Evaluation ideal and variety for a trio of error independentbinaryclassifiers

Introduction

This notebook will detail the algebraic geometry computations that take us from the “evaluation ideal” created from the voting patterns of a trio of binary classifiers to the “evaluation variety”. An evaluation ideal is a set of polynomials connecting observable voting pattern frequencies by the classifiers to unknown sample statistics of the ground truth that are our evaluation goal. We want to “grade” the classifiers using only the frequencies of their voting patterns.

That “grade” exists in sample statistics space. The test has already been taken. We have the decisions of the judges. We are faced with the task of grading them now. Not in the future, not in the past. This is another example of how the task of evaluation is much simpler than that of training. We have to estimate something that already exists, if you will. And there is only one time we have to do it. Training is much harder. You must create judges that, in the future, will behave correctly. And they have to do it many times. The task of evaluation is trivial in comparison. Why have we not conquered this much simpler space of the whole enterprise of learning?

Algebraic geometry of three error independent binary classifiers

The mathematics of algebraic evaluation is algebraic geometry. Every algebraic evaluation problem can be stated as a polynomial system relating observable decision events to unknown sample statistics. Here we are going to define that polynomial system assuming that the classifiers made errors independently on the sample. This is “the spherical cow” of Evaluation Land - the simplifying assumption that allows you to proceed forward and carry out computations that give you insight into the original problem. Workers in Training Land also have a preferred spherical cow - “consider an identically, independently drawn sample”. It may take some getting used to this new cow if you are a new visitor from Training Land.

The evaluation ideal of three error independent binary classifiers

Algebraic geometry is mainly the study of the connection between sets of polynomials and geometric objects in the variable space of those polynomials. The sets of polynomials are called “polynomial ideals". A set of linear equations is also a polynomial ideal. We define the "evaluation ideal" of our evaluation to be,

In[1]:=
```
Clear[MakeIndependentVotingIdeal]
MakeIndependentVotingIdeal[{i_, j_, k_}] :=
```
$$\{P_\alpha P_{i,\alpha} P_{j,\alpha} P_{k,\alpha} + (1-P_\alpha)(1-P_{i,\beta})(1-P_{j,\beta})(1-P_{k,\beta}) - f_{\alpha,\alpha,\alpha},$$
$$P_\alpha P_{i,\alpha} P_{j,\alpha}(1-P_{k,\alpha}) + (1-P_\alpha)(1-P_{i,\beta})(1-P_{j,\beta}) P_{k,\beta} - f_{\alpha,\alpha,\beta},$$
$$P_\alpha P_{i,\alpha}(1-P_{j,\alpha}) P_{k,\alpha} + (1-P_\alpha)(1-P_{i,\beta}) P_{j,\beta}(1-P_{k,\beta}) - f_{\alpha,\beta,\alpha},$$
$$P_\alpha P_{i,\alpha}(1-P_{j,\alpha})(1-P_{k,\alpha}) + (1-P_\alpha)(1-P_{i,\beta}) P_{j,\beta} P_{k,\beta} - f_{\alpha,\beta,\beta},$$
$$P_\alpha(1-P_{i,\alpha}) P_{j,\alpha} P_{k,\alpha} + (1-P_\alpha) P_{i,\beta}(1-P_{j,\beta})(1-P_{k,\beta}) - f_{\beta,\alpha,\alpha},$$
$$P_\alpha(1-P_{i,\alpha}) P_{j,\alpha}(1-P_{k,\alpha}) + (1-P_\alpha) P_{i,\beta}(1-P_{j,\beta}) P_{k,\beta} - f_{\beta,\alpha,\beta},$$
$$P_\alpha(1-P_{i,\alpha})(1-P_{j,\alpha}) P_{k,\alpha} + (1-P_\alpha) P_{i,\beta} P_{j,\beta}(1-P_{k,\beta}) - f_{\beta,\beta,\alpha},$$
$$P_\alpha(1-P_{i,\alpha})(1-P_{j,\alpha})(1-P_{k,\alpha}) + (1-P_\alpha) P_{i,\beta} P_{j,\beta} P_{k,\beta} - f_{\beta,\beta,\beta}\}$$

One convention in algebraic geometry may bother you. Ultimately we are interested in the geometrical object these polynomials define in the finite space needed for evaluating three independent binary classifiers. We want to consider the points in sample statistics space where all these equations are zero. We are really interested in these equations,

In[3]:=
```
MakeIndependentVotingIdeal[{1, 2, 3}] // Map[(# == 0) &, #] &
```

Out[3]=
$$\{P_\alpha P_{1,\alpha} P_{2,\alpha} P_{3,\alpha} + (1-P_\alpha)(1-P_{1,\beta})(1-P_{2,\beta})(1-P_{3,\beta}) - f_{\alpha,\alpha,\alpha} == 0,$$
$$P_\alpha P_{1,\alpha} P_{2,\alpha}(1-P_{3,\alpha}) + (1-P_\alpha)(1-P_{1,\beta})(1-P_{2,\beta}) P_{3,\beta} - f_{\alpha,\alpha,\beta} == 0,$$
$$P_\alpha P_{1,\alpha}(1-P_{2,\alpha}) P_{3,\alpha} + (1-P_\alpha)(1-P_{1,\beta}) P_{2,\beta}(1-P_{3,\beta}) - f_{\alpha,\beta,\alpha} == 0,$$
$$P_\alpha P_{1,\alpha}(1-P_{2,\alpha})(1-P_{3,\alpha}) + (1-P_\alpha)(1-P_{1,\beta}) P_{2,\beta} P_{3,\beta} - f_{\alpha,\beta,\beta} == 0,$$
$$P_\alpha(1-P_{1,\alpha}) P_{2,\alpha} P_{3,\alpha} + (1-P_\alpha) P_{1,\beta}(1-P_{2,\beta})(1-P_{3,\beta}) - f_{\beta,\alpha,\alpha} == 0,$$
$$P_\alpha(1-P_{1,\alpha}) P_{2,\alpha}(1-P_{3,\alpha}) + (1-P_\alpha) P_{1,\beta}(1-P_{2,\beta}) P_{3,\beta} - f_{\beta,\alpha,\beta} == 0,$$
$$P_\alpha(1-P_{1,\alpha})(1-P_{2,\alpha}) P_{3,\alpha} + (1-P_\alpha) P_{1,\beta} P_{2,\beta}(1-P_{3,\beta}) - f_{\beta,\beta,\alpha} == 0,$$
$$P_\alpha(1-P_{1,\alpha})(1-P_{2,\alpha})(1-P_{3,\alpha}) + (1-P_\alpha) P_{1,\beta} P_{2,\beta} P_{3,\beta} - f_{\beta,\beta,\beta} == 0\}$$

But it is a pain to carry all these equal to zero notation around. So we drop it, and prefer to work with,

In[4]:=
```
MakeIndependentVotingIdeal[{1, 2, 3}]
```

Out[4]=
$$\{P_\alpha P_{1,\alpha} P_{2,\alpha} P_{3,\alpha} + (1-P_\alpha)(1-P_{1,\beta})(1-P_{2,\beta})(1-P_{3,\beta}) - f_{\alpha,\alpha,\alpha},$$
$$P_\alpha P_{1,\alpha} P_{2,\alpha}(1-P_{3,\alpha}) + (1-P_\alpha)(1-P_{1,\beta})(1-P_{2,\beta}) P_{3,\beta} - f_{\alpha,\alpha,\beta},$$
$$P_\alpha P_{1,\alpha}(1-P_{2,\alpha}) P_{3,\alpha} + (1-P_\alpha)(1-P_{1,\beta}) P_{2,\beta}(1-P_{3,\beta}) - f_{\alpha,\beta,\alpha},$$
$$P_\alpha P_{1,\alpha}(1-P_{2,\alpha})(1-P_{3,\alpha}) + (1-P_\alpha)(1-P_{1,\beta}) P_{2,\beta} P_{3,\beta} - f_{\alpha,\beta,\beta},$$
$$P_\alpha(1-P_{1,\alpha}) P_{2,\alpha} P_{3,\alpha} + (1-P_\alpha) P_{1,\beta}(1-P_{2,\beta})(1-P_{3,\beta}) - f_{\beta,\alpha,\alpha},$$
$$P_\alpha(1-P_{1,\alpha}) P_{2,\alpha}(1-P_{3,\alpha}) + (1-P_\alpha) P_{1,\beta}(1-P_{2,\beta}) P_{3,\beta} - f_{\beta,\alpha,\beta},$$
$$P_\alpha(1-P_{1,\alpha})(1-P_{2,\alpha}) P_{3,\alpha} + (1-P_\alpha) P_{1,\beta} P_{2,\beta}(1-P_{3,\beta}) - f_{\beta,\beta,\alpha},$$
$$P_\alpha(1-P_{1,\alpha})(1-P_{2,\alpha})(1-P_{3,\alpha}) + (1-P_\alpha) P_{1,\beta} P_{2,\beta} P_{3,\beta} - f_{\beta,\beta,\beta}\}$$

Dropping the notation has no effect. Algebraic manipulations of the above set (multiplying them together, etc.) would be equivalent to polynomials of zero for points that satisfied the input set.

One “forest for the trees” note: the above ideal is one many possible ones. Algebraic evaluation is very much like data streaming algorithms. You are creating a sketch of the decisions by an ensemble of noisy judges. In the case of binary classification considered here, that “data sketch” is the frequency of their item-by-item voting patterns. Other polynomial systems are possible even for independent binary classifiers. We could be trying to evaluate sample statistics that look at how judges evaluated two different sample items, for example.

The evaluation variety of three error independent binary classifiers

The goal of algebraic evaluation is to obtain “grades” for the noisy judges on unlabeled data. We know the types of grades we want. We want their exact grades. Not spreads of where we think their grade is - no probabilistic solutions! That is not quite what we get in algebraic evaluation. The mathematical object you get as a grade is actually a geometric object in sample statistics space. The true grade lies on that geometric object. Mathematicians call the geometrical objects defined by polynomials - varieties. You will see that for error-independent binary classifiers that geometric object is almost what we want - a collection of two points. For correlated classifiers those points “bloom out”. It is an unsolved problem in algebraic evaluation to characterize the surface for correlated classifiers - but it does exist! The evaluation ideal for any set of correlated classifiers can be written down and it defines, by construction, an evaluation variety that is guaranteed to contain the true evaluation point for the classifiers. All that is left when you build the evaluation ideal is to figure out what that surface is and whether it would be useful to your AI safety task.

The ground truth for the performance of three noisy binary classifiers

Before continuing to present the formalism of algebraic evaluation, let’s do a quick end run to our goal - the grade for noisy binary classifiers. I’ll use the UCI Adult run that can be found in the Python code - AlgebraicEvaluation.py. The input to algebraic evaluation in our current application is the frequency of voting patterns by the classifiers. Here is how it looked for a single run of three binary classifiers on the UCI Adult dataset.

In[5]:=
```
singleEvaluationUCIAdult =
  <|0 → <|{0, 0, 0} → 715, {0, 0, 1} → 161, {0, 1, 0} → 2406, {0, 1, 1} → 455,
      {1, 0, 0} → 290, {1, 0, 1} → 94, {1, 1, 0} → 1335, {1, 1, 1} → 231|>,
    1 → <|{0, 0, 0} → 271, {0, 0, 1} → 469, {0, 1, 0} → 3395, {0, 1, 1} → 7517,
      {1, 0, 0} → 272, {1, 0, 1} → 399, {1, 1, 0} → 6377, {1, 1, 1} → 12 455|>|>
```

Out[5]=
```
<|0 → <|{0, 0, 0} → 715, {0, 0, 1} → 161, {0, 1, 0} → 2406, {0, 1, 1} → 455,
    {1, 0, 0} → 290, {1, 0, 1} → 94, {1, 1, 0} → 1335, {1, 1, 1} → 231|>,
  1 → <|{0, 0, 0} → 271, {0, 0, 1} → 469, {0, 1, 0} → 3395, {0, 1, 1} → 7517,
    {1, 0, 0} → 272, {1, 0, 1} → 399, {1, 1, 0} → 6377, {1, 1, 1} → 12 455|>|>
```

This is not a randomly selected run of binary classifiers on the UCI Adult dataset. It is being used for various reasons. It was engineered to be as close to error independence as possible. This will be discussed more later. For now, let's verify that, in fact, the classifiers are near error independence in this sample.

In[55]:=
```
evaluationGroundTruth = GTClassifiers[singleEvaluationUCIAdult]
```

Out[55]=

$$\left\langle\left| P_{\alpha} \to \frac{5687}{36\,842},\ P_{1,\alpha} \to \frac{3737}{5687},\ P_{2,\alpha} \to \frac{1260}{5687},\ P_{3,\alpha} \to \frac{4746}{5687},\ P_{1,\beta} \to \frac{6501}{10\,385},\ P_{2,\beta} \to \frac{29\,744}{31\,155},\ P_{3,\beta} \to \frac{4168}{6231},\ \Gamma_{1,2,\alpha} \to \frac{273\,192}{32\,341\,969},\ \Gamma_{1,3,\alpha} \to \frac{13\,325}{32\,341\,969},\ \Gamma_{2,3,\alpha} \to -\frac{264\,525}{32\,341\,969},\ \Gamma_{1,2,\beta} \to \frac{2\,204\,576}{323\,544\,675},\ \Gamma_{1,3,\beta} \to -\frac{79\,682}{12\,941\,787},\ \Gamma_{2,3,\beta} \to \frac{94\,508}{38\,825\,361},\ \Gamma_{1,2,3,\alpha} \to \frac{452\,568\,508}{183\,928\,777\,703},\ \Gamma_{1,2,3,\beta} \to -\frac{27\,265\,589}{134\,400\,457\,995} \right|\right\rangle$$

The evaluation ground truth is what we want. It was calculated above by cheating - we have the by-label counts so we can easily compute ALL the sample statistics required to explain exactly the frequency patterns we observe when we DO NOT have the knowledge of the true labels. Note that the evaluation ground truth are integer ratios. The exact sample statistics for evaluation are a subset of the real numbers. This is crucial. This allows algebraic evaluators to get closer to the true grades. Integer ratios are also in the field of algebraic numbers. But the field of algebraic numbers is less dense than reals!

This evaluation also reminds the reader of why evaluation is easier than training. There are no unknown unknowns. Evaluation computes sample statistics. The space of sample statistics required to explain observable voting patterns is finite and complete. You may not know what all these sample statistics are, but they are all you would need to know to describe the frequency of their observed decisions.

It is hard to compare integer ratios so let's get the floating point approximation to the evaluation to confirm the claim that these classifiers are nearly error independent.

In[56]:=
```
evaluationGroundTruth // N
```

Out[56]=

$$\left\langle\left| P_{\alpha} \to 0.154362,\ P_{1,\alpha} \to 0.657113,\ P_{2,\alpha} \to 0.221558,\ P_{3,\alpha} \to 0.834535,\ P_{1,\beta} \to 0.625999,\ P_{2,\beta} \to 0.95471,\ P_{3,\beta} \to 0.668913,\ \Gamma_{1,2,\alpha} \to 0.00844698,\ \Gamma_{1,3,\alpha} \to 0.000412003,\ \Gamma_{2,3,\alpha} \to -0.008179,\ \Gamma_{1,2,\beta} \to 0.00681382,\ \Gamma_{1,3,\beta} \to -0.00615695,\ \Gamma_{2,3,\beta} \to 0.00243418,\ \Gamma_{1,2,3,\alpha} \to 0.00246056,\ \Gamma_{1,2,3,\beta} \to -0.000202868 \right|\right\rangle$$

One can see that all the pair error correlation terms ($\Gamma_{i,j,\mathrm{label}}$) are less that 1% absolute. This is encouraging. Since these classifiers are already so near error independence on the sample, will the using an evaluation ideal that assumes they are error-independent work well enough? Let's try it by using Mathematica's built-in algebraic geometry algorithms.

