Vectors & Matrices

Problem Set 2

Matrices, linear maps and linear equations

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Matrices, linear maps and linear equations

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(a)

$$[A(BC)]_{ik} = A_{ij}(BC)_{jk} = A_{ij}B_{jl}C_{lk} \tag{1}$$

$$[(AB)C]_{ik} = (AB)_{ij}C_{jk} = A_{il}B_{lj}C_{jk} = A_{ij}B_{jl}C_{lk} = [A(BC)]_{ik} \tag{2}$$

with a change of dummy variable $l \to j$ and $j \to l$.

(b)

$$(A+B)^{\intercal}_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^{\intercal}_{ij} + B^{\intercal}_{ij} \tag{3}$$

$$(\alpha A)^{\intercal}_{ij} = (\alpha A)_{ji} = \alpha A_{ji} = \alpha(A^{\intercal}_{ij}) \tag{4}$$

$$(AB)^{\intercal}_{ij} = (AB)_{ji} = A_{jk}B_{ki} = (B^{\intercal})_{ik}(A^{\intercal})_{kj} = (B^{\intercal}A^{\intercal})_{ij} \tag{5}$$

(c)

$$(A+B)^{\dagger}_{ij} = (A+B)^{*}_{ji} = A^{*}_{ji} + B^{*}_{ji} = A^{\dagger}_{ij} + B^{\dagger}_{ij} \tag{6}$$

$$(\alpha A)^{\dagger}_{ij} = (\alpha A)^{*}_{ji} = \alpha^{*} A^{*}_{ji} = \alpha^{*}(A^{\dagger}_{ij}) \tag{7}$$

$$(AB)^{\intercal}_{ij} = (AB)^{*}_{ji} = A^{*}_{jk}B^{*}_{ki} = B^{\dagger}_{ik}A^{\dagger}_{kj} = (B^{\dagger}A^{\dagger})_{ij} \tag{8}$$

(d)

$$\begin{aligned}(AB)(AB)^{-1} &= I \\ B^{-1}A^{-1}(AB)(AB)^{-1} &= B^{-1}A^{-1} \\ (AB)^{-1} &= B^{-1}A^{-1}\end{aligned} \tag{9}$$

We have:

$$A^{\intercal}(A^{\intercal})^{-1} = I \tag{10}$$

Taking the transpose:

$$\left[(A^{\intercal})^{-1}\right]^{\intercal} A = I \tag{11}$$

Since the inverse of a matrix is unique:

$$\left[(A^\intercal)^{-1}\right]^\intercal = A^{-1} \tag{12}$$

Taking the transpose again:

$$(A^\intercal)^{-1} = (A^{-1})^\intercal \tag{13}$$

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$$A \to \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -2 & 1 \\ 3 & 2 & 0 & -4 \\ 1 & -2 & a & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & -1 & -1/2 \\ 0 & 0 & a-3 & 5/2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & a & 4 \end{pmatrix} \tag{14}$$

Thus, if $a = 8$, $\text{rank}(A) = 4$. If $a \neq 8$, $\text{rank}(A) = 3$.

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$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 3 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right) \tag{15}$$

Thus:

$$A^{-1} = \begin{pmatrix} -6 & 3 & 1 \\ -2 & 1 & 0 \\ -7 & 3 & 1 \end{pmatrix} \tag{16}$$

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We have $A' = \psi' f \psi = \psi' \phi \phi' f \phi \phi' \psi = PAP^{-1}$, where $A = \phi' f \phi$, $\phi = I$ and $P = \psi' \phi = \psi'$.

Thus:

$$P = \psi^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{17}$$

$$A' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \tag{18}$$

We have $A' = (\phi')^{-1} f \phi' = (\phi')^{-1} \phi \phi^{-1} f \phi \phi^{-1} \phi' = PAP^{-1}$, where $A = \phi^{-1} f \phi$, $\phi = I$ and $P = (\phi')^{-1}\phi = (\phi')^{-1}$.
Thus:

$$P = (\phi')^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{19}$$

$$A' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \tag{20}$$

P is the matrix that transforms the standard coordinate vector to the coordinate vector in the new basis.

