## 2. Convergence theorems

In this section we analyze the dynamics of integrabilty in the case when sequences of measurable functions are considered. Roughly speaking, a "convergence theorem" states that integrability is preserved under taking limits. In other words, if one has a sequence $(f_n)_{n=1}^\infty$ of integrable functions, and if $f$ is some kind of a limit of the $f_n$'s, then we would like to conclude that $f$ itself is integrable, as well as the equality $\int f = \lim_{n\to\infty} \int f_n$.

Such results are often employed in two instances:

A. *When we want to prove that some function $f$ is integrable.* In this case we would look for a sequence $(f_n)_{n=1}^\infty$ of integrable approximants for $f$.

B. *When we want to construct and integrable function.* In this case, we will produce first the approximants, and then we will examine the existence of the limit.

The first convergence result, which is somehow primite, but very useful, is the following.

Lemma 2.1. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, let $a \in (0, \infty)$ and let $f_n : X \to [0, a]$, $n \geq 1$, be a sequence of measurable functions satisfying*

*(a) $f_1 \geq f_2 \geq \cdots \geq 0$;*

*(b) $\lim_{n\to\infty} f_n(x) = 0$, $\forall\, x \in X$.*

*Then one has the equality*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = 0. \tag{1}$$

Proof. Let us define, for each $\varepsilon > 0$, and each integer $n \geq 1$, the set

$$A_n^\varepsilon = \{x \in X \, : \, f_n(x) \geq \varepsilon\}.$$

Obviously, we have $A_n^\varepsilon \in \mathcal{A}$, $\forall\, \varepsilon > 0$, $n \geq 1$. One key fact we are going to use is the following.

*Claim 1*: *For every $\varepsilon > 0$, one has the equality*

$$\lim_{n\to\infty} \mu(A_n^\varepsilon) = 0.$$

Fix $\varepsilon > 0$. Let us first observe that, using $(a)$, we have the inclusions

$$A_1^\varepsilon \supset A_2^\varepsilon \supset \dots \tag{2}$$

Second, using $(b)$, we clearly have the equality $\bigcap_{k=1}^\infty A_k^\varepsilon = \varnothing$. Since $\mu$ is finite, using the Continuity Property (Lemma III.4.1), we have

$$\lim_{n\to\infty} \mu(A_n^\varepsilon) = \mu\Big(\bigcap_{n=1}^\infty A_n^\varepsilon\Big) = \mu(\varnothing) = 0.$$

*Claim 2*: *For every $\varepsilon > 0$ and every integer $n \geq 1$, one has the inequality*

$$0 \leq \int_X f_n \, d\mu \leq a\mu(A_n^\varepsilon) + \varepsilon\mu(X).$$

Fix $\varepsilon$ and $n$, and let us consider the elementary function

$$h_n^\varepsilon = a\varkappa_{A_n^\varepsilon} + \varepsilon\varkappa_{B_n^\varepsilon},$$

where $B_n^\varepsilon = X \smallsetminus A^\varepsilon$. Obviously, since $\mu(X) < \infty$, the function $h_n^\varepsilon$ is elementary integrable. By construction, we clearly have $0 \leq f_n \leq h_n^\varepsilon$, so using the properties of integration, we get

$$0 \leq \int_X f_n\,d\mu \leq \int_X h_n^\varepsilon\,d\mu = a\mu(A_n^\varepsilon) + \varepsilon\mu(B^\varepsilon) \leq a\mu(A^\varepsilon) + \varepsilon\mu(X).$$

Using Claims 1 and 2, it follows immediately that

$$0 \leq \liminf_{n\to\infty} \int_X f_n\,d\mu \leq \limsup_{n\to\infty} \int_X f_n\,d\mu \leq \varepsilon\mu(X).$$

Since the last inequality holds for arbitrary $\varepsilon > 0$, the desired equality (1) immediately follows. □

We now turn our attention to a weaker notion of limit, for sequences of measurable functions.

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be a one of the symbols $\bar{\mathbb{R}}$, $\mathbb{R}$, or $\mathbb{C}$. Suppose $f_n : X \to \mathbf{K}$, $n \geq 1$, are measurable functions. Given a measurable function $f : X \to \mathbb{K}$, we say that the sequence $(f_n)_{n=1}^\infty$ *converges $\mu$-almost everywhere to* $f$, if there exists some set $N \in \mathcal{A}$, with $\mu(N) = 0$, such that

$$\lim_{n\to\infty} f_n(x) = f(x),\ \ \forall\, x \in X \smallsetminus N.$$

In this case we write

$$f = \mu\text{-a.e.-}\lim_{n\to\infty} f_n.$$

Remark 2.1. This notion of convergence has, among other things, a certain uniqueness feature. One way to describe this is to say that the limit of a $\mu$-a.e. convergent sequence is $\mu$-almost unique, in the sense that if $f$ anf $g$ are measurable functions which satisfy the equalities $f = \mu\text{-a.e.-}\lim_{n\to\infty} f_n$ and $g = \mu\text{-a.e.-}\lim_{n\to\infty} f_n$, then $f = g$, $\mu$-a.e. This is quite obvious, because there exist sets $M, N \in \mathcal{A}$, with $\mu(M) = \mu(N) = 0$, such that

$$\lim_{n\to\infty} f_n(x) = f(x),\ \ \forall\, x \in X \smallsetminus M,$$
$$\lim_{n\to\infty} f_n(x) = g(x),\ \ \forall\, x \in X \smallsetminus N,$$

then it is obvious that $\mu(M \cup N) = 0$, and

$$f(x) = g(x),\ \ \forall\, x \in X \smallsetminus [M \cup N].$$

Comment. The above definition makes sense if $\mathbf{K}$ is an arbitrary metric space. Any of the spaces $\bar{\mathbb{R}}$, $\mathbb{R}$, and $\mathbb{C}$ is in fact a *complete* metric space. There are instances where the requirement that $f$ is measurable is if fact redundant. This is somehow clarified by the the next two exercises.

*Exercise 1*.* Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be a complete separable metric space, and let $f_n : X \to \mathbf{K}$, $n \geq 1$, be measurable functions.

