Symmetry and Relativity

Problem Set 1

Kinematics of fluids

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1 Eulerian and Lagrangian descriptions

(a) Since the fluid is incompressible and steady, the density ρ is constant and the velocity field $\mathbf{v}$ is independent of time. By conservation of mass, the input from the circular area above equals the output from the side:

$$\rho v_0 \pi r^2 = \rho v_r (2\pi r) h \tag{1}$$

which gives $v_r = (r/2h)v_0$.

For incomprehensibility, we have $\nabla \cdot \mathbf{v} = 0$, which gives:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0 \tag{2}$$

where we have ignored any azimuthal component due to symmetry.

Substituting v_r into the equation and solving, we find $v_z = -(z/h)v_0 + f(r)$, where $f(r)$ is a function of r only. Since the fluid is at rest at $z = 0$, we have $f(r) = 0$. Thus, $v_z = -(z/h)v_0$.

To find the velocity potential, we demand:

$$\frac{v_0}{2h} r\hat{\mathbf{r}} - \frac{v_0}{h} z\hat{\mathbf{z}} = \nabla\phi \tag{3}$$

which gives:

$$\phi = \frac{v_0}{4h} r^2 - \frac{v_0}{2h} z^2 \tag{4}$$

(b) Consider $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = A_\theta \hat{\theta}$. We have the equations:

$$\begin{aligned} -\frac{\partial A_\theta}{\partial z} &= \frac{v_0}{2h} r \\ \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) &= -\frac{v_0}{h} z \end{aligned} \tag{5}$$

which gives $A_\theta = -(v_0/2h)rz$.

The streamlines are given by curves of constant A_θ. They are hyperbolae of the form $rz = \text{constant}$. Since the fluid is steady, the streamlines are also the trajectories of fluid particles.

The equipotential surfaces are given by curves of constant ϕ. They are paraboloids of the form $r^2 - 2z^2 = \text{constant}$.

(c) Consider the initial condition $\mathbf{r}(0) = (r_0, \theta_0, z_0)$. Due to symmetry, the θ component will stay constant. Focusing on the radial component, we have:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{v_0}{2h} r \tag{6}$$

which can be solved by separation of variables to give:

$$r(t) = r_0 e^{v_0 t/(2h)} \tag{7}$$

Similarly, the z component satisfies:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{v_0}{h} z \tag{8}$$

which gives:

$$z(t) = z_0 e^{-v_0 t/h} \tag{9}$$

It is easy to see that $r(t)z(t) = r_0 z_0 = \text{constant}$, which is the equation of the streamlines.

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2 Streamlines and trajectories

(a) The stream function A_z satisfies:

$$\begin{aligned} \frac{\partial A_z}{\partial y} &= v_0 \\ -\frac{\partial A_x}{\partial z} &= kt \end{aligned} \tag{10}$$

which gives $A_z = v_0 y - ktx$.

The streamlines are given by curves of constant A_z, which are straight lines of the form $y = (kt/v_0)x + \text{constant}$. The time component changes the gradient of the lines.

(b) The equations of motion can be trivially solved to give:

$$\begin{aligned} x(t) &= v_0 t \\ y(t) &= \frac{1}{2} k t^2 \end{aligned} \tag{11}$$

which gives $y(x) = (k/2v_0)x^2$, which is a parabola.

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3 Acceleration in a rotating frame

(a) First note the following relation:

$$\mathbf{v}_r = \mathbf{v}_f - \mathbf{\Omega} \times \mathbf{r} \tag{12}$$

which follows from simple velocity addition.

Consider:

$$\begin{aligned}\left(\frac{\mathrm{D}\mathbf{v}_f}{\mathrm{D}t}\right)_f &= \left(\frac{\mathrm{D}\mathbf{v}_f}{\mathrm{D}t}\right)_r + \mathbf{\Omega}\times\mathbf{v}_f \\ &= \left(\frac{\mathrm{D}\mathbf{v}_r}{\mathrm{D}t}\right)_r + \left[\frac{\mathrm{D}(\mathbf{\Omega}\times\mathbf{r})}{\mathrm{D}t}\right]_r + \mathbf{\Omega}\times\mathbf{v}_r\end{aligned} \tag{13}$$

where we have used the above relation.

