Ordinary Differential Equations

Problem Set 2

Second-Order ODEs, Part I

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Minimal Set

2.1 Homogeneous ODEs

(i) The characteristic equation is $r^2 + 2r - 15 = 0$ with the roots $r = 3$ and $r = -5$. Thus, the general solution is:

$$y = Ae^{3x} + Be^{-5x} \tag{1}$$

where A and B are arbitrary constants.

(ii) The characteristic equation is $r^2 - 6 + 9 = 0$ with the repeated root $r = 3$. Thus, the general solution is:

$$y = Ae^{3x} + Bxe^{3x} \tag{2}$$

where A and B are arbitrary constants.

Then we have $y' = 3Ae^{3x} + Be^{3x}(3x + 1)$. The initial conditions give us the equations:

$$\begin{aligned} A &= 3 \\ 3A + B &= 1 \end{aligned} \tag{3}$$

Thus, $A = 3$ and $B = -8$ and the solution is:

$$y = 3e^{3x} - 8xe^{3x} \tag{4}$$

(iii) The characteristic equation is $r^2 - 4r + 13 = 0$ with the roots $r = 2 + 3i$ and $r = 2 - 3i$. Thus, the general solution is:

$$y = e^{2x}(Ae^{3ix} + Be^{-3ix}) = e^{2x}(C \cos 3x + D \sin 3x) \tag{5}$$

where A, B, C, and D are arbitrary constants.

(iv) Assume a solution of the form $y = Ae^{rx}$. We have the characteristic equation $r^3 + 7r^2 + 7r - 15 = 0$ with the roots $r = 1$, $r = -3$ and $r = -5$. Thus, the general solution is:

$$y = Ae^{x} + Be^{-3x} + Ce^{-5x} \tag{6}$$

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2.2 Damped Oscillator

By Newton's second law, we have:

$$F = m\ddot{y} = -kx - \gamma\dot{x} \tag{7}$$

where m is the mass, k is the spring constant, γ is the damping constant.

$\omega_0 \equiv \sqrt{k/m}$ is the natural frequency of the oscillator.

(a) The differential equation can be simplified to:

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0 \tag{8}$$

whose characteristic equation is:

$$r^2 + \gamma r + \omega_0^2 = 0 \tag{9}$$

i First consider the case of over-damping where $\gamma > 2\omega_0$ such that there are two distinct real roots. The general solution is:

$$y(t) = Ae^{r_+ t} + Be^{r_- t} \tag{10}$$

where $r_+ = (-\gamma + \sqrt{\gamma^2 - 4\omega_0^2})/2$ and $r_- = (-\gamma - \sqrt{\gamma^2 - 4\omega_0^2})/2$.

The first derivative has the form:

$$\dot{y}(t) = Ar_+e^{r_+ t} + Br_-e^{r_- t} \tag{11}$$

The initial conditions give us the equations:

$$\begin{aligned} A + B &= y_0 \\ Ar_+ + Br_- &= 0 \end{aligned} \tag{12}$$

Thus, $A = y_0/(1 - r_+/r_-)$ and $B = y_0/(1 - r_-/r_+)$.

ii Next consider the case of critical damping where $\gamma = 2\omega_0$ such that there is one repeated real root $r = -\gamma/2$. The general solution is:

$$y(t) = e^{-\gamma t/2}(A + Bt) \tag{13}$$

The first derivative has the form:

$$\dot{y}(t) = e^{-\gamma t/2}\left[-\gamma(A+Bt)/2 + B\right] \tag{14}$$

The initial conditions give us the equations:

$$\begin{aligned} A &= y_0 \\ B - \gamma A/2 &= 0 \end{aligned} \tag{15}$$

Thus, $A = y_0$ and $B = \gamma y_0/2$.

iii Finally, consider the case of under-damping where $\gamma < 2\omega_0$ such that there are two distinct complex roots. The general solution is:

$$y(t) = e^{-\gamma t/2}(Ae^{i\omega t} + Be^{-i\omega t}) \tag{16}$$

where $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$.

The first derivative has the form:

$$\dot{y}(t) = e^{-\gamma t/2}\left[-\frac{\gamma}{2}(Ae^{i\omega t} + Be^{-i\omega t}) + i\omega(Ae^{i\omega t} - Be^{-i\omega t})\right] \tag{17}$$

The initial conditions give us the equations:

$$\begin{aligned} A + B &= y_0 \\ -\frac{\gamma}{2}(A+B) + i\omega(A-B) &= 0 \end{aligned} \tag{18}$$

Solving these equations yields:

$$\begin{aligned} A &= (1 + \frac{\gamma}{i2\omega})y_0 \\ B &= (1 - \frac{\gamma}{i2\omega})y_0 \end{aligned} \tag{19}$$

iv When $\gamma = 0$, the system is un-damped and pure harmonic oscillation occurs.

(b) The energy of the system is given by $E = m(\omega^2 y^2 + \dot{y}^2)/2$. Expand the expression in complex exponentials, noting that $\omega \approx \omega_0$ and ignoring all terms of order γ^2 and higher, we have:

$$E \approx 2me^{-\gamma t}AB\omega^2 \tag{20}$$

The energy lost in once period of oscillation can be approximated by:

$$\Delta E \approx \frac{\mathrm{d}E}{\mathrm{d}t}\frac{2\pi}{\omega} = -\frac{2\pi}{\omega}\gamma E \tag{21}$$

Thus, $Q = 2\pi E/\left|\Delta E\right| = \omega\gamma = \omega_0\gamma$

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2.3

Substituting $y = x + 1$ into the differential equation:

$$(x^2 - 1)(0) + (x + 1)(1) - (x + 1) = 0 \tag{22}$$

Thus $y = x + 1$ is a solution of the differential equation.

