Optics

Problem Set I

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(a) By law of refraction, we have the angle of refraction θ_1 after the first incidence:

$$\sin\theta_1 = \frac{n_1}{n_2}\sin\theta_i \tag{1}$$

which is just equal to θ_r.

By geometry, we see that the incident angle at point B is just θ_r, so that the angle of refraction θ_2 after the second incidence is:

$$\sin\theta_2 = \frac{n_2}{n_1}\sin\theta_r = \frac{n_2}{n_1}\sin\theta_1 = \sin\theta_i \tag{2}$$

This suggests that the two light rays are parallel to each other.

(b) Consider the length of OB:

$$OB = 2d\tan\theta_r \tag{3}$$

which means that the length of OA is:

$$OA = OB\cos(\pi/2 - \theta_i) = 2d\tan\theta_r\sin\theta_i = 2d\frac{n_2}{n_1}\tan\theta_r\sin\theta_r \tag{4}$$

which takes a time:

$$t_1 = \frac{OA}{c/n_1} = \frac{2d}{c}n_2\tan\theta_r\sin\theta_r \tag{5}$$

The light ray in the second medium travels a distance:

$$l = \frac{2d}{\cos\theta_r} \tag{6}$$

which takes a time:

$$t_2 = \frac{l}{c/n_2} = \frac{2d}{c}n_2\sec\theta_r \tag{7}$$

(c) The frequency of the light, which does not depend on the medium, is:

$$f = \frac{c/n_1}{\lambda} = k_0 \frac{c/n_1}{2\pi} \tag{8}$$

Taking the incident medium as vacuum, we set $n_1 = 1$. We have the angular frequency:

$$\omega = 2\pi f = ck_0 \tag{9}$$

The phase difference due to the path difference is:

$$\omega(t_2 - t_1) = \omega \frac{2d}{c} n_2 (\sec\theta_r - \tan\theta_r \sin\theta_r) = 2dk_0 n_2 \cos\theta_r \tag{10}$$

The actual phase difference has an additional term π due to the reflection at the second interface, so that the phase difference is:

$$\Delta\phi = 2dk_0 n_2 \cos\theta_r + \pi \tag{11}$$

(d) Confining θ_i to the range $[0, \pi/2]$, a larger θ_i corresponds to a larger θ_r and thus a lower $\cos\theta_r$. This means that the phase difference is smaller for larger θ_i.

(e) The superposed wave due to the two light rays has the form:

$$\begin{aligned} u &= u_0 \cos(\omega t + k_0 x) + u_0 \cos(\omega t + k_0 x + \Delta\phi) \\ &= 2u_0 \cos\left(\frac{\Delta\phi}{2}\right) \cos\left(\omega t + k_0 x + \frac{\Delta\phi}{2}\right) \\ &= 2u_0 \cos\left(\frac{\delta + \pi}{2}\right) \cos\left(\omega t + k_0 x + \frac{\delta + \pi}{2}\right) \\ &= 2u_0 \sin\left(\frac{\delta}{2}\right) \sin\left(\omega t + k_0 x + \frac{\delta}{2}\right) \end{aligned} \tag{12}$$

The (average) intensity of the superposed wave is:

$$I = 4u_0^2 \sin^2\left(\frac{\delta}{2}\right) \tag{13}$$

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(a) Following the standard derivation, it is easy to show that in a classical young's double-slit experiment, the path difference between the light rays from the two slits is:

$$\delta = \delta_1 - \delta_2 \approx d \sin\theta \approx \frac{dy}{D} \tag{14}$$

where y is the distance from the center of the screen to the point of interest.

The superposed wave has the form:

$$\begin{aligned} u &= u_0 e^{i(\omega t - k\delta_1)} + u_0 e^{i(\omega t - k\delta_2)} \\ &= u_0 e^{i\omega t}\left(e^{-ik\delta_1} + e^{-ik\delta_2}\right) \end{aligned} \tag{15}$$

The intensity of the superposed wave is:

$$\begin{aligned} I &= \frac{1}{2} u^* u \\ &= \frac{1}{2} u_0^2 \left(e^{ik\delta_1} + e^{ik\delta_2}\right)\left(e^{-ik\delta_1} + e^{-ik\delta_2}\right) \\ &= \frac{1}{2} u_0^2 \left(1 + e^{ik(\delta_1 - \delta_2)} + e^{ik(\delta_2 - \delta_1)} + 1\right) \\ &= u_0^2 \left(1 + 2\cos k\delta\right) \\ &= 2I_0 \left[1 + 2\cos\left(\frac{2\pi d}{\lambda}\sin\theta\right)\right] \end{aligned} \tag{16}$$

where we have defined the intensity at the center of the screen as $2I_0$.

