Mathematical Methods

Problem Sheet 3

Ordinary Differential Equations and Special Functions

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Ordinary Differential Equations and Special Functions

1 Green function

(a) Apparently for the given damped oscillator, we need $\omega = \sqrt{c^2 - 1}$.

(b) The Wronskian is:

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \omega e^{-2cx} \neq 0 \tag{1}$$

which means that the two solutions are linearly independent.

(c) The Green function is:

$$G(x,t) = \frac{e^{-c(x+t)}(\cos\omega t \sin\omega x - \cos\omega x \sin\omega t)}{\omega e^{-2ct}} = \frac{\sin\omega(x-t)}{\omega} e^{-c(x-t)} \tag{2}$$

(d) The general solution to the inhomogeneous equation is:

$$y(x) = \int_0^{2\pi/\omega} G(x,t) f(t) \,\mathrm{d}t \tag{3}$$

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2 Hermite polynomials again

(a) First consider the case $n \neq m$. We can assume $n > m$ without loss of generality so that:

$$\begin{aligned} \langle H_n, H_m \rangle &= \int_{\mathbb{R}} e^{-x^2} H_n(x) H_m(x) \,\mathrm{d}x \\ &= \int_{\mathbb{R}} D^{(n)} \left(e^{-x^2} \right) H_m(x) \,\mathrm{d}x \\ &= \left[D^{(n-1)} \left(e^{-x^2} \right) H_m(x) \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} D^{(n-1)} \left(e^{-x^2} \right) H_m'(x) \,\mathrm{d}x \\ &= (-1)^n \int_{\mathbb{R}} e^{-x^2} H_m^{(n)}(x) \,\mathrm{d}x \\ &= 0 \end{aligned} \tag{4}$$

as $H_m^{(n)}(x)$ is a polynomial of degree $m - n$.

Now consider the case $n = m$. We have:

$$\begin{aligned}
\langle H_n, H_n \rangle &= \int_{\mathbb{R}} e^{-x^2} H_n^2(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} D^{(n)}\left(e^{-x^2}\right) H_n(x)\,\mathrm{d}x \\
&= (-1)^n \int_{\mathbb{R}} e^{-x^2} H_n^{(n)}(x)\,\mathrm{d}x \\
&= (-1)^n k_n n! \int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x \\
&= (-1)^n k_n n! \sqrt{\pi}
\end{aligned} \tag{5}$$

where $k_n = 2^n$ is the coefficient of x^n in $H_n(x)$.

This means that H_n form an orthogonal set of functions with $\langle H_n, H_m \rangle = (-1)^n 2^n n! \sqrt{\pi} \delta_{nm}$.

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(b) Consider $H_{n+1}(x)$:

$$\begin{aligned}
H_{n+1}(x) &= -\left[(-1)^n e^{x^2} \frac{\mathrm{d}}{\mathrm{d}x}\left(D^{(n)} e^{-x^2}\right)\right] \\
&= -e^{x^2} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{H_n(x)}{e^{x^2}}\right] \\
&= -e^{x^2}\left[\frac{H_n'(x)}{e^{x^2}} - 2x\frac{H_n(x)}{e^{x^2}}\right] \\
&= -H_n'(x) + 2xH_n(x)
\end{aligned} \tag{6}$$

On the other hand, consider the inner product between $H_n'(x)$ and $H_{n-1}(x)$:

$$\begin{aligned}
\langle H_n', H_{n-1} \rangle &= \int_{\mathbb{R}} H_n'(x) D^{(n-1)} e^{-x^2}\,\mathrm{d}x \\
&= \left[H_n'(x) D^{(n-2)} e^{-x^2}\right]_{-\infty}^{\infty} - \int_{\mathbb{R}} H_n''(x) D^{(n-2)} e^{-x^2}\,\mathrm{d}x \\
&= (-1)^n \int_{\mathbb{R}} H_n^{(n)}(x) e^{-x^2}\,\mathrm{d}x \\
&= (-1)^n k_n n! \int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x \\
&= (-1)^n k_n n! \sqrt{\pi}
\end{aligned} \tag{7}$$

(c) We have:

$$H_n''(x) = D(2nH_{n-1}) = 4n(n-1)H_{n-2} \tag{8}$$

and:

$$H_{n-2} = \frac{2xH_{n-1} - H_n}{2(n-1)} \tag{9}$$

Combining the results leads to:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{10}$$

which shows that H_n satisfies the Hermite differential equation.

