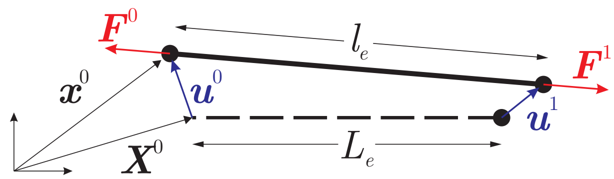
Project #3

assigned: Thursday, October 26th, 2017
due: Thursday, November 9th, 2017, 17:00
please hand in to one of the TAs in LEE N203



Review: variational problems, bars

Let us demonstrate the energy minimization principles discussed in class in a first computational example in 3D.

Consider a *2-node bar* undergoing finite deformations so its two end points move from $(\boldsymbol{X}^0, \boldsymbol{X}^1)$ to $(\boldsymbol{x}^0, \boldsymbol{x}^1)$ and $\boldsymbol{x}^i = \boldsymbol{X}^i + \boldsymbol{u}^i$. The bar stores elastic energy upon stretching with an energy density $W = W(\varepsilon)$ where ε is the axial bar strain, which we assume is constant (based on the linear interpolation discussed in class):

$$\varepsilon = \frac{l - L}{L} \qquad \text{with} \qquad \boldsymbol{l} = \boldsymbol{x}^1 - \boldsymbol{x}^0, \quad \boldsymbol{L} = \boldsymbol{X}^1 - \boldsymbol{X}^0, \quad \text{and} \quad l = |\boldsymbol{l}|, \quad L = |\boldsymbol{L}|.$$

The internal energy of a bar element with cross-sectional area A and initial length L is therefore

$$I_{\text{int},e} = A\,L\,W(\varepsilon).$$

As shown in class, we calculate the resulting (internal) force on each node as

$$\boldsymbol{F}^0_{\text{int},e} = \frac{\partial I_{\text{int},e}}{\partial \boldsymbol{u}^0} = A\,L\frac{\partial W}{\partial \varepsilon}\frac{\partial}{\partial \boldsymbol{u}^0}\frac{l-L}{L} = -A\,\sigma(\varepsilon)\frac{\boldsymbol{l}}{l} \qquad \text{and} \qquad \boldsymbol{F}^1_{\text{int},e} = -\boldsymbol{F}^0_{\text{int},e} = A\,\sigma(\varepsilon)\frac{\boldsymbol{l}}{l}.$$

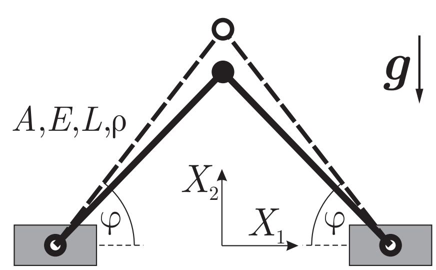
where $\sigma(\varepsilon) = \partial W/\partial\varepsilon$ is the axial stress in the bar.

A simple example of an external force is gravity. In case of a bar, gravitation produces potential energy

$$I_{\text{ext},e} = -\int_0^L \hat{\boldsymbol{t}}\cdot\boldsymbol{u}\,\mathrm{d}x = -\boldsymbol{F}^0_{\text{ext},e}\cdot\boldsymbol{u}^0_e - \boldsymbol{F}^1_{\text{ext},e}\cdot\boldsymbol{u}^1_e \qquad \text{with} \qquad \boldsymbol{F}^0_{\text{ext},e} = \boldsymbol{F}^1_{\text{ext},e} = -\frac{\rho A L \boldsymbol{g}}{2}$$

being the resulting (external) forces on each node. In case of several elements, the total potential energy and the total force on node a are, respectively,

$$I = \sum_e I_{\text{int},e} + I_{\text{ext},e}, \qquad \boldsymbol{F}^a = \sum_e \boldsymbol{F}^a_{\text{int},e} + \boldsymbol{F}^a_{\text{ext},e},$$



Let us investigate the simple nonlinear bar system shown on the right under its own weight (with $\varphi = \pi/4$). The two bottom nodes are fixed and only the x_2-coordinate of the central node's position is free to move.

As discussed in class, we will implement a *1D material model* class (for W and σ), a *2-node bar* class (for $I_{\text{int},e}$ and $\boldsymbol{F}_{\text{int},e}$), and a *constant body force* class (for $I_{\text{ext},e}$ and $\boldsymbol{F}_{\text{ext},e}$). By computing the total energy I and the total vertical force on the top node as functions of the vertical position of the top node, we demonstrate the bistable energy landscape and show that *energy extrema* indeed correspond to *force equilibria*.