(i) Prove that the set

$$L = \left\{ x \in X \,:\, \big(f_n(x)\big)_{n=1}^\infty \subset \mathbf{K} \text{ is convergent} \right\}$$

belongs to $\mathcal{A}$.

(ii) If we fix some point $\alpha \in \mathbf{K}$, and we define $\ell : X \to \mathbf{K}$ by

$$\ell(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } x \in L \\ \alpha & \text{if } x \in X \smallsetminus L \end{cases}$$

then $\ell$ is measurable.

In particular, if $\mu(X \smallsetminus L) = 0$, then $\ell = \mu\text{-a.e.-}\lim_{n\to\infty} f_n$.

Hints: If $d$ denotes the metric on $\mathbf{K}$, then prove first that, for every $\varepsilon > 0$ and every $m, n \geq 1$, the set

$$D^{\varepsilon}_{mn} = \big\{ x \in X : d\big(f_m(x), f_n(x)\big) < \varepsilon \big\}$$

belongs to $\mathcal{A}$ (use the results from III.3). Based on this fact, prove that, for every $p, k \geq 1$, the set

$$E^p_k = \big\{ x \in X \,;\, d\big(f_m(x), f_n(x)\big) < \tfrac{1}{p},\ \ \forall\, m, n \geq k \big\}$$

belongs to $\mathcal{A}$. Finally, use completeness to prove that

$$L = \bigcap_{p=1}^{\infty} \big( \bigcup_{k=1}^{\infty} E^p_k \big).$$

*Exercise 2.* Use the setting from Exercise 1. Prove that Let $(X, \mathcal{A}, \mu)$, $\mathbf{K}$, and $(f_n)_{n=1}^{\infty}$ be as in Exercise 1. Assume $f : X \to \mathbf{K}$ is an *arbitrary* function, for which there exists some set $N \in \mathcal{A}$ with $\mu(N) = 0$, and

$$\lim_{n\to\infty} f_n(x) = f(x),\ \ \forall\, x \in X \smallsetminus N.$$

Prove that, when $\mu$ is a *complete* measure on $\mathcal{A}$ (see III.5), the function $f$ is automatically measurable.

Hint: Use the results from Exercise 1. We have $X \smallsetminus N \subset L$, and $f(x) = \ell(x)$, $\forall\, x \in X \smallsetminus N$. Prove that, for a Borel set $B \subset \mathbf{K}$, one has the equality $f^{-1}(B) = \ell^{-1}(B) \triangle M$, for some $M \subset N$. By completeness, we have $M \in \mathcal{A}$, so $f^{-1}(B) \in \mathcal{A}$.

The following fundamental result is a generalization of Lemma 2.1.

Theorem 2.1 (Lebesgue Monotone Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $(f_n)_{n=1}^{\infty} \subset \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$ be a sequence with:*

- $f_n \leq f_{n+1}$, $\mu$-a.e., $\forall\, n \geq 1$;
- $\sup \big\{ \int_X f_n \, d\mu : n \geq 1 \big\} < \infty$.

*Assume $f : X \to [0, \infty]$ is a measurable function, with $f = \mu\text{-a.e.-}\lim_{n\to\infty} f_n$. Then $f \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$, and $\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$.*

Proof. Define $\alpha_n = \int_X f_n \, d\mu$, $n \geq 1$. First of all, we clearly have

$$0 \leq \alpha_1 \leq \alpha_2 \leq \dots,$$

so the sequence $(\alpha_n)_{n=1}^{\infty}$ has a limit $\alpha = \lim_{n\to\infty} \alpha_n$, and we have in fact the equality

$$\alpha = \sup \big\{ \alpha_n : n \geq 1 \big\} < \infty.$$

With these notations, all we need to prove is the fact that $f \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$, and that we have

$$\int_X f \, d\mu = \alpha. \tag{3}$$

Fix a set $M \in \mathcal{A}$, with $\mu(M) = 0$, and such that $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall\, x \in X \smallsetminus M$. For each $n$, we define the set

$$M_n = \{ x \in X : f_n(x) > f_{n+1}(x) \}.$$

Obviously $M_n \in \mathcal{A}$, and by assumption, we have $\mu(M_n) = 0$, $\forall\, n \geq 1$. Define the set $N = M \cup \big(\bigcup_{n=1}^{\infty} M_n\big)$. It is clear that $\mu(N) = 0$, and

- $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f(x)$, $\forall\, x \in X \smallsetminus N$;
- $f(x) = \lim_{n\to\infty} f_n(x)$, $\forall\, x \in X \smallsetminus N$.

So if we put $A = X \smallsetminus N$, and if we define the measurable functions $g_n = f_n \varkappa_A$, $n \geq 1$, and $g = f\varkappa_A$, then we have

(a) $0 \leq g_1 \leq g_2 \leq \cdots \leq g$ (everywhere!);
(b) $\lim_{n\to\infty} g_n(x) = g(x)$, $\forall\, x \in X$;
(c) $g_n = f_n$, $\mu$-a.e., $\forall\, n \geq 1$;
(d) $g = f$, $\mu$-a.e.;

Notice that property $(c)$ gives $g_n \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$ and $\int_X g_n\, d\mu = \alpha_n$, $\forall\, n \geq 1$. By property $(d)$, we see that we have the equivalence

$$f \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu) \Longleftrightarrow f \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu).$$

Moreover, if $g \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$, then we will have $\int_X g\, d\mu = \int_X f\, d\mu$. These observations show that it suffices to prove the theorem with $g$'s in place of the $f$'s. The advantage is now the fact that we have the slightly stronger properties $(a)$ and $(b)$ above. The first step in the proof is the following.

