

Mathematical Methods

Problem Sheet 2

Fourier Series and Fourier Integrals

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Fourier Series and Fourier Integrals

1 Fourier Series

(a) We have the coefficients for the cosine series as:

$$a_r = \frac{2}{2\pi} \int_0^\pi \sin x \cos rx \, dx \quad (1)$$

where apparently $a_0 = 2/\pi$.

For $r \geq 1$, the integral denoted as I_r can be evaluated by parts:

$$\begin{aligned} I_r &= [-\cos x \cos rx]_0^\pi - r \int_0^\pi \cos x \sin rx \, dx \\ &= [\cos x \cos rx]_\pi^0 - r \left\{ [\sin x \sin rx]_0^\pi - r \int_0^\pi \sin x \cos rx \, dx \right\} \\ &= 1 + \cos \pi r + r^2 I_r \end{aligned} \quad (2)$$

Therefore, the coefficients are:

$$a_r = \frac{1}{\pi} \frac{1 + \cos \pi r}{1 - r^2} = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 2/\pi(1 - r^2) & \text{if } r \text{ is even} \end{cases} \quad (3)$$

On the other hand, the coefficients for the sine series are:

$$b_r = \frac{2}{2\pi} \int_0^\pi \sin x \sin rx \, dx \quad (4)$$

which are zero except for $r = 1$:

$$b_1 = \frac{1}{\pi} \int_0^\pi \sin^2 x \, dx = \frac{1}{2} \quad (5)$$

Hence, the Fourier series is:

$$f(x) = \frac{1}{2} \sin x + \frac{2}{\pi} \sum_{\text{even } r \geq 0}^{\infty} \frac{1}{1 - r^2} \cos rx \quad (6)$$

(b) Since the function is even, we only need the coefficients for the cosine series:

$$\begin{aligned}
a_r &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos rx \, dx \\
&= \frac{2}{\pi} \int_0^{\pi} x^2 \cos rx \, dx \\
&= \frac{2}{\pi} \left\{ \left[x^2 \frac{1}{r} \sin rx \right]_0^{\pi} - \frac{2}{r} \int_0^{\pi} x \sin rx \, dx \right\} \\
&= -\frac{4}{r\pi} \left\{ \left[-x \frac{1}{r} \cos rx \right]_0^{\pi} + \frac{1}{r} \int_0^{\pi} \cos rx \, dx \right\} \\
&= \frac{4}{r^2} \cos r\pi = \frac{4}{r^2} (-1)^r
\end{aligned} \tag{7}$$

for $r \geq 1$.

Apparently, $a_0 = 2\pi^2/3$ and the Fourier series is:

$$f(x) = \frac{2\pi^2}{3} + \sum_{r=1}^{\infty} (-1)^r \frac{4}{r^2} \cos rx \tag{8}$$

(c) Consider the norm of the function $f(x) = x^2$ on the interval $[-\pi, \pi]$:

$$\|f\|^2 = \int_{-\pi}^{\pi} x^4 \, dx = \frac{2\pi^5}{5} \tag{9}$$

By Parseval's equation, we have:

$$\frac{\|f\|^2}{\pi} = \frac{1}{2} \left(\frac{\pi^2}{3} \right)^2 + \sum_{r=1}^{\infty} \frac{16}{r^4} \tag{10}$$

so that:

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{1}{16} \left(\frac{2\pi^4}{5} - \frac{\pi^4}{18} \right) \tag{11}$$

By Parseval's equation, we have:

$$\frac{\|f\|^2}{\pi} = \frac{1}{2} \left(\frac{2\pi^2}{3} \right)^2 + \sum_{r=1}^{\infty} \frac{16}{r^4} \quad (12)$$

so that:

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{1}{16} \left(\frac{2\pi^4}{5} - \frac{2\pi^4}{9} \right) = \frac{\pi^4}{90} \quad (13)$$

2 Sine and cosine Fourier series

(a) The coefficients for the cosine series are:

$$a_r = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos rx \, dx \quad (14)$$

For the case $r = 0$ and $r = 1$, we have:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx = \frac{2}{\pi} \left\{ [-x \cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx \right\} = 2 \quad (15)$$

and:

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx = \frac{1}{2\pi} \left\{ [-x \cos 2x]_0^{\pi} + \int_0^{\pi} \cos 2x \, dx \right\} = -\frac{1}{2} \quad (16)$$

For $r \geq 2$, the integral, which is denoted as I_r , can be evaluated by parts:

$$I_r = [-x \cos x \cos rx]_0^{\pi} + \int_0^{\pi} \cos x \cos rx \, dx - r \int_0^{\pi} x \cos x \sin rx \, dx \quad (17)$$

For the middle term, the only non-zero contribution is when $r = 1$ as $\cos rx$ are orthogonal. Therefore we may neglect the middle term for $r \geq 2$:

$$\begin{aligned} I_r &= \pi \cos r\pi - r \left\{ [x \sin x \sin rx]_0^{\pi} - \int_0^{\pi} \sin x \sin rx \, dx - r \int_0^{\pi} x \sin x \cos rx \, dx \right\} \\ &= \pi \cos r\pi + r^2 I_r \end{aligned} \quad (18)$$

where in the last step we have again used the orthogonality of $\sin rx$.

This means that for $r \geq 2$, the coefficients are:

$$a_r = (-1)^r \frac{2}{1-r^2} \quad (19)$$

Hence the cosine Fourier series is:

$$f(x) = 2 - \frac{1}{2} \cos x + \sum_{r=2}^{\infty} (-1)^r \frac{2}{1-r^2} \cos rx \quad (20)$$

(b) The coefficients for the sine series are:

$$b_r = \frac{2}{\pi} \int_0^{\pi} x \sin x \sin rx \, dx \quad (21)$$

where $b_1 = \pi/2$.

For $r \geq 2$, the integral, which is denoted as I_r , can be evaluated as:

$$\begin{aligned} I_r &= \frac{1}{2} \int_0^{\pi} x \cos(1+r)x \, dx - \frac{1}{2} \int_0^{\pi} x \cos(1-r)x \, dx \\ &= \frac{1}{2} \left[\frac{\cos(1+r)\pi - 1}{(1+r)^2} - \frac{\cos(1-r)\pi - 1}{(1-r)^2} \right] \end{aligned} \quad (22)$$

where we used the integral result:

$$\int_0^{\pi} x \cos kx \, dx = \frac{\cos k\pi - 1}{k^2} \quad (23)$$

This means that the coefficients are:

$$b_r = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 2[(1+r)^{-2} - (1-r)^{-2}] / \pi & \text{if } r \text{ is even} \end{cases} \quad (24)$$

Hence the sine Fourier series is:

$$f(x) = \frac{\pi}{2} \sin x + \sum_{\text{even } r \geq 2}^{\infty} \frac{2}{\pi} \left[\frac{1}{(1+r)^2} - \frac{1}{(1-r)^2} \right] \sin rx \quad (25)$$

(c) The coefficients for the cosine series are:

$$a_r = \frac{2}{\pi} \int_0^{\pi} x \cos rx \, dx = \frac{2 \cos k\pi - 1}{\pi k^2} = \begin{cases} -4/\pi r^2 & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases} \quad (26)$$

except for $r = 0$ where $a_0 = \pi$.

Hence the cosine Fourier series is:

$$f(x) = \pi - \frac{4}{\pi} \sum_{\text{odd } r \geq 1}^{\infty} \frac{1}{r^2} \cos rx \quad (27)$$

(d) The coefficients for the sine series are:

$$b_r = \frac{2}{\pi} \int_0^{\pi} x \sin rx \, dx = -\frac{2}{\pi} \frac{\cos r\pi}{r} = \begin{cases} 2/r & \text{if } r \text{ is odd} \\ -2/r & \text{if } r \text{ is even} \end{cases} \quad (28)$$

Hence the sine Fourier series is:

$$f(x) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{2}{r} \sin rx \quad (29)$$

The cosine series does not converge to $f(x)$ near zero because $f(x)$ is not even, whereas the sine series converges to zero.