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(a)

$$\begin{aligned} f(\alpha \mathbf{a} + \beta \mathbf{u}) &= (\alpha \mathbf{a} + \beta \mathbf{u}) - 2\left[\mathbf{n} \cdot (\alpha \mathbf{a} + \beta \mathbf{u})\right] \mathbf{n} \\ &= \alpha \mathbf{v} - 2\left(\mathbf{n} \cdot \alpha \mathbf{a}\right) \mathbf{n} + \beta \mathbf{u} - 2\left(\mathbf{n} \cdot \beta \mathbf{u}\right) \mathbf{n} \\ &= \alpha f(\mathbf{a}) + \beta f(\mathbf{u}) \end{aligned} \tag{21}$$

Thus f is linear.

$$\begin{aligned} f[f(\mathbf{v})] &= f[\mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n}] \\ &= \mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n} - 2\left\{\mathbf{n} \cdot [\mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n}]\right\} \\ &= \mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n} - 2\left\{\mathbf{n} \cdot \mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})(\mathbf{n} \cdot \mathbf{n})]\right\} \\ &= \mathbf{v} \end{aligned} \tag{22}$$

(b) Let $\mathbf{v} = (a, b, c)^\intercal$ and $\mathbf{n} = (\alpha, \beta, \gamma)$ relative to the standard basis. Then:

$$A\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} - 2(a\alpha + b\beta + c\gamma)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} (1-2\alpha^2)a - 2\alpha\beta b - 2\alpha\gamma c \\ -2\alpha\beta a + (1-2\beta^2)b - 2\beta\gamma c \\ -2\alpha\gamma a - 2\beta\gamma b + (1-2\gamma^2)c \end{pmatrix} \tag{23}$$

Thus:

$$A = I - 2\begin{pmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{pmatrix} = I - 2\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ \gamma & \gamma & \gamma \end{pmatrix}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{24}$$

(c) We have $\hat{A} = \psi' f \psi = \psi'\phi\phi' f \phi\phi'\psi = PAP^{-1}$, where $A = \phi' f \phi$, $\phi = I$ and $P = \psi'\phi = \psi' = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{n})^{-1}$. Thus:

$$\begin{aligned}
\hat{A} &= I - 2\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{n} \end{pmatrix}^{-1}\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ \gamma & \gamma & \gamma \end{pmatrix}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{n} \end{pmatrix} \\
&= I - 2\begin{pmatrix} u_{1x} & u_{1y} & u_{1z} \\ u_{2x} & u_{2y} & u_{2z} \\ \alpha & \beta & \gamma \end{pmatrix}\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ \gamma & \gamma & \gamma \end{pmatrix}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} u_{1x} & u_{2x} & \alpha \\ u_{1y} & u_{2y} & \beta \\ u_{1z} & u_{2z} & \gamma \end{pmatrix} \\
&= I - 2\begin{pmatrix} u_{1x} & u_{1y} & u_{1z} \\ u_{2x} & u_{2y} & u_{2z} \\ \alpha & \beta & \gamma \end{pmatrix}\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ 0 & 0 & \gamma \end{pmatrix} \\
&= I - 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned} \tag{25}$$

where the identity $\mathbf{u}_\alpha \cdot \mathbf{n} = 0$ has been used.

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(a) We have that $\text{tr}(A) = \delta_{ij}A_{ij}$. Thus

$$\text{tr}(AB) = \delta_{ij}(AB)_{ij} = \delta_{ij}A_{ik}B_{kj} = \delta_{ij}A_{jk}B_{ki} = \delta_{ij}B_{ik}A_{kj} = \text{tr}(BA) \tag{26}$$

(b)

$$\text{tr}\big(PAP^{-1}\big) = \text{tr}\big(AP^{-1}P\big) = \text{tr}(A) \tag{27}$$

A basis change of matrices should not affect the trace of a matrix.

(c) The trace is unity.

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(a) Because the equation is linear (with respect to the derivatives) and differentiation is a linear operation.

