Mathematical Methods

Problem Sheet 3

Partial Differential Equations

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Partial Differential Equations

1 Laplace eqaution in two dimensions

(a)

(b)

(c) Consider the Laplacian in polar coordinates:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \tag{1}$$

Assuming a solution of the form $V(r,\phi) = R(r)\Phi(\phi)$, we have the separated equations:

$$\begin{aligned} \Phi'' + k^2\Phi &= 0 \\ r^2R'' + rR' - k^2R &= 0 \end{aligned} \tag{2}$$

First consider non-zero k. The first equation has the solution:

$$\Phi(\phi) = A\cos k\phi + B\sin k\phi \tag{3}$$

The natural boundary condition $\Phi(0) = \Phi(2\pi)$ demands $k \in \mathbb{Z}$. The second solution can be solved with the Ansatz $R(r) = r^\lambda$ that leads to $\lambda = \pm k$ and the general solution:

$$R(r) = Cr^k + Dr^{-k} \tag{4}$$

In the case of $k = 0$, we have Φ is some constant and $R(r) = C\ln r + D$. The most general solution to the Laplace equation is then:

$$V(r,\phi) = \frac{a_0}{2} + \frac{\tilde{a}_0}{2}\ln r + \sum_{k=1}^{\infty}\left(a_k r^k + \tilde{a}_k r^{-k}\right)\cos k\phi + \sum_{k=1}^{\infty}\left(b_k r^k + \tilde{b}_k r^{-k}\right)\sin k\phi \tag{5}$$

where we have rewritten the arbitrary constants and taken the general solution as a sum of the solutions for $k \in \mathbb{Z}$.

(d) For the given boundary condition, we need:

$$\frac{a_0}{2} + \frac{\tilde{a}_0}{2}\ln a_\pm + \sum_{k=1}^{\infty}\left(a_k a_\pm^k + \tilde{a}_k a_\pm^{-k}\right)\cos k\phi + \sum_{k=1}^{\infty}\left(b_k a_\pm^k + \tilde{b}_k a_\pm^{-k}\right)\sin k\phi = g_\pm(\phi) \tag{6}$$

This is a Fourier series of $g_{\pm}(\phi)$ if we identify the coefficients:

$$c_{0,\pm} + \sum_{k=1}^{\infty} (c_{k,\pm} \cos k\phi + d_{k,\pm} \sin k\phi) = g_{\pm}(\phi) \tag{7}$$

where the coefficients are given by:

$$c_{k,\pm} = a_k a_{\pm}^k + \tilde{a}_k a_{\pm}^{-k} \quad \text{and} \quad d_{k,\pm} = b_k a_{\pm}^k + \tilde{b}_k a_{\pm}^{-k} \tag{8}$$

The coefficients can be solved by the orthogonality of the trigonometric functions:

$$\begin{aligned} c_{k,\pm} &= \frac{1}{\pi} \int_0^{2\pi} g_{\pm}(\phi) \cos k\phi \, \mathrm{d}\phi \\ d_{k,\pm} &= \frac{1}{\pi} \int_0^{2\pi} g_{\pm}(\phi) \sin k\phi \, \mathrm{d}\phi \end{aligned} \tag{9}$$

These leads to equations that can be solved for a_k, $\tilde{a}_k$, b_k, and $\tilde{b}_k$.

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2 Laplace equation in three dimensions

(a) Without loss of generality let the y-axis align with the vector pointing from the origin to the charge q at $(0, y, 0)$. We seek another charge q' at $(0, d, 0)$ such that the potential at the ball surface is zero. A direct calculation gives:

$$q' = -\frac{b}{y} q \tag{10}$$

and:

$$d = \frac{b^2}{y} \tag{11}$$

Let us define $\mathbf{d} = b^2 \mathbf{y}/y^3$, then the solution to the Laplace equation is:

$$V = q \left(\frac{1}{|\mathbf{r} - \mathbf{y}|} - \frac{b/y}{|\mathbf{r} - \mathbf{d}|} \right) \tag{12}$$

Generalising to the case of n charges q_i each at $\mathbf{y}_i$, we introduce image charges $q_i' = -bq_i/y_i$ at $\mathbf{d}_i = b^2 \mathbf{y}_i / y_i^3$ and the solution is:

$$V = \sum_{i=1}^{n} q_i \left(\frac{1}{|\mathbf{r} - \mathbf{y}_i|} - \frac{b/y_i}{|\mathbf{r} - \mathbf{d}_i|} \right) \tag{13}$$

