Symmetry and Relativity

Problem Set 1

Symmetries

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October 28, 2024

1 Conserved quantity under a Galilean transformation

(a) With the transformation $r \to r' = r + \epsilon v t$, the variation in the action is:

$$\begin{aligned}
\delta S &= \int \left(\frac{\partial L}{\partial r}\delta r + \frac{\partial L}{\partial \dot{r}}\delta \dot{r} \right) \mathrm{d}t \\
&= \int \epsilon \left(\frac{\partial L}{\partial r} v t + \frac{\partial L}{\partial \dot{r}} v \right) \mathrm{d}t \\
&= \int \epsilon v \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}} \right) t + \frac{\partial L}{\partial \dot{r}} \right] \mathrm{d}t \\
&= \left[\epsilon v \frac{\partial L}{\partial \dot{r}} t \right]_{t_1}^{t_2} + \epsilon v \int \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial \dot{r}} \,\mathrm{d}t \\
&= \epsilon v \left[\frac{\partial L}{\partial \dot{r}} t \right]_{t_1}^{t_2}
\end{aligned} \tag{1}$$

which is in general not zero.

But this demonstrates that the variation in L is a total time derivative:

$$\begin{aligned}
\int \delta L \,\mathrm{d}t &= \epsilon v \left[\frac{\partial L}{\partial \dot{r}} t \right]_{t_1}^{t_2} \\
&= \int \frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon v \frac{\partial L}{\partial \dot{r}} t \right) \mathrm{d}t
\end{aligned} \tag{2}$$

which means that the equations of motion are invariant and the transformation is a symmetry.

(b) Consider the transformation $q \to q' = q + \epsilon \eta(q,t)$. We demand that the variation in L is some total time derivative:

$$\delta L = \epsilon \left(\frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial \dot{q}}\dot{\eta} \right) = \epsilon \frac{\mathrm{d}f}{\mathrm{d}t} \tag{3}$$

For this to happen, consider integrating the equation:

$$\begin{aligned}
f &= \int \left(\frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial \dot{q}}\dot{\eta} \right) \mathrm{d}t \\
&= \frac{\partial L}{\partial \dot{q}}\eta + \text{constant}
\end{aligned} \tag{4}$$

We may change q to q_i and η to η_i without loss of generality. The equation becomes:

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\eta_i - f = \text{constant} \tag{5}$$

which is the Noether's theorem.

(c) Applying the theorem to the Galilean transformation, we identify $\eta = vt$ and $f = v(\partial L/\partial \dot{q}_i)t$. The conserved quantity is:

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} vt - v\frac{\partial L}{\partial \dot{q}_i} t \tag{6}$$

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2 Conservation of the Laplace-Runge-Lenz vector

(a) In the potential $V = -k/r$, the angular momentum can be written as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Consider the transformation $\mathbf{r} \to \mathbf{r}' = \mathbf{r} + \mathbf{a} \times \mathbf{L}$, where $\mathbf{a}$ is a constant vector. We have:

$$\delta\mathbf{r} = \mathbf{a} \times \mathbf{L} \tag{7}$$

and:

$$\delta\dot{\mathbf{r}} = \mathbf{a} \times \dot{\mathbf{L}} = \mathbf{a} \times (\mathbf{r} \times \mathbf{F}) + \mathbf{a} \times (\dot{\mathbf{r}} \times \mathbf{p}) = -\mathbf{a} \times (\mathbf{r} \times \nabla V) = \mathbf{0} \tag{8}$$

since $\nabla V = -k\mathbf{r}/r^3$.

The variation in the Lagrangian is:

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial \mathbf{r}} \cdot \delta\mathbf{r} + \frac{\partial L}{\partial \dot{\mathbf{r}}} \cdot \delta\dot{\mathbf{r}} \\ &= -\nabla V \cdot (\mathbf{a} \times \mathbf{L}) \\ &= -\frac{mk}{r^3}\mathbf{r} \cdot [\mathbf{a} \times (\mathbf{r} \times \dot{\mathbf{r}})] \\ &= -\frac{mk}{r^3}\mathbf{r} \cdot [(\mathbf{a} \cdot \dot{\mathbf{r}})\mathbf{r} - (\mathbf{a} \cdot \mathbf{r})\dot{\mathbf{r}}] \end{aligned} \tag{9}$$

Consider the quantity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathbf{r}}{r} = \frac{\dot{\mathbf{r}}r - \mathbf{r}\dot{r}}{r^2} = \frac{\dot{\mathbf{r}}r^2 - r\mathbf{r}\dot{r}}{r^3} \tag{10}$$

Comparing with the variation in the Lagrangian, we have:

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}\left(-mk\mathbf{a} \cdot \frac{\mathbf{r}}{r}\right) \tag{11}$$

(b) By Noether's theorem, the conserved quantity is:

$$\begin{aligned} \mathbf{A} &= \frac{\partial L}{\partial \dot{\mathbf{r}}} \times \eta + mk\frac{\mathbf{r}}{r} \\ &= \mathbf{p} \times \mathbf{L} + mk\hat{r} \end{aligned} \tag{12}$$

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3 Conservation of energy for fields

(a) Consider the time translation $t \to t' = t + \epsilon$. The variation in $\mathcal{L}$ is:

$$\delta\mathcal{L} = \epsilon\frac{\partial\mathcal{L}}{\partial t} = \epsilon\left[\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial\phi}{\partial t} + \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\frac{\partial^2\phi}{\partial^2 t} + \frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\frac{\partial^2\phi}{\partial t\partial r_i}\right] \tag{13}$$

Consider on the other hand the given expression:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\frac{\partial\phi}{\partial t} - \mathcal{L}\right] + \frac{\partial}{\partial r_i}\left[\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\frac{\partial\phi}{\partial t}\right] \\ &= \frac{\partial^2\mathcal{L}}{\partial(\partial_t\phi)\partial t}\frac{\partial\phi}{\partial t} + \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\frac{\partial^2\phi}{\partial^2 t} - \frac{\partial\mathcal{L}}{\partial t} + \frac{\partial\mathcal{L}}{\partial(\partial_i\phi)\partial t}\frac{\partial^2\phi}{\partial t\partial r_i} \end{aligned} \tag{14}$$

which is exactly the equation for $\delta\mathcal{L}$.

(b) Let us integrate the equation over space:

$$\begin{aligned} \text{constant} &= \int \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\frac{\partial\phi}{\partial t} - \mathcal{L}\right] + \frac{\partial}{\partial r_i}\left[\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\frac{\partial\phi}{\partial t}\right] \mathrm{d}^3 r \\ &= \left[\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\frac{\partial\phi}{\partial t}\right] + \int \frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\frac{\partial\phi}{\partial t} - \mathcal{L}\right] \mathrm{d}^3 r \end{aligned} \tag{15}$$

The boundary term vanishes because the fields vanish at infinity. The conserved quantity is then:

$$H \equiv \int \left[\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\frac{\partial\phi}{\partial t} - \mathcal{L}\right] \mathrm{d}^3 r \tag{16}$$

Then the Noether's theorem is a statement on the rate of change of the Hamiltonian of the field.

(c) Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho(\partial_t\phi)^2 - \frac{1}{2}T(\partial_x\phi)^2 - \mathcal{V}(\phi) \tag{17}$$

Then the Hamiltonian is trivially:

$$H = \int \left[\frac{1}{2}\rho(\partial_t\phi)^2 + \frac{1}{2}T(\partial_x\phi)^2 + \mathcal{V}(\phi)\right] \mathrm{d}x \tag{18}$$

4 Lagrangian for the electromagnetic field

(a) Consider the Lagrangian density:

$$\mathcal{L} = \frac{\epsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2 \tag{19}$$

Let us write the fields in terms of the potentials, using index notation. The electric term is:

$$\mathbf{E}^2 = (-\partial_i\phi - \partial_t A_i)^2 \tag{20}$$

The magnetic term is:

$$\begin{aligned}\mathbf{B}^2 &= \epsilon_{ijk}\partial_j A_k \epsilon_{ilm}\partial_l A_m \\ &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\partial_j A_k \partial_l A_m \\ &= \partial_j A_k \partial_j A_k - \partial_j A_k \partial_k A_j\end{aligned} \tag{21}$$

The Euler-Lagrange equations are:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi} &= \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} + \frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)} \\ \frac{\partial\mathcal{L}}{\partial A_j} &= \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial_t A_j)} + \frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial_i A_j)}\end{aligned} \tag{22}$$

For ϕ, we have:

$$\begin{aligned}0 &= \frac{\epsilon_0}{2}\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial(\partial_i\phi)}(\partial_i\phi + \partial_t A_i)^2\right] \\ &= \epsilon_0\partial_i(\partial_i\phi + \partial_t A_i) \\ &= -\epsilon_0\nabla\cdot\mathbf{E}\end{aligned} \tag{23}$$

which is the Gauss' law in vacuum.

