Calculus

Problem Sheet B

Integration

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Integration

B1 Practice in integration

All C appearing in the solutions are arbitrary constants unless otherwise stated.

(a)

i

$$\int \frac{x+a}{(1+2ax+x^2)^{3/2}}\,\mathrm{d}x = -\frac{1}{(1+2ax+x^2)^{1/2}} + C \tag{1}$$

ii

$$\int_0^{\pi/2} \cos x e^{\sin x}\,\mathrm{d}x = \left[e^{\sin x}\right]_0^{\pi/2} = e - 1 \tag{2}$$

iii

$$\int_0^{\pi/2} \cos^3 x\,\mathrm{d}x = \int_0^{\pi/2} \cos x(1-\sin^2 x)\,\mathrm{d}x = \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{2}{3} \tag{3}$$

iv

$$\int_{-2}^{2} |x|\,\mathrm{d}x = 2\int_0^2 x\,\mathrm{d}x = 2\left[\frac{x^2}{2}\right]_0^2 = 4 \tag{4}$$

(b)

i

$$\int \frac{1}{(3+2x-x^2)^{1/2}}\,\mathrm{d}x = \int \frac{1}{\left[4-(x-1)^2\right]^{1/2}}\,\mathrm{d}x \tag{5}$$

Let $z = x-1$ such that $\mathrm{d}x = \mathrm{d}z$:

$$\int \frac{1}{\left[4-(x-1)^2\right]^{1/2}}\,\mathrm{d}x = \int \frac{1}{\left(4-z^2\right)^{1/2}}\,\mathrm{d}z = \sin^{-1}\left(\frac{x-1}{2}\right) + C \tag{6}$$

where $-1 < x < 3$.

ii Let $t = \tan\theta/2$ such that $\mathrm{d}t = \sec^2(\theta/2)\mathrm{d}\theta/2$:

$$\int_0^{\pi} \frac{1}{5+3\cos\theta}\,\mathrm{d}\theta = \int_0^{\infty} \frac{2\cos^2(\theta/2)}{5+3\cos\theta}\,\mathrm{d}t = \int_0^{\infty} \frac{1+\cos\theta}{5+3\cos\theta}\,\mathrm{d}t \tag{7}$$

But $\cos\theta = 2/(t^2+1) - 1$, thus:

$$\int_0^{\pi} \frac{1}{5+3\cos\theta}\,\mathrm{d}\theta = \int_0^{\pi} \frac{\frac{2}{t^2+1}}{2+\frac{6}{t^2+1}}\,\mathrm{d}t = \int_0^{\infty} \frac{1}{t^2+4}\,\mathrm{d}t = \frac{\pi}{4} \tag{8}$$

(c)

$$\int \frac{1}{x(1+x^2)}\,\mathrm{d}x = \int \frac{1}{x} - \frac{x}{1+x^2}\,\mathrm{d}x = \ln|x| - \frac{1}{2}\ln\left|1+x^2\right| + C \tag{9}$$

(d)

i

$$\int x\sin x\,\mathrm{d}x = -x\cos x + \int \cos x\,\mathrm{d}x = -x\cos x + \sin x + C \tag{10}$$

ii

$$\int \ln x\,\mathrm{d}x = x\ln x - \int x\frac{1}{x}\,\mathrm{d}x = x\ln x - x + C \tag{11}$$

(e)

$$\int_0^\infty x^n e^{-x^2}\,\mathrm{d}x = \left[\frac{x^{n+1}}{n+1}e^{-x^2}\right]_0^\infty + \int_0^\infty \frac{2}{n+1}x^{n+2}e^{-x^2}\,\mathrm{d}x \tag{12}$$

Change the dummy variable from n to $n-2$ and let $I(n) = \int_0^\infty x^n e^{-x^2}\,\mathrm{d}x$:

$$I(n) = \frac{n-1}{2}I(n-2) \tag{13}$$

where $n \leq 2$.

