Symmetry and Relativity

Problem Set 3

Kinematics and Dynamics in Special Relativity

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November 25, 2024

1 Evaluation of derivatives for a four-vector field

We define the covariant derivative:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{1}$$

and the contravariant derivative:

$$\partial^\mu = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \tag{2}$$

We have the following calculations:

$$\partial_\lambda X^\lambda = 4 \tag{3}$$

$$\partial^\mu (X_\lambda X^\lambda) = 0 \tag{4}$$

as $X_\lambda X^\lambda$ is an invariant scalar.

$$\partial^\mu \partial_\mu X_\nu X^\nu = 0 \tag{5}$$

as $X_\nu X^\nu$ is an invariant scalar.

$$\partial^\mu X^\nu = \begin{pmatrix} \partial^0 X^0 & \partial^0 X^1 & \partial^0 X^2 & \partial^0 X^3 \\ \partial^1 X^0 & \partial^1 X^1 & \partial^1 X^2 & \partial^1 X^3 \\ \partial^2 X^0 & \partial^2 X^1 & \partial^2 X^2 & \partial^2 X^3 \\ \partial^3 X^0 & \partial^3 X^1 & \partial^3 X^2 & \partial^3 X^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta^{\mu\nu} \tag{6}$$

$$\begin{aligned} \partial_\lambda F^\lambda &= 2\partial_\lambda X^\lambda + \partial_\lambda \left[K^\lambda (X_\nu X^\nu)\right] \\ &= 8 \end{aligned} \tag{7}$$

since the second term is a constant scalar.

$$\partial^\mu (\partial_\lambda F^\lambda) = \partial^\mu (8) = 0 \tag{8}$$

$$\partial^\mu \partial_\mu \sin (K_\lambda X^\lambda) = 0 \tag{9}$$

as $\sin\left(K_\lambda X^\lambda\right)$ is an invariant scalar.

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2 Properties of spacetime intervals

(a) A time-like 4-vector satisfies $A^\mu A_\mu < 0$. We attempt to find a Lorentz transformation such that $A'^i = 0$ for $i = 1, 2, 3$. It is customary to align our axes such that the only non-zero components of A^μ are A^0 and A^1. Then we have:

$$A^\mu A_\mu = -A^0 A^0 + A^1 A^1 < 0 \tag{10}$$

Consider the Lorentz transformation along the x-axis:

$$\begin{aligned} A'^0 &= \gamma(A^0 - \beta A^1) \\ A'^1 &= \gamma(A^1 - \beta A^0) \\ A'^2 &= A^2 = 0 \\ A'^3 &= A^3 = 0 \end{aligned} \tag{11}$$

A'^1 can be made zero by choosing $\beta = A^1/A^0 < 1$. This can always be done since $A^0 > A^1$. This shows that there always exists a frame in which a time-like 4-vector has zero spatial components.

(b) Suppose that in a reference frame S, the two events A^μ and B^μ are simultaneous. Then we have $A^0 = B^0$ or their separation satisfies:

$$X^\mu = A^\mu - B^\mu = (0, X^1, X^2, X^3) \tag{12}$$

That is, the two events are separated by a space-like interval.

Suppose on the contrary that in a reference frame S, the two events A^μ and B^μ are separated by a time-like interval. Then their separation satisfies $X^\mu X_\mu > 0$. We attempt to find a Lorentz transformation such that $X'^0 = 0$. Again, we align our axes such that the only non-zero components of X^μ are X^0 and X^1. Then we have:

$$X^\mu X_\mu = -X^0 X^0 + X^1 X^1 > 0 \tag{13}$$

and a Lorentz transformation along the x-axis:

$$X'^0 = \gamma(X^0 - \beta X^1) \tag{14}$$

which is an equality satisfied by $\beta = X^0/X^1 < 1$.

This shows that two events are simultaneous if and only if they are separated by a space-like interval in some reference frame.

