Symmetry and Relativity

Problem Set 2

Symmetries

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1 Derivation of the rotation formula

Given that:

$$\hat{\mathbf{u}} \cdot \mathbf{J} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \tag{1}$$

we have its square:

$$\begin{aligned} (\hat{\mathbf{u}} \cdot \mathbf{J})^2 &= \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \\ &= \begin{bmatrix} -u_y^2 - u_z^2 & u_x u_y & u_x u_z \\ u_x u_y & -u_x^2 - u_z^2 & u_y u_z \\ u_x u_z & u_y u_z & -u_x^2 - u_y^2 \end{bmatrix} \end{aligned} \tag{2}$$

Let us explicitly check:

$$(\hat{\mathbf{u}} \cdot \mathbf{J})\mathbf{v} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix} = \mathbf{u} \times \mathbf{v} \tag{3}$$

and thus:

$$(\hat{\mathbf{u}} \cdot \mathbf{J})^2 \mathbf{v} = \mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}(\mathbf{u} \cdot \mathbf{v}) - \mathbf{u}^2 \mathbf{v} = \mathbf{u}(\mathbf{u} \cdot \mathbf{v}) - \mathbf{v} \tag{4}$$

This means:

$$\begin{aligned} R\mathbf{v} &= \mathbf{v} + (1 - \cos\theta)(\hat{\mathbf{u}} \cdot \mathbf{J})^2 \mathbf{v} + \sin\theta \hat{\mathbf{u}} \cdot \mathbf{J} \mathbf{v} \\ &= \mathbf{v} + (1 - \cos\theta)(\mathbf{u}(\mathbf{u} \cdot \mathbf{v}) - \mathbf{v}) + \sin\theta(\mathbf{u} \times \mathbf{v}) \\ &= \cos\theta \mathbf{v} + \sin\theta(\mathbf{u} \times \mathbf{v}) + (1 - \cos\theta)\mathbf{u}(\mathbf{u} \cdot \mathbf{v}) \end{aligned} \tag{5}$$

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2 Lorentz transformation for a boost in an arbitrary direction

(a) Consider the form of $\hat{\mathbf{u}} \cdot \mathbf{K}$:

$$\hat{\mathbf{u}} \cdot \mathbf{K} = \begin{bmatrix} 0 & n_x & n_y & n_z \\ n_x & 0 & 0 & 0 \\ n_y & 0 & 0 & 0 \\ n_z & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

Then:

$$\begin{aligned} (\hat{\mathbf{u}} \cdot \mathbf{K})^3 &= \begin{bmatrix} 0 & n_x & n_y & n_z \\ n_x & 0 & 0 & 0 \\ n_y & 0 & 0 & 0 \\ n_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & n_x^2 & n_x n_y & n_x n_z \\ 0 & n_x n_y & n_y^2 & n_y n_z \\ 0 & n_x n_z & n_y n_z & n_z^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & n_x & n_y & n_z \\ n_x & 0 & 0 & 0 \\ n_y & 0 & 0 & 0 \\ n_z & 0 & 0 & 0 \end{bmatrix} \\ &= \hat{\mathbf{u}} \cdot \mathbf{K} \end{aligned} \tag{7}$$

Thus all odd powers of $\hat{\mathbf{u}} \cdot \mathbf{K}$ are identical to the first power, and all even powers are identical to the second power. This means that:

$$\begin{aligned} \Lambda &= \exp\{-\zeta \hat{\mathbf{u}} \cdot \mathbf{K}\} \\ &= \sum_{n=0}^{\infty} \frac{(-\zeta)^n}{n!} (\hat{\mathbf{u}} \cdot \mathbf{K})^n \\ &= \mathbb{I} - \sum_{\text{odd } n} \frac{\zeta^n}{n!} (\hat{\mathbf{u}} \cdot \mathbf{K}) + \sum_{\text{even } n} \frac{\zeta^n}{n!} (\hat{\mathbf{u}} \cdot \mathbf{K})^2 \\ &= \mathbb{I} - (\sinh\zeta) \hat{\mathbf{u}} \cdot \mathbf{K} + (\cosh\zeta - 1)(\hat{\mathbf{u}} \cdot \mathbf{K})^2 \end{aligned} \tag{8}$$

