Vectors & Matrices

Problem Set 4

Eigenvectors, Eigenvalues and Diagonalization

Xin, Wenkang

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Eigenvectors, Eigenvalues and Diagonalization

1

The characteristic equation is given by:

$$\det \begin{pmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = (1-\lambda)(\lambda-4)(\lambda+1) = 0 \tag{1}$$

For $\lambda_1 = -1$, performing row reduction on $(A - \lambda_1 I)$ yields:

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

so $\hat{\mathbf{e}}_1 = (1, -1, 0)^\intercal/\sqrt{2}$.

Further, for $\lambda_2 = 1$, performing row reduction on $(A - \lambda_2 I)$ yields:

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

so $\hat{\mathbf{e}}_2 = (1, 1, -2)^\intercal/\sqrt{6}$.

Finally, for $\lambda_3 = 4$:

$$\begin{pmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

so $\hat{\mathbf{e}}_3 = (1, 1, 1)^\intercal/\sqrt{3}$.

Therefore we have $R^\intercal A R = \mathrm{diag}(-1, 1, 4)$, where:

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{5}$$

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2

The characteristic equation is given by:

$$\det(H - \lambda I) = (\lambda - 1)(\lambda - 11) = 0 \tag{6}$$

For $\lambda_1 = 1$:

$$\begin{pmatrix} 9 & 3i \\ -3i & 1 \end{pmatrix} \to \begin{pmatrix} 3 & i \\ 0 & 0 \end{pmatrix} \tag{7}$$

so $\hat{\mathbf{e}}_1 = (i, -3)^\intercal/\sqrt{10}$.

For $\lambda_2 = 11$:

$$\begin{pmatrix} -1 & 3i \\ -3i & -9 \end{pmatrix} \to \begin{pmatrix} 1 & -3i \\ 0 & 0 \end{pmatrix} \tag{8}$$

so $\hat{\mathbf{e}}_2 = (3i, 1)^\intercal/\sqrt{10}$.

Therefore, the required unitary matrix is given by:

$$U = \frac{1}{\sqrt{10}} \begin{pmatrix} i & 3i \\ -3 & 1 \end{pmatrix} \tag{9}$$

It is easy to verify that $U^\dagger U = I$.

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3

(a) Consider the equation $\mathbf{x}^\intercal A\mathbf{x} = x^2 + 3y^2 - 2xy$, where the matrix A has the form:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{10}$$

Expanding the matrix product yields the equation $ax^2 + dy^2 + (b + c)xy = x^2 + 3y^2 - 2xy$. We choose $(a, b, c, d) = (1, 0, -2, 3)$ for simplicity.

Performing the usual diagonalization procedure on A yields $\hat{\mathbf{e}}_1 = (1, 1)^\intercal/\sqrt{2}$ with $\lambda_1 = 1$ and $\hat{\mathbf{e}}_2 = (0, 1)^\intercal$ with $\lambda_2 = 3$. Therefore we have the equation:

$$\mathbf{x}^\intercal A\mathbf{x} = \mathbf{x}^\intercal P\hat{A}P^{-1}\mathbf{x} \tag{11}$$

where:

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \tag{12}$$

and $\hat{A} = \mathrm{diag}(1, 3)$.

Define the new coordinates with $\mathbf{x}' = P^{-1}\mathbf{x}$. In the new coordinate system, we have the equation:

$$x'^2 + 3y'^2 = 1 \tag{13}$$

which is an ellipse with semi-major axis 1 and semi-minor axis $\sqrt{3}$.

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4

(a) The characteristic equation is given by:

$$\det(R - \lambda I) = 1 - 2\lambda\cos\phi + \lambda^2 = \lambda^2 - (e^{i\phi} + e^{-i\phi})\lambda + 1 = 0 \tag{14}$$

Solving the quadratic equation for λ yields $\lambda_{1,2} = e^{\pm i\phi}$.

