Problem Set 0

Complex Numbers

Xin, Wenkang

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Complex Numbers

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(i)

$$-i = e^{-i\pi/2} \tag{1}$$

(ii)

$$\frac{1}{2} - \frac{\sqrt{3}i}{2} = e^{-i\pi/3} \tag{2}$$

(iii)

$$-3 - 4i = 5e^{i[\pi+\tan^{-1}(4/3)]} \tag{3}$$

(iv)

$$1 + i = e^{i\pi/4} \tag{4}$$

$$1 + i = \sqrt{2}e^{i\pi/4}$$

(v)

$$1 - i = e^{-i\pi/4} \tag{5}$$

$$1 + i = \sqrt{2}e^{-i\pi/4}$$

(vi)

$$(1 + i)/(1 - i) = e^{i\pi/2} \tag{6}$$

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(a)

$$\begin{aligned} z_1 + z_2 &= -2 + 3i \\ z_1 - z_2 &= 4 - i \\ z_1 z_2 &= -5 - i \\ z_1/z_2 &= -\frac{1}{13}(1 + 5i) \\ |z_1| &= \sqrt{2} \\ z_1^* &= 1 - i \end{aligned} \tag{7}$$

(b) Apparently $z_1 = 2 + 2i$ and $z_2 = -1 - i$.

$$\begin{aligned} z_1 + z_2 &= 1 + i \\ z_1 - z_2 &= 3 + 3i \\ z_1 z_2 &= 2e^{-i\pi/2} = -2i \\ z_1/z_2 &= 2e^{i\pi} = -2 \\ |z_1| &= 2 \\ z_1^* &= 2e^{-i\pi/4} \end{aligned} \tag{8}$$

Apparently $z_1 = \frac{2+2i}{\sqrt{2}}$ and $z_2 = \frac{-1-i}{\sqrt{2}}$.

$$\begin{aligned} z_1 + z_2 &= \frac{1+i}{\sqrt{2}} \\ z_1 - z_2 &= \frac{3+3i}{\sqrt{2}} \end{aligned}$$

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(i)

$$z^2 = (x^2 - y^2) + i(2xy) \tag{9}$$

(ii)

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \tag{10}$$

(iii)

$$i^{-5} = \frac{1}{i^4 \times i} = \frac{1}{i} = -i \tag{11}$$

(iv)

$$\frac{2+3i}{1+6i} = \frac{1}{37}(2+3i)(1-6i) = \frac{1}{37}(20-9i) \tag{12}$$

(v)

$$e^{i\pi/6} - e^{-i\pi/6} = i2\sin\frac{\pi}{6} = i \tag{13}$$

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By de Moivre's theorem:

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\ &= \cos^4\theta + i4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - i4\cos\theta\sin^3\theta + \sin^4\theta\end{aligned} \tag{14}$$

Taking the real part and simplifying:

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned} \tag{15}$$

Set $\theta = \pi/8$ so that $\cos 4\theta = \cos(\pi/2) = 0$:

$$\begin{aligned}8\cos^4(\pi/8) - 8\cos^2(\pi/8) + 1 &= 0 \\ \cos^2(\pi/8) = \frac{8 \pm \sqrt{64-32}}{16} &= \frac{2\pm\sqrt{2}}{4} \\ \cos(\pi/8) &= \sqrt{\frac{2+\sqrt{2}}{4}}\end{aligned} \tag{16}$$

Since $\theta = 3\pi/8$ yields the same equation, have:

$$\cos(3\pi/8) = \sqrt{\frac{2-\sqrt{2}}{4}} \tag{17}$$

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$$\begin{aligned}\sin^6\theta &= \left(\frac{1-\cos 2\theta}{2}\right)^3 \\ &= \frac{1}{8}(1 - 3\cos 2\theta + 3\cos^2 2\theta - \cos^3 2\theta) \\ &= \frac{1}{8}\left(1 - 3\cos 2\theta + \frac{3(1+\cos 4\theta)}{2} - \frac{\cos 3\theta + 3\cos\theta}{4}\right) \\ &= \frac{1}{8}\left(\frac{5}{2} - \frac{3}{4}\cos\theta - 3\cos 2\theta - \frac{1}{4}\cos 3\theta + \frac{3}{2}\cos 4\theta\right)\end{aligned} \tag{18}$$

