Example Material Model:

strain measure (deformation gradient):

$$\boldsymbol{F} = \boldsymbol{I} + \nabla \boldsymbol{u}$$

energy density (e.g., fiber material):

$$\begin{aligned} W(\boldsymbol{F}) &= \frac{k}{2}\lambda(\boldsymbol{N})^2 \\ &= \frac{k}{2}\boldsymbol{N} \cdot \boldsymbol{C}\boldsymbol{N} \end{aligned}$$

first Piola-Kirchhoff stress tensor:

$$P_{iJ} = \frac{\partial W}{\partial F_{iJ}} = k F_{iI} N_I N_J$$

$$\boldsymbol{P} = k\, \boldsymbol{F}\boldsymbol{N} \otimes \boldsymbol{N}$$

incremental tangent tensor:

$$\mathbb{C}_{iJkL} = \frac{\partial P_{iJ}}{\partial F_{kL}} = k\, \delta_{ik} N_J N_L$$

Note that we use **vector notation**:

$$\tilde{\boldsymbol{F}} = \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ \vdots \\ F_{33} \end{pmatrix}, \qquad \tilde{\boldsymbol{P}} = \begin{pmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{21} \\ \vdots \\ P_{33} \end{pmatrix}, \qquad \tilde{\mathbb{C}}_{ij} = \frac{\partial \tilde{P}_i}{\partial \tilde{F}_j}$$

important for the **derivative test**:

$$P_{iJ} = \frac{\partial W}{\partial (\nabla \boldsymbol{u})_{iJ}}, \qquad \mathbb{C}_{iJkL} = \frac{\partial P_{iJ}}{\partial (\nabla \boldsymbol{u})_{kL}}$$

example derivative test:

$$\tilde{P}_2 = \frac{\partial W}{\partial \tilde{F}_2} \approx \frac{W\left[\nabla \boldsymbol{u} + \begin{pmatrix} 0 \\ h \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right] - W\left(\nabla \boldsymbol{u}\right)}{h} \quad \text{and} \quad \tilde{\mathbb{C}}_{23} = \frac{\partial \tilde{P}_2}{\partial \tilde{F}_3} \approx \frac{\tilde{P}_2\left[\nabla \boldsymbol{u} + \begin{pmatrix} 0 \\ 0 \\ h \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right] - \tilde{P}_2\left(\nabla \boldsymbol{u}\right)}{h}$$