Calculus

Probelm Sheet C

Series and Limits

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Series and Limits

C1 Series Notation

(a)

$$
\begin{aligned}
a_n &= \left(-\frac{1}{2}\right)^{n+1} \\
b_n &= (-1)^n \left(\frac{1}{2}\right)^{n+2}
\end{aligned} \tag{1}
$$

$$
\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n+1} = \frac{1/4}{1+1/2} = \frac{1}{6} \tag{2}
$$

(b)

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \tag{3}
$$

(c)

$$
\begin{aligned}
&\left(\sum a_i\right)^2 \\
&\left(\sum a_i\right)\left(\sum b_i\right)
\end{aligned} \tag{4}
$$

•

C2 Maclaurin and Taylor series

(a)

i

$$
\begin{aligned}
e^x &= e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 + \dots \\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots
\end{aligned} \tag{5}
$$

ii

$$
\begin{aligned}
\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{2}\frac{1}{2!}x^2 - \frac{3}{2}\frac{1}{3!}x^3 - \dots \\
&= 1 + \frac{1}{2}x - \frac{1}{4}x^2 - \frac{1}{4}x^3 - \dots
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{4}\frac{1}{2!}x^2 + \frac{3}{8}\frac{1}{3!}x^3 - \dots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots
\end{aligned} \tag{7}
$$

iii

$$
\begin{aligned}
(\tan^{-1})'(x) &= \frac{1}{1+x^2} \\
(\tan^{-1})''(x) &= -\frac{2x}{(1+x^2)^2} \\
(\tan^{-1})'''(x) &= \frac{6x^2-2}{(1+x^2)^3}
\end{aligned} \tag{8}
$$

$$
\tan^{-1} x = x - \frac{1}{3}x^3 + \dots \tag{9}
$$

(b) Note that $1° = \pi/180$. Expanding $\sin x$ about $\pi/6$:

$$
\sin(\pi/180 + \pi/6) \approx \sin \pi/6 + \cos^{\pi/6} \frac{\pi}{180} - \frac{1}{2}\sin \pi/6 \left(\frac{\pi}{180}\right)^2 - \frac{1}{6}\cos^{\pi/6}\left(\frac{\pi}{180}\right)^3 = 0.51504 \tag{10}
$$

The forth term has a value around -7×10^{-7}, so the answer is accurate up to the 5th digit.

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C3 Manipulation of series

(a) Note that $\tan x$ is an odd function, so its expansion must have the form $\tan x = ax + bx^3 + cx^5 + \dots$. Using $\sin x = \cos x \tan x$ and comparing coefficients:

$$
\begin{aligned}
\sin x &= \cos x \tan x \\
\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right)\left(ax + bx^3 + cx^5 + \dots\right)
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
a &= 1 \\
b - \frac{a}{2} &= -\frac{1}{6} \\
c - \frac{b}{2} + \frac{a}{24} &= \frac{1}{120}
\end{aligned} \tag{12}
$$

Solving the equation yields $\tan x = x + x^3/3 + 2x^5/15 + \dots$.

(b)

$$\begin{aligned}
e^{\ln(1+x)} &= 1 + \ln(1+x) + \frac{\ln(1+x)^2}{2} + \frac{\ln(1+x)^3}{6} + \dots \\
&= 1 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) + \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)^2 + \frac{1}{6}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)^3 + \dots \\
&= 1 + x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^2 - x^3}{2} + \frac{x^3}{6} + \dots \\
&= 1 + x
\end{aligned} \tag{13}$$

as expected from the actual value of $e^{\ln(1+x)} = 1 + x$.

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C4 Integration of a power series

$$\begin{aligned}
\int_0^1 \frac{\sin x}{x}\,\mathrm{d}x &= \int_0^1 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\,\mathrm{d}x \\
&= \left[x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^8}{35280} + \dots\right]_0^1 \\
&= 0.9461
\end{aligned} \tag{14}$$

The next term yields a contribution of the value 3×10^{-7} so it does not affect the result up to four significant digits.

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C5 Continuity and differentiability

(a) The function is continuous but is not smooth or differentiable at $x = 0$.

(b) The function is continuous and smooth across the domain.

(c) The function is discontinuous at $x = 0$ and not differentiable.

(d) The function is continuous but is not smooth or differentiable at $x = 0$.

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C6 Limits

(a)

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right) = 1 \tag{15}$$

$$\lim_{x\to 0}\frac{1-\cos^2 x}{x^2}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=1 \tag{16}$$

$$\lim_{x\to 0}\frac{\sin x - x}{e^{-x}-1+x}=\lim_{x\to 0}\frac{-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots}{\frac{x^2}{2!}-\frac{x^3}{3!}+\dots}=0 \tag{17}$$

(b)

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1 \tag{18}$$

$$\lim_{x\to 0}\frac{1-\cos^2 x}{x^2}=\lim_{x\to 0}\frac{\sin 2x}{2x}=1 \tag{19}$$

$$\lim_{x\to 0}\frac{\sin x - x}{e^{-x}-1+x}=\lim_{x\to 0}\frac{\cos x-1}{-e^{-x}+1}=\lim_{x\to 0}\frac{-\sin x}{e^{-x}}=0 \tag{20}$$

(c) For $\lim_{x\to\infty}\sin x/x$, the numerator is bounded while the denominator goes to infinity, so the limit is zero.

For $\lim_{x\to\infty}(1-\cos^2 x)/x$, the same logic applies and the limit is zero.

For $\lim_{x\to\infty}(\sin x - x)/(e^{-x}-1+x)$, we may ignore all terms except for the $\pm x$ as x tends to infinity, and the limit is -1.

(d)

$$[\ln(1+x)]^2=\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)^2=x^2-x^3+\frac{11}{12}x^4+\dots \tag{21}$$

i

$$\begin{aligned}\cos 2x+[\ln(1+x)]^2&=1+\frac{x^2}{2}-\frac{x^3}{6}+x^4+\dots\\ \frac{\mathrm{d}}{\mathrm{d}x}\left\{\cos 2x+[\ln(1+x)]^2\right\}&=x-\frac{x^2}{2}+4x^3+\dots\\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{\cos 2x+[\ln(1+x)]^2\right\}&=1-x+12x^2+\dots\end{aligned} \tag{22}$$

Thus, at $x=0$, the first derivative is zero and the second derivative is unity. Therefore, this is a minimum point.

$$
\begin{aligned}
\cos 2x + [\ln(1+x)]^2 &= 1 - x^2 - x^3 + \frac{19}{12}x^4 + \dots \\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\cos 2x + [\ln(1+x)]^2\right\} &= -2x - 3x^2 + \frac{19}{3}x^3 + \dots \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{\cos 2x + [\ln(1+x)]^2\right\} &= -2 - 6x + 19x^2 + \dots
\end{aligned}
\tag{23}
$$

Thus, at $x = 0$, the first derivative is zero and the second derivative is -2. Therefore, this is a maximum point.

ii

$$
\frac{[\ln(1+x)]^2}{x(1-\cos x)} = \frac{x - x^2 + \frac{11}{12}x^3 + \dots}{\frac{x^2}{2} - \frac{x^3}{6} + \dots}
\tag{24}
$$

This tends to positive infinity as $x \to \infty$.