From Newton's Laws to Modelling Black Holes

The Power of Numerical Methods

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What is the **greatest** achievement of science?

Quantum mechanics? General relativity? The standard model?

Introduction

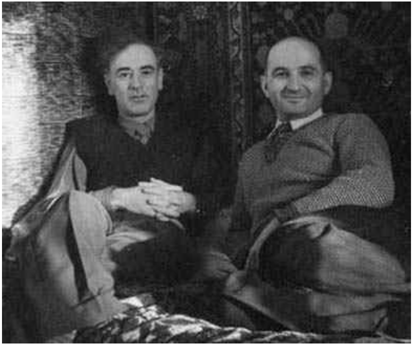
What is the **greatest** achievement of science?

Ability to predict the future,

using, for example, Newton's laws.

Contents

Newton's Laws



If all the co-ordinates and velocities (of a system) are simultaneously specified, it is known from experience that the state of the system is completely determined and that its subsequent motion can, **in principle**, *be calculated.*

[0] L.D. Landau and E.M. Lifshitz.

Newton's Laws

In principle, we can predict the future using Newton's second law:

$$\mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} \tag{1}$$

Once we know the force, we can solve the differential equation.

Newton's Laws

Imagine you are a rocket moving in space towards Mother Earth. Your dynamics is a constant updating of the state vector:

$$\begin{pmatrix} x \\ v \end{pmatrix} \xrightarrow{\delta t} \begin{pmatrix} x + v\delta t \\ v + a\delta t \end{pmatrix} \tag{2}$$

How do we know the acceleration a?

Numerical Methods

What we just did is called the **Euler's method**.

It is a general method of solving ordinary differential equations.

[0] Leonhard Euler (1707-1783).

Consider the following differential equation:

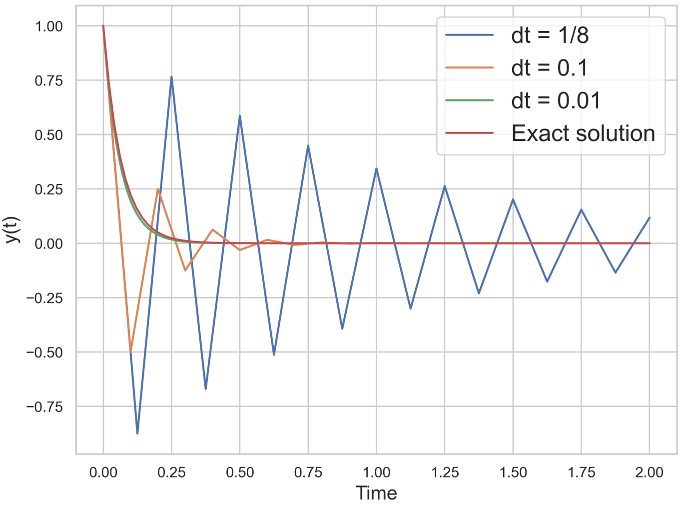
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -15y \quad y(0) = 1 \tag{3}$$

We know the solution is:

$$y(t) = e^{-15t} \tag{4}$$

Let's see how Euler's method performs with different step sizes.

Numerical Methods



Numerical Methods

There are at least two problems with the naive Euler's method:

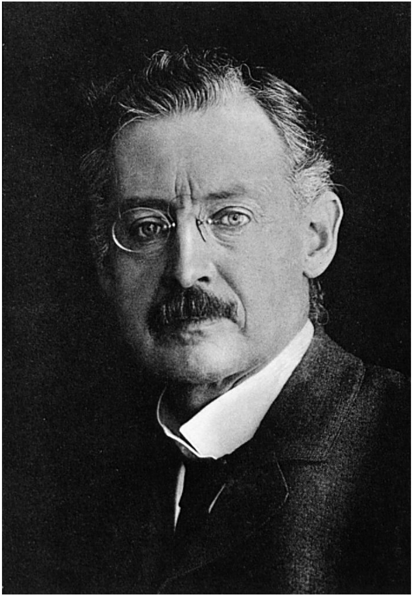
1. It is (very) **inaccurate** for large step sizes.
2. It becomes (very) **slow** for small step sizes.

