EDGE EFFECTS ON LOCAL STATISTICS IN LATTICE DIMERS: A STUDY OF THE AZTEC DIAMOND (FINITE CASE)

BY

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THESIS

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1. Introduction

A *tiling* of a checkerboard with dominoes is a way of putting dominoes on the board so that no square of the board is uncovered and no two dominoes overlap. Given a local pattern (see figure **??** for examples), a location in the board, and the shape and size of the board, how many tilings of the board have the given pattern at the given location? (Alternatively, we can substitute "bond" for "domino" and "particle" for "square", and ask for the probability of local patterns in a system of particles each of which bonds with exactly one of its neighbors.)

Suppose that the squares of the board are very small compared to the board itself. For some board shapes, the probability of finding a pattern at a given location will be the same for almost all locations. This is the case for the square board. (See figures **??** to **??**, where tiles are colored according to their direction and parity for the sake of clarity; see figure **??** for the coloring scheme.) There are some boards, however, for which the probability does depend on the location. Consider, for example, the *Aztec diamond*, that is, the board whose boundary is a square tilted 45 degrees (figures **??** and **??**). In random tilings of the Aztec diamond, we usually find brick-wall patterns outside the inscribed circles, and more complicated behavior inside the circle. (See figures **??** to **??**.)

The probabilities of local patterns in a rectangular board were computed recently [**?**]. Until now, there was no other board for which the probabilities of all local patterns were known. Many experiments and some important partial results [**?**] had shown that, as already stated, the probabilities of patterns in the Aztec diamond depend on location. This qualitative difference between the Aztec diamond and the rectangular board made the former as worthy of analysis as the latter. The main result of this work is an expression for the probability of any local pattern in a random tiling of the Aztec diamond. The expression is a determinant of size proportional to the number of squares in the pattern, just like Kenyon's expression [**?**] for the probabilities in the rectangural board,

Main Result 1. *The probability of a pattern covering white squares $v_1, v_2, \cdots v_k$ and black squares $w_1, w_2, \cdots w_k$ of an Aztec diamond of order n is equal to the absolute value of*

$$c(v_i, w_j)_{i,j=1,2,\cdots k}.$$

The coupling function $c(v,w)$ *at white square v and black square w is*

$$2^{-n} \sum_{j=0}^{x_i-1} \mathrm{kr}(j, n, y_i - 1)\, \mathrm{kr}(y\prime_i - 1, n-1, n-(j + x\prime_i - x_i))$$

for $x\prime_i > x_i$ and

$$-2^{-n} \sum_{j=x_i}^{n} \mathrm{kr}(j, n, y_i - 1)\, \mathrm{kr}(y\prime_i - 1, n-1, n-(j + x\prime_i - x_i))$$

for $x\prime_i \le x_i$, where (x_i, y_i) and $(x\prime_i, y\prime_i)$ are the coordinates of v and w, respectively, in the coordinate system in figure **??***, and the* Krawtchouk polynomial $\mathrm{kr}(a,b,c)$ *is the coefficient of x^a in $(1-x)^c \cdot (1+x)^{b-c}$.*

Our line of attack is as follows.

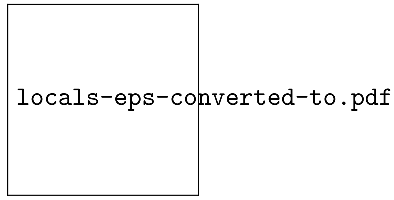

Figure 1. A few examples of local patterns

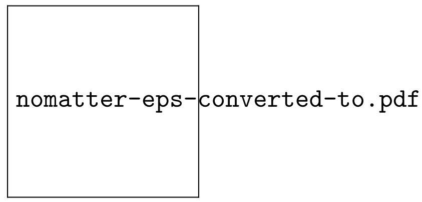

Figure 2. These two patterns have the same probability of being found in a random pattern at any given place

(1) Reduce the problem of finding probabilities of patterns to an enumerative problem;
(2) Reduce the enumerative problem to a simpler one involving Aztec diamonds with two holes rather than arbitrary even-area holes;
(3) Compute the weighted number of tilings of an Aztec diamond with two holes.

The first two steps involve known techniques, and were already considered to be a plausible strategy by other researchers. The third step is new.

Henry Cohn is currently analyzing the case of the board with infinitely small squares by approximating the sum of Krawtchouk polynomials in our main result as an integral for $n \to \infty$. His results will be presented in a later, joint version of this paper.

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FIGURE 3. Random tiling of square of side 20

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FIGURE 4. Random tiling of square of side 40

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FIGURE 5. Random tiling of square of side 80

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FIGURE 6. Shading chart

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FIGURE 7. Aztec diamond of order 4, as a board

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FIGURE 8. Aztec diamond of order 4, as a graph