Symmetry and Relativity

Problem Set 5

Dynamics and Electromagnetism in Special Relativity

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1 Magnetic dipole radiation

(a) Consider the magnetic potential:

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int \frac{I(t-R/c)}{R}\,\mathrm{d}\mathbf{l} \\ &= \hat{\phi}\frac{\mu_0}{4\pi}\int \frac{I(t-R/c)}{R} a\,\mathrm{d}\phi \\ &= \hat{\phi}\frac{\mu_0 a}{2}\frac{I(t-R/c)}{R}\end{aligned} \tag{1}$$

since there is no ϕ dependence in the integrand.

Several approximations can be made. First, from geometry:

$$\begin{aligned}R^2 &= a^2 + r^2 - 2ar\sin\theta \\ R &\approx \sqrt{r^2 - 2ar\sin\theta} \approx r\left(1 - \frac{a}{r}\sin\theta\right) \\ \frac{1}{R} &\approx \frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\right)\end{aligned} \tag{2}$$

Further:

$$\begin{aligned}I(t-R/c) &= I_0\cos\left(\omega t - R\frac{\omega}{c}\right) \\ &\approx I_0\cos\left[\omega(t-r/c) + \frac{a\omega}{c}\sin\theta\right] \\ &\approx I_0\cos\left[\omega(t-r/c)\right] - I_0\sin\left[\omega(t-r/c)\right]\frac{a\omega}{c}\sin\theta\end{aligned} \tag{3}$$

Then:

$$\frac{I(t-R/c)}{R} \approx \frac{I_0}{r}\left\{\cos\left[\omega(t-r/c)\right] + \cos\left[\omega(t-r/c)\right]\frac{a}{r}\sin\theta - \sin\left[\omega(t-r/c)\right]\frac{a\omega}{c}\sin\theta\right\} \tag{4}$$

Given that $I_0 = M_0/(\pi a^2)$, we have:

$$\mathbf{A}(\mathbf{r},t) \approx -\hat{\phi}\frac{\mu_0 M_0 \omega}{2\pi c}\frac{\sin\theta}{r}\sin\left[\omega(t-r/c)\right] \tag{5}$$

Consider the magnetic potential:

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int \frac{I(t-R/c)}{R}\,\mathrm{d}\mathbf{l} \\ &= \hat{\phi}\frac{\mu_0}{4\pi}\int \frac{I(t-R/c)}{R} a\cos\phi\,\mathrm{d}\phi\end{aligned} \tag{6}$$

where ϕ is the azimuthal angle between the current element and the radial vector $\mathbf{R}$. From geometry:

$$\begin{aligned}R^2 &= a^2\cos^2\phi + r^2 - 2ar\cos\phi\sin\theta \\ R &\approx \sqrt{r^2 - 2ar\cos\phi\sin\theta} \approx r\left(1 - \frac{a}{r}\cos\phi\sin\theta\right) \\ \frac{1}{R} &\approx \frac{1}{r}\left(1 + \frac{a}{r}\cos\phi\sin\theta\right)\end{aligned} \tag{7}$$

Further:

$$\begin{aligned}I(t-R/c) &= I_0\cos\left(\omega t - R\frac{\omega}{c}\right) \\ &\approx I_0\cos\left[\omega(t-r/c) + \frac{a\omega}{c}\cos\phi\sin\theta\right] \\ &\approx I_0\cos\left[\omega(t-r/c)\right] - I_0\sin\left[\omega(t-r/c)\right]\frac{a\omega}{c}\cos\phi\sin\theta\end{aligned} \tag{8}$$

Note that the first term integrates to zero, and the second term gives the correct result.

(b) For this case, since the magnetic potential has only a ϕ component, the magnetic field is perpendicular to the electric field, so that the Poynting vector is $S = E^2/c\mu_0$. The electric field is given by:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \hat{\phi}\frac{\mu_0 M_0\omega^2}{2\pi c}\frac{\sin\theta}{r}\cos\left[\omega(t-r/c)\right] \tag{9}$$

so that the magnitude of the Poynting vector is:

$$S = \frac{E^2}{c\mu_0} = \frac{\mu_0 M_0^2\omega^4}{8\pi^2 c^3}\frac{\sin^2\theta}{r^2}\cos^2\left[\omega(t-r/c)\right] \tag{10}$$

The radiated power is then:

$$\begin{aligned}P &= \int S\,\mathrm{d}A \\ &= \frac{\mu_0 M_0^2\omega^4}{8\pi^2 c^3}\cos^2\left[\omega(t-r/c)\right]\int \frac{\sin^2\theta}{r^2} r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{\mu_0 M_0^2\omega^4}{8\pi^2 c^3}\left(\frac{1}{2}\right)\left(\frac{4}{3}\right)(2\pi) \\ &= \frac{\mu_0 M_0^2\omega^4}{12\pi c^3}\end{aligned} \tag{11}$$

where the third line follows by time averaging the cosine squared function.

