Vacation Work

Problem Sheet B

Calculus

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Differentiation

1

(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \sin x + \ln x) = 2x \sin x + x^2 \cos x + \frac{1}{x} \tag{1}$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x} \tag{2}$$

(c)

$$\frac{\mathrm{d}}{\mathrm{d}x}(2^x) = \frac{\mathrm{d}}{\mathrm{d}x}(e^{x \ln 2}) = \ln 2(e^{x \ln 2}) \tag{3}$$

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2

(a)

$$F'(x) = 3 \cos x - 4 \sin x \tag{4}$$

$$F''(x) = -3 \sin x - 4 \cos x \tag{5}$$

(b)

$$y' = \frac{1}{x} \tag{6}$$

$$y'' = -\frac{1}{x^2} \tag{7}$$

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3

Given the parametric equations of x and y in terms of θ, we have by chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} / \frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{1 - \cos \theta}{\sin \theta} \tag{8}$$

A further differentiation by chain rule yields:

$$
\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}\theta}{\mathrm{d}x} \\
&= \frac{\sin\theta\sin\theta - \cos\theta(1-\cos\theta)}{\sin^2\theta}\frac{1}{a(1-\cos\theta)} \\
&= \frac{1}{a\sin^2\theta}
\end{aligned} \tag{9}
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Stationary points and graph sketching

4

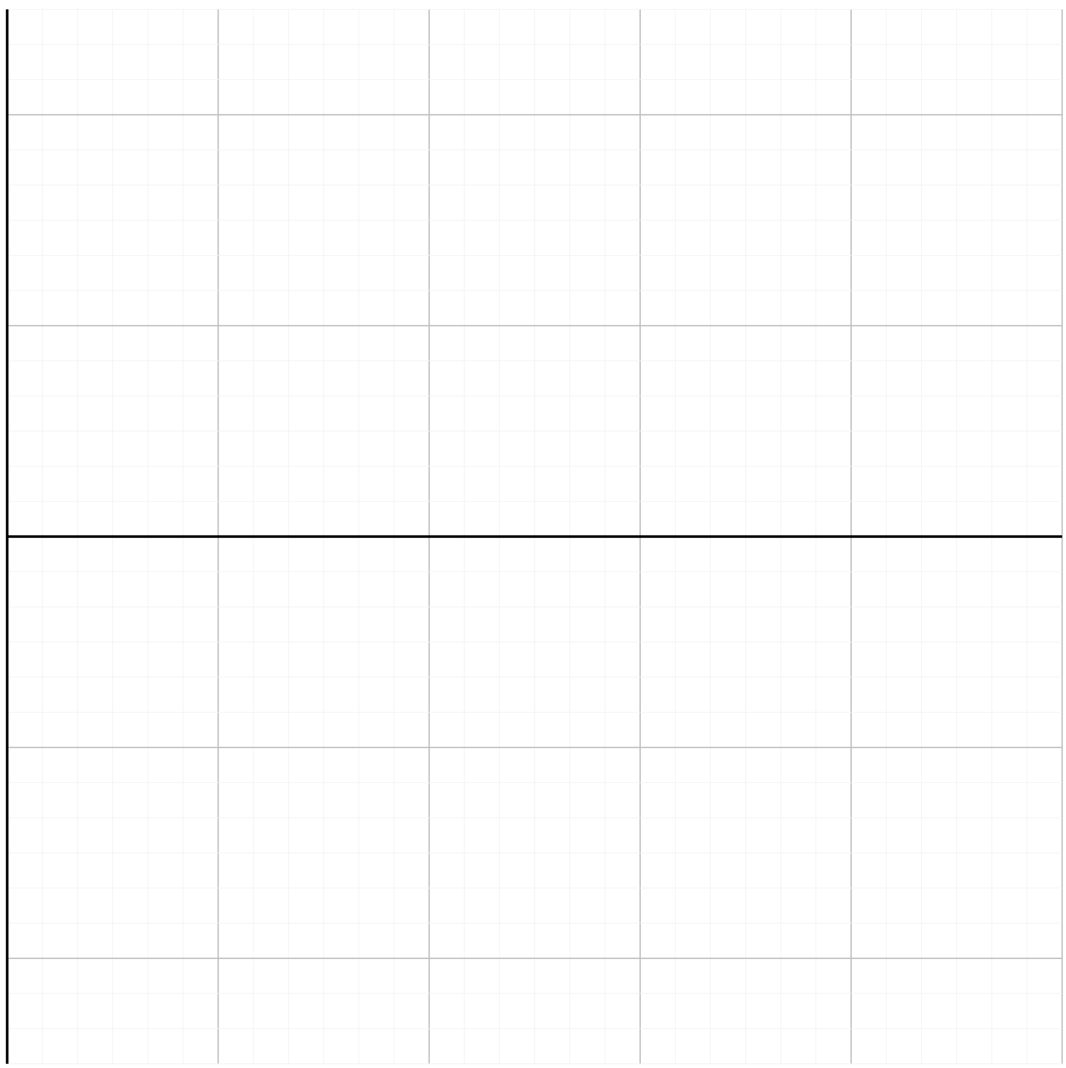
To find the stationary points of $E(r)$, differentiate it once to yield:

$$E'(r) = \frac{(1+r)^2(4) - 2(1+r)(4r)}{(1+r)^4} = \frac{4-4r^2}{(1+r)^4} \tag{10}$$

This leads to two stationary points at $r = 1$ and $r = -1$, but the function has a singularity at $r = -1$ so it is discarded. Differentiating further:

$$E''(r) = \frac{(1+r)^4(-8r) - 4(1+r)^3(4-4r^2)}{(1+r)^8} = \frac{8r-16}{(1+r)^4} \tag{11}$$

Since $E''(1) < 0$, and $r = 1$ is a local maximum.



