Optics

Problem Set II

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April 20, 2024

5

(a) Since the magnification of a telescope is given by:

$$M = -\frac{f_o}{f_s} \tag{1}$$

we choose $f_s = 18\,\mathrm{mm}$ for a better magnification.

(b) The magnification obtained is:

$$M = -\frac{30}{18} = -\frac{5}{3} \tag{2}$$

(c)

(d) The angular resolution is limited by diffraction via the objective lens:

$$\theta = 1.22\frac{\lambda}{D} \tag{3}$$

Taking $\lambda = 500\,\mathrm{nm}$ for typical visible light, we have:

$$\theta = 1.22 \times 10^{-5}\,\mathrm{rad} \tag{4}$$

The size of the diffraction pattern is thus:

$$\pi[\theta(f_o + f_s)]^2 = \pi(3.66 \times 10^{-6}\,\mathrm{m})^2 = 4.20 \times 10^{-11}\,\mathrm{m}^2 \tag{5}$$

Note that $3.66 \times 10^{-6}\,\mathrm{m}$ is around twice of the spacing between two photoreceptors in the human eye, so the diffraction pattern is not resolved by the human eye.

(e) Diffraction by the human pupil has the angular resolution:

$$\theta = 1.22\frac{\lambda}{D_p} = 6.1 \times 10^{-4}\,\mathrm{rad} \tag{6}$$

where we take $D_p = 1\,\mathrm{mm}$ for the diameter of the pupil.

The size of the diffraction pattern is thus:

$$\pi[\theta(f_e)]^2 = \pi(1.83 \times 10^{-5}\,\mathrm{m})^2 = 3.35 \times 10^{-10}\,\mathrm{m}^2 \tag{7}$$

where we take $f_e = 3\,\text{cm}$ for the focal length of the eye.

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6

(a) From geometry, the path difference between a ray hitting location x and the ray hitting the centre of the glass is:

$$\delta = x(\sin\theta_i - \sin\theta) \tag{8}$$

The diffraction pattern is then the Fourier transform of a constant transmission function:

$$\begin{aligned} u(x) &= \int_{-D/2}^{D/2} u_0 e^{ik\delta}\,\mathrm{d}x \\ &= u_0 \int_{-D/2}^{D/2} e^{ikx(\sin\theta_i - \sin\theta)}\,\mathrm{d}x \\ &= u_0 \frac{1}{k(\sin\theta_i - \sin\theta)} 2\sin\left[kD(\sin\theta_i - \sin\theta)/2\right] \\ &= u_0 D\,\mathrm{sinc}\left[kD(\sin\theta_i - \sin\theta)/2\right] \end{aligned} \tag{9}$$

The intensity of the diffraction pattern is then:

$$I(x) = I_0 D^2\,\mathrm{sinc}^2\left[kD(\sin\theta_i - \sin\theta)/2\right] \tag{10}$$

(b) The sinc function has its maximum when the argument is zero, i.e., $\sin\theta_i = \sin\theta$ which agrees with law of reflection.

(c) The angular displacement of the first minimum satisfies:

$$\sin\theta - \sin\theta_i = \frac{2\pi}{kD} = \frac{\lambda}{D} \tag{11}$$

Consider $\lambda = 500\,\text{nm}$ and $D = 5\,\text{cm}$, which gives:

$$\Delta\theta \approx \frac{\lambda}{D} = 1 \times 10^{-5}\,\text{rad} \tag{12}$$

which is too small an angle to be resolved by the human eye.

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7

(a) The transmission function of the grating is:

$$u_0 \sum_{r=0}^{N/2-1} \delta[x-(r+\frac{1}{2})d]+\delta[x+(r+\frac{1}{2})d] \tag{13}$$

The diffraction pattern is then:

$$\begin{aligned} u(x) &= u_0 \int \sum_{r=0}^{N/2-1} \delta[x-(r+\frac{1}{2})d]+\delta[x+(r+\frac{1}{2})d] e^{ikx\sin\theta}\,\mathrm{d}x \\ &= u_0 \sum_{r=0}^{N/2-1} \left[e^{ik(r+\frac{1}{2})d\sin\theta}+e^{-ik(r+\frac{1}{2})d\sin\theta}\right] \\ &= 2u_0 \sum_{r=0}^{N/2-1} \cos\left[(r+\frac{1}{2})\delta\right] \\ &= u_0 \frac{\sin(N\delta/2)}{\sin(\delta/2)} \end{aligned} \tag{14}$$

so that the intensity pattern is:

$$I(\delta) = \frac{I_0}{N^2}\frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)} \tag{15}$$

(b) The maxima satisfy:

$$\sin\theta_p = \left(n+\frac{1}{2}\right)\frac{\lambda}{Nd} \tag{16}$$

whereas the minima satisfy:

$$\sin\theta_m = n\frac{\lambda}{Nd} \tag{17}$$

This implies that the distance between adjacent maximum and minimum satisfies:

$$\sin(\theta_p+\Delta\theta)-\sin\theta_p = \frac{\lambda}{Nd} \tag{18}$$

Assuming small $\Delta\theta$, we have:

$$\Delta\theta = \frac{\lambda}{Nd\cos\theta_p} \tag{19}$$

(c) Compare this with the width from the previous problem λ/D, the extra factor of $1/N$ ensures that the central maximum is much smaller.

(d)

8

(a) For a reflection grating of distance d between lines, the path difference is given by:

$$d(\cos\theta_i - \cos\theta_r) \tag{20}$$

where θ_i and θ_r are the incident and reflected angles, respectively.

Second order maxima satisfy:

$$d(\cos\theta_i - \cos\theta_r) = \lambda \tag{21}$$

so that the reflected angle is:

$$\cos\theta_r = \cos\theta_i - \frac{\lambda}{d} \tag{22}$$

(b) If the diffracting elements are inclined at an angle θ_B, the reflection angle becomes $\theta_i - \theta_B$, so that the path difference becomes:

$$d[\cos\theta_i - \cos(\theta_i - \theta_B)] \tag{23}$$

The second order maxima satisfy:

$$d[\cos\theta_i - \cos(\theta_i - \theta_B)] = \lambda \tag{24}$$

(c) The maximum range of θ_i is $\pi/2$, so that the maximum range of the reflected angle is $\pi/2 - \theta_B$. The longest wavelength that can be measured in the second order is then:

$$\lambda_{\max} = -d\cos(\pi/2 - \theta_B) = d\sin\theta_B \tag{25}$$