Math 482: Linear Programming[1] Mikhail Lavrov

Lecture 6: The Two-Phase Simplex Method

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1 The problem we want to solve

In the previous lecture, we took a shortcut. When our feasible region has the form

$$\{A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

and $\mathbf{b} \geq \mathbf{0}$, the slack variables we add to put the program in equational form form our starting basis. In other words, when all our variables are required to be nonnegative, and the point $\mathbf{x} = \mathbf{0}$ is feasible, it is a basic feasible solution.

Suppose we have a problem such as this one:

$$\begin{array}{ll} \underset{\mathbf{x} \in \mathbb{R}^4}{\text{minimize}} & x_1 + x_2 - x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 + x_4 = 7 \\ & 2x_1 - x_2 - x_3 - 3x_4 = -1 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

We can definitely solve this problem with the simplex method… provided we have a starting basic feasible solution. But we don't know how to find one!

We could try guessing: try to see if the basic solution with x_1 and x_2 as the basic variables is feasible. If not, try x_1 and x_3, and so on. But this reduces to the naive approach to linear programming, where we try all possible bases. (We don't even know, necessarily, that this linear program has any feasible solutions.)

The solution is the two-phase simplex method. In this method, we:

1. Solve an auxiliary problem, which has a built-in starting point, to determine if the original linear program is feasible. If we succeed, we find a basic feasible solution to the orignal LP.

2. From that basic feasible solution, solve the linear program the way we've done it before.

2 Writing down the auxiliary problem

Let's assume we're given a program in equational form: the constraints are $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

We begin with a minor change to the linear program whose purpose will become clear in a bit. For every constraint $A_i\mathbf{x} = b_i$ with $b_i < 0$, we multiply both sides by -1 to make b_i positive rather

[1] This document comes from the Math 482 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/482-fall-2019.html

than negative. In our example, we multiply the second constraint by -1 and get:

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^4}{\text{minimize}} \quad & x_1 + x_2 - x_3 - x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + x_4 = 7 \\ & -2x_1 + x_2 + x_3 + 3x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

This is always valid when we're dealing with equations. (It's only with inequalities that multiplying by -1 could mess things up.)

Next, we add new nonnegative variables $x_1^a, x_2^a, \dots$ called *artificial variables*, one for every constraint. We modify the existing constraints by adding x_i^a to the left-hand side of the i^{th} constraint. So our constraints become:

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 + x_1^a \quad\quad &= 7 \\ -2x_1 + x_2 + x_3 + 3x_4 \quad\quad + x_2^a &= 1 \\ x_1, x_2, x_3, x_4, x_1^a, x_2^a &\geq 0. \end{aligned}$$

In general, $A\mathbf{x} = \mathbf{b}$ becomes $A\mathbf{x} + I\mathbf{x}^a = \mathbf{b}$, where $\mathbf{x}^a$ is the vector of artificial variables.

So, on the one hand, there are plenty of solutions to this new system that have nothing to do with the old one. On the other hand, there's no question of feasibility. If we set $\mathbf{x} = \mathbf{0}$ and $\mathbf{x}^a = \mathbf{b}$, then we get a feasible solution! (In the example, this means setting $x_1 = x_2 = x_3 = x_4 = 0$, $x_1^a = 7$, and $x_2^a = 1$.) This is why we wanted to make sure that $\mathbf{b} \geq \mathbf{0}$: this way, the nonnegativity constraints on $\mathbf{x}^a$ are satisfied when we do this.

Moreover, even though we've departed from the original problem a little, the new problem is not completely unrelated. When $x_1^a = x_2^a = 0$ (and in general, when $\mathbf{x}^a = \mathbf{0}$), the new constraints reduce to the old constraints. This is the ideal state we'd like to be in: if we can get here, we've found a feasible solution of the original problem.

