Mathematical Methods

Problem Sheet 2

Fourier Series and Fourier Integrals

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1 Fourier Series

(a) We have the coefficients for the cosine series as:

$$a_r = \frac{2}{2\pi} \int_0^\pi \sin x \cos rx \, \mathrm{d}x \tag{1}$$

where apparently $a_0 = 2/\pi$.

For $r \geq 1$, the integral denoted as I_r can be evaluated by parts:

$$\begin{aligned} I_r &= [-\cos x \cos rx]_0^\pi - r \int_0^\pi \cos x \sin rx \, \mathrm{d}x \\ &= [\cos x \cos rx]_\pi^0 - r \left\{ [\sin x \sin rx]_0^\pi - r \int_0^\pi \sin x \cos rx \, \mathrm{d}x \right\} \\ &= 1 + \cos \pi r + r^2 I_r \end{aligned} \tag{2}$$

Therefore, the coefficients are:

$$a_r = \frac{1}{\pi} \frac{1 + \cos \pi r}{1 - r^2} = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 2/\pi(1 - r^2) & \text{if } r \text{ is even} \end{cases} \tag{3}$$

On the other hand, the coefficients for the sine series are:

$$b_r = \frac{2}{2\pi} \int_0^\pi \sin x \sin rx \, \mathrm{d}x \tag{4}$$

which are zero except for $r = 1$:

$$b_1 = \frac{1}{\pi} \int_0^\pi \sin^2 x \, \mathrm{d}x = \frac{1}{2} \tag{5}$$

Hence, the Fourier series is:

$$f(x) = \frac{1}{2} \sin x + \frac{2}{\pi} \sum_{\text{even } r \geq 0}^{\infty} \frac{1}{1 - r^2} \cos rx \tag{6}$$

(b) Since the function is even, we only need the coefficients for the cosine series:

$$\begin{aligned}
a_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos rx \,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos rx \,\mathrm{d}x \\
&= \frac{2}{\pi}\left\{\left[x^2\frac{1}{r}\sin rx\right]_0^{\pi} - \frac{2}{r}\int_0^{\pi} x\sin rx \,\mathrm{d}x\right\} \\
&= -\frac{4}{r\pi}\left\{\left[-x\frac{1}{r}\cos rx\right]_0^{\pi} + \frac{1}{r}\int_0^{\pi} \cos rx \,\mathrm{d}x\right\} \\
&= \frac{4}{r^2}\cos r\pi = \frac{4}{r^2}(-1)^r
\end{aligned} \tag{7}$$

for $r \geq 1$.

Apparently, $a_0 = 2\pi^2/3$ and the Fourier series is:

$$f(x) = \frac{\pi^2}{3} + \sum_{r=1}^{\infty}(-1)^r\frac{4}{r^2}\cos rx \tag{8}$$

(c) Consider the norm of the function $f(x) = x^2$ on the interval $[-\pi, \pi]$:

$$\|f\|^2 = \int_{-\pi}^{\pi} x^4 \,\mathrm{d}x = \frac{2\pi^5}{5} \tag{9}$$

By Parseval's equation, we have:

$$\frac{\|f\|^2}{\pi} = \frac{1}{2}\left(\frac{\pi^2}{3}\right)^2 + \sum_{r=1}^{\infty}\frac{16}{r^4} \tag{10}$$

so that:

$$\sum_{r=1}^{\infty}\frac{1}{r^4} = \frac{1}{16}\left(\frac{2\pi^4}{5} - \frac{\pi^4}{18}\right) = \tag{11}$$

2 Sine and cosine Fourier series

(a) The coefficients for the cosine series are:

$$a_r = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos rx \,\mathrm{d}x \tag{12}$$

For the case $r = 0$ and $r = 1$, we have:

$$a_0 = \frac{2}{\pi}\int_0^\pi x\sin x \,\mathrm{d}x = \frac{2}{\pi}\left\{[-x\cos x]_0^\pi + \int_0^\pi \cos x \,\mathrm{d}x\right\} = 2 \tag{13}$$

and:

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x \,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi x\sin 2x \,\mathrm{d}x = \frac{1}{2\pi}\left\{[-x\cos 2x]_0^\pi + \int_0^\pi \cos 2x \,\mathrm{d}x\right\} = -\frac{1}{2} \tag{14}$$

