Calculus

Problem Sheet 0 & A

Hyperbolic Functions & Differnetiation

Xin, Wenkang

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Hyperbolic Functions

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(a) Done before in Induction problem set.

(b)

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{4e^x e^{-x}}{4} = 1 \tag{1}$$

$$\cosh^2 x + \sinh^2 x = \frac{2(e^{2x} + e^{-2x})}{4} = \cosh 2x \tag{2}$$

$$2\cosh x \sinh x = \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} = \sinh 2x \tag{3}$$

$$1 - \tanh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} = \operatorname{sech}^2 x \tag{4}$$

$$\coth^2 x - 1 = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x - e^{-x})^2} = \frac{4}{(e^x - e^{-x})^2} = \operatorname{csch}^2 x \tag{5}$$

(c) These identities are very similar to the trigonometric identities, except for some sign changes.

(d)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \frac{e^x - (-e^{-x})}{2} = \cosh x \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \frac{e^x + (-e^{-x})}{2} = \sinh x \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \frac{\cosh x \times \cosh x - \sinh x \times \sinh x}{\cosh^2 x} = \operatorname{sech}^2 x \tag{8}$$

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Differentiation

A1 Practice in differentiation

(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x e^{x^3} = \cos x e^{x^3} + \sin x e^{x^3}(3x^2) = e^{x^3}(\cos x + 3x^2\sin x) \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{x^3\sin x} = e^{x^3\sin x}\left[(3x^2)\sin x + x^3\cos x\right] = x^2 e^{x^3\sin x}(3\sin x + x\cos x) \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[\cosh(1/x)\right] = \frac{1}{\cosh(1/x)}\sinh(1/x)\left(-\frac{1}{x^2}\right) = -\frac{\tanh(1/x)}{x^2} \tag{11}$$

(b)

$$\begin{aligned}\cos y &= x\\ -\sin y\frac{\mathrm{d}y}{\mathrm{d}x} &= 1\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-x^2}}\end{aligned} \tag{12}$$

$$\begin{aligned}\tanh y &= \frac{x}{1+x}\\ \operatorname{sech}^2 y\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1+x-x}{(1+x)^2}\\ \left[1-\left(\frac{x}{1+x}\right)^2\right]\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{(1+x)^2}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1+2x}\end{aligned} \tag{13}$$

(c)

$$\begin{aligned}\ln y &= \cos x\ln x\\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos x}{x} - \sin x\ln x\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= x^{\cos x}\left(\frac{\cos x}{x} - \sin x\ln x\right)\end{aligned} \tag{14}$$

$$\begin{aligned}y &= \frac{2\ln x}{\ln 10}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2}{\ln 10}\frac{1}{x}\end{aligned} \tag{15}$$

(d)

$$\begin{aligned} ye^{y\ln x} &= x^2+y^2 \\ \frac{\mathrm{d}y}{\mathrm{d}x}e^{y\ln x} + ye^{y\ln x}\left(\ln x\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}\right) &= 2x+2y\frac{\mathrm{d}y}{\mathrm{d}x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x-y^2e^{y\ln x}/x}{e^{y\ln x}(1+y\ln x)-2y} \end{aligned} \tag{16}$$

Given $t = ax^2+bx+c$, differentiating with respect to time:

$$\begin{aligned} 1 &= 2ax\dot{x}+b\dot{x}+c \\ \dot{x} &= \frac{1-c}{2ax+b} \end{aligned} \tag{17}$$

Further differentiating:

$$\ddot{x} = -\frac{(1-c)(2a)\dot{x}}{(2ax+b)^2} = -\frac{(2a)(1-c)^2}{(2ax+b)^3} \propto \dot{x}^3 \tag{18}$$

(e)

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}/\frac{\mathrm{d}x}{\mathrm{d}\theta} = \coth\theta \tag{19}$$

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)/\frac{\mathrm{d}x}{\mathrm{d}\theta} \\ &= -\frac{\operatorname{sech}^2\theta}{\tanh^2\theta}\frac{1}{\sinh\theta} \\ &= -\frac{1}{\sinh^3\theta} \end{aligned} \tag{20}$$

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}/\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{t^{m-1}-t^{-m-1}}{1-t^{-2}} = m\frac{t^m-t^{-m}}{t-t^{-1}} \tag{21}$$

Therefore:

$$(x^2-4)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = m^2(t^2+2+t^{-2}-4)\frac{t^{2m}-2+t^{-2m}}{t^2-2+t^{-2}} = m^2(y^2-4) \tag{22}$$

Differentiating this result:

$$
\begin{aligned}
2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2(x^2-4)\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2m^2y\frac{\mathrm{d}y}{\mathrm{d}x} \\
(x^2-4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - m^2y &= 0
\end{aligned}
\tag{23}
$$

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A2 Differentiation from first principles

From the definition of differentiation:

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\begin{aligned}
\frac{\mathrm{d}(x^2)}{\mathrm{d}x} &= \lim_{\delta x\to 0}\frac{(x+\delta x)^2 - x^2}{\delta x} \\
&= \lim_{\delta x\to 0}\frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x} \\
&= \lim_{\delta x\to 0}(2x+\delta x) \\
&= 2x
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{\mathrm{d}(\sin x)}{\mathrm{d}x} &= \lim_{\delta x\to 0}\frac{(\sin x + \delta x - \sin x)}{\delta x} \\
&= \lim_{\delta x\to 0}\frac{\sin x\cos\delta x + \cos x\sin\delta x - \sin x}{\delta x} \\
&= \sin x\lim_{\delta x\to 0}\frac{\cos\delta x - 1}{\delta x} + \cos x\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x} \\
&= \cos x
\end{aligned}
\tag{25}
$$

where the following standard limit results have been used:

$$
\lim_{x\to 0}\frac{\cos x - 1}{x} = 0,\ \lim_{x\to 0}\frac{\sin x}{x} = 1 \tag{26}
$$