(Coding) Problem 1: 1D elastic material model (15 points).

Let us implement the material model for a 1D linear elastic bar. The energy density for *1D elasticity* is given by

$$W(\nabla u) = \frac{E}{2}(\nabla u)^2 \qquad \text{with} \qquad \varepsilon = \nabla u \in \mathbb{R} \quad \text{in 1D.}$$

We would like to implement this material model as a class that computes the energy density W, the axial stress $\sigma = \partial W / \partial \nabla u = E\,\nabla u$, and the associated axial stiffness $\mathbb{C} = \partial^2 W / \partial (\nabla u)^2 = E$.

We use the very same structure as for the 3D material models from Project #2. Note that for consistency we here define the scalar quanitities ∇u, σ and $\mathbb{C}$ as 1×1 matrices (so that we can still pass vectors back and forth between functions).

(Coding) Problem 2: bar element (30 points).

Let us implement the *2-node bar element* described above in our code as a new class `TwoNodeBar` with the following functionality in analogy to the material models:

- The class constructor receives the *undeformed nodal positions* $\{\boldsymbol{X}^0, \boldsymbol{X}^1\}$, a *1D material model* (which it can ask for W, σ, $\mathbb{C}$), and the *element properties* (here, simpy the cross-sectional area A and density ρ). The constructor can, e.g., compute and store L.
- The method `computeEnergy` turns nodal displacements $\{\boldsymbol{u}_e^0, \boldsymbol{u}_e^1\}$ into the total bar energy $I_{\text{int},e}$.
- The method `computeForces` turns nodal displacements $\{\boldsymbol{u}_e^0, \boldsymbol{u}_e^1\}$ into nodal forces $\left\{\boldsymbol{F}_{\text{int},e}^0, \boldsymbol{F}_{\text{int},e}^1\right\}$.

You can test the implementation of your bar class by using the available method `testElementDerivative`, which is analogous to the material model test you wrote for Project #2 and checks if forces are indeed the derivatives of the energy. Please use a general element for testing, i.e., pick some non-trivial nodal locations and not, e.g., simply $(0,0,0)$ and $(1,0,0)$ since the latter could accidentally lead to a passing of the test.

(Coding) Problem 3: external force element (25 points).

In close analogy to the above, let us implement the external forces due to gravity as a new class `BodyForce`, which does the following:

- The class constructor receives a *2-node bar element* (which can be asked for its length and properties) and the body force vector $\boldsymbol{g}$.
- The method `computeEnergy` turns nodal displacements $\{\boldsymbol{u}_e^0, \boldsymbol{u}_e^1\}$ into the potential energy $I_{\text{ext},e}$.
- The method `computeForces` turns nodal displacements $\{\boldsymbol{u}_e^0, \boldsymbol{u}_e^1\}$ into nodal forces $\left\{\boldsymbol{F}_{\text{ext},e}^0, \boldsymbol{F}_{\text{ext},e}^1\right\}$.

Again, you can test your body force class by using the available method `testElementDerivative`.

(Coding) Problem 4: a first "boundary value problem" (25 points).

Let us use the above classes to study the bistable two-bar system sketched above. To this end, in `Main`

(i) create a 1D elastic material model,
(ii) create the two bar elements with correct node locations (using the 1D model),
(iii) create an external force element for each bar (using the bar elements),
(iv) for varying x_2-position of the central node, compute the total energy I (of all bar/external elements) and plot the energy as a function of the top node's x_2-position,
(v) show that the energy I has in fact two minima (which are symmetric if $g = 0$),
(vi) compute the sum of all forces on the central node and show that it vanishes at all minima (and maxima) of the energy.

Hint: Use some positive numbers of your choice for all parameters.

(Coding) Problem 5: linearized kinematics check (5 points).

If the bars are linear (i.e., undergo very small displacements), we can calculate the effective stiffness of the system under vertical loading of the top node, which is

$$k_{\text{eff}} = \frac{EA}{L}.$$

By investigating the curve of the total force F on the top node vs. the top node's vertical position in the absence of gravity ($g = 0$), show that the *slope* is indeed k_{eff} in the undeformed position(s).

total: 100 points