*Claim 1*: *For every $t \in (0, \infty)$, one has the inequality $\mu\big(g^{-1}((t, \infty])\big) \leq \frac{\alpha}{t}$.*

Denote the set $g^{-1}((t, \infty])$ simply by $A_t$. For each $n \geq 1$, we also define the set $A_t^n = g_n^{-1}((t, \infty])$. Using property $(a)$ above, it is clear that we have the inclusions

$$A_1^t \subset A_t^2 \subset \cdots \subset A_t. \tag{4}$$

Using property $(b)$ above, we also have the equality $A_t = \bigcup_{n=1}^{\infty} A_t^n$. Using the continuity Lemma 4.1, we then have

$$\mu(A_t) = \lim_{n\to\infty} \mu(A_t^n),$$

so in order to prove the Claim, it suffices to prove the inequalities

$$\mu(A_t^n) \leq \frac{\alpha_n}{t}, \ \ \forall\, n \geq 1. \tag{5}$$

But the above inequality is pretty obvious, since we clearly have $0 \leq t\varkappa_{A_t^n} \leq g_n$, which gives

$$t\mu(A_t^n) = \int_X t\varkappa_{A_t^n}\, d\mu \leq \int_X g_n\, d\mu = \alpha_n.$$

*Claim 2*: *For any elementary function $h \in \mathcal{A}$-Elem$_{\mathbb{R}}(X)$, with $0 \leq h \leq g$, one has*
(i) *$h \in \mathfrak{L}^1_{\mathbb{R},elem}(X, \mathcal{A}, \mu)$;*
(ii) *$\int_X h\, d\mu \leq \alpha$.*

Start with some elementary function $h$, with $0 \leq h \leq g$. Assume $h$ is not identically zero, so we can write it as

$$h = \beta_1 \varkappa_{B_1} + \cdots + \beta_p \varkappa_{B_p},$$

with $(B_j)_{j=1}^p \subset \mathcal{A}$ pairwise disjoint, and $0 < \beta_1 < \cdots < \beta_p$. Define the set $B = B_1 \cup \cdots \cup B_p$. It is obvious that, if we put $t = \beta_1/2$, we have the inclusion $B \subset g^{-1}\big((t, \infty]\big)$, so by Claim 1, we get $\mu(B) < \infty$. This gives, of course $\mu(B_j) < \infty$, $\forall\, j = 1, \dots, p$, so $h$ is indeed elementary integrable. To prove the estimate (ii), we define the measurable functions $h_n : X \to [0, \infty]$ by $h_n = \min\{g_n, h\}$, $\forall\, n \geq 1$.

Since $0 \leq h_n \leq g_n$, $\forall\, n \geq 1$, it follows that, $h_n \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$, $\forall\, n \geq 1$, and we have the inequalities

$$\int_X h_n \, d\mu \leq \int_X g_n \, d\mu = \alpha_n, \ \ \forall\, n \geq 1. \tag{6}$$

It is obvious that we have

($*$) $0 \leq h_1 \leq h_2 \leq \cdots \leq h \leq \beta_p \varkappa_B$ (everywhere);
($**$) $h(x) = \lim_{n\to\infty} h_n(x)$, $\forall\, x \in X$.

Let us restrict everything to $B$. We consider the $\sigma$-algebra $\mathcal{B} = \mathcal{A}\big|_B$, and the measure $\nu = \mu\big|_B$. Consider the elementary function $\psi = h\big|_B \in \mathcal{B}\text{-Elem}_{\mathbb{R}}(B)$, as well as the measurable functions $\psi_n = h_n\big|_B : B \to [0, \infty]$, $n \geq 1$. It is clear that $\psi \in \mathfrak{L}^1_{\mathbb{R},elem}(B, \mathcal{B}, \nu)$, and we have the equality

$$\int_B \psi \, d\nu = \int_X h \, d\mu. \tag{7}$$

Likewise, using ($*$), which clearly forces $h_n\big|_{X \smallsetminus B} = 0$, it follows that, for each $n \geq 1$, the function $\psi_n$ belongs to $\mathfrak{L}^1_+(B, \mathcal{B}, \nu)$, and by (6), we have

$$\int_B \psi_n \, d\nu = \int_X h_n \, d\mu, \ \ \forall\, n \geq 1. \tag{8}$$

Let us analyze the differences $\varphi_n = \psi - \psi_n$. On the one hand, using ($*$), we have $\varphi_n(x) \in [0, \beta_p]$, $\forall\, x \in B$, $n \geq 1$. On the other hand, again by ($*$), we have $\varphi_1 \geq \varphi_2 \geq \dots$. Finally, by ($**$) we have $\lim_{n\to\infty} \varphi_n(x) = 0$, $\forall\, x \in B$. We can apply Lemma 2.1, and we will get $\lim_{n\to\infty} \int_B \varphi_n \, d\nu = 0$. This clearly gives,

$$\int_B \psi \, d\nu = \lim_{n\to\infty} \int_B \psi_n \, d\nu,$$

and then using (7) and (8), we get the equality

$$\int_X h \, d\mu = \lim_{n\to\infty} \int_X h_n \, d\mu.$$

Combining this with (6), immediately gives the desired estimate $\int_X h \, d\mu \leq \alpha$.

Having proven Claim 2, let us observe now that, using the definition of the positive integral, it follows immediately that $g \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$, and we have the inequality

$$\int_X g \, d\mu \leq \alpha.$$

The other inequality is pretty obvious, because the inequality $g \geq g_n$ forces

$$\int_X g \, d\mu \geq \int_X g_n \, d\mu = \alpha_n, \ \ \forall\, n \geq 1,$$

so we immediately get

$$\int_X g \, d\mu \geq \sup\{\alpha_n \,;\, n \geq 1\} = \alpha. \quad \square$$

Comment. In the previous section we introduced the convention which defines $\int_X f \, d\mu = \infty$, if $f : X \to [0, \infty]$ is measurable, but $f \notin \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$. Using this convention, the Lebesgue Monotone Convergence Theorem has the following general version.

Theorem 2.2 (General Lebesgue Monotone Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f, f_n : X \to [0, \infty]$, $n \geq 1$, be measurable functions, such that*

- $f_n \leq f_{n+1}$, *$\mu$-a.e., $\forall\, n \geq 1$;*
- $f = \mu\text{-a.e.-}\lim_{n\to\infty} f_n$.

