**Vacation Work**

# Problem Sheet C

Mechanics

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# Motion in one dimension

## 1

The total distance travelled $d$ during this uniform acceleration is given by:

$$d = \frac{1}{2}at^2 = \frac{1}{2}\frac{v}{t}t^2 = \frac{1}{2} \times \left(80 \times \frac{3600}{1000}\right) \mathrm{ms}^{-1} \times 10\,\mathrm{s} = \boxed{1440\,\mathrm{m}} \tag{1}$$

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## 2

The final speed $v$ is given by:

$$v = at = a\sqrt{\frac{2d}{a}} = \sqrt{2da} = \sqrt{2 \times 2\,\mathrm{m} \times 9.8\,\mathrm{ms}^{-2}} = \boxed{6\,\mathrm{ms}^{-1}} \tag{2}$$

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## 3

The final velocity $v_f$ is given by:

$$v_f = \sqrt{v_i^2 + 2ad} = \sqrt{(6\,\mathrm{ms}^{-1})^2 + 2 \times 3\,\mathrm{ms}^{-2} \times 20\,\mathrm{m}} = \boxed{12\,\mathrm{ms}^{-1}} \tag{3}$$

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# Work and energy

## 4

By conservation of energy, the work done $W_f$ by the frictional force $f$ is:

$$\begin{aligned} W_f &= E_k \\ fd &= \frac{1}{2}mv_0^2 \\ f &= \frac{mv_0^2}{2d} = \frac{1000\,\mathrm{kg} \times (15\,\mathrm{ms^{-1}})^2}{2 \times 30\,\mathrm{m}} = \boxed{3750\,\mathrm{N}} \end{aligned} \tag{4}$$

After the first 15 m of the skid, the speed $v$ of the car is:

$$\begin{aligned} \frac{1}{2}mv^2 &= \frac{1}{2}mv_0^2 - fs \\ v &= \sqrt{v_0^2 - \frac{2fs}{m}} = \sqrt{(15\,\mathrm{ms^{-1}})^2 - \frac{2 \times 3750\,\mathrm{N} \times 15\,\mathrm{m}}{1000\,\mathrm{kg}}} = \boxed{11\,\mathrm{ms^{-1}}} \end{aligned} \tag{5}$$

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## 5

Under the condition $h \ll R$ or $h/R \ll 1$, the gravitational potential energy $V$ can be approximated using the binomial expansion:

$$\begin{aligned} V &= -GMm\frac{1}{R+h} \\ &= -\frac{GMm}{R}\left(1 + \frac{h}{R}\right)^{-1} \\ &= -\frac{GMm}{R}\left[1 - \frac{h}{R} + \left(\frac{h}{R}\right)^2 - \dots\right] \\ &\approx mgh - \frac{GMm}{R} \end{aligned} \tag{6}$$

where second order terms and above are ignored and where $g \equiv GM/R^2$ as desired.

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## 6

For an object to just escape the earth's gravity, all of its kinetic energy must be converted to gravitational potential energy at infinity. By conservation of energy

$$\begin{aligned} E_k + V(R) &= V(\infty) \\ \frac{1}{2}mv_{\text{escape}}^2 - \frac{GMm}{R} &= 0 \\ v_{\text{escape}} &= \sqrt{\frac{2GM}{R}} \end{aligned} \tag{7}$$

If the initial velocity is $v_{\text{escape}}/2$, the maximum high is where all the kinetic energy is converted to potential energy:

$$\begin{aligned} E_k + V(R) &= V(h) \\ \frac{1}{2}m\left(\frac{v_{\text{escape}}}{2}\right)^2 - \frac{GMm}{R} &= -\frac{GMm}{r} \\ \frac{3}{4}\frac{GMm}{R} &= \frac{GMm}{r} \\ r &= \frac{4}{3}R \end{aligned} \tag{8}$$

Thus the maximum height is $r - R = \boxed{R/3}$.

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# Simple harmonic motion

## 7

Given $x(t) = A\sin(\omega t + \phi)$, the velocity $v(t)$ and acceleration $a(t)$ are given by its time derivatives:

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) = A\omega\cos(\omega t + \phi) \tag{9}$$

$$a(t) = \frac{\mathrm{d}}{\mathrm{d}t}v(t) = -A\omega^2\sin(\omega t + \phi) \tag{10}$$

If $\phi = \pi/6$ and $\omega t = \pi/6$, $\boxed{x = A/2}$. For acceleration to achieve maximum in magnitude, $\sin(\omega t + \phi) = \pm 1$, meaning that $\boxed{x = \pm A}$.

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## 8

Given $F = -kx$, the equation of motion of the particle is given by Newton's second law:

$$\begin{aligned} ma = F &= -kx \\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= -\frac{k}{m}x \end{aligned} \tag{11}$$

This second order differential equation has the general solution of the form $x(t) = A\cos(\omega t + \phi)$, because:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x \tag{12}$$

which shows that the form satisfies the equation of motion.

If the initial conditions are such that $x(0) = x_0$ and $v(0) = 0$, we have the equations:

$$\begin{aligned} x_0 &= A\cos\phi \\ 0 &= -A\sin\phi \end{aligned} \tag{13}$$

Apparently this means $\phi = 0$ and $A = x_0$, and hence:

$$\boxed{x(t) = x_0 \cos(\omega t)} \tag{14}$$