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\begin{aligned}
\sin^6\theta &= \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^6 \\
&= -\frac{1}{64}\left(e^{i6\theta}-6e^{i4\theta}+15e^{i2\theta}-20+15e^{-i2\theta}-6e^{-i4\theta}+e^{-i6\theta}\right) \\
&= \frac{1}{32}(10-15\cos 2\theta+6\cos 4\theta-\cos 6\theta)
\end{aligned}
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(i)

$$(1+i)^9 = \left(e^{i\pi/4}\right)^9 = e^{i9\pi/4} = e^{i\pi/4} = 1+i \tag{19}$$

$$(1+i)^9 = \left(\frac{e^{i\pi/4}}{\sqrt{2}}\right)^9 = \frac{e^{i9\pi/4}}{16\sqrt{2}} = \frac{e^{i\pi/4}}{16\sqrt{2}} = \frac{1+i}{16\sqrt{2}}$$

(ii)

$$(1-i)^9/(1+i)^9 = \left(\frac{e^{-i\pi/4}}{e^{i\pi/4}}\right) = e^{-i9\pi/2} = e^{-i\pi/2} = -i \tag{20}$$

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(i)

$$\sqrt[4]{\frac{-1-\sqrt{3}i}{2}} = \left[e^{i(-5/6+2k)\pi}\right]^{1/4} = e^{i(-5/24+k/2)\pi} \tag{21}$$

for $k=0,\pm1,2$.

$$\sqrt[4]{\frac{-1-\sqrt{3}i}{2}} = \left[e^{i(-2/3+2k)\pi}\right]^{1/4} = e^{i(-2/12+k/2)\pi}$$

for $k=0,\pm1,2$.

(ii)

$$(-8i)^{2/3} = \left[8e^{i(-1/2+2k)\pi}\right]^{2/3} = 4e^{i(-1/3+4k/3)\pi} \tag{22}$$

for $k=0,1$.

(iii)

$$\sqrt[8]{16} = \left[16e^{i2k\pi}\right]^{1/8} = 2^{1/4}e^{ik\pi/4} \tag{23}$$

for $k = 0, \pm1, \pm2, \pm3, -4$.

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(i) Consider the nth roots of a complex number $re^{i\theta}$. The sum of the roots is:

$$\sum_{k=0}^{n-1} r^{1/n}e^{i(\theta+2k\pi)/n} = r^{1/n}e^{i\theta/n}\sum_{k=0}^{n-1} e^{i2k\pi/n} \tag{24}$$

Focusing on the summation, which is a geometric series:

$$\sum_{k=0}^{n-1} e^{i2k\pi/n} = \frac{1-\left(e^{i2\pi/n}\right)^n}{1-e^{i2\pi/n}} = \frac{1-e^{i2\pi}}{1-e^{i2\pi/n}} = 0 \tag{25}$$

Therefore the summation of the nth roots always equals zero.

(ii)

$$\begin{aligned} z^{2n+1} &= -1 = e^{i(1+2k)\pi} \\ z &= e^{i(1+2k)\pi/(2n+1)} \end{aligned} \tag{26}$$

This essentially says that the possible values of z are $(2n+1)$th roots of -1. Then the desired summation is a sum of the (real parts) of the roots, which equates to zero according to the result derived above.

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First note that $z = \pm1$ do not satisfy the equation. Rearranging:

$$\begin{aligned} (z-1)^n &= -(z+1)^n \\ \left(\frac{z-1}{z+1}\right)^n &= e^{i(1+2k)\pi} \\ z-1 &= e^{i(1+2k)\pi/n}(z+1) \\ z &= \frac{1+e^{i(1+2k)\pi/n}}{1-e^{i(1+2k)\pi/n}} = \frac{e^{-i(1+2k)\pi/2n}+e^{i(1+2k)\pi/2n}}{e^{-i(1+2k)\pi/2n}-e^{i(1+2k)\pi/2n}} \\ z &= i\cot\frac{1+2k}{2n}\pi \end{aligned} \tag{27}$$

for $k \in \mathbb{Z}$.

for $k = 0, 1, \dots, n-1$.

For the equation, note that $x = -1$ is a root. Hence:

$$(x+1)(x^2 + 14x + 1) = 0 \tag{28}$$

Thus the solutions are $x = -1$ and $x = -7 \pm 4\sqrt{3}$.

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$$\begin{aligned}
\sum_{n=0}^{\infty} 2^{-n} \cos n\theta &= \operatorname{Re} \sum_{n=0}^{\infty} 2^{-n} e^{in\theta} \\
&= \operatorname{Re} \sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{2} \right) \\
&= \operatorname{Re} \frac{1}{1 - e^{i\theta}/2} \\
&= \operatorname{Re} \frac{2}{2 - \cos\theta - i \sin\theta} \\
&= \operatorname{Re} \frac{2(2 - \cos\theta + i \sin\theta)}{(2 - \cos\theta)^2 + \sin^2\theta} \\
&= \frac{2(2 - \cos\theta)}{5 - 4\cos\theta} \\
&= \frac{1 - \frac{1}{2}\cos\theta}{\frac{5}{4} - \cos\theta}
\end{aligned} \tag{29}$$