*Then*

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu. \tag{9}$$

Proof. As before, the sequence $(\alpha_n)_{n=1}^\infty \subset [0, \infty]$, defined by $\alpha_n = \int_X f_n\, d\mu$, $\forall\, n \geq 1$, is non-decreasing, and is has a limit

$$\alpha = \lim_{n\to\infty} \alpha_n = \sup\left\{ \int_X f_n\, d\mu \,:\, n \geq 1 \right\} \in [0, \infty].$$

There are two cases to analyze.

*Case I*: $\alpha = \infty$.

In this case the inequalities $f \geq f_n \geq 0$, $\mu$-a.e. will force

$$\int_X f\, d\mu \geq \int_X f_n\, d\mu = \alpha_n, \ \ \forall\, n \geq 1,$$

which will force $\int_X f\, d\mu \geq \alpha$, so we indeed get

$$\int_X f\, d\mu = \infty = \alpha.$$

*Case II*: $\alpha < \infty$.

In this case we apply directly Theorem 2.1. □

The following result provides an equivalent definition of integrability for non-negative functions (compare to the construction in Section 1).

Corollary 2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0, \infty]$ be a measurable function. The following are equivalent:*

(i) $f \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$;
(ii) *there exists a sequence $(h_n)_{n=1}^\infty \subset \mathfrak{L}^1_{\mathbb{R},elem}(X, \mathcal{A}, \mu)$, with*
  - $0 \leq h_1 \leq h_2 \dots$;
  - $\lim_{n\to\infty} h_n(x) = f(x)$, $\forall\, x \in X$;
  - $\sup\left\{ \int_X h_n\, d\mu \,:\, n \geq 1 \right\} < \infty$.

*Moreover, if $(h_n)_{n=1}^\infty$ is as in* (ii)*, then one has the equality*

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X h_n\, d\mu. \tag{10}$$

Proof. $(i) \Rightarrow (ii)$. Assume $f \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$. Using Theorem III.3.2, we know there exists a sequence $(h_n)_{n=1}^\infty \subset \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with

(*a*) $0 \leq h_1 \leq h_2 \leq \dots \leq f$;
(*b*) $\lim_{n\to\infty} h_n(x) = f(x)$, $\forall\, x \in X$.

Note the $(a)$ forces $h_n \in \mathfrak{L}^1_{\mathbb{R},elem}(X, \mathcal{A}, \mu)$, as well as the inequalities $\int_X h_n\, d\mu \leq \int_X f\, d\mu < \infty$, $\forall\, n \geq 1$, so the sequence $(h_n)_{n=1}^\infty$ clearly satisfies condition (ii).

The implication $(ii) \Rightarrow (i)$, and the equality (10) immediately follow from the General Lebesgue Monotone Convergence Theorem. □

Corollary 2.2 (Fatou Lemma). *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_n : X \to [0, \infty]$, $n \geq 1$, be a sequence of measurable functions. Define the function $f : X \to [0, \infty]$ by*

$$f(x) = \liminf_{n \to \infty} f_n(x), \quad \forall \, x \in X.$$

*Then $f$ is measurable, and one has the inequality*

$$\int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.$$

Proof. The fact that $f$ is measurable is already known (see Corollary III.3.5). Define the sequence $(\alpha_n)_{n=1}^\infty \subset [0, \infty]$ by $\alpha_n = \int_X f_n \, d\mu$, $\forall \, n \geq 1$.

Define, for each integer $n \geq 1$, the function $g_n : X \to [0, \infty]$ by

$$g_n(x) = \inf \big\{ f_k(x) \, : \, k \geq n \big\}, \quad \forall \, x \in X.$$

By Corollary III.3.4, we know that $g_n$, $n \geq 1$ are all measurable. Moreover, it is clear that

- $0 \leq g_1 \leq g_2 \leq \dots$;
- $f(x) = \lim_{n \to \infty} g_n(x)$, $\forall \, x \in X$.

By the General Lebesgue Monotone Convergence Theorem 2.2, it follows that

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X g_n \, d\mu. \tag{11}$$

Notice that, if we define the sequence $(\beta_n)_{n=1}^\infty \subset [0, \infty]$, by $\beta_n = \int_X g_n \, d\mu$, $\forall \, n \geq 1$, then the obvious inequalities $0 \leq g_n \leq f_n$ give $\int_X g \, d\mu \leq \int_X f_n \, d\mu$, so we get

$$\beta_n \leq \alpha_n, \quad \forall \, n \geq 1.$$

Using (11), we then get

$$\int_X f \, d\mu = \lim_{n \to \infty} \beta_n = \liminf_{n \to \infty} \beta_n \leq \liminf_{n \to \infty} \alpha_n. \quad \square$$

The following is an important application of Theorem 2.1, that deals with Riemann integration.

Corollary 2.3. *Let $a < b$ be real numbers. Denote by $\lambda$ the Lebesgue measure, and consider the Lebesgue space $\big([a, b], \mathfrak{M}_\lambda([a, b]), \lambda\big)$, where $\mathfrak{M}_\lambda([a, b])$ denotes the $\sigma$-algebra of all Lebesgue measurable subsets of $[a, b]$. Then every Riemann integrable function $f : [a, b] \to \mathbb{R}$ belongs to $\mathfrak{L}^1_{\mathbb{R}}([a, b], \mathfrak{M}_\lambda([a, b]), \lambda)$, and one has the equality*

$$\int_{[a,b]} f \, d\lambda = \int_a^b f(x) \, dx. \tag{12}$$

Proof. We are going to use the results from III.6. First of all, the fact that $f$ is Lebesgue integrable, i.e. $f$ belongs to $\mathfrak{L}^1_{\mathbb{R}}\big([a, b], \mathfrak{M}_\lambda([a, b]), \lambda\big)$, is clear since $f$ is Lebesgue measurable, and bounded. (Here we use the fact that the measure space $\big([a, b], \mathfrak{M}_\lambda([a, b]), \lambda\big)$ is *finite*.)