Evaluation with Mathematica's Solve function

Since we are trying to simulate evaluation on unlabeled data, we need to project the by-true-label counts into the counts that are observed when we have no knowledge of the true labels. This is easy. For binary classification, the observed counts for a voting pattern is the sum of the voting pattern counts when you know the true label. For example,

In[57]:=
```
sizeOfTestSet =
 singleEvaluationUCIAdult // Values // Map[Values, #] & // Flatten // Total
fα,β,α →
 Sum[singleEvaluationUCIAdult[label][{0, 0, 0}], {label, {0, 1}}] / sizeOfTestSet
```

Out[57]=

36 842

Out[58]=

$$f_{\alpha,\beta,\alpha} \to \frac{493}{18\,421}$$

So the “data sketch” for the evaluation of these three noisy binary classifiers is given by

In[60]:=
```
evaluationDataSketch =
 Transpose@{{fα,α,α, fα,α,β, fα,β,α, fα,β,β, fβ,α,α, fβ,α,β, fβ,β,α, fβ,β,β},
    Map[(Sum[singleEvaluationUCIAdult[label][#], {label, {0, 1}}] / sizeOfTestSet) &,
     {{0, 0, 0}, {0, 0, 1}, {0, 1, 0}, {0, 1, 1},
      {1, 0, 0}, {1, 0, 1}, {1, 1, 0}, {1, 1, 1}}]} //
  Rule @@ # & /@ # &
```

Out[60]=

$$\left\{f_{\alpha,\alpha,\alpha} \to \frac{493}{18\,421},\ f_{\alpha,\alpha,\beta} \to \frac{315}{18\,421},\ f_{\alpha,\beta,\alpha} \to \frac{5801}{36\,842},\ f_{\alpha,\beta,\beta} \to \frac{3986}{18\,421},\ f_{\beta,\alpha,\alpha} \to \frac{281}{18\,421},\ f_{\beta,\alpha,\beta} \to \frac{493}{36\,842},\ f_{\beta,\beta,\alpha} \to \frac{3856}{18\,421},\ f_{\beta,\beta,\beta} \to \frac{6343}{18\,421}\right\}$$

The goal of our current evaluation is to get estimates for the following sample statistics,

In[61]:=
```
evaluationVariables = MakeIndependentVotingIdeal[{1, 2, 3}] //
     Variables /@ # & // Flatten // DeleteDuplicates // Cases[#, Except[f__]] & // Sort
```

Out[61]=

$$\{P_{\alpha},\ P_{1,\alpha},\ P_{1,\beta},\ P_{2,\alpha},\ P_{2,\beta},\ P_{3,\alpha},\ P_{3,\beta}\}$$

Mathematica uses algebraic geometry under the hood of the Solve function to give us the grades for these classifiers.

In[64]:=
```
independentModelEvaluation = Solve[
   (MakeIndependentVotingIdeal[{1, 2, 3}] // Map[(# == 0) &, #] &) /.
    evaluationDataSketch,
   evaluationVariables] // Map[Association, #] &
```

Out[64]=

$$\left\{\left\langle\left| P_\alpha \to \frac{61\,316\,911\,076\,911\,789 - 2\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{122\,633\,822\,153\,823\,578}, \right.\right.\right.$$
$$P_{1,\alpha} \to \frac{197\,818\,302\,948\,040\,811 + 3\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{375\,985\,460\,508\,686\,570},$$
$$P_{1,\beta} \to \frac{3\left(59\,389\,052\,520\,215\,253 + \sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}\right)}{375\,985\,460\,508\,686\,570},$$
$$P_{2,\alpha} \to \frac{23\,470\,130\,463\,167\,807 + \sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{136\,288\,278\,313\,807\,950},$$
$$P_{2,\beta} \to \frac{112\,818\,147\,850\,640\,143 + \sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{136\,288\,278\,313\,807\,950},$$
$$P_{3,\alpha} \to \frac{209\,373\,072\,434\,759\,059 + 3\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{412\,438\,820\,078\,205\,386},$$
$$\left. P_{3,\beta} \to \frac{203\,065\,747\,643\,446\,327 + 3\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{412\,438\,820\,078\,205\,386} \right|\rangle,$$
$$\left\langle\left| P_\alpha \to \frac{61\,316\,911\,076\,911\,789 + 2\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{122\,633\,822\,153\,823\,578}, \right.\right.$$
$$P_{1,\alpha} \to \frac{197\,818\,302\,948\,040\,811 - 3\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{375\,985\,460\,508\,686\,570},$$
$$P_{1,\beta} \to \frac{3\left(59\,389\,052\,520\,215\,253 - \sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}\right)}{375\,985\,460\,508\,686\,570},$$
$$P_{2,\alpha} \to \frac{23\,470\,130\,463\,167\,807 - \sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{136\,288\,278\,313\,807\,950},$$
$$P_{2,\beta} \to \frac{112\,818\,147\,850\,640\,143 - \sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{136\,288\,278\,313\,807\,950},$$
$$P_{3,\alpha} \to \frac{209\,373\,072\,434\,759\,059 - 3\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{412\,438\,820\,078\,205\,386},$$
$$\left.\left. P_{3,\beta} \to \frac{203\,065\,747\,643\,446\,327 - 3\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{412\,438\,820\,078\,205\,386} \right|\rangle\right\}$$

Incredible! It is astonishing that more ML experts do not know about this. Consider what just happened. In essentially instantaneous time you are able evaluate these three noisy binary classifiers. Let's confirm that by timing the evaluation Mathematica carries out for us.

In[63]:=
```
Timing[Solve[
   (MakeIndependentVotingIdeal[{1, 2, 3}] // Map[(# == 0) &, #] &) /.
    evaluationDataSketch,
   evaluationVariables];]
```
Out[63]=
```
{0.027575, Null}
```

And look at the information we are immediately getting on the quality of the evaluation. Remember that the exact grades for these classifiers are integer ratios. The evaluation we got assuming that they were error independent is not telling us that. Consider the algebraic evaluator’s answer for the prevalence of the least likely label in the UCI Adult dataset - the alpha/0 label. We get two point answers for where the true prevalence must be,

In[65]:=
```
Map[#[Pα] &, independentModelEvaluation]
```
Out[65]=

$$\left\{\frac{61\,316\,911\,076\,911\,789 - 2\,\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{122\,633\,822\,153\,823\,578},\ \frac{61\,316\,911\,076\,911\,789 + 2\,\sqrt{416\,629\,916\,124\,502\,529\,599\,755\,188\,035\,849}}{122\,633\,822\,153\,823\,578}\right\}$$

So the evaluation ideal for error independent binary classifiers is represented in sample statistics space by a geometrical object that consists of two point solutions to the evaluation statistics we are looking for. This is the geometrical representation of the decoding ambiguity of evaluation. In fact, it is never possible to know, with true certainty, absent any other outside knowledge, the ground truth values for the evaluation. You can take this computation to be the proof of that. If error independent judges cannot do it, no judges will ever do so either.

This decoding ambiguity freezes academic ML researchers. The fact that two, not one solution, is returned by the evaluator confuses people that are not familiar with another area that shows a deep connection between pure mathematics and engineering: error-correcting codes. If you understand how error-correcting codes also have decoding ambiguity and yet are ubiquitous in signal processing engineering, you understand how decoding ambiguity in algebraic evaluation is a non-problem for engineers. This point is hard for computer science academics to get. Later, I’ll return to the practical significance that algebraic evaluation enables error-correcting algorithms. For now note that I have so far shown connections of Algebraic Evaluation with three other areas of research interest: algebraic geometry, data streaming algorithms and now error-correcting codes. I am not done. Algebraic evaluation has more goodies in store for us.

Algebraic evaluation connects mathematical field theory to AI safety. The number field you use to carry your AI safety computations matters. Algebraic numbers are more useful than real numbers. The evaluation above is a simple demonstration of this. The error independent evaluation model must be wrong. We can tell that it is wrong because it did not give us integer ratios. There are unresolved square roots in its output. This is gold in safety engineering contexts.

Knowing that you are flying blind is priceless

How good is the numerical estimate by the error independent evaluator?

So let’s see how well the error independent evaluator estimated the accuracy of the UCI Adult binary classifiers.

In[66]:=
```
N@independentModelEvaluation
```

Out[66]=

$$\{\langle|P_\alpha \to 0.167114, P_{1,\alpha} \to 0.688997, P_{1,\beta} \to 0.636731, P_{2,\alpha} \to 0.321977, P_{2,\beta} \to 0.977558, P_{3,\alpha} \to 0.656116, P_{3,\beta} \to 0.640823|\rangle, \langle|P_\alpha \to 0.832886, P_{1,\alpha} \to 0.363269, P_{1,\beta} \to 0.311003, P_{2,\alpha} \to 0.0224423, P_{2,\beta} \to 0.678023, P_{3,\alpha} \to 0.359177, P_{3,\beta} \to 0.343884|\rangle\}$$

Here is where we get to see why decoding is many times trivial in the real world. This is no different than decoding of error-correcting codes being trivial in the real world. You do here exactly what you would do for error correcting codes. When you decode in error-correcting codes you have multiple possible solutions if a detectable error has occurred. The default choice is ALWAYS least bit errors. Error-correcting codes are engineered so this is almost always true. It works for inter-planetary communications and your computer. Same thing in Algebraic Evaluation. Its engineering context usually has enough outside context to allow you to decode the right evaluation for noisy binary classifiers.

Let’s talk through one example of how this decoding choice could be done - you have very good knowledge of the prevalence of labels. For example, you could trying to evaluate DNA sequencers and you want to estimate their error rates. Since you are most likely handling Earth DNA, it would be a simple calibrating step to check which prevalence solution is closer to the known frequency distribution of DNA bases. That simple.

Another example applies to a business that is using AI to discover a rare, valuable thing. Like Google trying to find pages where users will click on ads. The click rate in the internet is about 1/1000 on a good website. The “it will be clicked” label will be rare in whatever ad campaign you run. This is the case in UCI Adult. The 0 label is the rare one. So we just choose the first solution.

In[71]:=
```
evaluationAlgebraicGuess = First@independentModelEvaluation;
N@evaluationAlgebraicGuess
```

Out[72]=

$$\langle|P_\alpha \to 0.167114, P_{1,\alpha} \to 0.688997, P_{1,\beta} \to 0.636731, P_{2,\alpha} \to 0.321977, P_{2,\beta} \to 0.977558, P_{3,\alpha} \to 0.656116, P_{3,\beta} \to 0.640823|\rangle$$

Now let’s look at the ground truth for this single run of three binary classifiers on a UCI Adult test set.

In[67]:=
```
N@evaluationGroundTruth
```

Out[67]=

$\langle| P_{\alpha} \to 0.154362, P_{1,\alpha} \to 0.657113, P_{2,\alpha} \to 0.221558, P_{3,\alpha} \to 0.834535, P_{1,\beta} \to 0.625999, P_{2,\beta} \to 0.95471, P_{3,\beta} \to 0.668913, \Gamma_{1,2,\alpha} \to 0.00844698, \Gamma_{1,3,\alpha} \to 0.000412003, \Gamma_{2,3,\alpha} \to -0.008179, \Gamma_{1,2,\beta} \to 0.00681382, \Gamma_{1,3,\beta} \to -0.00615695, \Gamma_{2,3,\beta} \to 0.00243418, \Gamma_{1,2,3,\alpha} \to 0.00246056, \Gamma_{1,2,3,\beta} \to -0.000202868 |\rangle$

Wow! Look at the closeness of the independent model estimates. Let’s pretty print the comparison.

In[87]:=
```
Column[{"Algebraic evaluation,assuming error
    independendence, of three binary classifiers on UCI Adult",
  Grid[Prepend[Transpose@{evaluationVariables, Map[{evaluationGroundTruth@#,
        N@evaluationGroundTruth@#} &, evaluationVariables],
      Map[{evaluationAlgebraicGuess@#, N@evaluationAlgebraicGuess@#} &,
       evaluationVariables]}, {"Evaluation Statistic", "Correct", "Estimated"}],
   Dividers → All]}, Dividers → All, Alignment → Center]
```

Out[87]=

Algebraic evaluation,assuming error independendence, of three binary classifiers on UCI Adult

Evaluation Statistic	Correct	Estimated
P_{α}	$\left\{\frac{5687}{36842}, 0.154362\right\}$	$\left\{\left(61316911076911789 - 2 \times \sqrt{416629916124502529599755188035849}\right) / 122633822153823578, 0.167114\right\}$
$P_{1,\alpha}$	$\left\{\frac{3737}{5687}, 0.657113\right\}$	$\left\{\frac{197818302948040811+3\sqrt{416629916124502529599755188035849}}{375985460508686570}, 0.688997\right\}$
$P_{1,\beta}$	$\left\{\frac{6501}{10385}, 0.625999\right\}$	$\left\{\frac{3\left(59389052520215253+\sqrt{416629916124502529599755188035849}\right)}{375985460508686570}, 0.636731\right\}$
$P_{2,\alpha}$	$\left\{\frac{1260}{5687}, 0.221558\right\}$	$\left\{\frac{23470130463167807+\sqrt{416629916124502529599755188035849}}{136288278313807950}, 0.321977\right\}$
$P_{2,\beta}$	$\left\{\frac{29744}{31155}, 0.95471\right\}$	$\left\{\frac{112818147850640143+\sqrt{416629916124502529599755188035849}}{136288278313807950}, 0.977558\right\}$
$P_{3,\alpha}$	$\left\{\frac{4746}{5687}, 0.834535\right\}$	$\left\{\frac{209373072434759059+3\sqrt{416629916124502529599755188035849}}{412438820078205386}, 0.656116\right\}$
$P_{3,\beta}$	$\left\{\frac{4168}{6231}, 0.668913\right\}$	$\left\{\frac{203065747643446327+3\sqrt{416629916124502529599755188035849}}{412438820078205386}, 0.640823\right\}$

The above table illustrates why algebraic numbers are more useful in an evaluation context than real ones. All real numbers look the same. Here we see that the correct answers look very much like the algebraic evaluation output when you represent them as floating point numbers. The difference between the correct performance of the classifiers (always an integer ratio) and the output of the error independent evaluator are obvious when you express them as algebraic numbers.

The exact polynomial formulation of voting patterns for arbitrarily correlated classifiers

The main topic of this notebook is the error independent evaluator. It is the easiest algebraic evaluator you can build. But the reader should not think that algebraic evaluation is inexact when you have correlated classifiers. Exact polynomial formulations of arbitrarily correlated classifiers exist. This is significant for anyone that worries about AI safety. There are no unknown unknowns in evaluations of finite samples. Unlike the much harder task of training noisy judges, evaluation of these judges is much easier. The algebraic evaluator is dumb. It has no knowledge of the world or the experts. It just has to estimate sample statistics. But these statistics exist in a finite space that can be universally characterized for ALL evaluations. Estimating a sample statistic is not as hard as, say, making future predictions about that statistic. All the statistics needed for three correlated binary classifiers are:

In[89]:=
```
Keys@evaluationGroundTruth // Grid[{#}, Dividers → All] &
```

Out[89]=

P_α	$P_{1,\alpha}$	$P_{2,\alpha}$	$P_{3,\alpha}$	$P_{1,\beta}$	$P_{2,\beta}$	$P_{3,\beta}$	$\Gamma_{1,2,\alpha}$	$\Gamma_{1,3,\alpha}$	$\Gamma_{2,3,\alpha}$	$\Gamma_{1,2,\beta}$	$\Gamma_{1,3,\beta}$	$\Gamma_{2,3,\beta}$	$\Gamma_{1,2,3,\alpha}$	$\Gamma_{1,2,3,\beta}$

Here is the polynomial set, based on these statistics, that generates the evaluation ideal for arbitrarily correlated classifiers,

