Vectors & Matrices

Problem Set 1

Vectors, vector spaces and geometry

Xin, Wenkang

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Vectors, vector spaces and geometry

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(a) is obviously a linearly independent set. To check whether the remaining sets are linearly independent, compute their determinant:

$$\det\left\{\begin{pmatrix}0 & 1 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\right\} = -1 \neq 0 \tag{1}$$

so (b) is linearly independent.

$$\det\left\{\begin{pmatrix}1 & 2 & 1\\ 0 & 3 & 6\\ 1 & 1 & -1\end{pmatrix}\right\} = 0 \tag{2}$$

so (c) is linearly dependent. The subset $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

For (d), suppose $\mathbf{v}_3 = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$. We have the equation system:

$$\begin{aligned} \alpha + 2\beta &= -3 \\ 2\alpha + \beta &= 6 \\ \beta &= -4 \\ -3\alpha - 4\beta &= 1 \end{aligned} \tag{3}$$

which has a solution $(\alpha, \beta) = (5, -4)$.

Thus (d) is linearly dependent

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2

(a) The set is spanned by a single vector $(2, 2, 1)^\intercal$ so it is closed under addition and multiplication and has a zero vector. Thus it is a sub space.

(b) There is no zero vector in this set so it is not a sub space.

(c) For $f : R \to R$ that satisfies $f(x) = f(-x)$, define the sum of two functions as:

$$(f_1 + f_2)(x) \equiv f_1(x) + f_2(x) \tag{4}$$

$f(x) = 0$ apparently belongs to the subset. Note that:

$$(\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x) = \alpha f_1(-x) + \beta f_2(-x) = (\alpha f_1 + \beta f_2)(-x) \tag{5}$$

Thus the subset is a vector space.

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3

The dimension of $\mathbf{M}_{3\times 3}$ is nine.

(a) Let $A = \sum A_{ij}, B = \sum B_{ij} \in V$, where V is the subset of symmetric 3×3 matrices. Apparently the 3×3 zero matrix belongs to V. Consider a linear combination of A and B:

$$(\alpha A + \beta B)_{ij} = \alpha A_{ij} + \beta B_{ij} = \alpha A_{ji} + \beta B_{ji} = (\alpha A + \beta B)_{ji} \tag{6}$$

Thus $(\alpha A + \beta B) \in V$, and V thus forms a sub vector space. It has a dimension of four with its basis is given by:

$$\left\{ I_3, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} \tag{7}$$

It has a dimension of six with its basis is given by:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} \tag{8}$$

(b) Let $A = \sum A_{ij}, B = \sum B_{ij} \in W$, where W is the subset of antisymmetric 3×3 matrices. Apparently the 3×3 zero matrix belongs to W. Consider a linear combination of A and B:

$$(\alpha A + \beta B)_{ij} = \alpha A_{ij} + \beta B_{ij} = -\alpha A_{ji} - \beta B_{ji} = -(\alpha A + \beta B)_{ji} \tag{9}$$

Thus $(\alpha A + \beta B) \in W$, and W thus forms a sub vector space. It has a dimension of three with its basis is given by:

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\} \tag{10}$$

(c) For symmetric matrices, the dimension is $(n^2 - n)/2 + 1$. For antisymmetric matrices, the dimension is $(n^2 - n)/2$. Their basis are given in a similar fashion as above.

For symmetric matrices, the dimension is $(n^2 - n)/2 + n$.

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$$\det \left\{ \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \right\} = 3 \neq 0 \tag{11}$$

so the set is linearly independent and thus is a basis of $\mathbb{R}^3$. For a general $\mathbf{v} \in \mathbb{R}^3$:

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 \tag{12}$$

where a, b, c are the coordinates of $\mathbf{v}$ relative to the basis determined by the equation system:

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{13}$$

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(a) Let $\mathbf{u}, \mathbf{v} \in U \cap W$. Apparently $\mathbf{0} \in U \cap W$ as $\mathbf{0} \in U$ and $\mathbf{0} \in W$. As $U \cap W$ is a sub set of V, $\mathbf{u}, \mathbf{v} \in V$. Consider a linear combination of $\mathbf{u}$ and $\mathbf{v}$:

$$\alpha\mathbf{u} + \beta\mathbf{v} \tag{14}$$

As $\mathbf{u}, \mathbf{v} \in U$ and U is a sub vector space, $\alpha\mathbf{u} + \beta\mathbf{v} \in U$. A similar argument can be made for W. Hence:

$$(\alpha\mathbf{u} + \beta\mathbf{v}) \in U \cap W \tag{15}$$

and thus $U \cap W$ is a sub vector space.