(b) Given the function:

$$f(x) = \delta(x - d/2) + \delta(x + d/2) \tag{17}$$

Its Fourier transform is:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, \mathrm{d}x = \frac{1}{\sqrt{2\pi}}\left(e^{-ikd/2} + e^{ikd/2}\right) = \sqrt{\frac{2}{\pi}} \cos\left(\frac{kd}{2}\right) \tag{18}$$

This is functionally equivalent to (varying part of) the intensity of the superposed wave in the previous part, which suggests that Fraunhofer diffraction is equivalent to a Fourier transform of the aperture function.

(c) Given the normalised square wave function:

$$f(x) = \begin{cases} 1/w & \text{if } |x| < w/2 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Its Fourier transform is:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-w/2}^{w/2} \frac{1}{w} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{ikw} \left(e^{ikw/2} - e^{-ikw/2} \right) = \sqrt{\frac{2}{\pi}} \frac{\sin(kw/2)}{kw/2} \tag{20}$$

(d) The convolution theorem states that the Fourier transform of the convolution of f and g is the product of their Fourier transforms:

$$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g) \tag{21}$$

(e) The Fourier transform of the convolution between two delta functions and a normalised square wave function is:

$$\sqrt{\frac{2}{\pi}} \cos\left(\frac{kd}{2}\right) \cdot \sqrt{\frac{2}{\pi}} \frac{\sin(kw/2)}{kw/2} = \frac{2}{\pi} \cos\left(\frac{kd}{2}\right) \frac{\sin(kw/2)}{kw/2} \tag{22}$$

This suggests that the intensity pattern due to a pair of finite slits of width w has the form:

$$I = I_0 \cos\left(\frac{kd}{2}\right) \frac{\sin(kw/2)}{kw/2} + C \tag{23}$$

where C is some constant.

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(a) A triangular wave function of height $1/a$ and width $2a$ in the range $[0, 2a]$ is:

$$g(x) = \begin{cases} x/a^2 & \text{if } 0 \le x \le a \\ (2a - x)/a^2 & \text{if } a \le x \le 2a \end{cases} \tag{24}$$

The convolution of a square wave function of width a with itself is:

$$(f * f)(y) = \int f(x) f(y - x)\, \mathrm{d}x = \int f(x) f(x - y)\, \mathrm{d}x \tag{25}$$

since the square wave function is symmetric about the origin.

This integration is zero if $|x| > a/2$, so that the convolution is non-zero only in the range $[-a, a]$. In the range $y \in [0, a]$, the convolution is:

$$(f * f)(y) = \int_{-a/2+y}^{a/2} \frac{1}{a^2}\,\mathrm{d}x = \frac{a-y}{a^2} \tag{26}$$

In the range $y \in [-a, 0]$, the convolution is:

$$(f * f)(y) = \int_{-a/2}^{a/2+y} \frac{1}{a^2}\,\mathrm{d}x = \frac{a+y}{a^2} \tag{27}$$

Overall, the convolution is:

$$(f * f)(y) = \begin{cases} \frac{a-y}{a^2} & \text{if } 0 \le y \le a \\ \frac{a+y}{a^2} & \text{if } -a \le y \le 0 \end{cases} \tag{28}$$

Consider the change of variable $y \to a - y$:

$$(f * f)(y) = \begin{cases} \frac{y}{a^2} & \text{if } 0 \le y \le a \\ \frac{2a-y}{a^2} & \text{if } a \le y \le 2a \end{cases} \tag{29}$$

which is the same as the previous triangular wave function.

This suggests that a square wave function of width $2a$ at location a can be convolved with itself to produce a triangular wave function.

(b) Consider the square wave function at location a:

$$f(x) = \begin{cases} 1/2a & \text{if } 0 \le x \le 2a \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

Its Fourier transform is:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_0^{2a} \frac{1}{2a} e^{-ikx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \frac{1}{2ika}\left(1 - e^{-2ika}\right) \tag{31}$$

This means that the Fourier transform of the triangular wave function is:

$$G(k) = [F(k)]^2 = -\frac{1}{2\pi}\frac{1}{4k^2a^2}\left(1 - e^{-2ika}\right)^2 \tag{32}$$

which is independent of a up to second order in ka.

(d) The Fourier transform of the triangular wave function located at $-a$ can be obtained by considering the square wave function located at $-a$ with its Fourier transform:

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{2ika} \left(1 - e^{2ika}\right) \tag{33}$$

The Fourier transform of the left triangular wave is:

$$G(k) = [F(k)]^2 = -\frac{1}{2\pi} \frac{1}{4k^2a^2} \left(1 - e^{2ika}\right)^2 \tag{34}$$

This suggests that the Farunhofer diffraction pattern of a pair of triangular slits has the form:

$$\begin{aligned} u &= -u_0 \frac{1}{8\pi^2(ka\sin\theta)^2} \left(1 - e^{2ika\sin\theta}\right)^2 - u_0 \frac{1}{8\pi^2(ka\sin\theta)^2} \left(1 - e^{-2ika\sin\theta}\right)^2 \\ &= -u_0 \frac{1}{8\pi^2\delta^2} \left[\left(1 - e^{2i\delta}\right)^2 + \left(1 - e^{-2i\delta}\right)^2\right] \\ &= -u_0 \frac{1}{4\pi^2\delta^2} \left(1 - 2\cos 2\delta + \cos 4\delta\right) \end{aligned} \tag{35}$$

where $\delta = ka\sin\theta$.