In[328]:=

unknownSideOfEvaluationIdealCorrelatedBinaryClassifiers =

$$\{P_\alpha\left(P_{1,\alpha}P_{2,\alpha}P_{3,\alpha}+P_{3,\alpha}\Gamma_{1,2,\alpha}+P_{2,\alpha}\Gamma_{1,3,\alpha}+P_{1,\alpha}\Gamma_{2,3,\alpha}+\Gamma_{1,2,3,\alpha}\right)+P_\beta\left((1-P_{1,\beta})(1-P_{2,\beta})(1-P_{3,\beta})+(1-P_{3,\beta})\Gamma_{1,2,\beta}+(1-P_{2,\beta})\Gamma_{1,3,\beta}+(1-P_{1,\beta})\Gamma_{2,3,\beta}-\Gamma_{1,2,3,\beta}\right),$$
$$P_\alpha\left(P_{1,\alpha}P_{2,\alpha}(1-P_{3,\alpha})+(1-P_{3,\alpha})\Gamma_{1,2,\alpha}-P_{2,\alpha}\Gamma_{1,3,\alpha}-P_{1,\alpha}\Gamma_{2,3,\alpha}-\Gamma_{1,2,3,\alpha}\right)+P_\beta\left((1-P_{1,\beta})(1-P_{2,\beta})P_{3,\beta}+P_{3,\beta}\Gamma_{1,2,\beta}-(1-P_{2,\beta})\Gamma_{1,3,\beta}-(1-P_{1,\beta})\Gamma_{2,3,\beta}+\Gamma_{1,2,3,\beta}\right),$$
$$P_\alpha\left(P_{1,\alpha}(1-P_{2,\alpha})P_{3,\alpha}-P_{3,\alpha}\Gamma_{1,2,\alpha}+(1-P_{2,\alpha})\Gamma_{1,3,\alpha}-P_{1,\alpha}\Gamma_{2,3,\alpha}-\Gamma_{1,2,3,\alpha}\right)+P_\beta\left((1-P_{1,\beta})P_{2,\beta}(1-P_{3,\beta})-(1-P_{3,\beta})\Gamma_{1,2,\beta}+P_{2,\beta}\Gamma_{1,3,\beta}-(1-P_{1,\beta})\Gamma_{2,3,\beta}+\Gamma_{1,2,3,\beta}\right),$$
$$P_\alpha\left(P_{1,\alpha}(1-P_{2,\alpha})(1-P_{3,\alpha})-(1-P_{3,\alpha})\Gamma_{1,2,\alpha}-(1-P_{2,\alpha})\Gamma_{1,3,\alpha}+P_{1,\alpha}\Gamma_{2,3,\alpha}+\Gamma_{1,2,3,\alpha}\right)+P_\beta\left((1-P_{1,\beta})P_{2,\beta}P_{3,\beta}-P_{3,\beta}\Gamma_{1,2,\beta}-P_{2,\beta}\Gamma_{1,3,\beta}+(1-P_{1,\beta})\Gamma_{2,3,\beta}-\Gamma_{1,2,3,\beta}\right),$$
$$P_\alpha\left((1-P_{1,\alpha})P_{2,\alpha}P_{3,\alpha}-P_{3,\alpha}\Gamma_{1,2,\alpha}-P_{2,\alpha}\Gamma_{1,3,\alpha}+(1-P_{1,\alpha})\Gamma_{2,3,\alpha}-\Gamma_{1,2,3,\alpha}\right)+P_\beta\left(P_{1,\beta}(1-P_{2,\beta})(1-P_{3,\beta})-(1-P_{3,\beta})\Gamma_{1,2,\beta}-(1-P_{2,\beta})\Gamma_{1,3,\beta}+P_{1,\beta}\Gamma_{2,3,\beta}+\Gamma_{1,2,3,\beta}\right),$$
$$P_\alpha\left((1-P_{1,\alpha})P_{2,\alpha}(1-P_{3,\alpha})-(1-P_{3,\alpha})\Gamma_{1,2,\alpha}+P_{2,\alpha}\Gamma_{1,3,\alpha}-(1-P_{1,\alpha})\Gamma_{2,3,\alpha}+\Gamma_{1,2,3,\alpha}\right)+P_\beta\left(P_{1,\beta}(1-P_{2,\beta})P_{3,\beta}-P_{3,\beta}\Gamma_{1,2,\beta}+(1-P_{2,\beta})\Gamma_{1,3,\beta}-P_{1,\beta}\Gamma_{2,3,\beta}-\Gamma_{1,2,3,\beta}\right),$$
$$P_\alpha\left((1-P_{1,\alpha})(1-P_{2,\alpha})P_{3,\alpha}+P_{3,\alpha}\Gamma_{1,2,\alpha}-(1-P_{2,\alpha})\Gamma_{1,3,\alpha}-(1-P_{1,\alpha})\Gamma_{2,3,\alpha}+\Gamma_{1,2,3,\alpha}\right)+P_\beta\left(P_{1,\beta}P_{2,\beta}(1-P_{3,\beta})+(1-P_{3,\beta})\Gamma_{1,2,\beta}-P_{2,\beta}\Gamma_{1,3,\beta}-P_{1,\beta}\Gamma_{2,3,\beta}-\Gamma_{1,2,3,\beta}\right),$$
$$P_\alpha\left((1-P_{1,\alpha})(1-P_{2,\alpha})(1-P_{3,\alpha})+(1-P_{3,\alpha})\Gamma_{1,2,\alpha}+(1-P_{2,\alpha})\Gamma_{1,3,\alpha}+(1-P_{1,\alpha})\Gamma_{2,3,\alpha}-\Gamma_{1,2,3,\alpha}\right)+P_\beta\left(P_{1,\beta}P_{2,\beta}P_{3,\beta}+P_{3,\beta}\Gamma_{1,2,\beta}+P_{2,\beta}\Gamma_{1,3,\beta}+P_{1,\beta}\Gamma_{2,3,\beta}+\Gamma_{1,2,3,\beta}\right)\};$$

We kept P_α and P_β to simplify the math. These are related by $P_\alpha + P_\beta = 1$ so we'll get rid of the P_β variable when we do our computation. We have this unknown side completely written in a gibberish of sample statistics. What is the value of all these polynomials when we plug in the true evaluation values for our working UCI Adult evaluation run?

In[329]:=

```
(unknownSideOfEvaluationIdealCorrelatedBinaryClassifiers /. {Pβ → (1 - Pα)}) /.
 evaluationGroundTruth
```

Out[329]=

$$\left\{\frac{493}{18421}, \frac{315}{18421}, \frac{5801}{36842}, \frac{3986}{18421}, \frac{281}{18421}, \frac{493}{36842}, \frac{3856}{18421}, \frac{6343}{18421}\right\}$$

This is encouraging. We know all the voting pattern frequencies are integer ratios. Do these polynomials get the exact right answer for these integer ratios? Yes. That is the claim that in Evaluation Land all evaluation ideals are exact.

In[330]:=

```
evaluationDataSketch
```

Out[330]=

$$\left\{f_{\alpha,\alpha,\alpha} \to \frac{493}{18421}, f_{\alpha,\alpha,\beta} \to \frac{315}{18421}, f_{\alpha,\beta,\alpha} \to \frac{5801}{36842}, f_{\alpha,\beta,\beta} \to \frac{3986}{18421}, f_{\beta,\alpha,\alpha} \to \frac{281}{18421}, f_{\beta,\alpha,\beta} \to \frac{493}{36842}, f_{\beta,\beta,\alpha} \to \frac{3856}{18421}, f_{\beta,\beta,\beta} \to \frac{6343}{18421}\right\}$$

Nothing like this exists in Training Land. It is impossible to devise training algorithms based on probability theory that are exact representations of all possible future data processed by AI agents. Not so in Evaluation Land. And for a trivial reason - methods of moments are always possible with finite statistics. Why is the AI community unaware of this trivial fact? Why are these exact polynomial representations not part of any textbook that claims to explain Machine Learning Theory? Evaluation is the forgotten twin of Learning. Learning is training + evaluation.

Consider what this means theoretically. We have the exact algebraic object that explains ALL evaluations of arbitrarily correlated classifiers. The unresolved problems in Algebraic Evaluation are not here. Exact representations will always be possible in the same way that moment expansions are always possible when describing sample statistics. The unresolved problems in Algebraic Evaluation lie in understanding the evaluation variety - the surface in sample statistics space that is universally guaranteed to contain the true evaluation values.

Computing the evaluation variety corresponding to the three error-independent evaluation ideal

Estimating sample label prevalence, P_α

Finally, after a long detour explaining how to calculate the Groebner basis for the independent algebraic evaluator, we get to a practical task. A simple, algebraic estimate of the unknown prevalence that we can the code back in AlgebraicEvaluation.py. We have advanced to the point of noticing that the unknown prevalence in the test sample is given by a quadratic,

$a(\ldots) * P_{\alpha}^{2} + b(\ldots) * P_{\alpha} + c(\ldots) == 0.$

In this section we want to obtain much simpler expressions for the a, b, and c coefficients. Before we start, let's pull out the a, b, and c polynomials so we can work with them directly.

In[355]:=

```
cRules = CoefficientRules[gb⟦2⟧, Pα] // Association
```

Out[355]=

$$\langle| \{2\} \to f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} + f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta}^2 -$$
$$2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \cdots 77 \cdots + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^3 +$$
$$2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^3 + f_{\beta,\beta,\beta}^4,$$
$$\{1\} \to - f_{\alpha,\beta,\beta}^2 f_{\beta,\cdots 1 \cdots \cdots 1 \cdots \alpha}^2 + \cdots 122 \cdots, \{0\} \to - f_{\alpha,\alpha,\beta}^2 f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\alpha}^2 + \cdots 1465 \cdots |\rangle$$

Size in memory: 0.5 MB | + Show more | Show all | Iconize | Store full expression in notebook

In[358]:=

```
a = cRules[{2}]
Length@a
```

Out[358]=

$$f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} + f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} -$$
$$2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} -$$
$$4 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} -$$
$$4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\alpha} + f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha}^2 - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}^2 + 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} -$$
$$2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} +$$
$$2 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} +$$
$$2 f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} +$$
$$2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} +$$
$$2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} +$$
$$2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} -$$
$$4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 -$$
$$2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 +$$
$$2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\beta}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 +$$
$$2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 +$$
$$4 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 -$$
$$2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 4 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 +$$
$$2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^3 +$$
$$2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^3 + f_{\beta,\beta,\beta}^4$$

Out[359]=

72

Our 1st set of simplifications relates to the frequency of votes from each classifiers. For example, the number of times that classifier 1 voted α is given by,

In[365]:=

```
votePatternFrequencies = Keys@evaluationDataSketch
votePatternFrequencies // Cases[#, f_{α,_,_}] & // Total
```

Out[365]=

$$\{f_{\alpha,\alpha,\alpha},\ f_{\alpha,\alpha,\beta},\ f_{\alpha,\beta,\alpha},\ f_{\alpha,\beta,\beta},\ f_{\beta,\alpha,\alpha},\ f_{\beta,\alpha,\beta},\ f_{\beta,\beta,\alpha},\ f_{\beta,\beta,\beta}\}$$

Out[366]=

$$f_{\alpha,\alpha,\alpha} + f_{\alpha,\alpha,\beta} + f_{\alpha,\beta,\alpha} + f_{\alpha,\beta,\beta}$$

Is this “polynomial” embedded in our complicated looking “a” coefficient?

In[367]:=

```
PolynomialReduce[a, {f_{α,α,α} + f_{α,α,β} + f_{α,β,α} + f_{α,β,β}}, votePatternFrequencies]
```

Out[367]=

$$\begin{aligned}
&\{\{0\},\ f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} + f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} - \\
&\quad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - \\
&\quad 4 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\alpha} + f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha}^2 - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}^2 + 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 - \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\beta}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + \\
&\quad 4 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 - \\
&\quad 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 4 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + \\
&\quad 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^3 + \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^3 + f_{\beta,\beta,\beta}^4\}
\end{aligned}$$

So, no. This linear expression is not allowing us to divide this coefficient. By algebraic symmetry, we would expect the same result for any of the other classifiers. Is this so?

In[368]:=

```
PolynomialReduce[a,
 votePatternFrequencies // Cases[#, f_,α,_] & // Total, votePatternFrequencies]
```

Out[368]=

$$
\begin{aligned}
&\{\{0\},\ f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} + f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} - \\
&\quad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - \\
&\quad 4 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\alpha} + f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha}^2 - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}^2 + 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 - \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\beta}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + \\
&\quad 4 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 - \\
&\quad 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 4 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + \\
&\quad 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^3 + \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^3 + f_{\beta,\beta,\beta}^4\}
\end{aligned}
$$

In[369]:=

```
PolynomialReduce[a,
 votePatternFrequencies // Cases[#, f_,_,α] & // Total, votePatternFrequencies]
```

Out[369]=

$$
\begin{aligned}
&\{\{0\},\ f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} + f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} - \\
&\quad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - \\
&\quad 4 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\alpha} + f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha}^2 - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}^2 + 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 - \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\beta}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + \\
&\quad 4 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 - \\
&\quad 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 4 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + \\
&\quad 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^3 + \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^3 + 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^3 + f_{\beta,\beta,\beta}^4\}
\end{aligned}
$$

Okay. Let’s move on to the other coefficients.

In[370]:=

```
b = cRules[{1}]
```

Out[370]=

$$\begin{aligned}
&-f_{\alpha,\beta,\beta}^{2} f_{\beta,\alpha,\alpha}^{2}+2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta}-f_{\alpha,\beta,\alpha}^{2} f_{\beta,\alpha,\beta}^{2}+2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha}+\\
&2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}-4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}+4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}+\\
&4 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}+4 f_{\alpha,\beta,\beta}^{2} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}+4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}+\\
&4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta}^{2} f_{\beta,\beta,\alpha}-f_{\alpha,\alpha,\beta}^{2} f_{\beta,\beta,\alpha}^{2}+4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}^{2}-4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}+\\
&2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}-\\
&2 f_{\alpha,\beta,\beta}^{2} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^{2} f_{\beta,\beta,\beta}+2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}-2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}-\\
&2 f_{\alpha,\beta,\alpha}^{2} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}-2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}-2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}-\\
&2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}-2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta}^{2} f_{\beta,\beta,\beta}+2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\alpha,\beta}^{2} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}-\\
&2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}-\\
&2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}+\\
&4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}-2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha}^{2} f_{\beta,\beta,\beta}-f_{\beta,\beta,\beta}^{2}+2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^{2}-f_{\alpha,\alpha,\beta}^{2} f_{\beta,\beta,\beta}^{2}+\\
&2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^{2}-2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^{2}-f_{\alpha,\beta,\alpha}^{2} f_{\beta,\beta,\beta}^{2}+2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^{2}-2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^{2}-\\
&2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^{2}-f_{\alpha,\beta,\beta}^{2} f_{\beta,\beta,\beta}^{2}+2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^{2}-2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^{2}-\\
&2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^{2}-4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^{2}-f_{\beta,\alpha,\alpha}^{2} f_{\beta,\beta,\beta}^{2}+2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^{2}-2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^{2}-\\
&4 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^{2}-2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^{2}-2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^{2}-f_{\beta,\alpha,\beta}^{2} f_{\beta,\beta,\beta}^{2}+\\
&2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^{2}-4 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^{2}-2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^{2}-2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^{2}-\\
&2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^{2}-2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^{2}-f_{\beta,\beta,\alpha}^{2} f_{\beta,\beta,\beta}^{2}+2 f_{\beta,\beta,\beta}^{3}-2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^{3}-\\
&2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^{3}-2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^{3}-2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^{3}-2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^{3}-2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^{3}-f_{\beta,\beta,\beta}^{4}
\end{aligned}$$

In[371]:=

```
PolynomialReduce[b,
 votePatternFrequencies // Cases[#, f_{_,α,_}] & // Total, votePatternFrequencies]
```

Out[371]=

$$
\begin{aligned}
&\{\{0\}, -f^2_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\alpha} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} - f^2_{\alpha,\beta,\alpha} f^2_{\beta,\alpha,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} + \\
&\quad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \\
&\quad 4 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f^2_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \\
&\quad 4 f_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - f^2_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\alpha} - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 2 f^2_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - \\
&\quad 2 f^2_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f^2_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} - f^2_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} + \\
&\quad 2 f_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} - f^2_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} - \\
&\quad 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} - f^2_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} - f^2_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - \\
&\quad 4 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - f^2_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} + \\
&\quad 2 f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - 4 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - \\
&\quad 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - f^2_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} + 2 f^3_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f^3_{\beta,\beta,\beta} - \\
&\quad 2 f_{\alpha,\beta,\alpha} f^3_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f^3_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\alpha} f^3_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\beta} f^3_{\beta,\beta,\beta} - 2 f_{\beta,\beta,\alpha} f^3_{\beta,\beta,\beta} - f^4_{\beta,\beta,\beta}\}
\end{aligned}
$$

This did not work out. Should we be looking at the frequency of β voting, instead?