(b) Let $\mathbf{a}, \mathbf{b} \in U + W$, such that $\mathbf{a} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{b} = \mathbf{u}_2 + \mathbf{w}_2$. Apparently $\mathbf{0} \in U + W$. Consider a linear combination of $\mathbf{a}$ and $\mathbf{b}$:

$$\alpha\mathbf{a} + \beta\mathbf{b} = (\alpha\mathbf{u}_1 + \beta\mathbf{u}_2) + (\alpha\mathbf{w}_1 + \beta\mathbf{w}_2) \tag{16}$$

The first term on the right-hand side belongs to U while the second term belongs to W due to them being vector spaces. Thus $(\alpha\mathbf{a} + \beta\mathbf{b}) \in U + W$ and $U + W$ is a sub vector space.

(c) This is a very loosely argued proof.

Let $U \cap W$ have a basis $\{\mathbf{v}_i\}$ of n elements, so that it has a dimension of n. Suppose that in order for $\{\mathbf{v}_i\}$ to be a basis of U, there needs to be an additional set $\{\mathbf{u}'_j\}$ of m vectors, where $0 \leq m \leq \dim U$. Also suppose that in order for $\{\mathbf{v}_i\}$ to be a basis of W, there needs to be an additional set $\{\mathbf{w}'_k\}$ of l vectors, where $0 \leq l \leq \dim W$. Then we can assert that the dimension of U is $n + m$ and the dimension of W is $n + l$.

An element in $U + W$ can be thus expressed as $\sum \alpha_i \mathbf{v}_i + \sum \beta_j \mathbf{u}'_j + \sum \gamma_k \mathbf{w}'_k$ where i, j, k take the values from 1 to n, l, m respectively. Since it has been assumed that $\{\mathbf{v}_i, \mathbf{u}'_j\}$ and $\{\mathbf{v}_i, \mathbf{w}'_k\}$ are bases, they must be linearly independent. Then it follows that:

$$\dim(U + W) = n + l + m = (n + m) + (n + l) - n = \dim U + \dim W - \dim(U \cap W) \tag{17}$$

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(a) This follows from the anticommutative property of the cross product.

(b)

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{a} \times \left(\sum_i \epsilon_{ijk} b_j c_k\right) \\ &= \epsilon_{lmi} a_m \epsilon_{ijk} b_j c_k \\ &= \epsilon_{ilm} \epsilon_{ijk} a_m b_j c_k \\ &= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}) a_m b_j c_k \\ &= a_m c_m b_l - a_m b_m c_l \\ &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \end{aligned} \tag{18}$$

(c)

$$\begin{aligned} (\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{c} \times \mathbf{d}) &= \epsilon_{ijk} a_j b_k \epsilon_{ilm} c_l d_m \\ &= (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}) a_j b_k c_l d_m \\ &= a_j b_k c_j d_k - a_j b_k c_k d_j \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \end{aligned} \tag{19}$$

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(a)

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) &= \mathbf{a} - \mathbf{b} \\ \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{a} \cdot (\mathbf{a} - \mathbf{b}) \\ 0 &= \mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) \\ \mathbf{a} &= \mathbf{b} \end{aligned} \tag{20}$$

provided that $\mathbf{a} \neq \mathbf{0}$.

(b)

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = 0 \tag{21}$$

(c) $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$ implies $(\mathbf{a} - \mathbf{b}) \times \mathbf{c} = 0$ and thus $(\mathbf{a} - \mathbf{b}) \perp \mathbf{c}$. Thus:

$$(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = \pm |\mathbf{a} - \mathbf{b}| \, |\mathbf{c}| \tag{22}$$

(d)

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{b} = [(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}]\mathbf{b} = \mathbf{b}[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})] \tag{23}$$

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(a) The magnitude of the cross product is twice the area of the triangle, so they are equal.

(b)

$$A = \frac{1}{2} |PQ \times PR| = \frac{1}{2} |(1, -4, -1)^\intercal \times (-2, -1, 1)^\intercal| = \frac{\sqrt{107}}{2} \text{ units}^2 \tag{24}$$

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(a) $\mathbf{a}$ and $\mathbf{b}$ form a plane with the normal vector $\mathbf{n} = \mathbf{a} \times \mathbf{b} = -7\mathbf{i} + 5\mathbf{j} + \mathbf{k}$, and the plane has the scalar equation:

$$\mathbf{r} \cdot \mathbf{n} = 0 \tag{25}$$

$\mathbf{c} \cdot \mathbf{n} = -7 + 10 - 3 = 0$, so $\mathbf{c}$ is on the plane, and thus they are coplanar.