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2 Electric field of a charge moving under a constant force

We have $\mathbf{r} = (\eta, y, 0)$, $\mathbf{r}(t_c) = (\sqrt{\eta^2 + c^2t_c^2}, 0, 0)$ and $\mathbf{R} = \mathbf{r} - \mathbf{r}(t_c)$. At $t = 0$, we have the relation:

$$\begin{aligned} ct_c &= R \\ c^2t_c^2 &= (\eta - \sqrt{\eta^2 + c^2t_c^2})^2 + y^2 \end{aligned} \tag{12}$$

Expanding the above equation gives us $2\eta\sqrt{\eta^2 + c^2t_c^2} = 2\eta^2 + y^2$, which leads to:

$$x_c = \eta + \frac{y^2}{2\eta} \tag{13}$$

Then:

$$\begin{aligned} v_c &= \frac{\mathrm{d}x_c}{\mathrm{d}t_c} = \frac{c^2t_c}{x_c} \\ a_c &= \frac{\mathrm{d}^2x_c}{\mathrm{d}t_c^2} = \frac{c^2\eta^2}{x_c^3} \end{aligned} \tag{14}$$

When $\eta = 1$ and $y = 2$, $x_c = 3$, $ct_c = \sqrt{8}$. This means $v_c/c = \sqrt{8}/3$ and $a_c/c^2 = 1/27$.

$v_c/c = -\sqrt{8}/3$ as the charge is moving in the negative x direction.

We have $\mathbf{R} = (-2, 2, 0)$ so $\hat{R} = (-1, 1, 0)/\sqrt{2}$. This means:

$$\begin{aligned} \hat{R} - \mathbf{v}_c/c &= \begin{pmatrix} -1/\sqrt{2} - \sqrt{8}/3 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \\ \hat{R} \times (\hat{R} \times \mathbf{a}_c) &= \begin{pmatrix} -1/54 \\ -1/54 \\ 0 \end{pmatrix} \end{aligned} \tag{15}$$

On the other hand, $\gamma^2 = 1 - v_c^2/c^2 = 1 - 8/9 = 1/9$ so the factor $\gamma^{-2}R^{-1}$ gives $9/\sqrt{8}$.

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3 Radiation losses in accelerators

(a) In the current case, the velocity is parallel to the acceleration, so that radiation power is given by:

$$P_{\text{rad}} = \frac{q^2}{6\pi\epsilon_0 c^3}\gamma^6 a^2 = \frac{q^2}{6\pi\epsilon_0 c^3} a_0^2 \tag{16}$$

where $a_0 = \gamma^6 a^2$ is the proper acceleration.

The 4-force is given by:

$$F^\mu = \gamma(P/c, f) = mA^\mu \tag{17}$$

Apparently the force is pure, so that $fu = \mathrm{d}E/\mathrm{d}t = P$. Then, taking the inner product of the above equation:

$$\begin{aligned} \gamma^2(-P^2/c^2 + f^2) &= m^2 a_0^2 \\ \gamma^2 P^2(-1/c^2 + 1/u^2) &= m^2 a_0^2 \\ P^2/u^2 &= m^2 a_0^2 \end{aligned} \tag{18}$$

Now consider:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}x} u \tag{19}$$

so that $\mathrm{d}E/\mathrm{d}x = P/u = ma_0$.

Finally, the ratio P_{rad}/P is:

$$\begin{aligned} \frac{P_{\text{rad}}}{P} &= \frac{q^2}{6\pi\epsilon_0 c^3}\frac{a_0^2}{P^2}P \\ &= \frac{q^2}{6\pi\epsilon_0 c^3}\frac{1}{m^2 u^2}\frac{\mathrm{d}E}{\mathrm{d}x}u \\ &\approx \frac{q^2}{6\pi\epsilon_0 m^2 c^4}\frac{\mathrm{d}E}{\mathrm{d}x} \end{aligned} \tag{20}$$

where the final approximation follows from the fact that $u \approx c$ for ultra-relativistic particles.