In[373]:=

```
PolynomialReduce[b,
 votePatternFrequencies // Cases[#, f_{_,β,_}] & // Total, votePatternFrequencies]
```

Out[373]=

$$\{\{2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} - f_{\alpha,\beta,\alpha} f^2_{\beta,\alpha,\beta} + f_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} - f_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} - f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - f^3_{\beta,\beta,\beta}\},$$

$$-f^2_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\alpha} - 2 f^2_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} - f^2_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 2 f_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - f^2_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\alpha} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\alpha} - f^2_{\beta,\alpha,\beta} f^2_{\beta,\beta,\alpha} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f^2_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} - f^2_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} - f^2_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta}\}$$

Yes. This is helping us factor the expression. Let’s mechanize it and define the three equations we want

In[418]:=

```
rewriteRules =
 Transpose@{{f_{1,β}, f_{2,β}, f_{3,β}}, {votePatternFrequencies // Cases[#, f_{β,_,_}] & // Total,
     votePatternFrequencies // Cases[#, f_{_,β,_}] & // Total,
     votePatternFrequencies // Cases[#, f_{_,_,β}] & // Total}} // Reverse /@ # & //
  Map[Rule @@ # &, #] &
```

Out[418]=

$$\{f_{\beta,\alpha,\alpha} + f_{\beta,\alpha,\beta} + f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta} \to f_{1,\beta}, f_{\alpha,\beta,\alpha} + f_{\alpha,\beta,\beta} + f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta} \to f_{2,\beta}, f_{\alpha,\alpha,\beta} + f_{\alpha,\beta,\beta} + f_{\beta,\alpha,\beta} + f_{\beta,\beta,\beta} \to f_{3,\beta}\}$$

Now comes the fun coding part of working in Machine Evaluation Land. We have to mechanize this polynomial factorization into these simpler polynomials. Nothing like this exists in Machine Training Land when working with measure theory and probability. These mechanical transformations are dumb, purely algebraic and guaranteed to yield simpler expressions that we should be able to understand better.

```
Clear[RewritePolynomials]
RewritePolynomials[poly_, rewriteRules_] := Module[
  {rulePolys, vars, newVars},
  rulePolys = First /@ rewriteRules;
  vars = Variables /@ rulePolys // Flatten // DeleteDuplicates // Sort;
  newVars = Last /@ rewriteRules;
  PolynomialReduce[poly, rulePolys, vars] //
    (* Start putting the polynomial back together *)
    {(* The remainder part *)
      Last@#,
      Transpose@{
        (* The simplyfing vars out of the voting pattern frequency space *)
        newVars,
        (* The quotients *)
        First@#}} & //
   (* Add it all up *)
   (First@# + Total@Map[Times @@ # &, Last@#]) &]
```

In[419]:=

```
RewritePolynomials[b, rewriteRules]
```

Out[419]=

$$
\begin{aligned}
&-4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - f^2_{\beta,\beta,\beta} - 4 f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} - 4 f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - 4 f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - \\
&\quad 4 f^3_{\beta,\beta,\beta} + f_{3,\beta} \big(- f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\alpha} - f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\alpha} - f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\alpha} + 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - \\
&\qquad 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - f^2_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\qquad 2 f^2_{\beta,\beta,\beta} - f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} - f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} - f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - f^3_{\beta,\beta,\beta}\big) + \\
&\quad f_{1,\beta} \big(- f^2_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + f^2_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} + \\
&\qquad f^2_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} - \\
&\qquad 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} - f^2_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - \\
&\qquad 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} - \\
&\qquad 2 f_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} - f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} - f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - f^3_{\beta,\beta,\beta}\big) + \\
&\quad f_{2,\beta} \big(- f_{\alpha,\beta,\alpha} f^2_{\beta,\alpha,\beta} - f_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \\
&\qquad f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\qquad 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f^2_{\beta,\beta,\beta} + \\
&\qquad 2 f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} - f_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} + f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} + f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} + f^3_{\beta,\beta,\beta}\big)
\end{aligned}
$$

The code is giving us back a polynomial that now exists in a larger parameter space. We haven’t stopped yet so let’s see what happens if we repeat this indefinitely.

In[420]:=

```
FixedPoint[RewritePolynomials[#, rewriteRules] &, b]
```

Out[420]=

$$-f_{1,\beta}^2 f_{\alpha,\beta,\beta}^2 + 2 f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} - f_{2,\beta}^2 f_{\beta,\alpha,\beta}^2 + 2 f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} + 2 f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - f_{3,\beta}^2 f_{\beta,\beta,\alpha}^2 - 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} + \left(2 f_{1,\beta} - 2 f_{1,\beta}^2 + 2 f_{1,\beta} f_{2,\beta} + 2 f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \left(2 f_{2,\beta} + 2 f_{1,\beta} f_{2,\beta} - 2 f_{2,\beta}^2 + 2 f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \left(2 f_{3,\beta} + 2 f_{1,\beta} f_{3,\beta} + 2 f_{2,\beta} f_{3,\beta} - 2 f_{3,\beta}^2\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \left(-1 + 2 f_{1,\beta} - f_{1,\beta}^2 + 2 f_{2,\beta} + 2 f_{1,\beta} f_{2,\beta} - f_{2,\beta}^2 + 2 f_{3,\beta} + 2 f_{1,\beta} f_{3,\beta} + 2 f_{2,\beta} f_{3,\beta} - f_{3,\beta}^2\right) f_{\beta,\beta,\beta}^2 - 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\beta,\beta}^3$$

Are the “a” and “c” polynomials similarly simplified by these algebraic rewrite rules?

In[423]:=

```
c = cRules[{0}]
```

Out[423]=

$$-f_{\alpha,\alpha,\beta}^2 f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\alpha}^2 + f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^2 - 2 f_{\alpha,\alpha,\beta}^2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^2 - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha}^2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^2 + f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 - f_{\alpha,\alpha,\beta}^2 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 + f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha}^2 - \cdots 1 \cdots + \cdots 1437 \cdots + 3 f^5_{\cdots 1 \cdots} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^5 - 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^5 - 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^5 - 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^5 - 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^5 - 4 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^5 - f_{\beta,\beta,\beta}^6$$

Size in memory: 0.4 MB | + Show more | Show all | Iconize | Store full expression in notebook

In[424]:=

```
FixedPoint[RewritePolynomials[#, rewriteRules] &, a]
FixedPoint[RewritePolynomials[#, rewriteRules] &, c]
```

Out[424]=

$$f_{1,\beta}^2 f_{\alpha,\beta,\beta}^2 - 2 f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + f_{2,\beta}^2 f_{\beta,\alpha,\beta}^2 - 2 f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} - 2 f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + f_{3,\beta}^2 f_{\beta,\beta,\alpha}^2 + 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} + \left(-2 f_{1,\beta} + 2 f_{1,\beta}^2 - 2 f_{1,\beta} f_{2,\beta} - 2 f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \left(-2 f_{2,\beta} - 2 f_{1,\beta} f_{2,\beta} + 2 f_{2,\beta}^2 - 2 f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \left(-2 f_{3,\beta} - 2 f_{1,\beta} f_{3,\beta} - 2 f_{2,\beta} f_{3,\beta} + 2 f_{3,\beta}^2\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \left(1 - 2 f_{1,\beta} + f_{1,\beta}^2 - 2 f_{2,\beta} - 2 f_{1,\beta} f_{2,\beta} + f_{2,\beta}^2 - 2 f_{3,\beta} - 2 f_{1,\beta} f_{3,\beta} - 2 f_{2,\beta} f_{3,\beta} + f_{3,\beta}^2\right) f_{\beta,\beta,\beta}^2 + 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\beta,\beta}^3$$

Out[425]=

$$-f_{1,\beta}^2 f_{2,\beta}^2 f_{3,\beta}^2 + f_{1,\beta}^2 f_{2,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} + f_{1,\beta} f_{2,\beta}^2 f_{3,\beta} f_{\beta,\alpha,\beta} - f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + f_{1,\beta} f_{2,\beta} f_{3,\beta}^2 f_{\beta,\beta,\alpha} - f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} - f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \left(f_{1,\beta}^2 f_{2,\beta} f_{3,\beta} + f_{1,\beta} f_{2,\beta}^2 f_{3,\beta} + f_{1,\beta} f_{2,\beta} f_{3,\beta}^2\right) f_{\beta,\beta,\beta} + \left(-f_{1,\beta} f_{2,\beta} - f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \left(-f_{1,\beta} f_{2,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \left(-f_{1,\beta} f_{3,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \left(-f_{1,\beta} f_{2,\beta} - f_{1,\beta} f_{3,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\beta}^3$$

Excellent! All the polynomials for the algebraic evaluation of the prevalence on the unlabeled test data are starting to look much simpler. Let’s spin our head around algebraically and reorganize these polynomials in terms of our new vars. It may reveal more simplifications

In[427]:=

```
rewrittenCoefficients =
  Map[FixedPoint[RewritePolynomials[#, rewriteRules] &, #] &, {a, b, c}];
rewrittenCoefficients // Column[#, Dividers → All] &
```

Out[428]=

$$\begin{aligned}
&f_{1,\beta}^2 f_{\alpha,\beta,\beta}^2 - 2 f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + f_{2,\beta}^2 f_{\beta,\alpha,\beta}^2 - 2 f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} - \\
&\quad 2 f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + f_{3,\beta}^2 f_{\beta,\beta,\alpha}^2 + \\
&\quad 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} + \left(-2 f_{1,\beta} + 2 f_{1,\beta}^2 - 2 f_{1,\beta} f_{2,\beta} - 2 f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \\
&\quad \left(-2 f_{2,\beta} - 2 f_{1,\beta} f_{2,\beta} + 2 f_{2,\beta}^2 - 2 f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \left(-2 f_{3,\beta} - 2 f_{1,\beta} f_{3,\beta} - 2 f_{2,\beta} f_{3,\beta} + 2 f_{3,\beta}^2\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad \left(1 - 2 f_{1,\beta} + f_{1,\beta}^2 - 2 f_{2,\beta} - 2 f_{1,\beta} f_{2,\beta} + f_{2,\beta}^2 - 2 f_{3,\beta} - 2 f_{1,\beta} f_{3,\beta} - 2 f_{2,\beta} f_{3,\beta} + f_{3,\beta}^2\right) f_{\beta,\beta,\beta}^2 + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\beta,\beta}^3
\end{aligned}$$

$$\begin{aligned}
&-f_{1,\beta}^2 f_{\alpha,\beta,\beta}^2 + 2 f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} - f_{2,\beta}^2 f_{\beta,\alpha,\beta}^2 + \\
&\quad 2 f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} + 2 f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - f_{3,\beta}^2 f_{\beta,\beta,\alpha}^2 - \\
&\quad 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} + \left(2 f_{1,\beta} - 2 f_{1,\beta}^2 + 2 f_{1,\beta} f_{2,\beta} + 2 f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \\
&\quad \left(2 f_{2,\beta} + 2 f_{1,\beta} f_{2,\beta} - 2 f_{2,\beta}^2 + 2 f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \left(2 f_{3,\beta} + 2 f_{1,\beta} f_{3,\beta} + 2 f_{2,\beta} f_{3,\beta} - 2 f_{3,\beta}^2\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad \left(-1 + 2 f_{1,\beta} - f_{1,\beta}^2 + 2 f_{2,\beta} + 2 f_{1,\beta} f_{2,\beta} - f_{2,\beta}^2 + 2 f_{3,\beta} + 2 f_{1,\beta} f_{3,\beta} + 2 f_{2,\beta} f_{3,\beta} - f_{3,\beta}^2\right) f_{\beta,\beta,\beta}^2 - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\beta,\beta}^3
\end{aligned}$$

$$\begin{aligned}
&-f_{1,\beta}^2 f_{2,\beta}^2 f_{3,\beta}^2 + f_{1,\beta}^2 f_{2,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} + f_{1,\beta} f_{2,\beta}^2 f_{3,\beta} f_{\beta,\alpha,\beta} - f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + \\
&\quad f_{1,\beta} f_{2,\beta} f_{3,\beta}^2 f_{\beta,\beta,\alpha} - f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} - f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \\
&\quad f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \left(f_{1,\beta}^2 f_{2,\beta} f_{3,\beta} + f_{1,\beta} f_{2,\beta}^2 f_{3,\beta} + f_{1,\beta} f_{2,\beta} f_{3,\beta}^2\right) f_{\beta,\beta,\beta} + \\
&\quad \left(-f_{1,\beta} f_{2,\beta} - f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \left(-f_{1,\beta} f_{2,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad \left(-f_{1,\beta} f_{3,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad \left(-f_{1,\beta} f_{2,\beta} - f_{1,\beta} f_{3,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\beta}^3
\end{aligned}$$

In[435]:=

```
newVars = Last /@ rewriteRules
Map[CoefficientRules[#, newVars] &, rewrittenCoefficients] //
  Map[Factor, #, {2}] & // Column[#, Dividers → All] &
```

Out[435]=

$$\{f_{1,\beta},\ f_{2,\beta},\ f_{3,\beta}\}$$

Out[436]=

$$\begin{aligned}
\{&\{2,0,0\}\to\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)^2,\ \{1,1,1\}\to 4\,f_{\beta,\beta,\beta},\\
&\{1,1,0\}\to -2\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right),\\
&\{1,0,1\}\to -2\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right),\\
&\{1,0,0\}\to -2\,f_{\beta,\beta,\beta}\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right),\ \{0,2,0\}\to\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right)^2,\\
&\{0,1,1\}\to -2\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right),\ \{0,1,0\}\to -2\,f_{\beta,\beta,\beta}\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right),\\
&\{0,0,2\}\to\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right)^2,\ \{0,0,1\}\to -2\,f_{\beta,\beta,\beta}\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right),\\
&\{0,0,0\}\to 4\,f_{\alpha,\beta,\beta}\,f_{\beta,\alpha,\beta}\,f_{\beta,\beta,\alpha}+4\,f_{\alpha,\beta,\beta}\,f_{\beta,\alpha,\beta}\,f_{\beta,\beta,\beta}+4\,f_{\alpha,\beta,\beta}\,f_{\beta,\beta,\alpha}\,f_{\beta,\beta,\beta}+\\
&\quad 4\,f_{\beta,\alpha,\beta}\,f_{\beta,\beta,\alpha}\,f_{\beta,\beta,\beta}+f^2_{\beta,\beta,\beta}+4\,f_{\alpha,\beta,\beta}\,f^2_{\beta,\beta,\beta}+4\,f_{\beta,\alpha,\beta}\,f^2_{\beta,\beta,\beta}+4\,f_{\beta,\beta,\alpha}\,f^2_{\beta,\beta,\beta}+4\,f^3_{\beta,\beta,\beta}\}
\end{aligned}$$

$$\begin{aligned}
\{&\{2,0,0\}\to -\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)^2,\ \{1,1,1\}\to -4\,f_{\beta,\beta,\beta},\\
&\{1,1,0\}\to 2\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right),\ \{1,0,1\}\to 2\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right),\\
&\{1,0,0\}\to 2\,f_{\beta,\beta,\beta}\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right),\ \{0,2,0\}\to -\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right)^2,\\
&\{0,1,1\}\to 2\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right),\ \{0,1,0\}\to 2\,f_{\beta,\beta,\beta}\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right),\\
&\{0,0,2\}\to -\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right)^2,\ \{0,0,1\}\to 2\,f_{\beta,\beta,\beta}\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right),\\
&\{0,0,0\}\to -4\,f_{\alpha,\beta,\beta}\,f_{\beta,\alpha,\beta}\,f_{\beta,\beta,\alpha}-4\,f_{\alpha,\beta,\beta}\,f_{\beta,\alpha,\beta}\,f_{\beta,\beta,\beta}-4\,f_{\alpha,\beta,\beta}\,f_{\beta,\beta,\alpha}\,f_{\beta,\beta,\beta}-\\
&\quad 4\,f_{\beta,\alpha,\beta}\,f_{\beta,\beta,\alpha}\,f_{\beta,\beta,\beta}-f^2_{\beta,\beta,\beta}-4\,f_{\alpha,\beta,\beta}\,f^2_{\beta,\beta,\beta}-4\,f_{\beta,\alpha,\beta}\,f^2_{\beta,\beta,\beta}-4\,f_{\beta,\beta,\alpha}\,f^2_{\beta,\beta,\beta}-4\,f^3_{\beta,\beta,\beta}\}
\end{aligned}$$

$$\begin{aligned}
\{&\{2,2,2\}\to -1,\ \{2,1,1\}\to f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta},\ \{1,2,1\}\to f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta},\\
&\{1,1,2\}\to f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta},\ \{1,1,0\}\to -\left(\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right)\right),\\
&\{1,0,1\}\to -\left(\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right)\right),\\
&\{0,1,1\}\to -\left(\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right)\right),\\
&\{0,0,0\}\to\left(f_{\alpha,\beta,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\alpha,\beta}+f_{\beta,\beta,\beta}\right)\left(f_{\beta,\beta,\alpha}+f_{\beta,\beta,\beta}\right)\}
\end{aligned}$$

Can we simplify this further? There is a long polynomial in the first two rows above that looks like it could be simplified. Let’s try it. First, some new code to

In[484]:=

```
Clear[ToFixedPointPolynomial]
ToFixedPointPolynomial[poly_, rewriteRules_] :=
 FixedPoint[RewritePolynomials[#, rewriteRules] &, poly]
```