Consider the norm of the function $f(x) = x$ on the interval $[0, \pi]$:

$$\|f\|^2 = \int_0^{\pi} x^2 \, dx = \frac{\pi^3}{3} \quad (30)$$

The Parseval's equation gives:

$$\frac{2\|f\|^2}{\pi} = \sum_{r=1}^{\infty} \frac{4}{r^2} \quad (31)$$

so that:

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6} \quad (32)$$

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3 Legendre polynomials as an orthogonal basis

(a) We know that the Legendre polynomials can be defined as:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \quad (33)$$

Consider the inner product $\langle P_n, x^k \rangle$ for $k < n$:

$$\begin{aligned}
 \langle P_n, x^k \rangle &\propto \int_{-1}^1 x^k \frac{d^l}{dx^l} (x^2 - 1)^l dx \\
 &= \left[x^k \frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l \right]_{-1}^1 - \int_{-1}^1 k x^{k-1} \frac{d^{l-1}}{dx^{l-1}} (x^2 - 1)^l dx \\
 &= \dots \\
 &\propto (-1)^k \int_{-1}^1 \frac{d^{l-k}}{dx^{l-k}} (x^2 - 1)^l dx \\
 &= 0
 \end{aligned} \tag{34}$$

where we have used the fact that $k < n$ and $l - k \geq 0$.

Any polynomial p of degree $k < n$ can be written as $p(x) = \sum_{r=0}^k a_r x^r$, so that:

$$\langle P_n, p \rangle = \sum_{r=0}^k a_r \langle P_n, x^r \rangle = 0 \tag{35}$$

(b) Using the above definition, the first five Legendre polynomials are:

$$\begin{aligned}
 P_0 &= 1 \\
 P_1 &= x \\
 P_2 &= \frac{1}{2}(3x^2 - 1) \\
 P_3 &= \frac{1}{2}(5x^3 - 3x) \\
 P_4 &= \frac{1}{8}(35x^4 - 30x^2 + 3)
 \end{aligned} \tag{36}$$

(c) Decomposing the function $f(x) = x^4$ in terms of the Legendre polynomials:

$$\begin{aligned}
a_0 &= \langle P_0, f \rangle = \int_{-1}^1 x^4 dx = \frac{2}{5} \\
a_1 &= \langle P_1, f \rangle = \int_{-1}^1 x^5 dx = 0 \\
a_2 &= \langle P_2, f \rangle = \int_{-1}^1 \frac{3x^6 - x^4}{2} dx = \frac{1}{35} \\
a_3 &= \langle P_3, f \rangle = \int_{-1}^1 \frac{5x^7 - 3x^5}{2} dx = 0 \\
a_4 &= \langle P_4, f \rangle = \int_{-1}^1 \frac{35x^8 - 30x^6 + 3x^4}{8} dx =
\end{aligned} \tag{37}$$

(d) The given function can be written as a generating function of the Legendre polynomials:

$$f(x) = \frac{1}{\sqrt{(1/2)^2 - x + 1}} = \sum_{r=0}^{\infty} P_r(x) \left(\frac{1}{2}\right)^r \tag{38}$$

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4 Examples of Fourier transforms

(a) For the given square pulse, its Fourier transform is:

$$\hat{\chi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-ik} - e^{ik}}{-ik} = \sqrt{\frac{2}{\pi}} \frac{\sin k}{k} \tag{39}$$

(b) Consider the following integral:

$$\begin{aligned}
\int_{(n-1)\pi}^{n\pi} |\hat{\chi}| dk &= \sqrt{\frac{2}{\pi}} \int_{(n-1)\pi}^{n\pi} \frac{|\sin k|}{k} dk \\
&\geq \sqrt{\frac{2}{\pi}} \frac{1}{n\pi} \int_{(n-1)\pi}^{n\pi} |\sin k| dk \\
&= \sqrt{\frac{2}{\pi}} \frac{2}{n\pi}
\end{aligned} \tag{40}$$

But this integral diverges as $n \rightarrow 0$, so that $\hat{\chi}$ is not integrable.

(c) The convolution is:

$$f(\mu) = \chi * \chi = \int_{\mathbb{R}} \chi(x) \chi(\mu - x) dx \quad (41)$$

But χ is symmetric about the origin and $\chi(\mu - x) = \chi(x - \mu)$, so that:

$$f(\mu) = \int_{-1}^1 \chi(x) \chi(x - \mu) dx \quad (42)$$

where the range of integration is restricted to $[-1, 1]$ because $\chi(x) = 0$ otherwise.

