Ordinary Differential Equations

Problem Set 1

First-Order ODEs

Xin, Wenkang

May 10, 2023

Minimal Set

1.1

(i) Second order linear DE.

(ii) Third order non-linear DE.

(iii) First order non-linear DE.

1.2

All C appearing in the following solutions are arbitrary constants unless otherwise stated.

(a) Let $\partial\Phi/\partial y = Q(x,y)$, such that:

$$\Phi(x,y) = y^2 \cos x + 2xy \cos x + \phi(x) \tag{1}$$

for some function $\phi(x)$.

Differentiating with respect to x and comparing with $P(x,y)$:

$$\begin{gathered} -y^2 \sin x + 2y(\cos x - x \sin x) + \phi'(x) = P(x,y) \\ \phi'(x) = 6x \cos x - 3x^2 \sin x \\ \phi(x) = 3x^2 \cos x + C \end{gathered} \tag{2}$$

Therefore, the solution $\Phi(x,y)$ is given by:

$$\Phi(x,y) = y^2 \cos x + 2xy \cos x + 3x^2 \cos x + C \tag{3}$$

(b) Consider the correction factor $\Lambda(x) = x^{-3}$.

(c)

i

$$\begin{aligned}\int_0^y \frac{1}{e^y}\,\mathrm{d}y &= \int_0^x \frac{x}{1+x^2}\,\mathrm{d}x \\ 1-e^{-y} &= \frac{\ln|1+x^2|}{2} \\ y &= \ln\left(\frac{2}{2-\ln|1+x^2|}\right)\end{aligned} \tag{4}$$

ii

$$\begin{aligned}y' &= \frac{x(2y^2+1)}{y(x^2-1)} \\ \int \frac{y}{2y^2+1}\,\mathrm{d}y &= \int \frac{x}{x^2-1}\,\mathrm{d}x \\ \frac{1}{4}\ln\left|2y^2+1\right| &= \frac{1}{2}\ln\left|x^2-1\right| + C \\ \ln\left|2y^2+1\right| &= \ln\left[C\left|x^2-1\right|^2\right]\end{aligned} \tag{5}$$

$$y = \sqrt{\frac{C(x^2-1)^2-1}{2}} \tag{6}$$

(d) Let $z = 2x + y$, such that $\mathrm{d}z/\mathrm{d}x = 2 + \mathrm{d}y/\mathrm{d}x$ and thus:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} - 2 &= 2z^2 \\ \int \frac{1}{z^2+1}\,\mathrm{d}z &= \int 2\,\mathrm{d}x \\ \tan^{-1}(z) &= 2x + C \\ y &= \tan(2x+C) - 2x\end{aligned} \tag{7}$$

(e) As the RHS is homogeneous, dividing the numerator and denominator by x^2 yields:

$$y' = \frac{y/x + (y/x)^2}{2} \tag{8}$$

Let $z = y/x$, such that $\mathrm{d}z/\mathrm{d}x = y'/x - z/x$ and thus:

$$\begin{aligned} x\frac{\mathrm{d}z}{\mathrm{d}x} + z &= \frac{z+z^2}{2} \\ 2\int\left(\frac{1}{z-1} - \frac{1}{z}\right)\mathrm{d}z &= \int \frac{1}{x}\,\mathrm{d}x \\ 2\ln|z-1| - 2\ln|z| &= \ln|x| + C \end{aligned} \tag{9}$$

$$y = \begin{cases} \frac{x}{1-C\sqrt{|x|}}, & z > 1 \text{ or } z < 0 \\ \frac{x}{1+C\sqrt{|x|}}, & 0 < z < 1 \end{cases} \tag{10}$$

where $x \neq 0$.

(f) Let $p = x - 3/2$ and $q = y + 1/2$, such that $y' = \mathrm{d}q/\mathrm{d}p$ and:

$$\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{p+q}{p-q} = \frac{1+q/p}{1-q/p} \tag{11}$$

Let $z = q/p$ so that following the standard procedure:

$$\begin{aligned} \int \frac{1}{\frac{1+z}{1-z} - z}\,\mathrm{d}z &= \ln|p| + C \\ \int \frac{1-z}{z^2+1}\,\mathrm{d}z &= \ln|p| + C \\ \tan^{-1}(z) - \frac{1}{2}\ln\left|z^2+1\right| &= \ln|p| + C \\ z &= \tan\left[\ln\left(C\,|p|\,\sqrt{|z^2+1|}\right)\right] \end{aligned} \tag{12}$$

where $z \neq 1$.

This equation gives an implicit relationship between x and y.