In[487]:=

```
rewrittenCoefficients
```

Out[487]=

$$
\begin{aligned}
&\{f_{1,\beta}^2 f_{\alpha,\beta,\beta}^2 - 2 f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + f_{2,\beta}^2 f_{\beta,\alpha,\beta}^2 - \\
&\quad 2 f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} - 2 f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + f_{3,\beta}^2 f_{\beta,\beta,\alpha}^2 + \\
&\quad 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} + \left(-2 f_{1,\beta} + 2 f_{1,\beta}^2 - 2 f_{1,\beta} f_{2,\beta} - 2 f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \\
&\quad \left(-2 f_{2,\beta} - 2 f_{1,\beta} f_{2,\beta} + 2 f_{2,\beta}^2 - 2 f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \left(-2 f_{3,\beta} - 2 f_{1,\beta} f_{3,\beta} - 2 f_{2,\beta} f_{3,\beta} + 2 f_{3,\beta}^2\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad \left(1 - 2 f_{1,\beta} + f_{1,\beta}^2 - 2 f_{2,\beta} - 2 f_{1,\beta} f_{2,\beta} + f_{2,\beta}^2 - 2 f_{3,\beta} - 2 f_{1,\beta} f_{3,\beta} - 2 f_{2,\beta} f_{3,\beta} + f_{3,\beta}^2\right) f_{\beta,\beta,\beta}^2 + \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + 4 f_{\beta,\beta,\beta}^3, \\
&\; -f_{1,\beta}^2 f_{\alpha,\beta,\beta}^2 + 2 f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} - f_{2,\beta}^2 f_{\beta,\alpha,\beta}^2 + 2 f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} + \\
&\quad 2 f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - f_{3,\beta}^2 f_{\beta,\beta,\alpha}^2 - \\
&\quad 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} + \left(2 f_{1,\beta} - 2 f_{1,\beta}^2 + 2 f_{1,\beta} f_{2,\beta} + 2 f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \\
&\quad \left(2 f_{2,\beta} + 2 f_{1,\beta} f_{2,\beta} - 2 f_{2,\beta}^2 + 2 f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad \left(2 f_{3,\beta} + 2 f_{1,\beta} f_{3,\beta} + 2 f_{2,\beta} f_{3,\beta} - 2 f_{3,\beta}^2\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad \left(-1 + 2 f_{1,\beta} - f_{1,\beta}^2 + 2 f_{2,\beta} + 2 f_{1,\beta} f_{2,\beta} - f_{2,\beta}^2 + 2 f_{3,\beta} + 2 f_{1,\beta} f_{3,\beta} + 2 f_{2,\beta} f_{3,\beta} - f_{3,\beta}^2\right) f_{\beta,\beta,\beta}^2 - \\
&\quad 4 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 - 4 f_{\beta,\beta,\beta}^3, \\
&\; -f_{1,\beta}^2 f_{2,\beta}^2 f_{3,\beta}^2 + f_{1,\beta}^2 f_{2,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} + f_{1,\beta} f_{2,\beta}^2 f_{3,\beta} f_{\beta,\alpha,\beta} - f_{1,\beta} f_{2,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + \\
&\quad f_{1,\beta} f_{2,\beta} f_{3,\beta}^2 f_{\beta,\beta,\alpha} - f_{1,\beta} f_{3,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} - f_{2,\beta} f_{3,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \\
&\quad f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + \left(f_{1,\beta}^2 f_{2,\beta} f_{3,\beta} + f_{1,\beta} f_{2,\beta}^2 f_{3,\beta} + f_{1,\beta} f_{2,\beta} f_{3,\beta}^2\right) f_{\beta,\beta,\beta} + \\
&\quad \left(-f_{1,\beta} f_{2,\beta} - f_{1,\beta} f_{3,\beta}\right) f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + \left(-f_{1,\beta} f_{2,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + \\
&\quad \left(-f_{1,\beta} f_{3,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + \\
&\quad \left(-f_{1,\beta} f_{2,\beta} - f_{1,\beta} f_{3,\beta} - f_{2,\beta} f_{3,\beta}\right) f_{\beta,\beta,\beta}^2 + f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 + f_{\beta,\beta,\beta}^3\}
\end{aligned}
$$

In[488]:=

$$
\begin{aligned}
&\mathbf{withPairRewriteRules = Map}\Big[\mathbf{ToFixedPointPolynomial}\Big[\#, \big\{ \left(f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta}\right) \to f_{1,2,\beta}, \\
&\qquad \left(f_{\beta,\alpha,\beta} + f_{\beta,\beta,\beta}\right) \to f_{1,3,\beta},\ \left(f_{\alpha,\beta,\beta} + f_{\beta,\beta,\beta}\right) \to f_{2,3,\beta}\big\}\Big]\ \&,\ \mathbf{rewrittenCoefficients},\ \{2\}\Big]
\end{aligned}
$$

Out[488]=

$$
\begin{aligned}
&\{f_{3,\beta}^2 f_{1,2,\beta}^2 - 2 f_{2,\beta} f_{3,\beta} f_{1,2,\beta} f_{1,3,\beta} + f_{2,\beta}^2 f_{1,3,\beta}^2 - 2 f_{1,\beta} f_{3,\beta} f_{1,2,\beta} f_{2,3,\beta} - 2 f_{1,\beta} f_{2,\beta} f_{1,3,\beta} f_{2,3,\beta} + \\
&\quad 4 f_{1,2,\beta} f_{1,3,\beta} f_{2,3,\beta} + f_{1,\beta}^2 f_{2,3,\beta}^2 + 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} - 2 f_{3,\beta}^2 f_{1,2,\beta} f_{\beta,\beta,\beta} + \\
&\quad \left(-2 f_{3,\beta} f_{1,2,\beta} - 2 f_{1,\beta} f_{3,\beta} f_{1,2,\beta} - 2 f_{2,\beta} f_{3,\beta} f_{1,2,\beta} + 2 f_{3,\beta}^2 f_{1,2,\beta}\right) f_{\beta,\beta,\beta} - 2 f_{2,\beta}^2 f_{1,3,\beta} f_{\beta,\beta,\beta} + \\
&\quad 4 f_{1,2,\beta} f_{1,3,\beta} f_{\beta,\beta,\beta} + \left(-2 f_{2,\beta} f_{1,3,\beta} - 2 f_{1,\beta} f_{2,\beta} f_{1,3,\beta} + 2 f_{2,\beta}^2 f_{1,3,\beta} - 2 f_{2,\beta} f_{3,\beta} f_{1,3,\beta}\right) f_{\beta,\beta,\beta} + \\
&\quad \left(2 f_{2,\beta} f_{3,\beta} f_{1,2,\beta} + 2 f_{2,\beta} f_{3,\beta} f_{1,3,\beta}\right) f_{\beta,\beta,\beta} - 2 f_{1,\beta}^2 f_{2,3,\beta} f_{\beta,\beta,\beta} + 4 f_{1,2,\beta} f_{2,3,\beta} f_{\beta,\beta,\beta} + \\
&\quad 4 f_{1,3,\beta} f_{2,3,\beta} f_{\beta,\beta,\beta} + \left(2 f_{1,\beta} f_{2,\beta} f_{1,3,\beta} + 2 f_{1,\beta} f_{2,\beta} f_{2,3,\beta}\right) f_{\beta,\beta,\beta} + \\
&\quad \left(-2 f_{1,\beta} f_{2,3,\beta} + 2 f_{1,\beta}^2 f_{2,3,\beta} - 2 f_{1,\beta} f_{2,\beta} f_{2,3,\beta} - 2 f_{1,\beta} f_{3,\beta} f_{2,3,\beta}\right) f_{\beta,\beta,\beta} + \\
&\quad \left(2 f_{1,\beta} f_{3,\beta} f_{1,2,\beta} + 2 f_{1,\beta} f_{3,\beta} f_{2,3,\beta}\right) f_{\beta,\beta,\beta} + \\
&\quad \left(-4 f_{1,2,\beta} f_{1,3,\beta} - 4 f_{1,2,\beta} f_{2,3,\beta} - 4 f_{1,3,\beta} f_{2,3,\beta}\right) f_{\beta,\beta,\beta} + f_{1,\beta}^2 f_{\beta,\beta,\beta}^2 - \\
&\quad 2 f_{1,\beta} f_{2,\beta} f_{\beta,\beta,\beta}^2 + f_{2,\beta}^2 f_{\beta,\beta,\beta}^2 - 2 f_{1,\beta} f_{3,\beta} f_{\beta,\beta,\beta}^2 - 2 f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta}^2 + \\
&\quad f_{3,\beta}^2 f_{\beta,\beta,\beta}^2 + \left(2 f_{1,\beta} - 2 f_{1,\beta}^2 + 2 f_{1,\beta} f_{2,\beta} + 2 f_{1,\beta} f_{3,\beta}\right) f_{\beta,\beta,\beta}^2 +
\end{aligned}
$$

$$\left(2\ f_{2,\beta}+2\ f_{1,\beta}\ f_{2,\beta}-2\ f_{2,\beta}^{2}+2\ f_{2,\beta}\ f_{3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+\left(2\ f_{3,\beta}+2\ f_{1,\beta}\ f_{3,\beta}+2\ f_{2,\beta}\ f_{3,\beta}-2\ f_{3,\beta}^{2}\right)\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(1-2\ f_{1,\beta}+f_{1,\beta}^{2}-2\ f_{2,\beta}-2\ f_{1,\beta}\ f_{2,\beta}+f_{2,\beta}^{2}-2\ f_{3,\beta}-2\ f_{1,\beta}\ f_{3,\beta}-2\ f_{2,\beta}\ f_{3,\beta}+f_{3,\beta}^{2}\right)\ f_{\beta,\beta,\beta}^{2}+$$
$$4\ f_{1,2,\beta}\ f_{\beta,\beta,\beta}^{2}+\left(-4\ f_{1,2,\beta}-4\ f_{1,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+4\ f_{1,3,\beta}\ f_{\beta,\beta,\beta}^{2}+\left(-4\ f_{1,2,\beta}-4\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(-4\ f_{1,3,\beta}-4\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+4\ f_{2,3,\beta}\ f_{\beta,\beta,\beta}^{2}+\left(4\ f_{1,2,\beta}+4\ f_{1,3,\beta}+4\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2},$$
$$-f_{3,\beta}^{2}\ f_{1,2,\beta}^{2}+2\ f_{2,\beta}\ f_{3,\beta}\ f_{1,2,\beta}\ f_{1,3,\beta}-f_{2,\beta}^{2}\ f_{1,3,\beta}^{2}+2\ f_{1,\beta}\ f_{3,\beta}\ f_{1,2,\beta}\ f_{2,3,\beta}+$$
$$2\ f_{1,\beta}\ f_{2,\beta}\ f_{1,3,\beta}\ f_{2,3,\beta}-4\ f_{1,2,\beta}\ f_{1,3,\beta}\ f_{2,3,\beta}-f_{1,\beta}^{2}\ f_{2,3,\beta}^{2}-4\ f_{1,\beta}\ f_{2,\beta}\ f_{3,\beta}\ f_{\beta,\beta,\beta}+$$
$$2\ f_{3,\beta}^{2}\ f_{1,2,\beta}\ f_{\beta,\beta,\beta}+\left(2\ f_{3,\beta}\ f_{1,2,\beta}+2\ f_{1,\beta}\ f_{3,\beta}\ f_{1,2,\beta}+2\ f_{2,\beta}\ f_{3,\beta}\ f_{1,2,\beta}-2\ f_{3,\beta}^{2}\ f_{1,2,\beta}\right)\ f_{\beta,\beta,\beta}+$$
$$2\ f_{2,\beta}^{2}\ f_{1,3,\beta}\ f_{\beta,\beta,\beta}-4\ f_{1,2,\beta}\ f_{1,3,\beta}\ f_{\beta,\beta,\beta}+\left(-2\ f_{2,\beta}\ f_{3,\beta}\ f_{1,2,\beta}-2\ f_{2,\beta}\ f_{3,\beta}\ f_{1,3,\beta}\right)\ f_{\beta,\beta,\beta}+$$
$$\left(2\ f_{2,\beta}\ f_{1,3,\beta}+2\ f_{1,\beta}\ f_{2,\beta}\ f_{1,3,\beta}-2\ f_{2,\beta}^{2}\ f_{1,3,\beta}+2\ f_{2,\beta}\ f_{3,\beta}\ f_{1,3,\beta}\right)\ f_{\beta,\beta,\beta}+$$
$$2\ f_{1,\beta}^{2}\ f_{2,3,\beta}\ f_{\beta,\beta,\beta}-4\ f_{1,2,\beta}\ f_{2,3,\beta}\ f_{\beta,\beta,\beta}-4\ f_{1,3,\beta}\ f_{2,3,\beta}\ f_{\beta,\beta,\beta}+$$
$$\left(-2\ f_{1,\beta}\ f_{2,\beta}\ f_{1,3,\beta}-2\ f_{1,\beta}\ f_{2,\beta}\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}+\left(-2\ f_{1,\beta}\ f_{3,\beta}\ f_{1,2,\beta}-2\ f_{1,\beta}\ f_{3,\beta}\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}+$$
$$\left(2\ f_{1,\beta}\ f_{2,3,\beta}-2\ f_{1,\beta}^{2}\ f_{2,3,\beta}+2\ f_{1,\beta}\ f_{2,\beta}\ f_{2,3,\beta}+2\ f_{1,\beta}\ f_{3,\beta}\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}+$$
$$\left(4\ f_{1,2,\beta}\ f_{1,3,\beta}+4\ f_{1,2,\beta}\ f_{2,3,\beta}+4\ f_{1,3,\beta}\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}-f_{1,\beta}^{2}\ f_{\beta,\beta,\beta}^{2}+$$
$$2\ f_{1,\beta}\ f_{2,\beta}\ f_{\beta,\beta,\beta}^{2}-f_{2,\beta}^{2}\ f_{\beta,\beta,\beta}^{2}+2\ f_{1,\beta}\ f_{3,\beta}\ f_{\beta,\beta,\beta}^{2}+2\ f_{2,\beta}\ f_{3,\beta}\ f_{\beta,\beta,\beta}^{2}-$$
$$f_{3,\beta}^{2}\ f_{\beta,\beta,\beta}^{2}+\left(-2\ f_{1,\beta}+2\ f_{1,\beta}^{2}-2\ f_{1,\beta}\ f_{2,\beta}-2\ f_{1,\beta}\ f_{3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(-2\ f_{2,\beta}-2\ f_{1,\beta}\ f_{2,\beta}+2\ f_{2,\beta}^{2}-2\ f_{2,\beta}\ f_{3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(-1+2\ f_{1,\beta}-f_{1,\beta}^{2}+2\ f_{2,\beta}+2\ f_{1,\beta}\ f_{2,\beta}-f_{2,\beta}^{2}+2\ f_{3,\beta}+2\ f_{1,\beta}\ f_{3,\beta}+2\ f_{2,\beta}\ f_{3,\beta}-f_{3,\beta}^{2}\right)\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(-2\ f_{3,\beta}-2\ f_{1,\beta}\ f_{3,\beta}-2\ f_{2,\beta}\ f_{3,\beta}+2\ f_{3,\beta}^{2}\right)\ f_{\beta,\beta,\beta}^{2}-4\ f_{1,2,\beta}\ f_{\beta,\beta,\beta}^{2}-4\ f_{1,3,\beta}\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(4\ f_{1,2,\beta}+4\ f_{1,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+\left(-4\ f_{1,2,\beta}-4\ f_{1,3,\beta}-4\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}-$$
$$4\ f_{2,3,\beta}\ f_{\beta,\beta,\beta}^{2}+\left(4\ f_{1,2,\beta}+4\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+\left(4\ f_{1,3,\beta}+4\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2},$$
$$-f_{1,\beta}^{2}\ f_{2,\beta}^{2}\ f_{3,\beta}^{2}+f_{1,\beta}\ f_{2,\beta}\ f_{3,\beta}^{2}\ f_{1,2,\beta}+f_{1,\beta}\ f_{2,\beta}^{2}\ f_{3,\beta}\ f_{1,3,\beta}-f_{2,\beta}\ f_{3,\beta}\ f_{1,2,\beta}\ f_{1,3,\beta}+$$
$$f_{1,\beta}^{2}\ f_{2,\beta}\ f_{3,\beta}\ f_{2,3,\beta}-f_{1,\beta}\ f_{3,\beta}\ f_{1,2,\beta}\ f_{2,3,\beta}-f_{1,\beta}\ f_{2,\beta}\ f_{1,3,\beta}\ f_{2,3,\beta}+$$
$$f_{1,2,\beta}\ f_{1,3,\beta}\ f_{2,3,\beta}-f_{1,\beta}^{2}\ f_{2,\beta}\ f_{3,\beta}\ f_{\beta,\beta,\beta}-f_{1,\beta}\ f_{2,\beta}^{2}\ f_{3,\beta}\ f_{\beta,\beta,\beta}-f_{1,\beta}\ f_{2,\beta}\ f_{3,\beta}^{2}\ f_{\beta,\beta,\beta}+$$
$$\left(f_{1,\beta}^{2}\ f_{2,\beta}\ f_{3,\beta}+f_{1,\beta}\ f_{2,\beta}^{2}\ f_{3,\beta}+f_{1,\beta}\ f_{2,\beta}\ f_{3,\beta}^{2}\right)\ f_{\beta,\beta,\beta}+\left(-f_{1,\beta}\ f_{3,\beta}\ f_{1,2,\beta}-f_{2,\beta}\ f_{3,\beta}\ f_{1,2,\beta}\right)\ f_{\beta,\beta,\beta}+$$
$$f_{1,2,\beta}\ f_{1,3,\beta}\ f_{\beta,\beta,\beta}+\left(-f_{1,\beta}\ f_{2,\beta}\ f_{1,3,\beta}-f_{2,\beta}\ f_{3,\beta}\ f_{1,3,\beta}\right)\ f_{\beta,\beta,\beta}+$$
$$\left(f_{2,\beta}\ f_{3,\beta}\ f_{1,2,\beta}+f_{2,\beta}\ f_{3,\beta}\ f_{1,3,\beta}\right)\ f_{\beta,\beta,\beta}+f_{1,2,\beta}\ f_{2,3,\beta}\ f_{\beta,\beta,\beta}+f_{1,3,\beta}\ f_{2,3,\beta}\ f_{\beta,\beta,\beta}+$$
$$\left(f_{1,\beta}\ f_{2,\beta}\ f_{1,3,\beta}+f_{1,\beta}\ f_{2,\beta}\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}+\left(-f_{1,\beta}\ f_{2,\beta}\ f_{2,3,\beta}-f_{1,\beta}\ f_{3,\beta}\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}+$$
$$\left(f_{1,\beta}\ f_{3,\beta}\ f_{1,2,\beta}+f_{1,\beta}\ f_{3,\beta}\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}+\left(-f_{1,2,\beta}\ f_{1,3,\beta}-f_{1,2,\beta}\ f_{2,3,\beta}-f_{1,3,\beta}\ f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}-$$
$$f_{1,\beta}\ f_{2,\beta}\ f_{\beta,\beta,\beta}^{2}-f_{1,\beta}\ f_{3,\beta}\ f_{\beta,\beta,\beta}^{2}-f_{2,\beta}\ f_{3,\beta}\ f_{\beta,\beta,\beta}^{2}+\left(f_{1,\beta}\ f_{2,\beta}+f_{1,\beta}\ f_{3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(-f_{1,\beta}\ f_{2,\beta}-f_{1,\beta}\ f_{3,\beta}-f_{2,\beta}\ f_{3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+\left(f_{1,\beta}\ f_{2,\beta}+f_{2,\beta}\ f_{3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(f_{1,\beta}\ f_{3,\beta}+f_{2,\beta}\ f_{3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+f_{1,2,\beta}\ f_{\beta,\beta,\beta}^{2}+\left(-f_{1,2,\beta}-f_{1,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+f_{1,3,\beta}\ f_{\beta,\beta,\beta}^{2}+$$
$$\left(-f_{1,2,\beta}-f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+\left(-f_{1,3,\beta}-f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}+f_{2,3,\beta}\ f_{\beta,\beta,\beta}^{2}+\left(f_{1,2,\beta}+f_{1,3,\beta}+f_{2,3,\beta}\right)\ f_{\beta,\beta,\beta}^{2}\}$$