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A3 Integration as the inverse of differentiation

A standard Riemann integral can be approximated by a Riemann sum, which is the summing of the areas of individual rectangular strips to approximate the area under a curve. Consider a infinitesimally thin rectangular strip of width δx, located at the interval $[x, x+\delta x)$. Let this strip take the height $f(x)$. The sum of all these strips approximates the area under curve. Then the individual strip leads to an infinitesimal contribution to the integral $\delta I(x) = f(x)\delta x$, or rearranging and taking the limit $\delta x \to 0$, $\mathrm{d}I(x)/\mathrm{d}x = f(x)$.

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A4 Derivatives of inverse functions

(a) Let $y = f(x)$ so that $x = f^{-1}(y)$, assuming that f has an inverse. Have:

$$\begin{aligned} f^{-1}[f(x)] &= x \\ \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}[f(x)] &= 1 \\ (f^{-1})'[f(x)]f'(x) &= 1 \\ f'(x) &= \frac{1}{(f^{-1})'[f(x)]} \end{aligned} \tag{27}$$

Or using the fact that $\mathrm{d}y/\mathrm{d}x = f'(x)$ and $\mathrm{d}x/\mathrm{d}y = (f^{-1})'(y) = (f^{-1})'[f(x)]$:

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1} \tag{28}$$

(b)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}y^2} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}\right]\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}\right]\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1} = -\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\Big/\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{3} \tag{29}$$

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A5 Changing variables in differential equations

(a) If $z = yx^2$, then:

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} &= x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy \\ \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} &= x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2(y+x)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y \end{aligned} \tag{30}$$

Substitution into the original differential equation yields:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + 3\frac{\mathrm{d}z}{\mathrm{d}x} + 2z = x \tag{31}$$

(b) If $t = \sqrt{x}$, then:

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{1}{2}x^{-1/2} = \frac{1}{2t} \\ \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} &= -\frac{1}{4}x^{-3/2} = -\frac{1}{4t^3} \end{aligned} \tag{32}$$

Substitution into the original equation yields:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} - 6y = e^{3t} \tag{33}$$

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A6 Leibnitz theorem

The Leibnitz theorem, which can be proven through induction, states that for n-times differentiable functions f and g, the derivatives of their product is given by:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \tag{34}$$

Given this result, the 8th derivative of $x^2 \sin x$ is simply:

$$\begin{aligned} \left(x^2 \sin x\right)^{(8)} &= (\sin x)^{(8)}(x^2) + 8(\sin x)^{(7)}(2x) + 28(\sin x)^{(6)}(2) \\ &= \left(x^2 - 56\right)\sin x - 16x\cos x \end{aligned} \tag{35}$$

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A7 Special points of a function

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+2x+6) - (x-1)(2x+2)}{(x^2+2x+6)^2} = \frac{-x^2+2x+8}{(x^2+2x+6)^2} = -\frac{(x-2)(x+4)}{(x^2+2x+6)^2} \tag{36}$$

Therefore, the two stationary points are at $(-4, -5/2)$ and $(2, 1/14)$. The function has no singularity as the denominator is always greater than zero. The root of the function is at $x = 1$. The function also approaches $0_\pm$ as x approaches $\pm\infty$. Hence it is inferred that $y(x)$ has the range $\boxed{[-5/2, 1/14]}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+2x+6)-(x-1)(2x+2)}{(x^2+2x+6)^2} = \frac{-x^2+2x+8}{(x^2+2x+6)^2} = -\frac{(x+2)(x-4)}{(x^2+2x+6)^2}$$

Therefore, the two stationary points are at $(-2, -1/2)$ and $(4, 1/10)$.

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3-2x}{(4+3x-x^2)^2} \tag{37}$$

Therefore, the single stationary point is at $(3/2, 4/25)$. The function has two singularities (vertical asymptotes) at $x = -4$ and $x = 2$ and it approaches 0_- as as x approaches $\pm\infty$. Hence it is inferred that $y(x)$ has the range $\boxed{(-\infty, 0) \cup [4/25, +\infty)}$.

The function has two singularities (vertical asymptotes) at $x = -1$ and $x = 4$.

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8\frac{(15+8\tan^2 x)\cos x - 16\tan x \sec^2 x \sin x}{(15+8\tan^2 x)^2} \tag{38}$$

Focusing on the numerator, to have zero derivative, we have the equation:

$$7\cos^4 x + 24\cos^2 x - 16 = 0 \tag{39}$$

with the condition $\cos x \neq 0$ and $15 + 8\tan^2 x \neq 0$.

Solving the equation yields $\cos x = \pm\sqrt{4/7}$, where the function takes the value $\frac{8}{21}\sqrt{3/7}$.

The function has zero points at $x = n\pi$ and approaches zero at $x = (1/2+n)\pi$, where n is an integer. In between these zero points, the function achieves extrema according to the previous quadratic equation. Therefore, the range of the function is $[-a, a]$, where $a = \frac{8}{21}\sqrt{3/7}$.