(b) Performing the usual diagonalization procedure on A yields $\hat{\mathbf{e}}_1 = (-1, 1, 0)^\intercal/\sqrt{2}$ with the two-fold degenerate $\lambda_1 = 0$ and $\hat{\mathbf{e}}_2 = (1, 0, 0)^\intercal$ with $\lambda_2 = 1$.

As the number of eigenvectors is less than three, the eigenvectors cannot form a basis for $\mathbb{R}^3$. Therefore, A is not diagonalisable.

Performing Gram-Schmidt procedure on $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. We have $\hat{\mathbf{e}}'_1 = \hat{\mathbf{e}}_1$ and:

$$\hat{\mathbf{e}}'_2 = \frac{\hat{\mathbf{e}}_2 - (\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}'_1)\hat{\mathbf{e}}'_1}{|\hat{\mathbf{e}}_2 - (\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}'_1)\hat{\mathbf{e}}'_1 =|} = (1, 1, 0)^\intercal/\sqrt{2} \tag{15}$$

But this is just $\hat{\mathbf{e}}_1 + 2\hat{\mathbf{e}}_2$. So there are only two linearly independent eigenvectors. Therefore, A is not diagonalisable.

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5

(a) Let us assume that M is diagonalisable so that we have:

$$\mathbf{v}^\intercal M\mathbf{v} = \mathbf{v}^\intercal P\hat{M}P^{-1}\mathbf{v} = \mathbf{v}'^\intercal\hat{M}\mathbf{v}' = \sum_{i=1}^{n}\lambda_i v_i'^2 \tag{16}$$

where $\mathbf{v}' = P^{-1}\mathbf{v}$ is the new coordinates.

Since $v_i'^2 \geq 0$, $\mathbf{v}^\intercal M \mathbf{v} > 0$ for all $\mathbf{v}$ if $\lambda_i > 0$, provided that $\mathbf{v} \neq \mathbf{0}$. Vice versa for the case where $\mathbf{v}^\intercal M \mathbf{v} < 0$.

(b) The characteristic polynomial is given by:

$$\chi(\lambda) = \lambda^2 - \operatorname{tr}(M)\lambda + \det(M) \tag{17}$$

Therefore, the two solutions of $\chi(\lambda) = 0$ satisfy $\lambda_1 + \lambda_2 = \operatorname{tr}(M)$ and $\lambda_1\lambda_2 = \det(M)$.

Another way to see this is to note that $\operatorname{tr}(M) = \operatorname{tr}\left(P\hat{M}P^{-1}\right) = \operatorname{tr}\left(\hat{M}\right) = \lambda_1 + \lambda_2$ and $\det(M) = \det\left(P\hat{M}P^{-1}\right) = \det\left(\hat{M}\right) = \lambda_1\lambda_2$.

(c)

$$M \text{ is } \begin{cases} \text{positive definite} & \text{if } \det(M) > 0 \text{ and } \operatorname{tr}(M) > 0 \\ \text{negative definite} & \text{if } \det(M) > 0 \text{ and } \operatorname{tr}(M) < 0 \\ \text{negative indefinite} & \text{if } \det(M) <= 0 \end{cases} \tag{18}$$

(d) M_1 is positive indefinite; M_2 is indefinite; M_3 is negative definite.

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(a)

$$\chi_A(\lambda) = \det(A - \lambda I) = (A_{11} - \lambda)(A_{22} - \lambda) - A_{12}A_{21} = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) \tag{19}$$

(b)

$$\begin{aligned} \chi_A(\lambda) &= (A_{11} - \lambda)\det\begin{pmatrix} A_{22} - \lambda & A_{23} \\ A_{32} & A_{33} - \lambda \end{pmatrix} - A_{12}\det\begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} - \lambda \end{pmatrix} + A_{13}\det\begin{pmatrix} A_{21} & A_{22} - \lambda \\ A_{31} & A_{32} \end{pmatrix} \\ &= -\lambda^3 + \operatorname{tr}(A)\lambda^2 - \frac{1}{2}[\operatorname{tr}(A)^2 - \operatorname{tr}\left(A^2\right)]\lambda + \det(A) \end{aligned} \tag{20}$$