(b) Consider the Laplacian in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{14}$$

Assuming a solution of the form $V(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, we have the separated equations:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r}\right) = l(l+1)R \\
&\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -m^2\Phi \\
&\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0
\end{aligned} \tag{15}$$

The radial solution can be solved with the Ansatz $R(r) = r^\lambda$ that leads to $\lambda = l$ or $\lambda = -l - 1$. The solution is then:

$$R(r) = Ar^l + Br^{-l-1} \tag{16}$$

The solution to Φ is:

$$\Phi(\phi) = e^{im\phi} \tag{17}$$

where we have dropped the arbitrary constant and the natural boundary condition $m \in \mathbb{Z}$.

The Θ equation is the associated Legendre equation with the solution:

$$\Theta(\theta) = P_l^m(\cos\theta) \tag{18}$$

Combining the solutions, we have:

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + B_{lm} r^{-l-1}\right) Y_{lm}(\theta, \phi) \tag{19}$$

where $Y_{lm}(\theta, \phi) = P_l^m(\cos\theta)e^{im\phi}$ are called the spherical harmonics.

If azimuthal symmetry can be assumed, then $m = 0$ and the solution is:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta) \tag{20}$$

(c) For the boundary condition $V(a,\theta,\phi) = \phi_0(1+\cos\theta)$, we have azimuthal symmetry and may discard B_l terms as they diverge at $r = 0$. We need:

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = \phi_0(1+\cos\theta) \tag{21}$$

Since $P_l(\cos\theta)$ are orthogonal, we have $A_0 = \phi_0$, $A_1 = \phi_0/a$ and $A_l = 0$ for $l \geq 2$. The solution is then:

$$V(r,\theta) = \phi_0 \left(1 + \frac{r}{a}\cos\theta\right) \tag{22}$$

(d) For the boundary condition $V(a,\theta,\phi) = \phi_0 \sin^2\theta$, we still have azimuthal symmetry and may discard A_l terms as they diverge at $r \to \infty$. We need:

$$\sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) = \phi_0 \sin^2\theta = \phi_0(1-\cos^2\theta) \tag{23}$$

Comparing the coefficients, we have $B_0 = 2\phi_0 b/3$, $B_1 = 0$, $B_2 = -2\phi_0 b^3/3$ and $B_l = 0$ for $l \geq 3$. The solution is then:

$$V(r,\theta) = \phi_0 \left[\frac{2b}{3r} - \frac{b^3}{3r^3}(3\cos^2\theta - 1)\right] \tag{24}$$

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3 Multiple expansion

(b) The first few spherical harmonics are:

$$\begin{aligned} Y_0^0 &= \sqrt{\frac{1}{4\pi}} \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_1^1 &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \\ Y_1^{-1} &= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \end{aligned} \tag{25}$$

(c) First consider the case of $l = 0$. The term q_{00} can be written as:

$$q_{00} = \int_{\mathbb{R}^3} \sqrt{\frac{1}{4\pi}} \rho(\mathbf{r}') \, \mathrm{d}^3 r' = \sqrt{\frac{1}{4\pi}} Q \tag{26}$$

so that the monopole term is:

$$4\pi \frac{q_{00}}{r} Y_0^0 = \frac{Q}{r} \tag{27}$$

Next consider the case of $l = 1$. The terms q_{10}, q_{11}, and q_{1-1} can be written as:

$$q_{10} = \int_{\mathbb{R}^3} \sqrt{\frac{3}{4\pi}} \cos\theta' \mathbf{r}' \rho(\mathbf{r}') \, \mathrm{d}^3 r' \tag{28}$$

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4 Strings

(a) Consider the wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial t^2} \tag{29}$$

Assuming a solution of the form $\psi(x,t) = X(x)T(t)$, separation of variables leads to the general solution:

$$\psi(t,x) = \sum_{\omega} C_\omega \sin(\omega x + \phi_x) \sin(\omega t + \phi_t) \tag{30}$$

The boundary condition $\psi(t,0) = \psi(t,a) = 0$ demands $\omega = n\pi/a$ for $n \in \mathbb{Z}$ and $\phi_x = 0$. Indexing via n, the solution is:

$$\psi(t,x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi t}{a} + \phi_n\right) \tag{31}$$

(b) Given the initial condition $\psi(0,x) = \psi_0 \sin(\pi x/a)$ and $\dot{\psi}(0,x) = 0$, we need:

$$\dot{\psi}(0,x) = \sum_{n=1}^{\infty} C_n \frac{n\pi}{a} \sin\left(\frac{n\pi x}{a}\right) \cos\phi_n = 0 \tag{32}$$

which gives $\phi_n = \pi/2$.