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$$\begin{aligned}
\sum_{r=1}^{n} \binom{n}{r} \sin 2r\theta &= \operatorname{Im} \sum_{r=1}^{n} \binom{n}{r} e^{i2r\theta} \\
&= \operatorname{Im} \left[\left(1 + e^{i2\theta}\right)^n - 1 \right] \\
&= \operatorname{Im} \left(1 + e^{i2\theta}\right)^n \\
&= \operatorname{Im} \left[e^{in\theta} \left(e^{-i\theta} + e^{i\theta}\right)^n \right] \\
&= 2^n \sin n\theta \cos^n \theta
\end{aligned} \tag{30}$$

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(i)

$$e^{3\ln 2 - i\pi} = -e^{3\ln 2} = -8 \tag{31}$$

(ii) If $y = \ln i$, $e^y = i = e^{i(1/2+2k)\pi}$. Thus:

$$\ln i = i(1/2 + 2k)\pi \tag{32}$$

(iii)

$$\begin{aligned} e^{\ln(-e)} &= -e = e^{1+i(1+2k)\pi} \\ \ln(-e) &= 1 + i(1+2k)\pi \end{aligned} \tag{33}$$

(iv)

$$(1+i)^{iy} = \left(e^{i\pi/4}\right)^{iy} = e^{-\pi y/4} \tag{34}$$

$$\boxed{(1+i)^{iy} = \left[\frac{e^{i(1/4+2k)\pi}}{\sqrt{2}}\right]^{iy} = \left[e^{i(1/4+2k)\pi} e^{-\ln(\sqrt{2})}\right]^{iy} = \left(\cos\ln(\sqrt{2})y - i\sin\ln(\sqrt{2})y\right) e^{-(1/4+2k)\pi}}$$

(v)

$$\sin i = \frac{e^{i^2} - e^{-i^2}}{2i} = -i\frac{e^{-1} - e}{2} = i\sinh 1 \tag{35}$$

(vi)

$$\cos(\pi - 2i\ln 3) = \frac{e^{i(\pi - 2i\ln 3)} + e^{-i(\pi - 2i\ln 3)}}{2} = -\frac{e^{2\ln 3} + e^{-2\ln 3}}{4} = -\frac{41}{9} \tag{36}$$

(vii)

$$\begin{aligned} \tanh(x+iy) &= \frac{e^{x+iy} - e^{-x-iy}}{e^{x+iy} + e^{-x-iy}} \\ &= \frac{e^{2x} - e^{-i2y}}{e^{2x} + e^{-i2y}} \\ &= \frac{1}{(e^{2x} + \cos 2y)^2 + \sin^2 2y}(e^{2x} - e^{-i2y})(e^{2x} + e^{i2y}) \\ &= \frac{1 + e^{4x} + i2e^{2x}\sin 2y}{1 + e^{4x} + 2e^{2x}\cos 2y} \end{aligned} \tag{37}$$

(viii) Let $y = \tan^{-1}(\sqrt{3}i)$:

$$\begin{aligned}
\tan y = \sqrt{3}i &= i\frac{e^{iy} - e^{-iy}}{e^{iy} + e^{-iy}} \\
(1-\sqrt{3})e^{iy} &= (1+\sqrt{3})e^{-iy} \\
e^{i2y} &= \frac{1+\sqrt{3}}{1-\sqrt{3}} = -2-\sqrt{3} = e^{i(1+2k)\pi + \ln(2+\sqrt{3})} \\
y &= \frac{1+2k}{2}\pi - i\frac{\ln(2+\sqrt{3})}{2}
\end{aligned} \tag{38}$$

(ix) Let $y = \sinh^{-1}(-1)$:

$$\begin{aligned}
\sinh y = \frac{e^y - e^{-y}}{2} &= -1 \\
e^{2y} - 1 + 2e^y &= 0 \\
e^y &= -1 \pm \sqrt{2}
\end{aligned} \tag{39}$$

For the positive case:

$$\begin{aligned}
y &= \ln 1 + \ln(\sqrt{2} - 1) \\
&= \ln e^{2k\pi} + \ln(\sqrt{2} - 1) \\
&= 2k\pi + \ln(\sqrt{2} - 1)
\end{aligned}$$

For the negative case

$$\begin{aligned}
y &= \ln -1 + \ln(\sqrt{2} + 1) \\
&= \ln e^{(1+2k)\pi} + \ln(\sqrt{2} + 1) \\
&= (1+2k)\pi + \ln(\sqrt{2} + 1)
\end{aligned}$$

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