Now focus on the second term:

$$\begin{aligned}\left[\frac{\mathrm{D}(\mathbf{\Omega}\times\mathbf{r})}{\mathrm{D}t}\right]_r &= \left[\frac{\mathrm{D}(\mathbf{\Omega}\times\mathbf{r})}{\mathrm{D}t}\right]_f - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) \\ &= \mathbf{\Omega}\times\mathbf{v}_f - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{f})\end{aligned} \tag{14}$$

On the other hand, the third term can be written as:

$$\mathbf{\Omega}\times\mathbf{v}_f = \mathbf{\Omega}\times\mathbf{v}_r + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) \tag{15}$$

where we use the fact that $\mathbf{\Omega}$ is constant.

Putting everything together, we find:

$$\left(\frac{\mathrm{D}\mathbf{v}_f}{\mathrm{D}t}\right)_f = \left(\frac{\mathrm{D}\mathbf{v}_r}{\mathrm{D}t}\right)_r + 2\mathbf{\Omega}\times\mathbf{v}_r + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) \tag{16}$$

(b) For a fluid element moving along a circle at constant speed, we have $\mathbf{v}_r = \mathbf{0}$ so $\mathbf{f}_{\text{cor}} = \mathbf{0}$ and $\mathbf{f}_{\text{cen}} = -\omega^2 r\hat{\mathbf{r}}$.

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4 Deformation of a fluid element

(a) We have the deformation tensor and its components:

$$\begin{aligned}D_{ij} &= \frac{\partial v_i}{\partial x_j} = \begin{pmatrix}\alpha & 2\beta\\ 0 & -\alpha\end{pmatrix} \\ e_{ij} &= \frac{1}{2}(D_{ij}+D_{ji}) = \begin{pmatrix}\alpha & \beta\\ \beta & -\alpha\end{pmatrix} \\ \omega_{ij} &= \frac{1}{2}(D_{ij}-D_{ji}) = \begin{pmatrix}0 & \beta\\ -\beta & 0\end{pmatrix}\end{aligned} \tag{17}$$

The diagonal components of e_{ij} is an expansion, the off-diagonal components is deformation, and ω_{ij} is a rotation.

(b) Labelling the points of the rectangle as A, B, C, D, starting from the top left and going clockwise, we have:

$$\begin{aligned}
\mathbf{v}_A &= D_{xx}(-\delta x)\hat{\mathbf{x}} + D_{yy}\delta y\hat{\mathbf{y}} = \alpha(-\delta x\hat{\mathbf{x}} - \delta y\hat{\mathbf{y}}) \\
\mathbf{v}_B &= D_{xx}\delta x\hat{\mathbf{x}} + D_{yy}\delta y\hat{\mathbf{y}} = \alpha(\delta x\hat{\mathbf{x}} - \delta y\hat{\mathbf{y}}) \\
\mathbf{v}_C &= D_{xx}\delta x\hat{\mathbf{x}} + D_{yy}(-\delta y)\hat{\mathbf{y}} = \alpha(\delta x\hat{\mathbf{x}} + \delta y\hat{\mathbf{y}}) \\
\mathbf{v}_D &= D_{xx}(-\delta x)\hat{\mathbf{x}} + D_{yy}(-\delta y)\hat{\mathbf{y}} = \alpha(-\delta x\hat{\mathbf{x}} + \delta y\hat{\mathbf{y}})
\end{aligned} \tag{18}$$

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5 Vorticity

(a) The vorticity is given by:

$$\begin{aligned}
\omega &= \nabla \times (\mathbf{\Omega} \times \mathbf{r}) \\
&= -(\mathbf{\Omega} \cdot \nabla)\mathbf{r} + \mathbf{\Omega}(\nabla \cdot \mathbf{r}) \\
&= 2\mathbf{\Omega}
\end{aligned} \tag{19}$$

since:

$$(\mathbf{\Omega} \cdot \nabla)\mathbf{r} = \Omega(\nabla \mathbf{r})_{3j} = \mathbf{\Omega} \tag{20}$$

The streamlines are concentric circles with the center at the origin.

(b) The vorticity is given by:

$$\begin{aligned}
\omega &= \nabla \times \mathbf{v} \\
&= -k\hat{\mathbf{z}}
\end{aligned} \tag{21}$$

The streamlines are lines parallel to the x with increasing length as y increases.