We may as well write the temporal part as cosine functions $\cos(n\pi t/a)$ with the same coefficients. The position initial condition gives:

$$\psi(0,x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) = \psi_0 \sin\left(\frac{\pi x}{a}\right) \tag{33}$$

so that $C_n = 0$ except for $C_1 = \psi_0$.

The solution is then:

$$\psi(t,x) = \psi_0 \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi t}{a}\right) \tag{34}$$

(c) Given the initial condition $\psi(0,x) = 0$ and $\dot{\psi}(0,x) = \psi_0 \sin(2\pi x/a)$, we need:

$$\psi(0,x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sin\phi_n = 0 \tag{35}$$

which gives $\phi_n = 0$.

The velocity initial condition gives:

$$\dot{\psi}(0,x) = \sum_{n=1}^{\infty} C_n \frac{n\pi}{a} \sin\left(\frac{n\pi x}{a}\right) = \psi_0 \sin\left(\frac{2\pi x}{a}\right) \tag{36}$$

so that $C_n = 0$ except for $C_2 = \psi_0 a/2\pi$.

The solution is then:

$$\psi(t,x) = \frac{\psi_0 a}{2\pi} \sin\left(\frac{2\pi x}{a}\right) \cos\left(\frac{2\pi t}{a}\right) \tag{37}$$

(d) Consider the skewed triangle wave initial condition:

$$\psi(0,x) = \begin{cases} \frac{hx}{b} & 0 \le x \le b \\ \frac{h(a-x)}{a-b} & b \le x \le a \end{cases} \tag{38}$$

and $\dot{\psi}(0,x) = 0$. From the velocity initial condition, we write the temporal part as cosine functions:

$$\psi(t,x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi t}{a}\right) \tag{39}$$

Then from the position initial condition, we have a Fourier sine series with the coefficients given by:

$$\begin{aligned} C_n &= \frac{2}{a}\left[\int_0^b \frac{hx}{b}\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x + \int_b^a \frac{h(a-x)}{a-b}\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x\right] \\ &= \frac{2ha^2}{b(a-b)(n\pi)^2}\sin\left(\frac{n\pi b}{a}\right) \end{aligned} \tag{40}$$

Consider C_n/h as a function of $(b/a) \equiv \lambda$:

$$\frac{C_n}{h} = \frac{2}{\lambda(1-\lambda)(n\pi)^2}\sin\left(n\pi\lambda\right) \tag{41}$$

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5 Membranes

Consider the two dimensional wave equation:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\psi}{\partial t^2} \tag{42}$$

Separation of variables leads to the general solution:

$$\psi = \sum_{\omega,k} C_{\omega,k}\sin\left(\omega t + \phi_t\right)\sin\left(kx + \phi_x\right)\sin\left(\sqrt{\omega^2 - k^2}y + \phi_y\right) \tag{43}$$

where we demand $\omega^2 > k^2$.

For the boundary conditions, we need $\phi_x = 0$ and $ka = n\pi$ for $n \in \mathbb{Z}$, which means $k = n\pi/a$. We also need $\phi_y = 0$ and $\sqrt{\omega^2 - k^2}b = m\pi$ for $m \in \mathbb{Z}$, which means $\omega^2 = (m\pi/b)^2 + (n\pi/a)^2$. Indexing via n and m, the general solution is:

$$\psi(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{m,n}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\omega_{m,n}t + \phi_{m,n}\right) \tag{44}$$

where we have defined the frequencies $\omega_{m,n} = \sqrt{(m\pi/b)^2 + (n\pi/a)^2}$.