(b) The normal vector is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. For a vector $\mathbf{r}$ to satisfy the given conditions, have $\mathbf{r} \cdot \mathbf{n} = 0$ and $\mathbf{r} \cdot \mathbf{a} = 0$. This leads to two linear equations:

$$\begin{aligned} x + 2y - 3z &= 0 \\ x + y + z &= 0 \end{aligned} \tag{26}$$

Solving this yields:

$$\mathbf{r} = \lambda \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} \tag{27}$$

for $\lambda \in \mathbb{R}$.

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(a)

$$\begin{aligned} \mathbf{v}_i \cdot \mathbf{v}_j' &= \frac{1}{V}[\mathbf{v}_i \cdot (\mathbf{v}_k \times \mathbf{v}_i)] = 0 \\ \mathbf{v}_i \cdot \mathbf{v}_i' &= \frac{1}{V}[\mathbf{v}_i \cdot (\mathbf{v}_j \times \mathbf{v}_k)] = 1 \end{aligned} \tag{28}$$

Let $\mathbf{w} = \sum_i \alpha_i \mathbf{v}_i$. Then:

$$\mathbf{w} \cdot \mathbf{v}_j' = \sum_i \alpha_i \mathbf{v}_i \cdot \mathbf{v}_j' = \sum_i \delta_{ij} \alpha_i = \alpha_j \tag{29}$$

(b)

$$\begin{aligned}
V' &= \frac{1}{V}(\mathbf{v}_2 \times \mathbf{v}_3) \cdot (\mathbf{v}_2' \times \mathbf{v}_3') \\
&= -\frac{1}{V^3}(\mathbf{v}_2 \times \mathbf{v}_3) \cdot [(\mathbf{v}_3 \times \mathbf{v}_1) \cdot (\mathbf{v}_2 \times \mathbf{v}_1)] \\
&= -\frac{1}{V^3}(\mathbf{v}_2 \times \mathbf{v}_3) \cdot [\mathbf{v}_1(\mathbf{v}_1 \cdot (\mathbf{v}_3 \times \mathbf{v}_2))] \\
&= \frac{[\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)]^2}{V^3} \\
&= \frac{1}{V}
\end{aligned} \tag{30}$$

(c)

$$\begin{aligned}
\frac{1}{V'}\mathbf{v}_i' \times \mathbf{v}_j' &= V\frac{1}{V^2}\left[(\mathbf{v}_j \times \mathbf{v}_k) \times (\mathbf{v}_k \times \mathbf{v}_i)\right] \\
&= -\frac{1}{V}\left[(\mathbf{v}_j \times \mathbf{v}_k) \times (\mathbf{v}_i \times \mathbf{v}_k)\right] \\
&= -\frac{1}{V}\mathbf{v}_k\left[\mathbf{v}_k \cdot (\mathbf{v}_j \times \mathbf{v}_i)\right] \\
&= \mathbf{v}_k
\end{aligned} \tag{31}$$

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The x-axis has the vector equation:

$$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{32}$$

where $\lambda \in \mathbb{R}$.

The distance is given by:

$$\begin{aligned}
d^2 &= (\mathbf{p} - \mathbf{r})^2 \\
&= \mathbf{p}^2 - 2\mathbf{p} \cdot \mathbf{r} + \mathbf{r}^2 \\
&= 29 - 4\lambda + \lambda^2 \\
&= (\lambda - 2)^2 + 25
\end{aligned} \tag{33}$$

Thus the minimum distance is 5 units.

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By inspection, the vector equation of the line is:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \tag{34}$$

where $\lambda \in \mathbb{R}$.

The distance is given by:

$$d^2 = \mathbf{r}^2 = 30 + 42\lambda + 29\lambda^2 \tag{35}$$

The minimum distance is thus $\sqrt{429/29}$ units.

