Mathematical Methods

Problem Sheet 1

Normed and Inner Product Vector Space

Xin, Wenkang

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1 Examples of function vector spaces

(a) For f and g in $\mathcal{F}$, the vector addition can be defined as:

$$(f+g)(x) = f(x) + g(x) \tag{1}$$

while the scalar multiplication can be defined as:

$$(\alpha f)(x) = \alpha f(x) \tag{2}$$

Apparently:

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) \tag{3}$$

There also exists a zero vector $0(x) = 0$. This means $\mathcal{F}$ is a vector space.

(b) Subspaces of $\mathcal{F}$ include:

$$\begin{aligned} \mathcal{P} &= \{f_n(x) = x^n, n \in \mathbb{N}\} \\ \mathcal{E} &= \{f_n(x) = e^{nx}, n \in \mathbb{N}\} \\ \mathcal{S} &= \{f_n(x) = \sin nx, n \in \mathbb{N}\} \end{aligned} \tag{4}$$

(c) A possible scalar product for $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{S}$ is:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x \tag{5}$$

with the associated norm:

$$\|f\| = \sqrt{\int_0^1 f(x)^2\,\mathrm{d}x} \tag{6}$$

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2 Polarisation identities

(a) Consider the following identities:

$$
\begin{aligned}
\langle v+w, v+w\rangle &= \langle v+w, v\rangle + \langle v+w, w\rangle \\
&= \langle v, v\rangle + \langle v, w\rangle^* + \langle w, v\rangle^* + \langle w, w\rangle \\
\langle v-w, v-w\rangle &= \langle v-w, v\rangle - \langle v-w, w\rangle \\
&= \langle v, v\rangle - \langle v, w\rangle^* - \langle w, v\rangle^* + \langle w, w\rangle
\end{aligned} \tag{7}
$$

Similarly

$$
\begin{aligned}
\langle v+iw, v+iw\rangle &= \langle v, v\rangle + i\langle v, w\rangle - i\langle w, v\rangle + \langle w, w\rangle \\
\langle v-iw, v-iw\rangle &= \langle v, v\rangle - i\langle v, w\rangle + i\langle w, v\rangle + \langle w, w\rangle
\end{aligned} \tag{8}
$$

Combining the above results, we have:

$$
\langle v+w, v+w\rangle - \langle v-w, v-w\rangle - i\langle v+iw, v+iw\rangle + i\langle v-iw, v-iw\rangle = 4\langle v, w\rangle \tag{9}
$$

as required.

(b) If V is a real inner product space, then:

$$
\begin{aligned}
\langle v+w, v+w\rangle &= \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \\
\langle v-w, v-w\rangle &= \langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle
\end{aligned} \tag{10}
$$

so that:

$$
\langle v+w, v+w\rangle - \langle v-w, v-w\rangle = 4\langle v, w\rangle \tag{11}
$$

(c) We have:

$$
\begin{aligned}
\langle v+w, T(v+w)\rangle &= \langle v, T(v)\rangle + \langle w, T(v)\rangle + \langle v, T(w)\rangle + \langle w, T(w)\rangle \\
\langle v-w, T(v-w)\rangle &= \langle v, T(v)\rangle - \langle w, T(v)\rangle - \langle v, T(w)\rangle + \langle w, T(w)\rangle \\
\langle v+iw, T(v+iw)\rangle &= \langle v, T(v)\rangle - i\langle w, T(v)\rangle + i\langle v, T(w)\rangle + \langle w, T(w)\rangle \\
\langle v-iw, T(v-iw)\rangle &= \langle v, T(v)\rangle + i\langle w, T(v)\rangle - i\langle v, T(w)\rangle + \langle w, T(w)\rangle
\end{aligned} \tag{12}
$$

Combining the above results, we have:

$$\langle v+w, T(v+w)\rangle - \langle v-w, T(v-w)\rangle - i\langle v+iw, T(v+iw)\rangle + i\langle v-iw, T(v-iw)\rangle = 4\langle v, T(w)\rangle \tag{13}$$

as required.

(d) If $\langle v, T(v)\rangle = 0$ for all $v \in V$, then:

$$\langle 2v, 2T(v)\rangle - i\langle v+iv, T(v+iv)\rangle + i\langle v-iv, T(v-iv)\rangle = 0 \tag{14}$$

However, we have:

$$\begin{aligned} \langle v+iv, T(v+iv)\rangle &= \langle v, T(v)\rangle - i\langle v, T(v)\rangle + i\langle v, T(v)\rangle + \langle v, T(v)\rangle = 2\langle v, T(v)\rangle \\ \langle v-iv, T(v-iv)\rangle &= \langle v, T(v)\rangle + i\langle v, T(v)\rangle - i\langle v, T(v)\rangle + \langle v, T(v)\rangle = 2\langle v, T(v)\rangle \end{aligned} \tag{15}$$

Therefore, the following equality holds:

$$(4 - 2i + 2i)\langle v, T(v)\rangle = 0 \tag{16}$$

which implies that $\langle v, T(v)\rangle = 0$ for all $v \in V$.