(c) Consider two events separated by a time-like interval $X^\mu X_\mu < 0$. We define the temporal order of the events as the sign of X^0. For this sign to be conserved under Lorentz transformations, we consider a general Lorentz transformation:

$$X'^0 = \gamma(X^0 - \beta_x X^1 - \beta_y X^2 - \beta_z X^3) \tag{15}$$

Again, we can align our axes such that the only non-zero components of X^μ are X^0 and X^1. Then we have $X'^0 = \gamma(X^0 - \beta_x X^1)$, where $\beta_x < 1$. Let us demand $\gamma = \Lambda^0_0 > 0$.

We have demanded X^μ to be time-like, that is $X^0X^0 > X^1X^1$. Say $X^0 > 0$. It is clear that there is no way to choose β_x or X^1 such that $X'^0 < 0$. The same argument can be made for $X^0 < 0$. This shows that the temporal order of two events separated by a time-like interval is always conserved under Lorentz transformations satisfying $\gamma > 0$. The complete opposite happens if $\gamma < 0$, which yields an improper Lorentz transformation.

(d) If two 4-vectors are orthogonal, they satisfy:

$$A^\mu B_\mu = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3 = 0 \tag{16}$$

Suppose that A^μ is time-like so that $A^0A_0 > A^1A_1 + A^2A_2 + A^3A_3$. We can orient our axes such that only A^0 and A^1 are non-zero. Then we have:

$$A^\mu B_\mu = A^0B^0 - A^1B^1 = 0 \tag{17}$$

This implies:

$$(B^0)^2 = \left(\frac{A^1}{A^0}\right)^2 (B^1)^2 < (B^1)^2 \tag{18}$$

which implies that B^μ must be space-like.

(e) Suppose instead that A^μ is light-like so that $A^0A_0 = A^1A_1$. We have:

$$(B^0)^2 = \left(\frac{A^1}{A^0}\right)^2 (B^1)^2 = (B^1)^2 \tag{19}$$

Unless $B^2 = B^3 = 0$, B^μ will be time-like. If $B^2 = B^3 = 0$, then B^μ is light-like.

(f) The world line of an observer is necessarily time-like. Thus, if some displacement vector is orthogonal to the world line, it must be space-like. By previous results, there must exist a frame in which the displacement vector has zero temporal component, i.e. the events are simultaneous. The position of the observer can be described by a time-like 4-vector A^μ and the displacement vector by a space-like 4-vector X^μ. Since they are orthogonal, they satisfy $A^\mu X_\mu = 0$. Let us align our axes such that only X^0 and X^1 are non-zero. Then we have:

$$A^\mu X_\mu = A^0 X^0 - A^1 X^1 = 0 \tag{20}$$

Then we choose a Lorentz transformation along the x-axis such that:

$$X'^0 = \gamma(X^0 - \beta X^1) = 0 \tag{21}$$

This is a condition on $\beta = X^0/X^1$. But from the orthogonality condition, we have $A^0 X^0 = A^1 X^1$ so that $\beta = A^1/A^0$. Let us consider transforming the observer's position vector:

$$\begin{aligned} A'^0 &= \gamma(A^0 - \beta A^1) \\ A'^1 &= \gamma(A^1 - \beta A^0) = 0 \\ A'^2 &= A^2 \\ A'^3 &= A^3 \end{aligned} \tag{22}$$

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3 Motion under a constant force

(a) The particle in the constant electric field has the momentum:

$$P^\mu = (E/c, qE_x t) \tag{23}$$

where $E = mc^2 + qE_x x$ is the total energy of the particle. Consider the invariant length of the momentum:

$$P^\mu P_\mu = -\frac{E^2}{c^2} + q^2 E_x^2 t^2 = -m^2 c^2 \tag{24}$$

Expanding the expression for E and simplifying, we have the equation of motion:

$$x^2 + \frac{2mc^2}{qE_x} x - c^2 t^2 = 0 \tag{25}$$

which is equivalent to:

$$\left(x + \frac{c^2}{\alpha}\right)^2 - c^2 t^2 = \frac{c^4}{\alpha^2} \tag{26}$$

where we claim that $\alpha = qE_x/m$ is the proper acceleration of the particle.