Next we prove the equality between the Riemann integral and the Lebesgue integral. Adding a constant, if necessary, we can assume that $f \geq 0$. For every partition $\Delta = (a = x_0 < x_1 < \dots < x_n = b)$ of $[a, b]$, we define the numbers

$$m_k = \inf_{t \in [x_{k-1}, x_k]} f(t), \quad \forall \, k = 1, \dots, n,$$

and we define the function

$$f_\Delta = m_1 \varkappa_{[x_0,x_1]} + m_2 \varkappa_{(x_1,x_2]} + \cdots + m_m \varkappa_{(x_{n-1},x_n]}.$$

Fix a sequence of partitions $(\Delta_p)_{p=1}^\infty$, with $\Delta_1 \subset \Delta_2 \subset \ldots$, and $\lim_{p\to\infty} |\Delta_p| = 0$, We know (see III.6) that we have

$$f = \lambda\text{-a.e.-}\lim_{p\to\infty} f_{\Delta_p}.$$

Clearly we have $0 \le f_{\Delta_1} \le f_{\Delta_2} \le \cdots \le f$, so by Theorem 2.1, we get

$$\int_{[a,b]} f \, d\lambda = \lim_{p\to\infty} \int_{[a,b]} f_{\Delta_p} \, d\lambda. \tag{13}$$

Notice however that

$$\int_{[a,b]} f_{\Delta_p} \, d\lambda = L(f, \Delta_p), \ \ \forall \, p \ge 1,$$

where $L(f, \Delta_p)$ denotes the lower Darboux sum. Combining this with (13), and with the well known properties of Riemann integration, we immediately get (12). $\square$

The following is another important convergence theorem.

Theorem 2.3 (Lebesgue Dominated Convergence Theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be one of the symbols $\bar{\mathbb{R}}$, $\mathbb{R}$, or $\mathbb{C}$, and let $(f_n)_{n=1}^\infty \subset \mathfrak{L}^1_{\mathbf{K}}(X, \mathcal{A}, \mu)$. Assume $f : X \to \mathbf{K}$ is a measurable function, such that*

(i) $f = \mu$-a.e.-$\lim_{n\to\infty} f_n$;

(ii) *there exists some function $g \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$, such that*

$$|f_n| \le g, \ \ \mu\text{-}a.e., \ \ \forall \, n \ge 1.$$

*Then $f \in \mathfrak{L}^1_{\mathbf{K}}(X, \mathcal{A}, \mu)$, and one has the equality*

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu. \tag{14}$$

Proof. The fact that $f$ is integrable follows from the following

*Claim*: $|f| \le g$, $\mu$-a.e.

To prove this fact, we define, for each $n \ge 1$, the set

$$M_n = \big\{ x \in X \, : \, |f_n(x)| > g(x) \big\}.$$

It is clear that $M_n \in \mathcal{A}$, and $\mu(M_n) = 0$, $\forall \, n \ge 1$. If we choose $M \in \mathcal{A}$ such that $\mu(M) = 0$, and

$$f(x) = \lim_{n\to\infty} f_n(x), \ \ \forall \, x \in X \smallsetminus M,$$

then the set $N = M \cup \big( \bigcup_{n=1}^\infty M_n \big) \in \mathcal{A}$ will satisfy

- $\mu(N) = 0$;
- $|f_n(x)| \le g(x)$, $\forall \, x \in X \smallsetminus N$;
- $f(x) = \lim_{n\to\infty} f_n(x)$, $\forall \, x \in X \smallsetminus N$.

We then clearly get

$$|f(x)| \le g(x), \ \ \forall \, x \in X \smallsetminus N,$$

and the Claim follows.

Having proven that $f$ is integrable, we now concentrate on the equality (14).

*Case I*: $\mathbf{K} = \bar{\mathbb{R}}$.

First of all, without any loss of generality, we can assume that $0 \leq g(x) < \infty$, $\forall\, x \in X$. (See Lemma 1.2.) Let us define the functions $g_n = \min\{f_n, g\}$ and $h_n = \max\{f_n, -g\}$, $n \geq 1$. Since we have $-g \leq f_n \leq g$, $\mu$-a.e., we immediately get

$$g_n = h_n = f_n,\ \ \mu\text{-a.e.},\ \ \forall\, n \geq 1, \tag{15}$$

thus giving the fact that $g_n, h_n \in \mathfrak{L}^1_{\mathbb{R}}(X, \mathcal{A}, \mu)$, $\forall\, n \geq 1$, as well as the equalities

$$\int_X g_n\, d\mu = \int_X h_n\, d\mu = \int_X f_n\, d\mu,\ \ \forall\, n \geq 1. \tag{16}$$

Define the measurable functions $\varphi, \psi : X \to \bar{\mathbb{R}}$ by

$$\varphi(x) = \liminf_{n\to\infty} h_n(x) \text{ and } \psi(x) = \limsup_{n\to\infty} g_n(x),\ \ \forall\, x \in X.$$

Using (15), we clearly have $f = \varphi = \psi$, $\mu$-a.e., so we get

$$\int_X f\, d\mu = \int_X \varphi\, d\mu = \int_X \psi\, d\mu. \tag{17}$$

Remark also that we have equalities

$$g(x) - \varphi(x) = \liminf_{n\to\infty} [g(x) - g_n(x)] \text{ and } g(x) + \psi(x) = \liminf_{n\to\infty} [g(x) + h_n(x)],\ \ \forall\, x \in X. \tag{18}$$

Since we clearly have

$$g - g_n \geq 0 \text{ and } g + h_n \geq 0,\ \ \forall\, n \geq 1,$$

using (18), and Fatou Lemma (Corollary 2.2) and we get the inequalities

$$\int_X (g - \varphi)\, d\mu \leq \liminf_{n\to\infty} \int_X (g - g_n)\, d\mu,$$
$$\int_X (g + \psi)\, d\mu \leq \liminf_{n\to\infty} \int_X (g + h_n)\, d\mu,$$

In other words, we get

$$\int_X g\, d\mu - \int_X \varphi\, d\mu \leq \liminf_{n\to\infty} \left[ \int_X g\, d\mu - \int_X g_n\, d\mu \right] = \int_X g\, d\mu - \limsup_{n\to\infty} \int_X g_n\, d\mu,$$
$$\int_X g\, d\mu + \int_X \psi\, d\mu \leq \liminf_{n\to\infty} \left[ \int_X g\, d\mu + \int_X h_n\, d\mu \right] = \int_X g\, d\mu + \liminf_{n\to\infty} \int_X h_n\, d\mu.$$