This means that $T(v) = 0$ because v is arbitrary.

(e) First suppose T is hermitian, then by definition:

$$\langle v, T(v)\rangle = \left\langle T^\dagger(v), v\right\rangle = \langle T(v), v\rangle \tag{17}$$

But $\langle v, T(v)\rangle = \langle T(v), v\rangle^*$ for a complex inner product space. This means $\langle v, T(v)\rangle \in \mathbb{R}$.

Now suppose $\langle v, T(v)\rangle \in \mathbb{R}$ for some T, then:

$$\langle v, T(v)\rangle = \langle v, T(v)\rangle^* = \langle T(v), v\rangle \tag{18}$$

On the other hand, we have:

$$\langle v, T(v)\rangle = \left\langle T^\dagger(v), v\right\rangle \tag{19}$$

which means:

$$\left\langle v, T(v) - T^\dagger(v)\right\rangle = 0 \tag{20}$$

for all $v \in V$. Therefore, $T = T^\dagger$ and T is hermitian.

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3 The normed vector space and the parallelogram identity

(a) We have:

$$\begin{aligned} &\langle v+w, v+w\rangle + \langle v-w, v-w\rangle \\ =&\langle v,v\rangle + \langle w,v\rangle + \langle v,w\rangle + \langle w,w\rangle + \langle v,v\rangle - \langle w,v\rangle - \langle v,w\rangle + \langle w,w\rangle \\ =&2\langle v,v\rangle + 2\langle w,w\rangle \end{aligned} \tag{21}$$

as required.

(b) Apparently the norm is positive definite because for a sequence (x_i) not all zero. Consider the linearity condition:

$$\|\alpha(x_i)\| = \left(\sum_{i=1}^{\infty} |\alpha x_i|^p\right)^{1/p} = |\alpha| \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} = |\alpha| \, \|x_i\| \tag{22}$$

To prove the triangle inequality, we use without proof the Holder's inequality:

$$\sum_{i=1}^{n} |v_i w_i| \le \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |w_i|^q\right)^{1/q} \tag{23}$$

where $1/p + 1/q = 1$.

Assuming a non-zero $\|x_i + y_i\|$, we have:

$$\begin{aligned} \|(x_i+y_i)\|^p &= \sum_{i=1}^{\infty} |x_i+y_i|^p \\ &= \sum_{i=1}^{\infty} |x_i+y_i|^{p-1} |x_i+y_i| \\ &\le \sum_{i=1}^{\infty} |x_i+y_i|^{p-1} (|x_i| + |y_i|) \\ &= \sum_{i=1}^{\infty} |x_i+y_i|^{p-1} |x_i| + \sum_{i=1}^{\infty} |x_i+y_i|^{p-1} |y_i| \end{aligned} \tag{24}$$

But by Holder's inequality:

$$\begin{aligned}\sum_{i=1}^{\infty} |x_i + y_i|^{p-1} |x_i| &\leq \left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1-1/p} \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \\ \sum_{i=1}^{\infty} |x_i + y_i|^{p-1} |y_i| &\leq \left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1-1/p} \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}\end{aligned} \tag{25}$$

Thus:

$$\|(x_i + y_i)\|^p \leq \left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1-1/p} (\|x_i\| + \|y_i\|) = \frac{\|(x_i + y_i)\|^p}{\|x_i + y_i\|} (\|x_i\| + \|y_i\|) \tag{26}$$

and the triangle inequality results.

(c) Suppose on the contrary that there exists some inner product $\langle \cdot, \cdot \rangle$ for some $p \neq 2$. Then the associated norm satisfies the parallelogram identity. With the proposed vectors, we have:

$$\|v + w\|^2 + \|v - w\|^2 = (2 \times (1^p))^{2/p} + (2 \times (1^p))^{2/p} \tag{27}$$

On the other hand:

$$2\|v\|^2 + 2\|w\|^2 = 2 \times (1^p) + 2 \times (1^p) \tag{28}$$

These are only equal when $p = 2$, which contradicts the assumption.