(b) Suppose $\mathbf{r} = \mathbf{v}t$ and $\mathbf{r}' = \mathbf{0}$. We demand that $\Lambda \mathbf{r} = \mathbf{r}'$ so that:

$$\begin{bmatrix} ct' \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} - (\sinh\zeta) \begin{bmatrix} 0 & n_x & n_y & n_z \\ n_x & 0 & 0 & 0 \\ n_y & 0 & 0 & 0 \\ n_z & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} + (\cosh\zeta - 1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & n_x^2 & n_x n_y & n_x n_z \\ 0 & n_x n_y & n_y^2 & n_y n_z \\ 0 & n_x n_z & n_y n_z & n_z^2 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{9}$$

Now focus on the x component of the equation:

$$\begin{aligned} 0 &= x - \sinh\zeta n_x ct + (\cosh\zeta - 1)n_x(n_x x + n_y y + n_z z) \\ &= x - \sinh\zeta n_x ct + (\cosh\zeta - 1)n_x(\mathbf{n}\cdot\mathbf{v})t \end{aligned} \tag{10}$$

Now note that $\mathbf{n} = \mathbf{v}/\left|\mathbf{v}\right|$ so that $v_x = vn_x$. Thus:

$$0 = vn_x t - \sinh\zeta n_x ct + (\cosh\zeta - 1)n_x vt \tag{11}$$

solving which gives $\tanh\zeta = v/c = \beta$.

Then:

$$\begin{aligned} \Lambda &= \mathbb{I} - \frac{\beta}{\sqrt{1-\beta^2}}\hat{\mathbf{u}}\cdot\mathbf{K} + \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)(\hat{\mathbf{u}}\cdot\mathbf{K})^2 \\ &= \begin{bmatrix} \gamma & -\gamma\beta n_x & -\gamma\beta n_y & -\gamma\beta n_z \\ -\gamma\beta n_x & 1+(\gamma-1)n_x^2 & (\gamma-1)n_x n_y & (\gamma-1)n_x n_z \\ -\gamma\beta n_y & (\gamma-1)n_x n_y & 1+(\gamma-1)n_y^2 & (\gamma-1)n_y n_z \\ -\gamma\beta n_z & (\gamma-1)n_x n_z & (\gamma-1)n_y n_z & 1+(\gamma-1)n_z^2 \end{bmatrix} \end{aligned} \tag{12}$$

3 Decomposing a Lorentz transformation into boost and rotation

Define a matrix L that satisfies $L^{\intercal}gL$ where $g = \operatorname{diag}(-1,1,1,1)$. It being in $SO(1,3)$, we need $L^{\intercal} = L$.

(a) Writing the relation in index notation, the inverse of L is defined as:

$$(L^{-1})_{ij} = L^{ij} = g^{ik}L_k^j = g^{ik}g^{jl}L_{kl} \tag{13}$$

From the relation, we have:

$$\begin{aligned} L_{ik}g^{kl}(L^{\intercal})_{lj} &= g_{ij} \\ L_{ik}g_{kl}L_{lj} &= g_{ij} \end{aligned} \tag{14}$$

First consider $i = j = 0$ so that the RHS is -1. The non-zero components of the LHS are obtained by setting $k = l$ so that:

$$-L_{00}L_{00} + \sum_{\alpha=1,2,3} L_{0\alpha}L_{0\alpha} = -1 \tag{15}$$

Consider $i \neq j$ so that the RHS is zero. The non-zero components of the LHS are obtained by setting $k = l$ so that:

$$-L_{i0}L_{0j} + \sum_{\alpha=1,2,3} L_{i\alpha}L_{\alpha j} = 0 \tag{16}$$

Now take $j = 0$ so that i takes $1, 2, 3$:

$$L_{00}L_{i0} - \sum_{\alpha=1,2,3} L_{i\alpha}L_{\alpha 0} = 0 \tag{17}$$

(b) We demand that transformations of coordinates between two frames is facilitated by L, i.e. $X'^{\mu} = L^{\mu}_{\nu}X^{\nu} = g^{\mu\rho}L_{\rho\nu}X^{\nu}$. Consider the origin of frame S' so that $X'^{\mu} = (ct', 0, 0, 0)$. Setting $\mu = 0$ demands $\rho = 0$, leading to:

$$ct' = -L_{0\nu}X^{\nu} \tag{18}$$

For $i = 1, 2, 3$:

$$0 = L_{i\nu}X^{\nu} = L_{i0}X^0 + L_{ij}X^j \tag{19}$$

Multiply the second equation by L_{00}:

$$\begin{aligned} 0 &= L_{00}L_{i0}X^0 + L_{00}L_{ij}X^j \\ &= L_{i\alpha}L_{\alpha 0}X^0 + L_{00}L_{ij}X^j \\ &= L_{ij}(L_{j0}X^0 + L_{00}X^j) \end{aligned} \tag{20}$$

where at the third equality we relabelled $\alpha \to j$.

But we know that X^{ν} has to be $(ct, \mathbf{v}t)$ so $X^0 = ct$ and $X^j = v^j t$. This implies:

$$\beta^i \equiv \frac{v_i}{c} = -\frac{L_{i0}}{L_{00}} \tag{21}$$

which determines L up to L_{00}.

Using the first equation we derived, we have:

$$\begin{aligned} -L_{00}L_{00} + \sum_{\alpha=1,2,3} L_{0\alpha}L_{0\alpha} &= -1 \\ -1 + \beta^2 &= -L_{00}^{-2} \end{aligned} \tag{22}$$

which immediately gives $L_{00} = \gamma \equiv (1-\beta^2)^{-1/2}$.

(c) To demonstrate $R = L\Lambda^{-1}$ is in $SO(1,3)$, it suffices to check the inner product condition:

$$R^\intercal g R = (\Lambda^{-1})^\intercal L^\intercal g L \Lambda^{-1} = (\Lambda^{-1})^\intercal g \Lambda^{-1} = g \tag{23}$$

since Λ preserves inner product.

From the previous problem we have:

$$\Lambda^{-1}(\beta) = \Lambda(-\beta) = \mathbb{I} + \gamma\beta \mathbf{n}\cdot\mathbf{K} + (\gamma-1)(\mathbf{n}\cdot\mathbf{K})^2 \tag{24}$$

Then:

$$\begin{aligned} R_{00} &= L_{0\mu}[\Lambda^{-1}(\beta)]_{\mu 0} \\ &= L_{0\mu}\left(\delta_{\mu 0} + \gamma\beta n_\mu + (\gamma-1)\delta_{0\mu}\right) \\ &= \gamma L_{00} + \gamma\beta L_{0i} n_i \\ &= \gamma^2 + \gamma\beta(-\gamma\beta^i) n_i \\ &= \gamma^2(1-\beta^2) \\ &= 1 \end{aligned} \tag{25}$$

since $n_i = v_i/|\mathbf{v}| = \beta_i/\beta$

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4 Lorentz transformations and their expressions through SU(2)

(a) Suppose $M^\dagger M = MM^\dagger = \mathbb{I}$. Consider $M = \mathbb{I} + X$. To first order in X, we need:

$$X^\dagger = -X \tag{26}$$

Assume a X in the form:

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{27}$$

We need the equations:

$$\begin{aligned} a + a^* = d + d^* = 0 \\ b + c^* = 0 \end{aligned} \tag{28}$$

The first line implies that a and d are purely imaginary but otherwise independent. For the second line, we write $c = -b^*$ so that c is determined by b. This gives three independent parameters, which means that there must be three independent generators.