Using the equalities (16) and (17), the above inequalities give

$$\int_X f\, d\mu = \int_X \varphi\, d\mu \geq \limsup_{n\to\infty} \int_X g_n\, d\mu = \limsup_{n\to\infty} \int_X f_n\, d\mu,$$
$$\int_X f\, d\mu = \int_X \psi\, d\mu \leq \liminf_{n\to\infty} \int_X h_n\, d\mu = \liminf_{n\to\infty} \int_X f_n\, d\mu.$$

In other words, we have

$$\int_X f\, d\mu \leq \liminf_{n\to\infty} \int_X f_n\, d\mu \leq \limsup_{n\to\infty} \int_X f_n\, d\mu \leq \int_X f\, d\mu,$$

thus giving the equality (14)

The case $\mathbf{K} = \mathbb{R}$ is trivial (it is in fact contained in case $\mathbf{K} = \bar{\mathbb{R}}$).

The case $\mathbf{K} = \mathbb{C}$ is also pretty clear, using real and imaginary parts, since for each $n \geq 1$, we clearly have

$$|\mathrm{Re}\, f_n| \leq g, \ \mu\text{-a.e.},$$
$$|\mathrm{Im}\, f_n| \leq g, \ \mu\text{-a.e.},. \qquad \square$$

*Exercise 3.* Give an example of a sequence of continuous functions $f_n : [0,1] \to [0,\infty)$, such that

(*a*) $\lim_{n\to\infty} f_n(x) = 0$, $\forall\, n \geq 1$;
(*b*) $\int_{[0,1]} f_n \, d\lambda = 1$, $\forall\, n \geq 1$.

(Here $\lambda$ denotes the Lebesgue measure). This shows that the Lebesgue Dominated Convergence Theorem fails, without the dominance condition (ii).

Hint: Consider the functions $f_n$ defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq 1/n \\ n(2 - nx) & \text{if } 1/n \leq x \leq 2/n \\ 0 & \text{if } 2/n \leq x \leq 1 \end{cases}$$

The Lebesgue Convergence Theorems 2.2 and 2.3 have many applications. They are among the most important results in Measure Theory. In many instances, these theorem are employed during proofs, at key steps. The next two results are good illustrations.

Proposition 2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0,\infty]$ be a measurable function. Then the map*

$$\nu : \mathcal{A} \ni A \longmapsto \int_A f \, d\mu \in [0,\infty]$$

*defines a measure on $\mathcal{A}$.*

Proof. It is clear that $\nu(\varnothing) = 0$. To prove $\sigma$-additivity, start with a pairwise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$, and put $A = \bigcup_{n=1}^\infty A_n$. For each integer $n \geq 1$, define the set $B_n = \bigcup_{k=1}^n A_k$, and the measurable function $g_n = f\varkappa_{B_n}$. Define also the function $g = f\varkappa_A$. It is obvious that

- $0 \leq g_1 \leq g_2 \leq \cdots \leq g$ (everywhere),
- $\lim_{n\to\infty} g_n(x) = g(x)$, $\forall\, x \in X$.

Using the General Lebesgue Monotone Convergence Theorem, it follows that

$$\nu(A) = \int_X f\varkappa_A \, d\mu = \int_X g \, d\mu = \lim_{n\to\infty} \int_X g_n \, d\mu. \tag{19}$$

Notice now that, for each $n \geq 1$, one has the equality

$$g_n = f\varkappa_{A_1} + \cdots + f\varkappa_{A_n},$$

so using Remark 1.7.C, we get

$$\int_X g_n \, d\mu = \sum_{k=1}^n \int_X f\varkappa_{A_k} \, d\mu = \sum_{k=1}^n \nu(A_k),$$

so the equality (19) immediately gives $\nu(A) = \sum_{n=1}^\infty \nu(A_n)$. $\square$

The next result is a version of the previous one for $\mathbb{K}$-valued functions.

Proposition 2.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be one of the symbols $\bar{\mathbb{R}}$, $\mathbb{R}$, or $\mathbb{C}$, and let $(A_n)_{n=1}^\infty \subset \mathcal{A}$ be a pairwise disjoint sequence with $\bigcup_{n=1}^\infty A_n = X$. For a function $f : X \to \mathbf{K}$, the following are equivalent.*

(i) $f \in \mathfrak{L}^1_{\mathbf{K}}(X, \mathcal{A}, \mu)$;
(ii) $f\big|_{A_n} \in \mathfrak{L}^1_{\mathbf{K}}(A_n, \mathcal{A}\big|_{A_n}, \mu\big|_{A_n})$, $\forall\, n \geq 1$, *and*

$$\sum_{n=1}^{\infty} \int_{A_n} |f| \, d\mu < \infty.$$

*Moreover, if $f$ satisfies these equivalent conditions, then*

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu.$$

Proof. $(i) \Rightarrow (ii)$. Assume $f \in \mathfrak{L}^1_{\mathbf{K}}(X, \mathcal{A}, \mu)$. Applying Proposition 2.1, to $|f|$, we immediately get

$$\sum_{n=1}^{\infty} \int_{A_n} |f| \, d\mu = \int_X |f| \, d\mu < \infty,$$

which clearly proves (ii).