Hyperbolic functions

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From the definitions of $\sinh x$ and $\tanh x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \left(\frac{1}{\cosh x}\right)^2 \tag{13}$$

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Integration

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(a)

$$\int 1 + 2x + 3x^2 \, \mathrm{d}x = x + x^2 + x^3 + C \tag{14}$$

(b)

$$\int \sin 2x - \cos 3x \, \mathrm{d}x = -\frac{1}{2}\cos 2x - \frac{1}{3}\sin 3x + C \tag{15}$$

(c)

$$\int e^t + \frac{1}{t^2} \, \mathrm{d}t = e^t - \frac{1}{t} + C \tag{16}$$

(d)

$$\int \mathrm{d}\omega = \omega + C \tag{17}$$

where all the C above are arbitrary constants.

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(a)

$$\int_{-1/4}^{1/4} \cos(2\pi x) \, \mathrm{d}x = \frac{1}{2\pi}\sin(2\pi x)\Big|_{-1/4}^{1/4} = \frac{1}{\pi} \tag{18}$$

(b)

$$\int_0^3 (2t-1)^2 \, \mathrm{d}t = \frac{1}{6}(2t-1)^3\Big|_0^3 = 21 \tag{19}$$

(c)

$$\int_1^2 \frac{(1+e^t)^2}{e^t} \, \mathrm{d}t = \int_1^2 e^t + 2 + e^{-t} \, \mathrm{d}t = (e^t - e^{-t} + 2t)\Big|_1^2 = e^2 - \frac{1}{e^2} - e + \frac{1}{e} + 2 \tag{20}$$

(d)

$$\int_4^9 \sqrt{x}(x - \frac{1}{x}) \, \mathrm{d}x = (\frac{2}{5}x^{5/2} - 2x^{1/2})\Big|_4^9 = \frac{412}{5} \tag{21}$$

(e) x^3 is an odd function as $x^3 = -(-x)^3$, so the definite integral equals zero.

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Integrating by parts:

$$\begin{aligned}\int x^2e^{-x}\,\mathrm{d}x &= -x^2e^{-x} + \int 2xe^{-x}\,\mathrm{d}x \\ &= -x^2e^{-x} - 2xe^{-x} + \int 2e^{-x}\,\mathrm{d}x \\ &= -x^2e^{-x} - 2xe^{-x} - 2e^{x} + C\end{aligned} \tag{22}$$

where C is an arbitrary constant.

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The integral is a standard form:

$$\int \sin x(1+\cos x)^4\,\mathrm{d}x = -\frac{1}{5}(1+\cos x)^5 + C \tag{23}$$

where C is an arbitrary constant.

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11

For the two curves to meet each other:

$$\begin{aligned}x^2+2 &= 5-2x \\ x^2+2x-3 &= 0 \\ (x+3)(x-1) &= 0\end{aligned} \tag{24}$$

Thus the integration takes place in the interval $[-3,1]$. The sought area is thus:

$$\begin{aligned}&\left|\int_{-3}^{1} x^2+2-(5-2x)\,\mathrm{d}x\right| \\ =&\left|\int_{-3}^{1} x^2+2x-3\,\mathrm{d}x\right| \\ =&\boxed{\frac{32}{3}}\end{aligned} \tag{25}$$

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(a) Slicing the volume of revolution into infinitesimal disks, each of radius $y = f(x)$, thickness $\mathrm{d}x$ and thus volume $\mathrm{d}V = \pi y^2 \, \mathrm{d}x$, the total volume of revolution V from a to b is represented by the integral:

$$V = \int \mathrm{d}V = \int_a^b \pi f(x)^2 \, \mathrm{d}x \tag{26}$$

For the function $y = \frac{1}{x}$, the volume of revolution from 1 to ∞ is represented by the improper integral:

$$\begin{aligned} V &= \lim_{n\to\infty} \int_1^n \pi \frac{1}{x^2} \, \mathrm{d}x \\ &= \pi \lim_{n\to\infty} \left(-\frac{1}{x}\right)\Big|_1^n \\ &= \pi \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) \\ &= \boxed{\pi \text{ units}^3} \end{aligned} \tag{27}$$

(b) By the definition of work done:

$$W = \int \mathrm{d}W = \int F \, \mathrm{d}x = \int_0^l kx^2 \, \mathrm{d}x = \frac{1}{3}kl^3 \tag{28}$$

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