This is overwhelming! it should be clear to the patient reader that this algebra is extremely complex. These are not trivial expressions that one can readily understand. So one must develop tools to analyze these algebraic objects that no human hand could construct. One very useful one is to just look at the variables involved in these polynomials. This inspection would also tell us whether we were done in our algebraic simplification of the coefficients involved in the estimate of the sample prevalence.

In[489]:=

```
Map[Variables, withPairRewriteRules] // Sort /@ # & // Column[#, Dividers → All] &
```

Out[489]=

$\{f_{1,\beta}, f_{2,\beta}, f_{3,\beta}, f_{1,2,\beta}, f_{1,3,\beta}, f_{2,3,\beta}, f_{\beta,\beta,\beta}\}$
$\{f_{1,\beta}, f_{2,\beta}, f_{3,\beta}, f_{1,2,\beta}, f_{1,3,\beta}, f_{2,3,\beta}, f_{\beta,\beta,\beta}\}$
$\{f_{1,\beta}, f_{2,\beta}, f_{3,\beta}, f_{1,2,\beta}, f_{1,3,\beta}, f_{2,3,\beta}, f_{\beta,\beta,\beta}\}$

So we are done. There are no more algebraic simplifications. As we see, the algebraic estimate of the prevalence is composed of finite moments of the single classifier, the pairs, and the single trio. How could it be otherwise? There are no unknown unknowns in sample statistics algebra. Finite samples have finite moment expansions and the algebra of evaluation of noisy judges cannot avoid that algebraic fact.

We continue by solving the quadratic polynomial for the prevalence. Let's do it in a simplified fashion. Let's erase the a, b, c variables and start from scratch.

In[491]:=

```
quadraticSolutions = Solve[a1 * Pα^2 + b1 * Pα + c1 == 0, Pα]
```

Out[491]=

$$\left\{\left\{P_\alpha \to \frac{-b1 - \sqrt{b1^2 - 4\,a1\,c1}}{2\,a1}\right\}, \left\{P_\alpha \to \frac{-b1 + \sqrt{b1^2 - 4\,a1\,c1}}{2\,a1}\right\}\right\}$$

We see that the radical term is going to be the one most difficult to handle. Let us see if we can simplify it before doing blind substitution into our quadratic polynomial solutions.

In[496]:=

```
((b1^2 - 4 a1 c1) /. Map[Rule @@ # &, Transpose@{{a1, b1, c1}, withPairRewriteRules}]) //
  Factor
```

Out[496]=

$$\left(-2\,f_{1,\beta}\,f_{2,\beta}\,f_{3,\beta} + f_{3,\beta}\,f_{1,2,\beta} + f_{2,\beta}\,f_{1,3,\beta} + f_{1,\beta}\,f_{2,3,\beta} - f_{\beta,\beta,\beta}\right)^2 \left(f_{3,\beta}^2\,f_{1,2,\beta}^2 - 2\,f_{2,\beta}\,f_{3,\beta}\,f_{1,2,\beta}\,f_{1,3,\beta} + f_{2,\beta}^2\,f_{1,3,\beta}^2 - 2\,f_{1,\beta}\,f_{3,\beta}\,f_{1,2,\beta}\,f_{2,3,\beta} - 2\,f_{1,\beta}\,f_{2,\beta}\,f_{1,3,\beta}\,f_{2,3,\beta} + 4\,f_{1,2,\beta}\,f_{1,3,\beta}\,f_{2,3,\beta} + f_{1,\beta}^2\,f_{2,3,\beta}^2 + 4\,f_{1,\beta}\,f_{2,\beta}\,f_{3,\beta}\,f_{\beta,\beta,\beta} - 2\,f_{3,\beta}\,f_{1,2,\beta}\,f_{\beta,\beta,\beta} - 2\,f_{2,\beta}\,f_{1,3,\beta}\,f_{\beta,\beta,\beta} - 2\,f_{1,\beta}\,f_{2,3,\beta}\,f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2\right)$$

This expression explains why algebraic evaluators can fail to return integer ratios as solutions for the prevalence. Any unresolved square root or imaginary value comes from this term. And as we see in the algebra above it has one perfect square and an irreducible polynomial that is not factorizable in this variable space. This has deeper theoretical significance but we continue with our algebraic goal - an expression for the alpha prevalence.

In[498]:=

```
Factor /@ withPairRewriteRules // Column[#, Dividers → All] &
```

Out[498]=

$f_{3,\beta}^2 f_{1,2,\beta}^2 - 2 f_{2,\beta} f_{3,\beta} f_{1,2,\beta} f_{1,3,\beta} + f_{2,\beta}^2 f_{1,3,\beta}^2 - 2 f_{1,\beta} f_{3,\beta} f_{1,2,\beta} f_{2,3,\beta} - 2 f_{1,\beta} f_{2,\beta} f_{1,3,\beta} f_{2,3,\beta} + 4 f_{1,2,\beta} f_{1,3,\beta} f_{2,3,\beta} + f_{1,\beta}^2 f_{2,3,\beta}^2 + 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} - 2 f_{3,\beta} f_{1,2,\beta} f_{\beta,\beta,\beta} - 2 f_{2,\beta} f_{1,3,\beta} f_{\beta,\beta,\beta} - 2 f_{1,\beta} f_{2,3,\beta} f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2$
$-f_{3,\beta}^2 f_{1,2,\beta}^2 + 2 f_{2,\beta} f_{3,\beta} f_{1,2,\beta} f_{1,3,\beta} - f_{2,\beta}^2 f_{1,3,\beta}^2 + 2 f_{1,\beta} f_{3,\beta} f_{1,2,\beta} f_{2,3,\beta} + 2 f_{1,\beta} f_{2,\beta} f_{1,3,\beta} f_{2,3,\beta} - 4 f_{1,2,\beta} f_{1,3,\beta} f_{2,3,\beta} - f_{1,\beta}^2 f_{2,3,\beta}^2 - 4 f_{1,\beta} f_{2,\beta} f_{3,\beta} f_{\beta,\beta,\beta} + 2 f_{3,\beta} f_{1,2,\beta} f_{\beta,\beta,\beta} + 2 f_{2,\beta} f_{1,3,\beta} f_{\beta,\beta,\beta} + 2 f_{1,\beta} f_{2,3,\beta} f_{\beta,\beta,\beta} - f_{\beta,\beta,\beta}^2$
$-\left(\left(f_{1,\beta} f_{2,\beta} - f_{1,2,\beta}\right) \left(f_{1,\beta} f_{3,\beta} - f_{1,3,\beta}\right) \left(f_{2,\beta} f_{3,\beta} - f_{2,3,\beta}\right)\right)$

Hmm. The “c” coefficient in the prevalence polynomial has factored out into an interesting product. Can we rewrite the other coefficients in terms of these new polynomials?

In[522]:=

```
fullySimplifiedQuadraticCoefficients =
  ToFixedPointPolynomial[#, {(f1,β f2,β - f1,2,β) → p12, (f1,β f3,β - f1,3,β) → p13,
      (f2,β f3,β - f2,3,β) → p23}] & /@ ((Factor /@ withPairRewriteRules) /.
     {(f1,β f2,β - f1,2,β) → p12, (f1,β f3,β - f1,3,β) → p13, (f2,β f3,β - f2,3,β) → p23}) //
   Factor /@ # &;
fullySimplifiedQuadraticCoefficients // Transpose@{
     {"a", "b", "c"},
     #,
     Variables /@ # // Sort /@ # &} & //
  Prepend[#,
    {"Quadratic coefficient", "Reduced Algebraic Estimate", "Variables"}] & //
 Grid[#, Dividers → All, Alignment → {Center, Center}] &
```

Out[523]=

Quadratic coefficient	Reduced Algebraic Estimate	Variables
a	$f_{3,\beta}^2 f_{1,2,\beta}^2 - 2\,p23\, f_{1,2,\beta} f_{1,3,\beta} + f_{2,\beta}^2 f_{1,3,\beta}^2 - 2\,p13\, f_{1,2,\beta} f_{2,3,\beta} - 2\,p12\, f_{1,3,\beta} f_{2,3,\beta} - 2 f_{1,2,\beta} f_{1,3,\beta} f_{2,3,\beta} + f_{1,\beta}^2 f_{2,3,\beta}^2 + 4\,p12\, f_{3,\beta} f_{\beta,\beta,\beta} + 2 f_{3,\beta} f_{1,2,\beta} f_{\beta,\beta,\beta} - 2 f_{2,\beta} f_{1,3,\beta} f_{\beta,\beta,\beta} - 2 f_{1,\beta} f_{2,3,\beta} f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2$	$\{p12, p13, p23, f_{1,\beta}, f_{2,\beta}, f_{3,\beta}, f_{1,2,\beta}, f_{1,3,\beta}, f_{2,3,\beta}, f_{\beta,\beta,\beta}\}$
b	$-f_{3,\beta}^2 f_{1,2,\beta}^2 + 2\,p23\, f_{1,2,\beta} f_{1,3,\beta} - f_{2,\beta}^2 f_{1,3,\beta}^2 + 2\,p13\, f_{1,2,\beta} f_{2,3,\beta} + 2\,p12\, f_{1,3,\beta} f_{2,3,\beta} + 2 f_{1,2,\beta} f_{1,3,\beta} f_{2,3,\beta} - f_{1,\beta}^2 f_{2,3,\beta}^2 - 4\,p12\, f_{3,\beta} f_{\beta,\beta,\beta} - 2 f_{3,\beta} f_{1,2,\beta} f_{\beta,\beta,\beta} + 2 f_{2,\beta} f_{1,3,\beta} f_{\beta,\beta,\beta} + 2 f_{1,\beta} f_{2,3,\beta} f_{\beta,\beta,\beta} - f_{\beta,\beta,\beta}^2$	$\{p12, p13, p23, f_{1,\beta}, f_{2,\beta}, f_{3,\beta}, f_{1,2,\beta}, f_{1,3,\beta}, f_{2,3,\beta}, f_{\beta,\beta,\beta}\}$
c	$-p12\, p13\, p23$	$\{p12, p13, p23\}$

It seems we are done. We finally have a fully algebraic expression for the estimate of the prevalence of the α label on the test sample. We can now solve equation,

$a(\ldots) * P_\alpha{}^2 + b(\ldots) * P_\alpha + c(\ldots) == 0.$

Not quite, we have to return to the voting frequency moments we observed in the sample since those are the counts we have observed empirically,

In[524]:=

```
votePatternFrequencies
```

Out[524]=

$$\{f_{\alpha,\alpha,\alpha},\ f_{\alpha,\alpha,\beta},\ f_{\alpha,\beta,\alpha},\ f_{\alpha,\beta,\beta},\ f_{\beta,\alpha,\alpha},\ f_{\beta,\alpha,\beta},\ f_{\beta,\beta,\alpha},\ f_{\beta,\beta,\beta}\}$$

In[527]:=

```
unwindRules = {
    Reverse /@ {(f_{1,β} f_{2,β} - f_{1,2,β}) → p12, (f_{1,β} f_{3,β} - f_{1,3,β}) → p13, (f_{2,β} f_{3,β} - f_{2,3,β}) → p23},
    Reverse /@ { (f_{β,β,α} + f_{β,β,β}) → f_{1,2,β},
       (f_{β,α,β} + f_{β,β,β}) → f_{1,3,β}, (f_{α,β,β} + f_{β,β,β}) → f_{2,3,β}},
    Reverse /@ {f_{β,α,α} + f_{β,α,β} + f_{β,β,α} + f_{β,β,β} → f_{1,β},
       f_{α,β,α} + f_{α,β,β} + f_{β,β,α} + f_{β,β,β} → f_{2,β}, f_{α,α,β} + f_{α,β,β} + f_{β,α,β} + f_{β,β,β} → f_{3,β}}} // Flatten
```

Out[527]=

$$\{p12 \to f_{1,\beta}\, f_{2,\beta} - f_{1,2,\beta},\ p13 \to f_{1,\beta}\, f_{3,\beta} - f_{1,3,\beta},\ p23 \to f_{2,\beta}\, f_{3,\beta} - f_{2,3,\beta},\ f_{1,2,\beta} \to f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta},$$
$$f_{1,3,\beta} \to f_{\beta,\alpha,\beta} + f_{\beta,\beta,\beta},\ f_{2,3,\beta} \to f_{\alpha,\beta,\beta} + f_{\beta,\beta,\beta},\ f_{1,\beta} \to f_{\beta,\alpha,\alpha} + f_{\beta,\alpha,\beta} + f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta},$$
$$f_{2,\beta} \to f_{\alpha,\beta,\alpha} + f_{\alpha,\beta,\beta} + f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta},\ f_{3,\beta} \to f_{\alpha,\alpha,\beta} + f_{\alpha,\beta,\beta} + f_{\beta,\alpha,\beta} + f_{\beta,\beta,\beta}\}$$

In[535]:=

```
prevalenceQuadraticGrid = ToFixedPointPolynomial[#, unwindRules] & /@
     fullySimplifiedQuadraticCoefficients //
    Factor /@ # & //
  Grid[Transpose@{{"a", "b", "c"}, #}, Dividers → All, Alignment → {Center, Center}] &
```

