**Classical Mechanics**

# Problem Set 2

Collisions in two dimensions & applications of the equation of motion

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# Collisions in 2-D

## 1 Elastic collisions in 2D

**(a)** Suppose $m_1$ has a velocity $\mathbf{v}$ in the centre of mass (CM) frame, such that $m_2$ has a velocity $-m_1\mathbf{v}/m_2$. Let $m_1$ have a velocity $\mathbf{v}_1$ after the collision and $m_2$ have a velocity $\mathbf{v}_2$. By conservation of momentum (COM):

$$\mathbf{0} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \tag{1}$$

By conservation of energy(COE):

$$\frac{1}{2}\left(m_1 v^2 + m_2\frac{m_1^2}{m_2^2}v^2\right) = \frac{1}{2}\left(1+\frac{m_1}{m_2}\right)m_1 v^2 = \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) \tag{2}$$

Substitute the first equation to the second:

$$\left(1+\frac{m_1}{m_2}\right)m_1 v^2 = \left(m_1 + \frac{m_1^2}{m_2}\right)v_1^2 \tag{3}$$

Thus $\mathbf{v}_1 = \pm\mathbf{v}$ and $\mathbf{v}_2 = \mp\frac{m_1}{m_2}\mathbf{v}$. It is seen that the magnitude of the velocities do not change.

**(b)** By COM:

$$m\mathbf{u}_1 = m\mathbf{v}_1 + m\mathbf{v}_2 \tag{4}$$

Squaring both sides:

$$u^2 = v_1^2 + 2\mathbf{v}_1\cdot\mathbf{v}_2 + v_2^2 \tag{5}$$

By COE:

$$\frac{1}{2}mu_1^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \tag{6}$$

Substituting yields:

$$v_1^2 + 2\mathbf{v}_1\cdot\mathbf{v}_2 + v_2^2 = v_1^2 + v_2^2 \tag{7}$$

Hence $\mathbf{v}_1\cdot\mathbf{v}_2 = 0$. This implies either $\mathbf{v}_1 = 0$, $\mathbf{v}_2 = 0$ or $\mathbf{v}_1 \perp \mathbf{v}_2$. But if $\mathbf{v}_2 = 0$ and $\mathbf{v}_1 = \mathbf{u}_1$, there is no change to the system and thus no collision. So $\mathbf{v}_2 = 0$ is ruled out.

Taking a dot product $\mathbf{u}_1\cdot\mathbf{v}_1$ gives:

$$\mathbf{u}_1 \cdot \mathbf{v}_1 = v_1^2 + \mathbf{v}_1 \cdot \mathbf{v}_2 = v_1^2 > 0 \tag{8}$$

But $\mathbf{u}_1 \cdot \mathbf{v}_1 = uv\cos\theta$ where $\theta$ is the scattering angle. Therefore, $\cos\theta > 0$ and $-90^\circ \leq \theta \leq 90^\circ$.

**(c)** In CM frame, $P_1$ has an initial velocity $\mathbf{u}_1/2$ and $P_2$ has $-\mathbf{u}_1/2$. COM gives

$$\mathbf{0} = m\mathbf{v}_1' + m\mathbf{v}_2' \tag{9}$$

COE gives:

$$m\frac{u_1^2}{4} = \frac{1}{2}mv_{1CM}'^2 + \frac{1}{2}mv_{2CM}'^2 \tag{10}$$

Substitution yields $v_1' = v_2' = u_1/2$ and $\mathbf{v}_1' = -\mathbf{v}_2'$. Thus:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (\mathbf{v}_1' + \mathbf{u}_1/2) \cdot (\mathbf{v}_2' + \mathbf{u}_1/2) = (\mathbf{v}_1' + \mathbf{u}_1/2) \cdot (-\mathbf{v}_1' + \mathbf{u}_1/2) = \frac{u_1^2}{4} - v_1'^2 = 0 \tag{11}$$

And:

$$\mathbf{u}_1 \cdot \mathbf{v}_1 = \mathbf{u}_1 \cdot (\mathbf{v}_1' + \mathbf{u}_1/2) = \frac{u_1^2}{2}(\cos\phi + 1) \geq 0 \tag{12}$$

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## 2 Collision of an alpha particle with a proton

Let the alpha particle have an initial velocity $\mathbf{v}$. Also let the alpha particle have a velocity $\mathbf{v}_1$ after the collision and the proton have $\mathbf{v}_2$. By COM:

$$4\mathbf{v} = 4\mathbf{v}_1 + \mathbf{v}_2 \tag{13}$$

or $4\mathbf{v} - 4\mathbf{v}_1 = \mathbf{v}_2$.

