Preliminary Examination 2023

CP 1

Classical Mechanics & Special Relativity

July 22, 2023

Section A

Section B

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(a) Given the central potential of the form $V(r) = \beta/r^2$, the effective potential is:

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \frac{\beta}{r^2} \tag{1}$$

where $L \equiv mr^2\dot{\theta}$ is the angular momentum of the particle.

(b) With $\beta > -L^2/2m$, the effective potential is always positive. We have the energy conservation equation:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{\beta}{r^2} \tag{2}$$

where E is the conserved energy of the particle.

Differentiating with respect to time, we have:

$$0 = m\dot{r}\ddot{r} - \frac{L^2}{mr^3}\dot{r} - \frac{2\beta}{r^3}\dot{r} \tag{3}$$

or, assuming non-zero $\dot{r}$:

$$\ddot{r} = \frac{L^2}{mr^3} + \frac{2\beta}{r^3} \tag{4}$$

Returning to Equation (2), with the substitution $\dot{r} = \dot{\theta}\mathrm{d}r/\mathrm{d}\theta$, we have:

$$\dot{r} = \dot{\theta}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \pm\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - \frac{\beta}{r^2}\right)} \tag{5}$$

But $\dot{\theta} = L/mr^2$, so that:

$$\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \pm\frac{1}{L}\sqrt{2m\left(E - \frac{L^2}{2mr^2} - \frac{\beta}{r^2}\right)} \tag{6}$$

Now use the substitution $u = 1/r$, so that:

$$-\frac{\mathrm{d}u}{\mathrm{d}\theta} = \pm\frac{1}{L}\sqrt{2mE - (L^2 + 2m\beta)u^2} \tag{7}$$

This is a separable differential equation with the solution:

$$\frac{L}{\sqrt{L^2 + 2m\beta}} \sin^{-1}(r_0 u) = \pm\theta + \theta_0 \tag{8}$$

where $r_0 = \sqrt{L^2/2mE + \beta/E}$.

The plus-minus sign corresponds to clock- and counter-clockwise orbits so let us choose the positive case for simplicity. We may set $\theta_0 = 0$ without loss of generality as this is just a rotation of the coordinate system. Further simplification gives:

$$\frac{1}{r} = \frac{1}{a}\sqrt{\frac{2mE}{L^2}} \sin(a\theta) \tag{9}$$

where $a^2 = 1 + 2m\beta/L^2$ as expected.

The minimum of r is apparently $r_{\min} = \sqrt{2mE/L^2}/a$.

If $\beta = 0$, $a = 1$ and the equation becomes:

$$\frac{1}{r} = \sqrt{\frac{2mE}{L^2}} \sin\theta \tag{10}$$

which is a straight line as expected for a free particle.

(c) With $\beta = -L^2/2m$, the effective potential is zero and Equation (7) becomes:

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \pm\frac{1}{L}\sqrt{2mE} \tag{11}$$

Taking the positive case, we have the solution:

$$r = \frac{1}{\theta}\sqrt{\frac{L^2}{2mE}} \tag{12}$$

Although for r to reach zero, θ must approach infinity, implying an infinite number of revolutions, this is still possible in finite time. To see this, consider Equation (4):

$$\ddot{r} = 0 \tag{13}$$

which means $\dot{r}$ is constant and if $\dot{r} < 0$ initially, r always reaches zero in finite time.

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(a) The Lagrangian of the system can be written as:

$$\mathcal{L} = \frac{1}{2}m\left(r^2\dot{\theta}^2 + \dot{r}^2\right) + \frac{1}{2}M\dot{r}^2 + Mg(l-r) \tag{14}$$

where the constant term Mgl can be ignored.

The Euler-Lagrange equation gives the equations of motion:

$$\begin{aligned} (m+M)\ddot{r} &= mr\dot{\theta}^2 - Mg \\ L \equiv mr^2\dot{\theta} &= \text{constant} \end{aligned} \tag{15}$$

For circular motion, we impose the conditions $\dot{r} = 0$ and $\ddot{r} = 0$ for some $r = r_0$ and $\dot{\theta} = \omega$. The equation for r gives us:

$$mr_0\omega^2 - Mg = 0 \tag{16}$$

This means that given some initial radius r_0, $\dot{\theta}$ must satisfy the above equation for circular motion to occur. Under this circular motion, the angular momentum is:

$$L = mr_0^2\omega = \sqrt{mMgr_0^3} \tag{17}$$

which is a constant.

Returning to the equation for r, we use the substitution $\dot{\theta} = L/mr_0^2$ to obtain:

$$(m+M)\ddot{r} = \frac{L^2}{mr^3} - Mg \tag{18}$$

We can expand the right-hand side as a Taylor series about $r = r_0$:

$$\frac{L^2}{mr^3} - Mg = \frac{L^2}{m}\left[\frac{1}{r_0^3} - \frac{3(r-r_0)}{r_0^4} + \cdots\right] - Mg \tag{19}$$

We may set the origin at r_0 so that $r' \equiv r - r_0$ and $\ddot{r}' = \ddot{r}$. Collecting the coefficients of r' and ignoring any constant and higher-order terms, we have:

$$\begin{aligned} (m+M)\ddot{r}' &= -\frac{3L^2}{mr_0^4}r' \\ \ddot{r}' &= -\frac{3M}{m}\frac{g}{r_0}r' \end{aligned} \tag{20}$$

which is simple harmonic motion with angular frequency:

$$\Omega = \sqrt{\frac{3M}{m}\frac{g}{r_0}} \tag{21}$$

For $m \gg M$, this tends to zero as the effect of M can be ignored; for $m \ll M$, small oscillation approximation is no longer true; and for $M = 2m$, this becomes $\sqrt{6g/r_0}$.

(b) The coordinates of the mass are $X(t) = l\sin\theta + A\cos\omega t$ and $Y(t) = -l\cos\theta$ so that the Lagrangian is:

$$\frac{\mathcal{L}}{ml^2} = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\kappa^2\sin^2\omega t - \kappa\sin\omega t\cos\theta\dot{\theta} + \frac{g}{l}\cos\theta \tag{22}$$

where $\kappa \equiv \omega A/l$.

The equation of motion is:

$$\ddot{\theta} = \kappa\omega\cos\omega t\cos\theta - \frac{g}{l}\sin\theta \tag{23}$$

For small oscillations, we may approximate $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ so that:

$$\ddot{\theta} = \kappa\omega\cos\omega t - \frac{g}{l}\theta \tag{24}$$

This is a forced harmonic oscillator with the complementary solution:

$$\theta_c = C\cos\left(\sqrt{\frac{g}{l}}t + \phi\right) \tag{25}$$

and the particular solution:

$$\theta_p = \frac{\kappa\omega}{g/l - \omega^2}\cos\omega t \tag{26}$$

Assuming that $\omega \neq \sqrt{g/l}$, the general solution is then:

$$\theta(t) = C\cos(\omega_0 t + \phi) + \frac{A}{l}\frac{\omega^2}{\omega_0^2 - \omega^2}\cos\omega t \tag{27}$$

where $\omega_0 \equiv \sqrt{g/l}$ is the natural frequency and C and ϕ are constants determined by the initial conditions.