(d) Consider the power series Ansatz:

$$y(x) = \sum_{k=0}^{\infty} a_k x^k \tag{11}$$

Substitution into the Hermite differential equation leads to:

$$\sum_{k=0}^{\infty} a_k \left[(k+2)(k+1) - 2k + 2n\right] x^k = 0 \tag{12}$$

which leads to the recurrence relation:

$$a_{k+2} = \frac{2(k-n)}{(k+2)(k+1)} a_k \tag{13}$$

For $k = 1$, we need.

(e) Given $2x$ as a solution, consider the variation $\tilde{y}(x) = 2xu(x)$. Substituting into the Hermite differential equation leads to a new equation for $u(x)$:

$$xu''(x) + 2(1-x^2)u'(x) = 0 \tag{14}$$

or equivalently:

$$z' + \frac{2(1-x^2)}{x} z = 0 \tag{15}$$

where $z(x) = u'(x)$.

Solving for z leads to:

$$z(x) = Cx^{-2}e^{x^2} = C\sum_{k=0}^{\infty}\frac{x^{2(k-1)}}{k!} \tag{16}$$

Choosing $C = 1$ and integrating term by term leads to:

$$u(x) = \sum_{k=0}^{\infty}\frac{x^{2k-1}}{(2k-1)k!} \tag{17}$$

so that the second solution is:

$$y(x) = \sum_{k=0}^{\infty}\frac{2x^{2k}}{(2k-1)k!} \tag{18}$$

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3 Hermitian and unitary operators

(a) Assuming $T \circ S$ is Hermitian, we have:

$$(T \circ S)^\dagger = S^\dagger \circ T^\dagger = S \circ T \tag{19}$$

so that T and S commute.

The other direction is trivial given the above result.

(b) Any operator T can be written as:

$$T = \frac{T + T^\dagger}{2} + \frac{T - T^\dagger}{2} \tag{20}$$

where it is easy to verify that the first term is Hermitian and the second term is anti-Hermitian.

(c) Consider the following calculations:

$$\left\langle i\frac{\mathrm{d}f}{\mathrm{d}x}, g\right\rangle = -\int_a^b i\frac{\mathrm{d}f^*}{\mathrm{d}x}g\,\mathrm{d}x = -\left[if^*g\right]_a^b + \int_a^b if^*\frac{\mathrm{d}g}{\mathrm{d}x}\,\mathrm{d}x = \left\langle f, i\frac{\mathrm{d}g}{\mathrm{d}x}\right\rangle \tag{21}$$

$$\langle x^k f, g\rangle = \int_a^b x^k f^* g\,\mathrm{d}x = \langle f, x^k g\rangle \tag{22}$$

which shows that $i\mathrm{d}/\mathrm{d}x$ and x^k are Hermitian operators.

It is trivial to show that $\mathrm{d}/\mathrm{d}x$ is anti-Hermitian, so that:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + x^2\right)^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + x^2 \tag{23}$$

and:

$$\left(i\frac{\mathrm{d}^3}{\mathrm{d}x^3}\right)^\dagger = -i\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\dagger\right]^3 = -i\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 = i\frac{\mathrm{d}^3}{\mathrm{d}x^3} \tag{24}$$

so that the operator is Hermitian.

Finally:

$$\left\langle f, ix\frac{\mathrm{d}g}{\mathrm{d}x}\right\rangle = \left[if^*xg\right]_a^b - i\int_a^b \left(\frac{\mathrm{d}f^*}{\mathrm{d}x}g + gf^*\right)\mathrm{d}x = \left\langle i\left(\frac{\mathrm{d}}{\mathrm{d}x}+1\right)f, g\right\rangle \tag{25}$$

so $ix\mathrm{d}/\mathrm{d}x$ has the hermitian conjugate $i(\mathrm{d}/\mathrm{d}x + 1)$.

(d) For a unitary operator U and its eigenfunction f, we have:

$$\begin{aligned} \langle Uf, Uf\rangle &= \langle f, f\rangle \\ |\lambda|^2 \langle f, f\rangle &= \langle f, f\rangle \end{aligned} \tag{26}$$

which means that $|\lambda| = 1$.

(e) We have:

$$\langle f, T_a(g)\rangle = \int_a^b f^*(x)g(x-a)\,\mathrm{d}x = \int_a^b f^*(x+a)g(x)\,\mathrm{d}x = \langle T_{-a}(f), g\rangle \tag{27}$$

so that $T_a^\dagger = T_{-a}$.