We can try to see if it's even possible to get to this state by minimizing the artificial objective function $z^a = x_1^a + x_2^a + \cdots + x_m^a$: the sum of all the artificial variables. In the example, this means solving the linear program

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^4, \mathbf{x}^a\in\mathbb{R}^2}{\text{minimize}} \quad & x_1^a + x_2^a \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + x_4 + x_1^a \quad\quad = 7 \\ & -2x_1 + x_2 + x_3 + 3x_4 \quad\quad + x_2^a = 1 \\ & x_1, x_2, x_3, x_4, x_1^a, x_2^a \geq 0. \end{aligned}$$

The smallest conceivable value of $x_1^a + x_2^a$ in the optimal solution to this linear program is 0, and that can only be achieved if $x_1^a = x_2^a = 0$. If we find an optimal solution to this linear program and its objective value is 0, great! In that case, we can take that optimal solution, throw away the artificial variables, and get a feasible solution to the original linear program.

What if the smallest possible value of $x_1^a + x_2^a$ turns out to be something positive? Well, in that case, we know that there can't be a feasible solution to the original linear program: any such solution would give a value of 0 in the new linear program. So in this case, you complain to whoever gave you that linear program, and tell them that you can't satisfy all of their constraints.

3 Setting up the tableau and solving

Recording all of this information in a tableau, we do things slightly differently. The first two rows are just the usual recording of the constraints; we make x_i^a the basic variable of the i^{th} constraint. However, we include rows for *both* objective functions: z, the original objective function, and z^a, the artificial objective function. We do this because eventually we'll want to switch back to the original linear program, and when we do, we'll want to keep z around.

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1^a	1	2	1	1	1	0	7
x_2^a	-2	1	1	3	0	1	1
$-z$	1	1	-1	-1	0	0	0
$-z^a$	0	0	0	0	1	1	0

Our first step here is always to finish the job of row-reducing. Everything in the artificial variables' columns is cleaned up, except for the $-z^a$ row. Subtract the x_1^a and x_2^a rows from that row, and we get:

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1^a	1	2	1	1	1	0	7
x_2^a	-2	1	1	3	0	1	1
$-z$	1	1	-1	-1	0	0	0
$-z^a$	1	-3	-2	-4	0	0	-8

We are minimizing z^a, so we want to bring in a variable with a negative entry in the $-z^a$ row. Let's use x_2. Of the ratios $\frac{7}{2}$ and $\frac{1}{1}$, both are nonnegative and the second is smaller, so x_2 is the new basic variable for the second row. We row-reduce:

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1^a	1	2	1	1	1	0	7
	-2	**1**	1	3	0	1	1
$-z$	1	1	-1	-1	0	0	0
$-z^a$	1	-3	-2	-4	0	0	-8

$\rightsquigarrow$

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1^a	5	0	-1	-5	1	-2	5
x_2	-2	**1**	1	3	0	1	1
$-z$	3	0	-2	-4	0	-1	-1
$-z^a$	-5	0	1	5	0	3	-5

Now x_1 has a negative entry in the $-z^a$ row, so we pivot to bring that in. Luckily, of the ratios $\frac{5}{5}$ and $\frac{1}{-2}$, the first is the smallest nonnegative one. Luckily, because it means x_1^a will also leave the basis and x_1 will become the basic variable for that row. We get:

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
	5	0	-1	-5	1	-2	5
x_2	-2	1	1	3	0	1	1
$-z$	3	0	-2	-4	0	-1	-1
$-z^a$	-5	0	1	5	0	3	-5

$\rightsquigarrow$

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1	**1**	0	$-1/5$	-1	$1/5$	$-2/5$	1
x_2	0	1	$3/5$	1	$2/5$	$1/5$	3
$-z$	0	0	$-7/5$	-1	$-3/5$	$1/5$	-4
$-z^a$	0	0	0	0	1	1	0

Now we have a basic feasible solution $(x_1, x_2, x_3, x_4) = (1, 3, 0, 0)$ to the original linear program! The artificial variables have all successfully been driven out.

To celebrate, let's rewrite the tableau, but erase all mention of anything artificial. This means:

- Deleting the x_1^a and x_2^a columns.

- Deleting the $-z^a$ row.

