Multiple Integrals & Vector Calculus

Problem Set 3

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\begin{aligned}
&\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \\
=&\sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) - \sin\theta\sin\phi(-r^2\sin^2\theta\sin\phi) \\
&+\cos\theta(r^2\cos^2\phi\cos\theta\sin\theta + r^2\sin^2\phi\cos\theta\sin\theta) \\
=&r^2\sin\theta
\end{aligned}
\tag{1}
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The volume of a sphere is given by $\{(r,\theta,\phi) \mid r\in[0,a],\theta\in[0,\pi/2],\phi\in[0,2\pi]\}$

$$M = \int_V kr^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \frac{2}{3}k\pi a^3 \tag{2}$$

By symmetry, $\bar{x} = \bar{y} = 0$. For $\bar{z}$:

$$\bar{z} = \frac{\int_V z\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi}{M} = \frac{\int_0^{\pi/2}\int_0^a r^3\cos\theta\sin\theta\,\mathrm{d}r\mathrm{d}\theta}{\int_0^{\pi/2}\int_0^a r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta} = \frac{3}{8}a \tag{3}$$

so that $(\bar{x},\bar{y},\bar{z}) = (0,0,\frac{3}{8}a)$.

For the principal moments of inertia:

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\begin{aligned}
I_{xx} &= \int_V kr^4(\sin^2\theta\cos^2\phi + \cos^2\theta)\sin\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \frac{4}{15}\pi ka^5 \\
I_{yy} &= I_{xx} = \frac{4}{15}\pi ka^5 \\
I_{zz} &= \int_V kr^4\sin^3\theta\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \frac{4}{15}\pi ka^5
\end{aligned}
\tag{4}
$$

For the product of inertia:

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\begin{aligned}
I_{xy} &= -\int_V kr^4\sin^3\theta\cos\phi\sin\phi\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = 0 \\
I_{yz} &= -\int_V kr^4\cos\theta\sin^2\theta\sin\phi\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = 0 \\
I_{xz} &= -\int_V kr^4\cos\theta\sin^2\theta\cos\phi\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = 0
\end{aligned}
\tag{5}
$$

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$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} \tag{6}$$

where $\mathbf{F} = (0.4/\sqrt{3})(-1, -1, 1)$.

We can divide the surface into three parts: a triangle bound by $y = x$, $y = 0$ and $x = 1$; a triangle formed by $(0, 0, 0)$, $(0, 1, 1)$ and $(1, 1, 0)$; a triangle formed by bound by $z = 1 - x$, $z = 1$ and $x = 1$. On the first triangle, the surface integral evaluates to $0.2/\sqrt{3}$. On the second triangle, which is formed by the surface $z = -x + y$

$$\int_A \frac{0.4}{\sqrt{3}} \,\mathrm{d}y\mathrm{d}y = \frac{0.2}{\sqrt{3}} \tag{7}$$

On the third triangle, the surface integral evaluates to $0.2/\sqrt{3}$. Therefore, the total volume of air flow per unit time is $0.2\sqrt{3}\mathrm{m}^3\mathrm{s}^{-1}$.

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(a)

$$\int_S \mathbf{r} \cdot \mathbf{n} \,\mathrm{d}S = \int_{x=1} x \,\mathrm{d}S + \int_{y=1} y \,\mathrm{d}S + \int_{z=1} z \,\mathrm{d}S = 3 \tag{8}$$

(b)

$$\int_S \mathbf{r} \cdot \mathbf{n} \,\mathrm{d}S = \int_S a \,\mathrm{d}S = 4\pi a^3 \tag{9}$$

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(a)

$$\int_S \mathbf{A} \cdot \mathbf{n} \,\mathrm{d}S = \int_V -1 \,\mathrm{d}V = -36 \tag{10}$$

Integrating on the surface:

$$\int_S \mathbf{A} \cdot \mathbf{n} \,\mathrm{d}S = \int_{2x+y=6} (2x + 2y) \,\mathrm{d}x\mathrm{d}z = 108 \tag{11}$$

(b)

$$\int_S \mathbf{A} \cdot \mathbf{n} \,\mathrm{d}S = -\int_{A_1} y^2 \,\mathrm{d}A - \int_{A_2} -2x \,\mathrm{d}A + \int_{A_3} \frac{2x^2 - 4xy - 2x + 12y}{2} \,\mathrm{d}A = 18 \tag{12}$$

where the regions are:

$$\begin{aligned} A_1 &= \{(x,y,z) \in \mathbb{R}^3 \mid x = 0, 0 \leq y \leq 6, 0 \leq z \leq 3 - y/2\} \\ A_2 &= \{(x,y,z) \in \mathbb{R}^3 \mid y = 0, 0 \leq x \leq 3, 0 \leq z \leq 1 - x\} \\ A_3 &= \{(x,y,z) \in \mathbb{R}^3 \mid z = 0, 0 \leq x \leq 3, 0 \leq y \leq 6 - 2x\} \end{aligned} \tag{13}$$

(c)

$$\int_S \mathbf{A} \cdot \mathbf{n} \, \mathrm{d}S = \int_V 1 \, \mathrm{d}V = 18\pi \tag{14}$$

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$$\int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S} = -\int_{y=0} x^2 \, \mathrm{d}S + \int_{z=1} yz \, \mathrm{d}S + \int_{x+y=1} (xy^2 + x^2) \, \mathrm{d}S = \frac{1}{4} \neq 0 \tag{15}$$

$$\int_V \nabla \cdot \mathbf{A} \, \mathrm{d}V = \int_V (y^2 + y) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{4} \tag{16}$$

This is a consequence of the divergence theorem.

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$$\begin{aligned} \nabla \cdot \mathbf{A} &= 3yz^2 + 6xy^2 - x^2y \\ \mathbf{A} \cdot \nabla\phi &= 3xyz^2 \times 6x + 2xy^3 \times (-z) - x^2yz \times (-y) = 18x^2yz^2 - 2xy^3z + 3x^2y^2z \\ \nabla \cdot (\phi\mathbf{A}) &= \phi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\phi = (3x^2 - yz)(3yz^2 + 6xy^2 - x^2y) + 18x^2yz^2 - 2xy^3z + 3x^2y^2z \\ \nabla \cdot (\nabla\phi) &= 6 \end{aligned} \tag{17}$$

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In spherical coordinates, the field is given by:

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \frac{1}{r} \hat{\phi} \tag{18}$$

so that the divergence is:

$$\nabla \cdot \mathbf{B} = \frac{\mu_0 I}{2\pi} \nabla \cdot (r^{-1}\hat{\phi}) = 0 \tag{19}$$

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