Numerical Methods - Go to Higher Orders

The Euler's method is naive in a sense that it is too ‘local’.

We could have 'scouted' ahead a bit and use the average of acceleration there and our current position.

Numerical Methods - Go to Higher Orders



Let us go as far as four steps ahead!

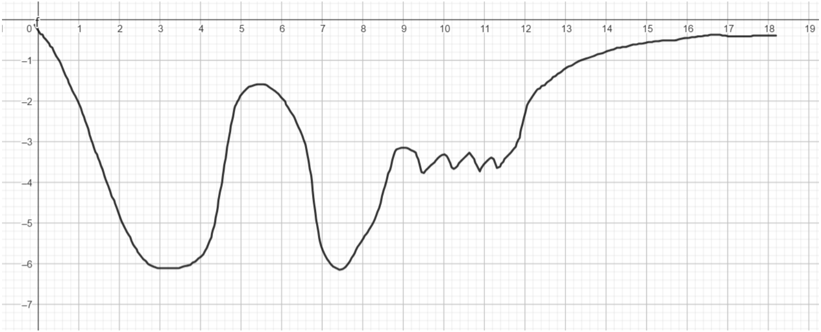
Runge-Kutta methods are a family of numerical methods for ODEs.

The most famous is the RK4 method.

[0] Carl David Tolmé Runge (1856-1927).

Numerical Methods - Adapt Your Step

The Euler's method is also too 'dumb' because it only knows a **fixed step size**.



Numerical Methods - Adapt Your Step

To know when and how to adapt the step size, we require a rough **estimate of the error**.

RK(F)45 method is a viable choice, which uses both 4th and 5th order results to estimate the error.

Foraging to Black Holes

Instead of a rocket, what if we are photons travelling towards a black hole?

What even is a black hole?

Foraging to Black Holes

John Michell first to proposed the existence of ‘black holes’.

Alas, he was too far ahead of his time.

Scientists then did not have the tools to investigate his ideas.

[0] John Michell (1724-1793).

Albert Einstein published the general theory of relativity in 1915 along with his field equations.

Karl Schwarzschild found the first solution to the equations in 1916.

From his solutions, the concept of a black hole emerged.

[0] Karl Schwarzschild (1873-1916).

Foraging to Black Holes

It was soon realised that BHs are very simple objects with only three properties:

1. Mass M
2. Charge Q (theorised to be zero)
3. Spin J

Foraging to Black Holes

BHs are often found in binary systems and a process called **accretion** occurs to give rise to **accretion disks**.

These disks get so hot ($\sim 10^7 K$) that they emit some of the most energetic radiation in the universe.

Foraging to Black Holes - Some GR

You have probably heard of mass 'bending' space in GR. How do we quantify this effect? Using a metric!

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -\left(1 - \frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1 - \frac{r_s}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Foraging to Black Holes - Some GR

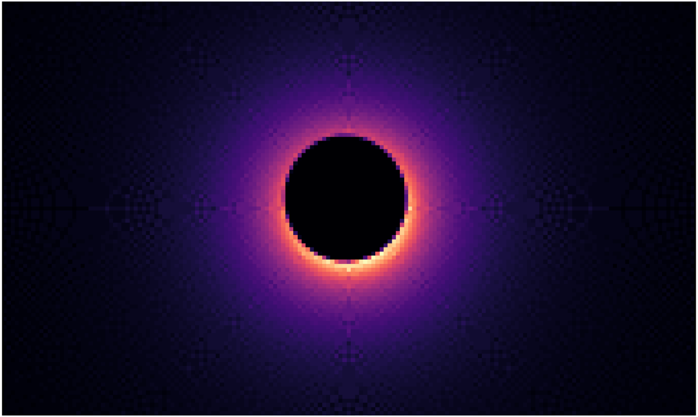
The motion of a photon is governed by the geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0$$

This can be numerically integrated!