Squaring the equation:

$$16v^2 - 32\mathbf{v} \cdot \mathbf{v}_1 + 16v_1^2 = v_2^2 \tag{14}$$

By COE:

$$4v^2 = 4v_1^2 + v_2^2 \tag{15}$$

Substituting:

$$\begin{aligned} 4v^2 &= 4v_1^2 + 16v^2 - 32\mathbf{v}\cdot\mathbf{v}_1 + 16v_1^2 \\ 3v^2 + 5v_1^2 &= 8\mathbf{v}\cdot\mathbf{v}_1 = 8vv_1\cos\theta \\ \cos\theta &= \frac{3v^2+5v_1^2}{8vv_1} \end{aligned} \tag{16}$$

where $\theta$ is the deflection angle.

Since $v$ is a constant, we can differentiate the equation with respect to $v_1$:

$$\frac{\mathrm{d}}{\mathrm{d}v_1}\cos\theta = \frac{5v_1^2 - 3v^2}{8vv_1^2} \tag{17}$$

Therefore, $v_1 = \pm\sqrt{3/5}v$ for a maximum $\cos\theta$. The maximum of $\theta$ is thus:

$$\theta_{max} = \cos^{-1}\left(\frac{6v^2}{\pm 8\sqrt{3/5}v^2}\right) = \pm 14.5^\circ \tag{18}$$

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## 3 Inelastic collision in 2-D

In the CM frame, $2m$ has an initial velocity $\mathbf{u}/3$ while $m$ has an initial velocity $-2\mathbf{u}/3$. Let them have velocities $\mathbf{v}_{1CM}$ and $\mathbf{v}_{2CM}$ respectively after the collision. By COM:

$$\mathbf{0} = 2\mathbf{v}_{1CM} + \mathbf{v}_{2CM} \tag{19}$$

By the definition of coefficient of restitution $\alpha$:

$$\begin{aligned} |\mathbf{v}_{1CM} - \mathbf{v}_{2CM}| &= \alpha u \\ |3\mathbf{v}_{1CM}| = |3\mathbf{v}_{2CM}/2| &= \alpha u \end{aligned} \tag{20}$$

or $v_{1CM} = \alpha u/3$ and $v_{2CM} = 2\alpha u/3$.

Given $\mathbf{v}_2 = \mathbf{v}_{2CM} + 2\mathbf{u}/3$:

$$
\begin{aligned}
v_2^2 &= v_{2CM}^2 - \frac{4}{3}v_{2CM}u\cos\phi + \frac{4}{9}u^2 \\
&= \frac{4}{9}u^2\left(\alpha^2 + 1 - 3\alpha\cos\phi\right)
\end{aligned} \tag{21}
$$

where $\phi$ is the angle between $\mathbf{v}_{2CM}$ and $\mathbf{u}$

But $v_{2CM}\sin\phi = v_2\sin\theta$, so:

$$
\begin{aligned}
v_{2CM}^2(1-\cos^2\phi) &= v_2^2(1-\cos^2\theta) \\
\cos^2\phi &= 1 - \frac{9v_2^2}{4\alpha^2u^2}(1-\cos^2\theta)
\end{aligned} \tag{22}
$$

This expression can be substituted back to the equation for $v^2$, and after some nasty algebra, we get:

$$
v_2 = \frac{2u}{3}\left(\cos\theta \pm \sqrt{\alpha^2 - \sin^2\theta}\right) \tag{23}
$$

A geometric approach is much simpler.

