Electromagnetism 2

Problem Sheet 2

Electric and Magnetic Fields in Matter

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Electric Field in Matter

2.1 Capacitance and dielectrics

(a) Using a small box-shape Gaussian surface, the field produced by a large plate carrying a uniform surface charge density σ is $\sigma/2\epsilon_0$ as computed via Gauss' law. The parallel plate capacitor has two such plates, so the field between the plates is a superposition of the fields produced by each plate:

$$E_0 = \frac{Q_0/A}{2\epsilon_0} - \left(\frac{-Q_0/A}{2\epsilon_0}\right) = \frac{Q_0}{A\epsilon_0} \tag{1}$$

As the field is uniform, the potential difference between the plates is simply $V_0 = E_0 d = Q_0 d/A\epsilon_0$. The capacitance is then $C = Q_0/V_0 = A\epsilon_0/d$.

The potential energy stored in the capacitor can be computed by integrating the energy density $\epsilon_0 E^2/2$ over the volume between the plates:

$$U_0 = \frac{\epsilon_0}{2}\int_V E_0^2\,\mathrm{d}V = \frac{\epsilon_0}{2}\frac{Q_0^2}{A^2\epsilon_0^2}\int_V \mathrm{d}V = \frac{Q_0^2 d}{2A\epsilon_0} \tag{2}$$

(b) Consider the relationship between the potential difference and the charge on the capacitor:

$$V = \frac{Qd}{\epsilon_0 A} \tag{3}$$

The dielectric material inserted causes a change in the permittivity $\epsilon_0 \to \epsilon_r\epsilon_0 > \epsilon_0$. Given constant V, the charge on the capacitor is then $Q \to \epsilon_r Q > Q$.

Apply Gauss' law for electric displacement $\mathbf{D}$ with a small box-shape Gaussian surface:

$$\begin{aligned} \oint_S \mathbf{D}\cdot\mathrm{d}\mathbf{S} &= Q_f \\ D\alpha &= \frac{Q\alpha}{A} \\ \epsilon_0 E + P &= \frac{Q}{A} \end{aligned} \tag{4}$$

which gives $E = (Q/A - P)/\epsilon_0$.

Assuming a linear dielectric with the polarization $P = \epsilon_0\chi_e E$, the electric field is then:

$$E = \frac{Q}{A\epsilon_0(1+\chi_e)} \tag{5}$$

Since the potential is kept constant, the field is also constant so that $E = E_0$, which gives the relation:

$$\frac{Q}{A\epsilon_0(1+\chi_e)} = \frac{Q_0}{A\epsilon_0} \tag{6}$$

or $Q = (1+\chi_e)Q_0$ as expected.

The capacitance is then $C = Q/V_0 = (1+\chi_e)C_0$, and the change in the potential energy is:

$$\Delta U = \frac{Q^2 d}{2A\epsilon_r\epsilon_0} - \frac{Q_0^2 d}{2A\epsilon_0} = \chi_e U_0 \tag{7}$$

(c) Still consider the previous relationship but keep the charge constant. The increase in permittivity causes a decrease in the potential difference $V \to V/\epsilon_r < V$.

The capacitance is a property of geometry and material (permittivity) so it is unchanged, i.e., $C = (1+\chi_e)C_0$. The change in the potential energy is:

$$\Delta U = \frac{Q_0^2 d}{2A\epsilon_r\epsilon_0} - \frac{Q_0^2 d}{2A\epsilon_0} = -\frac{\chi_e}{1+\chi_e}U_0 \tag{8}$$

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2.2 Capacitor half-filled with a dielectric

(a) Label the regions from 1 to 4 from left to right as depicted in the figure. Apparently the electric fields and polarisations in region 1 and 4 are zero. Applying Gauss' law in region 2 and 3 gives:

$$\begin{aligned} D_2\alpha &= \sigma\alpha = (\epsilon_0 E_2 + P)\alpha \\ D_3\alpha &= \sigma\alpha = \epsilon_0 E_3\alpha \end{aligned} \tag{9}$$

Solving the equations gives $E_2 = \sigma/\epsilon_0\epsilon_r$, $P = (1-1/\epsilon_r)\sigma$, and $E_3 = \sigma/\epsilon_0$. Thus:

$$\begin{aligned} \mathbf{D}_2 &= \mathbf{D}_3 = \sigma\hat{x} \\ \mathbf{E}_2 &= \frac{\sigma}{\epsilon_0\epsilon_r}\hat{x} \\ \mathbf{E}_3 &= \frac{\sigma}{\epsilon_0}\hat{x} \\ \mathbf{P} &= \left(1 - \frac{1}{\epsilon_r}\right)\sigma\hat{x} \end{aligned} \tag{10}$$

where $\hat{x}$ is the unit vector pointing from the positive plate to the negative one.

(b) Taking the negative plate as the reference point, the potential difference between the plates is:

$$
\begin{aligned}
V &= \int_0^d E_2 \, \mathrm{d}x + \int_d^{2d} E_3 \, \mathrm{d}x \\
&= \left(\frac{1}{\epsilon_0 \epsilon_r} + \frac{1}{\epsilon_0} \right) \sigma d
\end{aligned}
\tag{11}
$$

The capacitance is:

$$
C = \frac{\sigma A}{V} = \frac{A/d}{1/\epsilon_0 \epsilon_r + 1/\epsilon_0} \tag{12}
$$

Note that:

$$
\frac{1}{C} = \frac{d}{A} \left(\frac{1}{\epsilon_0 \epsilon_r} + \frac{1}{\epsilon_0} \right) = \frac{1}{C_1} + \frac{1}{C_2} \tag{13}
$$

where C_1 is the capacitance of the left half and C_2 is the capacitance of the right half.

This is as though the capacitance of two capacitors in series.

(c) As the polarisation is constant, there is no volume bound density since $\nabla \cdot \mathbf{P} = 0$. On the interface between region 1 and 2:

$$
\sigma_b = -p = -\left(1 - \frac{1}{\epsilon_r} \right) \sigma \tag{14}
$$

On the interface between region 2 and 3:

$$
\sigma_b = p = \left(1 - \frac{1}{\epsilon_r} \right) \sigma \tag{15}
$$

Using all surface charge densities, the electric field in four regions are:

$$
\begin{aligned}
\mathbf{E}_1 &= \frac{\sigma}{2\epsilon_0}\left[-\frac{1}{\epsilon_r} - \left(1 - \frac{1}{\epsilon_r}\right) + 1\right]\hat{x} = 0 \\
\mathbf{E}_2 &= \frac{\sigma}{2\epsilon_0}\left[\frac{1}{\epsilon_r} - \left(1 - \frac{1}{\epsilon_r}\right) + 1\right]\hat{x} = \frac{\sigma}{\epsilon_0\epsilon_r}\hat{x} \\
\mathbf{E}_3 &= \frac{\sigma}{2\epsilon_0}\left[\frac{1}{\epsilon_r} + \left(1 - \frac{1}{\epsilon_r}\right) + 1\right]\hat{x} = \frac{\sigma}{\epsilon_0}\hat{x} \\
\mathbf{E}_4 &= \frac{\sigma}{2\epsilon_0}\left[\frac{1}{\epsilon_r} + \left(1 - \frac{1}{\epsilon_r}\right) - 1\right]\hat{x} = 0
\end{aligned}
\tag{16}
$$

which are the same as the results in part (a).

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2.3 Force on a dielectric

Magnetic Field in Matter