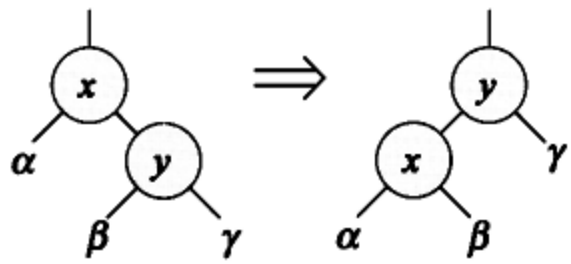
Homework #3
CSCI 4041: Algorithms and Data Structures

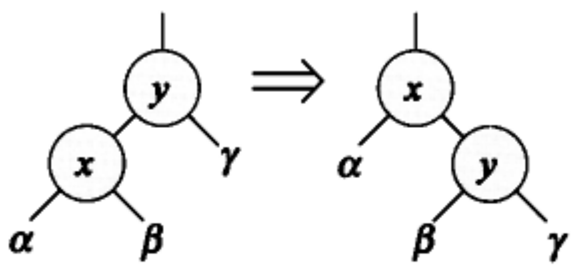
Last revision April 1, 2020

This homework assignment is about balanced binary search trees (BST's) and Optimal Binary Search Trees (OBST's). It is worth 35 points. It will be due in two weeks, at **11:55 PM** on **April 15, 2020.**

1. (5 points.) Suppose there is an algorithm that turns an unbalanced BST into a balanced BST. It is not the AVL algorithm *insert* that was discussed in a previous lecture. It is not any of the Red-Black algorithms that are discussed in Cormen. It uses these two rotations, called R_1 and R_2.

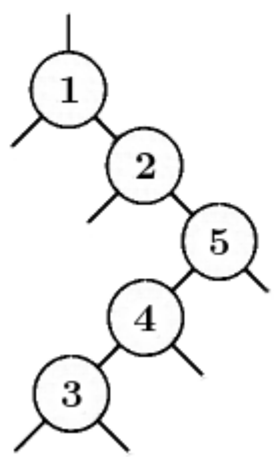


Rotation R_1



Rotation R_2

In the rotations, x and y are nodes; α, β, and γ are subtrees. Any or all of the subtrees may be empty. The rotations R_1 and R_2 are mirror images of each other. Now suppose that the algorithm is given the following BST. The numbers in the nodes are the BST's keys. Its values are not shown. The BST's root has key **1**.



Show how the algorithm might use R_1 and R_2 to turn this BST into one with the same keys, but with the minimum possible height. Do not write code or pseudocode.

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2. Questions **2a** and **2b** are about constructing an OBST, given some probabilities. Let $P(e)$ be the probability of e, and let m be an integer. The probabilities of m having certain values are shown below.

$$
\begin{aligned}
P(m \leq 0) &= 0.0125 \\
P(m = 1) &= 0.225 \\
P(m = 2) &= 0.0125 \\
P(m = 3) &= 0.5 \\
P(m = 4) &= 0.0125 \\
P(m = 5) &= 0.225 \\
P(m \geq 6) &= 0.0125
\end{aligned}
$$

2a. (10 points.) Using these probabilities, construct an OBST in which search succeeds when $m \in \{ 1, 3, 5 \}$, and fails otherwise.

2b. (10 points.) Prove that the OBST you constructed in **2a** is actually optimal.

Hints: Some of the probabilities describe success nodes, others describe failure nodes. You must decide which ones are which. It is simpler if you do not use the algorithm OPTIMAL-BST on page 402 of Cormen. Instead, use the definition of $E[S(T)]$ from the lectures. Questions **2a** and **2b** are closely related. If you can answer one, then you have what you need to answer the other.

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3. Questions **3a** and **3b** are about an OBST with n success keys and $n + 1$ failure keys. The probabilities of all the keys are equal to some constant c.

$$c = p_1 = p_2 \cdots = p_n = q_0 = q_1 \cdots = q_n$$

3a. (5 points.) What is c? Your answer must be $c =$ *some expression*. Briefly explain your answer.

3b. (5 points.) Suppose the procedure OPTIMAL-BST on page 402 of Cormen is given these probabilities. Prove that OPTIMAL-BST will produce an OBST with the minimum possible height. You need not show a rigorous proof. Part of your proof can be in English, but you will need to show some mathematics.

Hint: do not execute OPTIMAL-BST, but think about its purpose. What is it designed to do?