbf{u}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} = 0.547\mathbf{i} + 0.837\mathbf{j} \tag{1}$$

(b) Since $\hat{\mathbf{u}}$ and $\mathbf{r}$ are linearly independent, they span the whole $\mathbb{R}^2$.

(c) The direction of the particle's velocity changes while its magnitude stays constant, so the motion is accelerated.

(d)

$$\theta(t) = \frac{d(t)}{r} + \theta_0 = \frac{vt}{r} + \tan^{-1}(3/4) = 3t + 0.64\,\text{rad} \tag{2}$$

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2 Dimensional analysis

(a) Suppose a power law relationship between T, l, m, g and θ_0:

$$\begin{aligned} T &= kf(\theta_0)l^a m^b g^c \\ T &= L^a M^b (LT^{-2})^c \end{aligned} \tag{3}$$

where k is a dimensionless constant and $f(\theta_0)$ is an arbitrary function of θ_0, which is also dimensionless.

Solving the resulting system of linear equations yields $a = 1/2$, $b = 0$ and $c = -1/2$. Thus:

$$T = kf(\theta_0)\sqrt{\frac{l}{g}} \tag{4}$$

(b) Suppose a power law relationship between T, M, G and R. m is ignored because the system is inherently kinematic so only the acceleration of the satellite is of interest. We have:

$$\begin{aligned} T &= kM^a G^b R^c \\ T &= M^a (L^3 M^{-1} T^{-2})^b L^c \end{aligned} \tag{5}$$

Solving the resulting system of linear equations yields $a = b = -1/2$ and $c = 3/2$. Thus:

$$T^2 = k\frac{R^3}{GM} \tag{6}$$

(c) For vacuum permittivity:

$$\begin{aligned} F_E &= \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2} \\ [\epsilon_0] &= \frac{C^2}{L^2}\frac{1}{MLT^{-2}} = C^2T^2L^{-1}M^{-1} = I^2T^4L^{-3}M^{-1} \end{aligned} \tag{7}$$

For vacuum permeability:

$$\begin{aligned} F_M &= \frac{\mu_0}{4\pi}\frac{I_1I_2}{d}l \\ [\mu_0] &= MLT^{-2}\frac{L}{I^2L} = A^{-2}T^{-2}LM \end{aligned} \tag{8}$$

Thus:

$$\left[\sqrt{\frac{1}{\mu_0\epsilon_0}}\right] = \left(T^2L^{-2}\right)^{-1/2} = LT^{-1} \tag{9}$$

$(1/\mu_0\epsilon_0)^{1/2}$ is by definition the speed of light in vacuum based on wave equations for electromagnetic waves derived from Maxwell's equations.

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3 Energy conservation

(a) By conservation of energy:

$$\begin{aligned} \frac{1}{2}mv^2 &= mgl \\ v &= \sqrt{2gl} \end{aligned} \tag{10}$$

(b)

$$T = mg + m\frac{v^2}{l} = 3mg \tag{11}$$

(c) As the collision is inelastic:

$$v = v_{2m} - v_m \tag{12}$$

By conservation of momentum:

$$mv = 2mv_{2m} + mv_m \tag{13}$$

Solving the two linear equations yields $v_{2m} = 2v/3 = 2\sqrt{2gl}/3$ and $v_m = -v/3 = -\sqrt{2gl}/3$.

(d) By conservation of energy:

$$\begin{aligned} \frac{1}{2}mv_m^2 = mgh = mgl\cos\theta \\ \theta = \cos^{-1}\left(\frac{v_m^2}{2gl}\right) = 27.3° \end{aligned} \tag{14}$$

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4 The simple harmonic oscillator

(b)

$$U(x) - U(0) = \frac{1}{2}k\left[x_{\max}\cos\left(\omega_0 t + \phi\right)\right]^2 = \frac{1}{2}\omega_0^2 m\left[x_{\max}\cos\left(\omega_0 t + \phi\right)\right]^2 \tag{15}$$

The change is always positive, meaning that as long as the particle deviates from the origin, it possesses more potential energy than at the origin.

(c) First note that $v = \dot{x} = -\omega_0 x_{\max}\sin\left(\omega_0 t + \phi\right)$

$$\begin{aligned} E &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}m\omega_0^2 x_{\max}^2\sin^2\left(\omega_0 t + \phi\right) + \frac{1}{2}\omega_0^2 m x_{\max}^2\cos^2\left(\omega_0 t + \phi\right) \\ &= \frac{1}{2}m\omega_0^2 x_{\max}^2 = \frac{1}{2}mv_{\max}^2 \end{aligned} \tag{16}$$

which is a constant.

