Ordinary Differential Equations

Problem Set 3

Second-Order ODEs, Part II

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Minimal Set

3.1 Inhomogeneous ODEs

Given $y'' - 3y' + 2y = f(x)$. The complementary solution has the form:

$$y_c = \alpha e^{2x} + \beta e^{x} \tag{1}$$

(i) Let $y_p = Ax^2 + Bx + C$, so that:

$$y_p'' - 3y_p' + 2y_p = 2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2 \tag{2}$$

This gives us $A = 1$ and $B = C = 0$. Thus, $y_p = x^2$.

(ii) Let $y_p = Ae^{4x}$, so that:

$$y_p'' - 3y_p' + 2y_p = 16Ae^{4x} - 3(4Ae^{4x}) + 2Ae^{4x} = e^{4x} \tag{3}$$

This gives us $A = 1/6$. Thus, $y_p = e^{4x}/6$.

(iii) Let $y_p = Axe^{x}$, so that:

$$y_p'' - 3y_p' + 2y_p = Ae^{x}(x+2) - 3Ae^{x}(x+1) + 2Axe^{x} = e^{x} \tag{4}$$

This gives us $A = 1$. Thus, $y_p = xe^{x}$.

(iv) Let $y_p = Ae^{x} + Be^{-x}$, so that:

$$y_p'' - 3y_p' + 2y_p = Ae^{x} + Be^{-x} - 3(Ae^{x} - Be^{-x}) + 2(Ae^{x} + Be^{-x}) = \frac{e^{x} - e^{-x}}{2} \tag{5}$$

This gives us $B = -1/12$ and A can be incorporated into the complementary solution. Thus, $y_p = -e^{-x}/12$.

Let $y_p = Axe^{x} + Be^{-x}$, so that:

$$y_p'' - 3y_p' + 2y_p = Ae^{x}(x+2) + Be^{-x} - 3[Ae^{x}(x+1) - Be^{-x}] + 2(Axe^{x} + Be^{-x}) = \frac{e^{x} - e^{-x}}{2} \tag{6}$$

This gives us $B = -1/12$ and A is free. Thus, $y_p = Axe^{x} - e^{-x}/12$.

(v) Let $y_p = A\sin x + B\cos x$, so that:

$$y_p'' - 3y_p' + 2y_p = -A\sin x - B\cos x - 3(A\cos x - B\sin x) + 2(A\sin x + B\cos x) = \sin x \tag{7}$$

This gives us $A = 1/10$ and $B = 3/10$. Thus, $y_p = \sin x/10 + 3\cos x/10$.

(vi) Let $y_p = Ax\sin x + Bx\cos x + C\sin x + D\cos x$, so that:

$$\begin{aligned}
&y_p'' - 3y_p' + 2y_p \\
&= -(Ax + 2B)\sin x + (2A - Bx)\cos x - 3\left[(A - Bx)\sin x + (Ax + B)\cos x\right] + 2(Ax\sin x + Bx\cos x) \\
&= x\sin x
\end{aligned} \tag{8}$$

(vii) Let $y_p = Axe^{2x} + B\cos^2 x + C\sin^2 x$, so that:

$$y_p'' - 3y_p' + 2y_p = Ae^{2x} + 2B\cos^2 x + 2C\sin^2 x - 2(B - C)(\cos^2 x - \sin^2 x) + 6(B - C)\sin x\cos x \tag{9}$$

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3.2 Inhomogeneous ODEs

(i) The complementary solution is:

$$y_c = e^{-x/5}\left[\alpha\cos\left(\frac{2x}{5}\right) + \beta\sin\left(\frac{2x}{5}\right)\right] \tag{10}$$

where $\alpha = -1$ and $\beta = -2$

Let $y_p = Ax + B$, so that:

$$5y_p'' + 2y_p' + y_p = 5A + 2A + Ax + B = x \tag{11}$$

This gives us $A = 2$ and $B = -1$. Thus:

$$y(x) = -e^{-x/5}\left[\cos\left(\frac{2x}{5}\right) + 2\sin\left(\frac{2x}{5}\right)\right] + 2x - 1 \tag{12}$$