For A_j, we have:

$$\begin{aligned}0 &= \frac{1}{2}\left\{\epsilon_0\frac{\partial}{\partial t}\left[\frac{\partial}{\partial(\partial_t A_j)}(\partial_i\phi + \partial_t A_i)^2\right] + \frac{1}{\mu_0}\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial(\partial_i A_j)}(\partial_l A_m\partial_l A_m - \partial_l A_m\partial_m A_l)\right]\right\} \\ &= -\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{2\mu_0}\frac{\partial}{\partial x_i}\left[2\partial_i A_j - 2\partial_j A_i\right] \\ &= -\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\nabla\times\mathbf{B}\end{aligned} \tag{24}$$

which is the Faraday's law.

The other two Maxwell's equations have to be derived from continuity equations.

(b) Consider the gauge transformation:

$$\begin{aligned}\phi \to \tilde{\phi} &= \phi - \partial_t \Gamma \\ A_i \to \tilde{A}_i &= A_i + \partial_i \Gamma\end{aligned} \tag{25}$$

We choose Γ such that $\partial_t \Gamma = \phi$ and $\tilde{\phi} = 0$. The new electric field is:

$$\tilde{\mathbf{E}} = -\partial_t \tilde{\mathbf{A}} = -\partial_t A_i - \partial_t \partial_i \Gamma = -\partial_t A_i - \partial_i \phi = \mathbf{E} \tag{26}$$

The new magnetic field is:

$$\tilde{\mathbf{B}} = \nabla \times \tilde{\mathbf{A}} = \nabla \times \mathbf{A} + \nabla \times \nabla \Gamma = \mathbf{B} \tag{27}$$

Thus the fields are invariant under the gauge transformation.

We now remove the tildes and use the new gauge and its fields without loss of generality. The energy density is:

$$\begin{aligned}\varepsilon &= \frac{\partial \mathcal{L}}{\partial(\partial_t A_j)} \partial_t A_j - \mathcal{L} \\ &= \frac{\partial}{\partial(\partial_t A_j)} \left[\frac{\epsilon_0}{2} (\partial_t A_k)^2 \right] \partial_t A_j - \mathcal{L} \\ &= \epsilon_0 \mathbf{E}^2 - \mathcal{L} \\ &= \frac{\epsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2\end{aligned} \tag{28}$$

Then:

$$\begin{aligned}S_i &= \frac{\partial \mathcal{L}}{\partial(\partial_i A_j)} \partial_t A_j \\ &= \frac{\partial}{\partial(\partial_i A_j)} \left[-\frac{1}{2\mu_0} \epsilon_{klm} \partial_l A_m \epsilon_{kno} \partial_n A_o \right] \partial_t A_j \\ &= \frac{\partial}{\partial(\partial_i A_j)} \left[-\frac{1}{2\mu_0} (\partial_l A_m \partial_l A_m - \partial_l A_m \partial_m A_l) \right] \partial_t A_j \\ &= -\frac{1}{2\mu_0} \left(2\delta_{il}\delta_{jm} \partial_l A_m - \delta_{il}\delta_{jm} \partial_m A_l - \delta_{im}\delta_{jl} \partial_l A_m \right) \partial_t A_j \\ &= -\frac{1}{\mu_0} (\partial_i A_j - \partial_j A_i) \partial_t A_j\end{aligned} \tag{29}$$

But this is just $\mathbf{E} \times \mathbf{B}$, so:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \tag{30}$$

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5 Lagrangian for the Schrodinger equation

Consider components of the Euler-Lagrange equation for ψ:

$$\begin{aligned}
\partial_t \left[\frac{\partial \mathcal{L}}{\partial(\partial_t \psi)}\right] &= -\frac{i\hbar}{4\pi}\partial_t \psi^* \\
\partial_i \left[\frac{\partial \mathcal{L}}{\partial(\partial_i \psi)}\right] &= \partial_i \left[\frac{\hbar^2}{8\pi m}\nabla \psi^*\right] \\
\frac{\partial L}{\partial \psi} &= V\psi^*
\end{aligned} \tag{31}$$

which leads to the equation:

$$-\frac{\hbar^2}{8\pi m}\nabla^2 \psi^* + V\psi^* = -i\frac{\hbar}{4\pi}\partial_t \psi^* \tag{32}$$

Taking the complex conjugate of the equation, we obtain the Schrodinger equation.

The energy density is:

$$\begin{aligned}
\varepsilon &= \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)}\partial_t \psi + \frac{\partial \mathcal{L}}{\partial(\partial_t \psi^*)}\partial_t \psi^* - \mathcal{L} \\
&= \frac{i\hbar}{4\pi}(\psi \partial_t \psi^* - \psi^* \partial_t \psi) - \mathcal{L} \\
&= -\frac{\hbar^2}{8\pi m}\nabla \psi \cdot \nabla \psi^* - V\psi^* \psi
\end{aligned} \tag{33}$$

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