(g)

i Consider the following integrating factor:

$$\Lambda(x) = \exp\left(\int \frac{1}{x}\,\mathrm{d}x\right) = x \tag{13}$$

Multiplying both sides by $\Lambda(x)$:

$$\begin{aligned} xy' + y &= 3x \\ \frac{\mathrm{d}}{\mathrm{d}x}(yx) &= 3x \\ y &= \frac{3x}{2} + \frac{C}{x} \end{aligned} \tag{14}$$

where $x \neq 0$.

ii Consider the following integrating factor:

$$\Lambda(x) = \exp\left(\int \cos x \,\mathrm{d}x\right) = e^{\sin x} \tag{15}$$

Multiplying both sides by $\Lambda(x)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\sin x} y\right) &= 2e^{\sin x} \sin x \cos x \\ e^{\sin x} y &= 2\left(e^{\sin x} \sin x - \int e^{\sin x} \cos x \,\mathrm{d}x\right) \\ e^{\sin x} y &= 2\left(e^{\sin x} \sin x - e^{\sin x} + C\right) \\ y &= 2\sin x - 2 + Ce^{-\sin x} \end{aligned} \tag{16}$$

(h) For this Bernoulli's equation, divide the equation by $y^{2/3}$:

$$\begin{aligned} y' y^{-2/3} + y^{1/3} &= x \\ 3\left(\frac{\mathrm{d}y^{1/3}}{\mathrm{d}x}\right) + y^{1/3} &= x \end{aligned} \tag{17}$$

Let $z = y^{1/3}$ so that:

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} + \frac{z}{3} &= \frac{x}{3} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x/3} z\right) &= e^{x/3}\frac{x}{3} \\ e^{x/3} z &= xe^{x/3} - 3e^{x/3} + C \\ y &= \left(x - 3 + Ce^{-x/3}\right)^3 \end{aligned} \tag{18}$$

The trivial solution is $y = 0$.

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1.3

All C appearing in the following solutions are arbitrary constants unless otherwise stated.

(i) Let $\partial\Phi/\partial y = Q(x, y)$, such that:

$$\Phi(x, y) = y \sin x + \phi(x) \tag{19}$$

for some function $\phi(x)$.

Differentiating with respect to x and comparing with $P(x, y)$:

$$\begin{aligned} y \cos x + \phi'(x) &= P(x, y) \\ \phi'(x) &= -x \\ \phi(x) &= -\frac{x^2}{2} + C \end{aligned} \tag{20}$$

Therefore, the solution $\Phi(x, y)$ is given by:

$$\Phi(x, y) = y \sin x - \frac{x^2}{2} + C \tag{21}$$

where $x \neq 0$.

(ii)

$$\begin{aligned} \int \frac{1}{5y - 8} \, \mathrm{d}y &= \int \frac{1}{3x + x^2} \, \mathrm{d}x \\ \frac{1}{5} \ln |5y - 8| &= \frac{1}{3} \ln \left| \frac{x}{x+3} \right| + C \\ y &= \frac{1}{5} \left(\pm C \left| \frac{x}{x+3} \right|^{5/3} + 8 \right) \end{aligned} \tag{22}$$

where $x \neq 0, -3$.

The trivial solution is $y = 8/5$.

(iii)

$$y' = 3 - \frac{2x}{y} = \frac{3y - 2x}{y} = \frac{3(y/x) - 2}{y/x} \tag{23}$$

Let $z = y/x$ and following the standard procedure:

$$\begin{aligned}
\int \frac{z}{3z-2-z^2}\,\mathrm{d}z &= \ln x + C \\
\frac{1}{2}\ln\left(z^2-3z+2\right) - \frac{3}{2}\left[\ln(z-2) - \ln(z-1)\right] &= \ln\frac{C}{|x|} \\
\frac{y/x-1}{y/x-2} &= \frac{C}{|x|}
\end{aligned} \tag{24}$$

where $x \neq 0$.

This equation gives an implicit relationship between x and y.

The trivial solution is $y = x$.

(iv)

$$\begin{aligned}
y' + \frac{y}{x} &= 2x^{3/2}y^{1/2} \\
y^{-1/2}y' + \frac{y^{1/2}}{x} &= 2x^{3/2} \\
2\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{1/2}\right) + \frac{y^{1/2}}{x} &= 2x^{3/2}
\end{aligned} \tag{25}$$

Let $z = y^{1/2}$ and consider the integrating factor $\Lambda(x) = \sqrt{x}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}z\right) &= 2x^2 \\
\sqrt{x}z &= \frac{2}{3}x^3 + C \\
y &= \left(\frac{2}{3}x^{5/2} + Cx^{-1/2}\right)^2
\end{aligned} \tag{26}$$

where $x \neq 0$.