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4 Recap of Gram-Schmidt procedure

(a) With the basis $1, x, x^2$, we start from 1 and normalize it:

$$\hat{p}_1 = \frac{1}{\sqrt{\langle 1, 1 \rangle}} = \left(\int e^{-x^2}\, \mathrm{d}x\right)^{-1/2} = \pi^{-1/4} \tag{29}$$

Then the (un-normalised) second basis is:

$$p_2 = x - \langle x, \hat{p}_1 \rangle \hat{p}_1 = x - \frac{1}{\sqrt{\pi}} \int x e^{-x^2}\, \mathrm{d}x = x \tag{30}$$

Normalising it, we have:

$$\hat{p}_2 = \frac{x}{\sqrt{\langle x, x\rangle}} = x\left(\int x^2 e^{-x^2}\,\mathrm{d}x\right)^{-1/2} = \left(\frac{4}{\pi}\right)^{1/4} x \tag{31}$$

Finally, the (un-normalised) third basis is:

$$p_3 = x^2 - \left\langle x^2, \hat{p}_1\right\rangle \hat{p}_1 - \left\langle x^2, \hat{p}_2\right\rangle \hat{p}_2 = x^2 - \frac{1}{\sqrt{\pi}}\int x^2 e^{-x^2}\,\mathrm{d}x - \frac{4}{\pi}\int x^3 e^{-x^2}\,\mathrm{d}x = x^2 - \frac{1}{2} \tag{32}$$

Normalising it, we have:

$$\hat{p}_3 = \frac{x^2 - 1/2}{\sqrt{\langle x^2 - 1/2, x^2 - 1/2\rangle}} = \left(\int \left(x^2 - \frac{1}{2}\right)^2 e^{-x^2}\,\mathrm{d}x\right)^{-1/2} = \left(\frac{4}{\pi}\right)^{1/4}\left(x^2 - \frac{1}{2}\right) \tag{33}$$

(b) We will have:

$$q(x) = \sum_{k=0}^{2} b_k \hat{p}_k(x) \tag{34}$$

where $b_k = \langle q, \hat{p}_k\rangle$.

The coefficients should be:

$$b_k = \langle \hat{p}_k, q\rangle \tag{35}$$

(c) We have:

$$\langle q, q\rangle = \langle b_i \hat{p}_i, b_j \hat{p}_j\rangle = b_i b_j \left\langle \hat{p}_i, \hat{p}_j\right\rangle = b_i b_j \delta_{ij} = b_i^2 \tag{36}$$

(d) Given the linear operator, we can write the scalar product as:

$$\langle q, T(q)\rangle = \int q \frac{\mathrm{d}}{\mathrm{d}x}\left(w \frac{\mathrm{d}q}{\mathrm{d}x}\right)\mathrm{d}x \tag{37}$$

On the other hand, we have:

$$\langle T(q), q\rangle = \int \frac{\mathrm{d}}{\mathrm{d}x}\left(w \frac{\mathrm{d}q}{\mathrm{d}x}\right) q\,\mathrm{d}x = \langle q, T(q)\rangle \tag{38}$$

Therefore, T is hermitian.

We can compute the effect of T on the basis vectors:

$$
\begin{aligned}
T(\hat{p}_1) &= 0 \\
T(\hat{p}_2) &= \left(\frac{4}{\pi}\right)^{1/4}(-2x) = -2\hat{p}_2 \\
T(\hat{p}_3) &= \left(\frac{4}{\pi}\right)^{1/4}(-4x^2+2) = -4\hat{p}_3
\end{aligned}
\tag{39}
$$

Thus the matrix representation of T is:

$$
T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}
\tag{40}
$$

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5 Orthonormal basis

(a) Any vector v can be written as:

$$
v = \sum_i \langle v|\epsilon_i\rangle |\epsilon_i\rangle
\tag{41}
$$

so that the inner product between v and w is:

$$
\langle v|w\rangle = \sum_i \sum_j \langle v|\epsilon_i\rangle \langle \epsilon_j|w\rangle = \sum_i \langle v|\epsilon_i\rangle \langle \epsilon_i|w\rangle
\tag{42}
$$

as the basis is orthonormal.

(b) The entries of the hermitian conjugate of P are:

$$
P^\dagger_{ij} = P^*_{ji} = \left\langle \epsilon'_j \middle| \epsilon_i \right\rangle^* = \left\langle \epsilon_i \middle| \epsilon'_j \right\rangle
\tag{43}
$$

The matrix product $P^\dagger P$ is:

$$
(P^\dagger P)_{ij} = \sum_k P^\dagger_{ik} P_{kj} = \sum_k \langle \epsilon_i|\epsilon'_k\rangle \langle \epsilon'_k|\epsilon_j\rangle = \langle \epsilon_i|\epsilon_j\rangle = \delta_{ij}
\tag{44}
$$

which means $P^\dagger P = I$ or that P is unitary.