Therefore:

$$I(5) = 2I(1) = 2\int_0^\infty xe^{-x^2}\,\mathrm{d}x = 2\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^\infty = 1 \tag{14}$$

(f)

i

$$\begin{aligned}\int \left(\cos^5 x - \cos^3 x\right)\,\mathrm{d}x &= \int -\sin^2 x\cos^3 x\,\mathrm{d}x \\ &= -\frac{1}{3}\sin^3 x\cos^2 x - \int \frac{2}{3}\sin^4 x\cos x\,\mathrm{d}x \\ &= -\frac{1}{3}\sin^3 x\cos^2 x - \frac{2}{15}\sin^5 x + C \\ &= \frac{1}{5}\sin^5 x - \frac{1}{3}\sin^3 x + C\end{aligned} \tag{15}$$

ii

$$\begin{aligned}\int \sin^5 x \cos^4 x \, \mathrm{d}x &= -\frac{1}{5}\sin^4 x \cos^5 x + \int \frac{4}{5}\sin^3 x \cos^6 x \, \mathrm{d}x \\ &= -\frac{1}{5}\sin^4 x \cos^5 x - \frac{4}{35}\sin^2 x \cos^7 x + \int \frac{8}{35}\sin x \cos^8 x \, \mathrm{d}x \\ &= -\frac{1}{5}\sin^4 x \cos^5 x - \frac{4}{35}\sin^2 x \cos^7 x - \frac{8}{315}\cos^9 x + C\end{aligned} \tag{16}$$

iii

$$\int \sin^2 x \cos^4 x \, \mathrm{d}x = \int \cos^4 x - \cos^6 x \, \mathrm{d}x \tag{17}$$

Let $I(n) = \int \cos^n x \, \mathrm{d}x$. Have:

$$I(n) = \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, \mathrm{d}x \tag{18}$$

or:

$$nI(n) = \sin x \cos^{n-1} x + (n-1)I(n-2) \tag{19}$$

Then:

$$\begin{aligned} I(6) &= \frac{1}{6}\left[\sin x \cos^5 x + 5I(4)\right] \\ I(4) &= \frac{1}{4}\left[\sin x \cos^3 x + 3I(2)\right] \\ I(2) &= \frac{1}{2}(\sin x \cos x + x) + C \end{aligned} \tag{20}$$

Substituting:

$$\int \sin^2 x \cos^4 x \, \mathrm{d}x = -\frac{1}{6}\sin x \cos^5 x + \frac{1}{24}\sin x \cos^3 x + \frac{1}{6}\sin x \cos x + \frac{1}{6}x + C \tag{21}$$

Substituting:

$$\int \sin^2 x \cos^4 x \, \mathrm{d}x = -\frac{1}{192}\sin 6x - \frac{1}{64}\sin 4x + \frac{1}{64}\sin 2x + \frac{1}{16}x + C \tag{22}$$

(g)

i Let $x = 3\sec\theta$ such that $\mathrm{d}x = 3\tan\theta\sec\theta\mathrm{d}\theta$ and:

$$\begin{aligned}\int \frac{(x^2-9)^{1/2}}{x}\,\mathrm{d}x &= \int \frac{3\tan\theta}{3\sec\theta} 3\tan\theta\sec\theta\,\mathrm{d}\theta \\ &= 3\int \sec^2\theta - 1\,\mathrm{d}x \\ &= 3(\tan\theta - \theta) + C \\ &= \sqrt{x^2-9} - 3\cos^{-1}\left(\frac{3}{x}\right) + C\end{aligned} \tag{23}$$

ii Let $x = 4\sin\theta$ such that $\mathrm{d}x = 4\cos\theta\mathrm{d}\theta$ and:

$$\begin{aligned}\int \frac{1}{x^2(16-x^2)^{1/2}}\,\mathrm{d}x &= \int \frac{4\cos\theta}{16\sin^2\theta\cdot 4\cos\theta}\,\mathrm{d}\theta \\ &= \frac{1}{16}\int \csc^2\theta\,\mathrm{d}x \\ &= -\frac{1}{16}\cot\theta + C \\ &= -\frac{1}{16}\sqrt{\frac{16}{x^2} - 1} + C\end{aligned} \tag{24}$$

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B2 Properties of definite integrals

(a) The first and the third integrals are zero because their integrands are odd functions.