(b) Consider the acceleration of the particle in frame S.

$$\begin{aligned} A^\mu &= \frac{1}{m}\gamma_v \frac{dP^\mu}{dt} \\ &= \frac{1}{m}\gamma_v \left(\frac{qE_x}{c}\frac{\mathrm{d}x}{\mathrm{d}t}, qE_x\right) \end{aligned} \tag{27}$$

Now we have the proper acceleration:

$$a_0^2 = A^\mu A_\mu = \gamma_v^2 \left[-\frac{q^2E_x^2}{m^2c^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{q^2E_x^2}{m^2}\right] = \frac{q^2E_x^2}{m^2} \tag{28}$$

which confirms our previous claim.

(c) At point $P(ct, x)$, we construct an instantaneous inertial frame S' with its origin coinciding with that of S. As observed in S', the the point P has the coordinates:

$$P' = \gamma_v(ct - \beta x, x - \beta ct) \tag{29}$$

so $ct'_p = \gamma_v(ct - \beta x)$.

As observed in S', the point $A(0, -c^2/\alpha)$ has the coordinates:

$$A' = \gamma_v\left(\beta\frac{c^2}{\alpha}, -\frac{c^2}{\alpha}\right) \tag{30}$$

so $ct'_A = \gamma_v\beta(c^2/\alpha)$.

But we know from the equation of motion that $c^2/\alpha + x = (c^4/\alpha^2 + c^2t^2)^{1/2}$. Therefore, $ct'_A = -\gamma_v\beta(c^4/\alpha^2 + c^2t^2)^{1/2}$.

Let us compute the 'slope' of the line $P'A'$ in the S' frame:

$$\frac{ct'_P - ct'_A}{x'_P - x'_A} = \frac{ct - \beta x - \beta c^2/\alpha}{x - \beta ct + c^2/\alpha} \tag{31}$$

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4 Circular motion in a magnetic field

(a) For a pure magnetic field, the Lagrangian of a charged particle is:

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} + q\mathbf{v}\cdot\mathbf{A} \tag{32}$$

where $\mathbf{A}$ only has spatial components.

We identify the 4-momentum:

$$P^\mu = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \gamma m\mathbf{v} + q\mathbf{A} \tag{33}$$

The Hamiltonian (energy) of the particle is:

$$\begin{aligned}\mathcal{H} &= \mathbf{v}\cdot\frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \mathcal{L}\\ &= \gamma m v^2 + q\mathbf{v}\cdot\mathbf{A} + \frac{1}{\gamma}mc^2 - q\mathbf{v}\cdot\mathbf{A}\\ &= \gamma m v^2 + \frac{1}{\gamma}mc^2\\ &= \gamma mc^2\end{aligned} \tag{34}$$

The equation of motion is given by the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m\mathbf{v} + q\mathbf{A}\right) = \frac{\partial}{\partial \mathbf{r}}\left(q\mathbf{A}\cdot\mathbf{v}\right) \tag{35}$$

Consider the vector identity:

$$\nabla\left(\mathbf{A}\cdot\mathbf{v}\right) = (\mathbf{A}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{A}) = (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times(\nabla\times\mathbf{A}) \tag{36}$$

since the derivatives are taken with $\mathbf{v}$ held constant.

On the other hand:

$$\frac{\mathrm{d}A^i}{\mathrm{d}t} = \frac{\partial A^i}{\partial t} + \frac{\partial A^i}{\partial x^j}v^j = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} \tag{37}$$

Combining, we have the equation of motion:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -q\frac{\partial \mathbf{A}}{\partial t} + q\mathbf{v}\times(\nabla\times\mathbf{A}) \tag{38}$$

For a constant magnetic field $\mathbf{H}$, we can write $\mathbf{A} = \frac{1}{2}\mathbf{H} \times \mathbf{r}$ and $\mathbf{H} = \nabla \times \mathbf{A}$. The equation of motion becomes:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{H} \tag{39}$$

Finally, consider the time derivative of p^2:

$$\frac{\mathrm{d}p^2}{\mathrm{d}t} = 2\mathbf{p} \cdot \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0 \tag{40}$$

This is because in the expression for $\dot{\mathbf{p}}$, the right-hand-side is orthogonal to $\mathbf{v}$ and thus $\mathbf{p}$. Thus, we have proven that the particle has constant momentum and velocity. Since the energy is a function of γ, the energy is also constant.