(c) For a general A, the characteristic polynomial has the form:

$$\chi_A(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}\operatorname{tr}(A)\lambda^{n-1} + \cdots + \det(A) \tag{21}$$

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7

(a) For a given eigenvector of P, we have the equation $P\mathbf{v} = \lambda\mathbf{v}$. Apply P on both sides:

$$P^2\mathbf{v} = \lambda P\mathbf{v} = \lambda^2\mathbf{v} \tag{22}$$

But $P^2 = P$ so $\lambda^2\mathbf{v} = \lambda\mathbf{v}$. This shows that λ is either zero or unity. The diagonalisation of P is therefore:

$$\hat{P} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$

(b) P projects an arbitrary vector onto a line depending on the diagonalisation of P. The number of ones in $\hat{P}$ is the dimension of the space projected, and it is equal to $\mathrm{tr}(P)$

(c)

$$(Q^2)_{ij} = Q_{ik}Q_{kj} = n_i n_k n_k n_j = n_i n_j = Q_{ij} \tag{24}$$

as $\sum n_k^2 = 1$ for a unit vector.

Therefore, $Q^2 = Q$ and Q is a projector. $\mathrm{tr}(Q) = Q_{ii} = n_i n_i = 1$ so it projects to a one-dimensional space.

(d)

$$P^2 = (I - Q)^2 = I - 2Q + Q^2 = I - Q = P \tag{25}$$

We know that $P_{ij} = \delta_{ij} - n_i n_j$. Thus:

$$\mathrm{tr}(P) = \delta_{ij}(\delta_{ij} - n_i n_j) = 1 - n_i n_i = 0 \tag{26}$$

Therefore, P projects onto a zero-dimensional space.

$$\mathrm{tr}(P) = \delta_{ij}(\delta_{ij} - n_i n_j) = n - n_i n_i = n - 1 \tag{27}$$

Therefore, P projects onto a $n-1$-dimensional space.

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8

(a) For a given eigenvector of U, we have the equation $U\mathbf{v} = \lambda\mathbf{v}$. Taking the Hermitian of both sides:

$$
\begin{aligned}
\mathbf{v}^\dagger U^\dagger &= \lambda^* \mathbf{v}^\dagger \\
\mathbf{v}^\dagger U^\dagger U \mathbf{v} &= \lambda^* \lambda \mathbf{v}^\dagger \mathbf{v} \\
\mathbf{v}^\dagger \mathbf{v} &= |\lambda|^2 \mathbf{v}^\dagger \mathbf{v}
\end{aligned} \tag{28}
$$

Therefore, $|\lambda| = 1$.

(b) We have the equation $R\mathbf{v} = \lambda \mathbf{v}$. Taking the transpose of both sides:

$$
\begin{aligned}
\mathbf{v}^\intercal R^\intercal &= \lambda \mathbf{v}^\intercal \\
\mathbf{v}^\intercal R^\intercal R \mathbf{v} &= \lambda \lambda \mathbf{v}^\intercal \mathbf{v} \\
\mathbf{v}^\intercal \mathbf{v} &= \lambda^2 \mathbf{v}^\intercal \mathbf{v}
\end{aligned} \tag{29}
$$

This implies $\lambda = \pm 1$ so 1 is always an eigenvalue of R.