$(ii) \Rightarrow (i)$. Assume $f$ satisfies condition (ii). Define, for each $n \geq 1$, the set $B_n = A_1 \cup \dots \cup A_n$. First of all, since $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, and $\bigcup_{n=1}^{\infty} A_n = X$, it follows that $f$ is measurable. Consider the the functions $f_n = f \varkappa_{B_n}$ and $g_n = f \varkappa_{A_n}$, $n \geq 1$. Notice that, since $f\big|_{A_n} \in \mathfrak{L}^1_{\mathbf{K}}(A_n, \mathcal{A}\big|_{A_n}, \mu\big|_{A_n})$, it follows that $g_n \in \mathfrak{L}^1_{\mathbf{K}}(X, \mathcal{A}, \mu)$, $\forall\, n \geq 1$, and we also have

$$\int_X g_n \, d\mu = \int_{A_n} f \, d\mu, \ \ \forall\, n \geq 1.$$

In fact we also have

$$\int_X |g_n| \, d\mu = \int_{A_n} |f| \, d\mu, \ \ \forall\, n \geq 1.$$

Notice that we obviously have $f_n = g_1 + \dots + g_n$, and $|f_n| = |g_1| + \dots + |g_n|$, so if we define

$$S = \sum_{n=1}^{\infty} \int_{A_n} |f| \, d\mu,$$

we get

$$\int_X |f_n| \, d\mu = \sum_{k=1}^{n} \int_{A_k} |f| \, d\mu \leq S < \infty, \ \ \forall\, n \geq 1.$$

Notice however that we have $0 \leq |f_1| \leq |f_2| \leq \dots |f|$, as well as the equality $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall\, x \in X$. On the one hand, using the General Lebesgue Monotone Convergence Theorem, we will get

$$\int_X |f| \, d\mu = \lim_{n\to\infty} \int_X |f_n| \, d\mu = \lim_{n\to\infty} \left[ \sum_{k=1}^{n} \int_{A_k} |f| \, d\mu \right] = \sum_{n=1}^{\infty} \int_{A_n} |f| \, d\mu = S < \infty,$$

which proves that $|f| \in \mathfrak{L}^1_{+}(X, \mathcal{A}, \mu)$, so in particular $f$ belongs to $\mathfrak{L}^1_{\mathbf{K}}(X, \mathcal{A}, \mu)$. On the other hand, since we have $|f_n| \leq |f|$, by the Lebesgue Dominated Convergence Theorem, we get

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu = \lim_{n\to\infty} \left[ \sum_{k=1}^{n} \int_{A_k} f \, d\mu \right] = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu. \quad \square$$

Corollary 2.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathbf{K}$ be one of the symbols $\bar{\mathbb{R}}$, $\mathbb{R}$, or $\mathbb{C}$, and let $(X_n)_{n=1}^\infty \subset \mathcal{A}$ be sequence with $\bigcup_{n=1}^\infty X_n = X$, and $X_1 \subset X_2 \subset \dots$. For a function $f : X \to \mathbf{K}$ be a measurable function, the following are equivalent.*

(i) $f \in \mathfrak{L}^1_{\mathbf{K}}(X, \mathcal{A}, \mu)$;

(ii) $f\big|_{X_n} \in \mathfrak{L}^1_{\mathbf{K}}(X_n, \mathcal{A}\big|_{X_n}, \mu\big|_{X_n})$, $\forall\, n \geq 1$, *and*

$$\sup\left\{ \int_{X_n} |f|\, d\mu \,:\, n \geq 1 \right\} < \infty.$$

*Moreover, if $f$ satisfies these equivalent conditions, then*

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_{X_n} f\, d\mu.$$

Proof. Apply the above result to the sequence $(A_n)_{n=1}^\infty$ given by $A_1 = X_1$ and $A_n = X_n \smallsetminus X_{n-1}$, $\forall\, n \geq 2$. $\square$

Remark 2.2. Suppose $(X, \mathcal{A}, \mu)$ is a measure space, $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $f \in \mathfrak{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$. By Proposition 2.2, we get the fact that the map

$$\nu : \mathcal{A} \ni A \longmapsto \int_A f\, d\mu \in \mathbb{K}$$

is a $\mathbb{K}$-*valued measure on* $\mathcal{A}$. By Proposition 2.1, we also know that

$$\omega : \mathcal{A} \ni A \longmapsto \int_A |f|\, d\mu \in \mathbb{K}$$

is a finite "honest" measure on $\mathcal{A}$. Using Proposition 1.6, we clearly have

$$|\nu(A)| = \left| \int_A f\, d\mu \right| \leq \int_A |f|\, d\mu = \omega(A), \ \ \forall\, A \in \mathcal{A},$$

which by the results from III.8 gives the inequality $|\nu| \leq \omega$. (Here $|\nu|$ denotes the variation measure of $\nu$.) Later on (see Section 4) we are going to see that in fact we have the equality $|\nu| = \omega$.

Comment. It is important to understand the "sequential" nature of the convergence theorems discussed here. If we examine for instance the Mononotone Convergence Theorem, we could easily formulate a "series" version, which states the equality

$$\int_X \big( \sum_{n=1}^\infty f_n \big)\, d\mu = \sum_{n=1}^\infty \int_X f_n\, d\mu,$$

for any sequence measurable functions $f_n : X \to [0, \infty]$.

Suppose now we have an arbitrary family $f_j : X \to [0, \infty]$, $j \in J$ of measurable functions, and we define

$$f(x) = \sum_{j \in J} f_j(x), \ \ \forall\, x \in X.$$

(Here we use the summability convention which defines the sum as the supremum of all finite sums.) In general, $f$ is not always measurable. But if it is, one still cannot conclude that

$$\int_X f\, d\mu = \sum_{j \in J} \int_X f_i\, d\mu.$$

The following example illustrates this anomaly.

EXAMPLE 2.1. Take the measure space $([0,1], \mathfrak{M}_\lambda([0,1]), \lambda)$, and fix $J \subset [0,1]$ and arbitrary set. For each $j \in J$ we consider tha characteristic function $f_j = \varkappa_{\{j\}}$. It is obvious that the function $f : X \to [0,\infty]$, defined by

$$f(x) = \sum_{j \in J} f_j(x), \quad \forall \, x \in [0,1],$$

is equal to $\varkappa_J$ If $J$ is non-measurable, this already gives an example when $f = \sum_{j \in J} f_j$ is non-measurable. But even if $J$ were measurable, it would be impossible to have the equality

$$\int_X f \, d\lambda = \sum_{j \in J} \int_X f_j \, d\lambda,$$

simply because the right hand side is zero, while the left hand side is equal to $\lambda(J)$.

The next two exercises illustrate straightforward (but nevertheless interesting) applications of the convergence theorems to quite simple situations.

