Multiple Integrals & Vector Calculus

Problem Set 2

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$$\nabla(\ln r) = \frac{1}{r}\hat{r} \tag{1}$$

$$\nabla(1/r) = -\frac{1}{r^2}\hat{r} \tag{2}$$

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At $(1, 0, -2)$:

$$\nabla(F+G) = \nabla F + \nabla G = \begin{pmatrix} 2xz - ye^{y/x}/x^2 \\ e^{y/x}/x \\ x^2 \end{pmatrix} + \begin{pmatrix} -y^2 \\ 2z^2 - 2yx \\ 4zy \end{pmatrix} = \begin{pmatrix} -4 \\ 9 \\ 1 \end{pmatrix} \tag{3}$$

and

$$\nabla(FG) = f\nabla G + g\nabla F = \begin{pmatrix} 0 \\ -8 \\ 0 \end{pmatrix} \tag{4}$$

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(a) By chain rule, the total derivatives of x can be expressed as

$$\begin{aligned} \mathrm{d}x &= \frac{\partial x}{\partial u}\mathrm{d}u + \frac{\partial x}{\partial v}\mathrm{d}v \\ &= \frac{\partial x}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + \frac{\partial x}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right) \\ &= \left(\frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y \end{aligned} \tag{5}$$

Comparing the coefficients of $\mathrm{d}x$ and $\mathrm{d}y$, we have

$$\frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} = 1 \tag{6}$$

and

$$\frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{7}$$

Generalising this result to the elements of the product AB, we have

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{8}$$

(b) For polar coordinates, the matrices are given by:

$$\begin{aligned} A &= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ B &= \begin{pmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/(x^2+y^2) & -x/(x^2+y^2) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix} \end{aligned} \tag{9}$$

so that $AB = I$ as expected.

(c) Since $AB = I$, $B = A^{-1}$. But A is just the Jacobian matrix of the transformation from Cartesian to polar coordinates. Therefore:

$$\det\frac{\partial(u,v)}{\partial(x,y)} = \det(B) = \det\left(A^{-1}\right) = 1/\det\frac{\partial(x,y)}{\partial(u,v)} \tag{10}$$

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(a) We have $u = x + y$ and $v = x - y$ and $u \in [-1, 1]$, $v \in [-3, 1]$. The Jacobian is:

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = 2 \tag{11}$$

The integral evaluates to:

$$\begin{aligned} &\int_{-1}^{1}\int_{-1}^{1}\left[\frac{(u+v)^2}{4} + \frac{(u-v)^2}{4}\right] 2\,\mathrm{d}u\mathrm{d}v \\ =&\int_{-1}^{1}\int_{-1}^{1}(u^2+v^2)\,\mathrm{d}u\mathrm{d}v \\ =&\frac{8}{3} \end{aligned} \tag{12}$$

We have $u = x + y$ and $v = x - y$ and $u \in [-2, 0]$, $v \in [0, 2]$.
The integral evaluates to:

$$\begin{aligned} &\int_0^2 \int_{-2}^0 \left[\frac{(u+v)^2}{4} + \frac{(u-v)^2}{4} \right] 2 \, \mathrm{d}u \mathrm{d}v \\ =&\int_0^2 \int_{-2}^0 (u^2 + v^2) \, \mathrm{d}u \mathrm{d}v \\ =&\frac{8}{3} \end{aligned} \tag{13}$$

(b)

$$\frac{\partial(u,v)}{\partial(x,y)} = y\frac{1}{x} + x\frac{y}{x^2} = \frac{2y}{x} = 2v \tag{14}$$

The integral evaluates to:

$$\int_{1/2}^2 \int_0^1 e^{-u} 2v \, \mathrm{d}u \mathrm{d}v = \frac{15}{4}(1 - e^{-1}) \tag{15}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v} \tag{16}$$

The integral evaluates to:

$$\int_{1/2}^2 \int_0^1 e^{-u} \frac{1}{2v} \, \mathrm{d}u \mathrm{d}v = \ln 2(1 - e^{-1}) \tag{17}$$

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(a) On the straight line from $(0,0,0)$ to $(1,2,0)$, $y = 2x$ and $z = 0$. We also have $\mathrm{d}y = 2\mathrm{d}x$. The path integral is:

$$\int \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^1 4x - 16x^4 \, \mathrm{d}x + \int_0^1 (-32x^4 - 12x^2) 2 \, \mathrm{d}x = -22 \tag{18}$$

(b) Following the suggested path, the integral evaluates to:

$$\int \mathbf{A} \cdot \mathrm{d}\mathbf{l} = 4 + \int_0^1 4x \, \mathrm{d}x + \int_0^2 (-4y^3 - 3y^2) \, \mathrm{d}y - 4 = -22 \tag{19}$$

Let $V = 4z + \phi(x, y)$ for some function $\phi(x, y)$. Thee for $\mathbf{A} = \nabla V$, we need $\partial\phi/\partial x = 4x - y^4$, implying:

$$\phi(x, y) = 2x^2 - y^4 x + \psi(y) \tag{20}$$

for some function $\psi(y)$.