(c) The vorticity is given by:

$$\begin{aligned}
\omega &= \nabla \times \mathbf{v} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{k}{r}\right)\hat{\theta} \\
&= \mathbf{0}
\end{aligned} \tag{22}$$

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6 Stream function and flow rate

(a) First note that we may write the normal vector as:

$$\hat{n} = \frac{1}{\mathrm{d}l}(\mathrm{d}y\hat{\mathbf{x}} - \mathrm{d}x\hat{\mathbf{y}}) \tag{23}$$

Thus:

$$\begin{aligned} Q &= \int_A^B \mathbf{v} \cdot \hat{n}\,\mathrm{d}l \\ &= \int_A^B v_x\,\mathrm{d}y - v_y\,\mathrm{d}x \\ &= \int_A^B \frac{\partial \psi}{\partial y}\,\mathrm{d}y + \frac{\partial \psi}{\partial x}\,\mathrm{d}x \\ &= \int_A^B \mathrm{d}\psi \\ &= \psi(B) - \psi(A) \end{aligned} \tag{24}$$

Q quantifies how the velocity of the flow changes according to the distance between two streamlines.

(b) Note:

$$\begin{aligned} \nabla \cdot \mathbf{v} &= -\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) + \frac{\partial v_z}{\partial z} \\ &= -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) \\ &= 0 \end{aligned} \tag{25}$$

as required by incompressibility.

On streamlines, we have a change in ψ:

$$\begin{aligned} \delta\psi &= \frac{\partial \psi}{\partial r}\delta r + \frac{\partial \psi}{\partial z}\delta z \\ &= r v_z \delta r - r v_r \delta z \end{aligned} \tag{26}$$

But since on streamlines, $\delta r / v_r = \delta z / v_z$, we have $\delta\psi = 0$.

It can be easily checked that $\mathbf{v} = \nabla \times (\psi / r)\hat{\theta}$.

We can compute the required flow rate as the difference between two surface integrals:

$$\begin{aligned} Q &= \int_{S_+} \mathbf{v}\cdot \mathrm{d}\mathbf{S} - \int_{S_-} \mathbf{v}\cdot \mathrm{d}\mathbf{S} \\ &= \int_{S_+} \nabla\times\left(\frac{\psi}{r}\hat{\theta}\right)\cdot \mathrm{d}\mathbf{S} - \int_{S_-} \nabla\times\left(\frac{\psi}{r}\hat{\theta}\right)\cdot \mathrm{d}\mathbf{S} \\ &= \oint_{C_+} \frac{\psi}{r}\,\mathrm{d}l - \oint_{C_-} \frac{\psi}{r}\,\mathrm{d}l \\ &= 2\pi(\psi_+ - \psi_-) \end{aligned} \tag{27}$$

where the third equality follows from Stokes' theorem.

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7 Mass conservation

In a time interval δt, the mass of the fluid element changes according to:

$$m \to m - \rho(t)Q\delta t + \rho_0 Q\delta t \tag{28}$$

This means that the total density changes according to:

$$\rho \to \rho - \frac{Q}{V}(\rho - \rho_0)\delta t \tag{29}$$

where the second term is the change $\delta\rho$ in the density.

We then have the differential equation:

$$\frac{\partial \rho}{\partial t} = -\frac{Q}{V}(\rho - \rho_0) \tag{30}$$

which has the solution:

$$\rho(t) = \rho_0\left(1 + 0.025e^{-Qt/V}\right) \tag{31}$$

Demanding $\rho(t) = 0.99\rho(0)$, we find $t = 0.527V/Q$.

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8 Hydrostatic pressure

(a) The total force acting on a fluid element is given by the integral of the pressure over its surface:

$$\begin{aligned} \mathbf{F} &= -\int_S P\,\mathrm{d}\mathbf{S} \\ &= -\int_S \nabla P\,\mathrm{d}V \end{aligned} \tag{32}$$

which suggests that the force per unit volume is $-\nabla P$.

In a gravitational field, $\nabla P = -\rho g \hat{\mathbf{z}}$ so that $P = P_0 - \rho g z$.

(b) The force exerted on the cube immersed in the fluid is given by:

$$F = \int \rho g \mathbf{z} \, \mathrm{d}V = \rho g V \hat{\mathbf{z}} = m g \hat{\mathbf{z}} \tag{33}$$

where the last equality follows by balancing the forces.

Then we deduce the mass of the displaced fluid equals the mass of the cube, which gives the Archimedes' principle.

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