Quantum Mechanics

Problem Sheet 4

Transformations & Orbital Angular Momentum

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Transformations

4.1 Reflection symmetry around a point $\mathbf{x}_0$

Let $|\mathbf{x}_0+\mathbf{x}\rangle$ be a position eigenstate that yields $\mathbf{x}_0+\mathbf{x}$ upon measurement of position. On physical grounds, reflecting the eigenstate about the point $\mathbf{x}_0$ should yield the eigenstate $|\mathbf{x}_0-\mathbf{x}\rangle$:

$$\hat{P}_{\mathbf{x}_0}|\mathbf{x}_0+\mathbf{x}\rangle = |\mathbf{x}_0-\mathbf{x}\rangle \tag{1}$$

With this, consider the effect of $\hat{P}_{\mathbf{x}_0}\hat{x}\hat{P}_{\mathbf{x}_0}$ on a position eigenstate:

$$\begin{aligned}
\hat{P}_{\mathbf{x}_0}\hat{x}\hat{P}_{\mathbf{x}_0}|\mathbf{x}_0+\mathbf{x}\rangle &= \hat{P}_{\mathbf{x}_0}\hat{x}|\mathbf{x}_0-\mathbf{x}\rangle \\
&= (\mathbf{x}_0-\mathbf{x})\hat{P}_{\mathbf{x}_0}|\mathbf{x}_0-\mathbf{x}\rangle \\
&= (\mathbf{x}_0-\mathbf{x})|\mathbf{x}_0+\mathbf{x}\rangle \\
&= (2\mathbf{x}_0-\mathbf{x}_0-\mathbf{x})|\mathbf{x}_0+\mathbf{x}\rangle \\
&= (2\mathbf{x}_0\mathbb{I}-\hat{x})|\mathbf{x}_0+\mathbf{x}\rangle
\end{aligned} \tag{2}$$

Further consider $\hat{P}_{\mathbf{x}_0}\hat{p}\hat{P}_{\mathbf{x}_0}$. Apparently $\hat{p}$ anticommutes with $\hat{P}_{\mathbf{x}_0}$ so that $\hat{p}\hat{P}_{\mathbf{x}_0}=-\hat{P}_{\mathbf{x}_0}\hat{p}$. Thus:

$$\begin{aligned}
\hat{P}_{\mathbf{x}_0}\hat{p}\hat{P}_{\mathbf{x}_0} &= -\hat{P}_{\mathbf{x}_0}\hat{P}_{\mathbf{x}_0}\hat{p} \\
&= -\hat{p}
\end{aligned} \tag{3}$$

since two successive reflections about the same point is equivalent to no reflection at all.

Consider the position wave function after the reflection:

$$\begin{aligned}
\psi'(\mathbf{x}) &\equiv \langle\hat{x}|\hat{P}_{\mathbf{x}_0}|\psi\rangle \\
&= \int \langle\hat{x}|\hat{P}_{\mathbf{x}_0}|\mathbf{x}_0+\mathbf{x}'\rangle\langle\mathbf{x}_0+\mathbf{x}'|\psi\rangle\,\mathrm{d}^3x' \\
&= \int \langle\hat{x}|\mathbf{x}_0-\mathbf{x}'\rangle\,\psi(\mathbf{x}_0+\mathbf{x}')\,\mathrm{d}^3x' \\
&= \int (\mathbf{x}_0-\mathbf{x}')\psi(\mathbf{x}_0+\mathbf{x}')\,\mathrm{d}^3x'
\end{aligned} \tag{4}$$

Consider the change of variable $\mathbf{x}'\to\mathbf{x}_0-\mathbf{x}'$:

$$\psi'(\mathbf{x}) = \int \mathbf{x}'\psi(2\mathbf{x}_0-\mathbf{x}')\,\mathrm{d}^3x' = \psi(2\mathbf{x}_0-\mathbf{x}) \tag{5}$$

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4.2

For translation invariance, $\hat{H}$ must commute with $\hat{p}$. Since $\hat{x}$ and $\hat{p}$ generally do not commute, the only form $V(\hat{x})$ can take is a constant.