Exciting Science - Ray Tracing

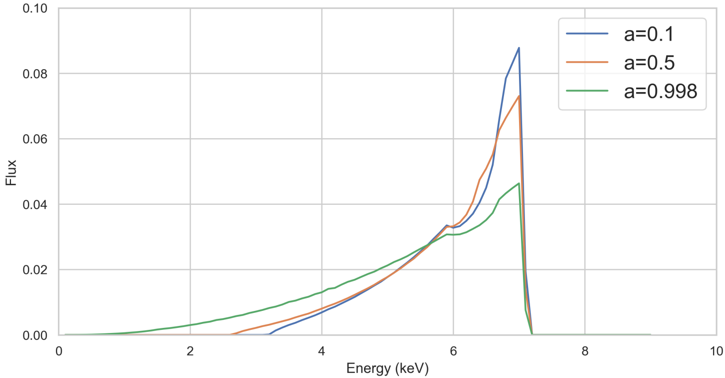
We can use C++ (for its speed) to **simulate** how photons from an accretion disk travel to an observer.



[0] spin = 0.95, inclination = 20 deg

Exciting Science - Actual Value

Besides simulating a BH on your PC, the algorithm also gives us a **spectrum** that contains key information about the BH.



Concluding Remarks

What have we learnt?

Concluding Remarks

Similar mathematical ideas arise from distinct physical principles.

Concluding Remarks

In tackling the most difficult questions, human ingenuity prevails over computational brute force.

Thank you!

```python
# Test time used
import time
start = time.time()
for i in range(100000):
    euler_method(f, y0, 0.01, T) # Test 1000 times
end = time.time()
print('Time used for dt = 0.01:', end - start)

start = time.time()
for i in range(100000):
    euler_method(f, y0, 0.1, T) # Test 1000 times
end = time.time()
print('Time used for dt = 0.1:', end - start)

start = time.time()
for i in range(100000):
    euler_method(f, y0, 1/8, T) # Test 1000 times
end = time.time()
print('Time used for dt = 1/8:', end - start)
```

[20] ✓ 4.5s Python

```
Time used for dt = 0.01: 3.7898566722869873
Time used for dt = 0.1: 0.4230952262878418
Time used for dt = 1/8: 0.35508084297180176
```

[0] Time comparison

$$
\begin{pmatrix}
-\frac{a^2+2\,r\,(-2\,M+r)+a^2\,\mathrm{Cos}[2\,\mathrm{theta}]}{a^2+2\,r^2+a^2\,\mathrm{Cos}[2\,\mathrm{theta}]} & 0 & 0 & -\frac{4\,a\,M\,r\,\mathrm{Sin}[\mathrm{theta}]^2}{a^2+2\,r^2+a^2\,\mathrm{Cos}[2\,\mathrm{theta}]} \\
0 & \frac{r^2+a^2\,\mathrm{Cos}[\mathrm{theta}]^2}{a^2-2\,M\,r+r^2} & 0 & 0 \\
0 & 0 & r^2+a^2\,\mathrm{Cos}[\mathrm{theta}]^2 & 0 \\
-\frac{4\,a\,M\,r\,\mathrm{Sin}[\mathrm{theta}]^2}{a^2+2\,r^2+a^2\,\mathrm{Cos}[2\,\mathrm{theta}]} & 0 & 0 & \frac{\left(r^2+a^2\,\mathrm{Cos}[\mathrm{theta}]^2\right)\,\mathrm{Sin}[\mathrm{theta}]^2\,\left(\left(a^2+r^2\right)^2-a^2\,\left(a^2+r\,(-2\,M+r)\right)\,\mathrm{Sin}[\mathrm{theta}]^2\right)}{\left(a^2+r^2-a^2\,\mathrm{Sin}[\mathrm{theta}]^2\right)^2}
\end{pmatrix}
$$