(b) We rewrite the equation of motion in terms of the velocity:

$$\frac{E}{c^2}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{H} \tag{41}$$

This is solved by:

$$\begin{aligned} v_x &= \frac{v_0}{\sqrt{2}}\cos(\omega t) \\ v_y &= \frac{v_0}{\sqrt{2}}\sin(\omega t) \end{aligned} \tag{42}$$

where $\omega = qc^2H/E$ and $v_0 = pc^2/E$.

Integrating the equations of motion, we have a condition on the radius of the circular motion:

$$r = \frac{v_0}{\omega} = \frac{pc^2}{qc^2H} = \frac{p}{qH} \tag{43}$$

(c) In S, the 4-velocity of the particle is $U^\mu = \gamma_v(c, v_x, v_y, 0)$. In some other frame S' moving at velocity u in the x-direction, the 4-velocity becomes $U'^\mu = \gamma_{v'}(c, v_x', v_y', 0)$. But by the Lorentz transformation:

$$U'^0 = \gamma_u\gamma_v(c - \beta_u v_x) = \gamma_{v'}c \tag{44}$$

Since v_x is varying, $\gamma_{v'}$ is also varying, implying that S' does not observe the particle to be moving at constant velocity. This is in a moving frame, the field is not purely magnetic so the particle may accelerate.

(d) For synchronisation, we need $\pi/\omega = (1/f)/2$, or:

$$2\pi f = \omega = \frac{qc^2 H}{E} \tag{45}$$

Since E is increasing after every period, we demand E to be increasing in tandem. This is achieved by having a non-uniform magnetic field $H(r)$ as a function of the energy E. The initial energy is ΔE so the magnetic field at the centre is:

$$H_i = \frac{2\pi f \Delta E}{qc^2} \tag{46}$$

The final energy is E_f, leading to:

$$H_f = \frac{2\pi f E_f}{qc^2} \tag{47}$$

Upon exiting the cyclotron, the momentum of the particle satisfies $p = rqH_f$ or:

$$p = rqH_f = rq\left(\frac{2\pi f E_f}{qc^2}\right) = 2\pi c^{-2} f r E_f \tag{48}$$

so that the exit velocity is:

$$v_f = c^2 \frac{p}{E_f} = 2\pi f r = 1.2 \times 10^{12}\,\mathrm{ms}^{-1} \tag{49}$$

Each revolution leads to $2\Delta E$ increase in energy, so the total number of revolutions is:

$$N = \frac{E_f}{2\Delta E} = 684 \tag{50}$$

The total time taken is:

$$T = \frac{N}{f} = 1.14 \times 10^{-6}\,\mathrm{s} \tag{51}$$

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5 Motion in a magnetic dipole

We have the equation of motion:

$$\frac{E}{c^2}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v} \times (\nabla \times \mathbf{A}) \tag{52}$$

Consider dotting the equation with $\hat{z}$:

$$\begin{aligned}\hat{z}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &\propto \hat{z}\cdot[\mathbf{v}\times(\nabla\times\mathbf{A})] \\ &= \mathbf{v}\cdot[(\nabla\times\mathbf{A})\times\hat{z}]\end{aligned} \tag{53}$$

Consider the following vector identity:

$$\nabla(\hat{z}\cdot\mathbf{A}) = (\hat{z}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\hat{z} + \hat{z}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\hat{z}) \tag{54}$$

Now, $\mathbf{A}$ does not have a z component, so the left-hand-side is zero. On the right-hand-side, the first term is zero since $\mathbf{A}$ does not have a z component. The second and last terms are zero since $\hat{z}$ is a constant vector. Thus, we must have $\mathbf{v}\cdot(\nabla\times\mathbf{A}) = 0$. This means that $\hat{z}\cdot\dot{\mathbf{v}} = 0$ and the motion is confined to the xy-plane.