(b) The differentiation operation can be represented by the matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{28}$$

Thus:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 2I = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \tag{29}$$

(c) Consider the augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \tag{30}$$

Thus:

$$\ker A = \left\{ p(x) | p(x) = \lambda(x^2 - 4x + 2), \lambda \in \mathbb{R} \right\} \tag{31}$$

(d) Note that:

$$D\left(x^2 - 4x + 2\right) = x(2) + (1 - x)(2x - 4) + 2x^2 - 8x + 4 = 0 \tag{32}$$

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(a)

$$\left(\begin{array}{ccc|c} 2-\lambda & 1 & 2 & 0 \\ 1 & 4-\lambda & -1 & 0 \\ 2 & -1 & 2-\lambda & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4-\lambda & -1 & 0 \\ 0 & 2\lambda-9 & 4-\lambda & 0 \\ 0 & \lambda^2-4\lambda-2 & 0 & 0 \end{array}\right) \tag{33}$$

Thus:

$$\operatorname{rank}(A) = \begin{cases} 2 & \text{if } \lambda = 4 \\ 3 & \text{else} \end{cases} \tag{34}$$

(b) The solution is either unique, corresponding to $\operatorname{rank}(A) = 3$ or infinite (a line), corresponding to $\operatorname{rank}(A) = 2$.

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$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \eta \\ 1 & 4 & 10 & \eta^2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta-1 \\ 0 & 0 & 0 & (\eta-1)(\eta-2) \end{array}\right) \tag{35}$$

Thus $\operatorname{rank}(A) = 2$ and the solution should be a line. A solution only exists if $\eta = 1$ or $\eta = 2$. If $\eta = 1$, then the solution is $(1, 0, 0)^\intercal + \lambda\mathbf{P}$. If $\eta = 2$, then the solution is $(0, 1, 0)^\intercal + \lambda\mathbf{P}$, where $\mathbf{P} = (2, -3, 1)^\intercal$.

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$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 10 \\ 5 & -1 & -4 & 17 \\ 1 & 5 & \alpha & \beta \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 5 & \alpha & \beta \\ 0 & -13 & -1-3\alpha & 10-3\beta \\ 0 & -26 & -4-5\alpha & 17-5\beta \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 5 & \alpha & \beta \\ 0 & -13 & -1-3\alpha & 10-3\beta \\ 0 & 0 & \alpha-2 & \beta-3 \end{array}\right) \tag{36}$$

If $\alpha \neq 2$, the solution is unique. If $\alpha = 2$, $\beta = 3$ then the solution is infinite (a line). If $\alpha = 2$ and $\beta \neq 3$, then there is no solution.

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(a) The essential reason is that addition is a linear operation.

(b) We only need to show that the dimension of the semi-magic squares space is five and that the matrices M_i are linearly independent. To prove the dimension, note that a 3×3 matrix can be represented by a vector of length nine:

$$\mathbf{v} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \equiv \begin{pmatrix} a & b & c & d & e & f & g & h & i \end{pmatrix}^\intercal \tag{37}$$

For a semi-magic square, the elements must satisfy:

$$a+b+c = d+e+f = g+h+i = a+d+g = b+e+h = c+f+i \tag{38}$$

This gives us the following system of equations:

$$A\mathbf{v} = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{39}$$

Performing Gaussian elimination on A gives us:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{40}$$

Thus, the nullity of A is four and $\operatorname{rank}(A) = 5$. It is also trivial to verify that the matrices M_i are linearly independent. Thus, M_i form a basis of the semi-magic squares space.

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(a) The equations can be rearranged into a system of n equations with the form:

$$x_i - \sum_{j \in L_i} \frac{x_j}{n_j} = 0 \tag{41}$$

Thus, the elements of the coefficient matrix A are:

$$A_{ij} = \delta_{ij} - \Delta_{jL_i} \frac{1}{n_j} \tag{42}$$

where:

$$\Delta_{jL_i} = \begin{cases} 1 \text{ if } j \in L_i \\ 1 \text{ if } j \notin L_i \end{cases} \tag{43}$$

Implicit in the definition is the fact that $i \notin L_i$ (a site does not have a link to itself).