Special Relativity

Problems 1

Lorentz Transformation, Velocity Addition and Energy & Mass

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Lorentz Transformation and Elementary Consequences

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In frame S, the two events have coordinates $(x_0/c, x_0, 0, 0)$ and $(x_0/2c, 2x_0, 0, 0)$ respectively. In frame S', we demand that they have coordinates $(t', x'_1, 0, 0)$ and $(t', x'_2, 0, 0)$. Thus, by Lorentz transformation:

$$t' = \gamma(\frac{x_0}{c} - \frac{vx_0}{c}) = \gamma(\frac{x_0}{2c} - \frac{2vx_0}{c}) \tag{1}$$

Hence, solving the equation yield $v = -c/2$ and $t' = \sqrt{3}x_0/c$.

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$$v = \omega R \sin 30^\circ = 3.3 \times 10^8\,\mathrm{ms}^{-1} \tag{2}$$

This does not violate the limitation of speed of light, as no new information is transmitted at a superluminal speed.

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In frame S_A, the first signal is produced at coordinates $(t, 0, 0, 0)$, which corresponds to $(t', x', 0, 0)$ in S_B.

After a time $\Delta t = |x'|/c$ in S_B, B receives the signal and after another $2t$, the return signal is produced. This event is marked as $(t' + \Delta t + 2t, 0, 0, 0)$ in S_B, which corresponds to $(t_A, x_A, 0, 0)$ in S_A.

Then the in S_A, the time T when A receives the return signal is given by:

$$\begin{aligned} T &= t_A + \frac{1}{c}|x_A| \\ &= \gamma(t' + \Delta t + 2t) + \frac{1}{c}\gamma v(t' + \Delta t + 2t) \\ &= \gamma(1 + \frac{v}{c})(t' + \Delta t + 2t) \end{aligned} \tag{3}$$

But by Lorentz transformation, $t' = \gamma t$ and $\Delta t = |x'|/c = \gamma vt/c$. Hence, substituting the values, we have:

$$T = (3 + 2\sqrt{3})t \tag{4}$$

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In the pilot's frame, the total distance to be travelled is $D' = D/\gamma = 2.52\,\text{ly}$ due to length contraction. Thus:

$$T = \frac{D'}{v} = \frac{D}{\gamma v} = 3.15\,\text{years} \tag{5}$$

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In the Galaxy frame, the energy of the proton is given by $E = \gamma mc^2$. We have the equation:

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \left(\frac{m_0 c^2}{E}\right)^2} \approx 1 \tag{6}$$

as the rest energy of the proton is too small compared to E.

Hence, the time to travel across the Galaxy measured in the Galaxy frame is $T_G = 10 \times 10^5$ years. Then by time dilation, the time measured in the proton's frame is:

$$T_P = \frac{1}{\gamma} T = \sqrt{1 - \beta^2}\, T_G = 4.93\,\text{min} \tag{7}$$

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(a) We have the relationship between proper time and time measured in a frame:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\gamma} \tag{8}$$

Then, as measured in the rest frame:

$$t = \int_0^\tau \gamma \,\mathrm{d}\tau = \gamma\tau = 38\,\text{ns} \tag{9}$$

(b)

$$D = \beta ct = 8.32\,\text{m} \tag{10}$$

(c)

$$D' = \beta c\tau = 5.69\,\mathrm{m} \tag{11}$$

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Assume that the 'dying' of muons follows a Poisson process, such that the time for one muon to die follows an exponential distribution with mean τ_0. We have:

$$T \sim \exp(1/\tau_0) \tag{12}$$

Since 1% of the muons survive, we have:

$$P(T > \tau) = 0.01 \tag{13}$$

where τ is the proper time for the muon to travel down to the ground.

Then, from the cumulative distribution function:

$$P(T \leq \tau) = 1 - e^{-\tau/\tau_0} = 1 - 0.01 \tag{14}$$

Solving this equation yields $\tau = \tau_0 \ln 100$. The distance travelled measured in muons' frame is thus $D_0 = v\tau = 3.01 \times 10^3\,\mathrm{m}$ and the distance travelled measured in the ground frame is $D = \gamma v\tau = 2.13 \times 10^4\,\mathrm{m}$.

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Addition of velocities; Energy & Mass

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Suppose an initial velocity 4-vector $U = \gamma_u(c, u, 0, 0)^\intercal$. Two successive Lorentz transformations yield:

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\begin{aligned}
U' &= \Lambda_{v_2}\Lambda_{v_1}U \\
&= \Lambda_{v_2}\gamma_u \begin{pmatrix} \gamma_{v_1}c - \gamma_{v_1}\beta_{v_1}u \\ -\gamma_{v_1}\beta_{v_1}c + \gamma_{v_1}u \\ 0 \\ 0 \end{pmatrix} \\
&= \gamma_u\gamma_{v_1}\gamma_{v_2} \begin{pmatrix} (1+\beta_{v_1}\beta_{v_2})c - (\beta_{v_1}+\beta_{v_2})u \\ (1-\beta_{v_1}\beta_{v_2})c - (\beta_{v_1}+\beta_{v_2})u \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
\tag{15}
$$

On the other hand, with $v = (v_1 + v_2)/(1 + v_1 v_2/c^2)$:

$$
\Lambda_v U = \gamma_u\gamma_v \begin{pmatrix} c - \beta_v u \\ -\beta_v c + u \\ 0 \\ 0 \end{pmatrix} \tag{16}
$$

To verify $\Lambda_{v_2}\Lambda_{v_1} = \Lambda_v$, we only need to show the following relationship:

$$
\frac{\gamma_{v_1}\gamma_{v_2}}{1+\beta_{v_1}\beta_{v_2}} \stackrel{?}{=} \gamma_v \tag{17}
$$

For γ_v:

$$
\gamma_v = \frac{1}{\sqrt{1-\frac{(v_1+v_2)^2}{c^2(1+v_1v_2/c^2)^2}}} = \frac{1}{1+\beta_{v_1}\beta_{v_2}} \frac{1+\beta_{v_1}\beta_{v_2}}{\sqrt{\left(1+\frac{\beta_{v_1}+\beta_{v_2}}{1+\beta_{v_1}\beta_{v_2}}\right)\left(1-\frac{\beta_{v_1}+\beta_{v_2}}{1+\beta_{v_1}\beta_{v_2}}\right)}} = \frac{\gamma_{v_1}\gamma_{v_2}}{1+\beta_{v_1}\beta_{v_2}} \tag{18}
$$

This verifies the proposed relationship.

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It is known that the molar mass of trinitrotoluene (TNT) is $M_{\mathrm{TNT}} = 227\,\mathrm{g\,mol^{-1}}$. Thus:

$$\Delta m = N\frac{\Delta E}{c^2} = N_A \frac{m_{\text{tot}}}{M_{\text{TNT}}}\frac{\Delta E}{c^2} = 2.6 \times 10^{34}\,\text{eVc}^{-2} = 47\,\text{g} \tag{19}$$

where N_A is the Avogadro constant.

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We have:

$$P = 4\pi R^2 I = \frac{\mathrm{d}E}{\mathrm{d}t} \tag{20}$$

Therefore:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E}{c^2}\right) = \frac{P}{c^2} = \frac{4\pi R^2 I}{c^2} = 4.4 \times 10^9\,\text{kgs}^{-1} \tag{21}$$

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