(v)

$$\begin{aligned}
2y' &= \frac{y}{x} + \frac{y^3}{x^3} \\
y' &= \frac{(y/x)^3 + (y/x)}{2}
\end{aligned} \tag{27}$$

Let $z = y/x$ and following the standard procedure:

$$
\begin{aligned}
\int \frac{2}{z^3 - z} \, \mathrm{d}z &= \ln|x| + C \\
\frac{1}{2} \ln\left|z^2 - 1\right| - \ln|z| &= \ln C\,|x| \\
\frac{\sqrt{y^2 - x^2}}{y} &= \pm Cx
\end{aligned}
\tag{28}
$$

where $x \neq 0$.

This equation gives an implicit relationship between x and y. The trivial solution is $y = x$.

(vi)

$$
\begin{aligned}
xyy' - y^2 &= (x + y)^2 e^{-y/x} \\
y' &= \frac{y}{x} \left(\frac{x}{y} + 2 + \frac{y}{x} \right) e^{-y/x}
\end{aligned}
\tag{29}
$$

Let $z = y/x$ and following the standard procedure:

$$
\begin{aligned}
\int \frac{e^z}{z + 2 + 1/z} \, \mathrm{d}z &= \ln x + C \\
\frac{e^{y/x}}{y/x + 1} &= \ln C\,|x|
\end{aligned}
\tag{30}
$$

This equation gives an implicit relationship between x and y.

(vii)

$$
\begin{aligned}
x(x - 1)y' + y &= x(x - 1)^2 \\
y' + \frac{y}{x(x - 1)} &= x - 1
\end{aligned}
\tag{31}
$$

Consider the integrating factor $\Lambda(x) = (x - 1)/x$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x - 1}{x} y \right) &= \frac{(x - 1)^2}{x} \\
\frac{x - 1}{x} y &= \frac{x^2}{2} - 2x + \ln|x| + C \\
y &= \frac{x}{x - 1} \left(\frac{x^2}{2} - 2x + \ln|x| + C \right)
\end{aligned}
\tag{32}
$$

where $x \neq 0, 1$.

(viii)

$$\begin{aligned} 2xy' - y &= x^2 \\ y' - \frac{y}{2x} &= \frac{x}{2} \end{aligned} \tag{33}$$

Consider the integrating factor $\Lambda(x) = 1/\sqrt{x}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x}}\right) &= \frac{\sqrt{x}}{2} \\ \frac{y}{\sqrt{x}} &= \frac{x^{3/2}}{3} + C \\ y &= \frac{x^2}{3} + C\sqrt{x} \end{aligned} \tag{34}$$

where $x \neq 0$.

(ix) Let $z = y + x$ so that:

$$\begin{aligned} z' &= \cos z + 1 \\ \int_{\pi/2}^{z} \frac{1}{\cos z + 1}\,\mathrm{d}z &= \int_0^x 1\,\mathrm{d}x \\ \tan z/2 - 1 &= x \\ y &= 2\tan^{-1}(x+1) - x \end{aligned} \tag{35}$$

(x) Let $z = x - y$, such that $\mathrm{d}z/\mathrm{d}x = 1 - \mathrm{d}y/\mathrm{d}x$:

$$\begin{aligned} 1 - \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{z}{z+1} \\ \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1}{z+1} \\ \frac{z^2}{2} + z &= x + C \\ x^2 - 2xy + y^2 - 2y + C &= 0 \\ y &= x + 1 \pm \sqrt{2x + 1 + C} \end{aligned} \tag{36}$$

(xi)

$$y' + \frac{y}{\tan x} = \cos 2x \tag{37}$$

Consider the integrating factor $\Lambda(x) = \sin x$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(y \sin x) &= \sin x \left(2\cos^2 x - 1\right) \\ y \sin x &= -\frac{2}{3}\cos^3 x + \cos x + C \\ y &= \frac{1}{\sin x}\left(\cos x - \frac{2}{3}\cos^3 x + C\right) \end{aligned} \tag{38}$$

Given the boundary conditions $y = 1/2$ at $x = \pi/2$, have:

$$y = \frac{1}{\sin x}\left(\cos x - \frac{2}{3}\cos^3 x + \frac{1}{2}\right) \tag{39}$$

where $x \neq 0$.