(c) We have:

$$(PTP^\dagger)_{ij} = P_{ik}T_{kl}P^\dagger_{lj} = \sum_{k,l} \langle \epsilon'_i|\epsilon_k\rangle \langle \epsilon_k|T|\epsilon_l\rangle \left\langle \epsilon_l\middle|\epsilon'_j\right\rangle = \left\langle \epsilon'_i\middle|T\middle|\epsilon'_j\right\rangle = T'_{ij} \tag{45}$$

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6 Rotations and unitary matrices

(a) Given that $R \approx I + iT$ is a rotation matrix, consider the product $R^\intercal R$:

$$\delta_{ij} = (R^\intercal R)_{ij} = R^\intercal_{ik}R_{kj} = (\delta_{ik} + iT_{ki})(\delta_{kj} + iT_{kj}) = \delta_{ij} + iT_{ij} + iT_{ji} - T_{ki}T_{kj} \approx \delta_{ij} + iT_{ij} + iT_{ji} \tag{46}$$

where at the last step we have ignored the second order terms.

This equation holds only if $T_{ij} = -T_{ji}$, which means T is anti-symmetric.

(b)

$$\begin{aligned}
[\tilde{T}_i, \tilde{T}_j]_{kl} &= \left(\tilde{T}_i\tilde{T}_j - \tilde{T}_j\tilde{T}_i\right)_{kl} \\
&= \left(\tilde{T}_i\right)_{km}\left(\tilde{T}_j\right)_{ml} - \left(\tilde{T}_j\right)_{km}\left(\tilde{T}_i\right)_{ml} \\
&= -\epsilon_{ikm}\epsilon_{jml} + \epsilon_{jkm}\epsilon_{iml} \\
&= -(\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}) + (\delta_{jl}\delta_{ki} - \delta_{ji}\delta_{kl}) \\
&= \delta_{jl}\delta_{ki} - \delta_{il}\delta_{kj}
\end{aligned} \tag{47}$$

On the other hand:

$$i\epsilon_{ijk}\tilde{T}_k = \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} = \delta_{jl}\delta_{ki} - \delta_{il}\delta_{kj} \tag{48}$$

which means $[\tilde{T}_i, \tilde{T}_j] = i\epsilon_{ijk}\tilde{T}_k$.

(c) Given that $U \approx I + iS$ is a unitary matrix, consider the product $U^\dagger U$:

$$\delta_{ij} = (U^\dagger U)_{ij} = U^\dagger_{ik}U_{kj} = (\delta_{ik} - iS^\dagger_{ik})(\delta_{kj} + iS_{kj}) = \delta_{ij} + iS_{ij} - iS^\dagger_{ij} - S^\dagger_{ik}S_{kj} \approx \delta_{ij} + iS_{ij} - iS^\dagger_{ij} \tag{49}$$

which means $S^\dagger_{ij} = S_{ij}$, which means S is hermitian.

On the other hand, suppose that S has the form:

$$S = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \tag{50}$$

where a and c must be real because S is hermitian.

The requirement of unit determinant means:

$$\det\{(I + iS)\} = (ia + 1)(ic + 1) + |b|^2 = 1 \tag{51}$$

or that:

$$-ac + i(a + c) + |b|^2 = 0 \tag{52}$$

which means that $a + c = 0$ or that S is traceless.

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7 Convergence and completeness

(a) Consider the difference $x^i - x^j$, where we assume without loss of generality that $j > i$ so that this difference is positive. Let us choose $\epsilon = a^s$ for some $s > 0$. Consider the integer $k = \lceil s \rceil + 1 > s$. We can choose $i = k$ and $j = k + n$, where n is an arbitrary positive integer. Then:

$$x^i - x^j = x^k(1 - x^n) < x^k < a^k < a^s = \epsilon \tag{53}$$

$$\boxed{x^i - x^j = x^k(1 - x^n) < x^k \leq a^k < a^s = \epsilon} \tag{54}$$

This shows that the sequence is Cauchy.

To show that it converges to 0, we still choose $\epsilon = a^s$ and consider the integer $k = \lceil s \rceil + 1 > s$. Then:

$$x^k - 0 = a^k < a^s = \epsilon \tag{55}$$

This shows that the sequence converges to 0.

(b) Consider the difference $s_j - s_i$, where we assume without loss of generality that $j > i$.

Let us choose $\epsilon = (1-a)^{-s}$ for some $s > 1$. Then:

$$s_j - s_i = \sum_{k=i+1}^{j} x^k \leq \sum_{k=i+1}^{j} a^k < \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} < \left(\frac{1}{1-a}\right)^s = \epsilon \tag{56}$$

This shows that the sequence is Cauchy.

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