(ii) The complementary solution is:

$$y_c = \alpha e^{2x} + \beta e^{-x} \tag{13}$$

Let $y_p = Axe^{2x}$, so that:

$$y_p'' - y_p' - 2y_p = Ae^{2x}(4x + 4 - 2x - 2x - 1) = e^{2x} \tag{14}$$

This gives us $A = 1/3$. Thus:

$$y(x) = \alpha e^{2x} + \beta e^{-x} + \frac{1}{3}xe^{2x} \tag{15}$$

(iii) The complementary solution is:

$$y_c = e^{x/2}(\alpha x + \beta) \tag{16}$$

where $\alpha = 1$ and $\beta = 0$.

Let $y_p = Ax^2e^{x/2}$, so that:

$$4y_p'' - 4y_p' + y_p = Ae^{x/2}(x^2 + x^2 + 8x + 8 - 2x^2 - 8x) = 8e^{x/2} \tag{17}$$

This gives us $A = 1$. Thus:

$$y(x) = xe^{x/2} + x^2e^{x/2} \tag{18}$$

(iv) The complementary solution is:

$$y_c = \alpha e^{-x} + \beta e^{-2x} \tag{19}$$

Let $y_p = Axe^{-x}$, so that:

$$y_p'' + 3y_p' + 2y_p = Ae^{-x}(x - 2 - 3x + 3 + 2x) = xe^{-x} \tag{20}$$

This gives us $A = 1$. Thus:

$$y(x) = \alpha e^{-x} + \beta e^{-2x} + xe^{-x} \tag{21}$$

Let $y_p = Axe^{-x} + Bx^2e^{-x}$, so that:

$$y_p'' + 3y_p' + 2y_p = Ae^{-x} + Be^{-x}(2x+2) = xe^{-x} \tag{22}$$

This gives us $A = -1$ and $B = 1/2$. Thus:

$$y(x) = \alpha e^{-x} + \beta e^{-2x} - xe^{-x} + \frac{1}{2}x^2e^{-x} \tag{23}$$

(v) The complementary solution is:

$$y_c = \alpha e^{x} + \beta e^{3x} \tag{24}$$

Let $y_p = A\cos x + B\sin x$, so that:

$$y_p'' - 4y_p' + 3y_p = 2(A-2B)\cos x + 2(2A+B)\sin x = 10\cos x \tag{25}$$

This gives us $A = 1$ and $B = -2$. Thus:

$$y(x) = \alpha e^{x} + \beta e^{3x} + \cos x - 2\sin x \tag{26}$$

(vi) The complementary solution is:

$$y_c = \alpha\sin 2x + \beta\cos 2x \tag{27}$$

Let $y_p = Ax^2 + Bx + C + Dx\cos(2x+\phi)$, so that:

$$y_p'' + 4y_p = 2A + Ax^2 + Bx + C + 4Dx\cos(2x+\phi) - 4D[2\sin(2x+\phi) + x\cos(2x+\phi)] \tag{28}$$

This gives us $A = 0$, $B = 1$, $C = 0$, $D = -1/4$, and $\phi = -\pi/2$. Thus:

$$y(x) = \alpha\sin 2x + \beta\cos 2x + x - \frac{1}{4}x\sin 2x \tag{29}$$

The initial conditions yields $\beta = 0$. Therefore:

$$y(x) = \alpha\sin 2x + x - \frac{1}{8}x\sin 2x \tag{30}$$

(vii) The complementary solution is:

$$y_c = e^x(\alpha \cos x + \beta \sin x) \tag{31}$$

Let $y_p = Axe^x \cos(x + \phi) + Be^x$, so that:

$$y_p'' - 2y_p' + 2y_p = Be^x + 2Ae^x[-\sin(x + \phi)] \tag{32}$$

This gives us $A = -1/2$, $B = 1$, and $\phi = 0$. Thus, with the initial conditions:

$$y(x) = -e^x(\cos x + \sin x) - \frac{1}{2}xe^x \cos x + e^x \tag{33}$$

(viii) The complementary solution is:

$$y_c = e^{-x}(\alpha x + \beta) \tag{34}$$

Let $y_p = Ax^3 + Bx^2 + Cx + D + Ex^2e^{-x} + F\sin(x + \phi)$, so that:

$$y_p'' + 2y_p' + y_p = Ax^3 + (6A + B)x^2 + (6A + 4B + C)x + (2B + 2C + D) + 2Ee^{-x} + 2F\cos(x + \phi) \tag{35}$$

