Circuit Theory

Problem Set 2

Response of Linear Circuits to Transients

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June 3, 2023

Response of Linear Circuits to Transients

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Let $q(t)$ be the (positive) charge on the upper plate of the capacitor. By KVL, we have:

$$\begin{aligned} V - R\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{q}{C} &= 0 \\ \int_0^q \frac{1}{VC - q}\,\mathrm{d}q &= \int_0^t \frac{1}{RC}\mathrm{d}t \\ q(t) &= VC(1 - e^{-t/RC}) \\ i(t) &= \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{V}{R}e^{-t/RC} \end{aligned} \tag{1}$$

For the energy dissipation:

$$Q_R = \int P\,\mathrm{d}t = \int_0^\infty i^2 R\,\mathrm{d}t = \frac{V^2}{R}\int_0^\infty e^{-2t/RC}\,\mathrm{d}t = \frac{CV^2}{2} = Q_C \tag{2}$$

Thus the total energy supplied by the battery is CV^2.

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Conduct a mesh analysis with the first loop on the left going clockwise and the second loop on the right going anticlockwise. We have the equations:

$$\begin{aligned} V - i_1R_1 - \frac{q}{C} &= 0 \\ -i_2R_2 - \frac{q}{C} &= 0 \\ i_1 + i_2 &= \frac{\mathrm{d}q}{\mathrm{d}t} \end{aligned} \tag{3}$$

Solving this system for $q(t)$ yields the equation:

$$\frac{R_2}{R_1 + R_2}V - \frac{q}{C} = \frac{R_1R_2}{R_1 + R_2}\frac{\mathrm{d}q}{\mathrm{d}t} \tag{4}$$

Solving this differential equation gives:

$$\begin{aligned} q(t) &= CV_0(1 - e^{-t/\tau}) \\ V_C(t) &= V_0(1 - e^{-t/\tau}) \end{aligned} \tag{5}$$

where $V_0 = R_2V/(R_1 + R_2) = 4.125\,\mathrm{V}$ and $\tau = CR_1R_2/(R_1 + R_2) = 137.5\,\mu\mathrm{s}$.

At steady state, $I = V/(R_1 + R_2)$. Thus:

$$\begin{aligned} P_1 &= I^2R_1 = 3.5 \times 10^{-4}\,\mathrm{W} \\ P_2 &= I^2R_2 = 7.7 \times 10^{-4}\,\mathrm{W} \\ Q &= \frac{1}{2}CV_0^2 = 1.7 \times 10^{-4}\,\mathrm{J} \end{aligned} \tag{6}$$

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Let $i(t)$ be the current flowing in the circuit. By KVL, we have:

$$V - Ri - L\frac{\mathrm{d}i}{\mathrm{d}t} = 0 \tag{7}$$

Solving this differential equation yields:

$$i(t) = \frac{V}{R}e^{-t/\tau} \tag{8}$$

where $\tau = L/R$.

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Let $q(t)$ be the (positive) charge on the upper plate of the capacitor. By KVL, we have:

$$-\dot{q}R - \ddot{q}L - \frac{q}{C} = 0 \tag{9}$$

or:

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{RC}q = 0 \tag{10}$$

Note that $1/(LC) \gg (R/2L)^2$. Thus for the characteristic equation:

$$r = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \equiv -\beta \pm i\omega \tag{11}$$

where $\beta \equiv (R/2L)$ and $\omega \equiv \sqrt{1/(LC) - \beta^2} \approx \omega_0 = \sqrt{1/(LC)}$

Then the solution is:

$$\begin{aligned} q(t) &= Ae^{-\beta t} \cos(\omega t + \phi) \\ i(t) &= -A\left[\beta e^{-\beta t} \cos(\omega t + \phi) + \omega e^{-\beta t} \sin(\omega t + \phi)\right] \end{aligned} \tag{12}$$

With the initial conditions $q(0) = V_0/C$ and $i(0) = 0$, we have $\tan\phi = -\beta/\omega \ll 1$ and $A \approx V_0/C$.

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Conduct a mesh analysis with the first loop on the left going clockwise and the second loop on the right going anticlockwise. We have the equations:

$$\begin{aligned} V - i_1 R_1 - \frac{q}{C} &= 0 \\ -L\dot{i}_2 - \frac{q}{C} &= 0 \\ i_1 + i_2 &= \dot{q} \end{aligned} \tag{13}$$

Solving this system for $q(t)$ yields the equation:

$$\ddot{q} + \frac{1}{RC}\dot{q} + \frac{1}{LC}q = 0 \tag{14}$$

Note that $\Delta = 1/(RC) - 4/(LC) < 0$ so the system is oscillatory. The solution is:

$$q(t) = Ae^{-\beta t} \cos(\omega t + \phi) \tag{15}$$

where $\beta = 1/(2RC)$ and $\omega = \sqrt{1/(LC) - \beta^2}$.

Thus the decay constant is $t_0 = 1/\beta = 2RC$.

The resonant (natural) frequency is $\omega_0 = \sqrt{1/(LC)} = 1.6 \times 10^5\,\mathrm{rad s^{-1}}$.

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Before the switch is closed, as the circuit has reached steady state, R_2 is shorted by the inductor and thus $V_A = V_B = 0$. The current in the steady state is V_0/R_1

When the switch is closed, there is current only in the loop (clockwise) between A and B. KVL yields:

$$\begin{aligned} -L\dot{i}_2 - i_2 R_2 = 0 \\ \int_{V_0/R_1}^{i_2} \frac{1}{i_2}\,\mathrm{d}i_2 = \int_0^t -\frac{R}{L}\,\mathrm{d}t \end{aligned} \tag{16}$$

which leads to the solution:

$$i_2(t) = \frac{V_0}{R_1} e^{-t/\tau} \tag{17}$$

where $\tau = L/R_2 = 10\,\mu\mathrm{s}$.

At all times, $V_A = V_0 = 10\,\mathrm{V}$. For V_B, note the relationship:

$$V_B(t) = V_A + i_2 R_2 = V_0 \left[\frac{R_2}{R_1} e^{-t/\tau} + 1\right] \tag{18}$$

Thus, $V_B(0) = 1010\,\mathrm{V}$ and $V_B(\infty) = 10\,\mathrm{V}$.

When the switch is again closed, there is an additional clockwise loop. A mesh analysis yields:

$$\begin{aligned} V_0 - i_1 R_1 - (i_1 - i_2) R_2 = 0 \\ -L\dot{i}_2 - (i_1 - i_2) R_2 = 0 \end{aligned} \tag{19}$$

Solving for i_1 yields:

$$V - i_1 R_1 - L\frac{R_1 + R_2}{R_2}\dot{i}_1 = 0 \tag{20}$$

This differential equation has the solution:

$$i_1(t) = \frac{V_0}{R_1}(1 - e^{-t/\kappa}) \tag{21}$$

where $\kappa = L(R_1 + R_2)/(R_1 R_2) = 1\,\mathrm{ms}$.

Thus the voltage across R_1, which is $i_1 R_1$, also rises exponentially.

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