Consider the case $0 < \mu < 2$, the integrand is non-zero only when $x \in [\mu - 1, 1]$. Therefore:

$$f(\mu) = \int_{\mu-1}^1 dx = 2 - \mu \quad (43)$$

Similarly, for $-2 < \mu \leq 0$:

$$f(\mu) = \int_{-1}^{\mu+1} dx = \mu + 2 \quad (44)$$

Thus, the convolution is:

$$f(\mu) = \begin{cases} 2 - \mu & \text{if } 0 < \mu < 2 \\ \mu + 2 & \text{if } -2 < \mu \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (45)$$

which is a triangular pulse.

(d) The Fourier transform of the convolution is:

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-2}^2 (\mu + 2) e^{-ik\mu} d\mu + \frac{1}{\sqrt{2\pi}} \int_0^2 (2 - \mu) e^{-ik\mu} d\mu \\ &= \frac{2}{\sqrt{2\pi}} \int_{-2}^2 (\mu + 2) \cos(k\mu) d\mu \\ &= \frac{4}{\sqrt{2\pi}} \frac{\sin^2 k}{k^2} \\ &= \sqrt{2\pi} [\hat{\chi}(k)]^2 \end{aligned} \quad (46)$$

as expected.

(e) Consider the inner product $\langle \mathcal{F}(f), \mathcal{F}(f) \rangle$:

$$\begin{aligned}
 \langle \mathcal{F}(f), \mathcal{F}(f) \rangle &= \langle \mathcal{F}^\dagger[\mathcal{F}(f)], f \rangle \\
 &= \langle f, f \rangle \\
 &= \int_{\mathbb{R}} f(x)f(x) \, dx \\
 &= \int_{-2}^0 (2 + \mu)^2 \, d\mu + \int_0^2 (2 - \mu)^2 \, d\mu \\
 &= \frac{16}{3}
 \end{aligned} \tag{47}$$

where the second step is due to the fact that \mathcal{F} is unitary.

On the other hand, the norm of $\mathcal{F}(f)$ is just:

$$\|\mathcal{F}(f)\|^2 = \frac{8}{\pi} \int_{\mathbb{R}} \frac{\sin^4 k}{k^4} \, dk \tag{48}$$

so that we have the result:

$$\int_{\mathbb{R}} \frac{\sin^4 k}{k^4} \, dk = \frac{2\pi}{3} \tag{49}$$

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5 Some properties of Fourier transforms

(a) Consider $\mathcal{F} \circ T_a$ acting on f :

$$\begin{aligned}
 \mathcal{F} \circ T_a(f) &= \mathcal{F}[f(x - a)] \\
 &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x - a) e^{-ikx} \, dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-ik(y+a)} \, dy \\
 &= e^{-ika} \mathcal{F}(f) \\
 &= E_{-a} \mathcal{F}(f)
 \end{aligned} \tag{50}$$

where we have used the substitution $y = x - a$ in the third step.

Further consider $\mathcal{F} \circ E_a$:

$$\begin{aligned}
\mathcal{F} \circ E_a(f) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iax} f(x) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i(k-a)x} dx \\
&= T_a(\hat{f})(k) \\
&= T_a \circ \mathcal{F}(f)
\end{aligned} \tag{51}$$

(b) Consider $\mathcal{F} \circ \mathcal{D}_a$ acting on f :

$$\begin{aligned}
\mathcal{F} \circ \mathcal{D}_a(f) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(ax) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{a} \int_{\mathbb{R}} f(y) e^{-iky/a} dy \\
&= \frac{1}{a} \hat{f}(k/a) \\
&= \frac{1}{a} \mathcal{D}_{1/a} \circ \mathcal{F}(f)
\end{aligned} \tag{52}$$

(c) Consider $\mathcal{F} \circ D_x$ acting on f :

$$\begin{aligned}
\mathcal{F} \circ D_x(f) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{df}{dx} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} [f(x) e^{-ikx}]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx \\
&= M_{ik} \circ \mathcal{F}(f)
\end{aligned} \tag{53}$$

where the boundary term vanishes because $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$ for f to be bounded.