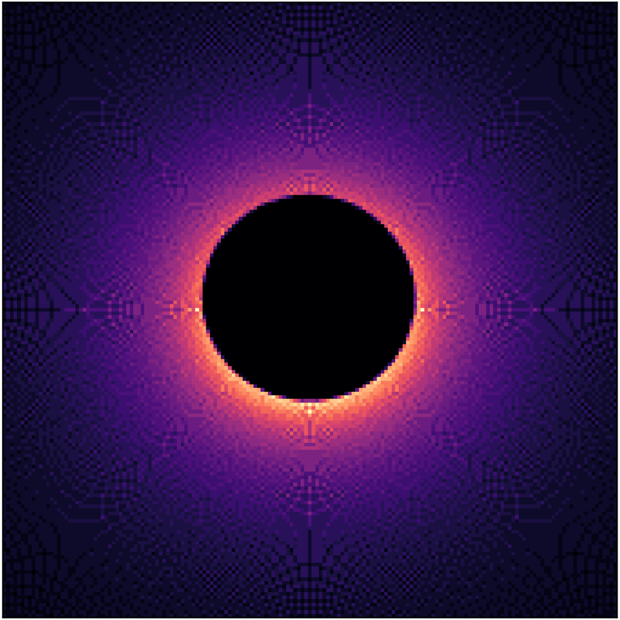
[0] Kerr metric

$$
\begin{pmatrix}
\begin{pmatrix} 0 \\ -\frac{2M\left(a^2+r^2\right)\left(a^2-2r^2+a^2\cos(2\theta)\right)}{\left(a^2+r(-2M+r)\right)\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ -\frac{4a^2Mr\sin(2\theta)}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ 0 \end{pmatrix}
&
\begin{pmatrix} -\frac{2M\left(a^2+r^2\right)\left(a^2-2r^2+a^2\cos(2\theta)\right)}{\left(a^2+r(-2M+r)\right)\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ 0 \\ 0 \\ \frac{2aM\left(a^4-3a^2r^2-6r^4+a^2\left(a^2-r^2\right)\cos(2\theta)\right)\sin(\theta)^2}{\left(a^2+r(-2M+r)\right)\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \end{pmatrix}
&
\begin{pmatrix} -\frac{4a^2Mr\sin(2\theta)}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ 0 \\ 0 \\ \frac{8a^3Mr\cos(\theta)\sin(\theta)^3}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \end{pmatrix}
&
\begin{pmatrix} 0 \\ \frac{2aM\left(a^4-3a^2r^2-6r^4+a^2\left(a^2-r^2\right)\cos(2\theta)\right)\sin(\theta)^2}{\left(a^2+r(-2M+r)\right)\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ \frac{8a^3Mr\cos(\theta)\sin(\theta)^3}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ 0 \end{pmatrix}
\\
\begin{pmatrix} -\frac{4M\left(a^2+r(-2M+r)\right)\left(a^2-2r^2+a^2\cos(2\theta)\right)}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^3} \\ 0 \\ 0 \\ \frac{4aM\left(a^2+r(-2M+r)\right)\left(a^2-2r^2+a^2\cos(2\theta)\right)\sin(\theta)^2}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^3} \end{pmatrix}
&
\begin{pmatrix} 0 \\ \frac{r\left(a^2-Mr\right)+a^2(M-r)\cos(\theta)^2}{\left(a^2+r(-2M+r)\right)\left(r^2+a^2\cos(\theta)^2\right)} \\ -\frac{a^2\cos(\theta)\sin(\theta)}{r^2+a^2\cos(\theta)^2} \\ 0 \end{pmatrix}
&
\begin{pmatrix} 0 \\ -\frac{a^2\cos(\theta)\sin(\theta)}{r^2+a^2\cos(\theta)^2} \\ -\frac{r\left(a^2+r(-2M+r)\right)}{r^2+a^2\cos(\theta)^2} \\ 0 \end{pmatrix}
&
\begin{pmatrix} \frac{4aM\left(a^2+r(-2M+r)\right)\left(a^2-2r^2+a^2\cos(2\theta)\right)\sin(\theta)^2}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^3} \\ 0 \\ 0 \\ -\frac{\left(a^2+r(-2M+r)\right)\left(a^4M+3a^4r-4a^2Mr^2+8a^2r^3+8r^5+4a^2r\left(a^2+r(M+2r)\right)\cos(2\theta)-a^4(M-r)\cos(4\theta)\right)\sin(\theta)^2}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^3} \end{pmatrix}