(xii) First consider the case where $n = 0$ and $y' + ky = \sin x$. Consider the integrating factor $\Lambda(x) = e^{kx}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{kx}\right) &= e^{kx}\sin x \\ ye^{kx} &= \int e^{kx}\sin x \,\mathrm{d}x \end{aligned} \tag{40}$$

Let $I(x) = \int e^{kx}\sin x\,\mathrm{d}x$. Integrating by parts:

$$\begin{aligned} I(x) &= \frac{1}{k}e^{kx}\sin x - \int \frac{1}{k}e^{kx}\cos x\,\mathrm{d}x \\ &= \frac{1}{k}e^{kx}\sin x - \frac{1}{k^2}e^{kx}\cos x - \int \frac{1}{k^2}e^{kx}\sin x\,\mathrm{d}x \\ &= \frac{1}{k}e^{kx}\sin x - \frac{1}{k^2}e^{kx}\cos x - I(x) \\ I(x) &= \frac{1}{k^2+1}e^{kx}\left(k\sin x - \cos x\right) + C \end{aligned} \tag{41}$$

Therefore:

$$y = \frac{1}{k^2+1}\left(k\sin x - \cos x\right) + Ce^{-kx} \tag{42}$$

Next consider the case where $n > 1$ and the equation is now a Bernoulli's equation. Dividing by y^n yields:

$$\begin{aligned} y^{-n}y' + ky^{1-n} &= \sin x \\ \frac{1}{1-n}\frac{\mathrm{d}}{\mathrm{d}x}\left(y^{1-n}\right) + ky^{1-n} &= \sin x \end{aligned} \tag{43}$$

Let $z = y^{1-n}$ and consider the integrating factor $\Lambda(x) = e^{(1-n)kx}$:

$$\begin{aligned} z' + (1-n)kz &= (1-n)\sin x \\ ze^{(1-n)kx} &= \int (1-n)e^{(1-n)kx}\sin x \,\mathrm{d}x \\ &= \frac{1-n}{(1-n)^2k^2+1}e^{(1-n)kx}\left[(1-n)k\sin x - \cos x\right] + C \\ y &= \left\{\frac{1-n}{(1-n)^2k^2+1}\left[(1-n)k\sin x - \cos x\right] + Ce^{(n-1)kx}\right\}^{1/(1-n)} \end{aligned} \tag{44}$$

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Supplementary Questions

Left for revision.

Extracurricular Questions

1.8 Integral curves and orthogonal curves

(a) Taking the differential:

$$
\begin{aligned}
\mathrm{d}y - \frac{1}{x}\sec^2[\ln(Cx)]\mathrm{d}x - \frac{1}{C}\sec^2[\ln(Cx)]\mathrm{d}C = 0 \\
\mathrm{d}y - \sec^2[\ln(Cx)]\left(\frac{1}{x}\mathrm{d}x + \frac{1}{C}\frac{\partial C}{\partial x}\mathrm{d}x + \frac{1}{C}\frac{\partial C}{\partial y}\mathrm{d}y\right) \\
\left(1 - \frac{\sec^2[\ln(Cx)]}{C}\frac{\partial C}{\partial y}\right)\mathrm{d}y - \sec^2[\ln(Cx)]\left(\frac{1}{x} + \frac{1}{C}\frac{\partial C}{\partial x}\right)\mathrm{d}x = 0
\end{aligned}
\tag{45}
$$

This ODE, with an arbitrary function $C(x, y)$, has the integral curves of the specified form.

(b) Let $F(x, y, y') = 0$ have the integral curves:

$$
f(x, y, C) = 0 \tag{46}
$$

where C is an arbitrary constant.

Then it must be the case that the equation:

$$
\left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial C}\frac{\partial C}{\partial y}\right)\mathrm{d}y + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial C}\frac{\partial C}{\partial x}\right)\mathrm{d}x = 0 \tag{47}
$$

is equivalent to $F(x, y, y') = 0$.

Making y' the subject

$$
y' = -\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial C}\frac{\partial C}{\partial x}\right) / \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial C}\frac{\partial C}{\partial y}\right) \tag{48}
$$

Then the curves $g(x, y, C) = 0$ orthogonal to $f(x, y, C) = 0$ must satisfy the ODE:

$$
y' = \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial C}\frac{\partial C}{\partial y}\right) / \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial C}\frac{\partial C}{\partial x}\right) \tag{49}
$$

But this is equivalent to $F(x, y, -1/y') = 0$. Thus, the integral curves of $F(x, y, -1/y') = 0$ are orthogonal to those of $F(x, y, y') = 0$.

(c)

$$
x + y'(y + 1) = 0 \tag{50}
$$

Then the orthogonal curves satisfy the ODE:

$$y' = \frac{y+1}{x} \tag{51}$$

The solution to the ODE is:

$$y = Dx - 1 \tag{52}$$

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1.9 Riccati equations

(a) Apparently, $y_0(x) = e^x$ is a particular solution to the ODE. Let $y(x) = z(x) + y_0(x)$:

$$\begin{aligned} y' &= z' + y_0' = z^2 + 2zy_0 + y_0^2 - 2e^x(z + y_0) + e^{2x} + e^x \\ z' &= z^2 \\ z &= -\frac{1}{x} + C \\ y &= e^x - \frac{1}{x} + C \end{aligned} \tag{53}$$

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