For a square membrane with $a = b$, we have $\omega_{m,n} = \pi\sqrt{m^2 + n^2}/a$. The ratio of the two lowest frequencies is:

$$\frac{\omega_{1,1}}{\omega_{0,1}} = \sqrt{2} \tag{45}$$

(b) For a triangular membrane with $a = b$, we impose the further condition that $\psi = 0$ along the line $y = x$. This means $m = n$ and $\omega_{m,n} = \pi\sqrt{2}m/a$. The ratio of the two lowest frequencies is:

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6 Eigenfunctions of the Laplacian

(a) Assuming a solution of the form $V(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, with the eigenvalue problem $-\nabla^2 V = EV$, we have the separated equations:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + Er^2R = l(l+1)R \\ &\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -m^2\Phi \\ &\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \end{aligned} \tag{46}$$

The angular parts are the same as the homogeneous Laplace equation. The solutions are the spherical harmonics $Y_{lm}(\theta, \phi) = P_l^m(\cos\theta)e^{im\phi}$. The radial part satisfies the differential equation:

$$r^2R'' + 2rR' + [Er^2 - l(l+1)]R = 0 \tag{47}$$

(b) Consider the change of variables $\rho = \sqrt{E}r$ and $u(\rho) = \sqrt{\rho}R(\rho)$. We have the relationships:

$$\begin{aligned} R' &= \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{u}{\sqrt{\rho}}\right) = \frac{u'}{\sqrt{\rho}} - \frac{u}{2\rho^{3/2}} \\ R'' &= \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{u'}{\sqrt{\rho}} - \frac{u}{2\rho^{3/2}}\right) = \frac{u''}{\sqrt{\rho}} - \frac{u'}{\rho^{3/2}} + \frac{3u}{4\rho^{5/2}} \end{aligned} \tag{48}$$

so that the radial equation becomes:

$$\begin{aligned} \frac{\rho^2}{E}\left(\frac{u''}{\sqrt{\rho}} - \frac{u'}{\rho^{3/2}} + \frac{3u}{4\rho^{5/2}}\right) + 2\frac{\rho}{\sqrt{E}}\left(\frac{u'}{\sqrt{\rho}} - \frac{u}{2\rho^{3/2}}\right) + \left[\rho^2 - l(l+1)\right]\frac{u}{\sqrt{\rho}} = 0 \\ \rho^2u'' + \rho u' + \left[\rho^2 - \left(l + \frac{1}{2}\right)^2\right]u = 0 \end{aligned} \tag{49}$$

which is the Bessel equation of order $l + 1/2$.

(c) Assuming integer l, the solution to $R(\rho)$ is the Bessel functions:

$$R(\rho) = \frac{1}{\sqrt{\rho}} \sum_{l=0}^{\infty} A_l J_{l+1/2}(\rho) + B_l J_{-l-1/2}(\rho) \tag{50}$$

The finiteness of $R(\rho)$ at $\rho = 0$ demands $B_l = 0$. We can now write the general solution as:

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m} \frac{J_{l+1/2}(\sqrt{E}r)}{(\sqrt{E}r)^{1/2}} Y_{lm}(\theta, \phi) \tag{51}$$

We further require $V(r = a) = 0$ so that:

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{l,m} J_{l+1/2}(\sqrt{E}a) = 0 \tag{52}$$

We invoke without proof the Bourget's hypothesis, which says that the Bessel functions (differing by integers) do not share common zeros except at the origin. This means $A_{a,b} = 0$ except for some $a = l$ and $b = m$ such that $J_{l+1/2}(\sqrt{E_l}a) = 0$. Thus given some E_l as the eigenvalue, we can uniquely determine all $A_{l,m}$.

The solution is then:

$$V(r, \theta, \phi) = \sum_{m=-l}^{l} A_l \frac{J_{l+1/2}(\sqrt{E_l}r)}{(\sqrt{E_l}r)^{1/2}} Y_{lm}(\theta, \phi) \tag{53}$$

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7 Heat equation

(a) The boundary conditions and the eigenfunction relation are trivial to show. To demonstrate orthogonality, consider the inner product of q_k and q_l:

$$\begin{aligned} \langle q_k, q_l \rangle &= \frac{2}{a} \int_0^a \sin\left[\frac{\pi}{a}(k + 1/2)x\right] \sin\left[\frac{\pi}{a}(l + 1/2)x\right] \mathrm{d}x \\ &= \frac{2}{\pi} \int_0^{\pi} \sin\left[(k + 1/2)y\right] \sin\left[(l + 1/2)y\right] \mathrm{d}y \\ &= \delta_{kl} \end{aligned} \tag{54}$$

where at the second step the substitution $y = \pi x/a$ is used and the last step follows from the orthogonality of sine functions.

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