Consider a new solution of the form $y = (x + 1)\phi(x)$ where $\phi(x)$ is an unknown function we seek. Substituting this into the differential equation and simplifying yields:

$$(x - 1)(x + 1)\phi'' + (3x - 1)\phi' = 0 \tag{23}$$

This is a separable differential equation for $\phi'(x)$:

$$\frac{\mathrm{d}\phi'}{\phi'} = \frac{1 - 3x}{(x - 1)(x + 1)}\mathrm{d}x \tag{24}$$

Integrating this yields:

$$\phi'(x) = \frac{C}{\left|x + 1\right|^2 \left|x - 1\right|} \tag{25}$$

where C is an arbitrary constant.

Integrating again gives:

$$\phi(x) = \frac{C}{4}\left(\frac{2}{x + 1} + \ln\left|\frac{x - 1}{x + 1}\right|\right) + D \tag{26}$$

where D is an arbitrary constant.

The new solution $(x + 1)\phi(x)$ is evidently linearly independent from $(x + 1)$, so their linear combination is the general solution to the differential equation. Combining the arbitrary constants, the general solution has the form:

$$y = D(x+1) + C\left(\frac{x+1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\right) \tag{27}$$

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2.4 Nonlinear ODEs

(a) Notice that $\mathrm{d}(yy')/\mathrm{d}y = yy'' + (y')^2$. Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}y}(yy') = -1 \tag{28}$$

We have $yy' = -y + C$ for some arbitrary constant C. This gives us a separable differential equation for y:

$$\frac{y}{-y+C}\mathrm{d}y = \mathrm{d}x \tag{29}$$

Integrating this gives us an implicit expression for y:

$$y + C\ln|y - C| = -x + D \tag{30}$$

where D is an arbitrary constant.

(b) The present differential equation is equivalent to a separable differential equation for y':

$$\frac{\mathrm{d}y'}{(y')^3 + y'} = \frac{\mathrm{d}x}{x} \tag{31}$$

which has the solution of the form:

$$\frac{y'}{(y')^2 + 1} = Cx \tag{32}$$

where C is an arbitrary constant and $y' \neq 0$.

Simplifying:

$$Cx(y')^2 - y' + Cx = 0 \tag{33}$$

and thus:

$$y' = \frac{1 \pm \sqrt{1-4C^2x^2}}{2Cx} \tag{34}$$

Integrating again yields the solution:

$$y = \frac{1}{2C}\left(\pm\sqrt{1-4C^2x^2} \mp \tan^{-1}\sqrt{1-4C^2x^2} + \ln x\right) + D \tag{35}$$

where D is an arbitrary constant.

(c) Using the identity $y'' = pp'$, we have:

$$p' = \frac{p-(y-1)^2}{y-1} \tag{36}$$

where $p = y' \neq 0$ and $y \neq 1$.

This becomes a first order differential equation for p:

$$\frac{\mathrm{d}p}{\mathrm{d}y} - \frac{1}{y-1}p = 1-y \tag{37}$$

Consider the integrating factor $\Lambda(x) = 1/(y-1)$:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{p}{y-1}\right) = -1 \tag{38}$$

Integrating this gives us:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p = (y-1)(C-y) \tag{39}$$

where C is an arbitrary constant.

Integrating the separable differential equation for y yields:

$$\left|\frac{y-1}{y-C}\right| = e^{(C-1)(x+D)} \tag{40}$$

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Supplementary Questions

2.5

Consider the substitution $z = y'/y$. We have $z' = y''/y - z^2$ and, after some simplification:

$$9xz' + 9xz^2 + (6+x)z + \lambda = 0 \tag{41}$$

Consider the case where $\lambda = 0$ and the equation becomes a Bernoulli's equation:

$$\begin{aligned} \frac{z'}{z^2} + \frac{6+x}{9x}z &= -1 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{z}\right) - \frac{6+x}{9x}z &= 1 \end{aligned} \tag{42}$$

Consider the integrating factor $\Lambda(x) = e^{-x/9}\left|x\right|^{-2/3}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x/9}\left|x\right|^{-2/3}\frac{1}{z}\right) = e^{-x/9}\left|x\right|^{-2/3} \tag{43}$$

Let $\int \Lambda(x) = \Delta(x) + C$ for some arbitrary constant C. Then:

$$\Lambda(x)\frac{y}{y'} = \Delta(x) + C \tag{44}$$

$$\ln|y| = \int \frac{\Lambda(x)}{\Delta(x)+C}\,\mathrm{d}x = \ln|\Delta(x)+C| + D \tag{45}$$

where D is an arbitrary constant.

Therefore:

$$y(x) = D\left[\Delta(x)+C\right] = D\left(\int e^{-x/9}\left|x\right|^{-2/3}\,\mathrm{d}x + C\right) \tag{46}$$

To make $y(x)$ such that $y \to 0$ as $x \to \pm\infty$, we incorporate the choice of C in to the limits of the integral:

$$y(x) = D\int_{-\infty}^{x} e^{-x/9}\left|x\right|^{-2/3}\,\mathrm{d}x \tag{47}$$

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