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4.3

We define the orbital angular momentum operator $\hat{L}_i$ as:

$$\hat{L}_i \equiv \epsilon_{ijk}\hat{x}_j\hat{p}_k \tag{6}$$

Its Hermitian conjugate is:

$$\hat{L}_i^\dagger = \epsilon_{ijk}\hat{p}_k^\dagger\hat{x}_j^\dagger = \epsilon_{ijk}\hat{p}_k\hat{x}_j \tag{7}$$

On the other hand, from the canonical commutation relation:

$$[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}\mathbb{I} \tag{8}$$

which implies that $\hat{x}_j$ and $\hat{p}_k$ commute if $j \neq k$.

Therefore:

$$\hat{L}_i^\dagger = \epsilon_{ijk}\hat{p}_k\hat{x}_j = \epsilon_{ijk}\hat{x}_j\hat{p}_k = \hat{L}_i \tag{9}$$

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4.4

For a central potential, we write the Hamiltonian as:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r}^2) \tag{10}$$

where we define the radial position operator $\hat{r}^2$ as:

$$\hat{r}^2 \equiv \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 \tag{11}$$

Let us write the potential as an expansion in terms of $\hat{r}^2$:

$$V(\hat{r}^2) = \sum_{n=0}^{\infty} a_n \hat{r}^{2n} \tag{12}$$

Consider the commutator $[\hat{H}, \hat{L}_i]$:

$$\begin{aligned}
[\hat{H}, \hat{L}_i] &= \frac{1}{2m}[\hat{p}^2, \hat{L}_i] + \sum_{n=0}^{\infty} a_n [\hat{r}^{2n}, \hat{L}_i] \\
&= \frac{1}{2m} \sum_{j=1,2,3} [\hat{p}_j^2, \hat{L}_i] + \sum_{n=0}^{\infty} a_n \sum_{j=1,2,3} [\hat{x}_j^{2n}, \hat{L}_i] \\
&= \frac{1}{2m} \epsilon_{ikl} \sum_{j=1,2,3} [\hat{p}_j^2, \hat{x}_k \hat{p}_l] + \sum_{n=0}^{\infty} a_n \epsilon_{ikl} \sum_{j=1,2,3} [\hat{x}_j^{2n}, \hat{x}_k \hat{p}_l]
\end{aligned} \tag{13}$$

Let us consider the commutators separately. Note the following commutation relations:

$$\begin{aligned}
[AB, C] &= A[B, C] + [A, C]B \\
[A, BC] &= [A, B]C + B[A, C]
\end{aligned} \tag{14}$$

For $[\hat{p}_j^2, \hat{x}_k \hat{p}_l]$:

$$\begin{aligned}
[\hat{p}_j^2, \hat{x}_k \hat{p}_l] &= \hat{p}_j [\hat{p}_j, \hat{x}_k \hat{p}_l] + [\hat{p}_j, \hat{x}_k \hat{p}_l] \hat{p}_j \\
&= \hat{p}_j [\hat{p}_j, \hat{x}_k] \hat{p}_l + \hat{p}_j \hat{x}_k [\hat{p}_j, \hat{p}_l] + [\hat{p}_j, \hat{x}_k] \hat{p}_l \hat{p}_j + \hat{x}_k [\hat{p}_j, \hat{p}_l] \hat{p}_j \\
&= \hat{p}_j [\hat{p}_j, \hat{x}_k] \hat{p}_l + [\hat{p}_j, \hat{x}_k] \hat{p}_l \hat{p}_j
\end{aligned} \tag{15}$$

where we have used the fact that $\hat{p}_j$ and $\hat{p}_l$ commute.