(b) Since J_i are generators of $SU(2)$, they satisfy $J_i^\dagger = -J_i$. On the other hand, if we write:

$$M = \mathbb{I} + iz_1 J_1 + iz_2 J_2 + iz_3 J_3 \tag{29}$$

we require:

$$\begin{aligned} M^\dagger M &= \mathbb{I} \\ i(z_1 J_1 - z_1^* J_1 \dagger) + i(z_2 J_2 - z_2^* J_2 \dagger) + i(z_3 J_3 - z_3^* J_3 \dagger) &= 0 \end{aligned} \tag{30}$$

Since $J_i^\dagger = -J_i$, we have $z_i = -z_i^*$, which means that z_i are purely imaginary. Thus:

$$K_i^\dagger = (z_i J_i)^\dagger = -z_i^* J_i^\dagger = z_i J_i = K_i \tag{31}$$

Suppose we write K_i in the form:

$$K_i = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{32}$$

For it to be Hermitian, we need $\alpha = \alpha^*$ and $\delta = \delta^*$, which means they are real. We also need $\beta = \gamma^*$, which means that β and γ are complex conjugates of each other. In a more symmetric form, we can write:

$$K_i = \begin{pmatrix} a+d & b-ic \\ b+ic & a-d \end{pmatrix} = a\mathbb{I} + b\sigma_1 + c\sigma_2 + d\sigma_3 \tag{33}$$

which gives:

$$M = \mathbb{I} - i(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3) \tag{34}$$

(c) We have demonstrated via K_i that any 2×2 Hermitian matrix S can be written as a linear combination of $\mathbb{I}$ and σ_i. Let us rename the coefficients of σ_i:

$$S = ct\mathbb{I} + x\sigma_1 + y\sigma_2 + z\sigma_3 \tag{35}$$

The determinant of S is:

$$\det(S) = c^2t^2 - x^2 - y^2 - z^2 \tag{36}$$

Consider the transformation $S' = LSL^\dagger$. We can show that S' is Hermitian:

$$(S')^\dagger = (LSL^\dagger)^\dagger = (L^\dagger)^\dagger S^\dagger L^\dagger = LSL^\dagger = S' \tag{37}$$

Thus L can be viewed as a Lorentz transformation and S and S' represent spacetimes events.

(d) Consider $L = \operatorname{diag}(a, b)$ with $\det(L) = ab = 1$. Then:

$$LSL^\dagger = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} ct & x \\ y & z \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2ct & abx \\ aby & b^2z \end{pmatrix} = \begin{pmatrix} ct' & x' \\ y' & z' \end{pmatrix} \tag{38}$$

This means that $t' = a^2t$, $x' = x$, $y' = y$, and $z' = z/a^2$. This is a boost in the z direction with a factor of a^2.

Consider another form of L:

$$L = (\cosh q)\mathbb{I} - (\sinh q)\sigma_3 = \begin{pmatrix} \cosh q - \sinh q & 0 \\ 0 & \cosh q + \sinh q \end{pmatrix} \tag{39}$$

which apparently satisfies $\det(L) = 1$.

(e) Consider $S = x\sigma_1 + y\sigma_2 + z\sigma_3$. Then:

$$\operatorname{tr}(S) = x \operatorname{tr}(\sigma_1) + y \operatorname{tr}(\sigma_2) + z \operatorname{tr}(\sigma_3) = 0 \tag{40}$$

Let us demand $\operatorname{tr}(S') = 0$ as well. Then:

$$\operatorname{tr}(S') = \operatorname{tr}\big(LSL^\dagger\big) = \operatorname{tr}\big(LL^\dagger S\big) = 0 = \operatorname{tr}(S) \tag{41}$$

which demands L to be unitary, i.e. L is in $SU(2)$.

We can thus write L in the form:

$$L = \exp[-i(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)] \tag{42}$$

Let $a_1 = a_2 = 0$ and $a_3 = \theta$, then:

$$\begin{aligned} L &= \exp[-i\theta\sigma_3] \\ &= \sum_{n=0}^{\infty} \frac{(-i\theta)^n}{n!}\sigma_3^n \end{aligned} \tag{43}$$

However, it is easy to check that σ_3^n is $\mathbb{I}$ for even n and σ_3 for odd n. Thus:

$$\begin{aligned} L &= \sum_{\text{even } n} \frac{(-i\theta)^n}{n!}\mathbb{I} + \sum_{\text{odd } n} \frac{(-i\theta)^n}{n!}\sigma_3 \\ &= \cos\theta\mathbb{I} - i\sin\theta\sigma_3 \\ &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \end{aligned} \tag{44}$$

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