\\
\begin{pmatrix} -\frac{8a^2Mr\sin(2\theta)}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^3} \\ 0 \\ 0 \\ \frac{8aMr\left(a^2+r^2\right)\sin(2\theta)}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^3} \end{pmatrix}
&
\begin{pmatrix} 0 \\ \frac{a^2\cos(\theta)\sin(\theta)}{\left(a^2+r(-2M+r)\right)\left(r^2+a^2\cos(\theta)^2\right)} \\ \frac{r}{r^2+a^2\cos(\theta)^2} \\ 0 \end{pmatrix}
&
\begin{pmatrix} 0 \\ \frac{r}{r^2+a^2\cos(\theta)^2} \\ -\frac{a^2\cos(\theta)\sin(\theta)}{r^2+a^2\cos(\theta)^2} \\ 0 \end{pmatrix}
&
\begin{pmatrix} \frac{8aMr\left(a^2+r^2\right)\sin(2\theta)}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^3} \\ 0 \\ 0 \\ -\frac{\left(3a^6+10a^4Mr+11a^4r^2+16a^2Mr^3+16a^2r^4+8r^6+4a^2\left(a^2+2r^2\right)\left(a^2+r(-2M+r)\right)\cos(2\theta)+a^4\left(a^2-2Mr+r^2\right)\cos(4\theta)\right)\sin(2\theta)}{2\left(a^2+2r^2+a^2\cos(2\theta)\right)^3} \end{pmatrix}
\\
\begin{pmatrix} 0 \\ -\frac{2aM\left(a^2-2r^2+a^2\cos(2\theta)\right)}{\left(a^2+r(-2M+r)\right)\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ -\frac{8aMr\cot(\theta)}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ 0 \end{pmatrix}
&
\begin{pmatrix} -\frac{2aM\left(a^2-2r^2+a^2\cos(2\theta)\right)}{\left(a^2+r(-2M+r)\right)\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ 0 \\ 0 \\ \frac{a^4M+3a^4r-12a^2Mr^2+8a^2r^3-16Mr^4+8r^5+4a^2r\left(a^2+r(-M+2r)\right)\cos(2\theta)-a^4(M-r)\cos(4\theta)}{2\left(a^2+r(-2M+r)\right)\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \end{pmatrix}
&
\begin{pmatrix} -\frac{8aMr\cot(\theta)}{\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ 0 \\ 0 \\ \frac{\left(3a^4+8a^2Mr+8a^2r^2+8r^4+4a^2\left(a^2+2r(-M+r)\right)\cos(2\theta)+a^4\cos(4\theta)\right)\cot(\theta)}{2\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \end{pmatrix}
&
\begin{pmatrix} 0 \\ \frac{a^4M+3a^4r-12a^2Mr^2+8a^2r^3-16Mr^4+8r^5+4a^2r\left(a^2+r(-M+2r)\right)\cos(2\theta)-a^4(M-r)\cos(4\theta)}{2\left(a^2+r(-2M+r)\right)\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ \frac{\left(3a^4+8a^2Mr+8a^2r^2+8r^4+4a^2\left(a^2+2r(-M+r)\right)\cos(2\theta)+a^4\cos(4\theta)\right)\cot(\theta)}{2\left(a^2+2r^2+a^2\cos(2\theta)\right)^2} \\ 0 \end{pmatrix}
\end{pmatrix}
$$