This gives us $A = 1$, $B = C = D = 0$, $E = 1$, $F = 1$, and $\phi = 0$. Thus:

$$y(x) = e^{-x}(\alpha x + \beta) + x^3 + x^2e^{-x} + \sin x \tag{36}$$

(ix) The complementary solution is:

$$y_c = e^x(\alpha x + \beta) \tag{37}$$

Let $y_p = Ax^2e^x$, so that:

$$y_p'' - 2y_p' + y_p = 2Ae^x = 3e^x \tag{38}$$

This gives us $A = 3/2$. Thus:

$$y(x) = e^x(\alpha x + \beta) + \frac{3}{2}x^2e^x \tag{39}$$

The initial conditions yields $\alpha = -\beta = -3$. Therefore:

$$y(x) = e^x \left(\frac{3}{2}x^2 - 3x + 3 \right) \tag{40}$$

(x) By inspection, $y_p(x) = x/2$ is a particular solution to the equation. For the homogeneous equation, the substitution $z = y_c'/y_c$ gives us a Riccati equation:

$$z' + z^2 + \frac{1}{x}z + \frac{1}{x^2} = 0 \tag{41}$$

Consider the substitution $g(x) = xz$. The equation becomes:

$$\frac{g'}{x} + \frac{1}{x^2}g^2 + \frac{1}{x^2} = 0 \tag{42}$$

which is a separable equation with the solution:

$$g = xz = -\tan\left[\ln\left(C\,|x|\right)\right] \tag{43}$$

This again is a separable equation for y_c:

$$\frac{\mathrm{d}y_c}{y_c} = -\frac{\tan\left[\ln\left(C\,|x|\right)\right]}{x}\mathrm{d}x \tag{44}$$

Making use of the substitution $u = 1/x$, the solution for y_c is:

$$y_c(x) = D\sec\left[\ln\left(C\,|x|\right)\right] \tag{45}$$

The full solution is therefore:

$$y(x) = D\sec\left[\ln\left(C\,|x|\right)\right] + \frac{x}{2} \tag{46}$$

3.3

Given that $\ddot{\theta} + \omega_0^2\theta = \cos\omega t$, the complementary solutions has the form:

$$\theta_c(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{47}$$

Let the particular solution take the form $\theta_p(t) = C\cos\omega t + D\sin\omega t$. Substitution gives:

$$-\omega^2(C\cos\omega t + D\sin\omega t) + \omega_0^2(C\cos\omega t + D\sin\omega t) = \cos\omega t \tag{48}$$

so that $C = 1/(\omega_0^2 - \omega^2)$, $D = 0$ and $\theta_p(t) = \cos\omega t/(\omega_0^2 - \omega^2)$.

We have:

$$\begin{aligned}\theta(t) &= A\cos\omega_0 t + B\sin\omega_0 t + \frac{1}{\omega_0^2-\omega^2}\cos\omega t \\ \dot{\theta}(t) &= -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t - \frac{\omega}{\omega_0^2-\omega^2}\sin\omega t\end{aligned} \tag{49}$$

Substituting the initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = 0$ gives $A = -1/(\omega_0^2 - \omega^2)$ and $B = 0$. Thus:

$$\theta(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2-\omega^2} \tag{50}$$

(a) The rms value of $\theta(t)$ is given by:

$$\begin{aligned}\left\langle\theta^2\right\rangle^{1/2} &= \sqrt{\frac{1}{T}\int_0^T\left(\frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2-\omega^2}\right)^2 \mathrm{d}t} \\ &= \frac{1}{\sqrt{T}\,|\omega_0^2-\omega^2|}\sqrt{\int_0^T(\cos^2\omega t + \cos^2\omega_0 t - 2\cos\omega t\cos\omega_0 t)\,\mathrm{d}t} \\ &= \frac{1}{|\omega_0^2-\omega^2|}\end{aligned} \tag{51}$$

(b) As the driving frequency approaches the natural frequency, the amplitude of the oscillation increases. At perfect resonance, the amplitude is infinite.