This gives us

	x_1	x_2	x_3	x_4	
x_1	1	0	$-1/5$	-1	1
x_2	0	1	$3/5$	1	3
$-z$	0	0	$-7/5$	-1	-4

Now we are left with an ordinary linear program to solve. Let's pivot to bring x_3 into the basis (since we're minimizing and $-\frac{7}{5} < 0$). The ratio $\frac{3}{3/5}$ is the smallest nonnegative one, so x_2 leaves the basis and we get:

	x_1	x_2	x_3	x_4	
x_1	1	$1/3$	0	$-2/3$	2
x_3	0	$5/3$	1	$5/3$	5
$-z$	0	$7/3$	0	$4/3$	3

Now all reduced costs are positive, so the current basic solution $(x_1, x_2, x_3, x_4) = (2, 0, 5, 0)$ achieves the minimum possible objective value of $z = -3$.

4 Complications

What happens when a linear program is infeasible? In this case, we will not be able to set all artificial variables to 0. We detect infeasibility by seeing that in the optimal solution to the stage one linear program, the artificial objective z^a is still positive.

There are several other complications that can occur.

Sometimes, when minimizing z^a, we may end with a basic solution where $z^a = 0$, and therefore $\mathbf{x}^a = \mathbf{0}$, and yet have some of the artificial variables still be basic. This would happen, for example, if we modified the first constraint in our example to $x_1 + 2x_2 + x_3 + x_4 = 2$. In that case, after the first pivot step, we'd have the tableau below (on the left):

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1^a	5	0	-1	-5	1	-2	0
x_2	-2	1	1	3	0	1	1
$-z$	3	0	-2	-4	0	-1	-1
$-z^a$	-5	0	1	5	0	3	0

$\rightsquigarrow$

	x_1	x_2	x_3	x_4	
	5	0	-1	-5	0
x_2	-2	1	1	3	1
$-z$	3	0	-2	-4	-1

Here, we've obtained a basic feasible solution $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$, but the tableau is missing a basic variable in the first row.

This can be fixed very easily. Just pick any of the variables with a nonzero entry in that row, and divide through by that entry to make that the basic variable. Then row-reduce. In this example, the easiest choice is x_3, because then we have to divide by -1, which is very little work. (It's fine that -1 is negative; normally, this would negate the right-hand side and ruin feasibility, but here, the right-hand side is 0 and doesn't care.)

In rare cases, this might be impossible, because the row with no basic variable is filled with 0 entries only. If this happens, it means we've discovered that one of the equations is redundant. Just delete that row (since it contains no useful information) and proceed.

5 Optional: a shortcut when dealing with slack variables

(We probably won't have time in class to talk about this topic, and it's not necessary to understand the two-phase simplex method, but in some cases it can save us a bit of work.)

Suppose that we're using the two-phase simplex method on a problem which started out in inequality form, where we added slack variables:

$$\begin{cases} x + y \leq 3 \\ x - 2y \leq 1 \\ -x - y \leq -1 \\ x, y \geq 0 \end{cases} \quad \rightsquigarrow \quad \begin{cases} x + y + s_1 = 3 \\ x - 2y + s_2 = 1 \\ -x - y + s_3 = -1 \\ x, y, s_1, s_2, s_3 \geq 0. \end{cases}$$

Our algorithm would tell us to multiply the last row by -1 (which is fine) and then to add artificial variables to each row (which seems wasteful). Instead, how about we only add an artificial variable x_3^a to the last row, and use the slack variables as the basic variables?

Now we have a stage-one problem

$$\begin{aligned} \underset{x,y,s_1,s_2,s_3,x_3^a}{\text{minimize}} \quad & x_3^a \\ \text{subject to} \quad & x + y + s_1 = 3 \\ & x - 2y + s_2 = 1 \\ & x + y - s_3 + x_3^a = 1 \\ & x, y, s_1, s_2, s_3, x_3^a \geq 0 \end{aligned}$$

which can start with the basic feasible solution $(x, y, s_1, s_2, s_3, x_3^a) = (0, 0, 3, 1, 0, 1)$. Only one artificial variable needs to be eliminated before we can start on the original problem.