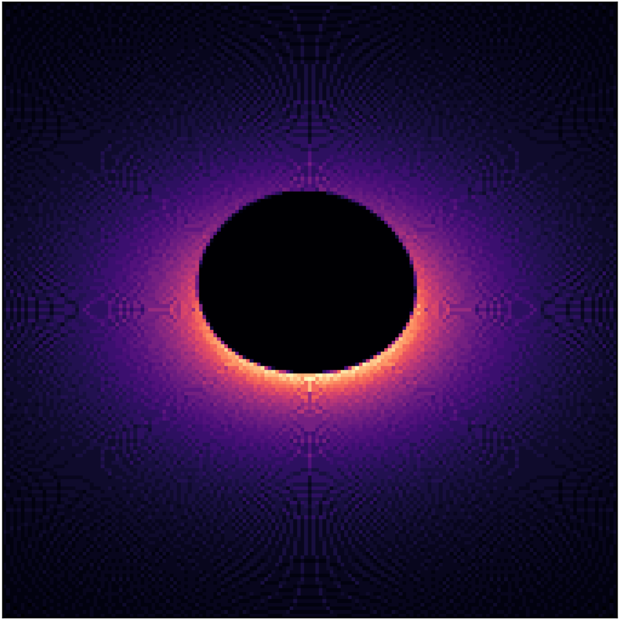
[0] Christoffel symbols of Kerr metric

$$-\frac{\left(a^2+r\,(-2\,M+r)\right)\,\left(a^4\,M+3\,a^4\,r-4\,a^2\,M\,r^2+8\,a^2\,r^3+8\,r^5+4\,a^2\,r\,\left(a^2+r\,(M+2\,r)\right)\,\text{Cos}[2\,\text{theta}]-a^4\,(M-r)\,\text{Cos}[4\,\text{theta}]\right)\,\text{Sin}[\text{theta}]^2}{\left(a^2+2\,r^2+a^2\,\text{Cos}[2\,\text{theta}]\right)^3}$$

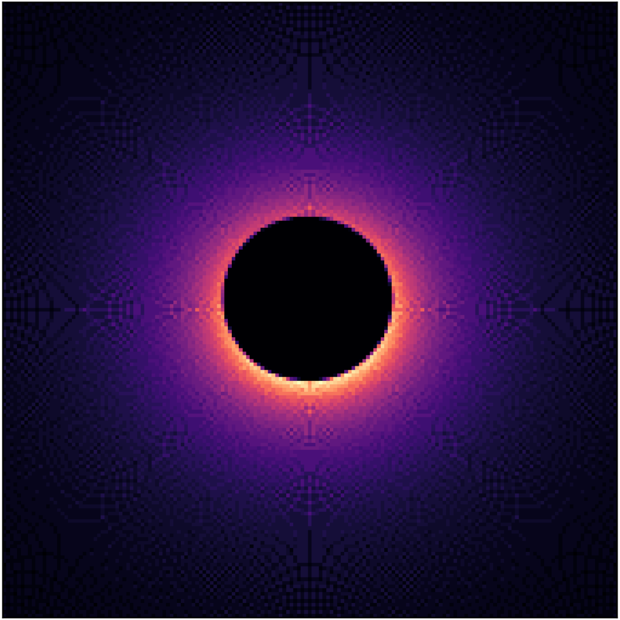
[0] An element of the Christoffel symbols