This commutator is nonzero only when $k = j$, in which case:

$$[\hat{p}_j^2, \hat{x}_k \hat{p}_l] = -2i\hbar \hat{p}_j \hat{p}_l \tag{16}$$

Then the first term in the commutator $[\hat{H}, \hat{L}_i]$ becomes:

$$\frac{1}{2m} \epsilon_{ijl} \sum_{j=1,2,3} (-2i\hbar \hat{p}_j \hat{p}_l) = \frac{1}{im} \epsilon_{ijl} \hat{p}_j \hat{p}_l \tag{17}$$

This is zero since $\hat{p}_j$ and $\hat{p}_l$ commute. We then consider the second commutator $[\hat{x}_j^{2n}, \hat{x}_k \hat{p}_l]$:

$$[\hat{x}_j^{2n}, \hat{x}_k\hat{p}_l] = [\hat{x}_j^{2n}, \hat{x}_k]\hat{p}_l + \hat{x}_k[\hat{x}_j^{2n}, \hat{p}_l] \tag{18}$$

where the first term is always zero since $\hat{x}_j^{2n}$ and $\hat{x}_k$ commute and the second term is nonzero only when $l = j$, in which case:

$$\begin{aligned}[\hat{x}_j^{2n}, \hat{p}_l] &= \hat{x}_k[\hat{x}_j^{2n}, \hat{p}_j] \\ &= \hat{x}_k\left\{\hat{x}_j^{2n-1}[\hat{x}_j, \hat{p}_j] + [\hat{x}_j^{2n-1}, \hat{p}_j]\hat{x}_j\right\} \\ &= \hat{x}_k\left\{\hat{x}_j^{2n-1}[\hat{x}_j, \hat{p}_j] + x_j^{2n-2}[\hat{x}_j, \hat{p}_j]\hat{x}_j + \cdots + [\hat{x}_j, \hat{p}_j]\hat{x}_j^{2n-1}\right\} \\ &= i\hbar(2n)\hat{x}_k\hat{x}_j^{2n-1}\end{aligned} \tag{19}$$

Therefore the second term in the commutator $[\hat{H}, \hat{L}_i]$ becomes:

$$\sum_{n=0}^{\infty} a_n\epsilon_{ikj} \sum_{j=1,2,3} i\hbar(2n)\hat{x}_k\hat{x}_j^{2n-1} \tag{20}$$

which is always zero since $\hat{x}_k$ and $\hat{x}_j^{2n-1}$ commute.

Therefore $[\hat{H}, \hat{L}_i] = 0$ for a central potential and the angular momentum is conserved.

Furthermore, consider a potential that has azimuthal symmetry, i.e. $V(\mathbf{x}) = V(\hat{x}_1^2 + \hat{x}_2^2)$. In this case, we can write the potential as:

$$V = \sum_{n=0}^{\infty} a_n(\hat{x}_1^2 + \hat{x}_2^2)^n \tag{21}$$

The change from previous results occurs on the second term, where we only let j run over 1 and 2. Due to the presence of the ϵ_{ikj} term, the sum is zero only for $i = 3$, since for the other two cases we will miss one term in the sum due to $j = 3$ missing. Therefore, the Hamiltonian only commutes with $\hat{L}_3$, which is the z-component of the angular momentum. The x- and y-components of the angular momentum are not conserved.

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4.5

Let us expand the expression using binomial theorem:

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\begin{aligned}
\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N &= \lim_{N\to\infty}\sum_{n=0}^{N}\binom{N}{n}\left(\frac{x}{N}\right)^n \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{N!}{n!(N-n)!}\frac{1}{N^n}x^n \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{N(N-1)(N-2)\cdots(N-n+1)}{N^n}\frac{1}{n!}x^n \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\frac{1}{n!}x^n \\
&= e^x
\end{aligned}
\tag{22}
$$

where at the last step we have used the fact that the sum is the Taylor series of e^x.

This can in some way be viewed as a definition of e^x. Indeed, the definition of exponential for an operator is just this limit:

$$
\exp\left(\hat{A}\right) \equiv \lim_{N\to\infty}\left(1+\frac{\hat{A}}{N}\right)^N = \left(1+\frac{\hat{A}}{N}\right)\left(1+\frac{\hat{A}}{N}\right)\cdots\left(1+\frac{\hat{A}}{N}\right) \tag{23}
$$

which can be viewed as applying the operator $(1+\hat{A}/N)$ to the state N times.

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4.6 Heisenberg equations of motion for the SHO