Comparing with $\partial\phi/\partial y = -4xy^3 - 3y^2$ gives $\psi(y) = -y^3 + C$. Therefore, a possible choice for V is:

$$V = 4z + 2x^2 - y^4 x - y^3 + C \tag{21}$$

This proves that $\mathbf{A}$ is conservative.

Alternatively, compute the curl of $\mathbf{A}$:

$$\nabla \times \mathbf{A} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x - y^4 & -4xy^3 - 3y^2 & 4 \end{pmatrix} = \mathbf{0} \tag{22}$$

This proves that $\mathbf{A}$ is conservative.

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(a) On the suggested parametric curve, the infinitesimal element of the path is:

$$\mathrm{d}\mathbf{l} = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \mathrm{d}t \\ 2t\,\mathrm{d}t \\ 3t^2\,\mathrm{d}t \end{pmatrix} \tag{23}$$

and the field has the form:

$$\mathbf{A} = \begin{pmatrix} 9t^2 \\ -14t^5 \\ 20t^7 \end{pmatrix} \tag{24}$$

The path integral is thus:

$$\int \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^1 9t^2 - 28t^6 + 60t^9\,\mathrm{d}t = 5 \tag{25}$$

(b) On the suggested path, the integral evaluates to:

$$\int \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^1 3x^2 \,\mathrm{d}x + \int_0^1 20z^2 \,\mathrm{d}z = \frac{23}{3} \tag{26}$$

(c) On the straight line from $(0,0,0)$ to $(1,1,1)$, $x = y = z$. The integral evaluates to:

$$\int \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^1 3x^2 + 6x - 14x^2 + 20x^3 \,\mathrm{d}x = \frac{13}{3} \tag{27}$$

Since the integral is path dependent, the field is not conservative.

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(a) A hemisphere of radius a is defined over the region $\{(r,\theta,\phi) \mid r = a, \theta \in [0,\pi], \phi \in [0,\pi]\}$. The surface area is given by the integral:

$$A = \int_0^\pi \int_0^\pi a^2 \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = 2\pi a^2 \tag{28}$$

(b) To compute the surface area on the x-y plane, we integrate infinitesimal area elements on the region $\{(r,\theta) \mid r = [0,a], \theta \in [0,2\pi]\}$. The infinitesimal area element can be expressed as

$$\mathrm{d}A = a^2 \mathrm{d}\beta\mathrm{d}\theta = \frac{a^2}{\sqrt{a^2 - r^2}} \mathrm{d}r\mathrm{d}\theta \tag{29}$$

where $\cos\beta = r/a$ is the polar angle.

Thus the surface area can be expressed as a double integral:

$$A = \int_0^{2\pi} \int_0^a \frac{a^2}{\sqrt{a^2 - r^2}} \mathrm{d}r\mathrm{d}\theta = 2\pi a^2 \tag{30}$$

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(a) Operating in cylindrical coordinates, the surface is defined by the region:

$$D = \{(r\cos\theta, r\sin\theta, z) \mid r \in [0,3], \theta \in [0,2\pi], z = (13 - r\cos\theta + 2r\sin\theta)/5\} \tag{31}$$

The Jacobian associated with the transformation $(x,y) \to (r,\theta)$ is:

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r \tag{32}$$

The surface area is given by the integral:

$$A = \int_D \sqrt{1 + (z_x)^2 + (z_y)^2}\,\mathrm{d}x\mathrm{d}y = \int_D \sqrt{\frac{6}{5}}\,\mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\int_0^3 \sqrt{\frac{6}{5}} r\,\mathrm{d}r\mathrm{d}\theta = 9\pi\sqrt{\frac{6}{5}} \tag{33}$$

(b) The surface area is given by the integral:

$$A = \int_D \sqrt{1 + (z_x)^2 + (z_y)^2}\,\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-x} \sqrt{3}\,\mathrm{d}y\mathrm{d}x = \frac{\sqrt{3}}{2} \tag{34}$$

The coordinates of the centre of mass are given by:

$$\begin{aligned} x_{CM} &= \frac{\int_0^1\int_0^{1-x}\sqrt{3}x\,\mathrm{d}y\mathrm{d}x}{\sqrt{3}/2} = \frac{1}{3} \\ y_{CM} &= \frac{\int_0^1\int_0^{1-x}\sqrt{3}y\,\mathrm{d}y\mathrm{d}x}{\sqrt{3}/2} = \frac{1}{3} \\ z_{CM} &= \frac{\int_0^1\int_0^{1-x}\sqrt{3}(1-x-y)\,\mathrm{d}y\mathrm{d}x}{\sqrt{3}/2} = \frac{1}{3} \end{aligned} \tag{35}$$

The surface can also be viewed as an isosceles triangle with side length $\sqrt{2}$ so that the area is:

$$A = \frac{1}{2}\sqrt{2}\frac{\sqrt{6}}{2} = \frac{\sqrt{3}}{2} \tag{36}$$

The coordinates of the centre of mass are the same because in the given equation $x + y + z = 1$, we can exchange any coordinate with others, leading to a high degree of symmetry.

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