(b) For circular motion, we have:

$$\begin{aligned} P_{\text{rad}} &= \frac{q^2}{6\pi\epsilon_0 c^3}\gamma^6\left(a^2 - \frac{v^2 a^2}{c^2}\right) \\ &\propto \gamma^4 a^2 \end{aligned} \tag{21}$$

On the other hand, $a = v^2/r$ and $T \propto r/v$. Now, the energy loss per turn is:

$$
\begin{aligned}
\delta E &\propto P_{\text{rad}} T \\
&\propto \gamma^4 \left(\frac{v^3}{r}\right)^2 \frac{r}{v} \\
&\propto \gamma^4 v^5 \frac{1}{r} \\
&\propto \frac{E^4}{r}
\end{aligned}
\tag{22}
$$

since for ultra-relativistic particles, $\gamma = E/mc^2$ and $v \approx c$.

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4 Radiation reaction force

(a) The work done by the radiation reaction force is:

$$
\begin{aligned}
\int_T \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\, \mathrm{d}t &= -\frac{\mu_0 q^2}{6\pi c} \int_T \mathbf{a} \cdot \mathbf{a}\, \mathrm{d}t \\
&= -\frac{\mu_0 q^2}{6\pi c} \int_T \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\, \mathrm{d}t \\
&= \left[-\frac{\mu_0 q^2}{6\pi c} \mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right]_T + \frac{\mu_0 q^2}{6\pi c} \int_T \mathbf{v} \cdot \frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2}\, \mathrm{d}t
\end{aligned}
\tag{23}
$$

The boundary term vanishes over a period, so that by equating the integrands, we have:

$$
\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} \tag{24}
$$

(b)

(c) To force $F^\mu U_\mu = 0$, we need to have:

$$
F^\mu U_\mu \propto \left[\frac{\mathrm{d}A^\mu}{\mathrm{d}\tau} U_\mu + \alpha \left(\frac{\mathrm{d}A^\nu}{\mathrm{d}\tau} U_\nu\right) U^\mu U_\mu\right] = 0 \tag{25}
$$

Since $U_\mu U^\mu = -c^2$, we simply need $\alpha = -1/c^2$ so that the two terms cancel each other out.

On the other hand, consider:

$$
\begin{aligned}
\frac{\mathrm{d}A^\nu}{\mathrm{d}\tau} U_\nu &= \int A^\nu U_\nu\, \mathrm{d}\tau - A^\nu \frac{\mathrm{d}U_\nu}{\mathrm{d}\tau} \\
&= -A^\nu A_\nu
\end{aligned}
\tag{26}
$$

where the last line follows from the fact that $A^\nu U_\nu = 0$.

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5 Radiation from a charge under a constant force

Using results from Problem 2, we have:

$$\begin{aligned} x &= \sqrt{\eta^2 + c^2 t^2} \\ v &= \frac{c^2 t}{x} \\ a &= \frac{c^2 \eta^2}{x^3} \end{aligned} \tag{27}$$

Since the velocity is parallel to the acceleration, the radiation power is given by:

$$\begin{aligned} P_{\text{rad}} &= \frac{q^2}{6\pi\epsilon_0 c^3}\gamma^6 a^2 \\ &= \frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{1}{1 - v^2/c^2}\right)^3 a^2 \\ &= \frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{1}{1 - c^2t^2/x^2}\right)^3 \frac{c^4\eta^4}{x^6} \\ &= \frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{1}{x^2 - c^2t^2}\right)^3 c^4\eta^4 \\ &= \frac{q^2}{6\pi\epsilon_0 c^3}\frac{c^4}{\eta^2} \end{aligned} \tag{28}$$

where c^4/η^2 is just the proper acceleration a_0.

From the previous question, we have the radiation reaction force:

$$\begin{aligned} F^{\mu}_{\text{rad}} &= \frac{\mu_0 q^2}{6\pi c}\left[-\frac{1}{c^2}(A_\nu A^\nu)U^\mu\right] \\ &= -\frac{q^2}{6\pi\epsilon_0 c^3}\left(\frac{c^4}{\eta^2}\right)\frac{U^\mu}{c^2} \end{aligned} \tag{29}$$

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