But $T_{-a} \circ T_a = T_a \circ T_{-a} = \mathbb{I}$, so T_a is unitary.

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4 Quantum harmonic oscillator

(a) We have the relation:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} = \frac{m\omega}{\hbar}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{28}$$

so that:

$$H = -\frac{1}{2}h\omega\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}\hbar\omega x^2 \tag{29}$$

We therefore have the equation:

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2\right)\psi(x) = \epsilon\psi(x) \tag{30}$$

The change of variable makes the independent variable dimensionless. Physically, this is equivalent to measuring the position in units of $\sqrt{\hbar/m\omega}$ and scaling the energy by $\hbar\omega$ correspondingly.

(b) Consider the Ansatz $\psi(x) = e^{-x^2/2}f(x)$. Substituting into the equation leads to:

$$-e^{-x^2/2}\left(y'' - xy' - y - xy' + x^2y\right) + e^{-x^2/2}x^2y = 2\epsilon e^{-x^2/2}y \tag{31}$$

Simplifying:

$$y'' - 2xy' + (2\epsilon - 1)y = 0 \tag{32}$$

which is the Hermite differential equation with $n = \epsilon - 1/2$.

(c) Since the above equation is the Hermite differential equation, the solutions are:

$$y_n(x) = H_n(x) \tag{33}$$

where $n \in \mathbb{N}$ and $\epsilon = n + 1/2$.

Therefore, the solutions to the original equation are:

$$\psi_n(x) = e^{-x^2/2}H_n(x) = h_n(x) \tag{34}$$

where we have ignored the normalisation constant.

(d) We have:

$$
\begin{aligned}
N(f) &= \frac{1}{2}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(xf + \frac{\mathrm{d}f}{\mathrm{d}x}\right) \\
&= \frac{1}{2}\left(x^2 f + x\frac{\mathrm{d}f}{\mathrm{d}x} - f - x\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right) \\
&= \mathcal{H}(f) - \frac{1}{2}f
\end{aligned} \tag{35}
$$

(e) Consider the ladder operator on $h_n(x) \equiv |n\rangle$:

$$
a^\dagger |n\rangle = \frac{1}{\sqrt{2}}\left[x h_n(x) - h_n'(x)\right] \tag{36}
$$

On the other hand:

$$
h_n'(x) = \frac{1}{A_n}\left[-xe^{-x^2/2}H_n(x) + e^{-x^2/2}H_n'(x)\right] = -xh_n(x) + 2n\frac{A_{n-1}}{A_n}h_{n-1}(x) \tag{37}
$$

so that:

$$
\begin{aligned}
a^\dagger |n\rangle &= \frac{1}{\sqrt{2}}\left[2xh_n(x) - 2n\frac{A_{n-1}}{A_n}h_{n-1}(x)\right] \\
&= \frac{1}{\sqrt{2}}e^{-x^2/2}\left[2x\frac{1}{A_n}H_n(x) - 2n\frac{1}{A_n}H_{n-1}(x)\right] \\
&= \frac{A_{n+1}}{A_n}\frac{1}{\sqrt{2}}e^{-x^2/2}H_{n+1}(x) \\
&= \sqrt{n+1}\,|n+1\rangle
\end{aligned} \tag{38}
$$

and:

$$
a|n\rangle = \frac{1}{\sqrt{2}}\left[2n\frac{A_{n-1}}{A_n}h_{n-1}(x)\right] = \sqrt{n}\,|n-1\rangle \tag{39}
$$

Further:

$$
N|n\rangle = a^\dagger\left(\sqrt{n}\,|n-1\rangle\right) = n\,|n\rangle \tag{40}
$$

Hence:

$$
\mathcal{H}|n\rangle = \left(n + \frac{1}{2}\right)|n\rangle \tag{41}
$$

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5 Sturm-Liouville operators

(a) The Sturm Liouville operator on a function f can be written as:

$$\begin{aligned} T_{\mathrm{SL}} f &= \frac{1}{w(x)}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}f}{\mathrm{d}x}\right] + q(x) f\right\} \\ &= \frac{p}{w}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{p'}{w}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{q}{w} f \end{aligned} \tag{42}$$

so that for any second order differential operator $T = \alpha_2 \mathrm{d}^2/\mathrm{d}x^2 + \alpha_1 \mathrm{d}/\mathrm{d}x + \alpha_0$, we have:

$$\begin{aligned} p &= C \exp\left(\int \frac{\alpha_1}{\alpha_2}\,\mathrm{d}x\right) \\ w &= \frac{p}{\alpha_2} \\ q &= \alpha_0 w \end{aligned} \tag{43}$$