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3.4 Forced and Damped Oscillator

(i) The stationary solution is the long-term behaviour of the full solution. It is given by the driving force:

$$\begin{aligned}y(t) &= A\cos(\omega t + \phi) \\ \dot{y}(t) &= -A\omega\sin(\omega t + \phi) \\ \ddot{y}(t) &= -A\omega^2\cos(\omega t + \phi)\end{aligned} \tag{52}$$

To determine A and ϕ, substitute them back to the equation:

$$-A\omega^2\cos(\omega t + \phi) - \gamma A\omega\sin(\omega t + \phi) + \omega_0^2 A\cos(\omega t + \phi) = \frac{F}{m}\cos\omega t \tag{53}$$

Solving this equation gives us:

$$\begin{aligned} A &= \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \\ \phi &= \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2-\omega^2}\right) \end{aligned} \tag{54}$$

where $f \equiv F/m$.

(iii) The resonant frequency is $\omega_{\text{res}} = \omega_0$.

(iv) At resonance, $A_{\max} = f/(\omega_0\gamma)$. For $A(\omega) = f/(2\omega_0\gamma)$, we have the equation:

$$\frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} = \frac{f}{2\omega_0\gamma} \tag{55}$$

Solving for ω^2 yields:

$$\omega^2 = \omega_0^2 + \frac{\gamma^2}{2}\left(\sqrt{1+\frac{12}{\chi^2}}-1\right) \tag{56}$$

where $\chi \equiv \gamma/\omega_0$.

Take $\Delta\omega = 2\sqrt{\omega^2}$. Further approximation with binomial expansion leads to:

$$\frac{\Delta\omega}{\omega_0} \approx \sqrt{2}\left(1+\frac{\sqrt{12}}{2}\chi\right) \tag{57}$$

(v) Consider the long-term behaviour of the solution. The energy stored is given by:

$$E = \frac{1}{2}m\dot{y} + \frac{1}{2}m\omega_0^2y^2 = \frac{1}{2}mA^2[\omega^2\sin^2(\omega t+\phi) + \omega_0^2\cos^2(\omega t+\phi)] \approx \frac{1}{2}mA^2\omega_0^2 \tag{58}$$

The power supplied to the system is given by:

$$P = \dot{y}F\cos\omega t = -FA\omega\sin(\omega t+\phi)\cos\omega t \approx -FA\omega\cos^2\omega t \tag{59}$$

But at steady state, the energy lost is all supplied by the driving force and we may use the average of P instead. Thus:

$$Q \approx 2\pi \frac{\frac{1}{2}mA^2(\omega^2+\omega_0^2)\cos^2(\omega t+\phi)}{\langle -P\rangle \frac{2\pi}{\omega}} = \frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \approx \frac{\omega_0}{\gamma} \tag{60}$$

We have that $Q = 1/\chi$, so that the a higher quality factor leads to a shaper resonance peak.

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3.5

(a) Given $y_1 = 1/x$, suppose that the other solution has the form $y_2(x) = \phi(x)/x$, so that substitution yields:

$$(x+1)\phi'' - (x+2)\phi' = 0 \tag{61}$$

Solving this leads to the solution $\phi(x) = xe^x$, so that the other homogeneous solution is $y_2(x) = e^x$.

Consider the particular solution $y_p = Ax + B$, substitution yields:

$$-2Ax^2 - 2Ax - Bx + 2A - 2B = x^2 + 2x + 1 \tag{62}$$

Solving this yields $A = -1/2$ and $B = -1$. Therefore, the general solution is:

$$y(x) = \frac{\alpha}{x} + \beta e^x - \frac{x}{2} - 1 \tag{63}$$

(b) If y_2 is known instead of y_1, we still employ the same variation of constant method and find y_1. The procedure and the outcome are the same.

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