We may now use cylindrical coordinates (r, ϕ, z) to describe the motion. Consider the vector potential:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{M\hat{z}\times\mathbf{r}}{r^3} = \frac{\mu_0 M}{4\pi r^2}\hat{\phi} \tag{55}$$

We can write the Lagrangian:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + q\mathbf{v}\cdot\mathbf{A} = -mc^2\left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{1/2} + \frac{q\mu_0 M}{4\pi}\frac{1}{r}\dot{\phi} \tag{56}$$

Apparently ϕ is a cyclic coordinate so the angular momentum is conserved:

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \gamma m r^2\dot{\phi} + \frac{q\mu_0 M}{4\pi}\frac{1}{r} = \text{constant} \tag{57}$$

Apparently the energy is also conserved, so we may write $E = \gamma mc^2$ as a constant. Now consider setting $\dot{r} = 0$ so that $v^2 = r^2\dot{\phi}^2$. We have from the energy conservation:

$$\begin{aligned}\frac{m^2c^4}{E^2} &= 1 - \frac{r^2\dot{\phi}^2}{c^2} \\ \frac{m^2c^4}{E^2} &= 1 - \frac{r^2}{c^2}\left[\frac{p_\phi - \frac{q\mu_0 M}{4\pi}\frac{1}{r}}{(E/c^2)r}\right]^2\end{aligned} \tag{58}$$

From the initial condition, we know $p_\phi = q\mu_0 M/(4\pi r_0)$ and $E = mc^2(1 - v_0^2/c^2)^{-1/2}$. We can solve for r:

$$r = r_0 \left(1 \pm \frac{4\pi}{q\mu_0 M}\frac{E}{c}\sqrt{1 - \frac{m^2c^4}{E^2}}\right)^{-1} \tag{59}$$

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6 Relativistic rocket

We know that rapidity is additive under Lorentz transformations. Consider the rocket in its instantaneous rest frame S' which moves at velocity v relative to the Earth S. In a short time interval $\mathrm{d}\tau$, the rocket acquires a velocity $\mathrm{d}v$ relative to S'. Consider its 4-momentum, which was initially $\mathbf{P} = (m(\tau)c, 0)$. Since the fuel is turned into photons, the 4-momentum becomes:

$$\mathbf{P} + \mathrm{d}\mathbf{P} = (mc - \mathrm{d}mc, \mathrm{d}mc) \tag{60}$$

Therefore, we have the increment in velocity:

$$\begin{aligned} \gamma_v(m - \mathrm{d}m)\mathrm{d}v &= \mathrm{d}mc \\ \mathrm{d}v &\approx \frac{c}{m}\mathrm{d}m \end{aligned} \tag{61}$$

On the other hand, the $\mathrm{d}\rho = \tanh^{-1}(\mathrm{d}v/c) \approx \mathrm{d}v/c$ so that:

$$\mathrm{d}\rho \approx \frac{\mathrm{d}m}{m} \tag{62}$$

where $\mathrm{d}m$ is a negative quantity.

This equation is also satisfied in the Earth frame as rapidity is additive. Changing to velocity in the Earth frame, we have:

$$\frac{\mathrm{d}M}{M} = -\frac{\mathrm{d}v/c}{1 - v^2/c^2} \tag{63}$$

Integrating, we have:

$$\rho = \tanh^{-1}\left(\frac{v}{c}\right) = -\ln\left(\frac{M_f}{M_i}\right) \tag{64}$$

The energy of the rocket is:

$$E = \frac{mc^2}{1-\rho^2} \tag{65}$$

When $M_f = 0$, we have $\rho \to \infty$ so the energy of the rocket tends to zero.