Further consider $iD_k \circ \mathcal{F}(f)$:

$$\begin{aligned}
iD_k \circ \mathcal{F}(f) &= \frac{i}{\sqrt{2\pi}} \frac{d}{dk} \left[\int_{\mathbb{R}} f(x) e^{-ikx} dx \right] \\
&= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \frac{d}{dk} e^{-ikx} dx \\
&= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) (-ix) e^{-ikx} dx \\
&= M_x \circ \mathcal{F}(f)
\end{aligned} \tag{54}$$

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6 More Fourier transforms

(a) Consider the Fourier transform of the standard Gaussian:

$$\begin{aligned}
 \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x^2/2+ikx)} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-((x+ik)^2-k^2)/2} dx \\
 &= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int_{\mathbb{R}} e^{-(x+ik)^2/2} dx \\
 &= e^{-k^2/2}
 \end{aligned} \tag{55}$$

so that the standard Gaussian is invariant under Fourier transform.

(b) We have $\mathcal{D}_a(f_a)(x) = \exp(-x^2/2)$ and:

$$\mathcal{F} \circ \mathcal{D}_a(f_a) = \frac{1}{a} \mathcal{D}_{1/a} \circ \mathcal{F}(f_a) \tag{56}$$

But we already established that $\mathcal{F}(f_a(ax)) = e^{-k^2/2}$, so that:

$$\mathcal{F}(f_a)(k) = \mathcal{D}_a \left(a e^{-k^2/2} \right) = a e^{-a^2 k^2/2} \tag{57}$$

(c) We have $\mathcal{D}_a \circ T_{-c}(f_a)(x) = \exp(-x^2/2)$ and:

$$\mathcal{F} \circ \mathcal{D}_a \circ T_{-c}(f_a) = \frac{1}{a} \mathcal{D}_{1/a} \circ \mathcal{F} \circ T_{-c}(f_a) = \frac{1}{a} \mathcal{D}_{1/a} \circ E_c \circ \mathcal{F}(f_a) \tag{58}$$

Therefore:

$$\mathcal{F}(f_a)(k) = a e^{-a^2 k^2/2 + ick} \tag{59}$$

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7 Hermite polynomials and Fourier transform

(a) We can rewrite the given Gaussian:

$$g(x) = \exp \left[- \left(\frac{x}{\sqrt{2}} - \sqrt{2}z \right)^2 + z^2 \right] = \exp(z^2) \exp \left[- \frac{(x - \sqrt{2}z)^2}{2} \right] \quad (60)$$

so that its Fourier transform is:

$$\begin{aligned} \hat{g}(k) &= \exp(z^2) \mathcal{F}(f_{2, \sqrt{2}z}) \\ &= \exp \left(-\frac{k^2}{2} - 2ikz + z^2 \right) \end{aligned} \quad (61)$$

where we have used the result from the previous problem.

(b) We know the generating function of the Hermite polynomials:

$$\exp(2yz - z^2) = \sum_{n=0}^{\infty} H_n(y) \frac{z^n}{n!} \quad (62)$$

Consider the Fourier transform of $h_n(x) = \exp(-x^2/2)H_n(x)$:

$$\begin{aligned} \mathcal{F}(h_n)(k) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left(-\frac{x^2}{2} - ikx \right) H_n(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left(\frac{x^2}{2} - ikx \right) \exp(-x^2) H_n(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\sum_j^{\infty} H_j(-ik/\sqrt{2}) \frac{(x/\sqrt{2})^j}{j!} \right] \exp(-x^2) H_n(x) dx \\ &= \sum_j^{\infty} H_j(-ik/\sqrt{2}) \frac{1}{j! 2^{j/2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-x^2) H_n(x) x^j dx \end{aligned} \quad (63)$$

Consider the integral term. We can always write x^j as a linear combination of $H_n(x)$:

$$x^j = \sum_{r=0}^j a_r H_r(x) \quad (64)$$

so that:

$$\int_{\mathbb{R}} \exp(-x^2) H_n(x) x^j dx = \sum_{r=0}^j a_r \int_{\mathbb{R}} \exp(-x^2) H_n(x) H_r(x) dx = a_n \sqrt{\pi} 2^n n! \quad (65)$$

where $a_n = \langle H_n, x^j \rangle$.