Ordinary Differential Equations

Problem Set 4

Systems of Linear ODEs

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Minimal Set

All C_i appearing in the following solutions are arbitrary constants unless otherwise stated.

4.1

Adding the equations yield first order equation in terms of $(x+y)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x+y)+(a-b)(x+y)=f \tag{1}$$

Solving this equation yields:

$$x+y=C_1e^{(b-a)t}-\frac{f}{b-a} \tag{2}$$

Subtracting the two equations yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}(x-y)+(a+b)(x-y)=f \tag{3}$$

and:

$$x-y=C_2e^{-(b+a)t}+\frac{f}{b+a} \tag{4}$$

Hence the solutions are:

$$\begin{aligned} x(t)&=C_1e^{(b-a)t}+C_2e^{-(b+a)t}+\frac{af}{a^2-b^2}\\ y(t)&=C_1e^{(b-a)t}-C_2e^{-(b+a)t}+\frac{bf}{a^2-b^2} \end{aligned} \tag{5}$$

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4.2

The system can be written in the following matrix form:

$$\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}y'\\z'\end{pmatrix}=\begin{pmatrix}-4&-10\\-1&1\end{pmatrix}\begin{pmatrix}y\\z\end{pmatrix}+\begin{pmatrix}2\\-3\end{pmatrix} \tag{6}$$

Uncoupling by left multiplying both sides with the inverse of the first matrix:

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 2 & 12 \\ -3 & -11 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} -8 \\ 5 \end{pmatrix} \tag{7}$$

Solving for the eigenvalues and eigenvectors of the coefficient matrix gives $\lambda_1 = -2$ corresponding to eigenvector $\mathbf{v}_1 = (-3, 1)^\intercal$ and $\lambda_2 = -7$ corresponding to eigenvector $\mathbf{v}_2 = (-4, 3)^\intercal$. By observation, a particular solutions is $(y, z) = (-2, 1)$. Thus the general solution has the form:

$$\begin{pmatrix} y \\ z \end{pmatrix} = C_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-2x} + C_2 \begin{pmatrix} -4 \\ 3 \end{pmatrix} e^{-7x} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{8}$$

Substituting the initial conditions $C_1 = 12/5$ and $C_2 = 1/5$. Thus the solutions are:

$$\begin{aligned} y(x) &= -\frac{36}{5}e^{-2x} - \frac{4}{5}e^{-7x} - 2 \\ z(x) &= \frac{12}{5}e^{-2x} + \frac{3}{5}e^{-7x} + 1 \end{aligned} \tag{9}$$

$\bullet$

4.3

(i) The system can be written in the following matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{10}$$

Solving for the eigenvalues and eigenvectors of the coefficient matrix gives $\lambda_1 = 0$ corresponding to eigenvector $\mathbf{v}_1 = (0, 1, 1)^\intercal$, $\lambda_2 = -1 + i$ corresponding to $\mathbf{v}_2 = (-2, i, 1)^\intercal$ and $\lambda_3 = -1 - i$ corresponding to $\mathbf{v}_3 = (-2, -i, 1)^\intercal$. Taking the Wronskian of the fundamental matrix at $t = 0$ yields a value of $4i$, verifying the linear independence of the solutions. Thus the solution has the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ i \\ 1 \end{pmatrix} e^{-(1-i)t} + C_3 \begin{pmatrix} -2 \\ -i \\ 1 \end{pmatrix} e^{-(1+i)t} \tag{11}$$

Taking only the reals part yields the solutions:

$$\begin{aligned} x(t) &= -2(C_2+C_3)e^{-t}\cos t \\ y(t) &= C_1 + (C_3 - C_2)e^{-t}\sin t \\ z(t) &= C_1 + (C_2+C_3)e^{-t}\cos t \end{aligned} \tag{12}$$

(ii) The system can be written in the following matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{13}$$