(b) For the Legendre differential equation, we have:

$$T_{\text{Legendre}} = (1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right] \tag{44}$$

For the Laguerre differential equation:

$$T_{\text{Laguerre}} = x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}}{\mathrm{d}x} = e^x \frac{\mathrm{d}}{\mathrm{d}x}\left(x e^{-x}\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{45}$$

For the Hermite differential equation:

$$T_{\text{Hermite}} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} = e^{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{46}$$

(c) The Legendre, Laguerre, Hermite differential equations are associated with the inner product spaces $\mathcal{L}^2([-1,1])$, $\mathcal{L}^2([0,\infty))$, $\mathcal{L}^2(\mathbb{R})$ respectively.

For any Sturm-Liouville operator T_{SL}, we have:

$$\begin{aligned}\langle f, T_{\mathrm{SL}} g\rangle &= \int_a^b f[D(pg') + qg]\,\mathrm{d}x \\ &= [fpg']_a^b - \int_a^b f'pg'\,\mathrm{d}x + \int_a^b fqg\,\mathrm{d}x \\ &= [fpg' - f'pg]_a^b + \int_a^b D(fp)g + fqg\,\mathrm{d}x \\ &= [fpg' - f'pg]_a^b + \langle T_{\mathrm{SL}} f, g\rangle\end{aligned} \tag{47}$$

which shows that T_{SL} is Hermitian as long as f and g satisfy the boundary conditions $fpg' - f'pg = 0$ at $x = a, b$.

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6 Bessel functions

(a) Consider the power series Ansatz $y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}$. Substituting into the differential equation leads to:

$$\sum_{k=2}^{\infty} \left\{\left[(k+\alpha)(k+\alpha-1) + (k+\alpha) - \nu^2\right] a_k + a_{k-2}\right\} x^{k+\alpha} + (1+\alpha)a_1 x^{1+\alpha} = 0 \tag{48}$$

This leads to the recurrence relation:

$$a_k = -\frac{1}{k^2 + 2k\alpha} a_{k-2} \tag{49}$$

and the condition $a_1 = 0$ so that all odd terms vanish.

(b) We can rewrite the recurrence relation as:

$$a_{2k} = \frac{-1}{4k(k+\alpha)} a_{2(k-1)} = \left(\frac{-1}{4}\right)^k \frac{\alpha!}{k!(k+\alpha)!} a_0 \tag{50}$$

With the given definition of a_0, we have:

$$a_{2k} = \left(\frac{-1}{2}\right)^{2k} \frac{1}{k!\Gamma(k+\alpha+1)} \left(\frac{1}{2}\right)^{\alpha} \tag{51}$$

so that with $\alpha = \nu$, we have the series solutions:

$$J_{\pm\nu} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k \pm \nu + 1)} \left(\frac{x}{2}\right)^{2k \pm \nu} \tag{52}$$

(c) We have:

$$\sqrt{\frac{x}{2}}J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+3/2)} \left(\frac{x}{2}\right)^{2k+1} \tag{53}$$

Consider $\Gamma(k+3/2)$:

$$\begin{aligned}\Gamma(k+3/2) &= (k+3/2)(k+1/2)(k-1/2)\cdots(3/2)(1/2)\Gamma(1/2) \\ &= \frac{(2k+1)(2k-1)\cdots(3)(1)}{2^{k+1}}\sqrt{\pi} \\ &= \frac{(2k+1)!}{2^{k+1}2k(2k-2)\cdots 2}\sqrt{\pi} \\ &= \frac{(2k+1)!}{2^{2k+1}k!}\sqrt{\pi}\end{aligned} \tag{54}$$

Thus:

$$\sqrt{\frac{x}{2}}J_{1/2}(x) = \sqrt{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{2^{2k+1}k!}{(2k+1)!}\left(\frac{x}{2}\right)^{2k+1} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} = \sqrt{\pi}\sin x \tag{55}$$

so that $J_{1/2}(x) = \sqrt{2/\pi x}\sin x$.

The proof for $J_{-1/2}(x)$ is similar.

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