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5 The potential energy function

(a) By definition, $F = -\mathrm{d}U/\mathrm{d}x$:

$$F = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{U_0 a^2}{x^2+a^2}\right) = U_0 a^2 \frac{2x}{(x^2+a^2)^2} \tag{17}$$

(c) F is always repulsive, as F is positive in the +x-axis and negative in the -x-axis.

(d) By conservation of energy:

$$\begin{aligned} U(0) &= U(\infty) + \frac{1}{2}mv^2 = \frac{1}{2}mv^2 \\ v &= \sqrt{\frac{2U(0)}{m}} = \sqrt{\frac{2U_0}{m}} \end{aligned} \tag{18}$$

(e) For the particle to reach $+\infty$, it just needs to overcome the potential barrier at $x = 0$, i.e., have a speed slightly larger than zero at $x = 0$. By conservation of energy:

$$\begin{aligned} \frac{1}{2}mv_0^2 &= U(0) \\ v_0 &= \sqrt{\frac{2U_0}{m}} \end{aligned} \tag{19}$$

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6 A two-particle problem in 1-D - the centre of mass system

(a) By Newton's second and third law:

$$\begin{aligned} m_1\ddot{x}_1 &= F_1 + F_{\text{int}} \\ m_2\ddot{x}_2 &= F_2 - F_{\text{int}} \end{aligned} \tag{20}$$

Adding the equations:

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = F_1 + F_2 \tag{21}$$

(b) Given that $F_1 = F_2 = 0$, have:

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0 \tag{22}$$

Integrating once with respect to time:

$$m_1\dot{x}_1 + m_2\dot{x}_2 = P \tag{23}$$

where P is an arbitrary constant.

This implies that the total momentum of the system is a constant, i.e., the momentum of the system is conserved.

(c)

$$X_{\text{CM}} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{24}$$

Differentiating with respect to time:

$$\begin{aligned} \dot{X}_{\text{CM}} &= \frac{m_1\dot{x}_1 + m_2\dot{x}_2}{m_1 + m_2} = \frac{P}{m_1 + m_2} \\ (m_1 + m_2)\dot{X}_{\text{CM}} &= P \end{aligned} \tag{25}$$

(d) Differentiating again with respect to time:

$$(m_1 + m_2)\ddot{X}_{\text{CM}} = m_1\ddot{x}_1 + m_2\ddot{x}_2 = F_1 + F_2 \tag{26}$$

If there is no external force such that $F_1 = F_2 = 0$, then $\ddot{X}_{\text{CM}} = 0$ and X_{CM} is either stationary or moving in a straight line at constant speed.

(e) Given:

$$\begin{aligned} m_1\ddot{x}_1 &= F_{\text{int}} \\ m_2\ddot{x}_2 &= -F_{\text{int}} \end{aligned} \tag{27}$$

Make the substitution $x_i = x_i' + X_{\text{CM}}$:

$$\begin{aligned} \ddot{x}_1' + \ddot{X}_{\text{CM}} &= \frac{F_{\text{int}}}{m_1} \\ \ddot{x}_2' + \ddot{X}_{\text{CM}} &= -\frac{F_{\text{int}}}{m_2} \end{aligned} \tag{28}$$

Subtracting:

$$\begin{aligned} \ddot{x}_1' - \ddot{x}_2' &= F_{\text{int}} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \\ \mu \ddot{x}' &= F_{\text{int}} \end{aligned} \tag{29}$$

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Additional Problems

10 Energy loss to rest

By conservation of energy, we have a relation between h_n and h_{n-1}:

$$mgh_n = (1-f)mgh_{n-1} \tag{30}$$

This recursive relation can be solved with the initial height $h_0 = h$, so that:

$$h_n = (1-f)^n h \tag{31}$$

The time taken from h_n to h_{n+1} is:

$$t_n = \sqrt{\frac{2h_n}{g}} + \sqrt{\frac{2h_{n+1}}{g}} = \left[(1-f)^{n/2} + (1-f)^{(n+1)/2}\right]\sqrt{\frac{2h}{g}} = (1+\sqrt{1-f})(1-f)^{n/2}\sqrt{\frac{2h}{g}} \tag{32}$$

This is a geometric series, so the total time is:

$$T = \sum_{n=0}^{\infty} t_n = \frac{1+\sqrt{1-f}}{1-\sqrt{1-f}}\sqrt{\frac{2h}{g}} \tag{33}$$

Given $h = 5\,\mathrm{m}$ and $f = 0.1$, we have $T = 38.3\,\mathrm{s}$.