Solving for the eigenvalues and eigenvectors of the coefficient matrix gives a two-fold degenerate $\lambda_1 = 3$ corresponding to eigenvectors $\mathbf{v}_{11} = (1,0,1)^\intercal$ and $\mathbf{v}_{12} = (1,1,0)^\intercal$ and $\lambda_2 = 2$ corresponding to $\mathbf{v}_2 = (1,1,1)^\intercal$. Thus, the solutions are:

$$\begin{aligned} x(t) &= (C_1+C_2)e^{3t} + C_3e^{2t} \\ y(t) &= C_2e^{3t} + C_3e^{2t} \\ z(t) &= C_1e^{3t} + C_3e^{2t} \end{aligned} \tag{14}$$

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4.4

(i) The system can be written in the following matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 4 & 3 & -3 \\ -3 & -2 & 3 \\ 3 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} e^{-t} \tag{15}$$

Solving for the eigenvalues and eigenvectors of the coefficient matrix gives a two-fold degenerate $\lambda_1 = 1$ corresponding to eigenvectors $\mathbf{v}_{11} = (1,0,1)^\intercal$ and $\mathbf{v}_{12} = (-1,1,0)^\intercal$ and $\lambda_2 = -2$ corresponding to $\mathbf{v}_2 = (1,-1,1)^\intercal$. Suppose that a particular solution has the form $(x,y,z) = (A,B,C)e^{-t}$. We have:

$$-\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 4 & 3 & -3 \\ -3 & -2 & 3 \\ 3 & 3 & -2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \tag{16}$$

Solving this system of equations gives $A = 3$, $B = -3$ and $C = 2$. Therefore, the solutions are:

$$\begin{aligned} x(t) &= (C_1 - C_2)e^t + C_3 e^{-2t} + 3e^{-t} \\ y(t) &= C_2 e^t - C_3 e^{-2t} - 3e^{-t} \\ z(t) &= C_1 e^t + C_3 e^{-2t} + 2e^{-t} \end{aligned} \tag{17}$$

(ii) The system can be written in the following matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -5 & 1 & -2 \\ -1 & -1 & 0 \\ 6 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \sinh t + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cosh t \tag{18}$$

or writing the hyperbolic functions in terms of exponentials:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -5 & 1 & -2 \\ -1 & -1 & 0 \\ 6 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1/2 \\ 3/2 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 1/2 \\ -1/2 \\ -1 \end{pmatrix} e^{-t} \tag{19}$$

Solving for the eigenvalues and eigenvectors of the coefficient matrix gives $\lambda_1 = -2$ corresponding to eigenvector $\mathbf{v}_1 = (-1, -1, 1)^\intercal$, $\lambda_2 = -1 + i$ corresponding to $\mathbf{v}_2 = (-1, -i, 2)^\intercal$ and $\lambda_3 = -1 - i$ corresponding to $\mathbf{v}_3 = (-1, i, 2)^\intercal$. Suppose that a particular solution has the form $(x, y, z) = (A, B, C)e^t + (D, E, F)e^{-t}$. We have:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -5 & 1 & -2 \\ -1 & -1 & 0 \\ 6 & -2 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} + \begin{pmatrix} 1/2 \\ 3/2 \\ -1 \end{pmatrix} \tag{20}$$

and

$$-\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} -5 & 1 & -2 \\ -1 & -1 & 0 \\ 6 & -2 & 2 \end{pmatrix} \begin{pmatrix} D \\ E \\ F \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \\ -1 \end{pmatrix} \tag{21}$$

Solving these two systems of equations gives $(A, B, C) = (-D, -E, -F) = (1/2, 1/2, -1)$. This implies that the particular solution is $(x, y, z) = (1, 1, -2)\sinh t$. Taking the real parts only, the solutions are:

$$\begin{aligned} x(t) &= -C_1 e^{-2t} - (C_2 + C_3)e^{-t} \cos t + \sinh t \\ y(t) &= -C_1 e^{-2t} + (C_2 + C_3)e^{-t} \cos t + \sinh t \\ z(t) &= C_1 e^{-2t} - 2(C_2 - C_3)e^{-t} \cos t - 2\sinh t \end{aligned} \tag{22}$$

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