Vacation Work

Problem Sheet A

Introductory Problems

Xin, Wenkang

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1

(a) Expanding the expression of $f(x)$ in terms of roots:

$$\begin{aligned} f(x) &= a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) \\ &= a_n[x^2-(\alpha_1+\alpha_2)x+\alpha_1\alpha_2](x-\alpha_3)\cdots \\ &= a_n[x^3-(\alpha_1+\alpha_2+\alpha_3)x^2+(\alpha_1\alpha_2-\alpha_1\alpha_3-\alpha_2\alpha_3)x-\alpha_1\alpha_2\alpha_3](x-\alpha_4)\cdots \end{aligned} \tag{1}$$

Therefore, it is concluded that the coefficient of x^{n-1} must be $-a_n\sum\alpha_k$, where k ranges from 1 to n. Hence, comparing with the coefficient expression:

$$\sum_{k=1}^{n}\alpha_k = -\frac{a_{n-1}}{a_n} \tag{2}$$

Further, note that the two expressions must be equal given any value of x. Taking $x=0$, have:

$$\begin{aligned} a_0 &= a_n\prod_{k=1}^{n}-\alpha_k \\ \prod_{k=1}^{n}\alpha_k &= (-1)^n\frac{a_0}{a_n} \end{aligned} \tag{3}$$

(b) $x=1$ is a root for obvious reason, and the sum and product of the two other roots are $\boxed{-7/4}$ and $\boxed{1/4}$ respectively.

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Employing the double angle formula:

$$\begin{aligned} \cos(4\theta) &= \cos^2(2\theta)-\sin^2(2\theta) \\ &= (\cos^2\theta-\sin^2\theta)^2-4\sin^2\theta\cos^2\theta \\ &= \cos^4\theta+\sin^4\theta-6\sin^2\theta\cos^2\theta \\ &= (1-\sin^2\theta)^2+\sin^4\theta-6\sin^2\theta(1-\sin^2\theta) \\ &= 8\sin^4\theta-8\sin^2\theta+1 \end{aligned} \tag{4}$$

Since $\cos(4\times\pi/8)=0$, $\theta=\pi/8$ is a root of the above equation, and thus $\sin(\pi/8)$ is a solution to $8s^2-8s+1=0$.

This can be regarded as a quadratic equation of s^2, with the solution:

$$s^2 = \frac{8 \pm \sqrt{64 - 32}}{16} = \frac{2 \pm \sqrt{2}}{4} \tag{5}$$

But $\sin(\pi/8) \leq 1$, so we retain the negative sign and yield:

$$\sin\frac{\pi}{8} = \sqrt{\frac{2-\sqrt{2}}{4}} \tag{6}$$

And hence:

$$\cos\frac{\pi}{8} = \sqrt{1 - \sin^2\frac{\pi}{8}} = \boxed{\sqrt{\frac{2+\sqrt{2}}{4}}} \tag{7}$$

$$\tan\frac{\pi}{8} = \sin\frac{\pi}{8} / \cos\frac{\pi}{8} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = \boxed{\sqrt{3-2\sqrt{2}}} \tag{8}$$

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Expanding the right hand side:

$$K\sin(\theta+\phi) = K(\sin\theta\cos\phi + \cos\theta\sin\phi) \tag{9}$$

Comparing the coefficients, we have:

$$K\cos\phi = a, \quad K\sin\phi = b \tag{10}$$

Hence, $K^2 = a^2 + b^2$ and $\phi = \arctan(b/a)$

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The equation can be rewritten as:

$$f(x,y) = (x+3)^2 + (y+4)^2 - 25 = 0 \tag{11}$$

This represents a circle centred at $(-3, -4)$ with a radius of 5 units.

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(a)

$$\frac{2x+1}{x^2+3x-10} = \frac{2x+1}{(x-2)(x+5)} = \frac{5/7}{x-2} + \frac{9/7}{x+5} \tag{12}$$

(b)

$$\frac{4}{x^2-3x} = \frac{4}{x(x-3)} = \frac{1}{x-3} - \frac{1}{x} \tag{13}$$

(c)

$$\frac{x^2+x-1}{x^2+x-2} = 1 + \frac{1}{x^2+x-2} = 1 + \frac{1}{(x-1)(x+2)} = 1 + \frac{1/3}{x-1} - \frac{1/3}{x+2} \tag{14}$$

(d)

$$\frac{2x}{(x+1)(x-1)^2} = -\frac{1/2}{x+1} + \frac{1/2}{x-1} + \frac{1}{(x+1)^2} \tag{15}$$

(e)

$$\frac{2+4x}{(x+2)(x^2+2)} = -\frac{1}{x+2} + \frac{x+2}{x^2+2} \tag{16}$$

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(a)

$$(1+x)^5 = 1 + 5x + 10x^2 + \dots \tag{17}$$

(b)

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - \dots \tag{18}$$

where $|x| < 1$.

(c)

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3}{8}x^2 - \dots \tag{19}$$

where $|x| < 1$.

First note that $\sqrt{4.2} = 2\sqrt{1.05} = 2\sqrt{1+0.05}$. Then using the above result:

$$\frac{1}{\sqrt{4.2}} \approx \frac{1}{2}(1 - \frac{0.05}{2} + \frac{3}{8} \times 0.05^2) = \boxed{0.488} \tag{20}$$

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(a) Let P_n denote the proposition that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ for some $n \in \mathbb{Z}$. P_1 is obviously true.

Now suppose that P_k is true for some $k \in \mathbb{Z}$, so that $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$. We have:

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1) = (\frac{1}{2}k + 1)(k+1) = \frac{1}{2}(k+1)(k+2) \tag{21}$$

Thus P_{k+1} is true if P_k is true. Since P_1 is true and P_k leads to P_{k+1}, P_n is true for all $n \in \mathbb{Z}$.

(b) Let P_n denote the proposition that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for some $n \in \mathbb{Z}$. P_1 is obviously true.

Now suppose that P_k is true for some $k \in \mathbb{Z}$, so that $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$. We have:

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3 = (\frac{1}{4}k^2 + k + 1)(k+1)^2 = \frac{1}{4}(k+1)^2(k+2)^2 \tag{22}$$

Thus P_{k+1} is true if P_k is true. Since P_1 is true and P_k leads to P_{k+1}, P_n is true for all $n \in \mathbb{Z}$.

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$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1} = S_n + r^{n+1} - 1 \tag{23}$$

Solving for S_n leads to:

$$S_n = \frac{1 - r^{n+1}}{1 - r} \tag{24}$$

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