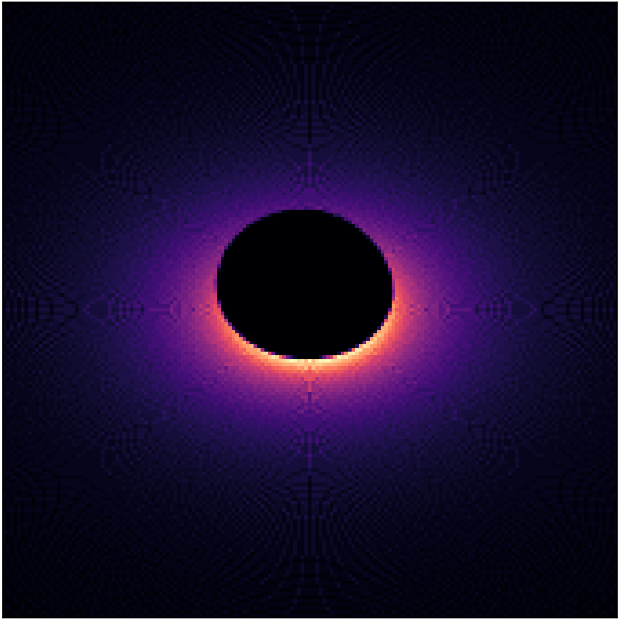
[0] spin = 0.5, inclination = 20 deg



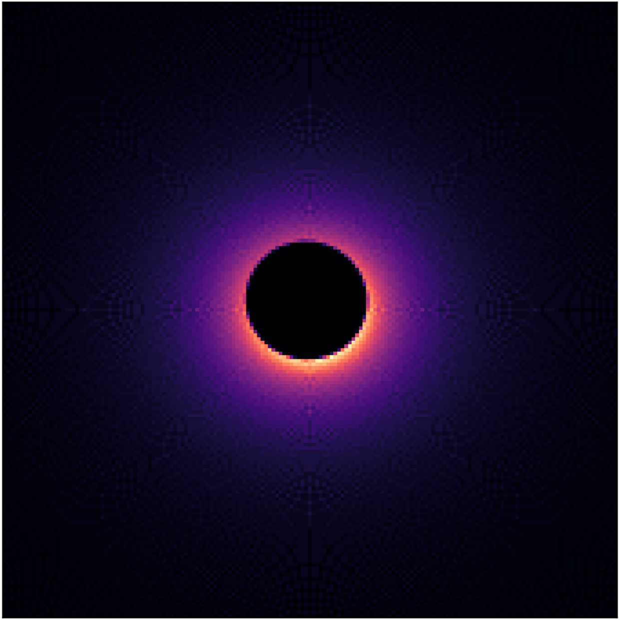
[0] spin = 0.5, inclination = 45 deg



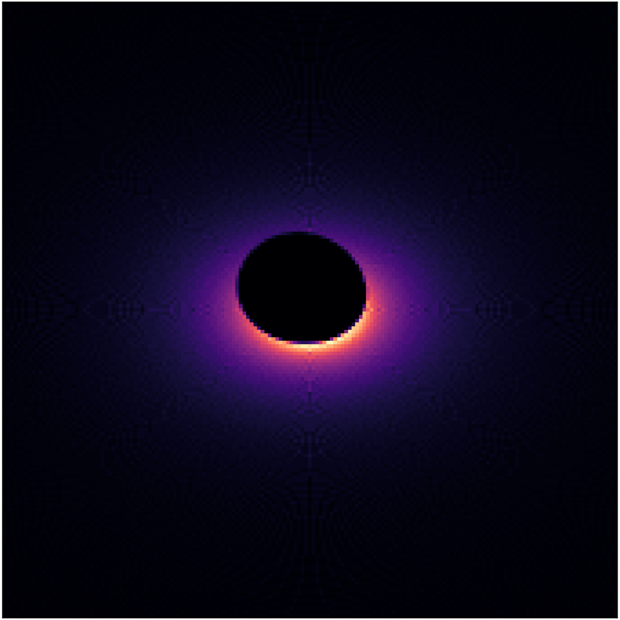
[0]spin = 0.75, inclination = 20 deg



[0] spin = 0.75, inclination = 45 deg



[0]spin = 0.95, inclination = 20 deg



[0] spin = 0.95, inclination = 45 deg