(b) As an odd function, $f(x)$ satisfies $f(x) = -f(-x)$. Then:

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_0^a f(x)\,\mathrm{d}x + \int_{-a}^{0} f(x)\,\mathrm{d}x = \int_0^a f(x)\,\mathrm{d}x - \int_a^0 f(-y)\,\mathrm{d}y = 0 \tag{25}$$

where the substitution $x = -y$ has been made.

(c) Without loss of generality, consider the case where $y > 1$ so that $xy > x$:

$$\begin{aligned}\ln xy &\equiv \int_1^{xy} \frac{1}{t}\,\mathrm{d}t \\ &= \int_1^{x} \frac{1}{t}\,\mathrm{d}t + \int_x^{xy} \frac{1}{t}\,\mathrm{d}t\end{aligned} \tag{26}$$

Make the substitution $z = t/x$ in the second integral:

$$\begin{aligned}\ln xy &= \int_1^x \frac{1}{t}\,\mathrm{d}t + \int_1^y \frac{1}{z}\,\mathrm{d}z \\ &= \ln x + \ln y\end{aligned} \tag{27}$$

The case where $xy < x$ follows a similar proof.

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B3 Arc length and area and volume of revolution

(a)

$$L = \int_0^1 \sqrt{1+\sinh^2 x}\,\mathrm{d}x = \int_0^1 \cosh x\,\mathrm{d}x = \sinh 1 \text{ units} \tag{28}$$

(b)

$$L = \int_0^{\pi/2} \sqrt{\sin^2 t + \cos^2 t}\,\mathrm{d}t = \frac{\pi}{2} \text{ units} \tag{29}$$

This parametric equation is a semi-circle.

(c)

$$A = 2\int_0^R 2\pi\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,\mathrm{d}x = 4\pi\int_0^R R\,\mathrm{d}x = 2\pi R^2 \text{ unit}^2 \tag{30}$$

$$\boxed{A = 2\int_{-R}^R 2\pi\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,\mathrm{d}x = 4\pi\int_{-R}^R R\,\mathrm{d}x = 4\pi R^2 \text{ unit}^2} \tag{31}$$

$$V = 2\int_0^R \pi(R^2-x^2)\,\mathrm{d}x = \frac{4}{3}\pi R^3 \tag{32}$$

as expected for the surface area and volume of a sphere.

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B4 Line integrals

(a)

$$\int_C x^2 + 2y\,\mathrm{d}x = \int_0^2 x^2 + 2x + 2\,\mathrm{d}x = \frac{32}{3} \tag{33}$$

(b)

$$\int_C xy\,\mathrm{d}x = \int_0^4 \sqrt{16-x^2}\,\mathrm{d}x = \frac{\pi}{2} \tag{34}$$

$$\int_C xy\,\mathrm{d}x = \int_0^4 x\sqrt{16-x^2}\,\mathrm{d}x = \frac{64}{3} \tag{35}$$

(c)

i On this line, $x = y = z$. Thus:

$$\int_c y^2\mathrm{d}x + xy\mathrm{d}y + zx\mathrm{d}z = \int_0^1 3x^2\,\mathrm{d}x = 1 \tag{36}$$

ii On the first segment, $x = y = \mathrm{d}x = \mathrm{d}y = 0$. On the second, $x = \mathrm{d}x = 0$ and $z = 1, \mathrm{d}z = 0$. On the third, $y = z = 1$. Thus, only the last segment contributes to the integral:

$$\int_c y^2\mathrm{d}x + xy\mathrm{d}y + zx\mathrm{d}z = \int_0^1 1\,\mathrm{d}x = 1 \tag{37}$$

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