The intensity pattern is:

$$I = u^* u = u_0^2 \frac{1}{16\pi^2\delta^4} \left(1 - 2\cos 2\delta + \cos 4\delta\right)^2 \approx u_0^2 \frac{1}{16\pi^2\delta^4} \left(16\delta^4 - \frac{224}{3}\delta^6 + \frac{656}{5}\delta^8\right) \tag{36}$$

Taking the derivative of the intensity pattern with respect to δ:

$$\frac{\partial I}{\partial \delta} \propto -\frac{446}{3}\delta + \frac{2624}{5}\delta^3 \tag{37}$$

which suggests that the secondary maximum is at $\delta \approx 0.532$.

(e) For zero intensity, we satisfy the equation:

$$1 - 2\cos 2\delta + \cos 4\delta = 2\cos^2 2\delta - 2\cos 2\delta = 0 \tag{38}$$

which suggests that the first minimum is at $\delta = \pi/4$.

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(a) Consider the two-dimensional square aperture:

$$T(x,y)=\begin{cases}1/a^2 & \text{if } |x|<a/2 \text{ and } |y|<a/2\\ 0 & \text{otherwise}\end{cases} \tag{39}$$

The Fourier transform of the aperture function is:

$$\begin{aligned}T(k_x,k_y)&=\frac{1}{2\pi}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\frac{1}{a^2}e^{-i(k_xx+k_yy)}\,\mathrm{d}x\,\mathrm{d}y\\ &=\frac{1}{2\pi}\frac{1}{a^2}\int_{-a/2}^{a/2}e^{-ik_xx}\,\mathrm{d}x\int_{-a/2}^{a/2}e^{-ik_yy}\,\mathrm{d}y\\ &=\frac{1}{2\pi}\frac{1}{a^2}\frac{1}{ik_x}\left(e^{ik_xa/2}-e^{-ik_xa/2}\right)\frac{1}{ik_y}\left(e^{ik_ya/2}-e^{-ik_ya/2}\right)\\ &=\frac{2}{\pi a^2k_xk_y}\sin\left(\frac{k_xa}{2}\right)\sin\left(\frac{k_ya}{2}\right)\end{aligned} \tag{40}$$

(b) The intensity pattern has the form:

$$I=\frac{I_0}{\delta_x^2\delta_y^2}\sin^2\left(\frac{\delta_x}{2}\right)\sin^2\left(\frac{\delta_y}{2}\right) \tag{41}$$

where $\delta_x=2\pi a\sin\theta_x/\lambda$ and $\delta_y=2\pi a\sin\theta_y/\lambda$.

The loci of the first minimum satisfy:

$$\frac{2\pi}{\lambda}a\sin\theta_x=\frac{2\pi}{\lambda}a\sin\theta_y=\pi \tag{42}$$

which suggests that the first minimum is at $\theta_x=\theta_y=\sin^{-1}(\lambda/2a)$.

The size of the central maximum is:

$$s_F=\pi\left(D\tan\theta_x\right)\left(D\tan\theta_y\right)=\pi D^2\frac{\lambda^2}{4a^2-\lambda^2} \tag{43}$$

(c) Based on geometric optics, the image of a small source due to the aperture is a square of side length:

$$l=\frac{u+D}{u}a \tag{44}$$

leading to a size of the image:

$$s_R=\frac{(u+D)^2}{u^2}a^2 \tag{45}$$

(d) Assuming that the actual size of the image follows the form:

$$s^2 = s_F^2 + s_R^2 = \pi^2 D^4 \frac{\lambda^4}{(4a^2 - \lambda^2)^2} + \frac{(u+D)^4}{u^4} a^4 \tag{46}$$

We can find the minimum of s^2 by taking the derivative with respect to a:

$$\frac{\partial s^2}{\partial a} = -16\pi^2 D^4 \frac{\lambda^4}{(4a^2 - \lambda^2)^3} a + 4\frac{(u+D)^4}{u^4} a^3 = 0 \tag{47}$$

This leads to the approximate solution in the limit $D \gg u$ and $a \gg \lambda$:

$$a \approx \left(\frac{\pi}{4}\right)^{1/4} \sqrt{\lambda u} \tag{48}$$

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