*Exercise 4.* Let $\mathcal{A}$ be a $\sigma$-algebra on a (non-empty) set $X$, and let $(\mu_n)_{n=1}^\infty$ be a sequence of signed measures on $\mathcal{A}$. Assume that, for each $A \in \mathcal{A}$, the sequence $\big(\mu_n(A)\big)_{n=1}^\infty$ has a limit denoted $\mu(A) \in [-\infty, \infty]$. Prove that the map $\mu : \mathcal{A} \to [0,\infty]$ defines a measure on $\mathcal{A}$, if the sequence $(\mu_n)_{n=1}^\infty$ satisfies one of the following hypotheses:

A. $0 \leq \mu_1(A) \leq \mu_2(A) \leq \dots, \forall \, A \in \mathcal{A}$;
B. there exists a *finite* measure $\omega$ on $\mathcal{A}$, such that $|\mu_n(A)| \leq \omega(A)$, $\forall \, n \geq 1$, $A \in \mathcal{A}$.

HINT: To prove $\sigma$-additivity, fix a pairwise disjoint sequence $(A_k)_{k=1}^\infty \subset \mathcal{A}$, and put $A = \bigcup_{k=1}^\infty A_k$. Treat the problem of proving the equality $\mu(A) = \sum_{k=1}^\infty \mu(A_k)$ as a convergence problem on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ - with $\nu$ the counting measure - for the sequence of functions $f_n : \mathbb{N} \to [0,\infty]$ defined by $f_n(k) = \mu_n(A_k)$, $\forall \, k \in \mathbb{N}$.

*Exercise 5*.* Let $\mathcal{A}$ be a $\sigma$-algebra on a (non-empty) set $X$, and let $(\mu_j)_{j \in J}$ be a family of signed measures on $\mathcal{A}$. Assume either of the following is true:

A. $\mu_j(A) \geq 0$, $\forall \, j \in J$, $A \in \mathcal{A}$.
B. There exists a *finite* measure $\omega$ on $\mathcal{A}$, such that $\sum_{j \in J} |\mu_j(A)| \leq \omega(A)$, $\forall \, A \in \mathcal{A}$.

Define the map $\mu : \mathcal{A} \to [0,\infty]$ by $\mu(A) = \sum_{j \in J} \mu_j(A)$, $\forall \, A \in \mathcal{A}$. (In Case A, the sum is defined as the supremum over finite sums. In case B, it follows that the family $\big(\mu_j(A)\big)_{j \in J}$ is summable.) Prove that $\mu$ is a measure on $\mathcal{A}$.

HINT: To prove $\sigma$-additivity, fix a pairwise disjoint sequence $(A_k)_{k=1}^\infty \subset \mathcal{A}$, and put $A = \bigcup_{k=1}^\infty A_k$. To prove the equality $\mu(A) = \sum_{k=1}^\infty \mu(A_k)$, analyze the following cases: (I) There is some $k \geq 1$, such that $\mu(A_k) = \infty$; (II) $\mu(A_k) < \infty$, $\forall \, k \geq 1$. The first case is quite trivial. In the second case reduce the problem to the previous exercise, by observing that, for each $k \geq 1$, the set $J(A_k) = \{j \in J : \mu_j(A_k) > 0\}$ must be countable. Then the set $J(A) = \{j \in J : \mu_j(A) > 0\}$ is also countable.

COMMENT. One of the major drawbacks of the theory of Riemann integration is illustrated by the approach to improper integration. Recall that for a function

$h : [a, b) \to \mathbb{R}$ the improper Riemann integral is defined as

$$\int_a^{b-} h(t)\,dt = \lim_{x \to b-} \int_a^x f(t)\,dt,$$

provided that

($a$) $h\big|_{[a,x]}$ is Riemann integrable, $\forall\, x \in (a, b)$, and

($b$) the above limit exists.

The problem is that although the improper integral may exist, and the function is actually defined on $[a, b]$, it may fail to be Riemann integrable, for example when it is unbounded.

In contrast to this situation, by Corollary 2.4, we see that if for example $h \geq 0$, then the Lebesgue integrability of $h$ on $[a, b]$ *is* equivalent to the fact that

(i) $h\big|_{[a,x]}$ is Lebesgue integrable, $\forall\, x \in (a, b)$, and

(ii) $\lim_{x \to b-} \int_{[a,x]} h\,d\lambda$ exists.

Going back to the discussion on improper Riemann integral, we can see that a sufficient condition for $h : [a, b) \to \mathbb{R}$ to be Riemann integrable in the improper sense, is the fact that $h$ has property ($a$) above, and $h$ is Lebesgue integrable on $[a, b)$. In fact, if $h \geq 0$, then by Corollary 2.4, this is also necessary.

Notation. Let $-\infty \leq a < b \leq \infty$, and let $f$ be a Lebesgue integrable function, defined on some interval $J$ which is one of $(a, b)$, $[a, b)$, $(a, b]$, or $[a, b]$. Then the Lebesgue integral $\int_J f\,d\lambda$ will be denoted simply by $\int_a^b f\,d\lambda$.

*Exercise 6**. Let $(X, \mathcal{A}, \mu)$ be a *finite* measure space. Prove that for every $f \in \mathfrak{L}^1_+(X, \mathcal{A}, \mu)$, one has the equality

$$\int_X f\,d\mu = \int_0^\infty \mu\big(f^{-1}([t, \infty])\big)\,dt,$$

where the second term is defined as improper Riemann integral.

Hint: The function $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = \mu\big(f^{-1}([t, \infty])\big)$, $\forall\, t \geq 0$, is non-increasing, so it is Riemann integrable on every interval $[0, a]$, $a > 0$. Prove the inequalities

$$\int_{X_a} f\,d\mu \leq \int_0^a \varphi(t)\,dt \leq \int_X f\,d\mu, \ \ \forall\, a > 0,$$

where $X_a = f^{-1}([0, a))$, by analyzing lower and upper Darboux sums of $\varphi\big|_{[0,a]}$. Use Corollary 2.4 to get $\lim_{a \to \infty} \int_{X_a} f\,d\mu = \int_X f\,d\mu$.