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# Work, potential energy and conservation of energy

## 4 The Work Energy Theorem

**(a)** Assuming that $m$ is a constant, by Newton's second law:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = mv\frac{\mathrm{d}v}{\mathrm{d}x} = F(x) \tag{24}$$

This has now become a separable differential equation. Integrating yields

$$\begin{aligned}\int_{v_a}^{v_b} mv\mathrm{d}v\,\mathrm{d}v &= \int_a^b F(x)\,\mathrm{d}x \\ \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 &= W_{ab} \\ W_{ab} &= T(b) - T(a)\end{aligned} \tag{25}$$

This expression apples to all forces.

**(b)** The term 'minimum approach' only makes sense if $x_0 > 0$ and $u < 0$, i.e., the negative velocity points at the origin. In this case, using the work energy theorem:

$$\begin{aligned}\int_{x_0}^{x_{\min}} \frac{k}{x^2}\,\mathrm{d}x &= 0 - \frac{1}{2}mu^2 \\ k\left(\frac{1}{x_{\min}} - \frac{1}{x_0}\right) &= \frac{1}{2}mu^2 \\ x_{\min} &= \frac{1}{mu^2/2k + 1/x_0}\end{aligned} \tag{26}$$

## 5 Gravitational potential of two point masses

**(a)** The y-components of the forces cancel, so we only consider the x-components:

$$F(x) = -2G\frac{Mm}{x^2+a^2}\frac{x}{\sqrt{x^2+a^2}} = -2G\frac{Mmx}{(x^2+a^2)^{3/2}} \tag{27}$$

**(b)** Treating infinity as the reference point, by the definition of potential energy:

$$V(x) = \int_{\infty}^{x} 2G\frac{Mmx}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = GMm\left[-2\frac{1}{(x^2+a^2)^{1/2}}\right]_{\infty}^{x} = \frac{-2GMm}{\sqrt{x^2+a^2}} \tag{28}$$

This is a symmetric potential well centred at $x = 0$. The minimum point has an energy $-2GMm/a$. Thus, for a particle trapped at the centre, it needs at least $2GMm/a$ amount of kinetic energy to be able to escape the well.

**(c)** By COE:

$$\begin{aligned}\frac{-2GMm}{\sqrt{25a^2/16}} &= \frac{-2GMm}{\sqrt{a^2}} + \frac{1}{2}mv_{\max}^2 \\ v_{\max} &= \sqrt{\frac{4GM}{5a}}\end{aligned} \tag{29}$$

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## 6 SHM about stable equilibrium

**(a)** Differentiating $U(r)$ with respect to $r$:

$$\frac{\mathrm{d}U}{\mathrm{d}r} = 12\epsilon\left(\frac{r_0^6}{r^7} - \frac{r_0^{12}}{r^{13}}\right) \tag{30}$$

For the minimum point, $r_0^6/r^7 = r_0^{12}/r^{13}$ or $r = r_0$. At this point, $U(r_0) = -\epsilon$.

**(b)**

$$U(r-r_0) \approx U(r_0) + U'(r_0)(r-r_0) + \frac{1}{2}U''(r_0)(r-r_0)^2 = \epsilon\left[36\frac{(r-r_0)^2}{r_0^2} - 1\right] \tag{31}$$

**(c)** By the definition of a force due to a potential:

$$F(r-r_0) = -\frac{\mathrm{d}U}{\mathrm{d}(r-r_0)} \approx -72\epsilon\frac{r-r_0}{r_0^2} \tag{32}$$

which shows that for small $(r - r_0)$, the motion is simple harmonic.

The frequency $\omega$ is:

$$\omega = \frac{6}{r_0}\sqrt{\frac{2\epsilon}{m}} \tag{33}$$

The frequency is $\omega = \sqrt{2k/m}$, as this is the interaction between two particles:

$$\omega = \frac{12}{r_0}\sqrt{\frac{\epsilon}{m}} \tag{34}$$

**(d)**

$$k = \omega^2 m = (2\pi f)^2 \, m \approx 1.7 \times 10^6 \, \mathrm{Nm}^{-1} \tag{35}$$

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# Applications of equation of motion and resistive forces

## 7 Projectile in 2D

The ball only experiences a force in the negative z-axis. The motion is restricted to the plane spanned by $-g\hat{\mathbf{z}}$ and $\mathbf{V}$. If the motion were to leave this plane, the ball must experiences a force that has a component perpendicular to the plane, which violates the first condition.