For $r \geq 2$, the integral, which is denoted as I_r, can be evaluated by parts:

$$I_r = [-x\cos x\cos rx]_0^\pi + \int_0^\pi \cos x\cos rx \,\mathrm{d}x - r\int_0^\pi x\cos x\sin rx \,\mathrm{d}x \tag{15}$$

For the middle term, the only non-zero contribution is when $r = 1$ as $\cos rx$ are orthogonal. Therefore we may neglect the middle term for $r \geq 2$:

$$\begin{aligned} I_r &= \pi\cos r\pi - r\left\{[x\sin x\sin rx]_0^\pi - \int_0^\pi \sin x\sin rx \,\mathrm{d}x - r\int_0^\pi x\sin x\cos rx \,\mathrm{d}x\right\} \\ &= \pi\cos r\pi + r^2 I_r \end{aligned} \tag{16}$$

where in the last step we have again used the orthogonality of $\sin rx$.

This means that for $r \geq 2$, the coefficients are:

$$a_r = (-1)^r \frac{2}{1-r^2} \tag{17}$$

Hence the cosine Fourier series is:

$$f(x) = 2 - \frac{1}{2}\cos x + \sum_{r=2}^{\infty}(-1)^r \frac{2}{1-r^2}\cos rx \tag{18}$$

(b) The coefficients for the sine series are:

$$b_r = \frac{2}{\pi}\int_0^\pi x\sin x\sin rx \,\mathrm{d}x \tag{19}$$

where $b_1 = \pi/2$.

For $r \geq 2$, the integral, which is denoted as I_r, can be evaluated as:

$$\begin{aligned} I_r &= \frac{1}{2}\int_0^\pi x\cos{(1+r)x}\,\mathrm{d}x - \frac{1}{2}\int_0^\pi x\cos{(1-r)x}\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\cos{(1+r)\pi}-1}{(1+r)^2} - \frac{\cos{(1-r)\pi}-1}{(1-r)^2}\right] \end{aligned} \tag{20}$$

where we used the integral result:

$$\int_0^\pi x\cos kx\,\mathrm{d}x = \frac{\cos k\pi - 1}{k^2} \tag{21}$$

This means that the coefficients are:

$$b_r = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 2\left[(1+r)^{-2} - (1-r)^{-2}\right]/\pi & \text{if } r \text{ is even} \end{cases} \tag{22}$$

Hence the sine Fourier series is:

$$f(x) = \frac{\pi}{2}\sin x + \sum_{\text{even } r\geq 2}^{\infty} \frac{2}{\pi}\left[\frac{1}{(1+r)^2} - \frac{1}{(1-r)^2}\right]\sin rx \tag{23}$$

(c) The coefficients for the cosine series are:

$$a_r = \frac{2}{\pi}\int_0^\pi x\cos rx\,\mathrm{d}x = \frac{2}{\pi}\frac{\cos k\pi - 1}{k^2} = \begin{cases} -4/\pi r^2 & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases} \tag{24}$$

except for $r = 0$ where $a_0 = \pi$.

Hence the cosine Fourier series is:

$$f(x) = \pi - \frac{4}{\pi}\sum_{\text{odd } r\geq 1}^{\infty} \frac{1}{r^2}\cos rx \tag{25}$$

(d) The coefficients for the sine series are:

$$b_r = \frac{2}{\pi}\int_0^\pi x\sin rx\,\mathrm{d}x = -\frac{2}{\pi}\frac{\pi\cos r\pi}{r} = \begin{cases} 2/r & \text{if } r \text{ is odd} \\ -2/r & \text{if } r \text{ is even} \end{cases} \tag{26}$$

Hence the sine Fourier series is:

$$f(x) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{2}{r} \sin rx \tag{27}$$

The cosine series does not converge to $f(x)$ near zero because $f(x)$ is not even, whereas the sine series converges to zero.

Consider the norm of the function $f(x) = x$ on the interval $[0, \pi]$:

$$\|f\|^2 = \int_0^\pi x^2 \, \mathrm{d}x = \frac{\pi^3}{3} \tag{28}$$

The Parseval's equation gives:

$$\frac{2\|f\|^2}{\pi} = \sum_{r=1}^{\infty} \frac{4}{r^2} \tag{29}$$

so that:

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6} \tag{30}$$