Out[535]=

a	$f^2_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\alpha} - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} + f^2_{\alpha,\beta,\alpha} f^2_{\beta,\alpha,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f^2_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + f^2_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\alpha} + 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f^2_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f^2_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f^2_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f^2_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} + f^2_{\alpha,\alpha,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} + f^2_{\alpha,\beta,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} + f^2_{\alpha,\beta,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} + f^2_{\beta,\alpha,\alpha} f^2_{\beta,\beta,\beta} - 2 f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} + 4 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} + f^2_{\beta,\alpha,\beta} f^2_{\beta,\beta,\beta} - 2 f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} + 4 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} + f^2_{\beta,\beta,\alpha} f^2_{\beta,\beta,\beta} - 2 f^3_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f^3_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f^3_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f^3_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\alpha} f^3_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\beta} f^3_{\beta,\beta,\beta} + 2 f_{\beta,\beta,\alpha} f^3_{\beta,\beta,\beta} + f^4_{\beta,\beta,\beta}$
	$- f^2_{\alpha,\beta,\beta} f^2_{\beta,\alpha,\alpha} + 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} - f^2_{\alpha,\beta,\alpha} f^2_{\beta,\alpha,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} +$

b	$2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} +$ $4 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} +$ $4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\alpha} - f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha}^2 + 4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha}^2 - 4 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} +$ $2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} -$ $2 f_{\alpha,\beta,\beta}^2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} -$ $2 f_{\alpha,\beta,\alpha}^2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} -$ $2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta} + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} -$ $2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} -$ $2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} +$ $4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta} - f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^2 - f_{\alpha,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 +$ $2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^2 - f_{\alpha,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 -$ $2 f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^2 - f_{\alpha,\beta,\beta}^2 f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 -$ $4 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^2 - f_{\beta,\alpha,\alpha}^2 f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 -$ $4 f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^2 - f_{\beta,\alpha,\beta}^2 f_{\beta,\beta,\beta}^2 +$ $2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 - 4 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 - 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 -$ $2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^2 - f_{\beta,\beta,\alpha}^2 f_{\beta,\beta,\beta}^2 + 2 f_{\beta,\beta,\beta}^3 - 2 f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta}^3 -$ $2 f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta}^3 - 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta}^3 - 2 f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta}^3 - 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta}^3 - 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta}^3 - f_{\beta,\beta,\beta}^4$
c	$-\big(\big(f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\alpha} - f_{\alpha,\beta,\beta} + f_{\alpha,\alpha,\beta} f_{\alpha,\beta,\beta} + f_{\alpha,\beta,\alpha} f_{\alpha,\beta,\beta} + f_{\alpha,\beta,\beta}^2 +$ $f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} + f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} + f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - f_{\beta,\beta,\beta} +$ $f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta} + f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta} + 2 f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2\big)$ $\big(f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\alpha} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} - f_{\beta,\alpha,\beta} + f_{\alpha,\alpha,\beta} f_{\beta,\alpha,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} + f_{\beta,\alpha,\alpha} f_{\beta,\alpha,\beta} +$ $f_{\beta,\alpha,\beta}^2 + f_{\alpha,\alpha,\beta} f_{\beta,\beta,\alpha} + f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} + f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} - f_{\beta,\beta,\beta} + f_{\alpha,\alpha,\beta} f_{\beta,\beta,\beta} +$ $f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + 2 f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2\big)$ $\big(f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\alpha} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\alpha} + f_{\alpha,\beta,\alpha} f_{\beta,\alpha,\beta} + f_{\alpha,\beta,\beta} f_{\beta,\alpha,\beta} - f_{\beta,\beta,\alpha} + f_{\alpha,\beta,\alpha} f_{\beta,\beta,\alpha} +$ $f_{\alpha,\beta,\beta} f_{\beta,\beta,\alpha} + f_{\beta,\alpha,\alpha} f_{\beta,\beta,\alpha} + f_{\beta,\alpha,\beta} f_{\beta,\beta,\alpha} + f_{\beta,\beta,\alpha}^2 - f_{\beta,\beta,\beta} + f_{\alpha,\beta,\alpha} f_{\beta,\beta,\beta} +$ $f_{\alpha,\beta,\beta} f_{\beta,\beta,\beta} + f_{\beta,\alpha,\alpha} f_{\beta,\beta,\beta} + f_{\beta,\alpha,\beta} f_{\beta,\beta,\beta} + 2 f_{\beta,\beta,\alpha} f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2\big)\big)$

Let us evaluate this step by step and see how the estimate for the prevalence is computed algebraically.

$a(\ldots) * P_\alpha^2 + b(\ldots) * P_\alpha + c(\ldots) == 0.$

In[537]:=

```
evaluationDataSketch
prevalenceQuadraticGrid /. evaluationDataSketch
```

Out[537]=

$$\left\{f_{\alpha,\alpha,\alpha} \to \frac{493}{18421},\ f_{\alpha,\alpha,\beta} \to \frac{315}{18421},\ f_{\alpha,\beta,\alpha} \to \frac{5801}{36842},\ f_{\alpha,\beta,\beta} \to \frac{3986}{18421},\ f_{\beta,\alpha,\alpha} \to \frac{281}{18421},\ f_{\beta,\alpha,\beta} \to \frac{493}{36842},\ f_{\beta,\beta,\alpha} \to \frac{3856}{18421},\ f_{\beta,\beta,\beta} \to \frac{6343}{18421}\right\}$$

Out[538]=

a	$\frac{29957776434081}{1842352775161025296}$
b	$-\frac{29957776434081}{1842352775161025296}$
c	$\frac{5659724374199502787125}{2500686153042940042298657344}$

In[574]:=

```
prevalenceAlgebraicEvaluation =
  (-(-29957776434081/1842352775161025296) - Sqrt[(-29957776434081/1842352775161025296)^2 -
        4 * 29957776434081/1842352775161025296 * 5659724374199502787125/2500686153042940042298657344]) /
    (2 * (29957776434081/1842352775161025296)) // Simplify // {#, N@#} &
```

Out[574]=

$$\left\{\frac{1}{2} - \frac{11187722681}{18421\sqrt{3328641826009}},\ 0.167114\right\}$$

The estimate for P_α one gets from majority voting is,

In[548]:=

```
majorityVotingEvaluation =
  (f_{α,α,α} + f_{α,α,β} + f_{α,β,α} + f_{β,α,α}) /. evaluationDataSketch // {#, N@#} &
```

Out[548]=

$$\left\{\frac{7979}{36842},\ 0.216573\right\}$$

And what is the sample value for the prevalence?

In[546]:=

```
evaluationGroundTruth[P_α] // {#, N@#} &
```

Out[546]=

$$\left\{\frac{5687}{36842},\ 0.154362\right\}$$

In[575]:=

```
comparisonTable = Transpose@{
      {"evaluationGroundTruth[Pα]",
       "majorityVotingEvaluation",
       "prevalenceAlgebraicEvaluation"},
      {evaluationGroundTruth[Pα] // {#, N@#} & // Reverse,
       majorityVotingEvaluation // Reverse,
       prevalenceAlgebraicEvaluation // Reverse
      }} // Grid[#, Dividers → All, Alignment → {Left}] & //
   Column[{
      StringJoin[
       {"Algebraic evaluation is not just
           different than majority voting, it is better.",
         "The majority vote estimate of the α
           label obtained using three binary classifiers",
         "tested on an unlabeled subset of UCI Adult
           is compared here to the actual prevalence",
         "of the label in the test sample. Majority voting is off by about 6%.",
       "In contrast, algebraic evaluation yields an estimate that is off by about
           1%."} // Riffle[#, "\n"] &], #}, Alignment → Center] &;
```

In[576]:=

```
comparisonTable
```

Out[576]=

Algebraic evaluation is not just different than majority voting, it is better.
The majority vote estimate of the α label obtained using three binary classifiers
tested on an unlabeled subset of
UCI Adult is compared here to the actual prevalence
of the label in the test sample. Majority voting is off by about 6%.
In contrast, algebraic evaluation yields an estimate that is off by about 1%.

evaluationGroundTruth[P_α]	$\left\{0.154362, \frac{5687}{36842}\right\}$
majorityVotingEvaluation	$\left\{0.216573, \frac{7979}{36842}\right\}$
prevalenceAlgebraicEvaluation	$\left\{0.167114, \frac{1}{2} - \frac{11187722681}{18421\sqrt{3328641826009}}\right\}$

Is algebraic evaluation just majority voting?

No. This is the single most common misconception people familiar with Training theory have. Inference and decision are two different things. The same applies in Evaluation land. There is a decision side to using the crowd - majority voting. There is an inference side to using the crowd - algebraic evaluation. This is vividly demonstrated by the algebra above that displays the estimate majority voting makes for the prevalence of the alpha label in the test sample. Along side it is the result of all our algebraic efforts preceding it. The two estimates are nothing like each other.

Estimating classifiers label accuracy

We are now ready to solve for the classifiers label accuracy. Our strategy follows the algebraic structure of the Groebner basis solution we obtained before. As we worked out above, the prevalence was solvable because it appeared as the only variable in a single equation in the computed basis. We then observe that we can relate any label accuracy to the prevalence because there are linear equations in the Groebner basis connecting the two. Let us see where they are,

In[598]:=

```
Map[Variables, gb] // Map[Cases[#, Except[f__]] &, #] & // Sort /@ # & //
  MapIndexed[{#1, #2⟦1⟧} &, #] & // Select[#, Length@First@# == 2 &] &
```

Out[598]=

$\{\{\{P_\alpha, P_{1,\alpha}\}, 3\}, \{\{P_\alpha, P_{1,\alpha}\}, 4\}, \{\{P_\alpha, P_{1,\alpha}\}, 5\}, \{\{P_\alpha, P_{1,\alpha}\}, 6\},$
$\{\{P_{1,\alpha}, P_{1,\beta}\}, 8\}, \{\{P_{1,\alpha}, P_{1,\beta}\}, 10\}, \{\{P_{1,\alpha}, P_{1,\beta}\}, 11\}, \{\{P_{1,\alpha}, P_{2,\alpha}\}, 12\},$
$\{\{P_{1,\alpha}, P_{2,\alpha}\}, 15\}, \{\{P_{1,\alpha}, P_{2,\alpha}\}, 16\}, \{\{P_{1,\beta}, P_{2,\alpha}\}, 18\}, \{\{P_{1,\alpha}, P_{2,\beta}\}, 19\},$
$\{\{P_{1,\alpha}, P_{2,\beta}\}, 21\}, \{\{P_{1,\beta}, P_{2,\beta}\}, 22\}, \{\{P_{2,\alpha}, P_{2,\beta}\}, 23\}, \{\{P_{2,\alpha}, P_{2,\beta}\}, 24\},$
$\{\{P_{1,\alpha}, P_{3,\alpha}\}, 25\}, \{\{P_{1,\alpha}, P_{3,\alpha}\}, 27\}, \{\{P_{1,\alpha}, P_{3,\alpha}\}, 28\}, \{\{P_{1,\beta}, P_{3,\alpha}\}, 30\},$
$\{\{P_{2,\alpha}, P_{3,\alpha}\}, 31\}, \{\{P_{2,\beta}, P_{3,\alpha}\}, 34\}, \{\{P_{1,\alpha}, P_{3,\beta}\}, 35\}, \{\{P_{1,\alpha}, P_{3,\beta}\}, 37\},$
$\{\{P_{1,\beta}, P_{3,\beta}\}, 38\}, \{\{P_{2,\alpha}, P_{3,\beta}\}, 39\}, \{\{P_{2,\beta}, P_{3,\beta}\}, 40\}, \{\{P_{3,\alpha}, P_{3,\beta}\}, 42\}\}$

Equations 3, 4, 5, and 6 in the Groebner basis relate $\{P_\alpha, P_{1,\alpha}\}$. Let's pick 3 arbitrarily

In[607]:=

```
labelAccuracyPolyCoefficients =
  gb⟦3⟧ // CoefficientRules[#, {Pα, P1,α}] & // Association;
Keys@labelAccuracyPolyCoefficients
```

Out[608]=

```
{{1, 0}, {0, 1}, {0, 0}}
```

This confirms that the third Groebner basis equation is a linear equation of the form,
d(...)*P_α + e(...)*$P_{1,\alpha}$ + g(...) == 0
Do we have enough transformation rules to simplify these coefficients like we did with the prevalence quadratic?

In[603]:=

$$\begin{aligned}&\texttt{simplifyRules} = \Big\{\\
&\quad\big\{\big(f_{1,\beta}\, f_{2,\beta} - f_{1,2,\beta}\big) \to \texttt{p12},\ \big(f_{1,\beta}\, f_{3,\beta} - f_{1,3,\beta}\big) \to \texttt{p13},\ \big(f_{2,\beta}\, f_{3,\beta} - f_{2,3,\beta}\big) \to \texttt{p23}\big\},\\
&\quad\big\{\ \big(f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta}\big) \to f_{1,2,\beta},\ \big(f_{\beta,\alpha,\beta} + f_{\beta,\beta,\beta}\big) \to f_{1,3,\beta},\ \big(f_{\alpha,\beta,\beta} + f_{\beta,\beta,\beta}\big) \to f_{2,3,\beta}\big\},\\
&\quad\big\{f_{\beta,\alpha,\alpha} + f_{\beta,\alpha,\beta} + f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta} \to f_{1,\beta},\\
&\quad\ f_{\alpha,\beta,\alpha} + f_{\alpha,\beta,\beta} + f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta} \to f_{2,\beta},\ f_{\alpha,\alpha,\beta} + f_{\alpha,\beta,\beta} + f_{\beta,\alpha,\beta} + f_{\beta,\beta,\beta} \to f_{3,\beta}\big\}\Big\}\ \texttt{//}\\
&\quad\texttt{Reverse // Flatten}\end{aligned}$$

Out[603]=

$$\begin{aligned}&\{f_{\beta,\alpha,\alpha} + f_{\beta,\alpha,\beta} + f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta} \to f_{1,\beta},\ f_{\alpha,\beta,\alpha} + f_{\alpha,\beta,\beta} + f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta} \to f_{2,\beta},\\
&\ f_{\alpha,\alpha,\beta} + f_{\alpha,\beta,\beta} + f_{\beta,\alpha,\beta} + f_{\beta,\beta,\beta} \to f_{3,\beta},\ f_{\beta,\beta,\alpha} + f_{\beta,\beta,\beta} \to f_{1,2,\beta},\ f_{\beta,\alpha,\beta} + f_{\beta,\beta,\beta} \to f_{1,3,\beta},\\
&\ f_{\alpha,\beta,\beta} + f_{\beta,\beta,\beta} \to f_{2,3,\beta},\ f_{1,\beta}\, f_{2,\beta} - f_{1,2,\beta} \to \text{p12},\ f_{1,\beta}\, f_{3,\beta} - f_{1,3,\beta} \to \text{p13},\ f_{2,\beta}\, f_{3,\beta} - f_{2,3,\beta} \to \text{p23}\}\end{aligned}$$

Let us get expressions for the terms d(...), e(...), and g(...)