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Following the usual procedure:

$$\begin{aligned} d^2 &= (\mathbf{r}_1 - \mathbf{r}_2)^2 \\ &= \mathbf{r}_1^2 - 2\mathbf{r}_1 \cdot r_2 + \mathbf{r}_2^2 \\ &= \mathbf{q}_1^2 + 2\lambda_1(\mathbf{q}_1 \cdot \mathbf{m}_1) + \mathbf{m}_1^2\lambda_1^2 - 2\left[\mathbf{q}_1\mathbf{q}_2 + \lambda_2(\mathbf{q}_1 \cdot \mathbf{m}_2) + \lambda_1(\mathbf{q}_2 \cdot \mathbf{m}_1) + \lambda_1\lambda_2\mathbf{m}_1\mathbf{m}_2\right] + \\ &\quad \mathbf{q}_2^2 + 2\lambda_1(\mathbf{q}_2 \cdot \mathbf{m}_2) + \mathbf{m}_2^2\lambda_2^2 \end{aligned} \tag{36}$$

To achieve minimum of $d^2(\lambda_1, \lambda_2)$:

$$\mathrm{d}d^2 = \frac{\partial d^2}{\partial \lambda_1}\mathrm{d}\lambda_1 + \frac{\partial d^2}{\partial \lambda_2}\mathrm{d}\lambda_2 = 0 \tag{37}$$

Each term yields an equation. Putting them together:

$$\begin{aligned} \mathbf{m}_1 \cdot (\mathbf{q}_1 + \mathbf{m}_1\lambda_1 - \mathbf{q}_2 - \mathbf{m}_2\lambda_2) = 0 \\ \mathbf{m}_2 \cdot (\mathbf{q}_2 + \mathbf{m}_2\lambda_2 - \mathbf{q}_1 - \mathbf{m}_1\lambda_1) = 0 \end{aligned} \tag{38}$$

This implies either $\mathbf{r}_1 = \mathbf{r}_2$, which is trivial, or $(\mathbf{r}_1 - \mathbf{r}_2)$ is perpendicular to both $\mathbf{m}_1$ and $\mathbf{m}_2$, i.e., $(\mathbf{r}_1 - \mathbf{r}_2) = k(\mathbf{m}_1 \times \mathbf{m}_2)$ for some non-zero $k \in \mathbb{R}$.

For the given two lines, $\mathbf{m}_1 \times \mathbf{m}_2 = (-3, 0, 3)^\intercal$. Then:

$$\begin{pmatrix} 4+2\lambda_1-\lambda_2 \\ 4+\lambda_1-2\lambda_2 \\ 2\lambda_1-\lambda_2 \end{pmatrix} = k \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \tag{39}$$

This can be transformed into the system of linear equations:

$$\begin{pmatrix} -2 & 1 & -3 \\ -1 & 2 & 0 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ k \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \tag{40}$$

the solution of which is $(\lambda_1, \lambda_2, k) = (0, 2, -2/3)$.

Thus the minimum distance is $2\sqrt{2}$ units.

Thus the minimum distance is $3\sqrt{2}$ units.

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For Cartesian description, note that:

$$\mathbf{n} = P_1P_2 \times P_1P_3 = \det\left\{ \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ -3 & 4 & 1 \end{pmatrix} \right\} = \begin{pmatrix} 11 \\ 5 \\ 13 \end{pmatrix} \tag{41}$$

Thus:

$$\mathbf{r} \cdot \mathbf{n} = OP_1 \cdot \mathbf{n} = 30 \tag{42}$$

or

$$11x + 5y + 13z = 30 \tag{43}$$

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For vector description, note that $\mathbf{b} - \mathbf{a} = (-4, 1, -3)^\intercal$ and $\mathbf{c} - \mathbf{a} = (-1, 0, 1)^\intercal$. Thus:

$$\mathbf{r} = \mathbf{a} + \lambda \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{44}$$

where $\lambda, \mu \in \mathbb{R}$.

For Cartesian description, note that:

$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \det \left\{ \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & -3 \\ -1 & 0 & 1 \end{pmatrix} \right\} = \begin{pmatrix} 1 \\ 7 \\ 1 \end{pmatrix} \tag{45}$$

Thus:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = 18 \tag{46}$$

or

$$x + 7y + z = 18 \tag{47}$$

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For the line:

$$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{48}$$

For the plane:

$$\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + \kappa \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{49}$$

For an intersection, we have the system of linear equations:

$$\begin{pmatrix} 1 & -2 & -1 \\ 1 & -4 & -1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \kappa \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \tag{50}$$

the solution of which is $(\lambda, \mu, \kappa) = (-1, -1, 2)$.

Thus the intersection is the point $(-1, -1, -1)^\intercal$.