**(a)**

$$T = \frac{2V\sin\theta}{g} \tag{36}$$

**(b)**

$$h = V\sin\theta\frac{T}{2} - \frac{1}{2}g\frac{T^2}{4} = \frac{V^2\sin^2\theta}{2g} \tag{37}$$

**(c)**

$$D = V\cos\theta T = \frac{V^2\sin 2\theta}{g} \tag{38}$$

**(d)**

$$\frac{1}{2}mV^2\cos^2\theta + mg\frac{V^2\sin^2\theta}{2g} = \frac{1}{2}mV^2 \tag{39}$$

The computed firing angles differ from the actual angle of 45°. This is due to possible air resistance.

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## 8

$$\begin{aligned} mgh &= \frac{1}{2}mv^2 + f_{\text{ave}}h \\ f_{\text{ave}} &= mg - \frac{1}{2h}mv^2 = 1.81\,\mathrm{N} \end{aligned} \tag{40}$$

The 'average force' here is a constant force that would have done the same negative work on the falling body as the actual air resistance, that is:

$$f_{\text{ave}}h = \int_0^h f_{\text{actual}}\,\mathrm{d}x \tag{41}$$

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## 9

### (a)

$$\begin{aligned} 6\pi a\eta v_f &= mg \\ v_f &= \frac{mg}{6\pi a\eta} = 102\,\mathrm{ms}^{-1} \end{aligned} \tag{42}$$

**(b)** By Newton's second law, at speed $v$:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - 6\pi a\eta v \tag{43}$$

or dividing by $m$:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{6\pi a\eta}{m}v \tag{44}$$

This is a separable differential equation:

$$\int_0^v \frac{1}{g - 6\pi a\eta v/m}\,\mathrm{d}v = \int_0^t \,\mathrm{d}t \tag{45}$$

Integrating this equation yields:

$$v(t) = \frac{g}{\lambda}(1 - \lambda t) \tag{46}$$

where $\lambda \equiv 6\pi a\eta/m$.

For $1 - \lambda t = 0.95$:

$$t = \frac{0.05}{\lambda} = 0.52\,\mathrm{s} \tag{47}$$

**(c)** Given an additional force, a quadratic equation results:

$$6\pi a\eta v_f + 0.87(av)^2 = mg \tag{48}$$

Solving the equation yields $v_f = 0.50\,\mathrm{ms}^{-1}$. The current model is more realistic.

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## 10

**(a)** Taking upwards as positive, for the upward motion:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{\alpha}{m}v^2 \tag{49}$$

For the downward motion:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g + \frac{\alpha}{m}v^2 \tag{50}$$

Focusing on the upwards equation, using the identity $\mathrm{d}v/\mathrm{d}t = v\mathrm{d}v/\mathrm{d}x$:

$$\begin{aligned} v\frac{\mathrm{d}v}{\mathrm{d}x} &= -g - \frac{\alpha}{m}v^2 \\ \int_{v_0}^{0} \frac{v}{-g - \alpha v^2/m}\,\mathrm{d}v &= \int_0^h \mathrm{d}x \\ h &= \frac{m}{2\alpha}\ln\left(1 + \frac{\alpha}{mg}v_0^2\right) = a\ln\left[1 + (v_0/v_l)^2\right] \end{aligned} \tag{51}$$

**(b)** Applying the same method on the downwards motion:

$$\begin{aligned} v\frac{\mathrm{d}v}{\mathrm{d}x} &= -g + \frac{\alpha}{m}v^2 \\ \int_{0}^{-v_r} \frac{v}{-g + \alpha v^2/m}\,\mathrm{d}v &= \int_h^0 \mathrm{d}x \\ v_r^2 &= \frac{mg}{\alpha}[1 - \exp(-2\alpha h/m)] = v_l^2[1 - \exp(-h/a)] \end{aligned} \tag{52}$$

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## Additional questions