Multiple Integrals & Vector Calculus

Problem Set 4

Xin, Wenkang

June 3, 2023

1

For the loop $OABCO$, we may break it down to two planes OAB, described by the normal $\mathbf{e}_1 = (0, -1, 1)^\intercal/\sqrt{2}$, and OBC, described by the normal $\mathbf{e}_2 = (-1, 0, 1)^\intercal/\sqrt{2}$. The vector area of the loop is then given by:

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 = \left(-1, -\frac{1}{2}, \frac{3}{2}\right)^\intercal \tag{1}$$

where $A_1 = |\overrightarrow{OA} \times \overrightarrow{OB}|/2$ and $A_2 = |\overrightarrow{OB} \times \overrightarrow{OC}|/2$.

We may also consider the projection of the loop onto different planes:

$$\begin{aligned} A_{xy} &= 3/2 \\ A_{xz} &= 1/2 \\ A_{yz} &= 1 \end{aligned} \tag{2}$$

Considering the orientation of the loop, we assign negative signs to A_{xz} and A_{yz}, and the results agree with the vector area. From the vector $\mathbf{e} = (0, -1, 1)^\intercal/\sqrt{2}$, the projected area is $\mathbf{S} \cdot \mathbf{e} = \sqrt{2}$.

The maximum projected area is $\mathbf{S} \cdot \hat{S} = 3$.

For the loop $OACBO$, we still break it down to two planes OAC, described by the normal $\mathbf{e}_3 = \hat{z}$, and CBO, described by the normal $\mathbf{e}_4 = (1, 0, -1)^\intercal/\sqrt{2}$. The vector area of the loop is then given by:

$$\mathbf{S} = \mathbf{S}_3 + \mathbf{S}_4 = A_3\hat{z} + A_4\mathbf{e}_4 = (2, 0, -1)^\intercal \tag{3}$$

$$\mathbf{S} = \mathbf{S}_3 + \mathbf{S}_4 = A_3\hat{z} + A_4\mathbf{e}_4 = (1, 0, 0)^\intercal \tag{4}$$

The projections are:

$$\begin{aligned} A_{xy} &= 1 \\ A_{xz} &= 0 \\ A_{yz} &= 2 \end{aligned} \tag{5}$$

From the vector $\mathbf{e} = (0, -1, 1)^\intercal/\sqrt{2}$, the projected area is $\mathbf{S} \cdot \mathbf{e} = 1/\sqrt{2}$.

$$
\begin{aligned}
A_{xy} &= 0 \\
A_{xz} &= 0 \\
A_{yz} &= 1
\end{aligned} \tag{6}
$$

From the vector $\mathbf{e} = (0, -1, 1)^\intercal/\sqrt{2}$, the projected area is $\mathbf{S} \cdot \mathbf{e} = 0$. The maximum projected area is $\mathbf{S} \cdot \hat{S} = 1$.

•

2

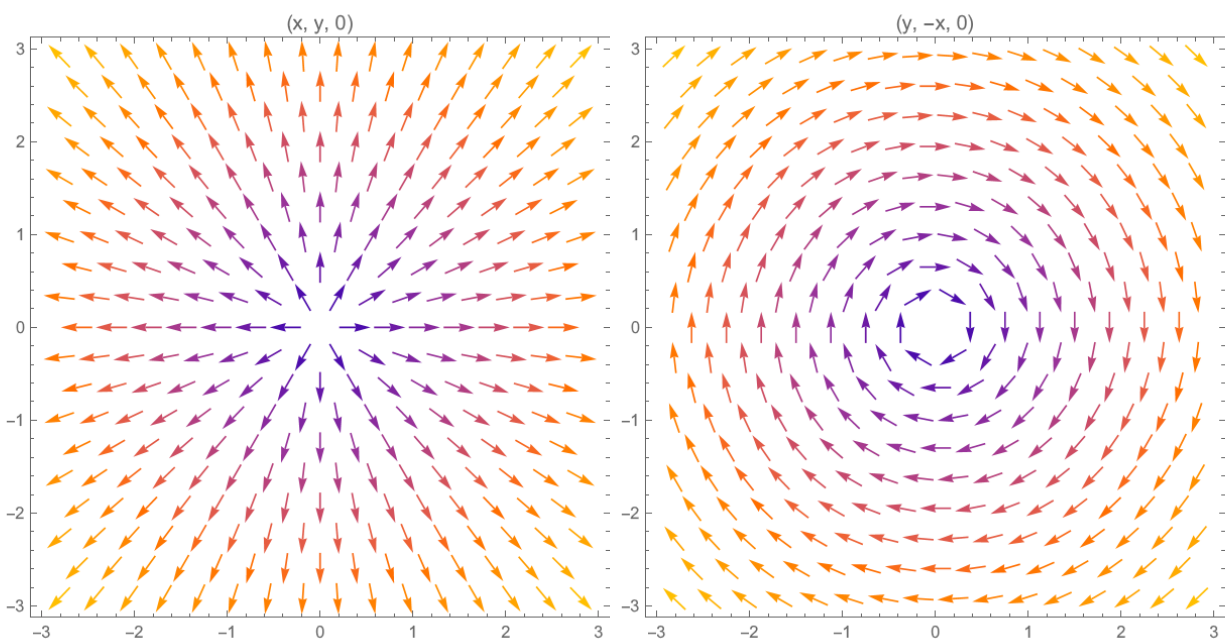
The area of the spherical cap subtended by a cone with half-angle α is given by:

$$A = \int_0^{2\pi} \int_0^{\alpha} r^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi = (1 - \cos\alpha) 2\pi r^2 \tag{7}$$

so that the solid angle is $\Omega = 2\pi(1 - \cos\alpha)$.

•

3



It is easy to calculate that

$$\nabla \cdot \mathbf{A} = 2, \quad \nabla \cdot \mathbf{B} = 0 \tag{8}$$

The curls are given by the matrix

$$\nabla \times \mathbf{A} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & 0 \end{pmatrix} = \mathbf{0}, \quad \nabla \times \mathbf{B} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{pmatrix} = (0, 0, -2)^\intercal \tag{9}$$

A is a curl-free vector field (like an electric field), and B is a divergence-free vector field (like a magnetic field).

•

4

The curl of $\mathbf{A}$ is:

$$\nabla \times \mathbf{A} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & z \end{pmatrix} = (0, 0, -2)^\intercal \tag{10}$$

The area integral of the curl on the hemisphere is given by:

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = -2 \int_S \mathrm{d}a_z = -2\pi \tag{11}$$

The curve C bounding the hemisphere is given by $\{(r, \theta, 0) | r = 1, \theta \in [0, 2\pi]\}$. The line integral of $\mathbf{A}$ along the curve is given by:

$$\int_C \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^{2\pi} -r^2 \sin^2\theta - r^2 \cos^2\theta \, \mathrm{d}\theta = -2\pi \tag{12}$$

as expected from Stokes' theorem.

•

5

Still using polar coordinates, the line integral is given by:

$$\int_C \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^{2\pi} -r \sin^2\theta - (3 + r\cos\theta) \cos\theta \, \mathrm{d}\theta = -2\pi r = -2\sqrt{2}\pi \tag{13}$$

$$\boxed{\int_C \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^{2\pi} -r^2 \sin^2\theta + (3 + r\cos\theta) r \cos\theta \, \mathrm{d}\theta = -4\pi} \tag{14}$$

•

6

Operating in cylindrical coordinates, a point at radius r has the velocity $\mathbf{v} = \omega r \hat{\phi}$. The divergence and curl of $\mathbf{v}$ are:

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \times \mathbf{v} = 2\omega \hat{z} \tag{15}$$

$\mathbf{v}$ being divergence-free means that the 'flow' is purely rotational, so a particle cannot change its r as expected from a rigid body. The flow cannot be represented by a potential field, as the curl is non-zero.

•

7

$$\nabla \times (\nabla \phi) = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \phi_x & \phi_y & \phi_z \end{pmatrix} = \mathbf{0} \tag{16}$$

as the double partial derivatives cancel.

$$\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \begin{pmatrix} \partial A_z/\partial y - \partial A_y/\partial z \\ \partial A_x/\partial z - \partial A_z/\partial x \\ \partial A_y/\partial x - \partial A_x/\partial y \end{pmatrix} = 0 \tag{17}$$

•

8

First express d$\mathbf{r}$ in terms of dt:

$$\mathrm{d}\mathbf{r} = (2t, 2, 2t^2)^\intercal \mathrm{d}t \tag{18}$$

The line integrals are given by:

$$\int_C \phi \, \mathrm{d}\mathbf{r} = \int_0^1 4t^9 (2t, 2, 2t^2)^\intercal \, \mathrm{d}\mathbf{r} = (8/11, 8/10, 8/12)^\intercal \tag{19}$$

$$\int_C \mathbf{F} \times \mathrm{d}\mathbf{r} = \int_0^1 2(-t^4 - t^5, -t^5, 2t^3 + t^4)^\intercal \, \mathrm{d}\mathbf{r} = (-11/15, -1/3, 7/5)^\intercal \tag{20}$$