(c)

$$
\begin{aligned}
\hat{R} &= \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{n} \end{pmatrix}^\intercal R \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{n} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{n} \end{pmatrix}^\intercal \begin{pmatrix} R\mathbf{u}_1 & R\mathbf{u}_2 & \mathbf{n} \end{pmatrix} \\
&= \begin{pmatrix} (R\mathbf{u}_1)\cdot \mathbf{u}_1 & (R\mathbf{u}_2)\cdot \mathbf{u}_1 & \mathbf{n}\cdot \mathbf{u}_1 \\ (R\mathbf{u}_1)\cdot \mathbf{u}_2 & (R\mathbf{u}_2)\cdot \mathbf{u}_2 & \mathbf{n}\cdot \mathbf{u}_2 \\ (R\mathbf{u}_1)\cdot \mathbf{n} & (R\mathbf{u}_2)\cdot \mathbf{n} & \mathbf{n}\cdot \mathbf{n} \end{pmatrix} \\
&= \begin{pmatrix} (R\mathbf{u}_1)\cdot \mathbf{u}_1 & (R\mathbf{u}_2)\cdot \mathbf{u}_1 & 0 \\ (R\mathbf{u}_1)\cdot \mathbf{u}_2 & (R\mathbf{u}_2)\cdot \mathbf{u}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{30}
$$

(d) From the form of R, we have $\operatorname{tr}\left(\hat{R}\right) = \operatorname{tr}(R) = 2\cos\phi + 1$ or:

$$
\cos\phi = \frac{\operatorname{tr}(R) - 1}{2} \tag{31}
$$

(e) Consider the determinant and the transpose of the matrix R:

$$
R = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 & 0 & 3 \\ -1 & -4 & 1 \\ 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix} \tag{32}
$$

We have $\det(R) = 1$ and $R^\intercal R = I$. Hence R is a rotation matrix. The characteristic polynomial of R is:

$$(1/\sqrt{2} - \lambda)[(4/\sqrt{2} + \lambda)(2/3 + \lambda) + 1/3] + 1/\sqrt{2}[\sqrt{2} + 2(4/(3\sqrt{2}) + \lambda)/3] = 0 \tag{33}$$

Substituting $\lambda = 1$ verifies that 1 is an eigenvalue of R. Further:

$$R - I = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 - 3\sqrt{2} & 0 & 3 \\ -1 & -4 - 3\sqrt{2} & 1 \\ 2\sqrt{2} & -\sqrt{2} & -5\sqrt{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 - \sqrt{2} \\ 0 & 1 & 3 - 2\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{34}$$

This gives us:

$$\mathbf{n} = \frac{1}{\sqrt{13 - 2\sqrt{2}}} \begin{pmatrix} 1 + \sqrt{2} \\ -3 + 2\sqrt{2} \\ 1 \end{pmatrix} \tag{35}$$

and

$$\cos\phi = \frac{\mathrm{tr}(R) - 1}{2} = -\frac{5\sqrt{2} + 1}{6\sqrt{2}} \tag{36}$$

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9

(a) Suppose that $M = P\hat{M}P^{-1}$ such that $\ddot{\mathbf{x}} = P\hat{M}P^{-1}\mathbf{x}$. Then:

$$\begin{aligned} P^{-1}\ddot{\mathbf{x}} &= \hat{M}P^{-1}\mathbf{x} \\ (P^{-1}\mathbf{x})_i^{\cdot\cdot} &= \hat{M}_{ii}(P^{-1}\mathbf{x})_i \end{aligned} \tag{37}$$

This is an linear second order ODE with constant coefficients for $(P^{-1}\mathbf{x})_i$ that can be easily solved.

(b) The eigenvalues of M are $\hat{M}_{ii} = \lambda_i$, which are the (square of) oscillating frequencies of the transformed coordinates (normal coordinates).

(c) The coefficient matrix has the form:

$$M = \begin{pmatrix} -k & -l \\ -l & -k \end{pmatrix} \tag{38}$$

Consider the characteristic equation:

$$\det\big(M - \Omega^2 I\big) = (k + \Omega^2)^2 - l^2 = 0 \tag{39}$$

Thus, the eigenfrequencies are $\Omega_{1,2} = \sqrt{\pm l - k}$. The eigenfrequencies can be real or imaginary depending on the sign of $(\pm l - k)$. For imaginary eigenfrequencies only, the solution is oscillatory. For real eigenfrequencies, the solution is exponentially decaying. For a combination of real and imaginary eigenfrequencies, the solution is a combination of oscillatory and exponential.