In[637]:=

```
thirdGBPolynomialCoefficients = Map[
   ToFixedPointPolynomial[labelAccuracyPolyCoefficients[#], simplifyRules] &,
   {{1, 0}, {0, 1}, {0, 0}}] // Transpose[{{d, e, g}, #}] & // Rule @@ # & /@ # &
```

Out[637]=

$$\begin{aligned}&\{d \to f_{3,\beta}^2\, f_{1,2,\beta}^2 - 2\,\text{p23}\, f_{1,2,\beta}\, f_{1,3,\beta} + f_{2,\beta}^2\, f_{1,3,\beta}^2 - 2\,\text{p13}\, f_{1,2,\beta}\, f_{2,3,\beta} -\\
&\quad 2\,\text{p12}\, f_{1,3,\beta}\, f_{2,3,\beta} - 2\, f_{1,2,\beta}\, f_{1,3,\beta}\, f_{2,3,\beta} + f_{1,\beta}^2\, f_{2,3,\beta}^2 + 4\,\text{p12}\, f_{3,\beta}\, f_{\beta,\beta,\beta} +\\
&\quad 2\, f_{3,\beta}\, f_{1,2,\beta}\, f_{\beta,\beta,\beta} - 2\, f_{2,\beta}\, f_{1,3,\beta}\, f_{\beta,\beta,\beta} - 2\, f_{1,\beta}\, f_{2,3,\beta}\, f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2,\\
&\ e \to 2\,\text{p12}\,\text{p23}\, f_{3,\beta} + \text{p23}\, f_{3,\beta}\, f_{1,2,\beta} - \text{p23}\, f_{2,\beta}\, f_{1,3,\beta} - \text{p12}\, f_{3,\beta}\, f_{2,3,\beta} -\\
&\quad f_{3,\beta}\, f_{1,2,\beta}\, f_{2,3,\beta} + f_{1,\beta}\, f_{2,3,\beta}^2 + \text{p23}\, f_{\beta,\beta,\beta},\\
&\ g \to -2\,\text{p12}\,\text{p23}\, f_{3,\beta} - \text{p23}\, f_{3,\beta}\, f_{1,2,\beta} + \text{p12}\, f_{3,\beta}^2\, f_{1,2,\beta} + \text{p12}\,\text{p23}\, f_{1,3,\beta} +\\
&\quad \text{p23}\, f_{2,\beta}\, f_{1,3,\beta} + \text{p23}\, f_{1,2,\beta}\, f_{1,3,\beta} - f_{2,\beta}^2\, f_{1,3,\beta}^2 - \text{p12}\,\text{p13}\, f_{2,3,\beta} + \text{p12}\, f_{3,\beta}\, f_{2,3,\beta} +\\
&\quad f_{3,\beta}\, f_{1,2,\beta}\, f_{2,3,\beta} + \text{p12}\, f_{1,3,\beta}\, f_{2,3,\beta} - f_{1,\beta}\, f_{2,3,\beta}^2 - \text{p23}\, f_{\beta,\beta,\beta} - 3\,\text{p12}\, f_{3,\beta}\, f_{\beta,\beta,\beta} -\\
&\quad f_{3,\beta}\, f_{1,2,\beta}\, f_{\beta,\beta,\beta} + 2\, f_{2,\beta}\, f_{1,3,\beta}\, f_{\beta,\beta,\beta} + f_{1,\beta}\, f_{2,3,\beta}\, f_{\beta,\beta,\beta} - f_{\beta,\beta,\beta}^2\}\end{aligned}$$

And let's get the expressions for the terms a(...), b(...), and c(...) that allowed us to solve for P_α

In[644]:=

```
prevalencePolynomialCoefficients = ToFixedPointPolynomial[#, simplifyRules] & /@
     fullySimplifiedQuadraticCoefficients //
   Factor /@ # & // Transpose[{{a, b, c}, #}] & // Rule @@ # & /@ # &
```

Out[644]=

$$\begin{aligned}&\{a \to f_{3,\beta}^2\, f_{1,2,\beta}^2 - 2\,\text{p23}\, f_{1,2,\beta}\, f_{1,3,\beta} + f_{2,\beta}^2\, f_{1,3,\beta}^2 - 2\,\text{p13}\, f_{1,2,\beta}\, f_{2,3,\beta} -\\
&\quad 2\,\text{p12}\, f_{1,3,\beta}\, f_{2,3,\beta} - 2\, f_{1,2,\beta}\, f_{1,3,\beta}\, f_{2,3,\beta} + f_{1,\beta}^2\, f_{2,3,\beta}^2 + 4\,\text{p12}\, f_{3,\beta}\, f_{\beta,\beta,\beta} +\\
&\quad 2\, f_{3,\beta}\, f_{1,2,\beta}\, f_{\beta,\beta,\beta} - 2\, f_{2,\beta}\, f_{1,3,\beta}\, f_{\beta,\beta,\beta} - 2\, f_{1,\beta}\, f_{2,3,\beta}\, f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2,\\
&\ b \to -f_{3,\beta}^2\, f_{1,2,\beta}^2 + 2\,\text{p23}\, f_{1,2,\beta}\, f_{1,3,\beta} - f_{2,\beta}^2\, f_{1,3,\beta}^2 + 2\,\text{p13}\, f_{1,2,\beta}\, f_{2,3,\beta} + 2\,\text{p12}\, f_{1,3,\beta}\, f_{2,3,\beta} +\\
&\quad 2\, f_{1,2,\beta}\, f_{1,3,\beta}\, f_{2,3,\beta} - f_{1,\beta}^2\, f_{2,3,\beta}^2 - 4\,\text{p12}\, f_{3,\beta}\, f_{\beta,\beta,\beta} - 2\, f_{3,\beta}\, f_{1,2,\beta}\, f_{\beta,\beta,\beta} +\\
&\quad 2\, f_{2,\beta}\, f_{1,3,\beta}\, f_{\beta,\beta,\beta} + 2\, f_{1,\beta}\, f_{2,3,\beta}\, f_{\beta,\beta,\beta} - f_{\beta,\beta,\beta}^2,\ c \to -\text{p12}\,\text{p13}\,\text{p23}\}\end{aligned}$$

In[685]:=

```
algebraicEstimate = ((-g - d * ((-b - Sqrt[b^2 - 4 * a * c]) / 2 / a)) / e) /. Join[
    prevalencePolynomialCoefficients, thirdGBPolynomialCoefficients] // Rationalize
```

Out[685]=

$$
\begin{aligned}
\Big(&2\,\mathrm{p12}\,\mathrm{p23}\,f_{3,\beta} + \mathrm{p23}\,f_{3,\beta}\,f_{1,2,\beta} - \mathrm{p12}\,f_{3,\beta}^2\,f_{1,2,\beta} - \mathrm{p12}\,\mathrm{p23}\,f_{1,3,\beta} - \mathrm{p23}\,f_{2,\beta}\,f_{1,3,\beta} - \\
&\mathrm{p23}\,f_{1,2,\beta}\,f_{1,3,\beta} + f_{2,\beta}^2\,f_{1,3,\beta}^2 + \mathrm{p12}\,\mathrm{p13}\,f_{2,3,\beta} - \mathrm{p12}\,f_{3,\beta}\,f_{2,3,\beta} - f_{3,\beta}\,f_{1,2,\beta}\,f_{2,3,\beta} - \\
&\mathrm{p12}\,f_{1,3,\beta}\,f_{2,3,\beta} + f_{1,\beta}\,f_{2,3,\beta}^2 + \mathrm{p23}\,f_{\beta,\beta,\beta} + 3\,\mathrm{p12}\,f_{3,\beta}\,f_{\beta,\beta,\beta} + f_{3,\beta}\,f_{1,2,\beta}\,f_{\beta,\beta,\beta} - \\
&2\,f_{2,\beta}\,f_{1,3,\beta}\,f_{\beta,\beta,\beta} - f_{1,\beta}\,f_{2,3,\beta}\,f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2 + \frac{1}{2}\Big(-f_{3,\beta}^2\,f_{1,2,\beta}^2 + 2\,\mathrm{p23}\,f_{1,2,\beta}\,f_{1,3,\beta} - \\
&\quad f_{2,\beta}^2\,f_{1,3,\beta}^2 + 2\,\mathrm{p13}\,f_{1,2,\beta}\,f_{2,3,\beta} + 2\,\mathrm{p12}\,f_{1,3,\beta}\,f_{2,3,\beta} + 2\,f_{1,2,\beta}\,f_{1,3,\beta}\,f_{2,3,\beta} - f_{1,\beta}^2\,f_{2,3,\beta}^2 - \\
&\quad 4\,\mathrm{p12}\,f_{3,\beta}\,f_{\beta,\beta,\beta} - 2\,f_{3,\beta}\,f_{1,2,\beta}\,f_{\beta,\beta,\beta} + 2\,f_{2,\beta}\,f_{1,3,\beta}\,f_{\beta,\beta,\beta} + 2\,f_{1,\beta}\,f_{2,3,\beta}\,f_{\beta,\beta,\beta} - \\
&\quad f_{\beta,\beta,\beta}^2 + \sqrt{\Big(\big(-f_{3,\beta}^2\,f_{1,2,\beta}^2 + 2\,\mathrm{p23}\,f_{1,2,\beta}\,f_{1,3,\beta} - f_{2,\beta}^2\,f_{1,3,\beta}^2 + 2\,\mathrm{p13}\,f_{1,2,\beta}\,f_{2,3,\beta} +} \\
&\qquad\quad 2\,\mathrm{p12}\,f_{1,3,\beta}\,f_{2,3,\beta} + 2\,f_{1,2,\beta}\,f_{1,3,\beta}\,f_{2,3,\beta} - f_{1,\beta}^2\,f_{2,3,\beta}^2 - 4\,\mathrm{p12}\,f_{3,\beta}\,f_{\beta,\beta,\beta} - \\
&\qquad\quad 2\,f_{3,\beta}\,f_{1,2,\beta}\,f_{\beta,\beta,\beta} + 2\,f_{2,\beta}\,f_{1,3,\beta}\,f_{\beta,\beta,\beta} + 2\,f_{1,\beta}\,f_{2,3,\beta}\,f_{\beta,\beta,\beta} - f_{\beta,\beta,\beta}^2\big)^2 + \\
&\qquad 4\,\mathrm{p12}\,\mathrm{p13}\,\mathrm{p23}\,\big(f_{3,\beta}^2\,f_{1,2,\beta}^2 - 2\,\mathrm{p23}\,f_{1,2,\beta}\,f_{1,3,\beta} + f_{2,\beta}^2\,f_{1,3,\beta}^2 - 2\,\mathrm{p13}\,f_{1,2,\beta}\,f_{2,3,\beta} - \\
&\qquad\quad 2\,\mathrm{p12}\,f_{1,3,\beta}\,f_{2,3,\beta} - 2\,f_{1,2,\beta}\,f_{1,3,\beta}\,f_{2,3,\beta} + f_{1,\beta}^2\,f_{2,3,\beta}^2 + 4\,\mathrm{p12}\,f_{3,\beta}\,f_{\beta,\beta,\beta} + \\
&\qquad\quad 2\,f_{3,\beta}\,f_{1,2,\beta}\,f_{\beta,\beta,\beta} - 2\,f_{2,\beta}\,f_{1,3,\beta}\,f_{\beta,\beta,\beta} - 2\,f_{1,\beta}\,f_{2,3,\beta}\,f_{\beta,\beta,\beta} + f_{\beta,\beta,\beta}^2\big)\Big)\Big)\Big) \Big/ \\
\Big(&2\,\mathrm{p12}\,\mathrm{p23}\,f_{3,\beta} + \mathrm{p23}\,f_{3,\beta}\,f_{1,2,\beta} - \mathrm{p23}\,f_{2,\beta}\,f_{1,3,\beta} - \mathrm{p12}\,f_{3,\beta}\,f_{2,3,\beta} - \\
&f_{3,\beta}\,f_{1,2,\beta} \\
&\; f_{2,3,\beta} + f_{1,\beta} \\
&\; f_{2,3,\beta}^2 + \mathrm{p23}\,f_{\beta,\beta,\beta}\Big)
\end{aligned}
$$

We leave it as an exercise for the reader to confirm that unwinding these transformations does not make this estimate simpler. Instead, let us compute all the moments of the voting pattern frequencies we used.

In[661]:=

```
momentsRules = simplifyRules // Reverse /@ # & // (# /. evaluationDataSketch) & //
   TakeDrop[#, -3] & //
  Join[(First@# /. Last@#), Last@#] &
```

Out[661]=

$$
\begin{aligned}
\Big\{&\mathrm{p12} \to -\frac{18\,432\,653}{1\,357\,332\,964},\ \mathrm{p13} \to -\frac{18\,272\,925}{1\,357\,332\,964},\ \mathrm{p23} \to -\frac{16\,803\,485}{1\,357\,332\,964}, \\
&f_{1,\alpha} \to \frac{15\,389}{36\,842},\ f_{2,\alpha} \to \frac{2671}{36\,842},\ f_{3,\alpha} \to \frac{15\,061}{36\,842},\ f_{1,\beta} \to \frac{21\,453}{36\,842},\ f_{2,\beta} \to \frac{34\,171}{36\,842}, \\
&f_{3,\beta} \to \frac{21\,781}{36\,842},\ f_{1,2,\beta} \to \frac{10\,199}{18\,421},\ f_{1,3,\beta} \to \frac{13\,179}{36\,842},\ f_{2,3,\beta} \to \frac{10\,329}{18\,421}\Big\}
\end{aligned}
$$

Finally, we can calculate the algebraic estimate for $P_{1,\alpha}$!

In[690]:=

```
{P1,α, (algebraicEstimate /.
      Join[prevalencePolynomialCoefficients, thirdGBPolynomialCoefficients]) //
    (# /. Join[momentsRules, evaluationDataSketch]) & // Simplify // {#, N@#} &,
 evaluationGroundTruth[P1,α] // {#, N@#} &}
```

Out[690]=

$$\left\{P_{1,\alpha}, \left\{\frac{17\,681\,731 + 3\,\sqrt{3\,328\,641\,826\,009}}{33\,606\,970}, 0.688997\right\}, \left\{\frac{3737}{5687}, 0.657113\right\}\right\}$$

And by symmetry arguments we can calculate $P_{2,\alpha}$, and $P_{3,\alpha}$.

In[688]:=

```
{P2,α,
 (algebraicEstimate /. {p12 → p23, p13 → p12, p23 → p13,
        f1,β → f2,β, f2,β → f3,β, f3,β → f1,β,
        f1,2,β → f2,3,β, f1,3,β → f1,2,β, f2,3,β → f1,3,β}) //
    (# /. Join[momentsRules, evaluationDataSketch]) & //
   Simplify // {{#, N@#}, evaluationGroundTruth[ P2,α] //
     {#, N@#} &} &}
```

Out[688]=

$$\left\{P_{2,\alpha}, \left\{\left\{\frac{2\,097\,847 + \sqrt{3\,328\,641\,826\,009}}{12\,181\,950}, 0.321977\right\}, \left\{\frac{1260}{5687}, 0.221558\right\}\right\}\right\}$$

In[689]:=

```
{P3,α,
 (algebraicEstimate /. {p12 → p13, p13 → p23, p23 → p12,
        f1,β → f3,β, f2,β → f1,β, f3,β → f2,β,
        f1,2,β → f1,3,β, f1,3,β → f2,3,β, f2,3,β → f1,2,β}) //
    (# /. Join[momentsRules, evaluationDataSketch]) & //
   Simplify // {{#, N@#}, evaluationGroundTruth[ P3,α] //
     {#, N@#} &} &}
```

Out[689]=

$$\left\{P_{3,\alpha}, \left\{\left\{\frac{18\,714\,539 + 3\,\sqrt{3\,328\,641\,826\,009}}{36\,865\,306}, 0.656116\right\}, \left\{\frac{4746}{5687}, 0.834535\right\}\right\}\right\}$$

When is algebraic evaluation warranted?

The example we have been using from a single test of three binary classifiers trained on UCI Adult was picked for various reasons. Let us summarize them:

1. We are using the UCI Adult dataset. If the prized label was common, why do AI to identify it? The economic utility of using AI in a business lies in detecting the rare, valuable thing. Common valuable things, like air, need no detector. They are there for the picking. UCI Adult has the “0” label with a prevalence of about 30% - the adults that had an income greater than $50K in the income tax returns pool that was used to build the detection features. This long tail skew is common in nature and technological settings. The converse, by the way, is possible. It may be that the rare thing is the dangerous thing. You are detecting carbon monoxide (CO) in a bedroom, for example. Mathematically both cases are essentially identically and it is just a matter of accounting how you transform one (detecting a valuable thing) to the other (detecting a dangerous thing). For the purposes of our discussion, let us continue with the rosier outlook - we are trying to detect a positive thing.

2. If you were looking for diamonds or gold, the prevalence would be even rarer. This rarity of the valuable label in binary classification means that average performance or the “exam grade” is not so useful. The best detectors are good at finding the rare thing without letting too much of the common stuff through. The run we have been using through out this notebooks has the feature that one of the detectors is malfunctioning - it has 21% accuracy of detecting the α label. This run mimics an important business failure mode - a detector has flipped on the rare label. This is the reason that the algebraic evaluation is much better than majority voting. If all the classifiers were working correctly then majority voting would be closer to the actual prevalence and algebraic evaluation would be marginally better. If everything is working okay, there is no advantage to using algebraic evaluation over using majority voting. Both algorithms would output assessments that would lead to marginally different outcomes that upon deployment may not be measurable or repeatable.

3. The test was successful. But only because we know what the ground truth answer is. We have confirmed, using the correct label for the dataset, that for this test run, the classifiers are nearly error independent. But this is still not quite what we would need to increase our safety upon deployments. How would we know, on a given test whether the classifiers were error independent enough for us to trust the outcome of this simple evaluation model? We need more that the independent model to solve this engineering problem. This is the subject of another notebook - ErrorDependencyAndHowToMeasureIt.nb

Where next?