•

9

$$\begin{aligned} a_i b_j c_i &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} \\ a_i b_j c_j d_i &= (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \\ \delta_{ij} a_i a_j &= \mathbf{a}\cdot\mathbf{a} \\ \delta_{ij}\delta_{ij} &= 1 \\ \epsilon_{ijk} a_i b_k &= \mathbf{b}\times\mathbf{a} \\ \epsilon_{ijk}\delta_{ij} &= 0 \end{aligned} \tag{21}$$

$$\boxed{\delta_{ij}\delta_{ij} = \delta_{ii} = 3} \tag{22}$$

•

10

$$\begin{aligned} \nabla r^n &= \sum_i \frac{\partial}{\partial x_i}\left(\sum_j x_j\right)^{n/2} \hat{e}_i \\ &= \sum_i \frac{n}{2}\left(\sum_j x_j\right)^{n/2-1} \frac{\partial \sum x_j^2}{\partial x_i}\hat{e}_i \\ &= \sum_i \frac{n}{2}\left(\sum_j x_j\right)^{n/2-1} 2x_i\hat{e}_i \\ &= nr^{n-2}\mathbf{r} \\ &= nr^{n-1}\hat{r} \end{aligned} \tag{23}$$

$$\nabla(\mathbf{a}\cdot\mathbf{r}) = \sum_i \frac{\partial}{\partial x_i}\left(\sum_j a_j x_j\right)\hat{e}_i = \sum_i a_i\hat{e}_i = \mathbf{a} \tag{24}$$

•

11

(a)

$$\nabla\cdot\mathbf{r} = \partial_{x_i} x_i = 3 \tag{25}$$

$$\nabla\times\mathbf{r} = \epsilon_{ijk}\partial_{x_j} x_k \hat{e}_i = \mathbf{0} \tag{26}$$

(b)

$$\begin{aligned}\nabla \cdot (r^n \mathbf{r}) &= \sum_i \partial_{x_i} \left[\left(\sum x_j^2 \right)^{n/2} x_i \right] \\ &= \sum_i \left(\sum x_j^2 \right)^{n/2} + x_i \frac{n}{2} \left(\sum x_j^2 \right)^{n/2-1} 2x_i \\ &= r^n + n r^{n-1}\end{aligned} \tag{27}$$

$$\begin{aligned}\nabla \cdot (r^n \mathbf{r}) &= \sum_i \partial_{x_i} \left[\left(\sum x_j^2 \right)^{n/2} x_i \right] \\ &= \sum_i \left(\sum x_j^2 \right)^{n/2} + x_i \frac{n}{2} \left(\sum x_j^2 \right)^{n/2-1} 2x_i \\ &= 3r^n + n r^n \\ &= (3+n) r^n\end{aligned} \tag{28}$$

$$\nabla \times (r^n \mathbf{r}) = \epsilon_{ijk} \partial_{x_j} (r^n x_k) \hat{e}_i = \epsilon_{ijk} r^{n-1} n x_j x_k \hat{e}_i = n r^{n-2} (\mathbf{r} \times \mathbf{r}) = \mathbf{0} \tag{29}$$

(c)

$$\nabla \cdot [(\mathbf{a} \cdot \mathbf{r}) \mathbf{b}] = \sum_i \partial_{x_i} \left(\sum_j a_j x_j \right) b_i = \sum_i a_i b_i = \mathbf{a} \cdot \mathbf{b} \tag{30}$$

$$\nabla \times [(\mathbf{a} \cdot \mathbf{r}) \mathbf{b}] = \epsilon_{ijk} \partial_{x_j} \left(\sum_l a_l x_l \right) b_k \hat{e}_i = \epsilon_{ijk} \delta_{lj} a l b_k \hat{e}_i = \mathbf{a} \times \mathbf{b} \tag{31}$$

(d)

$$\nabla \cdot (\mathbf{a} \times \mathbf{r}) = \partial_{x_i} \epsilon_{ijk} a_j x_k = 0 \tag{32}$$

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = \epsilon_{ijk} \partial_{x_j} (\epsilon_{klm} a_l x_m) \hat{e}_i = \epsilon_{kji} \epsilon_{klm} a_l \delta_{mj} \hat{e}_i = \delta_{il} \delta_{jm} a_l \delta_{mj} \hat{e}_i - \delta_{im} \delta_{jl} a_l \delta_{mj} \hat{e}_i = \mathbf{a} \tag{33}$$

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = \delta_{il} \delta_{jm} a_l \delta_{mj} \hat{e}_i - \delta_{im} \delta_{jl} a_l \delta_{mj} \hat{e}_i = 3a_i \hat{e}_i - a_i \hat{e}_i = 2\mathbf{a} \tag{34}$$

•

12

$$\nabla \times (\nabla \phi) = \nabla \times (\partial_{x_i} \phi_i \mathbf{e}_i) = \epsilon_{ijk} \partial_{x_j} \partial_{x_k} \phi_k \mathbf{e}_i \tag{35}$$

Note that $\epsilon_{ijk} \partial_{x_j} \partial_{x_k} = \epsilon_{ikj} \partial_{x_k} \partial_{x_j} = -\epsilon_{ikj} \partial_{x_k} \partial_{x_j} = 0$. Therefore, $\nabla \times (\nabla \phi) = \mathbf{0}$.

$$\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot (\epsilon_{ijk} \partial_{x_j} A_k \mathbf{e}_i) = \epsilon_{ijk} \partial_{x_i} \partial_{x_j} A_k = 0 \tag{36}$$